# The strong fractional choice number and the strong fractional paint number of graphs 

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#### Abstract

This paper studies the strong fractional choice number $c h_{f}^{s}(G)$ and the strong fractional paint number $\chi_{f, P}^{s}(G)$ of a graph $G$. We prove that these parameters of any finite graph are rational numbers. On the other hand, for any positive integers $p, q$ satisfying $2 \leq \frac{2 p}{2 q+1} \leq\left\lfloor\frac{p}{q}\right\rfloor$, there exists a graph $G$ with $c h_{f}^{s}(G)=\chi_{f, P}^{s}(G)=\frac{p}{q}$. The relationship between $\chi_{f, P}^{s}(G)$ and $c h_{f}^{s}(G)$ is explored. We prove that the gap $\chi_{f, P}^{s}(G)-c h_{f}^{s}(G)$ can be arbitrarily large. The strong fractional choice number of a family $\mathcal{G}$ of graphs is the supremum of the strong fractional choice number of graphs in $\mathcal{G}$. Let $\mathcal{P}$ denote the class of planar graphs and $\mathcal{P}_{k_{1}, \ldots, k_{q}}$ denote the class of planar graphs without $k_{i}$-cycles for $i=1, \ldots, q$. We prove that $3+\frac{1}{2} \leq c h_{f}^{s}\left(\mathcal{P}_{4}\right) \leq 4$, $c h_{f}^{s}\left(\mathcal{P}_{k}\right)=4$ for $k \in\{5,6\}, 3+\frac{1}{12} \leq c h_{f}^{s}\left(\mathcal{P}_{4,5}\right) \leq 4$ and $\operatorname{ch}_{f}^{s}(\mathcal{P}) \geq 4+\frac{1}{3}$. The last result improves the lower bound $4+\frac{2}{9}$ in [Zhu, multiple list colouring of planar graphs, Journal of Combin. Th. Ser. B,122(2017),794-799].


## 1 Introduction

Suppose $G$ is a graph, $f$ and $g$ are two functions from $V(G)$ to $\mathbb{N}$, with $f(v) \geq g(v)$ for every $v \in V(G)$. An $f$-assignment of $G$ is a mapping $L$ which assigns to each vertex $v$ of $G$ a set $L(v)$ of $f(v)$ integers as permissible colours. A $g$-fold colouring of $G$ is a mapping $S$ which assigns to each vertex $v$ of $G$ a set $S(v)$ of $g(v)$ colours such that for any two adjacent vertices $u$ and $v, S(u) \cap S(v)=\varnothing$.

[^0]Given a list assignment $L$ of $G$, an $(L, g)$-colouring of $G$ is a $g$-fold colouring $S$ of $G$ such that for each $v, S(v) \subseteq L(v)$. We say $G$ is $(L, g)$-colourable if there exists an $(L, g)$-colouring of $G$. For a positive integer $a$, we write $f \equiv a$ if $f$ is the constant function with $f(v)=a$ for every vertex $v$. If $g \equiv b$, then $(L, g)$-colourable is denoted by $(L, b)$-colourable. If $L(v)=\{1,2, \ldots, a\}$ for each $v \in V(G)$, then $(L, b)$-colourable is called $(a, b)$-colourable. The $b$-fold chromatic number $\chi_{b}(G)$ of $G$ is the least $k$ such that $G$ is $(k, b)$-colourable. The 1 -fold chromatic number of $G$ is also called the chromatic number of $G$ and denoted by $\chi(G)$. The fractional chromatic number of $G$ is defined as $\chi_{f}(G)=\inf \left\{\frac{a}{b}: G\right.$ is $(a, b)$-colourable $\}$.

Similarly, we say $G$ is $(f, g)$-choosable if for every $f$-list assignment $L, G$ is $(L, g)$ colourable.

- If $f \equiv a$ and $g \equiv b$, then $(f, g)$-choosable is called $(a, b)$-choosable.
- If $b=1$, then $(f, 1)$-choosable is called $f$-choosable.
- ( $a, 1$ )-choosable is also called $a$-choosable.

The choice number $\operatorname{ch}(G)$ of $G$ is the minimum $k$ such that $G$ is $k$-choosable. The $b$ fold choice number $\operatorname{ch}_{b}(G)$ of $G$ is the minimum $k$ such that $G$ is $(k, b)$-choosable. The fractional choice number of $G$ is defined as $c h_{f}(G)=\inf \left\{\frac{a}{b}: G\right.$ is $(a, b)$-choosable $\}$.

List colouring of graphs was introduced in the 1970s by Vizing [23] and independently by Erdős, Rubin and Taylor [8]. The subject offers a large number of challenging problems and has attracted an increasing attention since 1990. Readers are referred to [21] for a comprehensive survey on results and open problems.

The paint number of a graph is a variation of the choice number of a graph. Given two functions $f$ and $g$ from $V(G)$ to $N$, with $f(v) \geq g(v)$ for all $v \in V(G)$, the $(f, g)$-painting game on $G$ is played by two players: Lister and Painter. Initially, each vertex $v$ is given $f(v)$ tokens and is uncolourred. On each round, Lister selects a set $U$ of vertices and removes one token from each chosen vertex. Painter chooses an independent subset $I$ of $U$ and colours each vertex of $I$ with a new colour. If at the end of some round, there is a vertex $v$ with no tokens left and coloured with less than $g(v)$ colours, then Lister wins the game. If at the end of some round, each vertex $v$ is coloured with $g(v)$ colours, then Painter wins the game. We say $G$ is $(f, g)$-paintable if Painter has a winning strategy for the $(f, g)$-painting game.

- If $f \equiv a$ and $g \equiv b$, , then $(f, g)$-paintable is called $(a, b)$-paintable.
- If $b=1$, then $(f, b)$-paintable is called $f$-paintable.
- ( $a, 1$ )-paintable is also called $a$-paintable.

The $b$-fold paint number $\chi_{b, P}(G)$ is the minimum $k$ such that $G$ is $(k, b)$-paintable, and $\chi_{1, P}(G)$ is called the paint number (or the online choice number) of $G$, and denoted by $\chi_{P}(G)$. The fractional paint number of $G$ is defined as $\chi_{f, P}(G)=\inf \left\{\frac{a}{b}\right.$ : $G$ is ( $a, b$ )-paintable $\}$.

It follows from the definition that for any graph $G, \chi_{f}(G) \leq c h_{f}(G) \leq \chi_{f, P}(G)$. It was proved in [1] that $\chi_{f}(G)=c h_{f}(G)$ for every graph $G$, and proved in [10] that $\chi_{f}(G)=\chi_{f, P}(G)$ for every graph $G$. So the fractional chromatic number, the fractional choice number and the fractional paint number of a graph are a same invariant. As a variation of the fractional choice number, the concept of strong fractional choice number of a graph was introduced in [28].

Definition 1.1 Assume $G$ is a graph and $r$ is a real number. We say $G$ is strongly fractional $r$-choosable (respectively, strongly fractional $r$-paintable or strongly fractional $r$-colourable ) if $G$ is ( $a, b$ )-choosable (respectively, $(a, b)$-paintable, or $(a, b)$-colourable) for any $(a, b)$ for which $\frac{a}{b} \geq r$. The strong fractional choice number of $G$ is defined as

$$
\operatorname{ch}_{f}^{s}(G)=\inf \{r \in \mathbf{R}: G \text { is strongly fractional } r \text {-choosable }\} .
$$

The strong fractional paint number of $G$ is defined as

$$
\chi_{f, P}^{s}(G)=\inf \{r \in \mathbf{R}: G \text { is strongly fractional } r \text {-paintable }\} .
$$

We also define the strong fractional chromatic number of $G$ as

$$
\chi_{f}^{s}(G)=\inf \{r \in \mathbf{R}: G \text { is strongly fractional } r \text {-colourable }\} .
$$

The strong fractional choice number, the strong fractional paint number and the strong fractional chromatic number of a class $\mathcal{G}$ of graphs is defined as
$c h_{f}^{s}(\mathcal{G})=\sup \left\{c h_{f}^{s}(G): G \in \mathcal{G}\right\}, \chi_{f, P}^{s}(\mathcal{G})=\sup \left\{\chi_{f, P}^{s}(G): G \in \mathcal{G}\right\}, \chi_{f}^{s}(\mathcal{G})=\sup \left\{\chi_{f}^{s}(G): G \in \mathcal{G}\right\}$.
The paper studies basic properties of these parameters, and upper and lower bounds for these parameters for special families of graphs. In Section 2, we show that both $c h_{f}^{s}(G)$ and $\chi_{f, P}^{s}(G)$ are rational numbers. In Section 3, we study the problem as which rational numbers are the strong fractional choice number and strong fractional paint number of graphs. We conjecture that for every rational $r \geq 2$, there exists a graph $G$ with $c h_{f}^{s}(G)=r$ and a graph $G$ with $\chi_{f, P}^{s}(G)=r$, and prove that for any positive integers $p, q$, where $p \geq 2 q$ satisfying $\frac{2 p}{2 q+1} \leq\left\lfloor\frac{p}{q}\right\rfloor$, there exists a graph $G$ with $c h_{f}^{s}(G)=\chi_{f, P}^{s}(G)=\frac{p}{q}$. In Section 4, we show that the gap $\chi_{f, P}^{s}(G)-c h_{f}^{s}(G)$ can be arbitrarily large. In Section 5 , we study upper and lower bounds for the strong fractional choice number of planar graphs. Let $\mathcal{P}$ denote the family of planar graphs and for positive integers $k_{1}, k_{2}, \ldots, k_{q}$, let $\mathcal{P}_{k_{1}, \ldots, k_{q}}$ be the family of planar graphs without cycles of lengths $k_{i}$ for $i=1, \ldots, q$. It was proved in [28] that $\operatorname{ch}_{f}^{s}(\mathcal{P}) \geq 4+\frac{2}{9}$. We improve this result and prove that $c h_{f}^{s}(\mathcal{P}) \geq 4+\frac{1}{3}$. It is also shown that $3+\frac{1}{2} \leq c h_{f}^{s}\left(\mathcal{P}_{4}\right) \leq 4, c h_{f}^{s}\left(\mathcal{P}_{k}\right)=4$ for $k \in\{5,6\}$, and $3+\frac{1}{12} \leq c h_{f}^{s}\left(\mathcal{P}_{4,5}\right) \leq 4$. Some open problems are posed in Section 6.

## 2 Basic Properties

Lemma 2.1 gives an alternative definitions of $c h_{f}^{s}(G)$ and $\chi_{f, P}^{s}(G)$.
Lemma 2.1 For any graph $G$,

$$
\operatorname{ch}_{f}^{s}(G)=\sup \left\{\frac{\operatorname{ch}_{k}(G)-1}{k}: k \in \mathbf{N}\right\}, \chi_{f, P}^{s}(G)=\sup \left\{\frac{\chi_{k, P}(G)-1}{k}: k \in \mathbf{N}\right\} .
$$

Proof. Let $r=\sup \left\{\frac{c h_{k}(G)-1}{k}: k \in \mathbf{N}\right\}$. Then for any $\epsilon>0$, there is an integer $k$ such that $(r-\epsilon) k<c h_{k}(G)-1$. Thus $\lceil(r-\epsilon) k\rceil<c h_{k}(G)$ and $G$ is not $\left.\lceil(r-\epsilon) k\rceil, k\right)$-choosable. Therefore, $c h_{f}^{s}(G) \geq r-\epsilon$ for any $\epsilon>0$, which implies that $c h_{f}^{s}(G) \geq r$. On the other hand, for any $\epsilon>0$, for any integer $k,\lceil(r+\epsilon) k\rceil \geq c h_{k}(G)$. Hence $G$ is $(\lceil(r+\epsilon) k\rceil, k)$ choosable. So $c h_{f}^{s}(G) \leq r+\epsilon$ for any $\epsilon>0$, which implies that $c h_{f}^{s}(G) \leq r$. Therefore $c h_{f}^{s}(G)=r$. The other part of Lemma 2.1 is proved similarly.

The following lemma was proved in [10]. For the completeness of this paper, we present a short proof.

Lemma 2.2 Assume $G$ is a finite graph. Then for any $\epsilon>0$, there is a constant $k_{0}$ such that for any $k \geq k_{0}, \frac{c h_{k}(G)}{k} \leq \frac{\chi_{k, P}(G)}{k} \leq \chi_{f}(G)+\epsilon$.

Proof. Assume $\chi_{f}(G)=a / b$ and $\phi$ is a $b$-fold colouring of $G$ using colours $\{1,2, \ldots, a\}$ ( $a, b$ need not be coprime). Assume $k>\frac{a^{2 V(G) \mid}}{\epsilon}$ and let $m=k\left(\frac{a}{b}+\epsilon\right.$ ) (for simplicity, we may choose $\epsilon$ so that $k\left(\frac{a}{b}+\epsilon\right)$ is an integer). It suffices to show that Painter has a winning strategy for the ( $m, k$ )-painting game on $G$.

For $i=1,2, \ldots$, assume the set chosen by Lister at the $i$ th round is $U_{i}$. Let

$$
t_{i}=\left|\left\{j \leq i: U_{j}=U_{i}\right\}\right|,
$$

and let $\tau_{i} \in\{1,2, \ldots, a\}$ be the unique integer for which $\tau_{i} \cong t_{i} \bmod a$. Painter's strategy is to colour all the vertices in the set $I_{i}=\phi^{-1}\left(\tau_{i}\right) \cap U_{i}$ in the $i$ th round. As $\phi^{-1}\left(\tau_{i}\right)$ is an independent set, $I_{i}$ is an independent set.

We shall show that this is a winning strategy for Painter, i.e., when the game ends, every vertex will be coloured by at least $k$ colours.

Assume to the contrary that at the end of some round, say at the end of the $i$ th round, a vertex $v$ has no token left (hence $v$ has been chosen $m=k\left(\frac{a}{b}+\epsilon\right)$ ) times by Lister) and is coloured in $k(v)<k$ rounds.

For each subset $U$ of $V(G)$ and for each $t \in\{1,2, \ldots, a\}$, let

$$
(U, t)=\left\{j \leq i: U_{j}=U, \tau_{j}=t\right\}
$$

By the strategy, for each $j \leq i$, if $j \in(U, t), v \in U$ and $t \in \phi(v)$, then $v$ is coloured in round $j$. Therefore,

$$
k(v)=\sum_{v \in U, t \in \phi(v)}|(U, t)| .
$$

For each subset $U$ of $V(G)$, let $t_{U}=\left|\left\{j \leq i: U_{j}=U\right\}\right|$. It follows from the choice of colour $\tau_{j}$ that for any colour $t$, either $|(U, t)|=\left\lfloor\frac{t_{U}}{a}\right\rfloor$ or $|(U, t)|=\left\lceil\frac{t_{U}}{a}\right\rceil$. Therefore

$$
|(U, t)| \geq \frac{t_{U}}{a}-1
$$

Note that $\sum_{v \in U} t_{U}=m$ is the total number of times vertex $v$ is chosen by Lister. Since $\phi(v)$ is a $b$-subset of $\{1,2, \ldots, a\}$, we conclude that

$$
k(v)=\sum_{v \in U, t \epsilon \phi(v)}|(U, t)| \geq b \sum_{v \in U}\left(\frac{t_{U}}{a}-1\right) \geq \frac{b m}{a}-b 2^{|V(G)|}=\frac{k(a / b+\epsilon) b}{a}-b 2^{|V(G)|} \geq k,
$$

a contradiction.

Theorem 2.3 For any finite graph $G, c h_{f}^{s}(G)$ and $\chi_{f, P}^{s}(G)$ are rational numbers.
Proof. If $\frac{\chi_{k, P}(G)-1}{k} \leq \chi_{f}(G)$ for every positive integer $k$, then it follows from Lemma 2.1 that $\chi_{f, P}^{s}(G) \leq \chi_{f}(G)$. Since $\chi_{f}(G) \leq \chi_{f, P}^{s}(G)$, we conclude that $\chi_{f, P}^{s}(G)=\chi_{f}(G)$, which is a rational number.

Assume there is an integer $k_{0}$ such that $\frac{\chi_{k_{0}, P}(G)-1}{k_{0}}>\chi_{f}(G)$. Let $\epsilon=\frac{\chi_{k_{0}, P}(G)-1}{k_{0}}-\chi_{f}(G)>$ 0. By Lemma 2.2, there is a constant $k_{1} \geq k_{0}$ such that for $k \geq k_{1}, \frac{\chi_{k, P}(G)}{k} \leq \chi_{f}(G)+\epsilon$. Hence

$$
\sup \left\{\frac{\chi_{k, p}(G)-1}{k}: k \in \mathbf{N}, k \geq k_{1}\right\} \leq \frac{\chi_{k_{0}, P}(G)-1}{k_{0}} .
$$

Therefore

$$
\chi_{f, P}^{s}(G)=\sup \left\{\frac{\chi_{k, P}(G)-1}{k}: k \in \mathbf{N}\right\}=\max \left\{\frac{\chi_{k, P}(G)-1}{k}: 1 \leq k \leq k_{1}\right\}
$$

is a rational number. Moreover, the supremum in Lemma 2.1 is attained.
The part of the lemma concerning $c h_{f}^{s}(G)$ is proved similarly.
Lemma 2.1 gives an alternate definition of $c h_{f}^{s}(G)$ and $\chi_{f, P}^{s}(G)$. It follows from the proof of Theorem 2.3 that either $c h_{f}^{s}(G)=\chi_{f}(G)$ or the supremum in the definition $c h_{f}^{s}(G)=\sup \left\{\frac{c h_{k}(G)-1}{k}: k \in \mathbf{N}\right\}$ is attained. However, the infimum in the definition $c h_{f}^{s}(G)=\inf \{r: G$ is strongly fractional $r$-choosable $\}$ may be not attained even if $c h_{f}^{s}(G) \neq \chi_{f}(G)$. Similarly, either $\chi_{f, P}^{s}(G)=\chi_{f}(G)$ or the sumpremum in the definition $\chi_{f, P}^{s}(G)=\sup \left\{\frac{\chi_{k, P}(G)-1}{k}: k \in \mathbf{N}\right\}$ is attained. But the infimum in the definition $\chi_{f, P}^{s}(G)=\inf \{r: G$ is strongly fractional $r$-paintable $\}$ may be not attained even if $\chi_{f, P}^{s}(G) \neq \chi_{f}(G)$.

## 3 Constructing graphs with given $c h_{f}^{s}(G)$ and $\chi_{f, P}^{s}(G)$

By Theorem 2.3, for any finite graph $G, c h_{f}^{s}(G)$ and $\chi_{f, P}^{s}(G)$ are rational numbers. A natural question is whether every rational number $r \geq 2$ is the strong fractional choice (paint) numbers of a graph. We conjecture that the answer is yes. In this section, for some rational numbers $p / q$, we construct graphs $G$ with $\chi_{f, P}^{s}(G)=c h_{f}^{s}(G)=p / q$.

Given a graph $G$, a subset $S$ of $G$ and two graphs $H_{1}$ and $H_{2}$, let $G\left[S: H_{1}, H_{2}\right]$ be the graph with

$$
\begin{aligned}
V\left(G\left[S: H_{1}, H_{2}\right]\right) & =\left\{(u, v): u \in S \text { and } v \in V\left(H_{1}\right), \text { or } u \in V(G) \backslash S \text { and } v \in V\left(H_{2}\right)\right\}, \\
E\left(G\left[S: H_{1}, H_{2}\right]\right) & =\left\{(u, v)\left(u^{\prime}, v^{\prime}\right): u u^{\prime} \in E(G), \text { or } u=u^{\prime} \in S, v v^{\prime} \in E\left(H_{1}\right)\right. \\
& \text { or } \left.u=u^{\prime} \in V(G) \backslash S, v v^{\prime} \in E\left(H_{2}\right)\right\} .
\end{aligned}
$$

Note that if $S=V(G)$ or $H_{1}=H_{2}$, then $G\left[S: H_{1}, H_{2}\right]=G\left[H_{1}\right]$ is the lexicographic product of $G$ and $H_{1}$. In the rest of this section, we let $G_{n, m, k}$ denote the graph $C_{2 k+1}$ [I: $K_{n}, K_{m}$ ], where $I$ is a maximum independent set of $C_{2 k+1}$, see Fig. 1 for the example of $G_{6,4,3}$.


Fig. 1. $G_{6,4,3}$

Theorem 3.1 For any positive integer $n, m, k$ with $n \geq m$,

$$
c h_{f}^{s}\left(G_{n, m, k}\right)=\chi_{f, P}^{s}\left(G_{n, m, k}\right)=\chi_{f}\left(G_{n, m, k}\right)=n+m+\frac{m}{k} .
$$

Proof. Assume the vertices of $C_{2 k+1}$ are $\left(v_{0}, v_{1}, \ldots v_{2 k}\right)$ in this cyclic order, and assume that $I=\left\{v_{1}, v_{3}, \ldots, v_{2 k-1}\right\}$ and $G_{n, m, k}=C_{2 k+1}\left[I: K_{n}, K_{m}\right]$.

For $s \in\{0,1, \ldots, 2 k\}$, let

$$
V_{s}=\left\{(x, y) \in V\left(G_{n, m, k}\right): x=v_{s}\right\} .
$$

For any vertex set $S \subseteq V\left(G_{n, m, k}\right)$, let

$$
\partial(S)=\left\{s: S \cap V_{s} \neq \varnothing\right\} .
$$

It is clear that $\alpha\left(G_{n, m, k}\right)=\alpha\left(C_{2 k+1}\right)=k$. It is well known that $\chi_{f}(G) \geq \frac{|V(G)|}{\alpha(G)}$ for any graph $G$, so we have

$$
\chi_{f}\left(G_{n, m, k}\right) \geq \frac{n k+m(k+1)}{k}=n+m+\frac{m}{k} .
$$

Since $\chi_{f}(G) \leq c h_{f}^{s}(G) \leq \chi_{f, p}^{s}(G)$ for any graph $G$, it suffices to show that $\chi_{f, P}^{s}\left(G_{n, m, k}\right) \leq$ $n+m+\frac{m}{k}$. For this purpose, we will show that for any $\frac{a}{b} \geq n+m+\frac{m}{k}, G_{n, m, k}$ is ( $a, b$ )paintable. In the following, we present a winning strategy for Painter in the ( $a, b$ )painting game on $G_{n, m, k}$.

For simplicity, we assume that if a vertex $v$ has been coloured with $b$ colours, then Lister will not choose $v$ in later moves.

For $i=1,2, \cdots$, we denote by $U_{i}$ the set of vertices chosen by Lister, and by $X_{i}$ the independent set contained in $U_{i}$, coloured by Painter at the $i$ th round.

Assume Lister has chosen $U_{i}$. We describe a strategy for Painter to choose the independent set $X_{i}$.

We consider two cases.
Case $1 \partial\left(U_{i}\right)=\{0,1,2, \cdots, 2 k\}$.
Let $t_{i}=\mid\left\{j: j \leq i\right.$ and $\left.\partial\left(U_{j}\right)=\{0,1, \cdots, 2 k\}\right\} \mid$. Let $\tau_{i}$ be the unique integer in $\{0,1, \cdots, 2 k\}$ which is congruent to $t_{i}$ modulo $2 k+1$. Then $X_{i}$ is any independent set contained in $U_{i}$ with $\partial\left(X_{i}\right)=\left\{\tau_{i}, \tau_{i}+2, \cdots, \tau_{i}+2 k-2\right\}$ (the summation in the indices are modulo $2 k+1$ ).
Case $2 \partial\left(U_{i}\right) \neq\{0,1, \cdots, 2 k\}$.
Painter traverse the sets $V_{0}, V_{1}, \ldots, V_{2 k}$ of $G_{n, m, k}$ one by one in cyclic order along the clockwise direction, starting at an arbitrary set $V_{s}$ for which $s \notin \partial\left(U_{i}\right)$, and choose an independent set $X_{i}$ as follows: Initially, $X_{i}=\varnothing$ and vertices will be added to $X_{i}$ in the process. When we traverse to $V_{j}$, if $U_{i} \cap V_{j} \neq \varnothing$ and $X_{i} \cap V_{j-1}=\varnothing$, then add a vertex from $V_{j} \cap U_{i}$ to $X_{i}$. Otherwise, do not add any vertex from $V_{j}$ into $X_{i}$ (again the calculation in the indices are modulo $2 k+1$ ).

It follows from the definition that in both cases, the set $X_{i}$ is an independent set of $G$ contained in $U_{i}$. We shall show that this is a winning strategy for Painter. First we have the following claim.
Claim 3.1 Case 1 happens at most $\frac{m b(2 k+1)}{k}$ times.
Proof. Every $2 k+1$ times Case 1 happens, $k$ vertices (not necessarily distinct) in $V_{0}$ will be coloured. However, all the vertices from $V_{0}$ need to be coloured $m b$ times in total. Thus the claim holds.

Assume to the contrary that at the end of some round, say at the end of the $r$ th round, a vertex $v \in V_{j}$ has no token left and is coloured at most $b-1$ times.

Let $B_{r}(v)=\left\{i \leq r: v \in U_{r}-X_{r}\right\}$, which is the collection of rounds of the game that $v$ is chosen by Lister but not coloured by Painter at the end of $r$ th round. Then $\left|B_{r}(v)\right| \geq a-b+1$. It follows from the strategy that for each $i \in B_{r}(v)$, one of the following holds:

- $V_{j-1} \cap X_{i} \neq \varnothing$, i.e., some vertex in $V_{j-1}$ is coloured in this round.
- $\partial\left(U_{i}\right)=\{0,1, \cdots, 2 k\}$ and $\tau_{i}=j+1$.
- $V_{j} \cap X_{i} \neq \varnothing$ and $v \notin X_{i}$, i.e., some another vertex from $V_{j}$ is cloured at this round.

Since $\left|V_{j-1} \cup\left(V_{j} \backslash v\right)\right| \leq m+n-1$ and each vertex in $V_{j-1} \cup\left(V_{j} \backslash v\right)$ is coloured at most $b$ times, we conclude that

$$
\begin{align*}
\mid\left\{i \leq r: \partial\left(U_{i}\right)\right. & =\{0,1, \cdots, 2 k\}, \tau_{i}=j+1 \mid \\
& \geq a-b+1-(m+n-1) b \\
& =a-(m+n) b+1 . \tag{3.1}
\end{align*}
$$

It follows from the definition of $\tau_{i}$ that

$$
\begin{equation*}
\left\lvert\,\left\{i \leq r: \partial\left(U_{i}\right)=\{0,1, \cdots, 2 k\}, \tau_{i}=j+1 \left\lvert\, \leq\left\lceil\frac{t_{i}}{2 k+1}\right\rceil\right.\right.\right. \tag{3.2}
\end{equation*}
$$

By Claim 3.1, $t_{i} \leq \frac{(2 k+1) m b}{k}$. Hence

$$
\left\lceil\frac{t_{i}}{2 k+1}\right\rceil \leq\left\lceil\frac{(2 k+1) m b}{k(2 k+1)}\right\rceil=\left\lceil\frac{m b}{k}\right\rceil,
$$

Combining the inequality above with Inequalities (3.1) and (3.2), we have

$$
\frac{m b}{k}>a-(m+n) b
$$

that is

$$
\frac{a}{b}<m+n+\frac{m}{k},
$$

contrary to our assumption.
By setting $m=n=1$ in Theorem 3.1, we have
Corollary $3.2 \chi_{f}\left(C_{2 q+1}\right)=c h_{f}^{s}\left(C_{2 q+1}\right)=\chi_{f, P}^{s}\left(C_{2 q+1}\right)=2+\frac{1}{q}$.
Proposition 3.3 For any positive integers $p, q$ with $p \geq 2 q$ and $\frac{2 p}{2 q+1} \leq\left\lfloor\frac{p}{q}\right\rfloor$, there exists a graph $G$ such that

$$
\operatorname{ch}_{f}^{s}(G)=\chi_{f, p}^{s}(G)=\chi_{f}(G)=\frac{p}{q}
$$

Proof. Let $a=\left\lfloor\frac{p}{q}\right\rfloor(q+1)-p$ and $b=p-q\left\lfloor\frac{p}{q}\right\rfloor$. As $p \geq 2 q$, we first have that $a, b>0$. On the other hand, note that by the condition $\frac{2 p}{2 q+1} \leq\left\lfloor\frac{p}{q}\right\rfloor$, we have that

$$
a-b=(2 q+1)\left\lfloor\frac{p}{q}\right\rfloor-2 p \geq 0 .
$$

So by setting $a=n$ and $b=m$ in Theorem 3.1, we have that

$$
c h_{f}^{s}\left(G_{a, b, q}\right)=\chi_{f, P}^{s}\left(G_{a, b, q}\right)=a+b+\frac{b}{q}=\frac{p}{q} .
$$

This completes the proof.
According to Proposition 3.3, if $q \leq 2$, then for any $p \geq 2 q$, there exists a graph $G$ with $c h_{f}^{s}(G)=c h_{f, P}^{s}(G)=p / q$. If $q=3$, the only cases $p \geq 2 q$ unknown are $p=8$ and $p=11$.

## 4 Relation among $\chi_{f}^{s}(G), \operatorname{ch}_{f}^{s}(G)$ and $\chi_{f, P}(G)$

It follows from the definitions that for any graph $G$,

$$
\chi_{f}^{s}(G) \leq c h_{f}^{s}(G) \leq \chi_{f, P}^{s}(G) .
$$

The gap $c h_{f}^{s}(G)-\chi_{f}^{s}(G)$ can be arbitrarily large, as for complete bipartite graphs, we have $\chi_{f}^{s}\left(K_{n, n}\right)=2$ and $c h_{f}^{s}\left(K_{n, n}\right) \geq c h\left(K_{n, n}\right)-1 \geq \log _{2} n-(2+o(1)) \log _{2} \log _{2} n$.

In the following we show that the difference $\chi_{f, P}^{s}\left(K_{n, n}\right)-c h_{f}^{s}\left(K_{n, n}\right)$ also goes to infinity with $n$. It was proved in [4] that for $n \geq 2^{k+3}$, the graph $K_{n, n}$ is not $k$-paintable, so $\chi_{P}\left(K_{n, n}\right) \geq \log _{2} n-4$. Therefore, $\chi_{f, P}^{s}\left(K_{n, n}\right) \geq \chi_{P}\left(K_{n, n}\right)-1 \geq \log _{2} n-5$. So it suffices to show that $\log _{2} n-\chi_{f}^{s}\left(K_{n, n}\right)$ goes to infinity with $n$.

A $k$-uniform hypergraph $H=(V, E)$ consists of a vertex set $V$ and an edge set $E$, where each edge $e \in E$ is a $k$-subset of $V$. A proper $c$-colouring of $H$ is a mapping $\phi: V \mapsto\{1,2, \cdots, c\}$ such that no edge is monochromatic. Hypergraph 2-colourability, which is an alternate formulation of list colouring of complete bipartite graphs, is a central problem in combinatorics that has been studied in many papers (see [2, 7, 17], etc.) Corresponding to $b$-fold list colouring of complete bipartite graphs, we define a $b$-proper 2 -colouring of a hypergraph $H$ as a mapping $\phi: V(H) \rightarrow\{1,2\}$ such that for each edge $e$ of $H$, for each $i \in\{1,2\},\left|\phi^{-1}(i) \cap e\right| \geq b$, i.e., each edge contains at least $b$ vertices of each colour. We say that $H$ is $b$-proper 2 -colourable if $H$ has a $b$-proper 2 -colouring. Let $m(k, b)$ denote the minimum possible number of edges of a $k b$-uniform hypergraph which is not $b$-proper 2 -colourable.

Lemma 4.1 Every p-uniform hypergraph with $m$ edges satisfying $m \sum_{i=0}^{b-1}\binom{p}{i} \frac{1}{2^{p-1}}<1$ has a b-proper 2-colouring. As a result,

$$
m(k, b) \geq\left(\sum_{i=0}^{b-1}\binom{k s}{i}\right)^{-1} 2^{k b-1}
$$

Proof. Let $H=(V, E)$ be a $p$-uniform hypergraph satisfying the condition.
Colour the vertices of $H$ randomly by two colours with equal probability. We say an edge $e$ is bad if one colour is used on less than $b$ vertices in $e$. For each edge $e$, let $A_{e}$ be the event that $e$ is bad. Then

$$
\operatorname{Pr}\left(A_{e}\right)=2 \sum_{i=0}^{b-1}\binom{p}{i} \frac{1}{2^{p}}=\sum_{i=0}^{b-1}\binom{p}{i} \frac{1}{2^{p-1}} .
$$

Therefore,

$$
\operatorname{Pr}\left(\bigvee_{e \in E} A_{e}\right) \leq \sum_{e \in E} \operatorname{Pr}\left(A_{e}\right)=m \operatorname{Pr}\left(A_{e}\right)<1 .
$$

So there exists a colouring such that there is no bad edges.

Lemma 4.2 Let $G$ be a bipartite graph with $n$ vertices. When $n$ is big enough, the b-fold choice number $\operatorname{ch}_{b}(G)$ satisfies the following,

$$
\frac{c h_{b}(G)}{b}<\frac{1}{b} \log _{2} n+\left(1-\frac{1}{b}\right) \log _{2} \log _{2} n+O(1)
$$

Proof. Let $k=\frac{1}{b} \log _{2} n+\left(1-\frac{1}{b}\right) \log _{2} \log _{2} n+C$, for some constant $C$, we shall prove that $G$ is $(k b, b)$-choosable for any $b \geq 2$.

Assume $G=(X \cup Y, E)$ be a bipartite graph with $X$ and $Y$ being the two parts, and $L$ is a list assignment of $G$ with $|L(v)|=k b$ for each $v \in V(G)$. We construct a $k b$-uniform hypergraph $H$ with $V(H)=\bigcup_{v \in V(G)} L(v)$, and $E(H)=\{L(v): v \in V(G)\}$. So $|E(H)|=|V(G)|=n$.

Observe that if $H$ has a $b$-proper 2-colouring, then $G$ is $(L, m)$-colourable. Indeed, each vertex is either labeled with red or blue in the $b$-proper 2-colouring of $H$. Then for each vertex $v \in V(G)$, we can choose $b$ colours with label red for $v$ if $v \in X$, and choose $b$ colours with label blue for it if $v \in Y$.

Now, it suffices to prove that $H$ satisfies the condition in Lemma 4.1, so we only need to verify that

$$
n \sum_{i=0}^{b-1}\binom{k b}{i} \frac{1}{2^{k b-1}}<1
$$

The case that $b=1$ was proved in [7]. If $b=2$, then we have $k b=\log _{2} n+\log _{2} \log _{2} n+2 C$, so,

$$
\begin{aligned}
n \sum_{i=0}^{b-1}\binom{k b}{i} \frac{1}{2^{k b-1}} & =n\left(\log _{2} n+\log _{2} \log _{2} n+2 C+1\right) \frac{1}{2^{\log _{2} n+\log _{2} \log _{2} n+2 C-1}} \\
& =\left(\log _{2} n+\log _{2} \log _{2} n+2 C+1\right) \frac{1}{2^{\log _{2} \log _{2} n+2 C-1}} \\
& =\frac{2^{t}+t+2 C+1}{2^{t+2 C-1}},
\end{aligned}
$$

where $t=\log _{2} \log _{2} n$. When $n$ is large enough and hence $t$ is large enough, and $C \geq 1$, we have $n \sum_{i=0}^{b-1}\binom{k b}{i} \frac{1}{2^{k b-1}}<1$. Similarly, we can verify the case for $b=3$.

Assume $b \geq 4$. Using the inequality $\binom{n}{k}<\left(\frac{e n}{k}\right)^{k}$, we have

$$
\begin{aligned}
n \sum_{i=0}^{b-1}\binom{k b}{i} \frac{1}{2^{k b-1}} & \leq n b\left(\frac{e\left(\log _{2} n+(b-1) \log _{2} \log _{2} n+b C\right)}{b-1}\right)^{b-1} \frac{1}{2^{\log _{2} n+(b-1) \log _{2} \log _{2} n+b C-1}} \\
& =b\left(\frac{e\left(\log _{2} n+(b-1) \log _{2} \log _{2} n+b C\right)}{b-1}\right)^{b-1} \frac{1}{2^{(b-1) \log _{2} \log _{2} n+b C-1}} \\
& =\frac{b}{2^{b C-1}}\left(\frac{e\left(\log _{2} n+(b-1) \log _{2} \log _{2} n+b C\right)}{(b-1) \log _{2} n}\right)^{b-1} .
\end{aligned}
$$

Again when $n$ is large enough, we have $n \sum_{i=0}^{b-1}\binom{k b}{i} \frac{1}{2^{k b-1}}<1$. This finishes the proof of the lemma.

In 2000, Radhakrishnan and Srinivasan [17] actually gave a better lower bound for $m(k, 1)$, who showed that $m(k, 1)=\Omega\left(2^{k} \sqrt{\frac{k}{\ln k}}\right)$. This implies that $\frac{c h_{1}(G)}{1}=\operatorname{ch}(G) \leq$ $\log _{2} n-\left(\frac{1}{2}-o(1)\right) \log _{2} \log _{2} n$ if $G$ is a complete bipartite graph with $n$ vertices. Combing this fact and Lemma 4.2 and Lemma 2.1, we have the following proposition.

Corollary 4.3 Let $G$ be a bipartite graph with $n$ vertices. When $n$ is big enough,

$$
c h_{f}^{s}(G) \leq \log _{2} n-\left(\frac{1}{2}-o(1)\right) \log _{2} \log _{2} n .
$$

Consequently, $\chi_{f, P}^{s}\left(K_{n, n}\right)-c h_{f}^{s}\left(K_{n, n}\right)$ can be arbitrarily large.
Although the gaps $c h_{f}^{s}(G)-\chi_{f}^{s}(G)$ and $\chi_{f, P}(G)-c h_{f}^{s}(G)$ can be arbitrarily large, there are also many graphs $G$ for which the equality $c h_{f}^{s}(G)=\chi_{f}^{s}(G)$ and/or $\chi_{f, P}(G)=c h_{f}^{s}(G)$ hold. Recall that a graph $G$ is called chromatic-choosable if $\chi(G)=\operatorname{ch}(G)$. The study of chromatic choosable graphs attracted a lot of attention. The well-known list colouring conjecture asserts that line graphs are chromatic-choosable. This conjecture remains largely open, however, it was shown by Galvin [9] the the line graphs of bipartite graphs are chromatic-choosable. This result extends to strong fractional choice number and strong fractional paint number.

Theorem 4.4 If $G=L(H)$ is the line graph of a bipartite graph $H$, then

$$
\chi_{f}^{s}(G)=c h_{f}^{s}(G)=\chi_{f, P}^{s}(G)=\Delta(H)
$$

Proof. It is well-known that $\chi(G)=\omega(G)=\Delta(H)$. Hence $\chi_{f}^{s}(G) \geq \Delta(H)$. It remains to show that $\chi_{f, P}(G) \leq \Delta(H)$.

An orientation $D$ of $G$ is kernel perfect if any subset $X$ of $V(D)$ contains an independent set $I$ such that for any $v \in X-I, N_{D}^{+}(v) \cap I \neq \varnothing$. Here $N_{D}^{+}(v)$ is the set of out-neighbours of $v$. We set $N_{D}^{+}[v]=N_{D}^{+}(v) \cup\{v\}$. It was proved in [9] that $G$ has a kernel perfect orientation $D$ with $\Delta^{+}(D)=\Delta(H)$. On the other hand, for any kernel
perfect orientation $D$ of $G$, for any $f, g: V(D) \rightarrow \mathbb{N}$, if $f(v) \geq \sum_{u \in N_{D}^{+}[v]} g(u)$ for every vertex $v$, then it is easy to show by induction on $\sum_{v \in V(D)} f(v)$ that Painter has a winning strategy in the $(f, g)$-painting game.

Indeed, if Lister choose a subset $X$ of $V(G)$ in a round, then Painter chooses an independent set $I$ of $X$ for which $N_{D}^{+}(v) \cap I \neq \varnothing$ for all $v \in X-I$. Let

$$
f^{\prime}(v)= \begin{cases}f(v)-1, & \text { if } v \in X-I, \\ f(v), & \text { otherwise },\end{cases}
$$

and

$$
g^{\prime}(v)=\left\{\begin{array}{lc}
g(v)-1, & \text { if } v \in I \\
g(v), & \text { otherwise }
\end{array}\right.
$$

It follows from the definition that for any vertex $v$, we still have $f^{\prime}(v) \geq \sum_{u \in N_{D}^{+}[v]}^{\prime}(u)$. By induction hypothesis, Painter has a winning strategy in the $\left(f^{\prime}, g^{\prime}\right)$-painting game on $G$. Therefore Painter has a winning strategy for the $(f, g)$-painting game on $G$.

For any positive integer $m$, by letting $f(v)=m\left(d_{D}^{+}(v)+1\right) \leq \Delta(H) m$ and $g(v)=m$ for each vertex $v$, we have that $G$ is $(\Delta(H) m, m)$-paintable. Hence $\chi_{f, P}^{s}(G) \leq \Delta(H)$.

## $5 c h_{f}^{s}(G)$ for planar graphs

In this section, we study the strong fractional choice number of planar graphs. Let $\mathcal{P}$ denote the class of planar graphs and for integers $k_{1}, \ldots, k_{q} \geq 3$, let $\mathcal{P}_{k_{1}, \ldots, k_{q}}$ denote the class of planar graphs without $k_{i}$-cycles for $i=1, \ldots, q$. For example, $\mathcal{P}_{3,4,5}$ denotes planar graphs with girth 6.

It was shown in [28] that $4+\frac{2}{9} \leq c h_{f}^{s}(\mathcal{P}) \leq 5$. The following result improves the lower bound.

Proposition 5.1 For each positive integer $m$, there is a planar graph $G$ which is not $\left(4 m+\left\lfloor\frac{m-1}{3}\right\rfloor, m\right)$-choosable. Consequently, $c h_{f}^{s}(\mathcal{P}) \geq 4+\frac{1}{3}$.

Proof. Let $T$ be the graph as shown in Fig. 2, $\epsilon$ be a real number such that $\epsilon m=\left\lfloor\frac{m-1}{3}\right\rfloor$. Assume $A, B, C, D, E, F$ are pairwise disjoint sets of colours with $|A|=|B|=|C|=|D|=m$, $|E|=\epsilon m$ and $|F|=2 m$.

- $L(u)=A$ and $L(v)=B$.
- $L(x)=L(y)=A \cup B \cup F \cup E$.
- $L\left(u_{1}\right)=A \cup C \cup F \cup E$ and $L\left(v_{1}\right)=B \cup C \cup F \cup E$.
- $L\left(u_{2}\right)=A \cup D \cup F \cup E$. and $L\left(v_{2}\right)=B \cup D \cup F \cup E$.

$$
\text { - } L(z)=A \cup B \cup C \cup D \cup E \text {. }
$$



Fig. 2. The target graph $T$
Now we show that there is no $m$-fold $L$-colouring of $G$. Suppose to the contrary, $\phi$ is an $m$-fold $L$-colouring of $G$. Then $\phi(u)=A$ and $\phi(v)=B$. Note that $u_{1} v_{1} x$ is a clique, so each colour in $E \cup F$ can be used at most once in $u_{1}, v_{1}$ and $x$. As altogether, we use $3 m$ distinct colours in these three vertices, at least $(1-\epsilon) m$ colours in $C$ are used on vertex $u_{1}$ and $v_{1}$, which implies that at most $\epsilon m$ colours in $C$ can be used at vertex $z$. By symmetric, at most $\epsilon m$ colours in $D$ can be used at vertex $z$. Recall that $|E|=\epsilon m$, so for the vertex $z$,

$$
m=|\phi(z)|=|\phi(z) \cap C|+|\phi(z) \cap D|+|\phi(z) \cap E| \leq 3 \epsilon m<m
$$

a contradiction.
Let $p=\binom{(3+\epsilon) m}{m}^{2}$. Let $G$ be obtained from the disjoint union of $p$ copies of $T$, by identifying all the copies of $u$ into a single vertex, also named $u$, and identifying all the copies of $v$ into a single vertex named $v$. Let $L$ be the $(3+\epsilon) m$-list assignment of $G$ defined as follows: Let $L(u)=X$ and $L(v)=Y$, where $X, Y$ are two disjoint set of size $(3+\epsilon) m$. For each pair of $m$-sets $(A, B)$, where $A \subseteq X$ and $B \subseteq Y$, we associate a copy of $T_{A, B}$ of $T$ so that the lists of the vertices of this copy of $T_{A, B}$ is as given above. Then $G$ is not $m$-fold $L$-colourable, for otherwise, $u$ is coloured with an $m$-subset $A$ of $X, v$ is coloured with an $m$-subset $B$ of $Y$. However, by the argument above, $T_{A, B}$ has no $m$-fold $L$-colouring.

Next we consider the family $\mathcal{P}_{4}$.
Proposition 5.2 For each positive integer $m$, there is a planar graph $G$ without 4-cycle, which is not $\left(3 m+\left\lfloor\frac{m-1}{2}\right\rfloor, m\right)$-choosable. Consequently, $3+\frac{1}{2} \leq \operatorname{ch}_{f}^{s}\left(\mathcal{P}_{4}\right) \leq 4$.

Proof. Let $T$ be labeled as shown in Fig.3, $\epsilon$ be a real number such that $\epsilon m=\left\lfloor\frac{m-1}{2}\right\rfloor$. For any disjoint sets $A$ and $B$, we define a list assignment $L_{A, B}$ (when $A, B$ are clear, and there is no confusion, we write $L$ in short) of $T$ as follows: Assume $A, B, C, D, E$ are pairwise disjoint sets of colours with $|A|=|B|=|D|=m,|C|=2 m$ and $|E|=\epsilon m$.

- $L(u)=A$ and $L(v)=B$.
- $L\left(u_{1}\right)=L\left(u_{2}\right)=A \cup C \cup E$.
- $L\left(v_{1}\right)=L\left(v_{2}\right)=B \cup C \cup E$.
- $L(x)=L(y)=C \cup D \cup E$.


Fig. 3. For $\mathcal{P}_{4}$

By the same argument as in the proof of Theorem 2, it suffices to show that there is no $m$-fold $L$-colouring of $G$. Suppose to the contrary, $\phi$ is an $m$-fold $L$-colouring of $G$. Then $\phi(u)=A$ and $\phi(v)=B$. Note that $u_{1} v_{1} x$ is a clique, and we use $2 m$ colours in $C \cup E$ on $u_{1}$ and $v_{1}$. Therefore, only $\epsilon m$ colours in $C \cup E$ can be used at $u_{1}$. By symmetric, only $\epsilon m$ colours in $C \cup E$ can be used at $y$. Note that $|D|=m$, so we have

$$
2 m=|\phi(x)|+|\phi(y)| \leq 2 \epsilon m+|D|=2 \epsilon m+m<2 m,
$$

a contradiction.
It was proved in [12] that planar graphs without 4-cycles are $(4 m, m)$-choosable for any positive integer $m$. So $c h_{f}^{s}\left(\mathcal{P}_{4}\right) \leq 4$.

Observe that $K_{4}$ is a planar graph without $k$-cycle for any $k \geq 5$, and $c h_{f}^{s}\left(K_{4}\right)=4$. On the other hand, it was shown in [12] that every graph without $k$-cycle is $(4 m, m)$ choosable, where $k \in\{4,5,6\}$. We have the following.

Observation 5.3 For any $k \geq 5, \operatorname{ch}_{f}^{s}\left(\mathcal{P}_{k}\right) \geq 4$. In particular, $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{k}\right)=4$ when $k \in$ $\{5,6\}$.

The construction in Proposition 5.1 does not contain $k$-cycle for $k \geq 17$, which means that $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{k}\right)>4+\frac{1}{3}$ for $k \geq 17$. It remains an open question as what is the smallest $k$ such that $c h_{f}^{s}\left(\mathcal{P}_{k}\right)>4$ ?

The family of planar graphs without 4 -, 5 -cycles has been studied extensively in the literature, because of the well-known Steinberg's Conjecture, see [18]. The conjecture
asserts that every planar graph contains neither 4-cycle nor 5-cycle is 3-colourable. This conjecture was disproved [3]. The list version of this conjecture was disproved earlier by Voigt [25], who first constructed a non-3-choosable planar graph without 4-, 5-cycle with 344 vertices. A smaller one was given by Montassier [15] with 209 vertices.

However, the counterexample graph to Steinberg's Conjecture given in [3] is (6,2)colourable, see Appendix A, hence it has fractional chromatic number exactly 3 (The graph contains a triangle, so the lower bound is 3 ). Therefore, it is $(3 m, m)$-choosable for some $m$ by the main result in [1]. On the other hand, it is easily to check that all the non-3-choosable examples constructed in $[15,16,25,26]$ mentioned above are 3colourable, hence they are also $(3 m, m)$-choosable for some $m$. So before the present paper, it was unknown whether or not for every positive integer $m$, there is a planar graph without 4- and 5 -cycles which is not ( $3 m, m$ )-choosable. In the following, for each positive integer $m$, we construct a planar graph without cycles of length 4 and 5 which is not $\left(3 m+\left\lfloor\frac{m-1}{12}\right\rfloor, m\right)$-choosable. When $m=1$, the graph has 164 vertices, which is smaller than the counterexample graph found by Montassier in [15].

Proposition 5.4 For each positive integer $m$, there is a planar graph $G$ without 4-cycle and 5 -cycle, which is not $\left(3 m+\left\lfloor\frac{m-1}{12}\right\rfloor, m\right)$-choosable. Consequently, ch $f_{f}^{s}\left(\mathcal{P}_{4,5}\right) \geq 3+\frac{1}{12}$.

Proof. Let $T$ be labeled as shown in Fig.4, $\epsilon$ be a real number such that $\epsilon m=\left\lfloor\frac{m-1}{12}\right\rfloor$. Assume $A, B, C, D, E$ are pairwise disjoint sets of colours with $|A|=|B|=|C|=m$, $|D|=2 m$ and $|E|=\epsilon m$.

- $L(u)=A, L(v)=B$ and $L(w)=C \cup D \cup E$.
- $L\left(u_{1}\right)=L\left(u_{2}\right)=L\left(w_{1}\right)=L\left(x_{3}\right)=L\left(y_{1}\right)=L\left(y_{3}\right)=L\left(z_{1}\right)=L\left(z_{2}\right)=A \cup D \cup E$.
- $L\left(v_{1}\right)=L\left(v_{2}\right)=L\left(w_{2}\right)=L\left(x_{1}\right)=L\left(x_{2}\right)=L\left(y_{2}\right)=L\left(y_{3}\right)=B \cup D \cup E$.
- $L(x)=L(y)=L(z)=A \cup B \cup C \cup E$.

By the same argument as in the proof of Theorem 2, it suffices to show that there is no $m$-fold $L$-colouring of $G$. Suppose to the contrary, $\phi$ is an $m$-fold $L$-colouring of $G$. Then $\phi(u)=A$ and $\phi(v)=B$. Note that $u_{1} v_{1} z_{1}$ is a clique, so each colour in $D \cup E$ can be used only once in these vertices, which means that

$$
\left|\phi\left(z_{1}\right) \cap A\right| \geq 3 m-|D \cup E|=m-\epsilon m .
$$

Similarly, $\left|\phi\left(z_{2}\right) \cap B\right| \geq m-\epsilon m$. So $|\phi(z) \cap A| \leq \epsilon m$ and $|\phi(z) \cap B| \leq \epsilon m$. We assume that $|\phi(z) \cap A|=\alpha m,|\phi(z) \cap B|=\beta m$ and $|\phi(z) \cap E|=\gamma m$, it is clear that $\alpha, \beta, \gamma \leq \epsilon$, and $|\phi(z) \cap C|=(1-(\alpha+\beta+\gamma)) m$.

As $z z_{1} x$ is a clique, each colour in $A$ can be used once in these three vertices, so

$$
|\phi(x) \cap A| \leq|A|-\left|\phi\left(z_{1}\right) \cap A\right|-|\phi(z) \cap A| \leq(\epsilon-\alpha) m .
$$



Fig. 4. Target graph for $\mathcal{P}_{4,5}$

Similarly, by considering the edge $x z$, we have that $|\phi(x) \cap C| \leq(\alpha+\beta+\gamma) m$ and $|\phi(x) \cap E| \leq(\epsilon-\gamma) m$. Note that $\phi\left(z_{1}\right) \cap E$ might be empty. So we only have $|\phi(x) \cap E| \leq$ $|E|-|E \cap \phi(z)|$. Therefore, we have

$$
|\phi(x) \cap B| \geq m-|\phi(x) \cap A|-|\phi(x) \cap C|-|\phi(x) \cap E| \geq m-(2 \epsilon+\beta) m .
$$

This implies that $\left|\phi\left(x_{1}\right) \cap B\right| \leq(2 \epsilon+\beta) m$.
Since $x_{1} x_{2} x_{3}$ is a clique, each colour in $D \cup E$ can be used at most once on these three vertices, but we need $3 m$ colours for these vertices, so

$$
\left|\phi\left(x_{3}\right) \cap A\right| \geq 3 m-\left|\phi\left(x_{1}\right) \cap B\right|-\left|\left(\phi\left(x_{1}\right) \cup \phi\left(x_{2}\right) \cup \phi\left(x_{3}\right)\right) \cap(D \cup E)\right| \geq m-(3 \epsilon+\beta) m .
$$

Hence, $\left|\phi\left(w_{1}\right) \cap A\right| \leq(3 \epsilon+\beta) m$.
By symmetry, $\left|\phi\left(w_{2}\right) \cap A\right| \leq(3 \epsilon+\alpha) m$. On the other hand, $|\phi(w) \cap C| \leq m-|\phi(z) \cap C| \leq$ $(\alpha+\beta+\gamma) m$, so we have

$$
\begin{aligned}
3 m & =|\phi(w)|+\left|\phi\left(w_{1}\right)\right|+\left|\phi\left(w_{2}\right)\right| \\
& \leq\left|\phi\left(w_{1}\right) \cap A\right|+\left|\phi\left(w_{2}\right) \cap A\right|+|\phi(w) \cap C|+\left|\left(\phi(w) \cup \phi\left(w_{1}\right) \cup \phi\left(w_{2}\right)\right) \cap(D \cup E)\right| \\
& \leq 2 m+7 \epsilon m+2(\alpha+\beta) m+\gamma m \\
& \leq 2 m+12 \epsilon m<3 m,
\end{aligned}
$$

a contradiction.
It was proved in [13] that the strong fractional choice number of $K_{4}$-minor-free graphs with girth at least $g$ is $2+\frac{1}{[(g+1) / 4]}$. Thus the strong fractional choice number of the family of planar graphs of girth 5 or 6 is at least 3 , i.e., $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{3,4}\right) \geq 3$. On the other hand, extending the proofs in [19, 20], Voigt [24] proved that every planar graphs with girth 5 is $(3 m, m)$-choosable, so the family of planar graphs of girth 5 or 6 has strong fractional choice number at most 3 .

Proposition $5.5 \operatorname{ch}_{f}^{s}\left(\mathcal{P}_{3,4}\right)=\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{3,4,5}\right)=3$.
For the case of $\mathcal{P}_{3}$, the best known upper and lower bounds for their strong fractional chromatic number was obtained [11]: $3+\frac{1}{17} \leq c h_{f}^{s}\left(\mathcal{P}_{3}\right) \leq 4$.

## 6 Open problems

One basic unsolved problem concerning the strong fractional choice number is whether every rational $r \geq 2$ is the strong fractional choice number of a graph. We conjecture an affirmative answer.
Conjecture 6.1 For any rational number $r \geq 2$, there exists a graph $G$ such that $c h_{f}^{s}(G)=r$ and a graph $G^{\prime}$ with $\chi_{f, P}^{s}\left(G^{\prime}\right)=r$.

Erdős, Rubin and Taylor [8] characterized all the 2-choosable graphs. However, it seems to be a difficult problem to characterize all graphs $G$ with $c h_{f}^{s}(G)=2$. In a companion paper [27], we proved that every 3-choice critical bipartite graph $G$ (i.e., $G$ is not 2 -choosable, but every proper subgraph of $G$ is 2 -choosable) has strong fractioal choice number 2.
Question 6.2 Given a characterization of the class of graphs whose strong fractional choice number are 2.

It was asked by Erdős, Rubin and Taylor [8] that whether every $(a, b)$-choosable graph is also $(a m, b m)$-choosable for any positive integer $m$. The case $(a, b)=(2,1)$ was affirmed by Tuza and Voigt [22], but the case $a \geq 4$ and $b=1$ was negatived by Dvořák, Hu and Sereni [6] recently. For a relax and possibly correct version, we ask the following question.

Question 6.3 Is it true that $c h_{f}^{s}(G) \leq \operatorname{ch}(G)$ for any graph $G$ ?
Similarly, it was conjectured by Mahoney, Meng and Zhu [14] that every ( $a, b$ )paintable graph is also $(a m, b m)$-paintable for any positive integer $m$. We also ask the following weaker problem.

Question 6.4 Is it true that $\chi_{f, P}^{s}(G) \leq \chi_{P}(G)$ for any graph $G$ ?
Planar graph colouring is a central problem with respect to many colouring concepts. This is also the case for the strong fractional choice number of graphs.
Question 6.5 What is the exact value of $\operatorname{ch}_{f}^{s}(\mathcal{P})$ ? Is it true that $\operatorname{ch}_{f}^{s}(\mathcal{P})<5$ ?
Question 6.6 What is the exact value of $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{3}\right)$ ? Is it true that $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{3}\right)<4$ ?
Question 6.7 What is the exact value of $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{4,5}\right)$ ? Is it true that $\operatorname{ch}_{f}^{s}\left(\mathcal{P}_{4,5}\right)<4$ ?
Although Steinberg's conjecture is false, the fractional chromatic number and the strong fractional chromatic number of graphs in $\mathcal{P}_{4,5}$ is open. It was proved in [5] that for any $G \in \mathcal{P}_{4,5}, \chi_{f}(G) \leq 11 / 3$. The following question remains open.
Question 6.8 Is it true that every graph $G \in \mathcal{P}_{4,5}$ has $\chi_{f}(G) \leq 3$, or even has $\chi_{f}^{s}(G) \leq 3$ ?

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## Appendix A

In this part, we give a (6,2)-colouring $\phi$ of the counterexample to Steinberg's conjecture presented in [3]. The counterexample constructed in [3] is the graph depicted in Figure

5, where Figure 6 depicts two copies of $G_{2}$. We first pre-colour part of the graph in Fig.3. as follows: $\phi(a)=\{1,3\}, \phi(b)=\{5,6\}, \phi(c)=\{1,2\}, \phi(d)=\{2,3\}, \phi(e)=\{4,5\}$, $\phi(f)=\{1,6\}, \phi\left(c^{\prime}\right)=\{3,4\}, \phi\left(d^{\prime}\right)=\{1,4\}, \phi\left(e^{\prime}\right)=\{2,5\}$ and $\phi\left(f^{\prime}\right)=\{3,6\}$.


Fig. 5. The counterexample to Steinberg's Conjecture in [3]

We shall show that this partial colouring can be extended to a 2 -fold 6 -colouring of the whole graph. By symmetry, it suffices to extend the partial colouring to the left two copies of $G_{2}$, which is given in Figure 6.


Fig. 6. (6,2)-colourings of the left two copies of $G_{2}$


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