# Mean-Field Type FBSDEs under Domination-Monotonicity Conditions and Application to LQ Problems * 

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#### Abstract

This paper is concerned with a class of mean-field type coupled forward-backward stochastic differential equations (MF-FBSDEs, for short), in which the coupling appears in integral terms, terminal terms, and initial terms. Inspired by various mean-field type linear-quadratic (MF-LQ, for short) optimal control problems, we proposed a type of randomized domination-monotonicity conditions, under which and the usual Lipschitz condition, we obtain a well-posedness result on MF-FBSDEs in the sense of square integrability including the unique solvability, an estimate of the solution, and the related continuous dependence property of the solution on the coefficients. The result of MF-FBSDEs in turn extends MF-LQ problems in the literature to a general situation where the initial states or the terminal states are also controlled at the same time, and gives explicit expressions of the related unique optimal controls.


Key words: forward-backward stochastic differential equation, stochastic linear-quadratic problem, stochastic optimal control, mean-field, domination-monotonicity condition

AMS subject classification: 93E20, 60H10, 49N10

## 1 Introduction

Historically, as a kind of mean-field type stochastic differential equations (MF-SDE, for short), the research on McKean-Vlasov SDEs started from the pioneering works of Kac [10] and McKean [15]. In the past fifteen years, accompanying the research boom of mean-field games led by Huang et al. [9] and Lasry and Lions [11], the MF-SDEs gained a rapid development. Especially, mean-field type backward SDEs (MF-BSDEs, for short), which admit a typical structure different from the corresponding forward equations, were introduced and studied by Buckdahn et al. [3,4]. Due to the requirements of both theory and applications, a (forward) MF-SDE and an MF-BSDE were coupled together to form a system called an MF-FBSDE, and were studied by many scholars; see Carmona and Delarue [6, 7], Bensoussan et al. [2], Wei et al. [21], and so on.

In this paper, we consider the following MF-FBSDE:

$$
\left\{\begin{array}{l}
\mathrm{d} x(s)=b\left(s, \theta(s), \mathbb{E}_{t}[\theta(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} \sigma_{i}\left(s, \theta(s), \mathbb{E}_{t}[\theta(s)]\right) \mathrm{d} W_{i}(s), \quad s \in[t, T]  \tag{1.1}\\
\mathrm{d} y(s)=g\left(s, \theta(s), \mathbb{E}_{t}[\theta(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} z_{i}(s) \mathrm{d} W_{i}(s), \quad s \in[t, T] \\
x(t)=\Psi(y(t)), \quad y(T)=\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)
\end{array}\right.
$$

where $\mathbb{E}_{t}[\cdot] \equiv \mathbb{E}\left[\cdot \mid \mathcal{F}_{t}\right]$ is the conditional expectation with respect to $\mathcal{F}_{t}, \theta(\cdot):=\left(x(\cdot)^{\top}, y(\cdot)^{\top}, z(\cdot)^{\top}\right)^{\top}$ with $z(\cdot)=\left(z_{1}(\cdot)^{\top}, z_{2}(\cdot)^{\top}, \ldots, z_{d}(\cdot)^{\top}\right)^{\top}$ is the unknown process, and $W(\cdot):=\left(W_{1}(\cdot), W_{2}(\cdot), \ldots, W_{d}(\cdot)\right)^{\top}$

[^0]is a $d$-dimensional Brownian motion. Here and hereafter, the superscript $\top$ denotes the transpose of a vector or a matrix. For convenience, we also denote $\Gamma(\cdot):=\left(g(\cdot)^{\top}, b(\cdot)^{\top}, \sigma(\cdot)^{\top}\right)^{\top}$ and $\sigma(\cdot):=$ $\left(\sigma_{1}(\cdot)^{\top}, \sigma_{2}(\cdot)^{\top}, \ldots, \sigma_{d}(\cdot)^{\top}\right)^{\top}$. Then, all of the coefficients of MF-FBSDE (1.1) are collected by $(\Psi, \Phi, \Gamma)$.

To our best knowledge, in the literature on MF-FBSDEs, the coupling between forward variable $x(\cdot)$ and backward variables $(y(\cdot), z(\cdot))$ only appeared in the integral coefficient $\Gamma(\cdot)$ and the terminal coefficient $\Phi(\cdot)$, while the initial coefficient $\Psi(\cdot)$ was always in the decoupled form, i.e., $\Psi(\cdot) \equiv x_{t}$ which is a given $\mathcal{F}_{t}$-measurable random variable. In comparison, the coupling occurs in all three coefficients $(\Psi, \Phi, \Gamma)$ of (1.1). The present paper is the first time to consider this general situation.

It is well known that, if only the uniformly Lipschitz continuity of the coefficients is proposed, MFFBSDE (1.1) (even in the absence of initial coupling and/or mean-field terms) may be unsolvable on an arbitrarily large time interval (see a counterexample in Antonelli [1]). Carmona and Delarue [6] introduced some assumptions including the uniform ellipticity and the boundedness of the coefficient $\sigma(\cdot)$, and for the first time obtained an existence result on large intervals for coupled MF-FBSDEs. In their second paper [7], they introduced the convexity assumption to get a unique solvability result. Moreover, Bensoussan et al. [2] proposed a kind of monotonicity conditions and improve the result in [7]. In the present paper, we will introduce a type of domination-monotonicity conditions to ensure the global solvability of MF-FBSDEs. Such conditions can be regarded as a generalization of monotonicity conditions in [2]. We also note that a special and linear case was studied in Wei et al. [21].

The domination-monotonicity conditions are actually rooted in various MF-LQ optimal control problems; see Yong [23], Ni et al. [16], Sun [18], Li et al. [13], Wang [20], and Li et al. [12], for example. Let us explain this point in detail. When we apply the Pontryagin type maximum principle approach to study the necessary condition of an MF-LQ problem, by introducing an adjoint equation, the optimal control will be closely linked with a mean-field type Hamiltonian system (which is actually an MF-FBSDE, see (4.14) and (4.21)). Moreover, by imposing some uniformly positive definiteness condition (see Condition (MF-FLQ-PD) and Condition (MF-BLQ-PD) in Section 4), the necessary condition is also sufficient. Taking MF-LQ problems as an understanding background and intuitive guidance, we propose the domination-monotonicity conditions for the nonlinear MF-FBSDE (1.1). The features and novelties of such conditions are summarized as the following three points:
(i) The domination-monotonicity conditions (see Assumption (H4) and Remark 3.7) accurately correspond to the expression forms of optimal controls and the related uniformly positive definiteness conditions in the four classes of MF-LQ problems (see Proposition 4.2, Proposition 4.4 and Remark 4.5).
(ii) Compared with monotonicity conditions in [2], the introduction of matrices and matrix-valued functions in our domination-monotonicity conditions increases flexibility and extensiveness. In detail, due to their introduction, on the one hand, a new kind of domination conditions are created which strengthen the Lipschitiz condition; on the other hand, at the same time it also weakens the monotonicity conditions in [2]. The flexible selection of matrices and matrix-valued functions provides us with more possibilities.
(iii) Compared with the classical case without involving mean-field terms (see [25]), the dominationmonotonicity conditions in this paper present a randomized form (see (3.4) and (3.5)). This is a response to the difficulty caused by the mean-field terms. We notice that this randomization technique has been used in [2].

In this paper, we will develop the method of continuation, which is originally introduced by Hu and Peng [8], Yong [22], and Peng and Wu [17] when they studied the classical coupled FBSDEs without mean-field terms, and obtain the well-posedness of MF-FBSDE (1.1) in the sense of square integrability including the existence and uniqueness of the solution, an estimate, and the related continuous dependence property of the solution on the coefficients (see Theorem 3.4). In order to overcome the technical difficulty brought with the mean-field terms, besides the randomization technique stated in the above Point (iii), we will also employ a decomposition technique of random variables: $\xi=\left(\xi-\mathbb{E}_{t}[\xi]\right)+\mathbb{E}_{t}[\xi]$ which is originally introduced by Yong [23]. We notice that although this decomposition is simple, it has the advantage that its first part has zero conditional expectation and the second part is $\mathcal{F}_{t}$-measurable. Due to this, it is often adopted to deal with many mean-field type problems (see Lin et al. [14] and Tian et al. [19], for example). But, the difficulty has still not been completely overcome. Then, we have to assume that the matrices and matrix-valued processes involved in the domination-monotonicity conditions are
deterministic. We are finally able to overcome the difficulty and complete all analyses and proofs in this paper.

As an application of the unique solvability result of MF-FBSDE (1.1), we re-examined several MFLQ problems. It is worth noting that, in MF-LQ problems in the literature, control is only applied to the drift and diffusion terms of the controlled systems; while in the present paper, corresponding to the feature that the couplings appear in both $\Psi(\cdot)$ and $\Phi(\cdot)$ of MF-FBSDE (1.1), control can also be applied to the initial values of forward MF-LQ problems (see the forward controlled system (4.1), for example) and the terminal values of backward MF-LQ problems (see the backward controlled system (4.15)) at the same time. These extensions are interesting in their own right. As conclusions, the unique solvability of MF-FBSDE (1.1) under domination-monotonicity conditions implies that of Hamiltonian systems arising from these generalized MF-LQ problems, and further implies the existence and uniqueness of optimal controls.

The rest of this paper is organized as follows. In Section 2, we define some notations and give the preliminaries about MF-SDEs and MF-BSDEs. In Section 3, we rigorously state the dominationmonotonicity conditions in detail, then under them, we prove the well-posedness of MF-FBSDE (1.1). In Section 4, we apply the results of MF-FBSDEs obtained in the previous section to study some generalized MF-LQ problems, and obtain the explicit expressions of the unique optimal controls based on the related mean-field type Hamiltonian systems. Some proofs for the results of MF-SDEs and MF-BSDEs are put in Appendix A.

## 2 Notations and preliminaries

Let $\mathbb{R}^{n}$ be the $n$-dimensional Euclidean space equipped with the Euclidean inner product $\langle\cdot, \cdot\rangle$. The induced Euclidean norm is denoted by $|\cdot|$. Let $\mathbb{R}^{m \times n}$ be the set of all $(m \times n)$ matrices and $\mathbb{S}^{n} \subset \mathbb{R}^{n \times n}$ be the subset consisting of all $(n \times n)$ symmetric matrices. In this paper, we use the operator norm of matrices:

$$
\|A\|:=\sup _{\mathbb{R}^{n} \ni x \neq 0} \frac{|A x|}{|x|} \quad \text { for any } A \in \mathbb{R}^{m \times n}
$$

Let $0<T<\infty$ be a fixed time horizon. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a $d$-dimensional standard Brownian motion $W(\cdot)$ is defined and $\mathbb{F}=\left\{\mathcal{F}_{s}, 0 \leq s \leq T\right\}$ is the natural filtration of $W(\cdot)$ augmented by all $\mathbb{P}$-null sets. We also set $\mathcal{F}=\mathcal{F}_{T}$.

Let $0 \leq t<T$ and $q=1,2$. We introduce some Banach (or Hilbert in case of $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ or $\left.L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n}\right)\right)$ spaces as follows:

- $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ is the set of all $\mathcal{F}_{t}$-measurable random variables $\xi: \Omega \rightarrow \mathbb{R}^{n}$ such that

$$
\|\xi\|_{L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)}:=\left\{\mathbb{E}\left[|\xi|^{2}\right]\right\}^{1 / 2}<\infty
$$

- $L_{\mathbb{F}}^{2}\left(\Omega ; L^{q}\left([t, T] ; \mathbb{R}^{n}\right)\right)$ is the set of all $\mathbb{F}$-progressively measurable processes $f:[t, T] \times \Omega \rightarrow \mathbb{R}^{n}$ such that

$$
\|f(\cdot)\|_{L_{\mathbb{F}}^{2}\left(\Omega ; L^{q}\left([t, T] ; \mathbb{R}^{n}\right)\right)}:=\left\{\mathbb{E}\left[\left(\int_{t}^{T}|f(s)|^{q} \mathrm{~d} s\right)^{2 / q}\right]\right\}^{1 / 2}<\infty
$$

When $q=1$, we denote $L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right)=L_{\mathbb{F}}^{2}\left(\Omega ; L^{1}\left([t, T] ; \mathbb{R}^{n}\right)\right)$. When $q=2$, we denote $L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n}\right)=L_{\mathbb{F}}^{2}\left(\Omega ; L^{2}\left([t, T] ; \mathbb{R}^{n}\right)\right)$.

- $L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)$ is the set of all $\mathbb{F}$-progressively measurable processes $f:[t, T] \times \Omega \rightarrow \mathbb{R}^{n}$ such that for almost all $\omega \in \Omega, s \mapsto f(s, \omega)$ is continuous and

$$
\|f(\cdot)\|_{L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)}:=\left\{\mathbb{E}\left[\sup _{s \in[t, T]}|f(s)|^{2}\right]\right\}^{1 / 2}<\infty
$$

Now, we turn to consider an MF-SDE:

$$
\left\{\begin{array}{l}
\mathrm{d} x(s)=b\left(s, x(s), \mathbb{E}_{t}[x(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} \sigma_{i}\left(s, x(s), \mathbb{E}_{t}[x(s)]\right) \mathrm{d} W_{i}(s), \quad s \in[t, T]  \tag{2.1}\\
x(t)=x_{t}
\end{array}\right.
$$

For convenience, we denote $\sigma(\cdot):=\left(\sigma_{1}(\cdot)^{\top}, \sigma_{2}(\cdot)^{\top}, \ldots, \sigma_{d}(\cdot)^{\top}\right)^{\top}$. The coefficients $\left(x_{t}, b, \sigma\right)$ are assumed to satisfy the following

Assumption (H1). (i) For any $x, x^{\prime} \in \mathbb{R}^{n}$, the processes $b\left(\cdot, x, x^{\prime}\right)$ and $\sigma\left(\cdot, x, x^{\prime}\right)$ are $\mathbb{F}$-progressively measurable. Moreover, $x_{t} \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right), b(\cdot, 0,0) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right)$ and $\sigma(\cdot, 0,0) \in L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$.
(ii) The mappings $b$ and $\sigma$ are uniformly Lipschitz continuous with respect to $\left(x, x^{\prime}\right)$, i.e., there exists a constant $L>0$ such that

$$
\left|b\left(s, x, x^{\prime}\right)-b\left(s, \bar{x}, \bar{x}^{\prime}\right)\right|+\left|\sigma\left(s, x, x^{\prime}\right)-\sigma\left(s, \bar{x}, \bar{x}^{\prime}\right)\right| \leq L\left(|x-\bar{x}|+\left|x^{\prime}-\bar{x}^{\prime}\right|\right)
$$

for any $x, \bar{x}, x^{\prime}, \bar{x}^{\prime} \in \mathbb{R}^{n}$ and almost all $(s, \omega) \in[t, T] \times \Omega$.
We have the following result.
Proposition 2.1. Under Assumption (H1), MF-SDE (2.1) with coefficients ( $x_{t}, b, \sigma$ ) admits a unique solution $x(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)$. Moreover, we have the following estimate:

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|x(s)|^{2}\right] \leq K\left\{\left|x_{t}\right|^{2}+\mathbb{E}_{t}\left[\left(\int_{t}^{T}|b(s, 0,0)| \mathrm{d} s\right)^{2}+\int_{t}^{T}|\sigma(s, 0,0)|^{2} \mathrm{~d} s\right]\right\} \tag{2.2}
\end{equation*}
$$

where $K:=K(T-t, L)>0$ is a constant depending on $(T-t)$ and the Lipschitz constant of the mappings $b$ and $\sigma$. Furthermore, let $\left(\bar{x}_{t}, \bar{b}, \bar{\sigma}\right)$ be another set of coefficients, and assume that $\bar{x}(\cdot) \in$ $L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)$ is a solution to the MF-SDE with $\left(\bar{x}_{t}, \bar{b}, \bar{\sigma}\right)$. We continue to assume that $\bar{x}_{t} \in$ $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right), \bar{b}\left(\cdot, \bar{x}(\cdot), \mathbb{E}_{t}[\bar{x}(\cdot)]\right) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right)$ and $\bar{\sigma}\left(\cdot, \bar{x}(\cdot), \mathbb{E}_{t}[\bar{x}(\cdot)]\right) \in L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$. Then,

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|x(s)-\bar{x}(s)|^{2}\right] \leq K\left\{\left|x_{t}-\bar{x}_{t}\right|^{2}+\mathbb{E}_{t}\left[\left(\int_{t}^{T} \mid b\left(s, \bar{x}(s), \mathbb{E}_{t}[\bar{x}(s)]\right)\right.\right.\right.  \tag{2.3}\\
& \left.\left.\left.-\bar{b}\left(s, \bar{x}(s), \mathbb{E}_{t}[\bar{x}(s)]\right) \mid \mathrm{d} s\right)^{2}+\int_{t}^{T}\left|\sigma\left(s, \bar{x}(s), \mathbb{E}_{t}[\bar{x}(s)]\right)-\bar{\sigma}\left(s, \bar{x}(s), \mathbb{E}_{t}[\bar{x}(s)]\right)\right|^{2} \mathrm{~d} s\right]\right\}
\end{align*}
$$

where $K$ is the same constant as in (2.2).
The results in Proposition 2.1 are standard. We believe that they are not new. However, we are not able to find an exact reference, then we give a proof and put it in Appendix A. 1 for the readers' convenience.

Next, we consider an MF-BSDE as follows:

$$
\left\{\begin{array}{l}
\mathrm{d} y(s)=g\left(s, y(s), \mathbb{E}_{t}[y(s)], z(s), \mathbb{E}_{t}[z(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} z_{i}(s) \mathrm{d} W_{i}(s), \quad s \in[t, T]  \tag{2.4}\\
y(T)=y_{T}
\end{array}\right.
$$

where $z(\cdot):=\left(z_{1}(\cdot)^{\top}, z_{2}(\cdot)^{\top}, \ldots, z_{d}(\cdot)^{\top}\right)^{\top}$. For the coefficients $\left(y_{T}, g\right)$ of MF-BSDE (2.4), we introduce the following
Assumption (H2). (i) For any $y, y^{\prime} \in \mathbb{R}^{n}$ and $z, z^{\prime} \in \mathbb{R}^{n d}$, the process $g\left(\cdot, y, y^{\prime}, z, z^{\prime}\right)$ is $\mathbb{F}$-progressively measurable. Moreover, $y_{T} \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ and $g(\cdot, 0,0,0,0) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right)$.
(ii) The mapping $g$ is uniformly Lipschitz continuous with respect to ( $y, y^{\prime}, z, z^{\prime}$ ), i.e., there exists a constant $L>0$ such that

$$
\left|g\left(s, y, y^{\prime}, z, z^{\prime}\right)-g\left(s, \bar{y}, \bar{y}^{\prime}, \bar{z}, \bar{z}^{\prime}\right)\right| \leq L\left(|y-\bar{y}|+\left|y^{\prime}-\bar{y}^{\prime}\right|+|z-\bar{z}|+\left|z^{\prime}-\bar{z}^{\prime}\right|\right) .
$$

for any $y, \bar{y}, y^{\prime}, \bar{y}^{\prime} \in \mathbb{R}^{n}$, any $z, \bar{z}, z^{\prime}, \bar{z}^{\prime} \in \mathbb{R}^{n d}$ and almost all $(s, \omega) \in[t, T] \times \Omega$.

Proposition 2.2. Under Assumption (H2), MF-BSDE (2.4) with coefficients ( $y_{T}, g$ ) admits a unique solution $(y(\cdot), z(\cdot)) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$. Moreover, we have the following estimate:

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|y(s)|^{2}+\int_{t}^{T}|z(s)|^{2} \mathrm{~d} s\right] \leq K \mathbb{E}_{t}\left[\left|y_{T}\right|^{2}+\left(\int_{t}^{T}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right] \tag{2.5}
\end{equation*}
$$

where $K:=K(T-t, L)>0$ is a constant depending on $(T-t)$ and the Lipschitz constant of the mapping $g$. Furthermore, let $\left(\bar{y}_{t}, \bar{g}\right)$ be another set of coefficients, and assume that $(\bar{y}(\cdot), \bar{z}(\cdot)) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right) \times$ $L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$ is a solution to the MF-BSDE with $\left(\bar{y}_{T}, \bar{g}\right)$. We continue to assume that $\bar{y}_{T} \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ and $\bar{g}\left(\cdot, \bar{y}(\cdot), \mathbb{E}_{t}[\bar{y}(\cdot)], \bar{z}(\cdot), \mathbb{E}_{t}[\bar{z}(\cdot)]\right) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right)$. Then,

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|y(s)-\bar{y}(s)|^{2}+\int_{t}^{T}|z(s)-\bar{z}(s)|^{2} \mathrm{~d} s\right] \leq K \mathbb{E}_{t}\left\{\left|y_{T}-\bar{y}_{T}\right|^{2}\right. \\
& \left.+\left(\int_{t}^{T}\left|g\left(s, \bar{y}(s), \mathbb{E}_{t}[\bar{y}(s)], \bar{z}(s), \mathbb{E}_{t}[\bar{z}(s)]\right)-\bar{g}\left(s, \bar{y}(s), \mathbb{E}_{t}[\bar{y}(s)], \bar{z}(s), \mathbb{E}_{t}[\bar{z}(s)]\right)\right| \mathrm{d} s\right)^{2}\right\} \tag{2.6}
\end{align*}
$$

where $K$ is the same constant as in (2.5).
The proof of the above Proposition 2.2 can be found in Appendix A.2.
At the end of this section, we continue to introduce some other spaces which will be used in our following analysis.

- $L^{\infty}\left(t, T ; \mathbb{R}^{m \times n}\right)$ is the set of all Lebesgue measurable functions $A:[t, T] \rightarrow \mathbb{R}^{m \times n}$ such that

$$
\|A(\cdot)\|_{L^{\infty}\left(t, T ; \mathbb{R}^{m \times n}\right)}:=\operatorname{esssup}_{s \in[t, T]}\|A(s)\|<\infty
$$

Some product spaces are also introduced:

- $M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right):=L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$. For any $\theta(\cdot)=$ $\left(x(\cdot)^{\top}, y(\cdot)^{\top}, z(\cdot)^{\top}\right) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$, its norm is given by

$$
\|\theta(\cdot)\|_{M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)}:=\left\{\mathbb{E}\left[\sup _{s \in[t, T]}|x(s)|^{2}+\sup _{s \in[t, T]}|y(s)|^{2}+\int_{t}^{T}|z(s)|^{2} \mathrm{~d} s\right]\right\}^{1 / 2}
$$

- $\mathcal{M}_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right):=L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$. For any $\rho(\cdot)=$ $\left(\varphi(\cdot)^{\top}, \psi(\cdot)^{\top}, \gamma(\cdot)^{\top}\right)^{\top}$, its norm is

$$
\|\rho(\cdot)\|_{\mathcal{M}_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)}:=\left\{\mathbb{E}\left[\left(\int_{t}^{T}|\varphi(s)| \mathrm{d} s\right)^{2}+\left(\int_{t}^{T}|\psi(s)| \mathrm{d} s\right)^{2}+\int_{t}^{T}|\gamma(s)|^{2} \mathrm{~d} s\right]\right\}^{1 / 2}
$$

- $\mathcal{H}[t, T]:=L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times \mathcal{M}_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. For any $(\xi, \eta, \rho(\cdot)) \in \mathcal{H}[t, T]$, its norm is given by

$$
\|(\xi, \eta, \rho(\cdot))\|_{\mathcal{H}[t, T]}:=\left\{\|\xi\|_{L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)}^{2}+\|\eta\|_{{\mathcal{\mathcal { F } _ { T }}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)}^{2}+\|\rho(\cdot)\|_{\mathcal{M}_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)}^{2}\right\}^{1 / 2}
$$

## 3 MF-FBSDEs with domination-monotonicity conditions

In this section, we devote ourselves to investigating MF-FBSDE (1.1). Similar to MF-SDEs and MFBSDEs, we introduce the following assumptions on the coefficients $(\Psi, \Phi, \Gamma)$ :
 measurable. For any $\theta, \theta^{\prime} \in \mathbb{R}^{n(2+d)}, \Gamma\left(\cdot, \theta, \theta^{\prime}\right)$ is $\mathbb{F}$-progressively measurable. Moreover, $(\Psi(0), \Phi(0,0), \Gamma(\cdot, 0,0)) \in$ $\mathcal{H}[t, T]$.
(ii) The mappings $\Psi, \Phi$ and $\Gamma$ are uniformly Lipschitz continuous with respect to $y,\left(x, x^{\prime}\right)$ and $\left(\theta, \theta^{\prime}\right)$ respectively, i.e., the exists a constant $L>0$ such that

$$
\left\{\begin{array}{l}
|\Psi(y)-\Psi(\bar{y})| \leq L|y-\bar{y}| \\
\left|\Phi\left(x, x^{\prime}\right)-\Phi\left(\bar{x}, \bar{x}^{\prime}\right)\right| \leq L\left(|x-\bar{x}|+\left|x^{\prime}-\bar{x}^{\prime}\right|\right) \\
\left|\Gamma\left(s, \theta, \theta^{\prime}\right)-\Gamma\left(s, \bar{\theta}, \bar{\theta}^{\prime}\right)\right| \leq L\left(|\theta-\bar{\theta}|+\left|\theta^{\prime}-\bar{\theta}^{\prime}\right|\right)
\end{array}\right.
$$

for any $y, \bar{y}, x, \bar{x}, x^{\prime}, \bar{x}^{\prime} \in \mathbb{R}^{n}$, any $\theta, \bar{\theta}, \theta^{\prime}, \bar{\theta}^{\prime} \in \mathbb{R}^{n(2+d)}$ and almost all $(s, \omega) \in[t, T] \times \Omega$.

Besides the above Assumption (H3), the following domination-monotonicity conditions are also imposed:

Assumption (H4). There exist two constants $\mu \geq 0, \nu \geq 0$, three matrices $H \in \mathbb{R}^{m_{1} \times n}, P, \widetilde{P} \in \mathbb{R}^{m_{2} \times n}$, and six matrix-valued processes $A(\cdot), \widetilde{A}(\cdot), B(\cdot), \widetilde{B}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right), C(\cdot)=\left(C_{1}(\cdot), C_{2}(\cdot), \ldots, C_{d}(\cdot)\right)$, $\widetilde{C}(\cdot)=\left(\widetilde{C}_{1}(\cdot), \widetilde{C}_{2}(\cdot), \ldots, \widetilde{C}_{d}(\cdot)\right)$ with $C_{i}(\cdot), \widetilde{C}_{i}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right)$ (where $m_{1}, m_{2}, m_{3} \in \mathbb{N}$ are given) such that the following conditions hold:
(i) One of the following two cases holds. Case $A: \mu>0$ and $\nu=0$. Case B: $\mu=0$ and $\nu>0$.
(ii) (Domination conditions). For any $x, \bar{x}, x^{\prime}, \bar{x}^{\prime}, y, \bar{y}, y^{\prime}, \bar{y}^{\prime} \in \mathbb{R}^{n}$, any $z, \bar{z}, z^{\prime}, \bar{z}^{\prime} \in \mathbb{R}^{n d}$, and almost all $(s, \omega) \in[t, T] \times \Omega$, with the notations $\widehat{x}=x-\bar{x}, \widehat{x}^{\prime}=x^{\prime}-\bar{x}^{\prime}, \widehat{y}=y-\bar{y}, \widehat{y}^{\prime}=y^{\prime}-\bar{y}^{\prime}, \widehat{z}=z-\bar{z}, \widehat{z}^{\prime}=z^{\prime}-\bar{z}^{\prime}$, and $f=b, \sigma$, we have

$$
\left\{\begin{array}{l}
|\Psi(y)-\Psi(\bar{y})| \leq \frac{1}{\mu}|H \widehat{y}|  \tag{3.1}\\
\left|\Phi\left(x, x^{\prime}\right)-\Phi\left(\bar{x}, \bar{x}^{\prime}\right)\right| \leq \frac{1}{\nu}\left|\binom{P\left(\widehat{x}-\widehat{x}^{\prime}\right)}{\widetilde{P} \widehat{x}^{\prime}}\right| \\
\left|g\left(s, x, x^{\prime}, y, y^{\prime}, z, z^{\prime}\right)-g\left(s, \bar{x}, \bar{x}^{\prime}, y, y^{\prime}, z, z^{\prime}\right)\right| \leq \frac{1}{\nu}\left|\binom{A(s)\left(\widehat{x}-\widehat{x}^{\prime}\right)}{\widetilde{A}(s) \widehat{x}^{\prime}}\right| \\
\left|f\left(s, x, x^{\prime}, y, y^{\prime}, z, z^{\prime}\right)-f\left(s, x, x^{\prime}, \bar{y}, \bar{y}^{\prime}, \bar{z}, \bar{z}^{\prime}\right)\right| \leq \frac{1}{\mu}\left|\binom{B(s)\left(\widehat{y}-\widehat{y}^{\prime}\right)+C(s)\left(\widehat{z}-\widehat{z}^{\prime}\right)}{\widetilde{B}(s) \widehat{y}^{\prime}+\widetilde{C}(s) \widehat{z}^{\prime}}\right| .
\end{array}\right.
$$

Here, we have a bit of abusive notations, i.e., when $\mu=0$ (resp. $\nu=0$ ) , $1 / \mu($ resp. $1 / \nu$ ) means $+\infty$. In other words, if $\mu=0$ or $\nu=0$, the corresponding domination conditions will vanish.
(iii) (Monotonicity conditions). For any $y, \bar{y} \in \mathbb{R}^{n}$ and almost all $\omega \in \Omega$,

$$
\begin{equation*}
\langle\Psi(y)-\Psi(\bar{y}), \widehat{y}\rangle \leq-\mu|H \widehat{y}|^{2} . \tag{3.2}
\end{equation*}
$$

For any random variable $X$, we use the notations

$$
\begin{equation*}
X^{(1)}:=X-\mathbb{E}_{t}[X] \quad \text { and } \quad X^{(2)}:=\mathbb{E}_{t}[X] \tag{3.3}
\end{equation*}
$$

For any $X, \bar{X} \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\mathbb{E}_{t}\left[\left\langle\binom{\Phi\left(X, \mathbb{E}_{t}[X]\right)^{(1)}-\Phi\left(\bar{X}, \mathbb{E}_{t}[\bar{X}]\right)^{(1)}}{\Phi\left(X, \mathbb{E}_{t}[X]\right)^{(2)}-\Phi\left(\bar{X}, \mathbb{E}_{t}[\bar{X}]\right)^{(2)}},\binom{\widehat{X}^{(1)}}{\widehat{X}^{(2)}}\right\rangle\right] \geq \nu \mathbb{E}_{t}\left[\left|\binom{P \widehat{X}^{(1)}}{\widetilde{P} \widehat{X}^{(2)}}\right|^{2}\right] \tag{3.4}
\end{equation*}
$$

For almost all $s \in[t, T]$ and any $\Theta:=\left(X^{\top}, Y^{\top}, Z^{\top}\right)^{\top}, \bar{\Theta}:=\left(X^{\top}, Y^{\top}, Z^{\top}\right)^{\top} \in L_{\mathcal{F}_{s}}^{2}\left(\Omega ; \mathbb{R}^{n(2+d)}\right)$,

$$
\begin{align*}
& \mathbb{E}_{t}\left[\left\langle\binom{\Gamma\left(s, \Theta, \mathbb{E}_{t}[\Theta]\right)^{(1)}-\Gamma\left(s, \bar{\Theta}, \mathbb{E}_{t}[\bar{\Theta}]\right)^{(1)}}{\Gamma\left(s, \Theta, \mathbb{E}_{t}[\Theta]\right)^{(2)}-\Gamma\left(s, \bar{\Theta}, \mathbb{E}_{t}[\bar{\Theta}]\right)^{(2)}},\binom{\widehat{\Theta}^{(1)}}{\widehat{\Theta}^{(2)}}\right\rangle\right] \\
\leq & -\nu \mathbb{E}_{t}\left[\left|\binom{A(s) \widehat{X}^{(1)}}{\widetilde{A}(s) \widehat{X}^{(2)}}\right|^{2}\right]-\mu \mathbb{E}_{t}\left[\left|\binom{B(s) \widehat{Y}^{(1)}+C(s) \widehat{Z}^{(1)}}{\widetilde{B}(s) \widehat{Y}^{(2)}+\widetilde{C}(s) \widehat{Z}^{(2)}}\right|^{2}\right] . \tag{3.5}
\end{align*}
$$

The domination and monotonicity conditions in Assumption (H4), especially (3.4) and (3.5) in some form of randomization, are a bit complicated and not easy to understand. Therefore, we would like to give a remark and two examples to get some feeling. Firstly, the following remark shows a special case of Assumption (H4).
Remark 3.1. We introduce
Assumption (H4-S) (Coefficients without mean-field terms). Let the coefficients $\Phi$ and $\Gamma$ are independent of $x^{\prime}$ and $\theta^{\prime}$, respectively. There exist two constants $\mu \geq 0, \nu \geq 0$, two matrices $H \in \mathbb{R}^{m_{1} \times n}$, $P \in \mathbb{R}^{m_{2} \times n}$, and three matrix-valued processes $A(\cdot), B(\cdot), C(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right)$ (where $m_{1}, m_{2}, m_{3} \in \mathbb{N}$ are given) such that the following conditions hold:
(i) One of the following two cases holds. Case $A: \mu>0$ and $\nu=0$. Case $B: \mu=0$ and $\nu>0$.
(ii) (Domination conditions). For any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^{n}$, any $z, \bar{z} \in \mathbb{R}^{n d}$, and almost all $(s, \omega) \in[t, T] \times \Omega$, we have

$$
\left\{\begin{array}{l}
|\Psi(y)-\Psi(\bar{y})| \leq \frac{1}{\mu}|H \widehat{y}|  \tag{3.6}\\
|\Phi(x)-\Phi(\bar{x})| \leq \frac{1}{\nu}|P \widehat{x}| \\
|g(s, x, y, z)-g(s, \bar{x}, y, z)| \leq \frac{1}{\nu}|A(s) \widehat{x}| \\
|f(s, x, y, z)-f(s, x, \bar{y}, \bar{z})| \leq \frac{1}{\mu}|B(s) \widehat{y}+C(s) \widehat{z}|
\end{array}\right.
$$

(iii) (Monotonicity conditions). For any $x, \bar{x}, y, \bar{y} \in \mathbb{R}^{n}$, any $\theta, \bar{\theta} \in \mathbb{R}^{n(2+d)}$, and almost all $(s, \omega) \in$ $[t, T] \times \Omega$, we have

$$
\left\{\begin{array}{l}
\langle\Psi(y)-\Psi(\bar{y}), \widehat{y}\rangle \leq-\mu|H \widehat{y}|^{2}  \tag{3.7}\\
\langle\Phi(x)-\Phi(\bar{x}), \widehat{x}\rangle \geq \nu|P \widehat{x}|^{2} \\
\langle\Gamma(s, \theta)-\Gamma(s, \bar{\theta}), \widehat{\theta}\rangle \leq-\nu|A(s) \widehat{x}|^{2}-\mu|B(s) \widehat{y}+C(s) \widehat{z}|^{2}
\end{array}\right.
$$

A straightforward verification shows that the above Assumption (H4-S) implies Assumption (H4). We notice that Assumption (H4-S) and its special cases have been extensively studied in the literature, such as [8, 17, 22, 25] and so on. The present paper can be regarded as an extension of these studies in the mean-field case.

Now, we give an example of a decoupled linear MF-FBSDE where the coefficients satisfy Assumptions (H3) and (H4).

Example 3.2. Let $\mu \geq 0$ and $\nu \geq 0$ be two constants satisfying Assumption (H4)-(i), $H \in \mathbb{R}^{m_{1} \times n}, P, \widetilde{P} \in$ $\mathbb{R}^{m_{2} \times n}$ be three matrices, and $A(\cdot), \widetilde{A}(\cdot), B(\cdot), \widetilde{B}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right), C(\cdot)=\left(C_{1}(\cdot), C_{2}(\cdot), \ldots, C_{d}(\cdot)\right)$, $\widetilde{C}(\cdot)=\left(\widetilde{C}_{1}(\cdot), \widetilde{C}_{2}(\cdot), \ldots, \widetilde{C}_{d}(\cdot)\right)$ with $C_{i}(\cdot), \widetilde{C}_{i}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right)$ be six matrix-valued processes. For any $y, x, x^{\prime} \in \mathbb{R}^{n}$, any $\theta, \theta^{\prime} \in \mathbb{R}^{n(2+d)}$ and any $(s, \omega) \in[t, T] \times \Omega$, we define

$$
\left\{\begin{array}{l}
\Psi^{0}(y)=-\mu H^{\top} H y  \tag{3.8}\\
\Phi^{0}\left(x, x^{\prime}\right)=\nu\left\{P^{\top} P\left(x-x^{\prime}\right)+\widetilde{P}^{\top} \widetilde{P} x^{\prime}\right\} \\
g^{0}\left(s, \theta, \theta^{\prime}\right)=-\nu\left\{A(s)^{\top} A(s)\left(x-x^{\prime}\right)+\widetilde{A}(s)^{\top} \widetilde{A}(s) x^{\prime}\right\} \\
b^{0}\left(s, \theta, \theta^{\prime}\right)=-\mu\left\{B(s)^{\top}\left[B(s)\left(y-y^{\prime}\right)+C(s)\left(z-z^{\prime}\right)\right]+\widetilde{B}(s)^{\top}\left[\widetilde{B}(s) y^{\prime}+\widetilde{C}(s) z^{\prime}\right]\right\} \\
\sigma^{0}\left(s, \theta, \theta^{\prime}\right)=-\mu\left\{C(s)^{\top}\left[B(s)\left(y-y^{\prime}\right)+C(s)\left(z-z^{\prime}\right)\right]+\widetilde{C}(s)^{\top}\left[\widetilde{B}(s) y^{\prime}+\widetilde{C}(s) z^{\prime}\right]\right\}
\end{array}\right.
$$

Similarly, we denote $\Gamma^{0}:=\left(\left(g^{0}\right)^{\top},\left(b^{0}\right)^{\top},\left(\sigma^{0}\right)^{\top}\right)^{\top}$. When Case A in Assumption (H4)-(i) holds, we assume that

$$
\begin{equation*}
\frac{1}{\mu^{2}} \geq \max \left\{\|H\|,\left\|\binom{B(\cdot)}{\widetilde{B}(\cdot)}\right\|_{L^{\infty}\left(t, T ; \mathbb{R}^{2 m_{3} \times n}\right)},\left\|\binom{C(\cdot)}{\widetilde{C}(\cdot)}\right\|_{L^{\infty}\left(t, T ; \mathbb{R}^{2 m_{3} \times n d}\right)}\right\} \tag{3.9}
\end{equation*}
$$

Then Assumptions (H4)-(ii) and (H4)-(iii) also hold for $\left(\Psi^{0}, \Phi^{0}, \Gamma^{0}\right)$ which are defined by (3.8). Moreover, in this case, Assumption (H3) holds for $\left(\Psi^{0}, \Phi^{0}, \Gamma^{0}\right)$ with the Lipschitz constant

$$
\begin{equation*}
L \geq \mu \max \left\{\|H\|^{2},\|(B(\cdot), C(\cdot))\|_{L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n(1+d)}\right)}^{2}+\|(\widetilde{B}(\cdot), \widetilde{C}(\cdot))\|_{L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n(1+d)}\right)}^{2}\right\} \tag{3.10}
\end{equation*}
$$

When Case B in Assumption (H4)-(i) holds, we assume that

$$
\begin{equation*}
\frac{1}{\nu^{2}} \geq \max \left\{\left\|\binom{P}{\widetilde{P}}\right\|,\left\|\binom{A(\cdot)}{\widetilde{A}(\cdot)}\right\|_{L^{\infty}\left(t, T ; \mathbb{R}^{2 m_{3} \times n}\right)}\right\} \tag{3.11}
\end{equation*}
$$

Then Assumptions (H4)-(ii) and (H4)-(iii) also hold. Moreover, in this case, Assumption (H3) holds with the Lipschitz constant

$$
\begin{equation*}
L \geq \nu \max \left\{\|P\|^{2}+\|\widetilde{P}\|^{2},\|A(\cdot)\|_{L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right)}^{2}+\|\widetilde{A}(\cdot)\|_{L^{\infty}\left(t, T ; \mathbb{R}^{m_{3} \times n}\right)}^{2}\right\} \tag{3.12}
\end{equation*}
$$

The above Example 3.2 will be useful in our following analysis of this section. Another two coupled linear examples coming from the Hamiltonian systems of LQ control problems will be provided in Section 4. Next, we shall give a nonlinear example.

Example 3.3. Let $n=d=1$. Let $k_{1} \geq 1$ and $k_{2} \geq 1$ be two constants. For any $y, x, x^{\prime} \in \mathbb{R}$ and any $\theta, \theta^{\prime} \in \mathbb{R}^{3}$, we define

$$
\left\{\begin{array}{l}
\Psi(y)=-k_{1} y+\sin y, \quad b\left(\theta, \theta^{\prime}\right) \equiv b\left(y, y^{\prime}\right)=-k_{1} y+\sin y^{\prime}, \quad \sigma\left(\theta, \theta^{\prime}\right) \equiv \sigma\left(z, z^{\prime}\right)=-k_{1} z+\sin z^{\prime} \\
\Phi\left(x, x^{\prime}\right)=k_{2} x+\sin x^{\prime}, \quad g\left(\theta, \theta^{\prime}\right) \equiv g\left(x, x^{\prime}\right)=-k_{2} x+\sin x^{\prime}
\end{array}\right.
$$

Let $H=P=\widetilde{P}=1$ and

$$
A=\widetilde{A}=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right), \quad B=\widetilde{B}=\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right), \quad C=\widetilde{C}=\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)
$$

With the same notations in Assumption (H4), we firstly try to derive the monotonicity conditions:

$$
\begin{aligned}
& \left\langle\binom{\Gamma\left(s, \Theta, \mathbb{E}_{t}[\Theta]\right)^{(1)}-\Gamma\left(s, \bar{\Theta}, \mathbb{E}_{t}[\bar{\Theta}]\right)^{(1)}}{\Gamma\left(s, \Theta, \mathbb{E}_{t}[\Theta]\right)^{(2)}-\Gamma\left(s, \bar{\Theta}, \mathbb{E}_{t}[\bar{\Theta}]\right)^{(2)}},\binom{\widehat{\Theta}^{(1)}}{\widehat{\Theta}^{(2)}}\right\rangle \\
\leq & -k_{2}\left|\widehat{X}^{(1)}\right|^{2}-k_{1}\left[\left|\widehat{Y}^{(1)}\right|^{2}+\left|\widehat{Z}^{(1)}\right|^{2}\right]-\left(k_{2}-1\right)\left|\widehat{X}^{(2)}\right|^{2}-\left(k_{1}-1\right)\left[\left|\widehat{Y}^{(2)}\right|^{2}+\left|\widehat{Z}^{(2)}\right|^{2}\right] \\
\leq & -\left(k_{2}-1\right)\left[\left|\widehat{X}^{(1)}\right|^{2}+\left|\widehat{X}^{(2)}\right|^{2}\right]-\left(k_{1}-1\right)\left[\left.\widehat{Y}^{(1)}\right|^{2}+\left|\widehat{Z}^{(1)}\right|^{2}+\left|\widehat{Y}^{(2)}\right|^{2}+\left|\widehat{Z}^{(2)}\right|^{2}\right] \\
= & -\left(k_{2}-1\right)\left|\binom{A \widehat{X}^{(1)}}{\widetilde{A} \widehat{X}^{(2)}}\right|^{2}-\left(k_{1}-1\right)\left|\binom{B \widehat{Y}^{(1)}+C \widehat{Z}^{(1)}}{\widetilde{B} \widehat{Y}^{(2)}+\widetilde{C} \widehat{Z}^{(2)}}\right|^{2} .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \langle\Psi(y)-\Psi(\bar{y}), \widehat{y}\rangle \leq-\left(k_{1}-1\right)|H \widehat{y}|^{2}, \\
& \left\langle\binom{\Phi\left(X, \mathbb{E}_{t}[X]\right)^{(1)}-\Phi\left(\bar{X}, \mathbb{E}_{t}[\bar{X}]\right)^{(1)}}{\Phi\left(X, \mathbb{E}_{t}[X]\right)^{(2)}-\Phi\left(\bar{X}, \mathbb{E}_{t}[\bar{X}]\right)^{(2)}},\binom{\widehat{X}^{(1)}}{\widehat{X}^{(2)}}\right\rangle \geq\left(k_{2}-1\right)\left|\binom{P \widehat{X}^{(1)}}{\widetilde{P}^{(2)}}\right|^{2} .
\end{aligned}
$$

Secondly, we try to calculate the domination conditions:

$$
\left\{\begin{array}{l}
|\Psi(y)-\Psi(\bar{y})| \leq\left(k_{1}+1\right)|H \widehat{y}|, \\
\left|\Phi\left(x, x^{\prime}\right)-\Phi\left(\bar{x}, \bar{x}^{\prime}\right)\right| \leq 2\left(k_{2}+1\right)\left|\binom{P\left(\widehat{x}-\widehat{x}^{\prime}\right)}{\widetilde{P} \widehat{x}^{\prime}}\right| \\
\left|g\left(x, x^{\prime}\right)-g\left(\bar{x}, \bar{x}^{\prime}\right)\right| \leq 2\left(k_{2}+1\right)\left|\binom{A\left(\widehat{x}-\widehat{x}^{\prime}\right)}{\widetilde{A} \widehat{x}^{\prime}}\right| \\
\left|b\left(y, y^{\prime}\right)-b\left(\bar{y}, \bar{y}^{\prime}\right)\right|+\left|\sigma\left(z, z^{\prime}\right)-\sigma\left(\bar{z}, \bar{z}^{\prime}\right)\right| \leq 2\left(k_{1}+1\right)\left|\binom{B\left(\widehat{y}-\widehat{y}^{\prime}\right)+C\left(\widehat{z}-\widehat{z}^{\prime}\right)}{\widetilde{B} \widehat{y}^{\prime}+\widetilde{C} \widehat{z}^{\prime}}\right|
\end{array}\right.
$$

Based on the above calculation, we get the following results. On the one hand, when $k_{1}>1$ and $k_{2} \geq 1$, selecting $0<\mu \leq \min \left\{k_{1}-1,1 /\left(2\left(k_{1}+1\right)\right)\right\}$ and $\nu=0$ leads to Assumptions (H4)-(i)Case A, (H4)-(ii) and (H4)-(iii). On the other hand, when $k_{1} \geq 1$ and $k_{2}>1$, we choose $\mu=0$ and $0<\nu \leq \min \left\{k_{2}-1,1 /\left(2\left(k_{2}+1\right)\right)\right\}$ to get Assumptions (H4)-(i)-Case B, (H4)-(ii) and (H4)-(iii).

Now, we are in the position to give the main results of this section.

Theorem 3.4. Let Assumptions (H3) and (H4) hold for a set of coefficients ( $\Psi, \Phi, \Gamma$ ). Then MF-FBSDE (1.1) admits a unique solution $\theta(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. Moreover, the following estimate holds:

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|x(s)|^{2}+\sup _{s \in[t, T]}|y(s)|^{2}+\int_{t}^{T}|z(s)|^{2} \mathrm{~d} s\right] \leq K \mathbb{E}_{t}[\mathrm{I}], \tag{3.13}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{I}=|\Psi(0)|^{2}+|\Phi(0,0)|^{2}+\left(\int_{t}^{T}|g(s, 0,0)| \mathrm{d} s\right)^{2}+\left(\int_{t}^{T}|b(s, 0,0)| \mathrm{d} s\right)^{2}+\int_{t}^{T}|\sigma(s, 0,0)|^{2} \mathrm{~d} s \tag{3.14}
\end{equation*}
$$

and $K>0$ is a constant depending on $(T-t)$, the Lipschitz constant, $\mu$, $\nu$, and the bound of all $H$, $\underline{P}, \widetilde{P}, A(\cdot), \widetilde{A}(\cdot), B(\cdot), \widetilde{B}(\cdot), C(\cdot), \widetilde{C}(\cdot)$. Furthermore, Let $(\bar{\Psi}, \bar{\Phi}, \bar{\Gamma})$ be another set of coefficients and $\bar{\theta}(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{(n+2) d}\right)$ be a solution to the MF-FBSDE with the coefficients $(\bar{\Psi}, \bar{\Phi}, \bar{\Gamma})$. We assume that $\left(\bar{\Psi}(\bar{y}(t)), \bar{\Phi}\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right), \bar{\Gamma}\left(\cdot, \bar{\theta}(\cdot), \mathbb{E}_{t}[\bar{\theta}(\cdot)]\right)\right) \in \mathcal{H}[t, T]$. Then, we have

$$
\begin{equation*}
\left.\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}(s)|^{2} \mathrm{~d} s\right] \leq K \mathbb{E}_{t} \widehat{\mathrm{I}}\right], \tag{3.15}
\end{equation*}
$$

where $\widehat{x}(\cdot)=x(\cdot)-\bar{x}(\cdot), \widehat{y}(\cdot)=y(\cdot)-\bar{y}(\cdot), \widehat{z}(\cdot)=z(\cdot)-\bar{z}(\cdot)$,

$$
\begin{align*}
\widehat{\mathrm{I}}= & |\Psi(\bar{y}(t))-\bar{\Psi}(\bar{y}(t))|^{2}+\left|\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)-\bar{\Phi}\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)\right|^{2} \\
& +\left(\int_{t}^{T}\left|g\left(s, \bar{\theta}(s), \mathbb{E}_{t}[\bar{\theta}(s)]\right)-\bar{g}\left(s, \bar{\theta}(s), \mathbb{E}_{t}[\bar{\theta}(s)]\right)\right| \mathrm{d} s\right)^{2} \\
& +\left(\int_{t}^{T}\left|b\left(s, \bar{\theta}(s), \mathbb{E}_{t}[\bar{\theta}(s)]\right)-\bar{b}\left(s, \bar{\theta}(s), \mathbb{E}_{t}[\bar{\theta}(s)]\right)\right| \mathrm{d} s\right)^{2}  \tag{3.16}\\
& +\int_{t}^{T}\left|\sigma\left(s, \bar{\theta}(s), \mathbb{E}_{t}[\bar{\theta}(s)]\right)-\bar{\sigma}\left(s, \bar{\theta}(s), \mathbb{E}_{t}[\bar{\theta}(s)]\right)\right|^{2} \mathrm{~d} s
\end{align*}
$$

and $K$ is the same constant as in (3.13).
Next, we are going to prove Theorem 3.4 by virtue of the method of continuation. Let us introduce the convex combination of $(\Psi, \Phi, \Gamma)$ and $\left(\Psi^{0}, \Phi^{0}, \Gamma^{0}\right)$ (see (3.8)):

$$
\begin{equation*}
\left(\Psi^{\alpha}, \Phi^{\alpha}, \Gamma^{\alpha}\right):=\alpha(\Psi, \Phi, \Gamma)+(1-\alpha)\left(\Psi^{0}, \Phi^{0}, \Gamma^{0}\right), \quad \alpha \in[0,1] \tag{3.17}
\end{equation*}
$$

For any $(\xi, \eta, \rho(\cdot)) \in \mathcal{H}[t, T]$ with $\rho(\cdot)=\left(\varphi(\cdot)^{\top}, \psi(\cdot)^{\top}, \gamma(\cdot)^{\top}\right)^{\top}$ and $\gamma(\cdot)=\left(\gamma_{1}(\cdot)^{\top}, \gamma_{2}(\cdot)^{\top}, \ldots, \gamma_{d}(\cdot)^{\top}\right)^{\top}$, we continue to introduce a family of MF-FBSDEs parameterized by $\alpha \in[0,1]$ as follows:

$$
\left\{\begin{array}{l}
\mathrm{d} x^{\alpha}(s)=\left\{b^{\alpha}\left(s, \theta^{\alpha}(s), \mathbb{E}_{t}\left[\theta^{\alpha}(s)\right]\right)+\psi(s)\right\} \mathrm{d} s  \tag{3.18}\\
\quad+\sum_{i=1}^{d}\left\{\sigma_{i}^{\alpha}\left(s, \theta^{\alpha}(s), \mathbb{E}_{t}\left[\theta^{\alpha}(s)\right]\right)+\gamma_{i}(s)\right\} \mathrm{d} W_{i}(s), \quad s \in[t, T], \\
\mathrm{d} y^{\alpha}(s)=\left\{g^{\alpha}\left(s, \theta^{\alpha}(s), \mathbb{E}_{t}\left[\theta^{\alpha}(s)\right]\right)+\varphi(s)\right\} \mathrm{d} s+\sum_{i=1}^{d} z_{i}^{\alpha}(s) \mathrm{d} W_{i}(s), \quad s \in[t, T], \\
x^{\alpha}(t)=\Psi^{\alpha}\left(y^{\alpha}(t)\right)+\xi, \quad y^{\alpha}(T)=\Phi^{\alpha}\left(x^{\alpha}(T), \mathbb{E}_{t}\left[x^{\alpha}(T)\right]\right)+\eta .
\end{array}\right.
$$

Without loss of generality, we assume that the Lipschitz constant $L$ of the original coefficients $(\Psi, \Phi, \Gamma)$ is big enough and the constant $\mu$ in Assumption (H4)-(i)-Case A and the constant $\nu$ in Assumption (H4)-(i)-Case B are small enough such that the inequalities (3.9), (3.10), (3.11), (3.12) hold true. Then, a straightforward calculation shows that, for any $\alpha \in[0,1]$, the new coefficients ( $\Psi^{\alpha}, \Phi^{\alpha}, \Gamma^{\alpha}$ ) (see the definition (3.17)) also satisfy Assumptions (H3) and (H4) with the same Lipschitz constant $L, \mu, \nu, H$, $P, \widetilde{P}, A(\cdot), \widetilde{A}(\cdot), B(\cdot), \widetilde{B}(\cdot), C(\cdot)$, and $\widetilde{C}(\cdot)$ as the original coefficients $(\Psi, \Phi, \Gamma)$.

When $\alpha=0$, we substitute (3.8) into MF-FBSDE (3.18) to get

$$
\left\{\begin{array}{l}
\mathrm{d} x^{0}=\left\{-\mu\left[B^{\top}\left(B y^{0(1)}+C z^{0(1)}\right)+\widetilde{B}^{\top}\left(\widetilde{B} y^{0(2)}+\widetilde{C} z^{0(2)}\right)\right]+\psi\right\} \mathrm{d} s  \tag{3.19}\\
\quad+\sum_{i=1}^{d}\left\{-\mu\left[C_{i}^{\top}\left(B y^{0(1)}+C z^{0(1)}\right)+\widetilde{C}_{i}^{\top}\left(\widetilde{B} y^{0(2)}+\widetilde{C} z^{0(2)}\right)\right]+\gamma_{i}\right\} \mathrm{d} W_{i}, \quad s \in[t, T], \\
\mathrm{d} y^{0}=\left\{-\nu\left[A^{\top} A x^{0(1)}+\widetilde{A}^{\top} \widetilde{A} x^{0(2)}\right]+\varphi\right\} \mathrm{d} s+\sum_{i=1}^{d} z_{i}^{0} \mathrm{~d} W_{i}, \quad s \in[t, T], \\
x^{0}(t)=-\mu H^{\top} H y^{0}(t)+\xi, \quad y^{0}(T)=\nu\left[P^{\top} P x^{0}(T)^{(1)}+\widetilde{P}^{\top} \widetilde{P} x^{0}(T)^{(2)}\right]+\eta,
\end{array}\right.
$$

where the argument $s$ is suppressed and the decompositions (see (3.3)) of the corresponding random variables are used for simplicity of notations. Let us discuss in two situations. (i) When Assumption (H4)-(i)-Case A holds true (i.e., $\mu>0$ and $\nu=0$ ), MF-FBSDE (3.19) is in a decoupled form. In fact, we can solve the MF-BSDE first to get $\left(y^{0}(\cdot), z^{0}(\cdot)\right)$. Then we substitute $\left(y^{0}(\cdot), z^{0}(\cdot)\right)$ into the MF-SDE and solve $x^{0}(\cdot)$. (ii) When Assumption (H4)-(i)-Case B holds true (i.e., $\mu=0$ and $\nu>0$ ), MF-FBSDE (3.19) is also in a decoupled form. The difference is that, in this case, we first solve the MF-SDE and then MF-BSDE. In summary, under Assumptions (H3) and (H4), MF-FBSDE (3.19) admits a unique solution $\theta^{0}(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$.

When $\alpha=1$ and $(\xi, \eta, \rho(\cdot))$ vanish, MF-FBSDE (3.18) coincides with MF-FBSDE (1.1) that we care about. Next, we will show that if for some $\alpha_{0} \in[0,1)$, MF-FBSDE (3.18) is uniquely solvable for any $(\xi, \eta, \rho(\cdot)) \in \mathcal{H}[t, T]$, then there exists a fixed step length $\delta_{0}>0$ such that for any $\alpha \in\left[\alpha_{0}, \alpha_{0}+\delta_{0}\right]$, the same conclusion also holds. For this aim, we firstly establish a priori estimate for the solution to MF-FBSDE (3.18).

Lemma 3.5. Let Assumptions (H3) and (H4) hold for a given set of coefficients $(\Psi, \Phi, \Gamma)$. Let $\alpha \in[0,1]$ and $(\xi, \eta, \rho(\cdot)),(\bar{\xi}, \bar{\eta}, \bar{\rho}(\cdot)) \in \mathcal{H}[t, T]$. Suppose that $\theta(\cdot), \bar{\theta}(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$ satisfy MF-FBSDEs (3.18) with coefficients $\left(\Psi^{\alpha}+\xi, \Phi^{\alpha}+\eta, \Gamma^{\alpha}+\rho\right)$ and $\left(\Psi^{\alpha}+\bar{\xi}, \Phi^{\alpha}+\bar{\eta}, \Gamma^{\alpha}+\bar{\rho}\right)$, respectively. Then the following estimate holds:

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}(s)|^{2} \mathrm{~d} s\right] \leq K \mathbb{E}_{t}[\widehat{J}] \tag{3.20}
\end{equation*}
$$

where

$$
\begin{equation*}
\widehat{\mathrm{J}}=|\widehat{\xi}|^{2}+|\widehat{\eta}|^{2}+\left(\int_{t}^{T}|\widehat{\varphi}(s)| \mathrm{d} s\right)^{2}+\left(\int_{t}^{T}|\widehat{\psi}(s)| \mathrm{d} s\right)^{2}+\int_{t}^{T}|\widehat{\gamma}(s)|^{2} \mathrm{~d} s \tag{3.21}
\end{equation*}
$$

and $\widehat{x}(\cdot)=x(\cdot)-\bar{x}(\cdot), \widehat{\xi}=\xi-\bar{\xi}$, etc. Here $K>0$ is a constant depending on $(T-t)$, the Lipschitz constant $L, \mu, \nu$, and the bounds of all $H, P, \widetilde{P}, A(\cdot), \widetilde{A}(\cdot), B(\cdot), \widetilde{B}(\cdot), C(\cdot)$, and $\widetilde{C}(\cdot)$.

Proof. By the estimate (2.3) for MF-SDEs, we have (the argument $s$ is suppressed and the decompositions (3.3) are used for simplicity)

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}\right] \leq K \mathbb{E}_{t}\left\{\left|\alpha(\Psi(y(t))-\Psi(\bar{y}(t)))-(1-\alpha) \mu H^{\top} H \widehat{y}(t)+\widehat{\xi}\right|^{2}\right. \\
&+\left(\int_{t}^{T} \mid \alpha\left(b\left(\bar{x}, \mathbb{E}_{t}[\bar{x}], y, \mathbb{E}_{t}[y], z, \mathbb{E}_{t}[z]\right)-b\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right)\right. \\
&\left.-(1-\alpha) \mu\left[B^{\top}\left(B \widehat{y}^{(1)}+C \widehat{z}^{(1)}\right)+\widetilde{B}^{\top}\left(\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}\right)\right]+\widehat{\psi} \mid \mathrm{d} s\right)^{2}  \tag{3.22}\\
&+\int_{t}^{T} \mid \alpha\left(\sigma\left(\bar{x}, \mathbb{E}_{t}[\bar{x}], y, \mathbb{E}_{t}[y], z, \mathbb{E}_{t}[z]\right)-\sigma\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right) \\
&\left.-(1-\alpha) \mu\left[C^{\top}\left(B \widehat{y}^{(1)}+C \widehat{z}^{(1)}\right)+\widetilde{C}^{\top}\left(\widetilde{B} \widehat{y}^{(2)}+\widetilde{C}^{(2)}\right)\right]+\left.\widehat{\gamma}\right|^{2} \mathrm{~d} s\right\}
\end{align*}
$$

where the constant $K>0$ could be changed line to line. Similarly, applying the estimate (2.6) for

MF-BSDEs leads to

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right] \\
\leq & K \mathbb{E}_{t}\left\{\mid \alpha\left(\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)-\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)\right)\right. \\
& +(1-\alpha) \nu\left[P^{\top} P \widehat{x}(T)^{(1)}+\widetilde{P}^{\top} \widetilde{P} \widehat{x}(T)^{(2)}\right]+\left.\widehat{\eta}\right|^{2}  \tag{3.23}\\
& +\left(\int_{t}^{T} \mid \alpha\left(g\left(x, \mathbb{E}_{t}[x], \bar{y}, \mathbb{E}_{t}[\bar{y}], \bar{z}, \mathbb{E}_{t}[\bar{z}]\right)-g\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right)\right. \\
& \left.\left.-(1-\alpha) \nu\left[A^{\top} A \widehat{x}^{(1)}+\widetilde{A}^{\top} \widetilde{A} \widehat{x}^{(2)}\right]+\widehat{\varphi} \mid \mathrm{d} s\right)^{2}\right\}
\end{align*}
$$

Moreover, we also apply Itô's formula to $\langle\widehat{x}(\cdot), \widehat{y}(\cdot)\rangle$ to yield

$$
\begin{equation*}
\mathrm{I}_{1}+\mathbb{E}_{t}[\langle\widehat{\eta}, \widehat{x}(T)\rangle]=\mathrm{I}_{2}+\langle\widehat{\xi}, \widehat{y}(t)\rangle+\int_{t}^{T}\left\{\mathrm{I}_{3}+\mathbb{E}_{t}[\langle\widehat{\rho}, \widehat{\theta}\rangle]\right\} \mathrm{d} s \tag{3.24}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathrm{I}_{1}=\alpha \mathbb{E}_{t}\left[\left\langle\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)-\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right), \widehat{x}(T)\right\rangle\right] \\
+(1-\alpha) \nu \mathbb{E}_{t}\left[\left\langle P^{\top} P \widehat{x}(T)^{(1)}+\widetilde{P}^{\top} \widetilde{P} \widehat{x}(T)^{(2)}, \widehat{x}(T)\right\rangle\right], \\
\mathrm{I}_{2}=\alpha\langle\Psi(y(t))-\Psi(\bar{y}(t)), \widehat{y}(t)\rangle-(1-\alpha) \mu\left\langle H^{\top} H \widehat{y}(t), \widehat{y}(t)\right\rangle,
\end{gathered}
$$

and

$$
\begin{aligned}
\mathrm{I}_{3}= & \alpha \mathbb{E}_{t}\left[\left\langle\Gamma\left(\theta, \mathbb{E}_{t}[\theta]\right)-\Gamma\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right), \widehat{\theta}\right\rangle\right]-(1-\alpha) \mathbb{E}_{t}\left[\nu\left\langle A^{\top} A \widehat{x}^{(1)}+\widetilde{A}^{\top} \widetilde{A}^{(2)}, \widehat{x}\right\rangle\right. \\
& +\mu\left\langle B^{\top}\left(B \widehat{y}^{(1)}+C \widehat{z}^{(1)}\right)+\widetilde{B}^{\top}\left(\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}\right), \widehat{y}\right\rangle \\
& \left.+\mu\left\langle C^{\top}\left(B \widehat{y}^{(1)}+C \widehat{z}^{(1)}\right)+\widetilde{C}^{\top}\left(\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}\right), \widehat{z}\right\rangle\right]
\end{aligned}
$$

The monotonicity conditions(3.4), (3.2) and (3.5) lead to

$$
\begin{gathered}
\mathrm{I}_{1}=\alpha \mathbb{E}_{t}\left[\left\langle\binom{\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)^{(1)}-\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)^{(1)}}{\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)^{(2)}-\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)^{(2)}},\binom{\widehat{x}(T)^{(1)}}{\left.\widehat{x}(T)^{(2)}\right)}\right\rangle\right] \\
+(1-\alpha) \nu \mathbb{E}_{t}\left[\left|\binom{P \widehat{x}(T)^{(1)}}{\widetilde{P} \widehat{x}(T)^{(2)}}\right|^{2}\right] \geq \nu \mathbb{E}_{t}\left[\left|\binom{P \widehat{x}(T)^{(1)}}{\widetilde{P} \widehat{x}(T)^{(2)}}\right|^{2}\right] \\
\mathrm{I}_{2} \leq-\mu|H \widehat{y}(t)|^{2}
\end{gathered}
$$

and

$$
\begin{aligned}
\mathrm{I}_{3}= & \alpha \mathbb{E}_{t}\left[\left\langle\binom{\Gamma\left(\theta, \mathbb{E}_{t}[\theta]\right)^{(1)}-\Gamma\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)^{(1)}}{\Gamma\left(\theta, \mathbb{E}_{t}[\theta]\right)^{(2)}-\Gamma\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)^{(2)}},\binom{\widehat{\theta}^{(1)}}{\widehat{\theta}^{(2)}}\right\rangle\right] \\
& -(1-\alpha) \nu \mathbb{E}_{t}\left[\left|\binom{A \widehat{x}^{(1)}}{\widetilde{A} \widehat{x}^{(2)}}\right|^{2}\right]-(1-\alpha) \mu \mathbb{E}_{t}\left[\left|\binom{B \widehat{y}^{(1)}+C \widehat{z}^{(1)}}{\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}}\right|^{2}\right] \\
\leq & -\nu \mathbb{E}_{t}\left[\left|\binom{A \widehat{x}^{(1)}}{\widetilde{A} \widehat{x}^{(2)}}\right|^{2}\right]-\mu \mathbb{E}_{t}\left[\left|\binom{B \widehat{y}^{(1)}+C \widehat{z}^{(1)}}{\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}}\right|^{2}\right]
\end{aligned}
$$

Therefore, (3.24) is reduced to

$$
\begin{align*}
& \mathbb{E}_{t}\left\{\mu|H \widehat{y}(t)|^{2}+\nu\left|\binom{P \widehat{x}(T)^{(1)}}{\widetilde{P} \widehat{x}(T)^{(2)}}\right|^{2}+\int_{t}^{T}\left[\nu\left|\binom{A \widehat{x}^{(1)}}{\widetilde{A} \widehat{x}^{(2)}}\right|^{2}+\mu\left|\binom{B \widehat{y}^{(1)}+C \widehat{z}^{(1)}}{\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}}\right|^{2}\right] \mathrm{d} s\right\}  \tag{3.25}\\
& \leq \mathbb{E}_{t}\left\{\langle\widehat{\xi}, \widehat{y}(t)\rangle-\langle\widehat{\eta}, \widehat{x}(T)\rangle+\int_{t}^{T}\langle\widehat{\rho}, \widehat{\theta}\rangle \mathrm{d} s\right\}
\end{align*}
$$

We will divide the remaining proof into two cases according to Assumption (H4)-(i).
Case A: $\mu>0$ and $\nu=0$. By applying the domination conditions (3.1) to the estimate (3.22), we have

$$
\begin{align*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}\right] \leq & K \mathbb{E}_{t}\left\{|H \widehat{y}(t)|^{2}+\int_{t}^{T}\left|\binom{B \widehat{y}^{(1)}+C \widehat{z}^{(1)}}{\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}}\right|^{2} \mathrm{~d} s\right.  \tag{3.26}\\
& \left.+|\widehat{\xi}|^{2}+\left(\int_{t}^{T}|\widehat{\psi}| \mathrm{d} s\right)^{2}+\int_{t}^{T}|\widehat{\gamma}|^{2} \mathrm{~d} s\right\}
\end{align*}
$$

Applying the Lipschitz condition to (3.23) leads to

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right] \leq K \mathbb{E}_{t}\left\{\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+|\widehat{\eta}|^{2}+\left(\int_{t}^{T}|\widehat{\varphi}| \mathrm{d} s\right)^{2}\right\} . \tag{3.27}
\end{equation*}
$$

Combining (3.26) and (3.27) yields

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right]  \tag{3.28}\\
\leq & K \mathbb{E}_{t}\left\{|H \widehat{y}(t)|^{2}+\int_{t}^{T}\left|\binom{B \widehat{y}^{(1)}+C \widehat{z}^{(1)}}{\widetilde{B} \widehat{y}^{(2)}+\widetilde{C} \widehat{z}^{(2)}}\right|^{2} \mathrm{~d} s+\widehat{\mathrm{J}}\right\}
\end{align*}
$$

where $\widehat{J}$ is defined by (3.21). We continue to combine (3.28) and (3.25) to have

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right] \\
\leq & K_{1} \mathbb{E}_{t}\left\{\widehat{J}+\langle\widehat{\xi}, \widehat{y}(t)\rangle-\langle\widehat{\eta}, \widehat{x}(T)\rangle+\int_{t}^{T}\langle\widehat{\rho}, \widehat{\theta}\rangle \mathrm{d} s\right\}  \tag{3.29}\\
\leq & \mathbb{E}_{t}\left\{K_{1} \widehat{J}+K_{1}^{2}\left[|\widehat{\xi}|^{2}+|\widehat{\eta}|^{2}+\left(\int_{t}^{T}|\widehat{\varphi}| \mathrm{d} s\right)^{2}+\left(\int_{t}^{T}|\widehat{\psi}| \mathrm{d} s\right)^{2}\right]\right. \\
& \left.+\frac{K_{1}^{2}}{2} \int_{t}^{T}|\widehat{\gamma}|^{2} \mathrm{~d} s+\frac{1}{2}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right]\right\}
\end{align*}
$$

where the inequality $a b \leq \frac{1}{4 \varepsilon} a^{2}+\varepsilon b^{2}$ for any $\varepsilon>0$ and any $a, b \in \mathbb{R}$ was used. Clearly, the above (3.29) implies the desired estimate (3.20). We finish the proof in this case.

Case B: $\mu=0$ and $\nu>0$. In this case, we apply the domination conditions (3.1) to the estimate (3.23) to get

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right] \\
\leq & K \mathbb{E}_{t}\left\{\left|\binom{P \widehat{x}(T)^{(1)}}{\widetilde{P} \widehat{x}(T)^{(2)}}\right|^{2}+\int_{t}^{T}\left|\binom{A \widehat{x}^{(1)}}{\widetilde{A} \widehat{x}^{(2)}}\right|^{2} \mathrm{~d} s+|\widehat{\eta}|^{2}+\left(\int_{t}^{T}|\widehat{\varphi}| \mathrm{d} s\right)^{2}\right\} . \tag{3.30}
\end{align*}
$$

By the Lipschitz conditions on the coefficients $\Psi, b$ and $\sigma$, we deduce from (3.22) to have

$$
\begin{align*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}\right] \leq & K \mathbb{E}_{t}\left\{\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right. \\
& \left.+|\widehat{\xi}|^{2}+\left(\int_{t}^{T}|\widehat{\psi}| \mathrm{d} s\right)^{2}+\int_{t}^{T}|\widehat{\gamma}|^{2} \mathrm{~d} s\right\} \tag{3.31}
\end{align*}
$$

Then, (3.31) and (3.30) work together to yield

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right] \\
\leq & K \mathbb{E}_{t}\left\{\left|\binom{P \widehat{x}(T)^{(1)}}{\widetilde{P} \widehat{x}(T)^{(2)}}\right|^{2}+\int_{t}^{T}\left|\binom{A \widehat{x}^{(1)}}{\widetilde{A} \widehat{x}^{(2)}}\right|^{2} \mathrm{~d} s+\widehat{\mathrm{J}}\right\}, \tag{3.32}
\end{align*}
$$

where $\widehat{J}$ is defined by (3.21). Then, combining (3.32) and (3.25) leads to

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in[t, T]}|\widehat{x}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{y}(s)|^{2}+\int_{t}^{T}|\widehat{z}|^{2} \mathrm{~d} s\right]  \tag{3.33}\\
\leq & K_{2} \mathbb{E}_{t}\left\{\widehat{J}+\langle\widehat{\xi}, \widehat{y}(t)\rangle-\langle\widehat{\eta}, \widehat{x}(T)\rangle+\int_{t}^{T}\langle\widehat{\rho}, \widehat{\theta}\rangle \mathrm{d} s\right\}
\end{align*}
$$

By an argument similar to (3.29), we know that the above inequality implies the desired priori estimate (3.20). The whole proof of the lemma is completed.

With the help of Lemma 3.5, we provide the following continuation lemma.
Lemma 3.6. Let Assumptions (H3) and (H4) hold. If for some $\alpha_{0} \in[0,1$ ), MF-FBSDE (3.18) is uniquely solvable in $M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$ for any given $(\xi, \eta, \rho(\cdot)) \in \mathcal{H}[t, T]$, then there exists an absolute constant $\delta_{0}>0$ such that the same conclusion also holds for $\alpha=\alpha_{0}+\delta$ where $\delta \in\left(0, \delta_{0}\right]$ and $\alpha \leq 1$.

Proof. Let $\delta_{0}>0$ be determined below, and $\delta \in\left(0, \delta_{0}\right]$. For any $\theta(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$, we consider the following MF-FBSDE (compared to (3.18) with $\alpha=\alpha_{0}+\delta$ ):

$$
\left\{\begin{array}{l}
\mathrm{d} X(s)=\left\{b^{\alpha_{0}}\left(s, \Theta(s), \mathbb{E}_{t}[\Theta(s)]\right)+\widetilde{\psi}(s)\right\} \mathrm{d} s  \tag{3.34}\\
\quad+\sum_{i=1}^{d}\left\{\sigma_{i}^{\alpha_{0}}\left(s, \Theta(s), \mathbb{E}_{t}[\Theta(s)]\right)+\widetilde{\gamma}_{i}(s)\right\} \mathrm{d} W_{i}(s), \quad s \in[t, T] \\
\mathrm{d} Y(s)=\left\{g^{\alpha_{0}}\left(s, \Theta(s), \mathbb{E}_{t}[\Theta(s)]\right)+\widetilde{\varphi}(s)\right\} \mathrm{d} s+\sum_{i=1}^{d} Z_{i}(s) \mathrm{d} W_{i}(s), \quad s \in[t, T] \\
X(t)=\Psi^{\alpha_{0}}(Y(t))+\widetilde{\xi}, \quad Y(T)=\Phi^{\alpha_{0}}\left(X(T), \mathbb{E}_{t}[X(T)]\right)+\widetilde{\eta}
\end{array}\right.
$$

where

$$
\left\{\begin{array}{l}
\widetilde{\xi}:=\delta\left[\Psi(y(t))-\Psi^{0}(y(t))\right]+\xi  \tag{3.35}\\
\widetilde{\eta}:=\delta\left[\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)-\Phi^{0}\left(x(T), \mathbb{E}_{t}[x(T)]\right)\right]+\eta \\
\widetilde{\rho}(\cdot):=\delta\left[\Gamma\left(\cdot, \theta(\cdot), \mathbb{E}_{t}[\theta(\cdot)]\right)-\Gamma^{0}\left(\cdot, \theta(\cdot), \mathbb{E}_{t}[\theta(\cdot)]\right]+\rho(\cdot),\right.
\end{array}\right.
$$

and $\left(\Psi^{0}, \Phi^{0}, \Gamma^{0}\right)$ is given by (3.8). It is easy to verify that $(\widetilde{\xi}, \widetilde{\eta}, \widetilde{\rho}(\cdot)) \in \mathcal{H}[t, T]$. Then, by our assumption, MF-FBSDE (3.34) admits a unique solution $\Theta(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. Noting the arbitrariness of $\theta(\cdot)$, we have established a mapping

$$
\Theta(\cdot)=\mathscr{T}_{\alpha_{0}+\delta}(\theta(\cdot)): M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right) \rightarrow M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)
$$

In the following, we shall prove that this mapping is a contraction when $\delta$ is small.
Let $\theta(\cdot), \bar{\theta}(\cdot) \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$ and denote $\Theta(\cdot)=\mathscr{T}_{\alpha_{0}+\delta}(\theta(\cdot)), \bar{\Theta}(\cdot)=\mathscr{T}_{\alpha_{0}+\delta}(\bar{\theta}(\cdot))$. Moreover, denote $\widehat{\theta}(\cdot)=\theta(\cdot)-\bar{\theta}(\cdot), \widehat{\Theta}(\cdot)=\Theta(\cdot)-\bar{\Theta}(\cdot)$, etc. By applying Lemma 3.5, we have

$$
\begin{aligned}
&\|\widehat{\Theta}(\cdot)\|_{M_{\mathbb{E}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)}^{2}=\mathbb{E}\left[\sup _{s \in[t, T]}|\widehat{X}(s)|^{2}+\sup _{s \in[t, T]}|\widehat{Y}(s)|^{2}+\int_{t}^{T}|\widehat{Z}(s)|^{2} \mathrm{~d} s\right] \\
& \leq K \delta^{2} \mathbb{E}\left\{\left|[\Psi(y(t))-\Psi(\bar{y}(t))]-\left[\Psi^{0}(y(t))-\Psi^{0}(\bar{y}(t))\right]\right|^{2}\right. \\
&+\left|\left[\Phi\left(x(T), \mathbb{E}_{t}[x(T)]\right)-\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)\right]-\left[\Phi^{0}\left(x(T), \mathbb{E}_{t}[x(T)]\right)-\Phi^{0}\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)\right]\right|^{2} \\
&+\left(\int_{t}^{T}\left|\left[g\left(\theta, \mathbb{E}_{t}[\theta]\right)-g\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right]-\left[g^{0}\left(\theta, \mathbb{E}_{t}[\theta]\right)-g^{0}\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right]\right| \mathrm{d} s\right)^{2} \\
&+\left(\int_{t}^{T}\left|\left[b\left(\theta, \mathbb{E}_{t}[\theta]\right)-b\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right]-\left[b^{0}\left(\theta, \mathbb{E}_{t}[\theta]\right)-b^{0}\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right]\right| \mathrm{d} s\right)^{2} \\
&\left.+\int_{t}^{T}\left|\left[\sigma\left(\theta, \mathbb{E}_{t}[\theta]\right)-\sigma\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right]-\left[\sigma^{0}\left(\theta, \mathbb{E}_{t}[\theta]\right)-\sigma^{0}\left(\bar{\theta}, \mathbb{E}_{t}[\bar{\theta}]\right)\right]\right|^{2} \mathrm{~d} s\right\} .
\end{aligned}
$$

Then, due to the Lipschitz continuity of the coefficients $(\Psi, \Phi, \Gamma)$ and $\left(\Psi^{0}, \Phi^{0}, \Gamma^{0}\right)$, there exists a constant $K_{3}>0$ independent of $\alpha_{0}$ and $\delta$ such that

$$
\|\widehat{\Theta}(\cdot)\|_{M_{\mathbb{P}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)}^{2} \leq K_{3} \delta^{2}\|\widehat{\theta}(\cdot)\|_{M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)}^{2}
$$

Choose $\delta_{0}=1 /\left(2 \sqrt{K_{3}}\right)$. Then, for any $\delta \in\left(0, \delta_{0}\right]$, the above inequality implies that $\mathscr{T}_{\alpha_{0}+\delta}$ is a contraction mapping. Consequently, it admits a unique fixed point which is just the unique solution to MF-FBSDE (3.18) with $\alpha=\alpha_{0}+\delta$. The proof is completed.

Proof of Theorem 3.4. Firstly, the unique solvability of MF-FBSDE (1.1) in the space $M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$ is obtained by virtue of the unique solvability of MF-FBSDE (3.19) and Lemma 3.6. Secondly, by letting $\alpha=1,(\xi, \eta, \rho(\cdot))=(0,0,0)$ and

$$
\begin{aligned}
(\bar{\xi}, \bar{\eta}, \bar{\rho}(\cdot))= & \left(\bar{\Psi}(\bar{y}(t))-\Psi(\bar{y}(t)), \bar{\Phi}\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)-\Phi\left(\bar{x}(T), \mathbb{E}_{t}[\bar{x}(T)]\right)\right. \\
& \left.\bar{\Gamma}\left(\cdot, \bar{\theta}(\cdot), \mathbb{E}_{t}[\bar{\theta}(\cdot)]\right)-\Gamma\left(\cdot, \bar{\theta}(\cdot), \mathbb{E}_{t}[\bar{\theta}(\cdot)]\right)\right)
\end{aligned}
$$

we obtain (3.15) from (3.20). Finally, by selecting $(\bar{\Psi}, \bar{\Phi}, \bar{\Gamma})=(0,0,0)$, we get (3.13) from (3.15) and complete the proof.

Now, we give a remark to end this section.
Remark 3.7. There exists a symmetrical version of the monotonicity conditions in Assumption (H4)(iii) as follows:

Assumption (H4)-(iii)'. For any $y, \bar{y} \in \mathbb{R}^{n}$ and almost all $\omega \in \Omega$,

$$
\begin{equation*}
\langle\Psi(y)-\Psi(\bar{y}), \widehat{y}\rangle \geq \mu|H \widehat{y}|^{2} . \tag{3.36}
\end{equation*}
$$

For any $X, \bar{X} \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\mathbb{E}_{t}\left[\left\langle\binom{\Phi\left(X, \mathbb{E}_{t}[X]\right)^{(1)}-\Phi\left(\bar{X}, \mathbb{E}_{t}[\bar{X}]\right)^{(1)}}{\Phi\left(X, \mathbb{E}_{t}[X]\right)^{(2)}-\Phi\left(\bar{X}, \mathbb{E}_{t}[\bar{X}]\right)^{(2)}},\binom{\widehat{X}^{(1)}}{\widehat{X}^{(2)}}\right\rangle\right] \leq-\nu \mathbb{E}_{t}\left[\left|\binom{P \widehat{X}^{(1)}}{\widetilde{P} \widehat{X}^{(2)}}\right|^{2}\right] \tag{3.37}
\end{equation*}
$$

For almost all $s \in[t, T]$ and any $\Theta:=\left(X^{\top}, Y^{\top}, Z^{\top}\right)^{\top}, \bar{\Theta}:=\left(X^{\top}, Y^{\top}, Z^{\top}\right)^{\top} \in L_{\mathcal{F}_{s}}^{2}\left(\Omega ; \mathbb{R}^{n(2+d)}\right)$,

$$
\begin{align*}
& \mathbb{E}_{t}\left[\left\langle\binom{\Gamma\left(s, \Theta, \mathbb{E}_{t}[\Theta]\right)^{(1)}-\Gamma\left(s, \bar{\Theta}, \mathbb{E}_{t}[\bar{\Theta}]\right)^{(1)}}{\Gamma\left(s, \Theta, \mathbb{E}_{t}[\Theta]\right)^{(2)}-\Gamma\left(s, \bar{\Theta}, \mathbb{E}_{t}[\bar{\Theta}]\right)^{(2)}},\binom{\widehat{\Theta}^{(1)}}{\widehat{\Theta}^{(2)}}\right\rangle\right] \\
\geq & \nu \mathbb{E}_{t}\left[\left|\binom{A(s) \widehat{X}^{(1)}}{\widetilde{A}(s) \widehat{X}^{(2)}}\right|^{2}\right]+\mu \mathbb{E}_{t}\left[\left|\binom{B(s) \widehat{Y}^{(1)}+C(s) \widehat{Z}^{(1)}}{\widetilde{B}(s) \widehat{Y}^{(2)}+\widetilde{C}(s) \widehat{Z}^{(2)}}\right|\right. \tag{3.38}
\end{align*}
$$

In fact, it is easy to verify that, if $\theta(\cdot)=\left(x(\cdot)^{\top}, y(\cdot)^{\top}, z(\cdot)^{\top}\right)^{\top} \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$ is a solution to MF-FBSDE (1.1) with the coefficients $(\Psi, \Phi, \Gamma)$, then

$$
\begin{equation*}
\widetilde{\theta}(\cdot)=\left(\widetilde{x}(\cdot)^{\top}, \widetilde{y}(\cdot)^{\top}, \widetilde{z}(\cdot)^{\top}\right)^{\top}:=\left(x(\cdot)^{\top},-y(\cdot)^{\top},-z(\cdot)^{\top}\right)^{\top} \tag{3.39}
\end{equation*}
$$

is a solution to another MF-FBSDE with the coefficients

$$
\begin{gathered}
\widetilde{\Psi}(\widetilde{y}):=\Psi(-\widetilde{y}), \quad \widetilde{\Phi}\left(\widetilde{x}, \widetilde{x}^{\prime}\right):=-\Phi\left(\widetilde{x}, \widetilde{x}^{\prime}\right) \\
\widetilde{\Gamma}\left(s, \widetilde{\theta}, \widetilde{\theta}^{\prime}\right)=\left(-g\left(s, \widetilde{x},-\widetilde{y},-\widetilde{z}, \widetilde{x}^{\prime},-\widetilde{y}^{\prime},-\widetilde{z}^{\prime}\right)^{\top}, b\left(s, \widetilde{x},-\widetilde{y},-\widetilde{z}, \widetilde{x}^{\prime},-\widetilde{y}^{\prime},-\widetilde{z}^{\prime}\right)^{\top},\right. \\
\left.\sigma\left(s, \widetilde{x},-\widetilde{y},-\widetilde{z}, \widetilde{x}^{\prime},-\widetilde{y}^{\prime},-\widetilde{z}^{\prime}\right)^{\top}\right)^{\top}
\end{gathered}
$$

We continue to verify that, if the coefficients $(\Psi, \Phi, \Gamma)$ satisfy Assumption (H4)-(iii) (resp. Assumption (H4)-(iii)'), then the coefficients ( $\widetilde{\Psi}, \widetilde{\Phi}, \widetilde{\Gamma})$ will satisfy Assumption (H4)-(iii) (resp. Assumption (H4)(iii)). By virtue of the invertible transformation (3.39), all conclusions in this section are also valid when Assumption (H4)-(iii) is replaced by Assumption (H4)-(iii)'.

## 4 Application to MF-LQ problems

In this section, we shall consider some LQ optimal control problems driven by an MF-SDE and an MF-BSDE respectively. The Hamiltonian systems arising from these LQ problems will be found to be MF-FBSDEs with domination-monotonicity conditions, then they are uniquely solvable by Theorem 3.4. In fact, to study the (unique) solvability of Hamiltonian systems is one of our research motivations.

### 4.1 Forward MF-LQ control problem

In the first LQ problem, we consider the following linear controlled MF-SDE:

$$
\left\{\begin{array}{l}
\mathrm{d} x(s)=\left\{A(s) x(s)+\bar{A}(s) \mathbb{E}_{t}[x(s)]+B(s) u(s)+\bar{B}(s) \mathbb{E}_{t}[u(s)]+\alpha(s)\right\} \mathrm{d} s  \tag{4.1}\\
\quad+\sum_{i=1}^{d}\left\{C_{i}(s) x(s)+\bar{C}_{i}(s) \mathbb{E}_{t}[x(s)]+D_{i}(s) u(s)+\bar{D}_{i}(s) \mathbb{E}_{t}[u(s)]+\beta_{i}(s)\right\} \mathrm{d} W_{i}(s) \\
x(t)=H \xi+x_{t}
\end{array}\right.
$$

where $A(\cdot), \bar{A}(\cdot), C_{i}(\cdot), \bar{C}_{i}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{n \times n}\right), B(\cdot), \bar{B}(\cdot), D_{i}(\cdot), \bar{D}_{i}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{n \times m}\right), H \in \mathbb{R}^{n \times n}$, $\alpha(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right), \beta_{i}(\cdot) \in L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n}\right)$, and $x_{t} \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)(i=1,2, \ldots, d)$. The pair $(\xi, u(\cdot)) \in$ $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$ is called an admissible control. By Proposition 2.1, for any admissible control $(\xi, u(\cdot)) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right), \operatorname{MF}-\operatorname{SDE}(4.1)$ admits a unique solution $x(\cdot) \equiv x(\cdot ; \xi, u(\cdot)) \in$ $L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)$ which is called the admissible state process under $(\xi, u(\cdot))$. Moreover, $(x(\cdot), \xi, u(\cdot))$ is called an admissible triple. For convenience, we denote $C(\cdot):=\left(C_{1}(\cdot)^{\top}, C_{2}(\cdot)^{\top}, \ldots, C_{d}(\cdot)^{\top}\right)^{\top}$ and $D(\cdot):=\left(D_{1}(\cdot)^{\top}, D_{2}(\cdot)^{\top}, \ldots, D_{d}(\cdot)^{\top}\right)^{\top}$.

We notice that, in most of literature on LQ optimal control problems (see Yong and Zhou [24] for the problems driven by SDEs and $[12,16,18,20,23]$ for the problems driven by MF-SDEs), the initial condition is fixed to be $x(t)=x_{t}$. Clearly, this is a special case of our research, i.e. $H=0$. Here, we consider a general situation, i.e., we can select $\xi \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ to change the initial value $x(t)$. The introduction of matrix $H$ enables our formulation to be better meet the various actual applications. For example, if

$$
H=\left(\begin{array}{cccc}
1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right)
$$

then we can only change the value of the first component of $x(t)$.
Besides the controlled system (4.1), we are also given an objective functional in a quadratic form:

$$
\begin{align*}
J(\xi, u(\cdot))= & \frac{1}{2} \mathbb{E}_{t}\left\{\langle M \xi, \xi\rangle+\langle G x(T), x(T)\rangle+\left\langle\bar{G} \mathbb{E}_{t}[x(T)], \mathbb{E}_{t}[x(T)]\right\rangle\right. \\
& +\int_{t}^{T}\left[\langle Q(s) x(s), x(s)\rangle+\left\langle\bar{Q}(s) \mathbb{E}_{t}[x(s)], \mathbb{E}_{t}[x(s)]\right\rangle\right.  \tag{4.2}\\
& \left.\left.+\langle R(s) u(s), u(s)\rangle+\left\langle\bar{R}(s) \mathbb{E}_{t}[u(s)], \mathbb{E}_{t}[u(s)]\right\rangle\right] \mathrm{d} s\right\}
\end{align*}
$$

where $M, G, \bar{G} \in \mathbb{S}^{n}, Q(\cdot), \bar{Q}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{S}^{n}\right)$ and $R(\cdot), \bar{R}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{S}^{m}\right)$. Clearly, for any $(\xi, u(\cdot)) \in$ $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right), J(\xi, u(\cdot))$ is well-defined.

We propose a mean-field type forward LQ (MF-FLQ, for short) control problem as follows:
Problem (MF-FLQ). Find an admissible control $\left(\xi^{*}, u^{*}(\cdot)\right) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$ such that

$$
\begin{equation*}
J\left(\xi^{*}, u^{*}(\cdot)\right)=\underset{(\xi, u(\cdot)) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)}{\operatorname{essinf}} J(\xi, u(\cdot)) \tag{4.3}
\end{equation*}
$$

$\left(\xi^{*}, u^{*}(\cdot)\right)$ satisfying (4.3) is called an optimal control, $x^{*}(\cdot) \equiv x\left(\cdot ; \xi^{*}, u^{*}(\cdot)\right)$ is called the corresponding optimal state process, and $\left(x^{*}(\cdot), \xi^{*}, u^{*}(\cdot)\right)$ is called an optimal triple.

As usual, for a matrix $M \in \mathbb{S}^{n}$, when $M$ is positive semi-definite (resp. positive definite, negative semi-definite, negative definite), we denote $M \geq 0$ (resp. $>0, \leq 0,<0$ ). Moreover, for a mapping $M:[t, T] \rightarrow \mathbb{S}^{n}$, we denote $M(\cdot) \geq 0$ (resp. $>0, \leq 0,<0$ ) when $M(s) \geq 0$ (resp. $>0, \leq 0,<0$ ) for almost all $s \in[t, T]$. When there exists a constant $\delta>0$ such that $M(\cdot)-\delta I_{n} \geq 0\left(\right.$ resp. $\left.M(\cdot)+\delta I_{n} \leq 0\right)$, we denote $M(\cdot) \gg 0($ resp. $M(\cdot) \ll 0)$. Now, we introduce the following assumption called the uniformly positive definiteness condition ( PD , for short) for the weighting matrices in the objective functional:
Condition (MF-FLQ-PD). $M>0, G \geq 0, G+\bar{G} \geq 0, Q(\cdot) \geq 0, Q(\cdot)+\bar{Q}(\cdot) \geq 0, R(\cdot) \gg 0$, and $R(\cdot)+\bar{R}(\cdot) \gg 0$.

Lemma 4.1. Let Condition (MF-FLQ-PD) hold. Let $\left(x^{*}(\cdot), \xi^{*}, u^{*}(\cdot)\right)$ be an admissible triple. Denote by $(y(\cdot), z(\cdot)) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$ the unique solution to the following MF-BSDE:

$$
\left\{\begin{align*}
\mathrm{d} y(s) & =-\left\{A(s)^{\top} y(s)+\bar{A}(s)^{\top} \mathbb{E}_{t}[y(s)]+C(s)^{\top} z(s)+\bar{C}(s)^{\top} \mathbb{E}_{t}[z(s)]\right.  \tag{4.4}\\
& \left.+Q(s) x^{*}(s)+\bar{Q}(s) \mathbb{E}_{t}\left[x^{*}(s)\right]\right\} \mathrm{d} s+\sum_{i=1}^{d} z_{i}(s) \mathrm{d} W_{i}(s), \quad s \in[t, T] \\
y(T) & =G x^{*}(T)+\bar{G} \mathbb{E}_{t}\left[x^{*}(T)\right]
\end{align*}\right.
$$

Then, $\left(\xi^{*}, u^{*}(\cdot)\right)$ is an optimal control of Problem (MF-FLQ) if and only if

$$
\left\{\begin{array}{l}
M \xi^{*}+H^{\top} y(t)=0  \tag{4.5}\\
R(\cdot) u^{*}(\cdot)+\bar{R}(\cdot) \mathbb{E}_{t}\left[u^{*}(\cdot)\right]+B(\cdot)^{\top} y(\cdot)+\bar{B}(\cdot)^{\top} \mathbb{E}_{t}[y(\cdot)]+D(\cdot)^{\top} z(\cdot)+\bar{D}(\cdot)^{\top} \mathbb{E}_{t}[z(\cdot)]=0
\end{array}\right.
$$

Proof. Let $\left(x^{*}(\cdot), \xi^{*}, u^{*}(\cdot)\right)$ be an admissible triple. For any $(\xi, u(\cdot)) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$ and $\varepsilon \in$ $\mathbb{R}$, denote by $x^{\varepsilon}(\cdot)$ the admissible state process under $\left(\xi^{*}+\varepsilon \xi, u^{*}(\cdot)+\varepsilon u(\cdot)\right)$. Let $x_{1}(\cdot)=\left(x^{\varepsilon}(\cdot)-x^{*}(\cdot)\right) / \varepsilon$. Then it is the unique solution to the following MF-SDE:

$$
\left\{\begin{array}{l}
\mathrm{d} x_{1}(s)=\left\{A(s) x_{1}(s)+\bar{A}(s) \mathbb{E}_{t}\left[x_{1}(s)\right]+B(s) u(s)+\bar{B}(s) \mathbb{E}_{t}[u(s)]\right\} \mathrm{d} s  \tag{4.6}\\
+\sum_{i=1}^{d}\left\{C_{i}(s) x_{1}(s)+\bar{C}_{i}(s) \mathbb{E}_{t}\left[x_{1}(s)\right]+D_{i}(s) u(s)+\bar{D}_{i}(s) \mathbb{E}_{t}[u(s)]\right\} \mathrm{d} W_{i}(s), \quad s \in[t, T] \\
x_{1}(t)=H \xi
\end{array}\right.
$$

Applying Itô's formula to $\left\langle x_{1}(\cdot), y(\cdot)\right\rangle$ leads to (the argument $s$ is suppressed for simplicity)

$$
\begin{align*}
& \mathbb{E}_{t}\left\{\left\langle x_{1}(T), G x^{*}(T)+\bar{G} \mathbb{E}_{t}\left[x^{*}(T)\right]\right\rangle+\int_{t}^{T}\left\langle x_{1}, Q x^{*}+\bar{Q} \mathbb{E}_{t}\left[x^{*}\right]\right\rangle \mathrm{d} s\right\}  \tag{4.7}\\
= & \mathbb{E}_{t}\left\{\langle H \xi, y(t)\rangle+\int_{t}^{T}\left\langle u, B^{\top} y+\bar{B}^{\top} \mathbb{E}_{t}[y]+D^{\top} z+\bar{D}^{\top} \mathbb{E}_{t}[z]\right\rangle \mathrm{d} s\right\}
\end{align*}
$$

Now, we calculate the difference:

$$
J\left(\xi^{*}+\varepsilon \xi, u^{*}(\cdot)+\varepsilon u(\cdot)\right)-J\left(\xi^{*}, u^{*}(\cdot)\right)=\varepsilon \mathrm{I}_{1}+\frac{\varepsilon^{2}}{2} \mathrm{I}_{2}
$$

where

$$
\begin{aligned}
\mathrm{I}_{1}= & \mathbb{E}_{t}\left\{\left\langle M \xi^{*}, \xi\right\rangle+\left\langle G x^{*}(T), x_{1}(T)\right\rangle+\left\langle\bar{G} \mathbb{E}_{t}\left[x^{*}(T)\right], \mathbb{E}_{t}\left[x^{*}(T)\right]\right\rangle\right. \\
& \left.+\int_{t}^{T}\left[\left\langle Q x^{*}, x_{1}\right\rangle+\left\langle\bar{Q} \mathbb{E}_{t}\left[x^{*}\right], \mathbb{E}_{t}\left[x_{1}\right]\right\rangle+\left\langle R u^{*}, u\right\rangle+\left\langle\bar{R} \mathbb{E}_{t}\left[u^{*}\right], \mathbb{E}_{t}[u]\right\rangle\right] \mathrm{d} s\right\} \\
= & \mathbb{E}_{t}\left\{\left\langle M \xi^{*}, \xi\right\rangle+\left\langle G x^{*}(T)+\bar{G} \mathbb{E}_{t}\left[x^{*}(T)\right], x_{1}(T)\right\rangle\right. \\
& \left.+\int_{t}^{T}\left[\left\langle Q x^{*}+\bar{Q} \mathbb{E}_{t}\left[x^{*}\right], x_{1}\right\rangle+\left\langle R u^{*}+\bar{R} \mathbb{E}_{t}\left[u^{*}\right], u\right\rangle\right] \mathrm{d} s\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{I}_{2}= & \mathbb{E}_{t}\left\{\langle M \xi, \xi\rangle+\left\langle G x_{1}(T), x_{1}(T)\right\rangle+\left\langle\bar{G} \mathbb{E}_{t}\left[x_{1}(T)\right], \mathbb{E}_{t}\left[x_{1}(T)\right]\right\rangle\right. \\
& \left.+\int_{t}^{T}\left[\left\langle Q x_{1}, x_{1}\right\rangle+\left\langle\bar{Q} \mathbb{E}_{t}\left[x_{1}\right], \mathbb{E}_{t}\left[x_{1}\right]\right\rangle+\langle R u, u\rangle+\left\langle\bar{R} \mathbb{E}_{t}[u], \mathbb{E}_{t}[u]\right\rangle\right] \mathrm{d} s\right\} \\
= & \mathbb{E}_{t}\left\{\left\langle G\left(x_{1}(T)-\mathbb{E}_{t}\left[x_{1}(T)\right]\right), x_{1}(T)-\mathbb{E}_{t}\left[x_{1}(T)\right]\right\rangle+\left\langle(G+\bar{G}) \mathbb{E}_{t}\left[x_{1}(T)\right], \mathbb{E}_{t}\left[x_{1}(T)\right]\right\rangle\right. \\
& +\langle M \xi, \xi\rangle+\int_{t}^{T}\left[\left\langle Q\left(x_{1}-\mathbb{E}_{t}\left[x_{1}\right]\right), x_{1}-\mathbb{E}_{t}\left[x_{1}\right]\right\rangle+\left\langle(Q+\bar{Q}) \mathbb{E}_{t}\left[x_{1}\right], E_{t}\left[x_{1}\right]\right\rangle\right. \\
& \left.\left.+\left\langle R\left(u-\mathbb{E}_{t}[u]\right), u-\mathbb{E}_{t}[u]\right\rangle+\left\langle(R+\bar{R}) \mathbb{E}_{t}[u], \mathbb{E}_{t}[u]\right\rangle\right] \mathrm{d} s\right\} .
\end{aligned}
$$

By substituting (4.7) into the above expression of $\mathrm{I}_{1}$, we have

$$
\mathrm{I}_{1}=\mathbb{E}_{t}\left\{\left\langle\xi, M \xi^{*}+H^{\top} y(t)\right\rangle+\int_{t}^{T}\left\langle u, R u^{*}+\bar{R} \mathbb{E}_{t}\left[u^{*}\right]+B^{\top} y+\bar{B}^{\top} \mathbb{E}_{t}[y]+D^{\top} z+\bar{D}^{\top} \mathbb{E}_{t}[z]\right\rangle \mathrm{d} s\right\} .
$$

With the previous preparations, now we prove the sufficiency. In fact, when (4.5) hold, we have $\mathrm{I}_{1}=0$. Moreover, Condition (MF-FLQ-PD) implies that $\mathrm{I}_{2} \geq 0$. Therefore,

$$
\begin{equation*}
J\left(\xi^{*}+\varepsilon \xi, u^{*}(\cdot)+\varepsilon u(\cdot)\right)-J\left(\xi^{*}, u^{*}(\cdot)\right) \geq 0 \tag{4.8}
\end{equation*}
$$

By the arbitrariness of $(\xi, u(\cdot))$, we prove that $\left(\xi^{*}, u^{*}(\cdot)\right)$ is an optimal control of Problem (MF-FLQ).
Next, we will prove the necessity. When $\left(\xi^{*}, u^{*}(\cdot)\right)$ is optimal, the inequality $(4.8)$ hold for all $(\xi, u(\cdot)) \in$ $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$ and all $\varepsilon \in \mathbb{R}$. On the one hand, when $\varepsilon>0$, we have $\mathrm{I}_{1}+(\varepsilon / 2) \mathrm{I}_{2} \geq 0$. Sending $\varepsilon \rightarrow 0^{+}$leads to

$$
\begin{equation*}
\mathrm{I}_{1} \geq 0, \quad \text { for all }(\xi, u(\cdot)) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right) \tag{4.9}
\end{equation*}
$$

On the other hand, when $\varepsilon<0$, a similar analysis yields

$$
\begin{equation*}
\mathrm{I}_{1} \leq 0, \quad \text { for all }(\xi, u(\cdot)) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right) \tag{4.10}
\end{equation*}
$$

Combine (4.9) and (4.10) to get

$$
\mathrm{I}_{1}=0, \quad \text { for all }(\xi, u(\cdot)) \in L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)
$$

Finally, due to the arbitrariness of $(\xi, u(\cdot))$, the above equation implies (4.5). The proof is finished.
For convenience, we denote

$$
\begin{equation*}
\widetilde{R}(\cdot)=R(\cdot)+\bar{R}(\cdot), \quad \widetilde{B}(\cdot)=B(\cdot)+\bar{B}(\cdot), \quad \widetilde{D}(\cdot)=D(\cdot)+\bar{D}(\cdot) \tag{4.11}
\end{equation*}
$$

We notice that, Condition (MF-FLQ-PD) implies that the matrix $M$ and the matrix-valued processes $R(\cdot)$ and $\widetilde{R}(\cdot)$ are invertible. Moreover, $R(\cdot)^{-1}$ and $\widetilde{R}(\cdot)^{-1}$ are uniformly bounded. Then, we can solve (4.5) as

$$
\begin{equation*}
\xi^{*}=-M^{-1} H^{\top} y(t) \tag{4.12}
\end{equation*}
$$

and

$$
\begin{align*}
u^{*}(\cdot)= & -R(\cdot)^{-1}\left\{B(\cdot)^{\top}\left(y(\cdot)-\mathbb{E}_{t}[y(\cdot)]\right)+D(\cdot)^{\top}\left(z(\cdot)-\mathbb{E}_{t}[z(\cdot)]\right)\right\} \\
& -\widetilde{R}(\cdot)^{-1}\left\{\widetilde{B}(\cdot)^{\top} \mathbb{E}_{t}[y(\cdot)]+\widetilde{D}(\cdot)^{\top} \mathbb{E}_{t}[z(\cdot)]\right\} . \tag{4.13}
\end{align*}
$$

Now, we substitute (4.12) and (4.13) into the state equation (4.1), and form a system together with

MF-BSDE (4.4) (the argument $s$ is suppressed for simplicity):

$$
\left\{\begin{align*}
\mathrm{d} x^{*}= & \left\{A x^{*}+\bar{A} \mathbb{E}_{t}\left[x^{*}\right]-B R^{-1}\left\{B^{\top}\left(y-\mathbb{E}_{t}[y]\right)+D^{\top}\left(z-\mathbb{E}_{t}[z]\right)\right\}\right.  \tag{4.14}\\
& -{\left.\widetilde{B} \widetilde{R}^{-1}\left\{\widetilde{B}^{\top} \mathbb{E}_{t}[y]+\widetilde{D}^{\top} \mathbb{E}_{t}[z]\right\}+\alpha\right\} \mathrm{d} s}+\sum_{i=1}^{d}\left\{C_{i} x^{*}+\bar{C}_{i} \mathbb{E}_{t}\left[x^{*}\right]-D_{i} R^{-1}\left\{B^{\top}\left(y-\mathbb{E}_{t}[y]\right)+D^{\top}\left(z-\mathbb{E}_{t}[z]\right)\right\}\right. \\
& \left.-\widetilde{D}_{i} \widetilde{R}^{-1}\left\{\widetilde{B}^{\top} \mathbb{E}_{t}[y]+\widetilde{D}^{\top} \mathbb{E}_{t}[z]\right\}+\beta_{i}\right\} \mathrm{d} W_{i}, \quad s \in[t, T] \\
\mathrm{d} y= & -\left\{A^{\top} y+\bar{A}^{\top} \mathbb{E}_{t}[y]+C^{\top} z+\bar{C}^{\top} \mathbb{E}_{t}[z]+Q x^{*}+\bar{Q} \mathbb{E}_{t}\left[x^{*}\right]\right\} \mathrm{d} s \\
& +\sum_{i=1}^{d} z_{i} \mathrm{~d} W_{i}, \quad s \in[t, T], \quad \\
x^{*}(t)= & -H M^{-1} H^{\top} y(t)+x_{t}, \quad y(T)=G x^{*}(T)+\bar{G} \mathbb{E}_{t}\left[x^{*}(T)\right] .
\end{align*}\right.
$$

The above system is called a Hamiltonian system in the terminology of control theory, which is actually an MF-FBSDE.

Now, we give the main result of this subsection.
Proposition 4.2. Under Condition (MF-FLQ-PD), the Hamiltonian system (4.14) admits a unique solution $\theta(\cdot)=\left(x^{*}(\cdot)^{\top}, y(\cdot)^{\top}, z(\cdot)^{\top}\right)^{\top} \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. Moreover, $\left(\xi^{*}, u^{*}(\cdot)\right)$ defined by (4.12) and (4.13) is the unique optimal control of Problem (MF-FLQ).

Proof. Firstly, under Condition (MF-FLQ-PD), it is verified that the coefficients of MF-FBSDE (4.14) satisfy Assumptions (H3), (H4)-(i)-Case A, (H4)-(ii), and (H4)-(iii). Then, Theorem 3.4 shows that (4.14) admits a unique solution $\theta(\cdot)=\left(x^{*}(\cdot)^{\top}, y(\cdot)^{\top}, z(\cdot)^{\top}\right)^{\top} \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. Secondly, by Lemma 4.1, the unique solvability of (4.14) implies the existence and uniqueness of optimal control of Problem (MF-FLQ). Moreover, the unique optimal control must be given by (4.12) and (4.13).

### 4.2 Backward MF-LQ control Problem

In the second LQ problem, the controlled system is given by a linear MF-BSDE:

$$
\left\{\begin{align*}
\mathrm{d} y(s) & =\left\{A(s) y(s)+\bar{A}(s) \mathbb{E}_{t}[y(s)]+B(s) z(s)+\bar{B}(s) \mathbb{E}_{t}[z(s)]\right.  \tag{4.15}\\
& \left.+C(s) u(s)+\bar{C}(s) \mathbb{E}_{t}[u(s)]+\alpha(s)\right\} \mathrm{d} s+\sum_{i=1}^{d} z_{i}(s) \mathrm{d} W_{i}(s), \quad s \in[t, T] \\
y(T) & =P \eta+\bar{P} \mathbb{E}_{t}[\eta]+y_{T}
\end{align*}\right.
$$

In this subsection, we will adopt the decomposition $B(\cdot)=\left(B_{1}(\cdot), B_{2}(\cdot), \ldots, B_{d}(\cdot)\right)$ and $\bar{B}(\cdot)=\left(\bar{B}_{1}(\cdot), \bar{B}_{2}(\cdot), \ldots, \bar{B}_{d}(\cdot)\right)$. In (4.15), we assume that $A(\cdot), \bar{A}(\cdot), B_{i}(\cdot), \bar{B}_{i}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{n \times n}\right)(i=1,2, \ldots, d), C(\cdot), \bar{C}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{R}^{n \times m}\right)$, $P, \bar{P} \in \mathbb{R}^{n \times n}, \alpha(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left(t, T ; \mathbb{R}^{n}\right)\right)$, and $y_{T} \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right)$. The pair $(\eta, u(\cdot)) \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times$ $L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$ is called an admissible control. By Proposition 2.2 , for any admissible control $(\eta, u(\cdot)) \in$ $L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$, MF-BSDE (4.15) admits a unique solution $(y(\cdot), z(\cdot)) \equiv(y(\cdot ; \eta, u(\cdot)), z(\cdot ; \eta, u(\cdot))) \in$ $L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$ which is called the admissible state process under $(\eta, u(\cdot))$. Moreover, $(y(\cdot), z(\cdot), \eta, u(\cdot))$ is called an admissible quadruple. We notice that, the backward controlled system without terminal control $\eta$ has been studied by Li et al. [13]. Here, we consider the general situation (4.15). In order to evaluate admissible controls, we are also given an objective functional in a quadratic form:

$$
\begin{align*}
& J(\eta, u(\cdot))=\frac{1}{2} \mathbb{E}_{t}\left\{\langle M y(t), y(t)\rangle+\langle G \eta, \eta\rangle+\left\langle\bar{G} \mathbb{E}_{t}[\eta], \mathbb{E}_{t}[\eta]\right\rangle\right. \\
& \quad+\int_{t}^{T}\left[\langle Q(s) y(s), y(s)\rangle+\left\langle\bar{Q}(s) \mathbb{E}_{t}[y(s)], \mathbb{E}_{t}[y(s)]\right\rangle+\langle L(s) z(s), z(s)\rangle\right.  \tag{4.16}\\
& \left.\left.\quad+\left\langle\bar{L}(s) \mathbb{E}_{t}[z(s)], \mathbb{E}_{t}[z(s)]\right\rangle+\langle R(s) u(s), u(s)\rangle+\left\langle\bar{R}(s) \mathbb{E}_{t}[u(s)], \mathbb{E}[u(s)]\right\rangle\right] \mathrm{d} s\right\}
\end{align*}
$$

where $M, G, \bar{G} \in \mathbb{S}^{n}, Q(\cdot), \bar{Q}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{S}^{n}\right), L(\cdot):=\operatorname{diag}\left\{L_{1}(\cdot), L_{2}(\cdot), \ldots, L_{d}(\cdot)\right\}, \bar{L}(\cdot):=\operatorname{diag}\left\{\bar{L}_{1}(\cdot), \bar{L}_{2}(\cdot), \ldots, \bar{L}_{d}(\cdot)\right\}$ with $L_{i}(\cdot), \bar{L}_{i}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{S}^{n}\right)(i=1,2, \ldots, d)$, and $R(\cdot), \bar{R}(\cdot) \in L^{\infty}\left(t, T ; \mathbb{S}^{m}\right)$. Clearly, for any $(\eta, u(\cdot)) \in$ $L_{\mathcal{F}_{t}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right), J(\eta, u(\cdot))$ is well-defined. Now, we propose a mean-field type backward LQ (MF-BLQ, for short) control problem as follows:

Problem (MF-BLQ). Find an admissible control $\left(\eta^{*}, u^{*}(\cdot)\right) \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)$ such that

$$
\begin{equation*}
J\left(\eta^{*}, u^{*}(\cdot)\right)={\underset{(\eta, u(\cdot)) \in L_{\mathcal{F}_{T}}^{2}\left(\Omega ; \mathbb{R}^{n}\right) \times L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{m}\right)}{\operatorname{essinf}} J(\eta, u(\cdot)) . . . . . .} \tag{4.17}
\end{equation*}
$$

$\left(\eta^{*}, u^{*}(\cdot)\right)$ satisfying (4.17) is called an optimal control, $\left(y^{*}(\cdot), z^{*}(\cdot)\right) \equiv\left(y\left(\cdot ; \eta^{*}, u^{*}(\cdot)\right), z\left(\cdot ; \eta^{*}, u^{*}(\cdot)\right)\right)$ is called the corresponding optimal state process, and $\left(y^{*}(\cdot), z^{*}(\cdot), \eta^{*}, u^{*}(\cdot)\right)$ is called an optimal quadruple.

Similar to the previous subsection, we introduce the following uniformly positive definiteness condition for the weighting matrices in the objective functional:
Condition (MF-BLQ-PD). $M \geq 0, G>0, G+\bar{G}>0, Q(\cdot) \geq 0, Q(\cdot)+\bar{Q}(\cdot) \geq 0, L(\cdot) \geq 0$, $L(\cdot)+\bar{L}(\cdot) \geq 0, R(\cdot) \gg 0$, and $R(\cdot)+\bar{R}(\cdot) \gg 0$.

Lemma 4.3. Let Condition (MF-BLQ-PD) hold. Let $\left(y^{*}(\cdot), z^{*}(\cdot), \eta^{*}, u^{*}(\cdot)\right)$ be an admissible quadruple. Denote by $x(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)$ the unique solution to the following MF-SDE:

$$
\left\{\begin{array}{l}
\mathrm{d} x(s)=-\left\{A(s)^{\top} x(s)+\bar{A}(s)^{\top} \mathbb{E}_{t}[x(s)]+Q(s) y^{*}(s)+\bar{Q}(s) \mathbb{E}_{t}\left[y^{*}(s)\right]\right\} \mathrm{d} s  \tag{4.18}\\
\quad-\sum_{i=1}^{d}\left\{B_{i}(s)^{\top} x(s)+\bar{B}_{i}(s)^{\top} \mathbb{E}_{t}[x(s)]+L_{i}(s) z_{i}^{*}(s)+\bar{L}_{i}(s) \mathbb{E}_{t}\left[z_{i}^{*}(s)\right]\right\} \mathrm{d} W_{i}(s), \\
\\
x(t)=-M y^{*}(t) .
\end{array}\right.
$$

Then, $\left(\eta^{*}, u^{*}(\cdot)\right)$ is an optimal control of Problem (MF-BLQ) if and only if

$$
\left\{\begin{array}{l}
G \eta^{*}+\bar{G} \mathbb{E}_{t}\left[\eta^{*}\right]-P^{\top} x(T)-\bar{P}^{\top} \mathbb{E}_{t}[x(T)]=0  \tag{4.19}\\
R(\cdot) u^{*}(\cdot)+\bar{R}(\cdot) \mathbb{E}_{t}\left[u^{*}(\cdot)\right]+C(\cdot)^{\top} x(\cdot)+\bar{C}(\cdot)^{\top} \mathbb{E}_{t}[x(\cdot)]=0
\end{array}\right.
$$

The proof of the above Lemma 4.3 is similar to that of Lemma 4.1. Then we omit it.
Let us denote

$$
\widetilde{G}=G+\bar{G}, \quad \widetilde{P}=P+\bar{P}, \quad \widetilde{R}(\cdot)=R(\cdot)+\bar{R}(\cdot), \quad \widetilde{C}(\cdot)=C(\cdot)+\bar{C}(\cdot)
$$

Then, under Condition (MF-BLQ-PD), (4.19) can be rewritten in an explicit form:

$$
\left\{\begin{array}{l}
\eta^{*}=G^{-1} P^{\top}\left(x(T)-\mathbb{E}_{t}[x(T)]\right)+\widetilde{G}^{-1} \widetilde{P}^{\top} \mathbb{E}_{t}[x(T)],  \tag{4.20}\\
u^{*}(\cdot)=-R(\cdot)^{-1} C(\cdot)^{\top}\left(x(\cdot)-\mathbb{E}_{t}[x(\cdot)]\right)-\widetilde{R}(\cdot)^{-1} \widetilde{C}(\cdot)^{\top} \mathbb{E}_{t}[x(\cdot)]
\end{array}\right.
$$

Substituting (4.20) into the state equation (4.15) and combining with MF-SDE (4.18) yield the following Hamiltonian system (the argument $s$ is suppressed for simplicity):

$$
\left\{\begin{align*}
\mathrm{d} x= & -\left\{A^{\top} x+\bar{A}^{\top} \mathbb{E}_{t}[x]+Q y^{*}+\bar{Q} \mathbb{E}_{t}\left[y^{*}\right]\right\} \mathrm{d} s  \tag{4.21}\\
& -\sum_{i=1}^{d}\left\{B_{i}^{\top} x+\bar{B}_{i}^{\top} \mathbb{E}_{t}[x]+L_{i} z_{i}^{*}+\bar{L}_{i} \mathbb{E}_{t}\left[z_{i}^{*}\right]\right\} \mathrm{d} W_{i}, \quad s \in[t, T], \\
\mathrm{d} y^{*}= & \left\{A y^{*}+\bar{A} \mathbb{E}_{t}\left[y^{*}\right]+B z^{*}+\bar{B} \mathbb{E}_{t}\left[z^{*}\right]-C R^{-1} C^{\top}\left(x-\mathbb{E}_{t}[x]\right)\right. \\
& \left.-\widetilde{C} \widetilde{R}^{-1} \widetilde{C}^{\top} \mathbb{E}_{t}[x]+\alpha\right\} \mathrm{d} s+\sum_{i=1}^{d} z_{i}^{*} \mathrm{~d} W_{i}, \quad s \in[t, T], \\
x(t)= & -M y^{*}(t), \quad y^{*}(T)=P G^{-1} P^{\top}\left(x(T)-\mathbb{E}_{t}[x(T)]\right)+\widetilde{P} \widetilde{G}^{-1} \widetilde{P}^{\top} \mathbb{E}_{t}[x(T)] .
\end{align*}\right.
$$

We give the main result of this subsection.

Proposition 4.4. Under Condition (MF-BLQ-PD), the Hamiltonian system (4.21) admits a unique solution $\theta(\cdot)=\left(x(\cdot)^{\top}, y^{*}(\cdot)^{\top}, z^{*}(\cdot)^{\top}\right)^{\top} \in M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. Moreover, $\left(\eta^{*}, u^{*}(\cdot)\right)$ defined by (4.20) is the unique optimal control of Problem (MF-BLQ).

Proof. Under Condition (MF-BLQ-PD), we verify that the coefficients of MF-FBSDE (4.21) satisfy Assumptions (H3), (H4)-(i)-Case B, (H4)-(ii), and (H4)-(iii). Then, Theorem 3.4 shows that (4.21) admits a unique solution in the space $M_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n(2+d)}\right)$. Similar to the proof of Proposition 4.2 , the remaining of this proof is implied by Lemma 4.3.

As an echo of Remark 3.7, we give the following remark. Due to the similarity, the details will be omitted.

Remark 4.5. If the essential infimum in Problem (MF-FLQ) (resp. Problem (MF-BLQ)) is replaced with the essential supremum, and at the same time Condition (MF-FLQ-PD) (resp. Condition (MF$B L Q-P D)$ ) is replaced with the corresponding uniformly negative definiteness condition, then we will have similar conclusions to Proposition 4.2 (resp. Proposition 4.4). We notice the difference is: In the related assumptions satisfied by the Hamiltonian system arising from the new MF-LQ problem, Assumption (H4)-(iii) will be replaced by Assumption (H4)-(iii)'.

## References

[1] F. Antonelli, Backward-forward stochastic differential equations, Ann. Appl. Probab., 3 (1993), no. 3, 777-793.
[2] A. Bensoussan, S.C.P. Yam, and Z. Zhang, Well-posedness of mean-field type forward-backward stochastic differential equations, Stochastic Process. Appl., 125 (2015), no. 9, 3327-3354.
[3] R. Buckdahn, B. Djehiche, J. Li, and S. Peng, Mean-field backward stochastic differential equations: a limit approach, Ann. Probab., 37 (2009), no. 4, 1524-1565.
[4] R. Buckdahn, J. Li, and S. Peng, Mean-field backward stochastic differential equations and related partial differential equations, Stochastic Process. Appl., 119 (2009), no. 10, 3133-3154.
[5] R. Buckdahn, J. Li, S. Peng, and C. Rainer, Mean-field stochastic differential equations and associated PDEs, Ann. Probab., 45 (2017), no. 2, 824-878.
[6] R. Carmona and F. Delarue, Mean field forward-backward stochastic differential equations, sl Electron. Commun. Probab., 18 (2013), no. 68, 15 pp.
[7] R. Carmona and F. Delarue, Forward-backward stochastic differential equations and controlled McKean-Vlasov dynamics, Ann. Probab., 43 (2015), no. 5, 2647-2700.
[8] Y. Hu and S. Peng, Solution of forward-backward stochastic differential equations, Probab. Theory Related Fields, 103 (1995), no. 2, 273-283.
[9] M. Huang, R.P. Malhamé, and P.E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, Commun. Inf. Syst., 6 (3) (2006) 221-251.
[10] M. Kac, Foundations of kinetic theory, in: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. III, 1956, pp. 171-197.
[11] J.M. Lasry and P.L. Lions, Mean field games, Jpn. J. Math., 2 (1) (2007) 229-260.
[12] N. Li, X. Li, and Z. Yu, Indefinite mean-field type linear-quadratic stochastic optimal control problems, Automatica, 122 (2020), 109267, 10 pp.
[13] X. Li, J. Sun, and J. Xiong, Linear quadratic optimal control problems for mean-field backward stochastic differential equations, Appl. Math. Optim., 80 (2019), no. 1, 223-250.
[14] Y. Lin, X. Jiang, and W. Zhang, An open-loop Stackelberg strategy for the linear quadratic mean-field stochastic differential game, IEEE Trans. Automat. Control, 64 (2019), no. 1, 97-110.
[15] H.P. McKean, Propagation of chaos for a class of non-linear parabolic equations, Lect. Ser. Differ. Equ., 7 (1967) 41-57.
[16] Y. Ni, X. Li, and J. Zhang, Indefinite mean-field stochastic linear-quadratic optimal control: from finite horizon to infinite horizon, IEEE Trans. Automat. Control, 61 (2016), no. 11, 3269-3284.
[17] S. Peng and Z. Wu, Fully coupled forward-backward stochastic differential equations and applications to optimal control, SIAM J. Control Optim., 37 (1999), no. 3, 825-843.
[18] J. Sun, Mean-field stochastic linear quadratic optimal control problems: open-loop solvabilities, ESAIM Control Optim. Calc. Var., 23 (2017), no. 3, 1099-1127.
[19] R. Tian, Z. Yu, and R. Zhang, A closed-loop saddle point for zero-sum linear-quadratic stochastic differential games with mean-field type, Systems Control Lett., 136 (2020), 104624, 11 pp.
[20] T. Wang, Characterizations of equilibrium controls in time inconsistent mean-field stochastic linear quadratic problems. I, Math. Control Relat. Fields, 9 (2019), no. 2, 385-409.
[21] Q. Wei, J. Yong, and Z. Yu, Linear quadratic stochastic optimal control problems with operator coefficients: open-loop solutions, ESAIM Control Optim. Calc. Var., 25 (2019), Paper No. 17, 38 pp.
[22] J. Yong, Finding adapted solutions of forward-backward stochastic differential equations: method of continuation, Probab. Theory Related Fields, 107 (1997), no. 4, 537-572.
[23] J. Yong, Linear-quadratic optimal control problems for mean-field stochastic differential equations, SIAM J. Control Optim., 51 (2013), no. 4, 2809-2838.
[24] J. Yong and X. Zhou, Stochastic controls: Hamiltonian systems and HJB equations, Springer-Verlag, New York, 1999.
[25] Z. Yu, On forward-backward stochastic differential equations in a domination-monotonicity framework, submitted.

## Appendix A Proofs of Proposition 2.1 and Proposition 2.2

In this appendix, we devote ourselves to proving Proposition 2.1 and Proposition 2.2.

## A.1 Proof of Proposition 2.1

We split the whole proof into three steps.
Step 1: Unique solvability. Let $\left[t^{\prime}, T^{\prime}\right] \subset[t, T]$ and $x_{t^{\prime}} \in L_{\mathcal{F}_{t^{\prime}}\left(\Omega ; \mathbb{R}^{n}\right)}^{2}$. For any $x(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)$, we introduce the following SDE:

$$
\left\{\begin{array}{l}
\mathrm{d} X(s)=b\left(s, X(s), \mathbb{E}_{t}[x(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} \sigma_{i}\left(s, X(s), \mathbb{E}_{t}[x(s)]\right) \mathrm{d} W_{i}(s), \quad s \in\left[t^{\prime}, T^{\prime}\right]  \tag{A.1}\\
X\left(t^{\prime}\right)=x_{t^{\prime}}
\end{array}\right.
$$

Since the mappings $b$ and $\sigma$ are Lipschitz continuous with respect to $x^{\prime}, b(\cdot, 0,0) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left([t, T] ; \mathbb{R}^{n}\right)\right)$ and $\sigma(\cdot, 0,0) \in L_{\mathbb{F}}^{2}\left(t, T ; \mathbb{R}^{n d}\right)$, then it is easy to verify that $b\left(\cdot, 0, \mathbb{E}_{t}[x(\cdot)]\right) \in L_{\mathbb{F}}^{2}\left(\Omega ; L\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)$ and $\sigma\left(\cdot, 0, \mathbb{E}_{t}[x(\cdot)]\right) \in L_{\mathbb{F}}^{2}\left(t^{\prime}, T^{\prime} ; \mathbb{R}^{n d}\right)$. Therefore, the classical theory of SDEs implies that (A.1) admits a unique solution $X(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)$. Due to the arbitrariness of $x(\cdot)$, we have actually constructed a mapping $\mathscr{T}_{F}: L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right) \rightarrow L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)$ :

$$
\mathscr{T}_{F}(x(\cdot))=X(\cdot) .
$$

Let $x(\cdot), \bar{x}(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)$ and $X(\cdot)=\mathscr{T}_{F}(x(\cdot)), \bar{X}(\cdot)=\mathscr{T}_{F}(\bar{x}(\cdot))$. Denote $\widehat{x}(\cdot)=x(\cdot)-\bar{x}(\cdot)$ and $\widehat{X}(\cdot)=X(\cdot)-\bar{X}(\cdot)$. Then the continuous dependence property of the solution on the coefficients in
the classical theory of SDEs shows

$$
\begin{align*}
\mathbb{E}\left[\sup _{s \in\left[t^{\prime}, T^{\prime}\right]}|\widehat{X}(s)|^{2}\right] \leq & K_{F} \mathbb{E}\left\{\left(\int_{t^{\prime}}^{T^{\prime}}\left|b\left(s, \bar{X}(s), \mathbb{E}_{t}[x(s)]\right)-b\left(s, \bar{X}(s), \mathbb{E}_{t}[\bar{x}(s)]\right)\right| \mathrm{d} s\right)^{2}\right.  \tag{A.2}\\
& \left.+\int_{t^{\prime}}^{T^{\prime}}\left|\sigma\left(s, \bar{X}(s), \mathbb{E}_{t}[x(s)]\right)-\sigma\left(s, \bar{X}(s), \mathbb{E}_{t}[\bar{x}(s)]\right)\right|^{2} \mathrm{~d} s\right\}
\end{align*}
$$

where $K_{F}:=K_{F}(T-t, L)>0$ is a constant depending on $(T-t)$ and the Lipschitz constant of the mappings $b$ and $\sigma$ with respect to $x$. Moreover, the Lipschitz continuity of $b$ and $\sigma$ with respect ot $x^{\prime}$ was also used to yield

$$
\begin{aligned}
\mathbb{E}\left[\sup _{s \in\left[t^{\prime}, T^{\prime}\right]}|\widehat{X}(s)|^{2}\right] & \leq K_{F} L^{2} \mathbb{E}\left\{\left(\int_{t^{\prime}}^{T^{\prime}} \mathbb{E}_{t}[|\widehat{x}(s)|] \mathrm{d} s\right)^{2}+\int_{t^{\prime}}^{T^{\prime}}\left(\mathbb{E}_{t}[|\widehat{x}(s)|]\right)^{2} \mathrm{~d} s\right\} \\
& \leq 2 K_{F} L^{2}\left(T^{\prime}-t^{\prime}\right) \max \left\{\left(T^{\prime}-t^{\prime}\right), 1\right\} \mathbb{E}\left[\sup _{s \in\left[t^{\prime}, T^{\prime}\right]}|\widehat{x}(s)|^{2}\right]
\end{aligned}
$$

Let

$$
\begin{equation*}
\delta_{F}=\min \left\{1, \frac{1}{8 K_{F} L^{2}}\right\} \tag{A.3}
\end{equation*}
$$

Then, when $T^{\prime}-t^{\prime} \leq \delta_{F}$, the above inequality implies

$$
\|\widehat{X}(\cdot)\|_{L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)} \leq \frac{1}{2}\|\widehat{x}(\cdot)\|_{L_{\mathbb{F}}^{2}\left(\Omega ; C\left(\left[t^{\prime}, T^{\prime}\right] ; \mathbb{R}^{n}\right)\right)}
$$

i.e., the mapping $\mathscr{T}_{F}$ is a contraction. Due to Banach's contraction mapping theorem, it admits a unique fixed point which is the unique solution to the following MF-SDE:

$$
\left\{\begin{array}{l}
\mathrm{d} x(s)=b\left(s, x(s), \mathbb{E}_{t}[x(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} \sigma_{i}\left(s, x(s), \mathbb{E}_{t}[x(s)]\right) \mathrm{d} W_{i}(s), \quad s \in\left[t^{\prime}, T^{\prime}\right]  \tag{A.4}\\
x\left(t^{\prime}\right)=x_{t^{\prime}}
\end{array}\right.
$$

Now, we turn our attention to MF-SDE (2.1). Based on the previous study, we divide the whole interval into

$$
\begin{equation*}
N_{F}=\inf \left\{\bar{N} \in \mathbb{N} \left\lvert\, \bar{N} \geq \frac{T-t}{\delta_{F}}\right.\right\} \tag{A.5}
\end{equation*}
$$

sub-intervals with $\delta_{F}$ defined by (A.3) as the step length. Then, we derive that MF-SDE (2.1) admits a unique solution $x(\cdot) \in L_{\mathbb{F}}^{2}\left(\Omega ; C\left([t, T] ; \mathbb{R}^{n}\right)\right)$.

Step 2: Estimate (2.2). For any $s \in[t, T]$, the classical estimate of SDEs leads to

$$
\begin{aligned}
& \mathbb{E}_{t}\left[\sup _{r \in[t, s]}|x(r)|^{2}\right] \\
\leq & K_{F}\left\{\left|x_{t}\right|^{2}+\mathbb{E}_{t}\left[\left(\int_{t}^{s}\left|b\left(r, 0, \mathbb{E}_{t}[x(r)]\right)\right| \mathrm{d} r\right)^{2}+\int_{t}^{s}\left|\sigma\left(r, 0, \mathbb{E}_{t}[x(r)]\right)\right|^{2} \mathrm{~d} r\right]\right\} \\
\leq & M_{F}+2 K_{F} L^{2} \mathbb{E}_{t}\left[\left(\int_{t}^{s} \mathbb{E}_{t}[|x(r)|] \mathrm{d} r\right)^{2}+\int_{t}^{s}\left(\mathbb{E}_{t}[|x(r)|]\right)^{2} \mathrm{~d} r\right]
\end{aligned}
$$

where $K_{F}$ is the same constant as in (A.2) and

$$
\begin{equation*}
M_{F}:=2 K_{F}\left\{\left|x_{t}\right|^{2}+\mathbb{E}_{t}\left[\left(\int_{t}^{T}|b(r, 0,0)| \mathrm{d} r\right)^{2}+\int_{t}^{T}|\sigma(r, 0,0)|^{2} \mathrm{~d} r\right]\right\} \tag{A.6}
\end{equation*}
$$

We continue to derive

$$
\mathbb{E}_{t}\left[\sup _{r \in[t, s]}|x(r)|^{2}\right] \leq M_{F}+2 K_{F} L^{2}(T-t+1) \int_{t}^{s} \mathbb{E}_{t}\left[\sup _{u \in[t, r]}|x(u)|^{2}\right] \mathrm{d} r
$$

Then, Gronwall's inequality leads to the desired estimate (2.2).
Step 3: Estimate (2.3). Denote $\widehat{x}_{t}:=x_{t}-\bar{x}_{t}$ and

$$
\begin{aligned}
& \widehat{b}\left(s, x, x^{\prime}\right):=b\left(s, x+\bar{x}(s), x^{\prime}+\mathbb{E}_{t}[\bar{x}(s)]\right)-\bar{b}\left(s, \bar{x}(s), \mathbb{E}_{t}[\bar{x}(s)]\right) \\
& \widehat{\sigma}\left(s, x, x^{\prime}\right):=\sigma\left(s, x+\bar{x}(s), x^{\prime}+\mathbb{E}_{t}[\bar{x}(s)]\right)-\bar{\sigma}\left(s, \bar{x}(s), \mathbb{E}_{t}[\bar{x}(s)]\right)
\end{aligned}
$$

for all $\left(s, \omega, x, x^{\prime}\right) \in[t, T] \times \Omega \times \mathbb{R}^{n} \times \mathbb{R}^{n}$. It is easy to verify that the new defined coefficients $(\widehat{x} t, \widehat{b}, \widehat{\sigma})$ also satisfy Assumption (H1) with the same Lipschitz constant. Therefore, by the proved Step 1, the following MF-SDE

$$
\left\{\begin{array}{l}
\mathrm{d} \widehat{x}(s)=\widehat{b}\left(s, \widehat{x}(s), \mathbb{E}_{t}[\widehat{x}(s)]\right) \mathrm{d} s+\sum_{i=1}^{d} \widehat{\sigma}_{i}\left(s, \widehat{x}(s), \mathbb{E}_{t}[\widehat{x}(s)]\right) \mathrm{d} W_{i}(s), \quad s \in[t, T]  \tag{A.7}\\
\widehat{x}(t)=\widehat{x}_{t}
\end{array}\right.
$$

admits a unique solution. Moreover, we directly verify that the unique solution happens to be $\widehat{x}(\cdot)=$ $x(\cdot)-\bar{x}(\cdot)$. Finally, by the proved Step 2, the estimate (2.2) is applied to MF-SDE (A.7) to yield (2.3). The proof is finished.

## A. 2 Proof of Proposition 2.2

Similar to the proof of Proposition 2.1, here we also split the whole proof into three steps.
Step 1: Unique solvability. This step is similar to Step 1 in the proof of Proposition 2.1. Then, we would like to omit it.

Step 2: Estimate (2.5). For any $\left[t^{\prime}, T^{\prime}\right] \subset[t, T]$, by the classical estimate of BSDEs, we have

$$
\begin{aligned}
& \mathbb{E}_{t}\left[\sup _{s \in\left[t^{\prime}, T^{\prime}\right]}|y(s)|^{2}+\int_{t^{\prime}}^{T^{\prime}}|z(s)|^{2} \mathrm{~d} s\right] \\
\leq & K_{B} \mathbb{E}_{t}\left\{\left|y\left(T^{\prime}\right)\right|^{2}+\left(\int_{t^{\prime}}^{T^{\prime}}\left|g\left(s, 0, \mathbb{E}_{t}[y(s)], 0, \mathbb{E}_{t}[z(s)]\right)\right| \mathrm{d} s\right)^{2}\right\} \\
\leq & 3 K_{B} \mathbb{E}_{t}\left\{\left|y\left(T^{\prime}\right)\right|^{2}+\left(\int_{t^{\prime}}^{T^{\prime}}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right. \\
& \left.+L^{2}\left(T^{\prime}-t^{\prime}\right)^{2} \sup _{s \in\left[t^{\prime}, T^{\prime}\right]}|y(s)|^{2}+L^{2}\left(T^{\prime}-t^{\prime}\right) \int_{t^{\prime}}^{T^{\prime}}|z(s)|^{2} \mathrm{~d} s\right\}
\end{aligned}
$$

where $K_{B}:=K_{B}(T-t, L)>0$ is a constant depending on $(T-t)$ and the Lipschitz constant of $g$ with respect to $(y, z)$. Let

$$
\begin{equation*}
\delta_{B}:=\min \left\{1, \frac{1}{6 K_{B} L^{2}}\right\} . \tag{A.8}
\end{equation*}
$$

Then, when $T^{\prime}-t^{\prime} \leq \delta_{B}$, the above inequality implies

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in\left[t^{\prime}, T^{\prime}\right]}|y(s)|^{2}+\int_{t^{\prime}}^{T^{\prime}}|z(s)|^{2} \mathrm{~d} s\right] \leq 6 K_{B} \mathbb{E}_{t}\left\{\left|y\left(T^{\prime}\right)\right|^{2}+\left(\int_{t^{\prime}}^{T^{\prime}}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right\} \tag{A.9}
\end{equation*}
$$

Now, we divide the whole interval $[t, T]$ into

$$
\begin{equation*}
N_{B}=\inf \left\{\bar{N} \in \mathbb{N} \left\lvert\, \bar{N} \geq \frac{T-t}{\delta_{B}}\right.\right\} \tag{A.10}
\end{equation*}
$$

sub-intervals with $\delta_{B}$ defined by (A.8) as the step length. Denote the split points by $t=t_{0}<t_{1}<t_{2}<$ $\cdots<t_{N_{B}-1}<t_{N_{B}}=T$. Then, for each $k=0,1, \ldots, N_{B}-1$, from (A.9), we have

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in\left[t_{k}, t_{k+1}\right]}|y(s)|^{2}+\int_{t_{k}}^{t_{k+1}}|z(s)|^{2} \mathrm{~d} s\right] \leq 6 K_{B} \mathbb{E}_{t}\left\{\left|y\left(t_{k+1}\right)\right|^{2}+\left(\int_{t_{k}}^{t_{k+1}}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right\} \tag{A.11}
\end{equation*}
$$

Without loss of generality, we can assume that $6 K_{B} \geq 1$ (otherwise, we can use $\widetilde{K}_{B}:=\max \left\{K_{B}, 1 / 6\right\}$ instead of $K_{B}$ ). For $k=N_{B}-1, N_{B}-2, \ldots, 0$, by some recursive but straightforward calculations from (A.11), we have

$$
\begin{align*}
& \mathbb{E}_{t}\left[\sup _{s \in\left[t_{k}, t_{k+1}\right]}|y(s)|^{2}+\int_{t_{k}}^{t_{k+1}}|z(s)|^{2} \mathrm{~d} s\right] \\
\leq & \left(6 K_{B}\right)^{N_{B}-k} \mathbb{E}_{t}\left\{\left|y_{T}\right|^{2}+\left(\int_{t_{k}}^{T}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right\}  \tag{A.12}\\
\leq & \left(6 K_{B}\right)^{N_{B}-k} \mathbb{E}_{t}\left\{\left|y_{T}\right|^{2}+\left(\int_{t}^{T}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right\} .
\end{align*}
$$

Since

$$
\sup _{s \in[t, T]}|y(s)|^{2} \leq \sum_{k=0}^{N-1} \sup _{s \in\left[t_{k}, t_{k+1}\right]}|y(s)|^{2} \quad \text { and } \quad \int_{t}^{T}|z(s)|^{2} \mathrm{~d} s=\sum_{k=0}^{N-1} \int_{t_{k}}^{t_{k+1}}|z(s)|^{2} \mathrm{~d} s
$$

we sum up (A.12) from $k=0$ to $k=N_{B}-1$ to get

$$
\begin{equation*}
\mathbb{E}_{t}\left[\sup _{s \in[t, T]}|y(s)|^{2}+\int_{t}^{T}|z(s)|^{2} \mathrm{~d} s\right] \leq\left(\sum_{k=0}^{N_{B}-1}\left(6 K_{B}\right)^{N_{B}-k}\right) \mathbb{E}_{t}\left[\left|y_{T}\right|^{2}+\left(\int_{t}^{T}|g(s, 0,0,0,0)| \mathrm{d} s\right)^{2}\right] \tag{A.13}
\end{equation*}
$$

The estimate (2.5) for MF-BSDEs is proved.
Step 3: Estimate (2.6). This step is similar to Step 3 in the proof of Proposition 2.1. We omit it.


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