CONVERGENCE ANALYSIS OF THE NEWTON-SCHUR METHOD FOR THE SYMMETRIC ELLIPTIC EIGENVALUE PROBLEM *

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Abstract. In this paper, we consider the Newton-Schur method in Hilbert space and obtain quadratic convergence. For the symmetric elliptic eigenvalue problem discretized by the standard finite element method and non-overlapping domain decomposition method, we use the Steklov-Poincaré operator to reduce the eigenvalue problem on the domain Ω into the nonlinear eigenvalue subproblem on Γ , which is the union of subdomain boundaries. We prove that the convergence rate for the Newton-Schur method is $\epsilon_N \leq CH^2(1 + \ln(H/h))^2\epsilon^2$, where the constant C is independent of the fine mesh size h and coarse mesh size H, and ϵ_N and ϵ are errors after and before one iteration step respectively. Numerical experiments confirm our theoretical analysis.

 ${\bf Key}$ words. Eigenvalue problem, convergence analysis, Newton-Schur method, domain decomposition method

AMS subject classifications. 65N25, 65N30, 65N55

1. Introduction. It is well-known that the smallest eigenvalue problem is very important in scientific and engineering computations. Suppose V is a Hilbert space with inner product (\cdot, \cdot) , the eigenvalue problem can be defined as

(1.1)
$$a(v_{\lambda}, v) = \lambda (v_{\lambda}, v) \quad \forall v \in V,$$

where $a(\cdot, \cdot)$ is a symmetric bilinear form on $V \times V$.

There are many classical methods for computing the eigenvalue and its corresponding eigenvector in algebraic view [4,15,32,34,39]. However, traditional methods suffer from slow convergence for problems from fluid dynamics or electronic device simulation [33]. Therefore, preconditioning techniques are often necessary for converging fast. One of the most famous preconditioned method for eigenvalue problem is the Locally Optimal Block Preconditioned Conjugate Gradient (LOBPCG) method proposed by Knyazev et al. [23–25].

In PDE view, especially for symmetric elliptic eigenvalue problems, there are many effective methods. For the early important researches on computing the eigenpair, a multigrid method was proposed by Hackbusch in [17], a mesh refinement strategy was introduced by McCormick in [31] and a multilevel inverse iteration procedure was analyzed by Bank in [5]. As for the theoretical analysis, the standard Galerkin approximation scheme for computing the approximate eigenpair was analyzed by Babuška and Osborn in [1–3]. When it comes to the domain decomposition method, some two-domains decomposition methods for computing the smallest eigenpair were proposed by Lui in [28] and a Schwarz alternating method for many subdomains case was constructed by Maliassov in [29]. Another effective eigenvalue solver is the two grid method proposed by Xu and Zhou in [42,43]. There were some further study about it, such as [18,45,46]. Moreover, some methods based on correc-

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tion were also proposed for eigenvalue problem, such as [27,40]. Recently, a two-level overlapping hybrid domain decomposition method for solving the large scale elliptic eigenvalue problem by Jacobi-Davidson method was proposed by Wang and Xu in [36,37].

One important theoretical problem for these methods is to find conditions such that the algorithm is optimal [35], which means that there exists a constant C independent of fine mesh size h such that

 $\epsilon_N \leq C\epsilon$,

where ϵ_N and ϵ are errors after and before one iteration respectively. In early versions of two grid method, some conditions between h and H are needed to ensure the optimality, where H is the mesh size of the coarse space. The first method proposed by Xu and Zhou in [42,43] needs $\mathcal{O}(H^2) = h$, another two level methods based on inverse iteration can be optimal under the condition $\mathcal{O}(H^4) = h$, see [18,45]. Recently, the method proposed by Wang and Xu in [36,37] is optimal with no assumptions between h and H. For Maxwell eigenvalue problem, similar results can be found in [26].

One popular non-overlapping domain decomposition method for eigenvalue problem is the spectral Schur complements method proposed by Bekas and Saad in [6]. It can be regarded as a variation of Automated MultiLevel Substructuring (AMLS) method in [7], whose numerical implementation can be found in [14]. Recently, the spectral Schur complement method was developed into the Newton-Schur method in [19–22] by Kalantzis, Li and Saad. All these researches focused on algorithm design and numerical implementation in algebraic view. As for the convergence rate, since the Newton-Schur method is essentially Newton's method, it could be expected to converge quadratically, at least if a sufficiently accurate initial approximation is provided [21]. But a rigorous theoretical analysis is hard.

In this paper, we focus on the theoretical analysis of the convergence rate. The Newton-Schur method is studied in the abstract Hilbert space, and the quadratic convergence is obtained under some assumptions on the bilinear form $a(\cdot, \cdot)$ in (1.1). For symmetric elliptic eigenvalue problems discretized by the standard finite element method and non-overlapping domain decomposition method, we use the Steklov-Poincaré operator to reduce the eigenvalue problem in the domain Ω into the nonlinear eigenvalue subproblem on Γ , which is the union of subdomain boundaries. The assumptions on the bilinear form are verified and the convergence rate of the Newton-Schur method is

(1.2)
$$\epsilon_N \le CH^2 \left(1 + \ln(H/h)\right)^2 \epsilon^2,$$

where the constant C is independent of h and H. The theoretical results are confirmed by our numerical examples for both two-dimensional and three-dimensional elliptic problems. To the best of our knowledge, similar results are not found in the references.

The outline of this paper is organized as follows: we extend the Newton-Schur method into Hilbert space and provide some results about convergence in section 2. In section 3, we analyze an important problem, the symmetric elliptic eigenvalue problem discretized by the standard finite element method and non-overlapping domain decomposition method, we prove the rate of convergence in (1.2). Finally, numerical experiments are given in section 4. In the rest of this paper, we use notations in [41]. Let $A \leq B$ represent the statement that $A \leq cB$, where the constant c is positive and independent of h, H and the variables in A and B. The notation $A \gtrsim B$ means $B \leq A$ and $A \approx B$ means that $A \leq B$ and $B \leq A$.

2. Newton-Schur method in Hilbert space.

2.1. Setting the problem. Let W be a Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$, and $V \subset W$ be a closed subspace. Suppose $a(\cdot, \cdot)$ is a symmetric bilinear form on $V \times V$. Let V_I be a closed subspace of V and suppose $a(\cdot, \cdot)$ is coercive on V_I , i.e., there exists a constant $\alpha > 0$ such that

$$a(v,v) \ge \alpha \|v\|^2$$

for all $v \in V_I$ and with equality for some $v_I \in V_I$. Since $\alpha > 0$, $a(\cdot, \cdot)$ can be regarded as an inner product on V_I . For a scalar $\rho < \alpha$, let

(2.2)
$$a_{\rho}(\cdot, \cdot) \equiv a(\cdot, \cdot) - \rho(\cdot, \cdot),$$

then $a_{\rho}(\cdot, \cdot)$ is positive definite on V_I , and the a_{ρ} -orthogonal space of V_I is defined as

(2.3)
$$V_{B,\rho} \equiv \{ v \in V \mid a_{\rho}(v, v_I) = 0, \forall v_I \in V_I \}$$

By using the Lax-Milgram's lemma, we can get the following decomposition for V.

PROPOSITION 2.1. For any $\rho < \alpha$ and all $v \in V$, there exists a unique decomposition

$$v = v_I + v_B,$$

where $v_I \in V_I$ and $v_B \in V_{B,\rho}$.

Suppose W_{Γ} is another Hilbert space with one inner product $(\cdot, \cdot)_{\Gamma}$ and the corresponding norm $\|\cdot\|_{\Gamma}$, and $V_{\Gamma} \subset W_{\Gamma}$ is a closed subspace. Let \mathcal{H}_{ρ} be a bijective bounded extension from V_{Γ} to $V_{B,\rho}$ satisfying

$$(2.4) |||\mathcal{H}_{\rho}||| \le c_{\mathcal{H}}$$

for all $\rho < \alpha$, where $c_{\mathcal{H}}$ is a constant independent of ρ and

$$\||\mathcal{H}_{\rho}|| \equiv \sup_{0 \neq u \in V_{\Gamma}} \frac{||\mathcal{H}_{\rho}u||}{||u||_{\Gamma}}.$$

Moreover, for any ρ_1 , $\rho_2 < \alpha$, the extensions \mathcal{H}_{ρ_1} and \mathcal{H}_{ρ_2} satisfy

(2.5)
$$\mathcal{H}_{\rho_1} u - \mathcal{H}_{\rho_2} u \in V_I$$

for all $u \in V_{\Gamma}$. The following lemma describes the continuity of \mathcal{H}_{ρ} respect to ρ .

LEMMA 2.2. Let ρ_1 , $\rho_2 < \alpha$ and $\delta \mathcal{H} = \mathcal{H}_{\rho_1} - \mathcal{H}_{\rho_2}$, then

$$\|\delta \mathcal{H}u\| \leq \frac{|\rho_1 - \rho_2|}{\alpha - \rho_1} \|\mathcal{H}_{\rho_2}u\| \leq \frac{c_{\mathcal{H}}|\rho_1 - \rho_2|}{\alpha - \rho_1} \|u\|_{\Gamma}$$

for all $u \in V_{\Gamma}$.

Proof. According to $\delta \mathcal{H}u = \mathcal{H}_{\rho_1}u - \mathcal{H}_{\rho_2}u \in V_I$ and V_I is a_{ρ_1} -orthogonal to V_{B,ρ_1} ,

$$a_{\rho_1}(\delta \mathcal{H}u, \delta \mathcal{H}u) = a_{\rho_1}(\delta \mathcal{H}u, \mathcal{H}_{\rho_1}u - \mathcal{H}_{\rho_2}u) = -a_{\rho_1}(\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u).$$

Due to V_I is a_{ρ_2} -orthogonal to V_{B,ρ_2} and the linearity of a_{ρ} respect to ρ ,

$$a_{\rho_1}(\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u) = a_{\rho_2}(\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u) + (\rho_2 - \rho_1)\left(\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u\right) = (\rho_2 - \rho_1)\left(\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u\right).$$

Therefore, by combining these two equations above,

(2.6)
$$a_{\rho_1}(\delta \mathcal{H}u, \delta \mathcal{H}u) = (\rho_1 - \rho_2) (\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u).$$

Since $a_{\rho_1}(\cdot, \cdot)$ is coercive on V_I and $\delta \mathcal{H} u \in V_I$,

$$(\alpha - \rho_1) \|\delta \mathcal{H}u\|^2 \le (\rho_1 - \rho_2) (\delta \mathcal{H}u, \mathcal{H}_{\rho_2}u) \le |\rho_1 - \rho_2| \|\delta \mathcal{H}u\| \|\mathcal{H}_{\rho_2}u\|.$$

By eliminating $\|\delta \mathcal{H}u\|$ on both sides of the equation above and using the bound for $\||\mathcal{H}_{\rho_2}\|\|$,

$$\|\delta \mathcal{H}u\| \leq \frac{|\rho_1 - \rho_2|}{\alpha - \rho_1} \|\mathcal{H}_{\rho_2}u\| \leq \frac{c_{\mathcal{H}} |\rho_1 - \rho_2|}{\alpha - \rho_1} \|u\|_{\Gamma}$$

This inequality means $\mathcal{H}_{\rho_1}u - \mathcal{H}_{\rho_2}u$ goes to zero when $|\rho_1 - \rho_2| \to 0$, which leads to the continuity.

The Steklov-Poincaré operator $S_{\rho} \colon V_{\Gamma} \mapsto (V_{\Gamma})'$ can be defined as

(2.7)
$$\langle S_{\rho}u_1, u_2 \rangle \equiv a_{\rho}(\mathcal{H}_{\rho}u_1, \mathcal{H}_{\rho}u_2)$$

for all $u_1, u_2 \in V_{\Gamma}$, where $(V_{\Gamma})'$ is the dual space of V_{Γ} and the bilinear form $\langle \cdot, \cdot \rangle$ is the duality pairing.

2.2. The smallest eigenvalue problem and the Newton-Schur method. We are interested in the smallest eigenvalue problem of $a(\cdot, \cdot)$ in V, which is to find the smallest λ and $||v_{\lambda}|| = 1$ such that

(2.8)
$$a(v_{\lambda}, v) = \lambda (v_{\lambda}, v)$$

for all $v \in V$. In the rest of this paper, we assume that the smallest eigenvalue of $a(\cdot, \cdot)$ is simple and $-\infty < \lambda < \alpha$. Actually, due to the variation principle of eigenvalues (see Equation 2.1 in Section 3 of [38]),

$$\lambda = \min_{0 \neq v \in V} \frac{a(v, v)}{(v, v)} \le \min_{0 \neq v \in V_I} \frac{a(v, v)}{(v, v)} = \alpha,$$

with equality only when $v_{\lambda} \in V_I$. So $\lambda < \alpha$ means that the eigenvector v_{λ} corresponding to the smallest eigenvalue λ is not in V_I . If there exists a scalar λ and $u_{\lambda} \in V_{\Gamma}$, $u_{\lambda} \neq 0$ such that

(2.9)
$$\langle S_{\lambda}u_{\lambda},u\rangle = 0$$

for all $u \in V_{\Gamma}$, by using the definition of Steklov-Poincaré operator in (2.7),

$$a_{\lambda}(\mathcal{H}_{\lambda}u_{\lambda},\mathcal{H}_{\lambda}u) = \langle S_{\lambda}u_{\lambda},u \rangle = 0$$

for all $u \in V_{\Gamma}$. On the one hand, as \mathcal{H}_{λ} is injective on $V_{B,\rho}$, $u_{\lambda} \neq 0$ leads to $\mathcal{H}_{\lambda}u_{\lambda} \neq 0$. On the other hand, since \mathcal{H}_{λ} is surjective to $V_{B,\lambda}$ and $V_{B,\lambda}$ is the a_{λ} -orthogonal complement of V_{I} , by Proposition 2.1,

(2.10)
$$a_{\lambda}(\mathcal{H}_{\lambda}u_{\lambda},v) = 0$$

for all $v \in V$, i.e., λ is the eigenvalue of $a(\cdot, \cdot)$ and $\mathcal{H}_{\lambda}u_{\lambda}$ is parallel to the corresponding eigenvector. It is easy to verify that if λ is the smallest root of (2.9), then λ is the

smallest eigenvalue of $a(\cdot, \cdot)$. So we can use Steklov-Poincaré operator to reduce the smallest eigenvalue problem of $a(\cdot, \cdot)$ into the smallest root finding problem of S_{ρ} respect to ρ . The Newton-Schur method in [6,19–22] is in this framework of algebraic view. In this paper, we take a look at the method in Hilbert space and extend the Newton-Schur method to infinite dimension space. First, let us consider the eigenvalue problem of S_{ρ} :

(2.11)
$$\langle S_{\rho}u_{\rho}, u \rangle = \theta_{\rho} (u_{\rho}, u)_{\Gamma} \quad \forall u \in V_{\Gamma},$$

where $(\theta_{\rho}, u_{\rho})$ is the smallest eigenpair of S_{ρ} and $||u_{\rho}||_{\Gamma} = 1$. The root of (2.9) is found if we can find a λ such that $\theta_{\lambda} = 0$. Therefore, the root-finding problem (2.9) can be transformed into the nonlinear eigenvalue problem (2.11). In order to apply the Newton-Schur method for (2.11) to find ρ such that $\theta_{\rho} = 0$, the first order Fréchet derivative $S'_{\rho} \equiv S'(\rho)$ with respect to ρ needs to be calculated first. By $\mathcal{H}'_{\rho}u \in V_{I}$ for all $u \in V_{\Gamma}$, we have the following proposition.

PROPOSITION 2.3. The linear operator $S'_{\rho} \colon V_{\Gamma} \to (V_{\Gamma})'$ can be expressed as

$$\langle S'_{\rho}u_1, u_2 \rangle = -(\mathcal{H}_{\rho}u_1, \mathcal{H}_{\rho}u_2) \quad \forall u_1, u_2 \in V_{\Gamma}.$$

LEMMA 2.4. Assume $(\theta_{\rho}, u_{\rho})$ is the smallest eigenpair of S_{ρ} as (2.11). If $\rho < \alpha$, then the first order derivative $\theta'_{\rho} \equiv \theta'(\rho)$ satisfies

$$\theta'_{\rho} = \frac{\langle S'_{\rho} u_{\rho}, u_{\rho} \rangle}{(u_{\rho}, u_{\rho})_{\Gamma}} = -\frac{(\mathcal{H}_{\rho} u_{\rho}, \mathcal{H}_{\rho} u_{\rho})}{(u_{\rho}, u_{\rho})_{\Gamma}} < 0$$

Proof. By taking the derivative of (2.11) first:

(2.12)
$$\langle S'_{\rho}u_{\rho},u\rangle + \langle S_{\rho}u'_{\rho},u\rangle = \theta'_{\rho}(u_{\rho},u)_{\Gamma} + \theta_{\rho}(u'_{\rho},u)_{\Gamma}.$$

Let $u = u_{\rho}$, since $(\theta_{\rho}, u_{\rho})$ is the eigenpair of S_{ρ} and S_{ρ} is symmetric,

$$\langle S_{\rho}u'_{\rho}, u_{\rho} \rangle = \langle S_{\rho}u_{\rho}, u'_{\rho} \rangle = \theta_{\rho} (u_{\rho}, u'_{\rho})_{\Gamma} = \theta_{\rho} (u'_{\rho}, u_{\rho})_{\Gamma}$$

The lemma is proved by combining these two equations above with Proposition 2.3.

The Newton iteration $\rho_N = \rho - \theta_\rho / \theta'_\rho$ becomes

$$\rho_N = \rho - \frac{\langle S_\rho u_\rho, u_\rho \rangle}{\langle S'_\rho u_\rho, u_\rho \rangle}$$

by Lemma 2.4 and (2.11). Furthermore, let us take $v_{\rho} = \mathcal{H}_{\rho} u_{\rho}$, since

$$\langle S_{\rho}u_{\rho}, u_{\rho} \rangle = a_{\rho}(\mathcal{H}_{\rho}u_{\rho}, \mathcal{H}_{\rho}u_{\rho}) = a(v_{\rho}, v_{\rho}) - \rho(v_{\rho}, v_{\rho})$$

and $\langle S'_{\rho}u_{\rho}, u_{\rho}\rangle = -(v_{\rho}, v_{\rho})$, the Newton iteration becomes

(2.13)
$$\rho_N = \frac{a(v_\rho, v_\rho)}{(v_\rho, v_\rho)}.$$

Based on the iteration (2.13), the Newton-Schur method in Hilbert space can be sketched as Algorithm 1.

Al	gorithm	1:	Newton-S	Schur	method	in	Hilbert	space
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Input: initial point ρ .

Output: The approximation for the smallest eigenvalue ρ .

1 repeat

- **2** Solve the smallest eigenvalue problem (2.11) in V_{Γ} to get u_{ρ} ;
- **3** Extend u_{ρ} to V by $v_{\rho} = \mathcal{H}_{\rho} u_{\rho}$;
- 4 Update ρ with (2.13):

$$\rho = \frac{a(v_{\rho}, v_{\rho})}{(v_{\rho}, v_{\rho})};$$

5 until Convergence;

Since each iteration point ρ is a Rayleigh quotient of $a(\cdot, \cdot)$ and λ is the smallest eigenvalue of $a(\cdot, \cdot)$, $\rho \geq \lambda$ always holds during the algorithm. Let ϵ and ϵ_N be errors before and after one step of Newton iteration respectively:

(2.14)
$$\epsilon = \rho - \lambda$$
 and $\epsilon_N = \rho_N - \lambda$.

The following proposition can guarantee the convergence of Algorithm 1.

PROPOSITION 2.5. Let $\rho_0 \in (\lambda, \alpha)$ be the initial point of Algorithm 1, where α is the constant for coercivity and λ is the smallest eigenvalue of $a(\cdot, \cdot)$. The Newton-Schur method is convergent and at each iteration

$$0 \leq \epsilon_N < \epsilon$$

Proof. According to (2.13) and (2.14),

$$\epsilon_N = \epsilon - \frac{\theta_\rho}{\theta'_\rho}$$

Due to Lemma 2.4, we know $\theta_{\rho} < \theta_{\lambda} = 0$. By combining it with $\theta'_{\rho} < 0$,

$$\epsilon_N = \epsilon - \frac{\theta_\rho}{\theta'_\rho} < \epsilon.$$

On the other hand, as ρ_N is a Rayleigh quotient of $a(\cdot, \cdot)$, we have $\epsilon_N = \rho_N - \lambda \ge 0.\Box$

2.3. Convergence factor for the Newton-Schur method. In order to analyze the convergence, we need to use the well-known result about Newton's method.

PROPOSITION 2.6. Let ϵ and ϵ_N be errors before and after one step of Newton iteration. Suppose the iterative point ρ is in (λ, α) , the error ϵ_N satisfies

$$\epsilon_N = \frac{\theta_\xi''}{2\theta_\rho'} \,\epsilon^2,$$

where $\lambda \leq \xi \leq \rho$ and $\theta_{\xi}^{\prime\prime} \equiv \theta^{\prime\prime}(\xi)$.

The Newton-Schur method will converge quadratically if $\theta_{\xi}''/2\theta_{\rho}'$ is bounded. In this paper, we refer to the upper bound of $\theta_{\xi}''/2\theta_{\rho}'$ as convergence factor η , i.e.,

$$\eta = \sup_{\lambda \le \xi \le \rho \le \rho_0} \frac{\theta_{\xi}''}{2\theta_{\rho}'},$$

then errors satisfy $\epsilon_N \leq \eta \epsilon^2$.

LEMMA 2.7. Assume $(\theta_{\rho}, u_{\rho})$ is the smallest eigenpair of S_{ρ} as (2.11). If $\rho < \alpha$, then the second order derivative θ_{ρ}'' respect to ρ satisfies

$$\theta_{\rho}^{\prime\prime} \leq 0.$$

Proof. First, let us take the derivative in (2.12) respect to ρ again,

$$\langle S''_{\rho}u_{\rho},u\rangle + 2\,\langle S'_{\rho}u'_{\rho},u\rangle + \langle S_{\rho}u''_{\rho},u\rangle = \theta''_{\rho}\,(u_{\rho},u)_{\Gamma} + 2\,\theta'_{\rho}\,(u'_{\rho},u)_{\Gamma} + \theta_{\rho}\,(u''_{\rho},u)_{\Gamma}.$$

Let $u = u_{\rho}$ and by using (2.11), it becomes

$$\theta_{\rho}^{\prime\prime}(u_{\rho},u_{\rho})_{\Gamma} = \langle S_{\rho}^{\prime\prime}u_{\rho},u_{\rho}\rangle - 2\,\theta_{\rho}^{\prime}\,(u_{\rho}^{\prime},u_{\rho})_{\Gamma} + 2\,\langle S_{\rho}^{\prime}u_{\rho}^{\prime},u_{\rho}\rangle.$$

Let $u = u'_{\rho}$ in (2.12),

(2.15)
$$\langle S'_{\rho}u_{\rho}, u'_{\rho} \rangle + \langle S_{\rho}u'_{\rho}, u'_{\rho} \rangle = \theta'_{\rho}(u_{\rho}, u'_{\rho})_{\Gamma} + \theta_{\rho}(u'_{\rho}, u'_{\rho})_{\Gamma}$$

Combining these two equations above, we know

(2.16)
$$\theta_{\rho}^{\prime\prime}(u_{\rho}, u_{\rho})_{\Gamma} = \langle S_{\rho}^{\prime\prime}u_{\rho}, u_{\rho} \rangle + 2 \theta_{\rho} (u_{\rho}^{\prime}, u_{\rho}^{\prime})_{\Gamma} - 2 \langle S_{\rho}u_{\rho}^{\prime}, u_{\rho}^{\prime} \rangle.$$

Since θ_{ρ} is the smallest eigenvalue of S_{ρ} , we have $\theta_{\rho} (u'_{\rho}, u'_{\rho})_{\Gamma} \leq \langle S_{\rho}u'_{\rho}, u'_{\rho} \rangle$. Thus, the θ''_{ρ} can be bounded by

(2.17)
$$\theta_{\rho}^{\prime\prime}(u_{\rho}, u_{\rho})_{\Gamma} \leq \langle S_{\rho}^{\prime\prime}u_{\rho}, u_{\rho} \rangle.$$

By taking the derivative of S'_{ρ} in Lemma 2.4,

$$\langle S_{\rho}''u_{\rho}, u_{\rho} \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big((\mathcal{H}_{\rho}u_{\rho}, \mathcal{H}_{\rho}u_{\rho}) - (\mathcal{H}_{\rho+\varepsilon}u_{\rho}, \mathcal{H}_{\rho+\varepsilon}u_{\rho}) \Big).$$

Let $\delta \mathcal{H} = \mathcal{H}_{\rho+\varepsilon} - \mathcal{H}_{\rho}$, then

$$\langle S_{\rho}'' u_{\rho}, u_{\rho} \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big((\mathcal{H}_{\rho} u_{\rho} - \mathcal{H}_{\rho+\varepsilon} u_{\rho}, \mathcal{H}_{\rho} u_{\rho}) + (\mathcal{H}_{\rho} u_{\rho} - \mathcal{H}_{\rho+\varepsilon} u_{\rho}, \mathcal{H}_{\rho+\varepsilon} u_{\rho}) \Big)$$

(2.18)
$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(- (\delta \mathcal{H} u_{\rho}, \mathcal{H}_{\rho} u_{\rho}) - (\delta \mathcal{H} u_{\rho}, \mathcal{H}_{\rho+\varepsilon} u_{\rho}) \Big)$$
$$= -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(2 \left(\delta \mathcal{H} u_{\rho}, \mathcal{H}_{\rho+\varepsilon} u_{\rho} \right) - \left(\delta \mathcal{H} u_{\rho}, \delta \mathcal{H} u_{\rho} \right) \Big).$$

According to (2.6),

$$a_{\rho}(\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho}) = \varepsilon(\delta \mathcal{H}u_{\rho}, \mathcal{H}_{\rho+\varepsilon}u_{\rho}).$$

Substituting it into (2.18), we have

(2.19)
$$\langle S''_{\rho}u_{\rho}, u_{\rho}\rangle = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(\frac{2}{\varepsilon} a_{\rho}(\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho}) - (\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho})\Big).$$

Since $a_{\rho}(\cdot, \cdot)$ is coercive on V_I , then

(2.20)
$$(\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho}) \leq \frac{1}{\alpha - \rho} a_{\rho} (\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho}).$$

Therefore, by combining (2.17), (2.19), and (2.20),

$$(2.21) \quad \theta_{\rho}^{\prime\prime} \leq \frac{\langle S_{\rho}^{\prime\prime} u_{\rho}, u_{\rho} \rangle}{(u_{\rho}, u_{\rho})_{\Gamma}} \leq \frac{-1}{(u_{\rho}, u_{\rho})_{\Gamma}} \lim_{\varepsilon \to 0} \left(\left(\frac{2}{\varepsilon^2} - \frac{1}{\varepsilon (\alpha - \rho)} \right) a_{\rho} (\delta \mathcal{H} u_{\rho}, \delta \mathcal{H} u_{\rho}) \right) \leq 0.$$

The lemma is proved.

 $Remark~2.8.~{\rm Lemma~2.7~also~holds~for~other~eigenvalues~of~}S_{\rho}~{\rm if}~\rho<\alpha.$ Proposition 2.6 can be relaxed to

$$\epsilon_N \leq \sup_{\xi \in (\lambda, \rho)} \frac{\theta_{\xi}''}{2\theta_{\xi}'} \epsilon^2$$

based on Lemma 2.7.

THEOREM 2.9. Suppose $\rho_0 \in (\lambda, \alpha)$ is the initial point of Algorithm 1, denote g_{ρ} the gap between the smallest two eigenvalues of S_{ρ} , if there exists a constant $c_g > 0$ such that

for all $\rho \in (\lambda, \rho_0)$, then Algorithm 1 is quadratic convergence, and

$$\epsilon_N \leq \left(\frac{1}{\alpha - \rho_0} + \frac{c_{\mathcal{H}}^2}{c_g}\right) \epsilon^2.$$

Proof. Suppose $||u_{\rho}||_{\Gamma} = 1$, according to (2.16), $(\theta_{\rho}''/2\theta_{\rho}')$ can be divided into two parts:

(2.23)
$$-\langle S''_{\rho} u_{\rho}, u_{\rho} \rangle \le C_1 \, \|v_{\rho}\|^2,$$

and

(2.24)
$$\langle S_{\rho}u'_{\rho}, u'_{\rho}\rangle - \theta_{\rho}(u'_{\rho}, u'_{\rho})_{\Gamma} \leq C_2 \|v_{\rho}\|^2,$$

where $v_{\rho} = \mathcal{H}_{\rho} u_{\rho}$. According to (2.18),

$$-\langle S_{\rho}''u_{\rho}, u_{\rho} \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(2 \left(\delta \mathcal{H}u_{\rho}, \mathcal{H}_{\rho+\varepsilon}u_{\rho} \right) - \left(\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho} \right) \Big) \\ = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(2 \left(\delta \mathcal{H}u_{\rho}, \mathcal{H}_{\rho}u_{\rho} \right) + \left(\delta \mathcal{H}u_{\rho}, \delta \mathcal{H}u_{\rho} \right) \Big) \\ \leq \lim_{\varepsilon \to 0} \frac{\|\delta \mathcal{H}u_{\rho}\|}{|\varepsilon|} \left(2 \|\mathcal{H}_{\rho}u_{\rho}\| + \|\delta \mathcal{H}u_{\rho}\| \right)$$

Due to Lemma 2.2,

$$\begin{aligned} -\langle S_{\rho}''u_{\rho}, u_{\rho} \rangle &\leq \lim_{\varepsilon \to 0} \frac{\|\mathcal{H}_{\rho}u_{\rho}\|}{(\alpha - \rho - \varepsilon)} \Big(2\,\|\mathcal{H}_{\rho}u_{\rho}\| + \frac{|\varepsilon|}{\alpha - \rho}\|\mathcal{H}_{\rho}u_{\rho}\| \Big) \\ &= \frac{2}{\alpha - \rho}\|v_{\rho}\|^{2} \leq \frac{2}{\alpha - \rho_{0}}\|v_{\rho}\|^{2}. \end{aligned}$$

Then the constant C_1 in estimation (2.23) can be $2(\alpha - \rho_0)^{-1}$. For the estimation (2.24), an orthogonality about u_ρ and u'_ρ is needed. By taking the derivative on both

sides of $||u_{\rho}||_{\Gamma}^2 = 1$, we have $(u_{\rho}, u'_{\rho})_{\Gamma} = 0$, which means u_{ρ} and u'_{ρ} are orthogonal in $(\cdot, \cdot)_{\Gamma}$. Let

(2.25)
$$V_{\Gamma,\rho} \equiv \{ u \in V_{\Gamma} \mid (u, u_{\rho})_{\Gamma} = 0 \}.$$

Since g_{ρ} is the gap between θ_{ρ} and the smallest eigenvalue of S_{ρ} in $V_{\Gamma,\rho}$, and $u'_{\rho} \in V_{\Gamma,\rho}$,

(2.26)
$$\langle S_{\rho}u'_{\rho}, u'_{\rho} \rangle - \theta_{\rho} (u'_{\rho}, u'_{\rho})_{\Gamma} \ge g_{\rho} \|u'_{\rho}\|_{\Gamma}^{2}$$

Let $u = u'_{\rho}$ in (2.12), by combining the orthogonality of u_{ρ} and u'_{ρ} ,

(2.27)
$$\langle S_{\rho}u'_{\rho}, u'_{\rho} \rangle - \theta_{\rho} (u'_{\rho}, u'_{\rho})_{\Gamma} = -\langle S'_{\rho}u_{\rho}, u'_{\rho} \rangle + \theta'_{\rho} (u_{\rho}, u'_{\rho}) = -\langle S'_{\rho}u_{\rho}, u'_{\rho} \rangle.$$

According to the Cauchy-Schwarz inequality and Proposition 2.3,

(2.28)
$$-\langle S'_{\rho}u_{\rho}, u'_{\rho}\rangle = (\mathcal{H}_{\rho}u_{\rho}, \mathcal{H}_{\rho}u'_{\rho}) \le \|v_{\rho}\| \, \|\mathcal{H}_{\rho}u'_{\rho}\| \le c_{\mathcal{H}}\|v_{\rho}\| \, \|u'_{\rho}\|_{\Gamma}.$$

By combining the inequalities (2.26)–(2.28) and eliminating $\|u'_{\rho}\|_{\Gamma}$ on both sides,

(2.29)
$$\|u_{\rho}'\|_{\Gamma} \leq \frac{c_{\mathcal{H}}}{g_{\rho}} \|v_{\rho}\|.$$

And combining the inequalities (2.27)-(2.29), we have

$$\langle S_{\rho}u'_{\rho}, u'_{\rho}\rangle - \theta_{\rho} \left(u'_{\rho}, u'_{\rho}\right)_{\Gamma} \leq \frac{c_{\mathcal{H}}^2}{g_{\rho}} \|v_{\rho}\|^2 \leq \frac{c_{\mathcal{H}}^2}{c_g} \|v_{\rho}\|^2,$$

which means C_2 in estimation (2.24) can be set as $c_{\mathcal{H}}^2 c_g^{-1}$. Therefore, the constant in Theorem 2.9 can be $((\alpha - \rho_0)^{-1} + c_{\mathcal{H}}^2 c_g^{-1})$, which is independent of ρ .

The condition (2.22) in Theorem 2.9 may be difficult to verified directly, here we give a lemma for the existence of c_g .

LEMMA 2.10. Let $\hat{\lambda}$ be the second smallest eigenvalue of $a(\cdot, \cdot)$. If the initial point $\rho_0 \in (\lambda, \hat{\lambda})$ and $\hat{\lambda} < \alpha$, then the constant c_q in Theorem 2.9 exists and

$$c_g \geq \frac{\widehat{\lambda} - \rho_0}{\widehat{\lambda} - \lambda} \, \theta_{\lambda}^{(2)}$$

where $\theta_{\lambda}^{(2)}$ is the second smallest eigenvalue of S_{λ} .

Proof. Suppose $\theta_{\rho}^{(2)}$ is the second smallest eigenvalue of S_{ρ} , since $\theta_{\rho}^{(2)}$ is concave,

$$\frac{\theta_{\rho}^{(2)} - \theta_{\lambda}^{(2)}}{\rho - \lambda} \ge \frac{\theta_{\rho}^{(2)} - \theta_{\widehat{\lambda}}^{(2)}}{\rho - \widehat{\lambda}}$$

for $\lambda < \rho < \hat{\lambda}$. Since $\hat{\lambda}$ is the second smallest eigenvalue of $a(\cdot, \cdot)$, we know that $\theta_{\hat{\lambda}}^{(2)} = 0$. By using $\theta_{\lambda}^{(2)} > \theta_{\hat{\lambda}}^{(2)} = 0$ and

$$\theta_{\rho}^{(2)} \geq \frac{\widehat{\lambda} - \rho}{\widehat{\lambda} - \lambda} \, \theta_{\lambda}^{(2)} \geq \frac{\widehat{\lambda} - \rho_0}{\widehat{\lambda} - \lambda} \, \theta_{\lambda}^{(2)}$$

the proof is finished.

If the bilinear form $a(\cdot, \cdot)$ satisfies some more conditions, the convergence factor η can be estimated more specifically.

LEMMA 2.11. Suppose $\rho_0 \in (\lambda, \widehat{\lambda})$ is the initial point, where λ and $\widehat{\lambda}$ are the smallest two eigenvalues of $a(\cdot, \cdot)$ respectively. If there exists a constant $c_t > 0$ such that

(2.30)
$$a(\mathcal{H}_{\lambda}u, \mathcal{H}_{\lambda}u) \ge c_t \|u\|_{\Gamma}^2$$

holds for all $u \in V_{\Gamma}$, then there is a lower bound for the constant c_g , i.e.,

$$c_g \ge \frac{\widehat{\lambda} - \rho_0}{\widehat{\lambda}} \, c_s c_t,$$

where $0 < c_s = \theta_{\lambda}^{(2)} / \theta_{\lambda}^{(3)} \leq 1$ and $\theta_{\lambda}^{(2)}$, $\theta_{\lambda}^{(3)}$ are the second and third smallest eigenvalue of S_{λ} respectively, including multiplicities.

Proof. Let θ_{λ} , $\theta_{\lambda}^{(2)}$ and $\theta_{\lambda}^{(3)}$ be the smallest three eigenvalues of S_{λ} with corresponding eigenvectors u_{λ} , \hat{u}_2 and \hat{u}_3 , where $||u_{\lambda}||_{\Gamma} = ||\hat{u}_2||_{\Gamma} = ||\hat{u}_3||_{\Gamma} = 1$ and $(u_{\lambda}, \hat{u}_2)_{\Gamma} = (\hat{u}_2, \hat{u}_3)_{\Gamma} = (\hat{u}_3, u_{\lambda})_{\Gamma} = 0$. Let

$$\widehat{u}_{\lambda} = \begin{cases} \frac{(\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{2})\,\widehat{u}_{3} - (\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{3})\,\widehat{u}_{2}}{\|(\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{2})\,\widehat{u}_{3} - (\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{3})\,\widehat{u}_{2}\|_{\Gamma}} & \text{if } (\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{2}) \neq 0, \\ \widehat{u}_{2} & \text{if } (\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{2}) = 0, \end{cases}$$

then it is easy to be verified that $\|\widehat{u}_{\lambda}\|_{\Gamma} = 1$ and $(\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{\lambda}) = 0$. Since $\mathcal{H}_{\lambda}u_{\lambda}$ is the eigenvector corresponding to λ of $a(\cdot, \cdot)$, $(\mathcal{H}_{\lambda}u_{\lambda}, \mathcal{H}_{\lambda}\widehat{u}_{\lambda}) = 0$ leads to $\mathcal{H}_{\lambda}\widehat{u}_{\lambda}$ lies in the eigenspace corresponding to the eigenvalue at least $\widehat{\lambda}$, then

$$\begin{aligned} a_{\lambda}(\mathcal{H}_{\lambda}\widehat{u}_{\lambda},\mathcal{H}_{\lambda}\widehat{u}_{\lambda}) &= a(\mathcal{H}_{\lambda}\widehat{u}_{\lambda},\mathcal{H}_{\lambda}\widehat{u}_{\lambda}) - \lambda\left(\mathcal{H}_{\lambda}\widehat{u}_{\lambda},\mathcal{H}_{\lambda}\widehat{u}_{\lambda}\right) \\ &\geq \left(1 - \frac{\lambda}{\overline{\lambda}}\right)a(\mathcal{H}_{\lambda}\widehat{u}_{\lambda},\mathcal{H}_{\lambda}\widehat{u}_{\lambda}) \geq c_{t}\left(1 - \frac{\lambda}{\overline{\lambda}}\right). \end{aligned}$$

Since \hat{u}_{λ} is the linear combination of \hat{u}_2 and \hat{u}_3 , and $\|\hat{u}_{\lambda}\|_{\Gamma} = 1$,

$$\theta_{\lambda}^{(2)} \leq \langle S_{\lambda} \widehat{u}_{\lambda}, \widehat{u}_{\lambda} \rangle \leq \theta_{\lambda}^{(3)}.$$

Therefore, the second smallest eigenvalue of S_{λ} satisfies

$$\theta_{\lambda}^{(2)} = \frac{\theta_{\lambda}^{(2)}}{\theta_{\lambda}^{(3)}} \; \theta_{\lambda}^{(3)} \ge c_s \left\langle S_{\lambda} \widehat{u}_{\lambda}, \widehat{u}_{\lambda} \right\rangle = c_s \, a_{\lambda} (\mathcal{H}_{\lambda} \widehat{u}_{\lambda}, \mathcal{H}_{\lambda} \widehat{u}_{\lambda}) \ge (1 - \frac{\lambda}{\widehat{\lambda}}) \, c_s c_t.$$

Then the constant c_g can be bounded by

$$c_g \geq \frac{\widehat{\lambda} - \rho_0}{\widehat{\lambda}} \, c_s c_t$$

by using Lemma 2.10.

COROLLARY 2.12. Let λ and $\widehat{\lambda}$ be the smallest two eigenvalues of $a(\cdot, \cdot)$ satisfying $\lambda < \widehat{\lambda} < \alpha$. Suppose the constants c_s and c_t are defined as Lemma 2.11, if the initial point $\rho_0 \in (\lambda, \widehat{\lambda})$, then the rate of convergence for Algorithm 1 is

$$\epsilon_N \leq \left(\frac{1}{\alpha - \rho_0} + \frac{c_{\mathcal{H}}^2 \widehat{\lambda}}{c_s c_t (\widehat{\lambda} - \rho_0)}\right) \epsilon^2.$$

3. Finite element method for the symmetric elliptic eigenvalue problem. In this section, we focus on the smallest eigenvalue problem of the symmetric elliptic operator. The problem is discretized by finite element method and solved by the Newton-Schur method, the space V_I , $V_{B,\rho}$ and V_{Γ} are constructed by nonoverlapping domain decomposition method.

3.1. Elliptic eigenvalue problem and finite element method. Let $\Omega \subset \mathbb{R}^d$, where d = 2 or 3 be a bounded convex polygonal domain, the smallest eigenvalue problem of the symmetric elliptic operator is to find the smallest $\lambda \in \mathbb{R}$ and sufficiently smooth v_{λ} such that

(3.1)
$$\begin{aligned} Av_{\lambda} &= \lambda v_{\lambda} \quad \text{in } \Omega, \\ v_{\lambda} &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

where

$$Av \equiv -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial v(x)}{\partial x_j} \right)$$

and $\int_{\Omega} v_{\lambda}^2 dx = 1$. Assume the matrix $\{a_{ij}(x)\}_{i,j=1}^d$ is symmetric and uniformly positive definite and $a_{ij}(x) \in C^{0,1}(\overline{\Omega})$ for $i, j = 1, \ldots, d$. Let $W = L^2(\Omega)$, (\cdot, \cdot) be L^2 inner product on Ω and V be the Sobolev space $H_0^1(\Omega)$, then the variational form of (3.1) is

(3.2)
$$a(v_{\lambda}, v) \equiv \sum_{i,j=1}^{d} \int_{\Omega} a_{ij} \frac{\partial v_{\lambda}}{\partial x_{i}} \frac{\partial v}{\partial x_{j}} \, \mathrm{d}x = \lambda \left(v_{\lambda}, v \right) \equiv \lambda \int_{\Omega} v_{\lambda} v \, \mathrm{d}x \quad \forall v \in V.$$

Let λ be the smallest eigenvalue of (3.2), it is well-known that λ is simple (see Theorem 2 in Section 6.5 of [13]). Moreover, we assume that $a(\cdot, \cdot)$ is equivalent to the square of the H_0^1 norm, i.e.,

(3.3)
$$|v|_a \equiv \left(a(v,v)\right)^{1/2} \approx \left(\int_{\Omega} |\nabla v|^2 \,\mathrm{d}x\right)^{1/2}.$$

We construct continuous and piecewise linear element spaces $V^H \subset V^h \subset V$ based on quasi-uniform triangular partitions \mathcal{T}^H and \mathcal{T}^h , where 0 < h < H < 1 are mesh sizes of \mathcal{T}^H and \mathcal{T}^h respectively and \mathcal{T}^h is refined by \mathcal{T}^H . By using the finite element discretization, the variational form of (3.2) becomes

(3.4)
$$a(v_{\lambda}^{h}, v^{h}) = \lambda^{h} (v_{\lambda}^{h}, v^{h}) \quad \forall v^{h} \in V^{h},$$

where $v_{\lambda}^{h} \in V^{h}$. Then the discrete elliptic operator A^{h} is defined as

$$(A^{h}v_{1}^{h}, v_{2}^{h}) \equiv a(v_{1}^{h}, v_{2}^{h}) \quad \forall v_{1}^{h}, v_{2}^{h} \in V^{h}.$$

The convergence of the discrete eigenvalues can be found in [1].

PROPOSITION 3.1. Suppose A^h and A^H are discrete elliptic operators with mesh sizes h and H respectively, then, following properties hold.

(3.5)
$$\lambda^h - \lambda \approx h^2, \quad \widehat{\lambda}^h - \widehat{\lambda} \approx h^2 \quad and \quad \lambda^H - \lambda \approx H^2,$$

where λ , λ^h and λ^H are the smallest eigenvalues of A, A^h and A^H , $\hat{\lambda}$ and $\hat{\lambda}^h$ are the second smallest eigenvalues of A and A^h respectively.

From Proposition 3.1 and the variational principle of eigenvalues (see Equation 2.1 in Section 3 of [38]), the gap between λ^h and λ^H can be bounded by

(3.6)
$$0 \le \lambda^H - \lambda^h \lesssim H^2.$$

In the rest of this paper, we take the initial point $\rho_0 = \lambda^H$, thus $0 \le \rho_0 - \lambda^h \le H^2$.

3.2. Non-overlapping domain decomposition methods. Suppose Ω is divided into N non-overlapping convex polygonal subdomains $\{\Omega_k\}_{k=1}^N$ with diameter no more than H and the union of the boundaries are denoted by $\Gamma = \bigcup_{k=1}^N \partial \Omega_k$. Let $W_{\Gamma} = L^2(\Gamma), \ (\cdot, \cdot)_{\Gamma}$ be an inner product on Γ , whose corresponding norm $\|\cdot\|_{\Gamma}$ is spectral equivalent to $\|\cdot\|_{L^2(\Gamma)}$, and $V_{\Gamma} = H_*^{1/2}(\Gamma)$, where $u \in H_*^{1/2}(\Gamma)$ means that the restriction of u on $\partial \Omega_k$ belongs to $H^{1/2}(\Omega_k)$ for all $k = 1, \ldots, N$. Let

$$\|u^{h}\|_{H^{1/2}_{*}(\Gamma)} \equiv \left(\sum_{k=1}^{N} \|u^{h}\|_{H^{1/2}(\partial\Omega_{k})}^{2}\right)^{1/2},$$

where the scaled full-norm (see, for examples, [35, 44]) in Ω_k is defined as

$$\|u^{h}\|_{H^{1}(\Omega_{k})} \equiv \left(|u^{h}|^{2}_{H^{1}(\Omega_{k})} + H^{-2}\|u^{h}\|^{2}_{L^{2}(\Omega_{k})}\right)^{1/2},$$

$$\|u^{h}\|_{H^{1/2}(\partial\Omega_{k})} \equiv \left(|u^{h}|^{2}_{H^{1/2}(\partial\Omega_{k})} + H^{-1}\|u^{h}\|^{2}_{L^{2}(\partial\Omega_{k})}\right)^{1/2}.$$

Then we can define V_{Γ}^{h} as the trace space of V^{h} on Γ and V_{I}^{h} as the subspace of V^{h} whose members vanish at Γ , i.e.,

$$\begin{split} V_{\Gamma}^{h} &\equiv \{u^{h} \in H^{1/2}_{*}(\Gamma) \mid \exists v^{h} \in V^{h}, \, u^{h} = \operatorname{Tr}(v^{h})\}, \\ V_{I}^{h} &\equiv \{v^{h} \in V^{h} \mid \operatorname{Tr}(v^{h}) = 0\}, \end{split}$$

where $\operatorname{Tr}: H^1(\Omega) \mapsto H^{1/2}_*(\Gamma)$ is the trace map defined as $\operatorname{Tr}(v) = v$. In order to define the extension, a lemma for the coercivity of $a(\cdot, \cdot)$ in V_I^h is needed.

LEMMA 3.2. Suppose α is the smallest eigenvalue of $a(\cdot, \cdot)$ in V_I^h , i.e.,

$$\alpha = \min_{v^h \in V_I^h} \frac{a(v^h, v^h)}{(v^h, v^h)},$$

then we have $\alpha \gtrsim H^{-2}$.

Proof. For every $v^h \in V_I^h$, it can be decomposed as $v^h = \sum_{k=1}^N v_k^h$, where $\operatorname{supp}(v_k) \subset \overline{\Omega}_k$. Since $a(v_i^h, v_j^h) = (v_i^h, v_j^h) = 0$ for all $i \neq j$, we know

$$\frac{a(v^h, v^h)}{(v^h, v^h)} = \frac{\sum_{k=1}^N a(v^h_k, v^h_k)}{\sum_{k=1}^N (v^h_k, v^h_k)} \ge \min_{1 \le k \le N} \frac{a(v^h_k, v^h_k)}{(v^h_k, v^h_k)} \ge \min_{1 \le k \le N} \lambda^h_k,$$

where λ_k^h refers to the smallest eigenvalue of $a(\cdot, \cdot)$ restricted on Ω_k . Since the diameter of Ω_k is no more than H, then $\alpha \gtrsim H^{-2}$ holds due to the Poincaré inequality.

For all $\rho < \alpha$, denote $a_{\rho}(\cdot, \cdot) \equiv a(\cdot, \cdot) - \rho(\cdot, \cdot)$, the a_{ρ} -orthogonal space of V_{I}^{h} can be defined as

$$V^h_{B,\rho} \equiv \{ v^h \in V^h \mid a_\rho(v^h, v^h_I) = 0, \, \forall \, v^h_I \in V^h_I \}.$$

Then the discrete a_{ρ} -harmonic extension $\mathcal{H}^{h}_{\rho} \colon V^{h}_{\Gamma} \mapsto V^{h}_{B,\rho}$ can be defined. For all $u^{h} \in V^{h}_{\Gamma}$, let $\mathcal{H}^{h}_{\rho}u^{h}$ be the solution of

(3.7)
$$\begin{aligned} a_{\rho}(\mathcal{H}^{h}_{\rho}u^{h}, v^{h}_{I}) &= 0 \quad \forall v^{h}_{I} \in V^{h}_{I}, \\ \mathcal{H}^{h}_{\rho}u^{h} &= u^{h} \quad \text{on } \Gamma. \end{aligned}$$

Here are some propositions about the extension \mathcal{H}^h_{ρ} (see [9]).

PROPOSITION 3.3. The extension \mathcal{H}^h_{ρ} is bijective since the functions in $V^h_{B,\rho}$ are completely determined by their values on Γ .

PROPOSITION 3.4. For any $u^h \in V_{\Gamma}^h$ and $v^h \in V^h$, if u^h is the trace of v^h on Γ , *i.e.*, $\operatorname{Tr}(v^h) = u^h$, then $a_{\rho}(\mathcal{H}^h_{\rho}u^h, \mathcal{H}^h_{\rho}u^h) \leq a_{\rho}(v^h, v^h)$ holds for all $\rho < \alpha$.

PROPOSITION 3.5. For any $u^h \in V_{\Gamma}^h$ and $\lambda^h \leq \rho \leq \rho_0$, $H \|u^h\|_{\Gamma}^2 \lesssim |\mathcal{H}_{\rho}^h u^h|_a^2$.

LEMMA 3.6. Suppose $D \subset \mathbb{R}^d$, where d = 2 or 3, is a convex polygonal domain with unit diameter. Let

$$\mathcal{A}(v_1, v_2) = \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial v_1}{\partial x_i} \frac{\partial v_2}{\partial x_j} - c(x) v_1 v_2 \, \mathrm{d}x$$

be a symmetric positive definite bilinear form on $H^1(D)$ satisfying

(3.8)
$$\mathcal{A}(v,v) \approx \|v\|_{H_1(D)}^2,$$

where $a_{ij}(x)$ and $c(x) \in C^{0,1}(\overline{D})$ for i, j = 1, ..., d. For all $u \in H^{1/2}(\partial D)$, let $\mathcal{H}u$ be the solution of

(3.9)
$$\mathcal{A}(\mathcal{H}u, v) = 0 \quad \forall v \in H_0^1(D),$$
$$\mathcal{H}u = u \quad on \ \partial D,$$

then the following estimation holds:

$$\|\mathcal{H}u\|_{L^2(D)} \lesssim \|u\|_{H^{-1/2}(\partial D)}.$$

Proof. For any $v \in L^2(D)$, let $\psi \in H^1_0(D)$ be the solution of

$$\mathcal{A}(w,\psi) - \langle \frac{\partial \psi}{\partial \nu}, w \rangle_{\partial D} = (v,w)_{L^2(D)} \quad \forall w \in H^1(D),$$

where

$$\frac{\partial \psi}{\partial \nu} = \sum_{i,j=1}^{d} a_{ij}(x) \cos(\mathbf{n}, x_i) \frac{\partial \psi}{\partial x_j}$$

with **n** is the outer normal to the boundary ∂D and

$$\langle \frac{\partial \psi}{\partial \nu}, w \rangle_{\partial D} = \int_{\partial D} \frac{\partial \psi}{\partial \nu} w \, \mathrm{d}s.$$

By using Aubin-Nitsche's trick, for all $u \in H^{1/2}(\partial D)$

$$\|\mathcal{H}u\|_{L^2(D)} = \sup_{0 \neq v \in L^2(D)} \frac{(\mathcal{H}u, v)_{L^2(D)}}{\|v\|_{L^2(D)}} = \sup_{0 \neq v \in L^2(D)} \frac{-\langle \mathcal{H}u, \frac{\partial \psi}{\partial \nu} \rangle_{\partial D} + \mathcal{A}(\mathcal{H}u, \psi)}{\|v\|_{L^2(D)}}.$$

Since $\psi \in H_0^1(D)$ and $\mathcal{H}u$ is the solution of (3.9), $\mathcal{A}(\mathcal{H}u, \psi) = 0$, then

(3.10)
$$\|\mathcal{H}u\|_{L^2(D)} = \sup_{0 \neq v \in L^2(D)} \frac{-\langle \mathcal{H}u, \frac{\partial \psi}{\partial \nu} \rangle_{\partial D}}{\|v\|_{L^2(D)}}.$$

By using the Cauchy-Schwarz inequality and trace theorem,

$$(3.11) \qquad \left| \langle \mathcal{H}u, \frac{\partial \psi}{\partial \nu} \rangle_{\partial D} \right| \le \|u\|_{H^{-1/2}(\partial D)} \left\| \frac{\partial \psi}{\partial \nu} \right\|_{H^{1/2}(\partial D)} \lesssim \|u\|_{H^{-1/2}(\partial D)} \|\psi\|_{H^{2}(D)}.$$

Due to the H^2 regularity (see Theorem 3.2.1.2 in [16]),

(3.12)
$$\|\psi\|_{H^2(D)} \lesssim \|v\|_{L^2(D)}.$$

By combining (3.10)–(3.12), the proof is finished.

COROLLARY 3.7. If the diameter of domain D is H, it can be verified that

$$\|\mathcal{H}u\|_{L^2(D)} \lesssim \|u\|_{H^{-1/2}(\partial D)} \lesssim H^{1/2} \|u\|_{L^2(\partial D)}$$

by a scaling argument and the Sobolev inequality.

The following lemma gives a finite element method version of Lemma 3.6.

LEMMA 3.8. Suppose D and A are defined as Lemma 3.6. Let $V_D^h \subset H^1(D)$ be the continuous and piecewise linear elements space based on the quasi-uniform triangular partition \mathcal{T}_D with mesh size h and $V_{\partial D}^h$ is the trace space of V_D^h on ∂D . For all $u^h \in V_{\partial D}^h$, let $\mathcal{H}^h u^h$ be the solution of

(3.13)
$$\mathcal{A}(\mathcal{H}^{h}u^{h}, v^{h}) = 0 \quad \forall v^{h} \in H^{1}_{0}(D) \cap V^{h}_{D},$$
$$\mathcal{H}^{h}u^{h} = u^{h} \quad on \; \partial D,$$

then the following estimation holds:

$$\|\mathcal{H}^{h}u^{h}\|_{L^{2}(D)} \lesssim \|u^{h}\|_{H^{-1/2}(\partial D)}.$$

Proof. According to Lemma 3.6, it is enough to prove that

$$\|\mathcal{H}^h u^h - \mathcal{H} u^h\|_{L^2(D)} \lesssim \|u^h\|_{H^{-1/2}(\partial D)}.$$

By the L^2 estimation for $\mathcal{H}^h u^h$ (see Theorem 5.4.8 in [10]),

(3.14)
$$\|\mathcal{H}^h u^h - \mathcal{H} u^h\|_{L^2(D)} \lesssim h \, \|\mathcal{H}^h u^h - \mathcal{H} u^h\|_{H^1(D)}.$$

Since $\mathcal{H}^h u^h - \mathcal{H} u^h$ vanishes on ∂D and $\mathcal{H} u^h$ is the solution of (3.13),

$$\mathcal{A}(\mathcal{H}u^h, \mathcal{H}^h u^h - \mathcal{H}u^h) = 0.$$

By using the norm equivalence (3.8) and the Cauchy-Schwarz inequality,

$$\begin{split} \|\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}\|_{H^{1}(D)}^{2} &\approx \mathcal{A}(\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}, \mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}) \\ &= \mathcal{A}(\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}, \mathcal{H}^{h}u^{h}) \\ &\lesssim \|\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}\|_{H^{1}(D)}\|\mathcal{H}^{h}u^{h}\|_{H^{1}(D)}. \end{split}$$

Then eliminating $\|\mathcal{H}^h u^h - \mathcal{H} u^h\|_{H^1(D)}$ on both sides, we have

(3.15)
$$\|\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}\|_{H^{1}(D)} \lesssim \|\mathcal{H}^{h}u^{h}\|_{H^{1}(D)}.$$

Since $\mathcal{H}^h u^h$ is the solution of (3.13), by using Proposition 3.4 and the extension theorem (see Lemma 3.77 in [30]),

(3.16)
$$\|\mathcal{H}^{h}u^{h}\|_{H^{1}(D)} \lesssim \|u^{h}\|_{H^{1/2}(\partial D)}.$$

And by combining (3.14)-(3.16),

$$\|\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}\|_{L^{2}(D)} \lesssim h \|u^{h}\|_{H^{1/2}(\partial D)}.$$

Since u^h is piecewise linear on ∂D , by using the inverse estimation (see Theorem 4.1 and Theorem 4.6 in [12]),

$$\|\mathcal{H}^{h}u^{h} - \mathcal{H}u^{h}\|_{L^{2}(D)} \lesssim h \|u^{h}\|_{H^{1/2}(\partial D)} \lesssim \|u^{h}\|_{H^{-1/2}(\partial D)},$$

which finishes the proof.

COROLLARY 3.9. If the diameter of domain D is H, it can be verified that

$$\|\mathcal{H}^{h}u^{h}\|_{L^{2}(D)} \lesssim \|u^{h}\|_{H^{-1/2}(\partial D)} \lesssim H^{1/2}\|u^{h}\|_{L^{2}(\partial D)}$$

by a scaling argument and the Sobolev inequality.

Suppose $\mathcal{H}^{h}_{\rho,k}u^{h}$ is the solution of

$$a_{\rho}(\mathcal{H}^{h}_{\rho,k}u^{h}, v^{h}_{i,k}) = 0 \quad \forall v^{h}_{i,k} \in V^{h}_{I}, \operatorname{supp}(v^{h}_{i,k}) \subset \overline{\Omega}_{k},$$
$$\mathcal{H}^{h}_{\rho,k}u^{h} = u^{h} \quad \text{on } \partial\Omega_{k},$$

then $\mathcal{H}^{h}_{\rho,k}u^{h}$ is well-defined in Ω_{k} and $\mathcal{H}^{h}_{\rho}u^{h} = \mathcal{H}^{h}_{\rho,k}u^{h}$ in Ω_{k} for all $1 \leq k \leq N$, where $\mathcal{H}^{h}_{\rho}u^{h}$ is the solution of (3.7). In other words, the extension $\mathcal{H}^{h}_{\rho}u^{h}$ can be computed in each subdomain by the boundary value problem separately.

LEMMA 3.10. Assume $\lambda \leq \rho \leq \rho_0 < \alpha$, the extension operator \mathcal{H}_{ρ} is bounded with norm

$$\left\|\left|\mathcal{H}_{\rho}^{h}\right|\right\| \equiv \sup_{0 \neq u^{h} \in V_{\Gamma}^{h}} \frac{\left\|\mathcal{H}_{\rho}^{h}u^{h}\right\|}{\left\|u^{h}\right\|_{\Gamma}},$$

and the bound for $\||\mathcal{H}^h_{\rho}|||$ is $c_{\mathcal{H}}$, where $c_{\mathcal{H}} \lesssim H^{1/2}$.

Proof. For all $u^h \in V^h_{\Gamma}$, according to Corollary 3.9,

$$\|\mathcal{H}_{\rho}^{h}u^{h}\|^{2} = \int_{\Omega} |\mathcal{H}_{\rho}^{h}u^{h}|^{2} \,\mathrm{d}x = \sum_{k=1}^{N} \int_{\Omega_{k}} |\mathcal{H}_{\rho}^{h}u^{h}|^{2} \,\mathrm{d}x = \sum_{k=1}^{N} \int_{\Omega_{k}} |\mathcal{H}_{\rho,k}^{h}u^{h}|^{2} \,\mathrm{d}x \lesssim H \,\|u^{h}\|_{\Gamma}^{2}.$$

Therefore $c_{\mathcal{H}} = \max_{\lambda \leq \rho \leq \rho_0} \left\| \left| \mathcal{H}_{\rho}^h \right| \right\| \lesssim H^{1/2}.$

The Steklov-Poincaré operator $S^h_\rho\colon V^h_\Gamma\mapsto (V^h_\Gamma)'$ can be defined as

$$\langle S^h_{\rho} u^h_1, u^h_2 \rangle \equiv a_{\rho}(\mathcal{H}^h_{\rho} u^h_1, \mathcal{H}^h_{\rho} u^h_2),$$

and the eigenvalue problem on Γ is

(3.17)
$$\langle S^h_{\rho} u^h_{\rho}, u^h \rangle = \theta^h_{\rho} (u^h_{\rho}, u^h)_{\Gamma} \quad \forall u^h \in V^h_{\Gamma}.$$

Now we can use Algorithm 1 to calculate the elliptic eigenvalue problem (3.1).

Remark 3.11. Suppose $\{\phi_j\}_{j=1}^n$ is the basis of V_{Γ}^h , then the discrete L^2 inner product on Γ can be defined as

$$(u, u^*)_{l^2(\Gamma)} \equiv h^{d-1} \sum_{j=1}^n u_j u_j^*$$

for $u = \sum_{j=1}^{n} u_j \phi_j$ and $u^* = \sum_{j=1}^{n} u_j^* \phi_j$. Since $||u||_{L^2(\Gamma)} \approx ||u||_{l^2(\Gamma)}$ (see Equation 2.2 in [8]), the convergence analysis also holds for the discrete L^2 norm. Moreover, the mass matrix for (3.17) becomes a scalar matrix, which makes it easy to compute.

LEMMA 3.12. For all $\lambda^h \leq \rho \leq \rho_0$, $\|\mathcal{H}^h_{\rho} u^h_{\rho}\| \approx H^{1/2} \|u^h_{\rho}\|_{\Gamma}$.

Proof. On the one hand, by Lemma 3.10, we have $\|\mathcal{H}_{\rho}^{h}u_{\rho}^{h}\| \lesssim H^{1/2}\|u_{\rho}^{h}\|_{\Gamma}$. On the other hand, since u_{ρ}^{h} is the eigenvector of (3.17) and $\theta_{\rho}^{h} \leq 0$,

$$a_{\rho}(\mathcal{H}^{h}_{\rho}u^{h}_{\rho},\mathcal{H}^{h}_{\rho}u^{h}_{\rho}) = \langle S^{h}_{\rho}u^{h}_{\rho},u^{h}_{\rho} \rangle = \theta^{h}_{\rho} \|u^{h}_{\rho}\|_{\Gamma}^{2} \leq 0.$$

The lemma is obtained by using Proposition 3.5.

3.3. Convergence analysis. In order to analyze the convergence factor of Algorithm 1, some estimations on the eigenvalue of S^h_{ρ} should be calculated first. Similar to Lemma 3.10, a bounded bijective extension \mathcal{H}_{ρ} from V_{Γ} to $V_{B,\rho}$ can be defined as

(3.18)
$$a_{\rho}(\mathcal{H}_{\rho}u, v_{I}) = 0 \quad \forall v_{I} \in V_{I} \\ \mathcal{H}_{\rho}u = u \quad \text{on } \Gamma,$$

where V_{Γ} is the trace space of V on Γ , and V_I is the subspace of V whose members vanish at Γ and $V_{B,\rho}$ is the a_{ρ} -orthogonal space of V_I , i.e.,

$$V_I \equiv \{ v \in V \mid \operatorname{Tr}(v) = 0 \},\$$

$$V_{B,\rho} \equiv \{ v \in V \mid a_\rho(v, v_I) = 0, \forall v_I \in V_I \}$$

The Steklov-Poincaré operator $S_{\rho} \colon V_{\Gamma} \mapsto (V_{\Gamma})'$ can be defined as

$$\langle S_{\rho}u_1, u_2 \rangle \equiv a_{\rho}(\mathcal{H}_{\rho}u_1, \mathcal{H}_{\rho}u_2),$$

and its eigenvalue problem is

$$\langle S_\rho u_\rho, u\rangle = \theta_\rho \, (u_\rho, u)_\Gamma \quad \forall \, u \in V_\Gamma.$$

Now we can generalize Theorem 3.1 in [1] to estimate the gap between eigenvalues of S_{ρ} and S_{ρ}^{h} .

LEMMA 3.13. Suppose $\theta_1^h \leq \theta_2^h \leq \theta_3^h$ and $\theta_1 < \theta_2 \leq \theta_3$ are the smallest three eigenvalues of S_{ρ}^h and S_{ρ} respectively, where $\lambda^h \leq \rho \leq \rho_0$, then

$$\lim_{h \to 0} |\theta_j^h - \theta_j| = 0 \quad j = 1, 2, 3.$$

Proof. Since θ_2 may be equal to θ_3 , we need to consider both cases. According to the well-known variational principles of eigenvalue (see Theorem 2.3 in Section 3 of [38]), for j = 1, 2, 3,

(3.19)
$$\theta_{j} = \min_{\substack{U_{j} \subset V_{\Gamma} \ u \in U_{j} \\ \dim U_{j} = j}} \max_{\substack{(u, u)_{\Gamma} \\ (u, u)_{\Gamma}}} \quad \text{and} \quad \theta_{j}^{h} = \min_{\substack{U_{j}^{h} \subset V_{\Gamma}^{h} \ u^{h} \in U_{j}^{h} \\ \dim U_{j}^{h} = j}} \max_{\substack{(u^{h}, u^{h})_{\Gamma}}} \frac{\langle S_{\rho}^{h} u^{h}, u^{h} \rangle}{(u^{h}, u^{h})_{\Gamma}}.$$

Suppose the first minimal of θ_j and θ_j^h in (3.19) are reached by U_j and U_j^h respectively. Let

(3.20)
$$\bar{\theta}_{j}^{h} \equiv \max_{u^{h} \in U_{j}^{h}} \frac{\langle S_{\rho} u^{h}, u^{h} \rangle}{(u^{h}, u^{h})_{\Gamma}}$$

then the error $|\theta_j - \theta_j^h|$ can be decomposed as

$$(3.21) \qquad |\theta_j - \theta_j^h| \le |\theta_j - \theta_j^h| + |\theta_j^h - \theta_j^h| \\ \le \left| \max_{u \in U_j} \frac{\langle S_\rho u, u \rangle}{(u, u)_{\Gamma}} - \max_{u^h \in U_j^h} \frac{\langle S_\rho u^h, u^h \rangle}{(u^h, u^h)_{\Gamma}} \right| + \max_{u^h \in U_j^h} \left| \frac{\langle S_\rho u^h, u^h \rangle - \langle S_\rho^h u^h, u^h \rangle}{(u^h, u^h)_{\Gamma}} \right|.$$

For the first term, it can be regarded as the error between the eigenvalue of S_{ρ} in V_{Γ} and V_{Γ}^{h} . Let

$$\langle S_{\rho,\beta}u_1, u_2 \rangle \equiv \langle S_{\rho}u_1, u_2 \rangle + \beta \, (u_1, u_2)_{\Gamma_1}$$

it is obvious that eigenvalues of $S_{\rho,\beta}$ are eigenvalues of S_{ρ} plus β in both V_{Γ} and V_{Γ}^{h} , so it is enough to analyze the error between the eigenvalue of $S_{\rho,\beta}$ in V_{Γ} and V_{Γ}^{h} . When $\beta = \rho_0 c_{\mathcal{H}}^2 + H^{-1}$, by the trace theorem and the norm equivalence (3.3),

$$\begin{aligned} \|u\|_{H^{1/2}_{*}(\Gamma)}^{2} &\lesssim \sum_{k=1}^{N} |u|_{H^{1/2}(\partial\Omega_{k})}^{2} + \frac{1}{H} \|u\|_{\Gamma}^{2} \lesssim |\mathcal{H}_{\rho}u|_{a}^{2} + (\beta - \rho |||\mathcal{H}_{\rho}|||^{2}) ||u||_{\Gamma}^{2} \\ &\leq |\mathcal{H}_{\rho}u|_{a}^{2} + \beta ||u||_{\Gamma}^{2} = \langle S_{\rho,\beta}u, u \rangle. \end{aligned}$$

On the other hand, by the definition of $a_{\rho}(\cdot, \cdot)$, Proposition 3.4 and (3.3),

$$\begin{split} \langle S_{\rho,\beta}u,u\rangle &= a_{\rho}(\mathcal{H}_{\rho}u,\mathcal{H}_{\rho}u) + \beta \left(u,u\right)_{\Gamma} \leq a_{\rho}(\mathcal{H}u,\mathcal{H}u) + \beta \|u\|_{\Gamma}^{2} \\ &\leq |\mathcal{H}u|_{a}^{2} + \beta \|u\|_{\Gamma}^{2} \approx \sum_{k=1}^{N} |\mathcal{H}u|_{H^{1}(\Omega_{k})}^{2} + \beta \|u\|_{\Gamma}^{2} \\ &\lesssim \sum_{k=1}^{N} |u|_{H^{1/2}(\partial\Omega_{k})}^{2} + \beta \|u\|_{\Gamma}^{2} \lesssim \|u\|_{H^{1/2}(\Gamma)}^{2}, \end{split}$$

where $\mathcal{H}u$ is an extension satisfying $|\mathcal{H}u|_{H^1(\Omega_k)} \lesssim |u|_{H^{1/2}(\partial\Omega_k)}$ for all $1 \leq k \leq N$ (see Lemma 3.78 in [30]). So $S_{\rho,\beta}$ is bounded in $H^{1/2}_*(\Gamma)$. Let

$$\|u\|_{\beta} \equiv \left(\langle S_{\rho,\beta} u, u \rangle \right)^{1/2} \approx \|u\|_{H^{1/2}_{*}(\Gamma)},$$

by using the Theorem 3.1 in [1],

(3.22)
$$\left| \left(\max_{u \in U_j} \frac{\langle S_{\rho} u, u \rangle}{(u, u)_{\Gamma}} - \max_{u^h \in U_j^h} \frac{\langle S_{\rho} u^h, u^h \rangle}{(u^h, u^h)_{\Gamma}} \right) \right| \lesssim \delta_{j,h}^2,$$

where

$$\delta_{j,h} = \inf_{\substack{u \in M(\theta_j) \ u^h \in V_{\Gamma}^h \\ \|u\|_{\beta} = 1}} \inf_{u^h \in V_{\Gamma}^h} \|u - u^h\|_{\beta} \quad j = 1, 2,$$

and

$$\delta_{3,h} = \begin{cases} \inf_{\substack{u \in M(\theta_3) \ u^h \in V_{\Gamma}^h}} \|u - u^h\|_{\beta} & \text{if } \theta_2 \neq \theta_3, \\ \|u\|_{\beta} = 1 & \\ \inf_{\substack{u \in M(\theta_2) \ u^h \in V_{\Gamma}^h \\ \|u\|_{\beta} = 1 \\ (u,u_2)_{\beta} = 0 \\ (u,u_2)_{\beta} = 0 \\ \end{cases}} \|u - u^h\|_{\beta} & \text{if } \theta_2 = \theta_3, \end{cases}$$

where u_2 is the choice of the first infimum of $\delta_{2,h}$ and $M(\theta_j)$ is the eigenspace corresponding to θ_j . Due to Theorem 3.2.3 of [11], we have $\lim_{h\to 0} \delta_{j,h} = 0$, thus, the first term in (3.21) goes to zeros as $h \to 0$, i.e.,

(3.23)
$$\lim_{h \to 0} |\theta_j - \bar{\theta}_j^h| = 0.$$

For the second term,

$$\langle S_{\rho}u^{h}, u^{h}\rangle - \langle S_{\rho}^{h}u^{h}, u^{h}\rangle = a_{\rho}(\mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}, \mathcal{H}_{\rho}u^{h} + \mathcal{H}_{\rho}^{h}u^{h}).$$

Since $\mathcal{H}_{\rho}u^h - \mathcal{H}^h_{\rho}u^h$ vanishes on Γ , due to $\mathcal{H}_{\rho}u^h$ is the solution of (3.18),

$$a_{\rho}(\mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}, \mathcal{H}_{\rho}u^{h}) = 0.$$

Since $a_{\rho}(\cdot, \cdot)$ is positive definite on V_I , combining these two equations above, we have

$$(3.24) \qquad \left| \langle S_{\rho}u^{h}, u^{h} \rangle - \langle S_{\rho}^{h}u^{h}, u^{h} \rangle \right| = \left| a_{\rho}(\mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}, \mathcal{H}_{\rho}u^{h} + \mathcal{H}_{\rho}^{h}u^{h}) \right| = \left| a_{\rho}(\mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}, -\mathcal{H}_{\rho}u^{h} + \mathcal{H}_{\rho}^{h}u^{h}) \right| = a_{\rho}(\mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}, \mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}).$$

Similar to Céa's lemma, for all $v^h \in V^h_{B,\rho}$ with trace u^h on Γ ,

$$(3.25) \quad a_{\rho}(\mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}, \mathcal{H}_{\rho}u^{h} - \mathcal{H}_{\rho}^{h}u^{h}) \leq a_{\rho}(\mathcal{H}_{\rho}u^{h} - v^{h}, \mathcal{H}_{\rho}u^{h} - v^{h}) \lesssim |\mathcal{H}_{\rho}u^{h} - v^{h}|_{H^{1}}^{2}$$

due to the norm equivalence (3.3) and the Poincaré inequality. Combining (3.21), (3.24), and (3.25) we know

(3.26)
$$\begin{aligned} |\bar{\theta}_{j}^{h} - \theta_{j}^{h}| &\lesssim \max_{u^{h} \in U_{j}^{h}} \inf_{v^{h} \in V_{B,\rho}^{h}} \frac{|\mathcal{H}_{\rho}u^{h} - v^{h}|_{H^{1}}^{2}}{\|u^{h}\|_{\Gamma}^{2}} \\ &= \max_{u^{h} \in U_{j}^{h}} \left\{ \frac{|\mathcal{H}_{\rho}u^{h}|_{H^{1}}^{2}}{\|u^{h}\|_{\Gamma}^{2}} \inf_{v^{h} \in V_{B,\rho}^{h}} \frac{|\mathcal{H}_{\rho}u^{h} - v^{h}|_{H^{1}}^{2}}{|\mathcal{H}_{\rho}u^{h}|_{H^{1}}^{2}} \right\} \end{aligned}$$

According to the definition of $\bar{\theta}_{j}^{h}$ (3.20) and the norm equivalence (3.3),

$$\bar{\theta}_j^h \geq \frac{\langle S_\rho u^h, u^h \rangle}{(u^h, u^h)_{\Gamma}} = \frac{a_\rho(\mathcal{H}_\rho u^h, \mathcal{H}_\rho u^h)}{(u^h, u^h)_{\Gamma}} \approx \frac{|\mathcal{H}_\rho u^h|_{H^1}^2 - \rho ||\mathcal{H}_\rho u^h||^2}{\|u^h\|_{\Gamma}^2}$$

for all $u^h \in U_j^h$. So

(3.27)
$$\frac{|\mathcal{H}_{\rho}u^{h}|_{H^{1}}^{2}}{\|u^{h}\|_{\Gamma}^{2}} \lesssim \bar{\theta}_{j}^{h} + \rho_{0}H \lesssim \theta_{j} + \delta_{j,h}^{2} + \rho_{0}H$$

due to Lemma 3.10 and (3.22). By combining (3.26) and (3.27),

$$(3.28) \qquad \lim_{h \to 0} |\bar{\theta}_j^h - \theta_j^h| \lesssim \lim_{h \to 0} \left(\max_{u^h \in U_j^h} \left\{ \frac{|\mathcal{H}_{\rho} u^h|_{H^1}^2}{\|u^h\|_{\Gamma}^2} \inf_{v^h \in V_{B,\rho}^h} \frac{|\mathcal{H}_{\rho} u^h - v^h|_{H^1}^2}{|\mathcal{H}_{\rho} u^h|_{H^1}^2} \right\} \right) = 0.$$

Combining (3.23) and (3.28), we finish the proof.

COROLLARY 3.14. Let $(\theta_{\lambda}^{h})^{(j)}$ and $\theta_{\lambda}^{(j)}$ be the *j*th smallest eigenvalue of S_{λ}^{h} and S_{λ} respectively, then

$$\lim_{h \to 0} |c_s^h - c_s| = 0$$

where $c_s^h = \frac{(\theta_\lambda^h)^{(2)}}{(\theta_\lambda^h)^{(3)}}$ and $c_s = \frac{\theta_\lambda^{(2)}}{\theta_\lambda^{(3)}}$.

Now we can give the convergence factor for Algorithm 1.

THEOREM 3.15. Suppose that the coarse mesh size H is small enough, there exists a constant C independent of h and H such that

$$\epsilon_N \leq C\epsilon^2$$
,

where ϵ and ϵ_N are errors of the eigenvalue before and after one iteration.

Proof. This proof is mainly based on Corollary 2.12. First, some conditions in Lemma 2.11 need to be verified. According to Proposition 3.5, the constant c_t in Lemma 2.11 satisfies $c_t \gtrsim H$. Due to Proposition 3.1, Lemma 2.11, and Corollary 3.14, the constant c_g satisfies

$$(3.29) c_g = \min_{\rho \in (\lambda^h, \rho_0)} \left(\theta_{\rho}^{(3)} - \theta_{\rho}^{(2)} \right) \ge \frac{\widehat{\lambda}^h - \rho_0}{\widehat{\lambda}^h} c_s^h c_t$$
$$= \frac{\widehat{\lambda} - \lambda + (\widehat{\lambda}^h - \widehat{\lambda}) + (\lambda - \rho_0)}{\widehat{\lambda} + (\widehat{\lambda}^h - \widehat{\lambda})} c_s^h c_t$$
$$\gtrsim \frac{\widehat{\lambda} - \lambda + h^2 + H^2}{\widehat{\lambda} + h^2} c_s H \gtrsim H.$$

By using Lemmas 3.2 and 3.10, Theorem 2.9, and Proposition 3.1, the convergence factor is bounded by

(3.30)
$$\frac{\epsilon_N}{\epsilon^2} \le \frac{1}{\alpha - \rho_0} + \frac{c_{\mathcal{H}}^2}{c_g} \lesssim H^2 + 1,$$

which finishes the proof.

3.4. A sharper bound for the convergence factor. After carefully checking the numerical results in next section, we find the estimation in Theorem 3.15 can be further improved. In this part, we will give a sharper estimation for the convergence

factor by using a special norm $\|\cdot\|_{\Gamma}$ on Γ , which is spectral equivalent to $\|\cdot\|_{L^2(\Gamma)}$. Let $V_{\Gamma,\rho}^h$ be the S_{ρ}^h -orthogonal space of u_{ρ}^h in V_{Γ}^h , for all $u^h \in V_{\Gamma,\rho}^h$, we have

$$\langle S^h_\rho u^h, u^h \rangle - \theta^h_\rho (u^h, u^h)_\Gamma \gtrsim H \, (u^h, u^h)_\Gamma$$

due to the estimation for c_g in (3.29). In $V^h_{\Gamma,\rho}$, the operator $(S^h_{\rho} - \theta^h_{\rho})^{-1}$ is symmetric positive definite, and it is spectral equivalent to $(S_0^h)^{-1}$, which means

(3.31)
$$\left\langle (S^h_{\rho} - \theta^h_{\rho} I^h)^{-1} u^h, u^h \right\rangle \approx \left\langle (S^h_0)^{-1} u^h, u^h \right\rangle$$

holds for all $u^h \in V^h_{\Gamma,\rho}$, where I^h is the identity operator in V^h_{Γ} . Now we need some important results in non-overlapping domain decomposition methods, whose detailed proof can be found in many papers and books, for examples, [30, 35, 44].

PROPOSITION 3.16. For $\Omega \in \mathbb{R}^d$, where d = 2 or 3, there exists a decomposition

$$V_{\Gamma}^{h} = R_{H}^{\mathsf{T}} V_{\Gamma}^{H} + \sum_{i=1}^{n_{l}} R_{i}^{\mathsf{T}} V_{\Gamma}^{i},$$

- where R_H^{T} and R_i^{T} 's are interpolation operators, such that following properties hold. The space $R_H^{\mathsf{T}}V_{\Gamma}^H$ is a global coarse space, and all functions in $R_H^{\mathsf{T}}V_{\Gamma}^H$ are linear in coarse meshes. Other spaces, i.e., $R_i^{\mathsf{T}}V_{\Gamma}^{\mathsf{T}}$'s, are local spaces.
 - Let M^h be the preconditioner defined as

$$M^{h} \equiv R_{H}^{\mathsf{T}} (R_{H} S_{0}^{h} R_{H}^{\mathsf{T}})^{-1} R_{H} + \sum_{i=1}^{n_{l}} R_{i}^{\mathsf{T}} (R_{i} S_{0}^{h} R_{i}^{\mathsf{T}})^{-1} R_{i},$$

then for all $u^h \in V_{\Gamma}^h$,

$$\langle S_0^h u^h, u^h \rangle \lesssim \left(1 + \ln(H/h)\right)^2 \langle S_0^h u^h, M^h S_0^h u^h \rangle$$

• For all $u^h \in V_{\Gamma}$,

(3.32)
$$\langle M^h u^h, u^h \rangle \lesssim H^{-1} \| R_H u^h \|_*^2 + H \| u^h \|_{\Gamma}^2,$$

where $||R_H u^h||_*$ is defined as

$$||R_H u^h||_* \equiv \sup_{0 \neq u_H^h \in R_H^T V_\Gamma^H} \frac{(u^h, u_H^h)_\Gamma}{||u_H^h||_\Gamma}.$$

Remark 3.17. The estimation (3.32) can be obtained by the Poincaré inequality and scaling arguments when $R_i^{\mathsf{T}} V_{\Gamma}^i$'s are local spaces with diameters no more than H.

Let Q_H be the $L^2(\Gamma)$ projection from V_{Γ}^h to $R_H^{\mathsf{T}}V_{\Gamma}^H$, i.e.,

(3.33)
$$(Q_H u^h, u^h_H)_{L^2(\Gamma)} = (u^h, u^h_H)_{L^2(\Gamma)} \quad \forall u^h_H \in R^{\mathsf{T}}_H V^H_{\Gamma}.$$

From the definition we know

(3.34)
$$\|R_H r_{\rho}^h\|_* = \sup_{0 \neq u_H^h \in R_H^\mathsf{T} V_\Gamma^\mathsf{T}} \frac{(r_{\rho}^h, u_H^h)_\Gamma}{\|u_H^h\|_\Gamma} = \|Q_H r_{\rho}^h\|_\Gamma.$$

Now, we can define a bilinear form $(\cdot, \cdot)_{\Gamma}$ as follows:

$$(3.35) \qquad (u_1^h, u_2^h)_{\Gamma} \equiv H^{-1}(\mathcal{H}_0^h Q_H u_1^h, \mathcal{H}_0^h Q_H u_2^h) + (u_1^h - Q_H u_1^h, u_2^h - Q_H u_2^h)_{L^2(\Gamma)}.$$

It can be verified that $(\cdot, \cdot)_{\Gamma}$ is an inner product on V_{Γ}^h . Moreover, Q_H is also a $(\cdot, \cdot)_{\Gamma}$ - projection, and

(3.36)
$$H^{1/2} \|Q_H u^h\|_{\Gamma} = \|\mathcal{H}_0^h Q_H u^h\|_{\Gamma}$$

holds for all $u^h \in V^h_{\Gamma}$, where $\|\cdot\|_{\Gamma}$ is the norm corresponding to $(\cdot, \cdot)_{\Gamma}$.

LEMMA 3.18. For all $u^h \in V_{\Gamma}^h$, $\|u^h\|_{\Gamma} \approx \|u^h\|_{L^2(\Gamma)}$.

Proof. By (3.35), it is sufficient to prove that for all $u_H^h \in R_H^\mathsf{T} V_\Gamma^h$,

$$\|u_H^h\|_{L^2(\Gamma)} \approx \|u_H^h\|_{\Gamma} = H^{-1/2} \|\mathcal{H}_0^h u_H^h\|_{\Gamma}$$

On the one hand, by Lemma 3.10, we have

$$\|\mathcal{H}_0^h u_H^h\| \lesssim H^{1/2} \|u_H^h\|_{L^2(\Gamma)}.$$

On the other hand, according to Theorem 1.6.6 of $\left[10\right]$ and the Cauchy-Schwarz inequality,

$$\begin{aligned} \|u_{H}^{h}\|_{L^{2}(\Gamma)}^{2} &\lesssim \sum_{k=1}^{N} \|\mathcal{H}_{0}^{h}u_{H}^{h}\|_{L^{2}(\Omega_{k})} \|\mathcal{H}_{0}^{h}u_{H}^{h}\|_{H^{1}(\Omega_{k})} \\ &\leq \|\mathcal{H}_{0}^{h}u_{H}^{h}\| (|\mathcal{H}_{0}^{h}u_{H}^{h}|_{H^{1}}^{2} + H^{-2}\|\mathcal{H}_{0}^{h}u_{H}^{h}\|^{2})^{1/2}. \end{aligned}$$

By the inverse estimation, Proposition 3.4 and (3.3),

$$|\mathcal{H}_0^h u_H^h|_{H^1}^2 \lesssim \sum_{k=1}^N |u_H^h|_{H^{1/2}(\partial\Omega_k)}^2 \lesssim H^{-1} ||u_H^h||_{L^2(\Gamma)}^2.$$

Combining these two inequalities above, we have

$$|u_{H}^{h}||_{L^{2}(\Gamma)} \lesssim H^{-1/2} ||\mathcal{H}_{0}^{h}u_{H}^{h}||_{L^{2}(\Gamma)}$$

which finishes the proof.

LEMMA 3.19. Let Π_H denote the interpolation operator associated with the coarse space $R_H^{\mathsf{T}} V_{\Gamma}^H$, for all $\lambda^h \leq \rho \leq \rho_0$,

$$\left| \|\Pi_H u_{\rho}^h\|_{\Gamma} - \|u_{\rho}^h\|_{\Gamma} \right| \lesssim H \|u_{\rho}^h\|_{\Gamma}.$$

Proof. By the estimation of interpolation and the trace theorem, we have

$$\left\|\|\Pi_{H}u_{\rho}^{h}\|_{\Gamma} - \|u_{\rho}^{h}\|_{\Gamma}\right\|^{2} \leq \|\Pi_{H}u_{\rho}^{h} - u_{\rho}^{h}\|_{\Gamma}^{2} \leq H\sum_{k=1}^{N} |u_{\rho}^{h}|_{H^{1/2}(\partial\Omega_{k})}^{2} \leq H |\mathcal{H}_{\rho}^{h}u_{\rho}^{h}|_{H^{1}(\Omega)}^{2}$$

According to (3.3) and Lemma 3.10,

$$|\mathcal{H}^h_\rho u^h_\rho|^2_{H^1} \lesssim |\mathcal{H}^h_\rho u^h_\rho|^2_a = \theta^h_\rho \|u^h_\rho\|^2_\Gamma + \rho \|\mathcal{H}^h_\rho u^h_\rho\|^2 \lesssim H \, \|u^h_\rho\|^2_\Gamma$$

The lemma is proved by combining these two inequalities above.

COROLLARY 3.20. Since Q_H is a projection, $||Q_H u_{\rho}^h||_{\Gamma} - ||u_{\rho}^h||_{\Gamma}| \lesssim H ||u_{\rho}^h||_{\Gamma}$. LEMMA 3.21. By using notations in Theorem 2.9, in the finite element space,

$$\left\langle (S^h_\rho - \theta^h_\rho I^h)(u^h_\rho)', (u^h_\rho)' \right\rangle \lesssim H^2 \left(1 + \ln(H/h) \right)^2 \left\| v^h_\rho \right\|^2$$

holds for all $\lambda^h \leq \rho \leq \rho_0$, where $v_{\rho}^h = \mathcal{H}_{\rho}^h u_{\rho}^h$ and $\|u_{\rho}^h\|_{\Gamma} = 1$.

Proof. Let $r_{\rho}^{h} = (\theta_{\rho}^{h})' u_{\rho}^{h} - (S_{\rho}^{h})' u_{\rho}^{h}$, according to (3.31) and (3.34) and Proposition 3.16, we have

$$\begin{split} \left\langle (S_{\rho}^{h} - \theta_{\rho}^{h}I^{h})(u_{\rho}^{h})', (u_{\rho}^{h})' \right\rangle &= \left\langle (S_{\rho}^{h} - \theta_{\rho}^{h}I^{h})^{-1}r_{\rho}^{h}, r_{\rho}^{h} \right\rangle \approx \left\langle (S_{0}^{h})^{-1}r_{\rho}^{h}, r_{\rho}^{h} \right\rangle \\ &= \left\langle S_{0}^{h}(S_{0}^{h})^{-1}r_{\rho}^{h}, (S_{0}^{h})^{-1}r_{\rho}^{h} \right\rangle \lesssim \left(1 + \ln(H/h)\right)^{2} \left\langle M^{h}r_{\rho}^{h}, r_{\rho}^{h} \right\rangle \\ &\lesssim \left(1 + \ln(H/h)\right)^{2} \left(H^{-1} \|Q_{H}r_{\rho}^{h}\|_{\Gamma}^{2} + H \|r_{\rho}^{h}\|_{\Gamma}^{2}\right). \end{split}$$

By Lemmas 2.4 and 3.12 and Proposition 2.3,

$$\|r_{\rho}^{h}\|_{\Gamma} \leq |(\theta_{\rho}^{h})'|\|u_{\rho}^{h}\|_{\Gamma} + \|v_{\rho}^{h}\|\left\|\left|\left|\mathcal{H}_{\rho}^{h}\right|\right|\right\| \lesssim H.$$

Combining these two inequalities above, we have

(3.37)
$$\left\langle (S^h_{\rho} - \theta^h_{\rho} I^h)(u^h_{\rho})', (u^h_{\rho})' \right\rangle \lesssim \left(1 + \ln(H/h) \right)^2 \left(H^{-1} \|Q_H r^h_{\rho}\|_{\Gamma}^2 + H^3 \right).$$

According to Proposition 2.3, Lemma 3.10, and Corollary 3.20,

(3.38)
$$\|Q_{H}r_{\rho}^{h}\|_{\Gamma} \leq \|(\theta_{\rho}^{h})'Q_{H}u_{\rho}^{h} - Q_{H}(S_{\rho}^{h})'Q_{H}u_{\rho}^{h}\|_{\Gamma} + \|Q_{H}(S_{\rho}^{h})'(u_{\rho}^{h} - Q_{H}u_{\rho}^{h})\|_{\Gamma} \\ \lesssim \|(\theta_{\rho}^{h})'Q_{H}u_{\rho}^{h} - Q_{H}(S_{\rho}^{h})'Q_{H}u_{\rho}^{h}\|_{\Gamma} + H^{2}.$$

For the first term, due to Lemmas 2.2 and 3.10,

$$(3.39) \qquad \begin{aligned} & \left\| (\theta_{\rho}^{h})' Q_{H} u_{\rho}^{h} - Q_{H} (S_{\rho}^{h})' Q_{H} u_{\rho}^{h} \right\|_{\Gamma} \\ & \leq \left\| (\theta_{\rho}^{h})' Q_{H} u_{\rho}^{h} - Q_{H} (S_{0}^{h})' Q_{H} u_{\rho}^{h} \right\|_{\Gamma} + \left\| Q_{H} \left((S_{0}^{h})' - (S_{\rho}^{h})' \right) Q_{H} u_{\rho}^{h} \right\|_{\Gamma} \\ & \lesssim \left\| (\theta_{\rho}^{h})' Q_{H} u_{\rho}^{h} - Q_{H} (S_{0}^{h})' Q_{H} u_{\rho}^{h} \right\|_{\Gamma} + H^{3}. \end{aligned}$$

By Proposition 2.3 and (3.35)

$$\begin{aligned} \left\| (\theta_{\rho}^{h})'Q_{H}u_{\rho}^{h} - Q_{H}(S_{0}^{h})'Q_{H}u_{\rho}^{h} \right\|_{\Gamma}^{2} \\ (3.40) &= \left((\theta_{\rho}^{h})' \right)^{2} \|Q_{H}u_{\rho}^{h}\|_{\Gamma}^{2} + 2(\theta_{\rho}^{h})' \|\mathcal{H}_{0}^{h}Q_{H}u_{\rho}^{h}\|^{2} + \sup_{0 \neq u^{h} \in V_{\Gamma}^{h}} \frac{\left| (\mathcal{H}_{0}^{h}Q_{H}u_{\rho}^{h}, \mathcal{H}_{0}^{h}Q_{H}u^{h}) \right|^{2}}{\|u^{h}\|_{\Gamma}^{2}} \\ &\leq \left(\left((\theta_{\rho}^{h})' \right)^{2} + 2(\theta_{\rho}^{h})'H + H^{2} \right) \|Q_{H}u_{\rho}^{h}\|_{\Gamma}^{2} \leq \left((\theta_{\rho}^{h})' + H \right)^{2}. \end{aligned}$$

Using Lemmas 2.2, 2.4 and 3.10 and (3.36), we know that

(3.41)
$$|(\theta_{\rho}^{h})' + H| \leq |||\mathcal{H}_{\rho}^{h}u_{\rho}^{h}||^{2} - ||\mathcal{H}_{\rho}^{h}Q_{H}u_{\rho}^{h}||^{2}| + |||\mathcal{H}_{\rho}^{h}Q_{H}u_{\rho}^{h}||^{2} - ||\mathcal{H}_{0}^{h}Q_{H}u_{\rho}^{h}||^{2}| + H||Q_{H}u_{\rho}^{h}||_{\Gamma}^{2} - ||u_{\rho}^{h}||_{\Gamma}^{2}| \lesssim H^{2}.$$

Thus the lemma is proved by Lemma 3.12 and (3.37)-(3.41).

Combining Lemma 3.21 with Theorem 2.9, we obtain a sharper estimation for the convergence factor.

THEOREM 3.22. Suppose the coarse mesh size H is small enough, if the inner product $(\cdot, \cdot)_{\Gamma}$ is defined as (3.35), there exists a constant C independent of h and H such that

$$\epsilon_N \le CH^2 \left(1 + \ln(H/h)\right)^2 \epsilon^2,$$

where ϵ and ϵ_N are errors of the eigenvalue before and after one iteration.

Remark 3.23. In this part, we only prove that when $(\cdot, \cdot)_{\Gamma}$ is defined as (3.35), the rate of convergence is $\epsilon_N \leq CH^2(1 + \ln(H/h))^2\epsilon^2$. For other inner products, whose corresponding norms are spectral equivalent to $\|\cdot\|_{L^2(\Gamma)}$, whether similar results can be obtained is still unknown. We do not know similar results before, and the discussions along this direction is quite interesting.

4. Numerical experiments. In this section, we present some numerical results to support our theoretical analysis above. We compute some second order symmetric elliptic eigenvalue problems in 2D and 3D by Algorithm 1. Assume \mathbf{K}^h and \mathbf{M}^h are the stiffness matrix and mass matrix generated by the finite element method as section 3 respectively. Due to the non-overlapping domain decomposition method, \mathbf{K}^h and \mathbf{M}^h can be partitioned as

$$\mathbf{K}^{h} = \begin{bmatrix} \mathbf{K}_{II}^{h} & \mathbf{K}_{IB}^{h} \\ \mathbf{K}_{BI}^{h} & \mathbf{K}_{BB}^{h} \end{bmatrix} \text{ and } \mathbf{M}^{h} = \begin{bmatrix} \mathbf{M}_{II}^{h} & \mathbf{M}_{IB}^{h} \\ \mathbf{M}_{BI}^{h} & \mathbf{M}_{BB}^{h} \end{bmatrix}$$

where $\mathbf{K}_{IB}^{h} = (\mathbf{K}_{BI}^{h})^{\mathsf{T}}$ and $\mathbf{M}_{IB}^{h} = (\mathbf{M}_{BI}^{h})^{\mathsf{T}}$, indices I are associated with the nodes in $\bigcup_{i=1}^{N} \Omega_{i}$ while B are associated with the nodes on Γ . Let

(4.1)
$$\mathbf{S}_{\rho}^{h} \equiv (\mathbf{K}_{BB}^{h} - \rho \,\mathbf{M}_{BB}^{h}) - (\mathbf{K}_{BI}^{h} - \rho \,\mathbf{M}_{BI}^{h}) (\mathbf{K}_{II}^{h} - \rho \,\mathbf{M}_{II}^{h})^{-1} (\mathbf{K}_{IB}^{h} - \rho \,\mathbf{M}_{IB}^{h}),$$

the eigenvalue problem $\mathbf{K}^{h}\mathbf{v}^{h} = \lambda^{h} \mathbf{M}^{h}\mathbf{v}^{h}$ can be rewritten as $\mathbf{S}_{\lambda^{h}}^{h}\mathbf{u}^{h} = 0$, where \mathbf{u}^{h} is the restriction of \mathbf{v}^{h} on Γ . Suppose $\mathbf{M}_{\Gamma}^{h} = h^{d-1} \mathbf{I}_{\Gamma}^{h}$ is the mass matrix on Γ , where \mathbf{I}^{h} is identity matrix on Γ , the nonlinear eigenvalue problem can be written as

(4.2)
$$\mathbf{S}^{h}_{\rho}\mathbf{u}^{h}_{\rho} = \theta^{h}_{\rho}\mathbf{M}^{h}_{\Gamma}\mathbf{u}^{h}_{\rho}.$$

By defining the extension operator from Γ to Ω as

$$\mathbf{H}_{\rho}^{h} = \begin{bmatrix} -(\mathbf{K}_{II}^{h} - \rho \, \mathbf{M}_{II}^{h})^{-1} (\mathbf{K}_{IB}^{h} - \rho \, \mathbf{M}_{IB}^{h}) \\ \mathbf{I}^{h} \end{bmatrix},$$

we get the matrix version of Algorithm 1. Our numerical experiments were performed in Matlab 2020b, the real solution λ^h is calculated from solving the smallest eigenvalue of $(\mathbf{K}^h, \mathbf{M}^h)$ by the MATLAB function "**eigs**" with tolerance 10^{-15} , the stopping criterion is chosen as $\epsilon < 10^{-12}$ and the subproblem (4.2) is solved by the Matlab function "**eigs**" with tolerance 10^{-12} . Let ρ_k and η_k be the approximated eigenvalue and convergence factor after k steps of Newton's method respectively, i.e.,

$$\epsilon_k = \rho_k - \lambda^h$$
 and $\eta_k = \frac{\epsilon_{k+1}}{\epsilon_k^2}$.

In our experiments, Algorithm 1 converges after few steps, so we only consider η_0 .

NIAN SHAO AND WENBIN CHEN



Fig. 4.1: A partition for non-overlapping domain decomposition. Red points are nodes for coarse mesh. Meshes in \mathcal{T}^h with same color belong to a same subdomain.

4.1. The Laplacian eigenvalue problem in 3D. In this subsection, the domain Ω is the unit cube $[0,1]^3$ in 3D, and a partition is shown in Figure 4.1a. We consider the relationship between the convergence factor η and the fine mesh size h or the coarse mesh size H separately. For the relationship with the fine mesh size, the coarse mesh size is fixed as $H = 2^{-1}$ and fine mesh sizes h are chosen as 2^{-j} for $j = 2, \ldots, 5$. For the relationship with the coarse mesh size is fixed as $h = 2^{-5}$ and the coarse mesh sizes H are chosen as 2^{-j} for $j = 1, \ldots, 4$. Table 4.1 and Figure 4.2 show that the Newton-Schur method converges quadratically. We can see that the convergence factor η_0 decreases in $\mathcal{O}(H^2)$ from Figure 4.3, which means the logarithmic factor may be removed.

h	ϵ_0	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
2^{-2}	0.6000	0.0220	2.5235e-05	3.2885e-11	3.7896e-16	\checkmark
2^{-3}	0.9031	0.1442	0.0030	1.2326e-06	2.1061e-13	\checkmark
2^{-4}	0.9943	0.2411	0.0119	2.7295e-05	1.4354e-10	4.3693e-15
2^{-5}	1.0183	0.2799	0.0183	7.4511e-05	1.2260e-09	2.2229e-14

Η	ϵ_0	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5
2^{-1}	1.0183	0.2799	0.0183	7.4511e-05	1.2260e-09	2.2229e-14
2^{-2}	0.2614	0.0029	3.5313e-07	2.7726e-14	\checkmark	\checkmark
2^{-3}	0.0605	2.7729e-05	5.8386e-12	2.2946e-14	\checkmark	
2^{-4}	0.0120	1.0350e-07	2.1750e-14		\checkmark	

(a) Various fine mesh sizes

(b) Various coarse mesh sizes

Table 4.1: Relative errors for approximated eigenvalues by Algorithm 1 for the 3D Laplacian eigenvalue problem. The " $\sqrt{}$ " entry means that the algorithm converged.



Fig. 4.2: Convergence history of the 3D Laplacian eigenvalue problem. The green dashed lines are references lines with $y_0 = 0.3$.



Fig. 4.3: Convergence factor for the 3D Laplacian eigenvalue problem.

4.2. The Laplacian eigenvalue problem on a 2D L-shaped domain. In this subsection, the domain Ω is an L-shaped domain in 2D, which is shown in Figure 4.1b. Similar to the previous subsection, we also consider the relationship between the convergence factor η and the fine mesh size h or the coarse mesh size H separately. For the relationship with the fine mesh size, the coarse mesh size is fixed as $H = 2^{-2}$ and fine mesh sizes h are chosen as 2^{-j} for $j = 3, \ldots, 9$. For the relationship with the coarse mesh size, the fine mesh size is fixed as $h = 2^{-8}$ and the coarse mesh sizes H are chosen as 2^{-j} for $j = 2, \ldots, 6$. Table 4.2 and Figure 4.4 show that the Newton-Schur method converges quadratically. We can see that the convergence factor η_0 decreases in $\mathcal{O}(H^2)$ from Figure 4.5, which is similar to the 3D case, even though the domain Ω is no longer convex.

5. Conclusions. In this paper, we study the Newton-Schur method in Hilbert space and obtain some sufficient conditions for quadratic convergence. Moreover, we

NIAN SHAO AND WENBIN CHEN

h	ϵ_0	ϵ_1	ϵ_2	ϵ_3
2^{-3}	0.0811	5.5397e-04	2.5657e-08	1.7824e-16
2^{-4}	0.1061	0.0011	1.2506e-07	1.4172e-16
2^{-5}	0.1139	0.0014	2.0185e-07	2.0201e-15
2^{-6}	0.1164	0.0015	2.4069e-07	1.4172e-14
2^{-7}	0.1172	0.0015	2.5877e-07	6.8337e-14
2^{-8}	0.1175	0.0015	2.6718e-07	2.8761e-13
2^{-9}	0.1176	0.0016	2.7114e-07	1.1677e-12

(a) Various fine mesh sizes

H	ϵ_0	ϵ_1	ϵ_2	ϵ_3
2^{-2}	0.1175	0.0015	2.6718e-07	2.8761e-13
2^{-3}	0.0337	2.8695e-05	2.0531e-11	2.9258e-13
2^{-4}	0.0103	6.5277e-07	2.9424e-13	\checkmark
2^{-5}	0.0033	1.5976e-08	3.0769e-13	\checkmark
2^{-6}	0.0010	3.6742e-10	2.9516e-13	\checkmark

(b) Various coarse mesh sizes

Table 4.2: Relative errors for approximated eigenvalues by Algorithm 1 for the 2D Laplacian eigenvalue problem on an L-shaped domain. The " $\sqrt{}$ " entry means that the algorithm converged.



Fig. 4.4: Convergence history of the 2D Laplacian eigenvalue problem on an L-shaped domain. The black dashed lines are references lines with $y_0 = 0.01$.

analyze the Newton-Schur method for symmetric elliptic eigenvalue problems discretized by the standard finite element method and non-overlapping domain decomposition method. Theoretical analysis shows that the rate of convergence is $\epsilon_N \leq CH^2(1 + \ln(H/h))^2\epsilon^2$, which is supported by our numerical results.

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Fig. 4.5: Convergence factor for the 2D Laplacian eigenvalue problem on an L-shaped domain.

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REFERENCES

- I. BABUŠKA AND J. OSBORN, Estimates for the errors in eigenvalue and eigenvector approximation by Galerkin methods, with particular attention to the case of multiple eigenvalues, SIAM Journal on Numerical Analysis, 24 (1987), pp. 1249–1276, https://doi.org/10.1137/ 0724082.
- I. BABUŠKA AND J. OSBORN, Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems, Mathematics of Computation, 52 (1989), pp. 275– 297, https://doi.org/10.1090/S0025-5718-1989-0962210-8.
- [3] I. BABUŠKA AND J. OSBORN, Handbook of Numerical Analysis, vol. 2, Elsevier Science B.V., 1991, https://doi.org/10.1016/S1570-8659(05)80042-0.
- [4] Z. BAI, J. DEMMEL, J. DONGARRA, A. RUHE, AND H. VAN DER VORST, eds., Templates for the Solution of Algebraic Eigenvalue Problems: A Practical Guide, SIAM Philadelphia, 2000, https://doi.org/10.1137/1.9780898719581.
- [5] R. E. BANK, Analysis of a multilevel inverse iteration procedure for eigenvalue problems, SIAM Journal on Numerical Analysis, 19 (1982), pp. 886–898, https://doi.org/10.1137/0719064.
- C. BEKAS AND Y. SAAD, Computation of smallest eigenvalues using spectral Schur complements, SIAM Journal on Scientific Computing, 27 (2005), pp. 458–481, https://doi.org/ 10.1137/040603528.
- [7] J. K. BENNIGHOF AND R. LEHOUCO, An automated multilevel substructuring method for eigenspace computation in linear elastodynamics, SIAM Journal on Scientific Computing, 25 (2004), pp. 2084–2106, https://doi.org/10.1137/S1064827502400650.
- [8] J. H. BRAMBLE AND J. XU, Some estimates for a weighted L² projection, Mathematics of Computation, 56 (1991), pp. 463–476, https://doi.org/10.1090/S0025-5718-1991-1066830-3.
- S. C. BRENNER, The condition number of the Schur complement in domain decomposition, Numerische Mathematik, 83 (1999), pp. 187–203, https://doi.org/10.1007/s002110050446.
- [10] S. C. BRENNER AND L. R. SCOTT, The Mathematical Theory of Finite Element Methods, Springer, New York, 3 ed., 2008, https://doi.org/10.1007/978-0-387-75934-0.
- P. G. CIARLET, The finite element method for elliptic problems, SIAM Philadelphia, 2002, https://doi.org/10.1137/1.9780898719208.
- [12] W. DAHMEN, B. FAERMANN, I. GRAHAM, W. HACKBUSCH, AND S. SAUTER, Inverse inequalities on non-quasi-uniform meshes and application to the mortar element method,

Mathematics of Computation, 73 (2004), pp. 1107–1138, https://doi.org/10.1090/S0025-5718-03-01583-7.

- [13] L. C. EVANS, Partial Differential Equations, American Mathematical Society, Providence, Rhode Island, 2010.
- [14] W. GAO, X. S. LI, C. YANG, AND Z. BAI, An implementation and evaluation of the AMLS method for sparse eigenvalue problems, ACM Transactions on Mathematical Software (TOMS), 34 (2008), pp. 1–28, https://doi.org/10.1145/1377596.1377600.
- [15] G. GOLUB AND C. F. VAN LOAN, Matrix computations, The Johns Hopkins University Press, 4th ed., 2013.
- P. GRISVARD, *Elliptic problems in nonsmooth domains*, SIAM Philadelphia, 2011, https://doi. org/10.1007/BF01059054.
- [17] W. HACKBUSCH, On the computation of approximate eigenvalues and eigenfunctions of elliptic operators by means of a multi-grid method, SIAM Journal on Numerical Analysis, 16 (1979), pp. 201–215, https://doi.org/10.1137/0716015.
- [18] X. HU AND X. CHENG, Acceleration of a two-grid method for eigenvalue problems, Mathematics of computation, 80 (2011), pp. 1287–1301, https://doi.org/10.1090/ S0025-5718-2011-02458-0.
- [19] V. KALANTZIS, Domain Decomposition Algorithms for the Solution of Sparse Symmetric Generalized Eigenvalue Problems, PhD thesis, University of Minnesota, 2018.
- [20] V. KALANTZIS, A domain decomposition Rayleigh-Ritz algorithm for symmetric generalized eigenvalue problems, SIAM Journal on Scientific Computing, 42 (2020), pp. C410-C435, https://doi.org/10.1137/19M1280004.
- [21] V. KALANTZIS, A spectral Newton-Schur algorithm for the solution of symmetric generalized eigenvalue problems, Electronic Transactions on Numerical Analysis, 52 (2020), pp. 132– 153, https://doi.org/10.1553/etna_vol52s132.
- [22] V. KALANTZIS, R. LI, AND Y. SAAD, Spectral Schur complement techniques for symmetric eigenvalue problems, Electronic Transactions on Numerical Analysis, 45 (2016), pp. 305– 329.
- [23] A. V. KNYAZEV, Toward the optimal preconditioned eigensolver: Locally optimal block preconditioned conjugate gradient method, SIAM Journal on Scientific Computing, 23 (2001), pp. 517–541, https://doi.org/10.1137/S1064827500366124.
- [24] A. V. KNYAZEV, M. ARGENTATI, I. LASHUK, AND E. OVTCHINNIKOV, Block locally optimal preconditioned eigenvalue xolvers (BLOPEX) in Hypre and PETSc, SIAM Journal on Scientific Computing, 29 (2007), pp. 2224–2239, https://doi.org/10.1137/060661624.
- [25] A. V. KNYAZEV AND K. NEYMEYR, A geometric theory for preconditioned inverse iteration iii: A short and sharp convergence estimate for generalized eigenvalue problems, Linear Algebra and its Applications, 358 (2003), pp. 95–114, https://doi.org/10.1016/S0024-3795(01) 00461-X.
- [26] Q. LIANG AND X. XU, A two-level preconditioned Helmholtz-Jacobi-Davidson method for the Maxwell eigenvalue problem, Mathematics of Computation, 91 (2022), pp. 623–657, https: //doi.org/10.1090/mcom/3702.
- [27] Q. LIN AND H. XIE, A multi-level correction scheme for eigenvalue problems, Mathematics of Computation, 84 (2015), pp. 71–88, https://doi.org/10.1090/S0025-5718-2014-02825-1.
- [28] S. LUI, Domain decomposition methods for eigenvalue problems, Journal of Computational and Applied Mathematics, 117 (2000), pp. 17–34, https://doi.org/10.1016/S0377-0427(99) 00326-X.
- [29] S. MALIASSOV, On the Schwarz alternating method for eigenvalue problems, Russian Journal of Numerical Analysis and Mathematical Modelling, 13 (1998), pp. 45–56, https://doi.org/ 10.1515/rnam.1998.13.1.45.
- [30] T. P. A. MATHEW, Domain Decomposition Methods for the Numerical Solution of Partial Differential Equations, Springer Berlin Heidelberg, 2008, https://doi.org/10.1007/ 978-3-540-77209-5.
- [31] S. F. MCCORMICK, A mesh refinement method for $Ax = \lambda Bx$, Mathematics of Computation, 36 (1981), pp. 485–498, https://doi.org/10.2307/2007654.
- [32] B. N. PARLETT, The symmetric eigenvalue problem, SIAM Philadelphia, 1998, https://doi.org/ 10.1137/1.9781611971163.
- [33] Y. SAAD, Iterative Methods for Sparse Linear Systems, SIAM Philadelphia, second ed., 2003, https://doi.org/10.1137/1.9780898718003.
- [34] Y. SAAD, Numerical methods for large eigenvalue problems: revised edition, SIAM Philadelphia, 2011, https://doi.org/10.1137/1.9781611970739.
- [35] A. TOSELLI AND O. WIDLUND, Domain Decomposition Methods Algorithms and Theory, Springer Berlin Heidelberg, 2005, https://doi.org/10.1007/b137868.

- [36] W. WANG AND X. XU, A two-level overlapping hybrid domain decomposition method for eigenvalue problems, SIAM Journal on Numerical Analysis, 56 (2018), pp. 344–368, https://doi.org/10.1137/16M1088302.
- [37] W. WANG AND X. XU, On the convergence of a two-level preconditioned Jacobi-Davidson method for eigenvalue problems, Mathematics of Computation, 88 (2019), pp. 2295–2324, https://doi.org/10.1090/mcom/3403.
- [38] H. F. WEINBERGER, Variational methods for eigenvalue approximation, SIAM Philadelphia, 1974, https://doi.org/10.1137/1.9781611970531.
- [39] J. H. WILKINSON, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, UK, 1965, https://doi.org/10.1017/S0013091500012104.
- [40] H. XIE, L. ZHANG, AND H. OWHADI, Fast eigenpairs computation with operator adapted wavelets and hierarchical subspace correction, SIAM Journal on Numerical Analysis, 57 (2019), pp. 2519–2550, https://doi.org/10.1137/18M1194079.
- [41] J. XU, Theory of multilevel methods, PhD thesis, Cornell University, 1989.
- [42] J. XU AND A. ZHOU, A two-grid discretization scheme for eigenvalue problems, Mathematics of Computation, 70 (2001), pp. 17–25, https://doi.org/10.1090/s0025-5718-99-01180-1.
- [43] J. XU AND A. ZHOU, Local and parallel finite element algorithms for eigenvalue problems, Acta Mathematicae Applicatae Sinica, 18 (2002), pp. 185–200, https://doi.org/10.1007/ s102550200018.
- [44] J. XU AND J. ZOU, Some nonoverlapping domain decomposition methods, SIAM review, 40 (1998), pp. 857–914, https://doi.org/10.1137/S0036144596306800.
- [45] Y. YANG AND H. BI, Two-grid finite element discretization schemes based on shifted-inverse power method for elliptic eigenvalue problems, SIAM Journal on Numerical Analysis, 49 (2011), pp. 1602–1624, https://doi.org/10.1137/100810241.
- [46] J. ZHOU, X. HU, L. ZHONG, S. SHU, AND L. CHEN, Two-grid methods for Maxwell eigenvalue problems, SIAM Journal on Numerical Analysis, 52 (2014), pp. 2027–2047, https://doi. org/10.1137/130919921.