

Exchange properties of finite set-systems

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Abstract

In a recent breakthrough, Adiprasito, Avvakumov, and Karasev constructed a triangulation of the n -dimensional real projective space with a subexponential number of vertices. They reduced the problem to finding a small downward closed set-system \mathcal{F} covering an n -element ground set which satisfies the following condition: For any two disjoint members $A, B \in \mathcal{F}$, there exist $a \in A$ and $b \in B$ such that either $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \setminus \{a\} \in \mathcal{F}$, or $A \cup \{b\} \in \mathcal{F}$ and $B \cup \{a\} \setminus \{b\} \in \mathcal{F}$. Denoting by $f(n)$ the smallest cardinality of such a family \mathcal{F} , they proved that $f(n) < 2^{O(\sqrt{n} \log n)}$, and they asked for a nontrivial lower bound. It turns out that the construction of Adiprasito *et al.* is not far from optimal; we show that $2^{(1.42+o(1))\sqrt{n}} \leq f(n) \leq 2^{(1+o(1))\sqrt{2n \log n}}$.

We also study a variant of the above problem, where the condition is strengthened by also requiring that for any two disjoint members $A, B \in \mathcal{F}$ with $|A| > |B|$, there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$. In this case, we prove that the size of the smallest \mathcal{F} satisfying this stronger condition lies between $2^{\Omega(\sqrt{n} \log n)}$ and $2^{O(n \log \log n / \log n)}$.

1 Introduction

It is an old problem to find a triangulation of the n -dimensional real projective space with as few vertices as possible. Recently, Adiprasito, Avvakumov, and Karasev [1] broke the exponential barrier by finding a construction of size $2^{O(\sqrt{n} \log n)}$. For the proof, they considered the following problem in extremal set theory.

What is the *minimum* cardinality of a system \mathcal{F} of subsets of $[n] = \{1, 2, \dots, n\}$, which satisfies three conditions:

1. \mathcal{F} is *atomic*, that is, $\emptyset \in \mathcal{F}$ and $\{a\} \in \mathcal{F}$ for every $a \in [n]$;

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2. \mathcal{F} is *downward closed*, that is, if $A \in \mathcal{F}$, then $A' \in \mathcal{F}$ for every $A' \subset A$;
3. for any two disjoint members $A, B \in \mathcal{F} \setminus \{\emptyset\}$, there exist $a \in A$ and $b \in B$ such that
either $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \setminus \{a\} \in \mathcal{F}$,
or $A \cup \{b\} \in \mathcal{F}$ and $B \cup \{a\} \setminus \{b\} \in \mathcal{F}$.

Letting $f(n)$ denote the minimum size of a set-system \mathcal{F} with the above three properties, Adiprasito *et al.* proved

$$f(n) \leq 2^{(1/2+o(1))\sqrt{n} \log n}, \quad (1)$$

where \log always denotes the base 2 logarithm. They used the following construction. Let $s, t > 0$ be integers, $n = st$. Fix a partition $[n] = X_1 \cup \dots \cup X_t$ of the ground set into t parts of equal size, $|X_1| = \dots = |X_t| = s$. Let

$$\mathcal{F} = \cup_{i=1}^t \mathcal{F}_i, \quad \text{where } \mathcal{F}_i = \{F \subseteq [n] : |F \cap X_j| \leq 1 \text{ for every } j \neq i\}, \quad (2)$$

for $1 \leq i \leq t$. (In the definition of \mathcal{F}_i , there is no restriction on the size of $F \cap X_i$.) It is easy to verify that \mathcal{F} meets the requirements. We have

$$|\mathcal{F}| = (t2^s - (s+1)(t-1))(s+1)^{t-1} < 2^{s+t \log(s+1) + \log t}.$$

Substituting $s = (1/\sqrt{2} + o(1))\sqrt{n \log n}$ and $t = (\sqrt{2} + o(1))\sqrt{n/\log n}$, we obtain that

$$f(n) \leq 2^{(1/\sqrt{2}+o(1))\sqrt{n \log n} + (\sqrt{2}+o(1))\sqrt{n/\log n} \cdot \log \sqrt{n}} = 2^{(1+o(1))\sqrt{2n \log n}}. \quad (3)$$

This is slightly better than (1). (The authors of [1] remarked that their bound can be improved by a ‘‘subpolynomial factor.’’) Any further improvement on the upper bound would result in a smaller triangulation of the $(n-1)$ -dimensional projective space.

Our first theorem implies that (3) is not far from optimal.

The *rank* of a set-system \mathcal{F} , denoted by $\text{rk}(\mathcal{F})$, is the size of the largest set $F \in \mathcal{F}$; see, *e.g.*, [2].

We denote by $\lfloor x \rfloor$ the integer closest to x . (We will use this notation only for $x = \sqrt{2n}$, in which case $\lfloor x \rfloor$ is uniquely determined.)

Theorem 1. *Let \mathcal{F} be an atomic system of subsets of $[n]$, such that for any two disjoint members $A, B \in \mathcal{F}$, either there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$, or there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. Then we have*

- (i) $|\mathcal{F}| \geq e^{(2e^{-1/\sqrt{2}}+o(1))\sqrt{n}} \geq 2^{(1.42+o(1))\sqrt{n}}$;
- (ii) $\text{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rfloor$, and this bound is best possible.

If we also assume that \mathcal{F} is downward closed, then the inequality $\text{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rfloor$ immediately implies that $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rfloor}$. This is only slightly weaker than the lower bound $f(n) \geq 2^{(1.42+o(1))\sqrt{n}}$, which follows from part (i).

Remark. We remark that the assumptions of Theorem 1 are weaker than those made by Adiprasito *et al.*, in two different ways: we do not require that \mathcal{F} is downward closed (which is their condition

2), and the exchange condition between two disjoint sets is also less restrictive than condition 3. Nevertheless, we know no significantly smaller set-systems satisfying these weaker conditions than the ones described in (2), for which $|\mathcal{F}| = 2^{(1+o(1))\sqrt{2n \log n}}$. We can, however, further weaken the conditions under which Theorem 1 holds; instead of the exchange property, it is sufficient to assume the following:

For any two disjoint members $A, B \in \mathcal{F}$ with $|A| = |B|$, there is a set $C \in \mathcal{F}$ such that $C \subset A \cup B$ and $|C| = |A| + 1$.

This answers Question 3.7 of Adiprasito *et al.* [1]: from Claim 3.1 of [1], one cannot obtain a significantly better construction, using another family. To see that condition (3) in Claim 3.1 implies our condition above, apply it with the unit vector X identified with $A \cup B$.

While part (ii) of Theorem 1 is tight, we suspect that part (i) and the lower bound $f(n) \geq 2^{\Omega(\sqrt{n})}$ can be improved. As a first step, we slightly strengthen the assumptions of Theorem 1, in order to obtain a better lower bound on $|\mathcal{F}|$.

Theorem 2. *Let \mathcal{F} be an atomic system of subsets of $[n]$, such that for any two disjoint members $A, B \in \mathcal{F}$, either there exists $a \in A$ such that $B \cup \{a\} \in \mathcal{F}$, or there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. Moreover, suppose that if $|A| < |B|$, then the second option is true.*

Then we have $|\mathcal{F}| \geq 2^{(1/2+o(1))\sqrt{n \log n}}$.

This lower bound exceeds the upper bound in (3). Therefore, construction (2) cannot satisfy the stronger assumptions in Theorem 2. For example, set

$$A = \{a_1\} \cup \{a_2, a'_2\} \cup \{\emptyset\} \cup \dots \cup \{\emptyset\} \in \mathcal{F}_2 \subset \mathcal{F},$$

$$B = (X_1 \setminus \{a_1\}) \cup \emptyset \cup \{\emptyset\} \cup \dots \cup \{\emptyset\} \in \mathcal{F}_1 \subset \mathcal{F},$$

where $a_1 \in X_1$ and $a_2, a'_2 \in X_2$. If $s > 4$, then $|A| < |B|$, but there is no element of B that can be added to A such that the resulting set also belongs to \mathcal{F} . If $s \leq 4$, then the conditions of Theorem 2 are satisfied, but the construction is uninteresting, as $|\mathcal{F}| = 2^{\Theta(n)}$ and $\text{rk}(\mathcal{F}) = \Theta(n)$.

Our next result provides a nontrivial construction.

Theorem 3. *There exists an atomic downward closed set-system $\mathcal{F} \subset 2^{[n]}$ with the property that for any two disjoint members $A, B \in \mathcal{F}$ with $|A| \leq |B|$, there is $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$, and*

(i) $|\mathcal{F}| \leq 2^{(2+o(1))n \log \log n / \log n},$

(ii) $\text{rk}(\mathcal{F}) \leq (2 + o(1))n / \log n.$

The proofs of Theorems 1, 2, and 3 are presented in Sections 2, 3, and 4, respectively.

2 Proof of Theorem 1

We start with a statement which immediately implies the inequality in part (ii).

Lemma 2.1. *Let $k \geq 1$ be an integer, $n > \binom{k}{2}$, and let \mathcal{F} be an atomic family of subsets of $[n]$ satisfying the exchange property in Theorem 1, or the condition in the Remark.*

Then there is a set $F \in \mathcal{F}$ such that $|F| = k$. This bound cannot be improved: there are families satisfying the conditions, for which $\text{rk}(\mathcal{F}) = k$.

Proof. By induction on k . For $k = 1$, the claim is trivial. Suppose that $k > 1$ and that the lemma has already been proved for $k - 1$.

Let $\mathcal{F} \subset 2^{[n]}$ be a family satisfying the conditions, where $n > \binom{k+1}{2}$. By the induction hypothesis, there is a member $A \in \mathcal{F}$ such that $|A| = k$. Consider the family $\mathcal{F}' = \{F \in \mathcal{F} : F \cap A = \emptyset\}$. Obviously, \mathcal{F}' satisfies the conditions on the ground set $[n] \setminus A$, and we have $|[n] \setminus A| > \binom{k+1}{2} - k = \binom{k}{2}$. Hence, we can apply the induction hypothesis to \mathcal{F}' to find a set $B \in \mathcal{F}'$ of size k which is disjoint from A . Using the exchange property in Theorem 1, or the condition in the Remark, for the sets A and B , we can conclude that \mathcal{F} has a member with $k + 1$ elements.

Now we show the tightness of Lemma 2.1. Let X_1, \dots, X_{k-1} be pairwise disjoint sets with $|X_i| = i$, for every i . Then $V = X_1 \cup \dots \cup X_{k-1}$ is a set of $\binom{k}{2}$ elements. For $i = 1, \dots, k - 1$, define

$$\mathcal{F}_i = \{F \subseteq V : |F \cap X_j| = 0 \text{ for every } j < i \text{ and } |F \cap X_j| \leq 1 \text{ for every } j > i\}. \quad (4)$$

In the definition of \mathcal{F}_i , there is no restriction on the size of $F \cap X_i$. Let $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_{k-1}$. Obviously, every member of \mathcal{F}_i has at most $|X_i| + k - 1 - i = k - 1$ elements, which yields that $\text{rk}(\mathcal{F}) = \max_{i=1}^k \text{rk}(\mathcal{F}_i) = k - 1$. Furthermore, \mathcal{F} is atomic and any two disjoint members of \mathcal{F} satisfy the exchange condition in Theorem 1. Hence, the lemma is tight. \square

We remark that the maximal sets in the above \mathcal{F} form the same hypergraph as the one defined in Example 3 of [4] for $v = 1$.

To prove the inequality $\text{rk}(\mathcal{F}) \geq \lfloor \sqrt{2n} \rfloor$ in part (ii) of Theorem 1, we have to find the largest k for which we can apply Lemma 2.1. It is easy to verify by direct computation that

$$\max\{k : \binom{k}{2} < n\} = \lfloor \sqrt{2n} \rfloor.$$

If $n = \binom{k}{2}$ for some $k \geq 1$, then the tightness of part (ii) of Theorem 1 follows from the tightness of Lemma 2.1. Suppose next that $\binom{k}{2} < n < \binom{k+1}{2}$. Let X_1, \dots, X_k be pairwise disjoint sets with $|X_i| = i$ for every $i < k$ and let $|X_k| = n - \binom{k}{2}$. Set $V = X_1 \cup \dots \cup X_k$. For $i = 1, \dots, k$, define \mathcal{F}_i as in (4), and let $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$. Then \mathcal{F} has the exchange property and $\text{rk}(\mathcal{F}) = k = \lfloor \sqrt{2n} \rfloor$. This proves part (ii) of Theorem 1.

It remains to establish part (i). Let \mathcal{F} be a family of subsets of $[n]$ satisfying the conditions. Let \mathcal{F}' denote the k -uniform hypergraph (i.e., family of k -element sets) consisting of all k -element sets in \mathcal{F} , i.e., $\mathcal{F}' = \{F \in \mathcal{F} : |F| = k\}$.

The *independence number* $\alpha(\mathcal{H})$ of a hypergraph \mathcal{H} is the maximum cardinality of a subset of its ground set which contains no element (hyperedge) of \mathcal{H} . It follows from Lemma 2.1 that any subset $S \subseteq [n]$ of size $|S| = \binom{k}{2} + 1$ contains at least one element of \mathcal{F} whose size is k . Therefore, any such set contains at least one element of \mathcal{F}' , which means that $\alpha(\mathcal{F}') \leq \binom{k}{2}$.

We need a result of Katona, Nemetz, and Simonovits [3] which is a generalization of Turán's theorem to k -uniform hypergraphs.

Lemma 2.2. [3] *Let \mathcal{H} be a k -uniform hypergraph on an n -element ground set. If the independence number of \mathcal{H} is at most α , then we have*

$$|\mathcal{H}| \geq \binom{n}{k} / \binom{\alpha}{k}.$$

Applying Lemma 2.2 to the hypergraph $\mathcal{H} = \mathcal{F}'$ with $k = (\sqrt{2}e^{-1/\sqrt{2}} + o(1))\sqrt{n}$ and $\alpha = \binom{k}{2}$, we obtain

$$|\mathcal{F}| \geq |\mathcal{F}'| \geq e^{(2e^{-1/\sqrt{2}} + o(1))\sqrt{n}} \geq 2^{(1.42 + o(1))\sqrt{n}},$$

completing the proof of part (i). This bound is slightly better than the inequality $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rfloor}$, which immediately follows from part (ii), under the stronger assumption that \mathcal{F} is downward closed.

3 Proof of Theorem 2

Let \mathcal{F} be an atomic set-system on an n -element ground set X , where n is large, and let s and t be two positive integers to be specified later. We describe a procedure to identify $\sum_{i=0}^t s^i$ distinct members of \mathcal{F} . To explain this procedure, we fix an s -ary tree T of depth t . At the end, each of the s^t root-to-leaf paths in T will correspond to a unique member of \mathcal{F} .

Each *non-leaf vertex* v will be associated with an s -element subset $X(v) \subset X$ such that along every root-to-leaf path $p = v_0v_1 \dots v_t$, the sets $X(v_0), X(v_1), \dots, X(v_{t-1})$, associated with the root and with the internal vertices of p , will be pairwise disjoint. See Figure 1 for an example.

Each *edge* $e = vu$ of T , where u is a child of v , will be labelled with an element $x(e) \in X(v)$ in such a way that every edge from v to one of its s children gets a different label. Thus,

$$\{x(vu) : u \text{ is a child of } v\} = X(v).$$

Denoting the root by v_0 , we choose $X(v_0)$ to be an arbitrary s -element subset of the ground set X , and set $F(v_0) = \emptyset \in \mathcal{F}$. For any non-root vertex v , let

$$F(v) = \{x(e) : e \text{ is an edge along the root-to-}v \text{ path}\}.$$

We will choose $X(v)$ such that $F(v) \in \mathcal{F}$ for every v . All of the sets $F(v)$ will be distinct, as any two different paths starting from the root diverge somewhere, unless one contains the other.

Suppose that we have already determined the set $X(u)$ for all ancestors of some non-leaf vertex v at level $\ell < t$ of T . At this point, we already know the set $F(v) \in \mathcal{F}$, where $|F(v)| = \ell$, and we want to determine $X(v)$. The next lemma guarantees that there is a good choice for $X(v)$.

Lemma 3.1. *There is an s -element subset $X(v) \subset X$ such that for every $x \in X(v)$, we have $F(v) \cup \{x\} \in \mathcal{F}$.*

Proof. Let $Z = \cup\{X(u) : u \text{ lies on the root-to-}v \text{ path, } u \neq v\}$. If v is at level $\ell < t$, we have $|Z| = \ell s \leq (t-1)s$. Let

$$Y = \{y \in X \setminus Z : F(v) \cup \{y\} \notin \mathcal{F}\}.$$

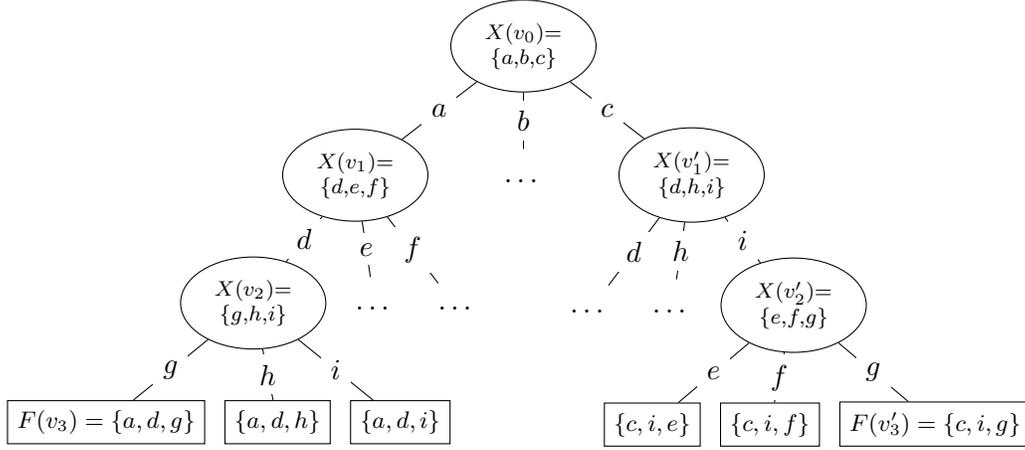


Figure 1: Construction of the auxiliary tree T for the proof of Theorem 2.

In other words, Y consists of all elements of $X \setminus Z$ that cannot be added to $F(v)$ to obtain a set in \mathcal{F} .

Consider the family $\mathcal{F}' = \{F \in \mathcal{F} : F \subseteq Y\}$. If $|Y| > \binom{t}{2}$, then Lemma 2.1 implies that there is a set $B \in \mathcal{F}'$ with $|B| = t > |F(v)|$. In this case, we can apply the exchange condition in Theorem 2 to the sets $F(v)$ and B , to conclude that there exists $b \in B$ for which $F(v) \cup \{b\} \in \mathcal{F}$. However, this contradicts the fact that $b \in Y$.

Thus, we can assume that $|Y| \leq \binom{t}{2}$. Now we have

$$|(X \setminus Z) \setminus Y| \geq n - (t-1)s - \binom{t}{2}.$$

If the right-hand side of this inequality is at least s , there is a proper choice for the set $X(v)$. For this, it is enough if $n \geq \frac{t^2}{2} + ts$, or, equivalently, $2 \geq (\frac{t}{\sqrt{n}})^2 + \frac{t}{\sqrt{n}} \frac{2s}{\sqrt{n}}$. To achieve this, let n be large, $s = \lfloor \sqrt{n} / \log^2 n \rfloor$, and $t = \lfloor (1 - 1/\log n) \sqrt{2n} \rfloor$. \square

By the above procedure, we can recursively assign a different set $F(v) \in \mathcal{F}$ to each vertex v of T . This gives

$$|\mathcal{F}| \geq \sum_{i=0}^t s^i \geq (n^{1/2} / \log^2 n)^{(1-o(1))\sqrt{2n}} = n^{\sqrt{(1/2+o(1))n}},$$

as desired.

4 Proof of Theorem 3

Assume for simplicity that n is a multiple of k , and fix a partition $[n] = X_1 \cup \dots \cup X_{n/k}$ into n/k parts, each of size k . That is, let $|X_1| = \dots = |X_{n/k}| = k$, where k is the largest number for which $2^{k-2} \leq n/k$; this gives $k = (1 + o(1)) \log n$. We will also assume $n, k \geq 3$.

For any $A \subseteq [n]$ and $0 \leq i \leq k$, let $p_A(i)$ and $s_A(i)$ denote the number of parts X_t which intersect A in *precisely* i elements and in *at least* i elements, respectively. Thus, we have $s_A(i) = \sum_{j=i}^k p_A(j)$ and $|A| = \sum_{i=1}^k i p_A(i) = \sum_{i=1}^k s_A(i)$. Define the *profile vector* of A , as

$$p_A = (p_A(k), p_A(k-1), \dots, p_A(0)),$$

and let

$$s_A = (s_A(k), s_A(k-1), \dots, s_A(0)).$$

That is, $p_A(0)$ is the number of parts that are disjoint from A , while $s_A(0)$ is always equal to n/k . We claim that the set-system

$$\mathcal{F} = \{A \subseteq [n] : s_A(k) \leq 1 \text{ and } s_A(i) \leq 2^{k-1-i} \text{ for every } 2 \leq i \leq k-1\}$$

meets the requirements of the theorem. Notice that for $i = 0$ and $i = 1$, there is no restriction on $s_A(i)$ other than the trivial bounds $0 \leq s_A(i) \leq n/k$. In particular, if A is an element of \mathcal{F} with maximum cardinality, we have

$$s_A = (1, 1, 2^1, 2^2, 2^3, \dots, 2^{k-3}, n/k, n/k), \text{ and}$$

$$p_A = (1, 0, 1, 2^1, 2^2, \dots, 2^{k-4}, n/k - 2^{k-3}, 0).$$

Here, we used that $2^{k-3} \leq n/k$, by assumption. Thus, the size of such a largest set is

$$\text{rk}(\mathcal{F}) = |A| = \sum_{i=1}^k s_A(i) = n/k + \sum_{i=1}^{k-1} 2^{k-i-1} + 1 = n/k + 2^{k-2} \leq 2n/k = (2 + o(1))n/\log n.$$

This also gives the following simple bound on the size of the family.

$$|\mathcal{F}| \leq \sum_{i=0}^{\text{rk}(\mathcal{F})} \binom{n}{i} \leq \sum_{i=0}^{2n/k} \binom{n}{i} \leq \left(\frac{en}{2n/k}\right)^{2n/k} \leq \left(\frac{e \log n}{2}\right)^{(2+o(1))n/\log n} \leq 2^{(2+o(1))n \log \log n / \log n}.$$

To prove that \mathcal{F} meets the requirements of the theorem, we consider any pair of disjoint sets $A, B \in \mathcal{F}$ such that there exists no $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. We need to show that in this case we have $|A| > |B|$. Suppose for contradiction that this is not the case, and fix a *counterexample* for which $|B| \geq |A|$ and $|B| - |A|$ is as large as possible. We refer to such a counterexample as a *maximal counterexample*. We say that $s_A(i)$ is *saturated* if $s_A(i) = 2^{k-1-i}$ for $1 < i < k$, if $s_A(i) = 1$ for $i = k$, and if $s_A(i) = n/k$ for $i = 1$.

Claim 4.1. *For any part $X \in \{X_1, \dots, X_{n/k}\}$, if $B \cap X \neq \emptyset$, then $A \cap X \neq \emptyset$. Moreover, if $|A \cap X| = i$, then $s_A(i+1)$ is saturated. Hence, $s_B(k) = 0$.*

Proof. Otherwise, we could add one element from $B \cap X$ to A , as $s_A(i+1)$ was not saturated. As A and B are disjoint, this also implies $s_B(k) = 0$. \square

Claim 4.2. *There is a maximal counterexample $A, B \in \mathcal{F}$ such that, for every $X \in \{X_1, \dots, X_{n/k}\}$,*

- (i) $A \cap X \neq \emptyset$;
- (ii) *if $|A \cap X| = 1$, then $B \cap X \neq \emptyset$.*

Proof. Among all maximal counterexamples $A, B \in \mathcal{F}$, choose one for which $s_A(1)$ is as large as possible. Let $j \geq 1$ be the smallest positive integer for which there is a part X with $|A \cap X| = j$. Thus, we have $s_A(j) = s_A(j-1) = \dots = s_A(1) \geq s_B(1)$, where the last inequality follows from Claim 4.1. This implies that $s_A(i)$ cannot be saturated for any positive integer $i < j$.

Suppose first that $j > k/2$. Then we have

$$|A| = \sum_{i=1}^{n/k} |A \cap X_i| > \sum_{X_i \cap A \neq \emptyset} k/2 \geq \sum_{\substack{X_i \cap A \neq \emptyset \\ X_i \cap B \neq \emptyset}} k/2 \geq \sum_{\substack{X_i \cap A \neq \emptyset \\ X_i \cap B \neq \emptyset}} |B \cap X_i| = \sum_{i=1}^{n/k} |B \cap X_i| = |B|,$$

where the second inequality follows from Claim 4.1, the third inequality from the disjointness of A and B and $j > k/2 = |X_i|/2$, and the final equality again from Claim 4.1. Hence, in this case, the pair A, B did not constitute a counterexample.

From now on, we can assume $j \leq k/2$. If for any part X , we have $|A \cap X| = j$ and $B \cap X = \emptyset$, then we claim that $A' = A \setminus X$ and B would also form a counterexample. To see this, assume for a contradiction that some b can be added to A' . Then b could also be added to A , unless $b \in X'$ for some part X' such that $|A' \cap X'| = j-1$. But this is impossible for $j \geq 2$, because of the minimality of j . It is also impossible for $j = 1$, because, by Claim 4.1, we have $A' \cap X' \neq j-1$. For the counterexample formed by A' and B , we have $|B| - |A'| > |B| - |A|$, contradicting the maximality of $|B| - |A|$. This proves (ii) because $|A \cap X| = 1$ implies $j = 1$ by the definition of j . To prove (i), note that by the ‘moreover’ part of Claim 4.1, we can conclude that $s_A(j+1)$ is saturated.

Next, we show that $s_A(j)$ is also saturated. Otherwise, if (i) does not hold, pick a part X for which $A \cap X = \emptyset$. By Claim 4.1, this implies that $B \cap X = \emptyset$. Add any $j \leq k/2$ elements of X to A and $j \leq k/2$ other elements of X to B , and denote the resulting sets by A' and B' , so that $|B'| - |A'| = |B| - |A|$. In view of Claim 4.1, we have $s_{B'}(j) = s_B(j) + 1 \leq s_B(1) + 1 \leq s_A(1) + 1 = s_A(j) + 1 = s_{A'}(j)$. Thus, A' and B' belong to \mathcal{F} , as we assumed that $s_A(j)$ is not saturated. To show that they constitute a counterexample, we need to prove that for every $b \in B'$, we have $A' \cup \{b\} \notin \mathcal{F}$. That is, if $b \in X'$ for some part X' and $|A' \cap X'| = i$, then we need to prove that $s_{A'}(i+1)$ is saturated. We know that $i \geq j$ as the intersection of A' with any part is at least j . If $i > j$, then $X' \neq X$ and so we could also add b to A from B to obtain $A \cup \{b\} \in \mathcal{F}$, contradicting that they formed a counterexample. If $i = j$, then $s_{A'}(j+1) = s_A(j+1)$ is saturated because of the previous paragraph. Since $s_{A'}(1) > s_A(1)$, this contradicts the maximal choice of A made at the very beginning of this proof.

Moreover, $j = 1$ must hold. Otherwise, if (i) does not hold, pick a part X disjoint from A and B , and add any $j-1$ elements of X to A and $j-1$ other elements of X to B . Denote the resulting sets by A' and B' , so that $|B'| - |A'| = |B| - |A|$. Similarly as before, these new sets also belong to \mathcal{F} , as $s_A(j)$ was saturated. We get a contradiction again, because we have $s_{A'}(1) > s_A(1)$. Since $s_A(1)$ is saturated, we have $s_A(1) = n/k$. This proves part (i). \square

Claim 4.3. *There is a maximal counterexample $A, B \in \mathcal{F}$ for which Claim 4.2 holds and $|B \cap X| > 1$ implies $|A \cap X| = 1$, for every $X \in \{X_1, \dots, X_{n/k}\}$.*

Proof. There is a maximal counterexample $A, B \in \mathcal{F}$ for which Claim 4.2 holds. By definition, we have

$$s_B(2) \leq 2^{k-3} \leq n/k - 2^{k-3} \leq n/k - s_A(2) = p_A(0) + p_A(1) = p_A(1),$$

where the second inequality follows from our assumption $n/k \geq 2^{k-2}$.

Therefore, if $|B \cap X| > 1$ and $|A \cap X| > 1$ for some part X , then there exists another part X' for which $|B \cap X'| = 1$ and $|A \cap X'| \leq 1$. By Claim 4.2 (i) (or by Claim 4.1), $|A \cap X'| = 1$. Choose any $|B \cap X| - 1$ elements of $B \cap X$, and remove them from B . Choose the same number of elements of $X' \setminus A$, and add them to B . By a repeated application of this procedure, we can achieve that the condition in the claim is satisfied. \square

From now on, we consider a counterexample $A, B \in \mathcal{F}$ satisfying the condition in Claim 4.3. Let j be the smallest positive integer with the property that for every part X with $|A \cap X| = j$, we have $B \cap X = \emptyset$. To see that there exists at least one such j , notice that if for some j there is no part X with $|A \cap X| = j$, then j meets the requirement. This implies that the number $k - 1$ has the desired property: If there is a part X such that $|A \cap X| = k - 1$, then since $s_A(k - 1) = 1$, there can be no part X' such that $|A \cap X'| = k$, so if $B \cap X \neq \emptyset$, we could add $B \cap X$ to A , contradicting that they are a counterexample. It follows from Claim 4.2 (ii) that $j = 1$ is not possible, so $1 < j < k$.

For each part X intersecting A in more than j elements, remove $|A \cap X| - j$ elements of $A \cap X$ from A and remove all elements in $B \cap X$ from B . Denote the sets obtained this way by A' and B' , respectively. As $s_{A'}(i) = s_A(i)$ for every $i \leq j$, and if $A' \cap X = j$, then $B \cap X = \emptyset$, the sets A' and B' are a counterexample. According to Claim 4.3, $|B'| - |A'| \geq |B| - |A|$, and it is easy to see that Claims 4.2 and 4.3 remain valid. We summarize these properties in the below claim.

Claim 4.4. *There exists a maximal counterexample A, B which satisfies Claims 4.2 and 4.3, and the following three properties:*

- (i) $s_A(j + 1) = 0$;
- (ii) $p_B(0) \geq s_A(j)$;
- (iii) $s_A(i) = 2^{k-1-i}$ for every $2 \leq i \leq j$.

Proof. Parts (i) and (ii) follow directly from the construction, while (iii) follows from the ‘moreover’ part of Claim 4.1 using $j < k$. \square

Now we can easily complete the proof of Theorem 3. We have

$$|A| = \sum_{i=1}^k s_A(i) = \sum_{i=2}^j 2^{k-1-i} + n/k.$$

On the other hand, $s_B(k) = 0$ holds, by Claim 4.1, and $s_B(1) \leq n/k - s_A(j) = n/k - 2^{k-1-j}$, by Claim 4.4 (iii). Thus,

$$|B| = \sum_{i=1}^k s_B(i) \leq n/k - 2^{k-1-j} + \sum_{i=2}^{k-1} 2^{k-1-i},$$

$$|A| - |B| \geq 2^{k-1-j} - \sum_{i=j+1}^{k-1} 2^{k-1-i} = 1.$$

This means that $|A| > |B|$, contradicting our assumption that $A, B \in \mathcal{F}$ is a counterexample.

5 Concluding remarks

If we strengthen the condition of our results by requiring that for any two non-empty disjoint members $A, B \in \mathcal{F}$, there exist $a \in A$ and $b \in B$ such that $B \cup \{a\} \in \mathcal{F}$ and $A \cup \{b\} \in \mathcal{F}$ both hold, then the problem becomes trivial. Any atomic set-system $\mathcal{F} \subset 2^{[n]}$ with this property must contain all subsets of $[n]$. Indeed, every set $F = \{x_1, \dots, x_k\}$ can be built up, sequentially applying the condition to the sets $\{x_1, \dots, x_i\}$ and $\{x_{i+1}\}$, for $i = 1, \dots, k - 1$.

In Theorems 1 and 2, we only assume that \mathcal{F} is atomic. However, our best constructions have the stronger property that \mathcal{F} is downward closed. Could we substantially strengthen these results under the stronger assumption? The proof of the bound $|\mathcal{F}| \geq 2^{\lfloor \sqrt{2n} \rfloor}$, which is only slightly weaker than Theorem 1 (i), becomes much easier if we assume that \mathcal{F} is downward closed, and the proof of Theorem 2 can also be simplified if \mathcal{F} is downward closed.

The property of the set-system described in Theorems 2 and 3 is reminiscent of the *independent set exchange property* of matroids; see [5]. A common generalization of these two properties would be to require that for any two members $A, B \in \mathcal{F}$, if either $|A| = |B|$ and $A \cap B = \emptyset$, or $|A| < |B|$ (but they are not necessarily disjoint), then there exists $b \in B$ such that $A \cup \{b\} \in \mathcal{F}$. A downward closed set-system \mathcal{F} has this property if and only if \mathcal{F} is the family of independent sets in a matroid in which no subspace has two disjoint generators A and B , i.e., $A \cap B = \emptyset$ and $\text{rk}(A) = \text{rk}(B) = \text{rk}(A \cup B)$ is forbidden. We do not know whether this question has been studied before.

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