# SHADOWS OF 3-UNIFORM HYPERGRAPHS UNDER A MINIMUM DEGREE CONDITION 

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#### Abstract

We prove a minimum degree version of the Kruskal-Katona theorem for triple systems: given $d \geq 1 / 4$ and a triple system $\mathcal{F}$ on $n$ vertices with minimum degree $\delta(\mathcal{F}) \geq$ $d\binom{n}{2}$, we obtain asymptotically tight lower bounds for the size of its shadow. Equivalently, for $t \geq n / 2-1$, we asymptotically determine the minimum size of a graph on $n$ vertices, in which every vertex is contained in at least $\binom{t}{2}$ triangles. This can be viewed as a variant of the Rademacher-Turán problem.


## 1. Introduction

Given a set $X$ and a family $\mathcal{F}$ of $k$-subsets of $X$, the shadow $\partial \mathcal{F}$ of $\mathcal{F}$ is the family of all $(k-1)$-subsets of $X$ contained in some member of $\mathcal{F}$. The Kruskal-Katona theorem [12, 13] is one of the most important results in extremal set theory - it gives a tight lower bound for the size of shadows of all $k$-uniform families of a given size. The following is a version due to Lovász [17], where $\binom{t}{k}=\frac{t(t-1) \cdots(t-k+1)}{k!}$ for a real number $t$. Note that it is tight when $t$ is an integer by considering the family of all $k$-subsets of a set of $t$ vertices.

Theorem 1 (Kruskal-Katona theorem). If $\mathcal{F}$ is a family of $k$-sets with $|\mathcal{F}| \geq\binom{ t}{k}$ for some real number $t$, then $|\partial \mathcal{F}| \geq\binom{ t}{k-1}$.

A family $\mathcal{F}$ of $k$-subsets of $X$ is often regarded as a $k$-uniform hypergraph, or $k$-graph $(X, \mathcal{F})$ with $X$ as the vertex set and $\mathcal{F}$ as the edge set. For every $x \in X$, define $\mathcal{F}_{x}=\{F \backslash x: x \in F$ and $F \in \mathcal{F}\}$. The minimum (vertex) degree of $\mathcal{F}$ is denoted by $\delta(\mathcal{F}):=\min _{x}\left|\mathcal{F}_{x}\right|$. The following minimum degree version of the Kruskal-Katona theorem has not been studied before but emerged naturally when Han, Zang, and Zhao [9] investigated a packing problem for 3graphs.

Problem 2. Given $k \geq 3$ and $0<d<1$, let $X$ be a set of $n$ vertices and $\mathcal{F}$ be a family of $k$-subsets of $X$ with $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$ How small can $|\partial \mathcal{F}|$ be?

Problem 2 belongs to an area of active research on extremal problems under maximum or minimum degree conditions. Two early examples are the work of Bollobás, Daykin, and Erdős [1], who studied the minimum degree version of the Erdős matching conjecture, and of Frankl [6], who studied the Erdős-Ko-Rado theorem under maximum degree conditions. More recent examples include the minimum (co)degree Turán's problems [15, 18, the minimum degree version of the Erdős-Ko-Rado theorem [8, 10, 14], and the minimum degree version of Hilton-Milner

[^0]theorem [7, 14]. Recently Jung [11] studied minimum $|\partial \mathcal{F}| /|\mathcal{F}|$ among all $k$-graphs $\mathcal{F}$ with maximum degree $\Delta(\mathcal{F}) \leq d$.

Since $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$ implies that $|\mathcal{F}| \geq d\binom{n}{k}$, we could apply Theorem $\mathbb{1}$ to $\mathcal{F}$ but will not obtain a tight bound for $|\partial \mathcal{F}|$. A better approach is applying Theorem 1 to $\mathcal{F}_{x}$ for each vertex $x$.
 Consequently,

$$
\begin{equation*}
|\partial \mathcal{F}|=\sum_{x} \frac{\left|\partial \mathcal{F}_{x}\right|}{k-1} \geq \frac{n}{k-1} d^{\frac{k-2}{k-1}}\binom{n}{k-2}+O\left(n^{k-2}\right) \geq d^{\frac{k-2}{k-1}}\binom{n}{k-1}+O\left(n^{k-2}\right) . \tag{1}
\end{equation*}
$$

This bound is tight (up to the error term) when the first inequality in (1) is asymptotically an equality, which occurs when $\mathcal{F}_{x}$ is a clique of order $d^{\frac{1}{k-1}} n$ for every $x$. Thus, the bound in (11) is asymptotically tight when $\mathcal{F}$ consists of $d^{\frac{1}{1-k}}$ vertex-disjoint cliques of order $d^{\frac{1}{k-1}} n$, in particular, when $d=\ell^{1-k}$ for some $\ell \in \mathbb{N}$.

In this paper we improve (1) and answer Problem 2 asymptotically for $k=3$ and $d \geq 1 / 4$. Two overlapping cliques of order about $\sqrt{d} n+1$ is a natural candidate for extremal hypergraphs - the following theorem confirms this for $\frac{1}{4} \leq d<\frac{47-5 \sqrt{57}}{24} \approx 0.385$. However, there is a different extremal hypergraph for larger values of $d$.
Theorem 3. Let $1 / 4 \leq d<1$ and $n \in \mathbb{N}$ be sufficiently large. If $\mathcal{F}$ is a triple system on $n$ vertices with $\delta(\mathcal{F}) \geq d\binom{n}{2}$, then

$$
|\partial \mathcal{F}| \geq \begin{cases}(4 \sqrt{d}-2 d-1)\binom{n}{2} & \text { if } \frac{1}{4} \leq d<\frac{47-5 \sqrt{57}}{24} \\ \left(\frac{1}{2}+\sqrt{\frac{4 d-1}{12}}\right)\binom{n}{2} & \text { if } d \geq \frac{47-5 \sqrt{57}}{24}\end{cases}
$$

These bounds are best possible up to an additive term of $O(n)$.
Although seemingly technical, Theorem 3 has an interesting application on 3-graph packing and covering. Given positive integers $a, b, c$, let $K_{a, b, c}^{3}$ denote the complete 3-partite 3-graph with parts of size $a, b$, and $c$. Answering a question of Mycroft [19, Han, Zang, and Zhao (9] determined the minimum $\delta(H)$ of a 3-graph $H$ that forces a perfect $K_{a, b, c}^{3}$-packing in $H$ for any given $a, b, c^{2}$ One of the main steps in their proof is determining the smallest $\delta(H)$ of a 3 -graph $H$ that guarantees that every vertex of $H$ is contained in a copy of $K_{a, b, c}^{3}$ (this is necessary for $H$ containing a perfect $K_{a, b, c}^{3}$-packing).
Corollary 4. [9, Lemma 3.7] Let $d_{0}=6-4 \sqrt{2} \approx 0.343$. For any $\gamma>0$, there exists $\eta>0$ such that the following holds for sufficiently large $n$. If $H$ is an $n$-vertex 3 -graph with $\delta(H) \geq$ $\left(d_{0}+\gamma\right)\binom{n}{2}$, then each vertex of $H$ is contained in at least $\eta n^{a+b+c-1}$ copies of $K_{a, b, c}^{3}$.

It was shown [9, Construction 2.6] that $d_{0}$ in Corollary 4 is best possible. We give a proof outline of Corollary 4 at the end of Section 2 - a complete proof can be found in 9].

Our approach towards Theorem 3 is viewing it as an extremal problem on graphs. The following is an equivalent form of Problem 2, in which $K_{t}^{k}$ denotes the complete $k$-graph on $t$ vertices (and we omit the superscript when $k=2$ ).
Problem 5. Given $a(k-1)$-graph $G$ on $n$ vertices such that every vertex is contained in at least $d\binom{n}{k-1}$ copies of $K_{k}^{k-1}$, how many edges must $G$ have?

[^1]To see why Problems 2 and 5 are equivalent, let $m_{1}$ be the minimum $|\partial \mathcal{F}|$ for Problem 2 and $m_{2}$ be the minimum $e(G)$ for Problem 5. To see why $m_{1} \geq m_{2}$, consider a $k$-uniform family $\mathcal{F}$ with $\delta(\mathcal{F}) \geq d\binom{n}{k-1}$. Let $G=(V(\mathcal{F}), \partial \mathcal{F})$ be the $(k-1)$-graph of its shadow. Since every member of $\mathcal{F}$ gives rise to a copy of $K_{k}^{k-1}$ in $G, \delta(\mathcal{F}) \geq d\binom{n}{k-1}$ implies that every vertex is contained in at least $d\binom{n}{k-1}$ copies of $K_{k}^{k-1}$. Thus $|\partial \mathcal{F}|=e(G) \geq m_{2}$. To see why $m_{2} \geq m_{1}$, consider a $(k-1)$-graph $G$ such that every vertex is contained in at least $d\binom{n}{k-1}$ copies of $K_{k}^{k-1}$. Let $\mathcal{F}$ be the family of $k$-subsets of $V(G)$ that span copies of $K_{k}^{k-1}$ in $G$. Then $\partial F \subseteq G$ and for every $v \in V(G)$, we have $\left|\mathcal{F}_{v}\right| \geq d\binom{n}{k-1}$. Thus $e(G) \geq|\partial \mathcal{F}| \geq m_{1}$ as desired.

In order to prove Theorem [3, we solve the $k=3$ case of Problem [5 with $d \geq 1 / 4$. For convenience, we assume that every vertex of $G$ is contained in at least $\binom{t}{2}$ triangles. There are essentially two extremal graphs: the first one consists of two copies of $K_{t+1}$ that share $2 t+2-n$ vertices; the second one is obtained from two disjoint copies of $K_{n / 2}$ by adding a regular bipartite graph between them. The size of these two extremal graphs can be conveniently represented by a quadratic function $f(x)$, which arises naturally from a lower bound for $e(G)$ in Proposition 7 ,
Theorem 6. Let $n \in \mathbb{N}, t, r \in \mathbb{R}$ such that $n / 2 \leq t+1 \leq n, r \geq 0$, and

$$
\begin{equation*}
\binom{\frac{n}{2}-1}{2}+3\binom{r}{2}=\binom{t}{2} . \tag{2}
\end{equation*}
$$

Define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
f(x)=\binom{t}{2}+x(n-x)-\binom{n-x-1}{2} . \tag{3}
\end{equation*}
$$

If $G$ is an n-vertex graph such that each vertex is contained in at least $\binom{t}{2}$ triangles, then

$$
e(G) \geq \begin{cases}f(t) & \text { if } r+t \leq \frac{5 n}{6} \text { or approximately } t \leq 0.6208 n,  \tag{4}\\ f\left(\frac{n}{2}+r-1\right) & \text { otherwise. }\end{cases}
$$

Furthermore, these bounds are tight when $n / 2, t, r$ are integers, and tight up to an additive $O(n)$ in general.

Theorem 6 can be viewed as a variant of the well-studied Rademacher-Turán problem. Starting with the work of Rademacher (unpublished) and of Erdős [4, the Rademacher-Turán problem studies the minimum number of triangles in a graph with given order and size. Instead of the total number of triangles in a graph, one may ask for the maximum or minimum number of triangles containing a fixed vertex. Given a graph $G$, we define the triangle-degree of a vertex as the number of triangles that contain this vertex. Let $\Delta_{K_{3}}(G)$ and $\delta_{K_{3}}(G)$ denote the maximum and minimum triangle-degree in $G$, respectively. The contrapositive of Theorem 6 states that if $G$ is a graph on $n$ vertices that fails (4), then $\delta_{K_{3}}(G)<\binom{t}{2}$. Correspondingly, the maximum triangle-degree version of Rademacher-Turán problem was recently studied by Falgas-Ravry, Markström, and Zhao [5]. In addition, Theorem 6 looks similar to the question of Erdős and Rothschild [3] on the book size of graphs: in the complementary form, it asks for the maximum size of a graph on $n$ vertices, in which every edge is contained in at most $d$ triangles.

We prove Theorem [6 and Theorem 3] in the next section. When $t<n / 2-1$, it is reasonable to speculate that an extremal graph is a disjoint union of copies of $K_{t+1}$ and an extremal graph for Theorem 6] Unfortunately we cannot verify this. We provide some evidence for this speculation in the last section.
Notation. Given a family $\mathcal{F}$ of sets, $|\mathcal{F}|$ is the size of $\mathcal{F}$, namely, the number of sets in $\mathcal{F}$. A $k$-uniform hypergraph $H$, or $k$-graph, consists of a vertex set $V(H)$ and an edge set $E(H)$,
which is a family of $k$-subsets of $V(H)$. Given a vertex set $S$, denote by $e_{H}(S)$ the number of edges of $H$ induced on $S$. Suppose $G$ is a graph. For a vertex $v \in V(G)$, let $N_{G}(v)$ denote the neighborhood of $v$, the set of vertices adjacent to $v$, and let $d_{G}(v)=\left|N_{G}(v)\right|$ be the degree of $v$. Let $N_{G}[v]:=N_{G}(v) \cup\{v\}$ denote the closed neighborhood of $v$. When the underlying (hyper)graph is clear from the context, we omit the subscript in these notations.

## 2. Proofs of Theorem 6 and Thereom 3

Suppose that $G=(V, E)$ is a graph on $n$ vertices such that each vertex is contained in at least $\binom{t}{2}$ triangles, in other words,

$$
\begin{equation*}
\forall v \in V, \quad e(N(v)) \geq\binom{ t}{2} \tag{5}
\end{equation*}
$$

where $t$ is a positive real number. Trivially $t \leq \delta(G) \leq n-1$ because $e(N(v)) \leq\binom{ d(v)}{2}$ for every vertex $v \in V$. Therefore

$$
e(G) \geq \frac{\delta(G) n}{2} \geq \frac{t n}{2}
$$

When $t+1$ divides $n$, this bound is tight because $G$ can be a disjoint union of $\frac{n}{t+1}$ copies of $K_{t+1}$. Below we often assume that $t \leq n-2$ because when $t=n-1$, we must have $G=K_{n}$.

Let us derive another lower bound for $e(G)$ by using the function $f$ defined in (3).
Proposition 7. If $G=(V, E)$ is a graph on $n$ vertices satisfying (5), then $e(G) \geq f(\delta(G))$, and the equality holds if and only if there exists $v_{0} \in V$ such that $e\left(N\left(v_{0}\right)\right)=\binom{t}{2}, d(v)=\delta(G)$ for all $v \notin N\left(v_{0}\right)$, and $V \backslash N\left[v_{0}\right]$ induces a clique.
Proof. Suppose $\delta(G)=\delta$ and $v_{0} \in V$ satisfies $d\left(v_{0}\right)=\delta$. Since we may partition $E(G)$ into the edges induced on $N\left(v_{0}\right)$ and the edges incident to some vertex $v \notin N\left(v_{0}\right)$, we have

$$
e(G)=e\left(N\left(v_{0}\right)\right)+\left(\sum_{v \notin N\left(v_{0}\right)} d(v)\right)-e\left(V \backslash N\left(v_{0}\right)\right) .
$$

Because of (5), $d(v) \geq \delta$ for all $v \notin N\left(v_{0}\right)$, and $e\left(V \backslash N\left(v_{0}\right)\right) \leq\binom{ n-\delta-1}{2}$ (note that $v_{0}$ has no neighbor outside $N\left(v_{0}\right)$ ), we derive that $e(G) \geq\binom{ t}{2}+\delta(n-\delta)-\binom{n-\delta-1}{2}$. Furthermore, the equality holds exactly when $e\left(N\left(v_{0}\right)\right)=\binom{t}{2}, d(v)=\delta(G)$ for all $v \notin N\left(v_{0}\right)$, and $V \backslash N\left[v_{0}\right]$ induces a clique.

Let us construct three graphs satisfying (5). Note that, if $r$ satisfies (2), then $r \leq n / 2$ because $\binom{n / 2-1}{2}+3\binom{n / 2}{2}=\binom{n-1}{2} \geq\binom{ t}{2}$.
Construction 8. Suppose $t, r \in \mathbb{R}$ satisfy $\frac{n}{2}-1 \leq t \leq n-2, r \geq 0$, and (2).
(1) Let $G_{1}$ be the union of two copies of $K_{\lceil t\rceil+1}$ sharing $2\lceil t\rceil+2-n$ vertices.
(2) When $n$ is even, let $G_{2}$ be the n-vertex graph obtained from two disjoint copies of $K_{n / 2}$ by adding an $\lceil r\rceil$-regular bipartite graph between two cliques.
(3) When $n$ is odd, let $r^{\prime} \in \mathbb{R}^{+}$satisfy $\left(\frac{n-3}{2}\right)+3\binom{r^{\prime}}{2}=\binom{t}{2}$. Let $G_{2}^{\prime}$ be the $n$-vertex graph obtained from two disjoint copies of $K_{(n-1) / 2}$ by adding an $\left\lceil r^{\prime}\right\rceil$-regular bipartite graph between them, and a new vertex whose adjacency is the exactly the same as one of the existing vertices.
It is easy to see that $G_{1}, G_{2}, G_{2}^{\prime}$ all satisfy (5). For example, consider a vertex $x \in V\left(G_{2}\right)$. Let $A$ and $B$ denote the vertex sets of the two copies of $K_{n / 2}$ of $G_{2}$ and assume $x \in A$. Then
$N(x)$ contains $\binom{n / 2-1}{2}$ edges from $A,\binom{\lceil r\rceil}{ 2}$ edges from $B$, and $\lceil r\rceil(\lceil r\rceil-1)$ edges between $A$ and $B$. Hence $e(N(x))=\binom{\frac{n}{2}-1}{2}+3\binom{[r\rceil}{ 2} \geq\binom{ t}{2}$.

The following proposition gives the sizes of $G_{1}, G_{2}$, and $G_{2}^{\prime}$.
Proposition 9. Suppose $n \in \mathbb{N}, t, r \geq 0$ satisfy $\frac{n}{2}-1 \leq t \leq n-1$ and (2). If all $n / 2, t, r$ are integers, then $e\left(G_{1}\right)=f(t)$ and $e\left(G_{2}\right)=f(n / 2+r-1)$, otherwise $e\left(G_{1}\right) \leq f(t)+n$ and $e\left(G_{2}\right) \leq f(n / 2+r-1)+n / 2$. Furthermore, $e\left(G_{2}^{\prime}\right)=f(n / 2+r-1)+O(n)$ when $r^{\prime}, r=\Omega(n)$.
Proof. First, by the definition of $f(x)$, it is easy to see that

$$
\begin{equation*}
f(t)=\binom{n}{2}-(n-1-t)^{2} \tag{6}
\end{equation*}
$$

(alternatively when $t \in \mathbb{Z}$, we can apply Proposition 7 by letting $v_{0}$ be any vertex not in the intersection of the two cliques). We know that

$$
e\left(G_{1}\right)=\binom{n}{2}-(n-1-\lceil t\rceil)^{2} \geq\binom{ n}{2}-(n-1-t)^{2}=f(t)
$$

and equality holds when $t \in \mathbb{Z}$. In addition, we have $e\left(G_{1}\right) \leq f(t)+n$ because

$$
(n-1-\lceil t\rceil)^{2}-(n-1-t)^{2}=(2(n-1)-(\lceil t\rceil+t))(\lceil t\rceil-t) \leq n
$$

by using $t+1 \geq\lceil t\rceil \geq t \geq n / 2-1$.
Second, using the definitions of $f(x)$ and $r$, it is not hard to see that

$$
\begin{equation*}
f\left(\frac{n}{2}+r-1\right)=\frac{n}{2}\left(\frac{n}{2}+r-1\right) \tag{7}
\end{equation*}
$$

It follows that

$$
e\left(G_{2}\right)=\frac{n}{2}\left(\frac{n}{2}+\lceil r\rceil-1\right) \leq f\left(\frac{n}{2}+r-1\right)+\frac{n}{2}
$$

and equality holds when $r \in \mathbb{Z}$.
Third, it is easy to see that

$$
e\left(G_{2}^{\prime}\right)=\frac{n+1}{2}\left(\frac{n-1}{2}+\left\lceil r^{\prime}\right\rceil-1\right)
$$

By the definitions of $r$ and $r^{\prime}$, we have $\binom{r^{\prime}}{2}-\binom{r}{2}=\frac{2 n-7}{24}$. When $r, r^{\prime}=\Omega(n)$, we have $r^{\prime}-r=O(1)$ and consequently,

$$
\begin{aligned}
e\left(G_{2}^{\prime}\right)-f\left(\frac{n}{2}+r-1\right) & \leq \frac{n+1}{2}\left(\frac{n-1}{2}+r^{\prime}-1\right)-\frac{n}{2}\left(\frac{n}{2}+r-1\right) \\
& =\frac{n}{2}\left(r^{\prime}-r\right)+\frac{r^{\prime}}{2}-\frac{3}{4}=O(n)
\end{aligned}
$$

We compare $f(t)$, the approximate size of $G_{1}$, with $f\left(\frac{n}{2}+r-1\right)$, the approximate size of $G_{2}$ and $G_{2}^{\prime}$, in the next proposition.

Proposition 10. Suppose $\frac{n}{2}-1 \leq t \leq n-1, f(x)$ and $r$ are defined as in (3) and (2), respectively. We have $f(t) \leq f\left(\frac{n}{2}+r-1\right)$ if and only if $r+t \leq \frac{5 n}{6}$, equivalently,

$$
\begin{equation*}
t \leq \frac{5}{4} n-\frac{\sqrt{57 n^{2}-72 n}}{12}-1 \approx 0.6208 n \tag{8}
\end{equation*}
$$

To prove Proposition 10, we need a simple fact on quadratic functions.
Fact 11. Suppose $g(x)$ is a quadratic function with a maximum at $x=a$ and assume $x_{1} \leq x_{2}$. Then $g\left(x_{1}\right) \leq g\left(x_{2}\right)$ if and only if $x_{1}+x_{2} \leq 2 a$.

Proof of Proposition 10. First note that

$$
f(x)=-\frac{3}{2} x^{2}+\frac{4 n-3}{2} x-\frac{n^{2}}{2}+\binom{t}{2}+\frac{3}{2} n-1
$$

is a quadratic function with a maximum at $x=\frac{2 n}{3}-\frac{1}{2}$. Second, since $r \leq \frac{n}{2}$, it follows that

$$
\binom{\frac{n}{2}+r-1}{2}=\binom{\frac{n}{2}-1}{2}+\left(\frac{n}{2}-1\right) r+\binom{r}{2} \geq\binom{\frac{n}{2}-1}{2}+3\binom{r}{2}=\binom{t}{2} .
$$

Consequently $\frac{n}{2}+r-1 \geq t$. By Fact [11, $f(t) \leq f\left(\frac{n}{2}+r-1\right)$ if and only if $t+\frac{n}{2}+r-1 \leq \frac{4 n}{3}-1$ or $r+t \leq \frac{5 n}{6}$. By (2), this is equivalent to

$$
\binom{\frac{n}{2}-1}{2}+3\binom{\frac{5 n}{6}-t}{2} \geq\binom{ t}{2} \quad \text { or } \quad(t+1)^{2}-\frac{5}{2}(t+1) n+\frac{7}{6} n^{2}+\frac{n}{2} \geq 0
$$

which holds exactly when $t+1 \leq \frac{5}{4} n-\frac{\sqrt{57 n^{2}-72 n}}{12}$ (because $t<n$ ).
We are ready to prove Theorem 6.
Proof of Theorem 6. Assume that $\delta=\delta(G)$. We separate two cases.
Case 1: $r+t \leq \frac{5 n}{6}$, equivalently, (8).
First assume that $\delta \geq \frac{4}{3} n-t-1$. Since $t \leq \frac{5 n}{6}-r$, we have $\delta \geq \frac{n}{2}+r-1$ and consequently,

$$
e(G) \geq \frac{n}{2}\left(\frac{n}{2}+r-1\right)=f\left(\frac{n}{2}+r-1\right) \geq f(t)
$$

by (7) and Proposition 10 ,
Second assume that $\delta<\frac{4}{3} n-t-1$. By Proposition 7, we have $e(G) \geq f(\delta)$. Recall that (5) forces $t \leq \delta$. Since $t \leq \delta<\frac{4}{3} n-t-1$ and $f(x)$ is a quadratic function maximized at $\frac{2 n}{3}-\frac{1}{2}$, we derive from Fact 11 that $f(\delta) \geq f(t)$. Hence $e(G) \geq f(\delta) \geq f(t)$.
Case 2: $r+t>\frac{5 n}{6}$.
If $\delta \geq \frac{n}{2}+r-1$, then $e(G) \geq \frac{n}{2}\left(\frac{n}{2}+r-1\right)=f\left(\frac{n}{2}+r-1\right)$ by (7). Otherwise $\delta<\frac{n}{2}+r-1$. Note that

$$
\delta+\frac{n}{2}+r-1 \geq t+\frac{n}{2}+r-1>\frac{5 n}{6}+\frac{n}{2}-1=\frac{4 n}{3}-1 .
$$

Since the quadratic function $f(x)$ is maximized at $\frac{2 n}{3}-\frac{1}{2}$, we derive from Fact 11 that $f(\delta) \geq$ $f\left(\frac{n}{2}+r-1\right)$. By Proposition 7, we have $e(G) \geq f(\delta) \geq f\left(\frac{n}{2}+r-1\right)$.

By Proposition 9 , when $n / 2, t, r$ are all integers, we have $e\left(G_{1}\right)=f(t)$ and $e\left(G_{2}\right)=f\left(\frac{n}{2}+r-1\right)$. In other cases, we have $e\left(G_{1}\right) \leq f(t)+n$ and $e\left(G_{2}\right) \leq f\left(\frac{n}{2}+r-1\right)+n / 2$. When $n$ is odd and $r+t>5 n / 6$, we have $r, r^{\prime}=\Omega(n)$ and thus $e\left(G_{2}^{\prime}\right)=f\left(\frac{n}{2}+r-1\right)+O(n)$.
Remark 12. When $n / 2, t, r$ are all integers, we actually learn the following about extremal graphs from the proof of Theorem 6. Suppose that $G$ is an extremal graph. We claim that $G=G_{1}$ when $r+t<5 n / 6$, and $G$ is $(n / 2+r-1)$-regular when $r+t>5 n / 6$,.

Indeed, first assume $r+t<5 n / 6$. If $\delta \geq \frac{4}{3} n-t-1$, then $\delta>\frac{n}{2}+r-1$ and consequently, $e(G)>\frac{n}{2}\left(\frac{n}{2}+r-1\right)=f(t)$, a contradiction. Following the second case of Case 1, we obtain that $e(G)=f(\delta)=f(t)$ and consequently, $\delta=t$. Using Proposition 7 , we can derive that $G=G_{1}$. When $r+t>5 n / 6$, the second case of Case 2 shows that $e(G) \geq f(\delta)>f\left(\frac{n}{2}+r-1\right)$, a contradiction. Thus $\delta \geq \frac{n}{2}+r-1$ and $e(G)=\frac{n}{2}\left(\frac{n}{2}+r-1\right)$, which forces $G$ to be $(n / 2+r-1)$ regular.

We now prove Theorem 3 by applying Theorem 6 and the arguments that show the equivalence of Problems 2 and 5 in Section 1.

Proof of Theorem 圂. Suppose $1 / 4 \leq d<1$ and $n \in \mathbb{N}$ is sufficiently large. Choose $t \in \mathbb{R}^{+}$such that $\binom{t}{2}=d\binom{n}{2}$. Since $\binom{\sqrt{d} n}{2}<d\binom{n}{2}<\binom{\sqrt{d} n+1}{2}$, we have $\sqrt{d} n<t<\sqrt{d} n+1$.

Suppose $\mathcal{F}$ is a triple system on $n$ vertices with $\delta(\mathcal{F}) \geq d\binom{n}{2}$. Let $G=(V(\mathcal{F}), \partial \mathcal{F})$ be the graph whose edge set is the shadow $\partial \mathcal{F}$. For every $x \in V(G)$, we have $e_{G}(N(x)) \geq d\binom{n}{2}$.
Case 1: $\frac{1}{4} \leq d<\frac{47-5 \sqrt{57}}{24}$.
Thus $\frac{1}{2} \leq \sqrt{d}<\frac{15-\sqrt{57}}{12}$. Since $n$ is sufficiently large, we have $\sqrt{d} n \leq \frac{15-\sqrt{57}}{12} n-2$. Since $\sqrt{d} n<t<\sqrt{d} n+1$, it follows that

$$
\frac{n}{2}<t<\frac{15-\sqrt{57}}{12} n-1<\frac{5}{4} n-\frac{\sqrt{57 n^{2}-72 n}}{12}-1 .
$$

This allows us to apply the first case of Theorem 6 and (6) to derive that

$$
\begin{aligned}
e(G) & \geq f(t)=\binom{n}{2}-(n-1-t)^{2} \geq\binom{ n}{2}-(n-1-\sqrt{d} n)^{2} \\
& =(4 \sqrt{d}-2 d-1)\binom{n}{2}+n-d n-1 \\
& \geq(4 \sqrt{d}-2 d-1)\binom{n}{2} \quad \text { as } d<1 \text { and } n \text { is sufficiently large. }
\end{aligned}
$$

Case 2: $d \geq \frac{47-5 \sqrt{57}}{24}$.
Thus $\sqrt{d} \geq \frac{15-\sqrt{57}}{12}$. Since $t>\sqrt{d} n$, it follows that

$$
t+1>\frac{15-\sqrt{57}}{12} n+1>\frac{5}{4} n-\frac{\sqrt{57 n^{2}-72}}{12}
$$

because $\sqrt{57 n^{2}-72}>\sqrt{57 n^{2}}-6$ for $n \geq 2$. Since (8) fails, we will apply the second case of Theorem 6. Since $\binom{t}{2}=d\binom{n}{2}$ and $r \geq 0$, we can obtain from (2) that

$$
r=\frac{1}{6}(3+\sqrt{3(n-1)((4 d-1) n+5)})=\frac{1}{2}+\frac{n}{2} \sqrt{\frac{4 d-1}{3}}+h(n),
$$

where

$$
h(n)=\frac{1}{2 \sqrt{3}}\left(\sqrt{(4 d-1) n^{2}+(6-4 d) n-5}-\sqrt{4 d-1} n\right) .
$$

It is easy to see that $0 \leq h(n)=O(1)$. Theorem 6 thus gives that

$$
\begin{align*}
e(G) & \geq f\left(\frac{n}{2}+r-1\right)=\frac{n}{2}\left(\frac{n}{2}+r-1\right) \\
& =\frac{n}{2}\left(\frac{n}{2}-\frac{1}{2}+\frac{n}{2} \sqrt{\frac{4 d-1}{3}}+h(n)\right) \\
& =\binom{n}{2}\left(\frac{1}{2}+\sqrt{\frac{4 d-1}{12}}\right)+\frac{n}{4} \sqrt{\frac{4 d-1}{3}}+\frac{n}{2} h(n)  \tag{9}\\
& \geq\binom{ n}{2}\left(\frac{1}{2}+\sqrt{\frac{4 d-1}{12}}\right) .
\end{align*}
$$

To see why these bounds are asymptotically tight, for every graph $G \in\left\{G_{1}, G_{2}, G_{2}^{\prime}\right\}$, we construct a triple system $\mathcal{F}_{G}$ whose members are all triangles of $G$. Then $\partial \mathcal{F}_{G} \subseteq E(G)$ and $\delta\left(\mathcal{F}_{G}\right) \geq\binom{ t}{2}=d\binom{n}{2}$.

Proposition 9 gives that $\left|\partial \mathcal{F}_{G_{1}}\right| \leq e\left(G_{1}\right) \leq f(t)+n$. By (6) and the assumption $t \leq \sqrt{d} n+1$,

$$
\begin{aligned}
\left|\partial \mathcal{F}_{G_{1}}\right| & \leq f(t)+n \leq\binom{ n}{2}-(n-2-\sqrt{d} n)^{2}+n \\
& =(4 \sqrt{d}-2 d-1)\binom{n}{2}+(3-2 \sqrt{d}-d) n-4+n \\
& =(4 \sqrt{d}-2 d-1)\binom{n}{2}+O(n)
\end{aligned}
$$

When $n$ is even, we apply Proposition 9 and (9) obtaining that

$$
\left|\partial \mathcal{F}_{G_{2}}\right| \leq e\left(G_{2}\right) \leq f\left(\frac{n}{2}+r-1\right)+\frac{n}{2}=\binom{n}{2}\left(\frac{1}{2}+\sqrt{\frac{4 d-1}{12}}\right)+O(n) .
$$

When $n$ is odd, we assume $r+t>5 n / 6$ and thus $r, r^{\prime}=\Omega(n)$. By Proposition 9 and (9), we conclude that

$$
\left|\partial \mathcal{F}_{G_{2}^{\prime}}\right| \leq e\left(G_{2}^{\prime}\right)=f\left(\frac{n}{2}+r-1\right)+O(n)=\binom{n}{2}\left(\frac{1}{2}+\sqrt{\frac{4 d-1}{12}}\right)+O(n)
$$

We outline the proof of Corollary 4 emphasizing how Theorem 3 is applied. In a 3-graph, the degree of a pair $p$ of vertices is the number of the edges that contains $p$.

Proof Outline of Corollary 4 Assume $\eta \ll \gamma$ and $\varepsilon=\gamma / 12$. Let $H$ be an $n$-vertex 3 -graph and $x$ be a vertex of $H$. In order to find $\eta n^{a+b+c-1}$ copies of $K_{a, b, c}^{3}$, it suffices to find $\frac{\gamma}{2}\binom{n}{2}$ pairs of vertices of $H_{x}$ with degree at lease $\varepsilon^{2} n$ - this follows from standard counting arguments in extremal (hyper)graph theory, or conveniently [16, Lemma 4.2] of Lo and Markström.

Suppose $\delta_{1}(H) \geq\left(d_{0}+\gamma\right)\binom{n}{2}$ with $d_{0}=6-4 \sqrt{2} \approx 0.343$. As shown in [9, Lemma 3.3], it is easy to find a set $V_{0}$ of at most $3 \varepsilon n$ vertices and a subgraph $H^{\prime}$ of $H$ on $V \backslash V_{0}$ such that $\delta\left(H^{\prime}\right) \geq d_{0}\binom{n^{\prime}}{2}$, where $n^{\prime}=\left|V \backslash V_{0}\right|$, and every pair in $\partial H^{\prime}$ has degree at least $\varepsilon^{2} n$ in $H$. Since $\frac{1}{4}<d_{0}<\frac{47-5 \sqrt{57}}{24} \approx 0.385$, by the first case of Theorem 3, we have

$$
\left|\partial H^{\prime}\right| \geq\left(4 \sqrt{d_{0}}-2 d_{0}-1\right)\binom{n^{\prime}}{2} \geq\left(4 \sqrt{d_{0}}-2 d_{0}-1-\frac{\gamma}{2}\right)\binom{n}{2}
$$

For every vertex $x \in V(H)$, since $d(x) \geq\left(d_{0}+\gamma\right)\binom{n}{2}$ and crucially $4 \sqrt{d_{0}}-2 d_{0}-1=1-d_{0}$, at least $\frac{\gamma}{2}\binom{n}{2}$ pairs in $H_{x}$ are also in $\partial H^{\prime}$ thus having degree at lease $\varepsilon^{2} n$, as desired.

## 3. Concluding remarks

Let us restate the $k=3$ case of Problem [5,
Problem 13. Let $G$ be a graph on $n$ vertices such that each vertex is contained in at least $\binom{t}{2}$ triangles, where $t$ is a positive real number. How many edges must $G$ have?

Our Theorem 6 (asymptotically) answers Problem 13 for $n / 2 \leq t+1 \leq n$. The following proposition shows that for larger $n$, all but $O\left(t^{3}\right)$ vertices of an extremal graph are contained in isolated copies of $K_{t+1}$.
Proposition 14. When $n>(t+1)^{2}(t+2) / 4$, every extremal graph for Problem 13 contains an isolated copy of $K_{t+1}$.

Proof. Let $G=(V, E)$ be an extremal graph with $|V|=n$. Since every vertex lies in at least $\binom{t}{2}$ triangles, it suffices to show that $G$ contains a vertex of degree $t$ and all of its neighbors also have degree $t$ (thus inducing an isolated copy of $K_{t+1}$ ).

Suppose $n=a(t+1)+b$, where $0<b \leq t$. Let $G^{\prime}$ be the disjoint union of $a-1$ copies of $K_{t+1}$ together with two copies of $K_{t+1}$ sharing $t+1-b$ vertices. Since $G$ is extremal, we have

$$
2 e(G) \leq 2 e\left(G^{\prime}\right)=t n+(t+1-b) b \leq t n+(t+1)^{2} / 4
$$

Partition $V(G)$ into $A \cup B$ such that $A$ consists of all vertices of degree greater than $t$ and $B$ consists of all vertices of degree exactly $t$. Then

$$
\sum_{v \in A}\left(d_{G}(v)-t\right)=\sum_{v \in V}\left(d_{G}(v)-t\right)=2 e(G)-t n \leq(t+1)^{2} / 4 .
$$

This implies that $|A| \leq(t+1)^{2} / 4$. Let $e(A, B)$ denote the number of edges (of $G$ ) between $A$ and $B$. It follows that

$$
e(A, B) \leq \sum_{v \in A} d(v) \leq \frac{1}{4}(t+1)^{2}+t|A| \leq \frac{1}{4}(t+1)^{3}
$$

Let $B^{\prime}$ consists of the vertices of $B$ that are adjacent to some vertex of $A$. Then $\left|B^{\prime}\right| \leq e(A, B) \leq$ $(t+1)^{3} / 4$. If $n>(t+1)^{2}(t+2) / 4$, then $n>|A|+\left|B^{\prime}\right|$ and consequently, there exists a vertex of $B$ whose $t$ neighbors are all in $B$, as desired.

The $t=2$ case of Problem 13 assumes that every vertex in an $n$-vertex graph is contained in a triangle. Since $\delta(G) \geq 2$, it follows that $e(G) \geq n$, which is best possible when 3 divides $n$. Recently, Chakraborti and Loh [2] determined the minimum number of edges an $n$-vertex graph in which every vertex is contained in a copy of $K_{s}$, for arbitrary $s \leq n$. Their extremal graph is the union of copies of $K_{s}$, all but two of which are isolated.

Finally, using careful case analysis, we can answer Problem 13 exactly when $t$ is very close to $n$. This falls into the $r+t>5 n / 6$ case of Theorem 6 but $G_{2}$ defined in Construction 8 is not necessarily extremal (unless both $r$ and $n / 2$ are integers).

- When $n=t+2$, the (unique) extremal graph is $K_{n}^{-}$, the complete graph on $n$ vertices minus one edge.
- When $n=t+3$ is even, the (unique) extremal graph is $K_{n}$ minus a perfect matching (provided $t>5$ ). When $n=t+3$ is odd, $K_{n}$ minus a matching of size $\frac{n-1}{2}$ is an extremal graph (provided $t>6$ ).
- When $n=t+4$, the complement of any $K_{3}$-free 2-regular graph on $n$ vertices is an extremal graph. Note that $r=n / 2-2$ in this case and thus $G_{2}$ is one of the extremal graphs when $n$ is even.


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    ${ }^{1}$ It is more natural to assume $\delta(\mathcal{F}) \geq d\binom{n-1}{k-1}$ as $\binom{n-1}{k-1}$ is the largest possible degree. However, since we are mainly interested in the asymptotics of $|\partial \mathcal{F}|$, we choose the simpler looking condition $\delta(\mathcal{F}) \geq d\binom{n-1}{k-1}$.

[^1]:    ${ }^{2}$ Given hypergraphs $H$ and $F$, a perfect $F$-packing in $H$ is a spanning subgraph of $H$ that consists of vertexdisjoint copies of $F$.

