Tight Revenue Gaps among Multi-Unit Mechanisms*

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Abstract

This paper considers Bayesian revenue maximization in the k-unit setting, where a monopolist seller has k copies of an indivisible item and faces n unit-demand buyers (whose value distributions can be non-identical). Four basic mechanisms among others have been widely employed in practice and widely studied in the literature: Myerson Auction, Sequential Posted-Pricing, $(k+1)$-th Price Auction with Anonymous Reserve, and Anonymous Pricing. Regarding a pair of mechanisms, we investigate the largest possible ratio between the two revenues (a.k.a. the revenue gap), over all possible value distributions of the buyers.

Divide these four mechanisms into two groups: (i) the discriminating mechanism group, Myerson Auction and Sequential Posted-Pricing, and (ii) the anonymous mechanism group, Anonymous Reserve and Anonymous Pricing. Within one group, the involved two mechanisms have an asymptotically tight revenue gap of $1+\Theta(1/\sqrt{k})$. In contrast, any two mechanisms from the different groups have an asymptotically tight revenue gap of $\Theta(\log k)$.

*A preliminary version of this work [JJLZ21] appears in Proceedings of the 22nd ACM Conference on Economics and Computation (EC'21). The proof for the "Anonymous Reserve vs. Anonymous Pricing" problem is omitted there yet is included in the current full version.

Funding: The first author is supported by NSF IIS-1838154, NSF CCF-1703925, NSF CCF-1814873 and NSF CCF-1563155. The second/fourth authors are supported by NSF CAREER award CCF-1844887. The third author is supported by Science and Technology Innovation 2030 – "New Generation of Artificial Intelligence" Major Project No.(2018AAA0100903), NSFC grant 61922052 and 61932002, Innovation Program of Shanghai Municipal Education Commission, Program for Innovative Research Team of Shanghai University of Finance and Economics (IRTSHUFE) and the Fundamental Research Funds for the Central Universities.

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1 Introduction

"Simple vs. optimal" is one of the central themes in Bayesian mechanism design. The revenue-optimal mechanisms are more of theoretical significance, but usually are complicated and are hard to implement. On the other hand, most daily-life mechanisms are much simpler, although sacrificing a (small) amount of revenue. This trade-off motivates the study on how well simple mechanisms can approximate the optimal mechanisms.

For example, consider selling a number of identical copies of a product on Amazon. (This scenario is characterized by the *multi-unit* model; see Section 1.1 for details.) The seller ideally would like to extract optimal revenues by using the remarkable Myerson's auction [Mye81], but often abandons it out of the following and other practical concerns:

- Myerson' auction as a centralized *auction scheme* requires coordination between the seller and all potential buyers, which is inconvenient or even impossible in most real business.
- It individually charges the winning buyers *different payments* – such price discrimination incurs fairness issues to the buyers and usually is illegal.
- It requires a complete profile of all potential buyers' value distributions. In practice, this means a full access to historical transaction records and other personal information, thus incurring privacy concerns to the buyers.

Instead, the seller simply posts a price. Each buyer on arrival makes a take-it-or-leave-it decision by himself, depending on whether the price is acceptable (and whether the product has not been sold out). This simple and prevalent mechanism, called *anonymous pricing*, clearly settles or at least mitigates the above issues. To justify the legitimacy of anonymous pricing, the remaining consideration is its revenue guarantees against Myerson's auction.

The last two decades have seen extensive progress on the "simple versus optimal" trade-off [BK+96, GHW01, BHW02, GHK+05, HR09, Ala14, CGL14, CGL15, FILS15, DFK16, CFH+17, AHN+19, JLTX20, JLQ+19b, JLQ19a, and the reference therein]. By now we can say that it constitutes a subfield within mechanism design. In this work, we will study this trade-off in the multi-unit model.

1.1 Background

Let us first review the previous results. In the most basic *single-item* model, four fundamental mechanisms among others are widely studied. Denote by $\mathbf{F} = \{F_j\}_{j\in[n]}$ the independent value distributions of buyers $j \in [n]$. These four mechanisms work as follows (see Section 2.2 for the formal definitions).

- Anonymous Pricing (AP): This mechanism treats all buyers equally by posting a price p. On arrival, a buyer will pay this price p and take the item, when his value $b_j \sim F_j$ is higher than p (and the item is still available). If the seller knows the value distributions $\{F_j\}_{j\in[n]}$, she would select a particular price p to maximize her expected revenue among all Anonymous Pricing mechanisms.
- Sequential Posted Pricing (SPM): This mechanism selects an array of prices $\{p_j\}_{j\in[n]}$ and an ordering $\sigma : [n] \mapsto [n]$. The buyers join in the mechanism *sequentially* $\sigma(1), \cdots, \sigma(n)$, and each index-$\sigma(j)$ buyer must pay the *order-specific* price p_j if winning. This discrimination can give better revenue than Anonymous Pricing.
- Anonymous Reserve (AR): This is a variant of the Second-Price Auction. The seller ignores the buyers whose bids b_j are below an anonymous reserve r. The winner (which exists only if the highest bid $b_{(1)}$ is above the reserve r) is the highest of the remaining buyers, and his payment is the bigger one between the second highest bid $b_{(2)}$ and the reserve r.

- Myerson Auction (OPT): A generic auction $\mathcal{A} : \{b_j\}_{j\in[n]} \mapsto (\mathbf{x}, \boldsymbol{\pi})$ is a mapping from the bids/values to the allocations $\mathbf{x} = (x_j)_{j\in[n]}$ and the payments $\boldsymbol{\pi} = (\pi_j)_{j\in[n]}$. In the single-item case, Myerson Auction is the optimal one among those mappings [Mye81]. (When the distributions $\{F_j\}_{j\in[n]}$ are identical, Myerson Auction degenerates to Anonymous Reserve.)

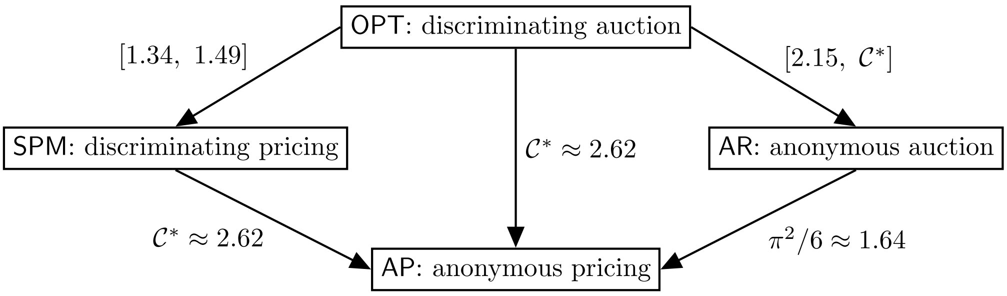


Figure 1: Demonstration for the **previous results** in the **single-item** setting with asymmetric regular buyers, where an interval indicates the best known lower/upper bounds, and a number indicates a tight bound. For the references of these results and further discussions, one can refer to [JLQ+19c, Section 6] and [Har13, Chapter 4].

These four mechanisms together form the hierarchy in Figure 1, where each arrow goes from a more complicated mechanism with higher revenue to a simpler mechanism with lower revenue. There are two notable distinctions among the four mechanisms.

- Anonymity (AP and AR) vs. Discrimination (SPM and OPT). We say a mechanism is *discriminating* if, when different buyers become the winner, the required payments can be different. Otherwise we say the mechanism is *anonymous*. Intuitively, discrimination gives a mechanism more power to extract revenue.

- Pricing (AP and SPM) vs. Auction (AR and OPT). In a pricing scheme, the buyers simply make take-it-or-leave-it decisions based on the given prices. In contrast, an auction is an arbitrary mapping from the bids to the allocations and the payments. Auctions can gain higher revenues than pricing schemes by further leveraging the competition among buyers.

Because SPM is a discriminating pricing scheme and AR is an anonymous auction, they have different powers and are incomparable. Accordingly, there are five comparable mechanism pairs (i.e., the five arrows in Figure 1).

To understand the relative powers of those mechanisms, the very first question is how large the *revenue gap* between any two mechanisms can be. We characterize the revenue gap as the *approximation ratio*[1] between the two revenues. Formally, for a more complicated mechanism $\mathcal{M}_1$ and a simpler mechanism $\mathcal{M}_2$, their approximation ratio is given by

$$\Re_{\mathcal{M}_1/\mathcal{M}_2} \overset{\text{def}}{=} \sup\left\{\frac{\textsc{Rev}_{\mathcal{M}_1}(\mathbf{F})}{\textsc{Rev}_{\mathcal{M}_2}(\mathbf{F})} \;\middle|\; \mathbf{F} \in \mathcal{F}\right\},$$

where $\textsc{Rev}_{\mathcal{M}}(\mathbf{F})$ denotes the revenue from a mechanism $\mathcal{M}$ on an input instance $\mathbf{F} = \{F_j\}_{j\in[n]}$, and the supremum is taken over a certain family of distributions $\mathbf{F} \in \mathcal{F}$.

For the single-item model, the known results are shown in Figure 1. Notice that all these revenue gaps are universal constants, and most of them have matching lower and upper bounds.

From Single Unit to Multiple Units. In this work, we focus on the *k-unit setting*, where the seller has $k \geq 1$ identical copies of an item, and aims to sell them to n unit-demand buyers.

[1]The earlier "mechanism design for digit goods" literature [GHW01, BHW02, GHK+05, CGL14, CGL15], due to technical reasons, often uses the term "competitive ratio" rather than "approximation ratio".

This setting is much more realistic and common in real business. Further, it is of intermediate complexity in comparison with the (more restricted) single-item setting and the (more general) multi-item setting.[2] Nonetheless, the "simple vs. optimal" trade-offs are much less understood in this setting than in the single-item setting.

Since the k-unit setting is still a *single-parameter* setting, Myerson Auction remains revenue-optimal [Mye81]. In addition, both of Anonymous Pricing and Sequential Posted Pricing can be naturally extended to this setting. For Anonymous Reserve, the counterpart auction is no longer "second-price-type", but is the $(k+1)$-th Price Auction with Anonymous Reserve.

1.2 An overview of our results

In the k-unit setting, previously only the revenue gap $\mathfrak{R}_{\mathsf{OPT/SPM}}$ between OPT and SPM is well understood [Yan11, Ala14], whereas the other four gaps are widely open. By exploring the relative power of those mechanisms systematically, in this work we establish the (asymptotically) tight ratios of all previously unknown revenue gaps. We formalize our new results as the next two theorems and demonstrate them in Figure 2. Here, the regularity assumption is very standard in the mechanism design literature [Mye81].[3]

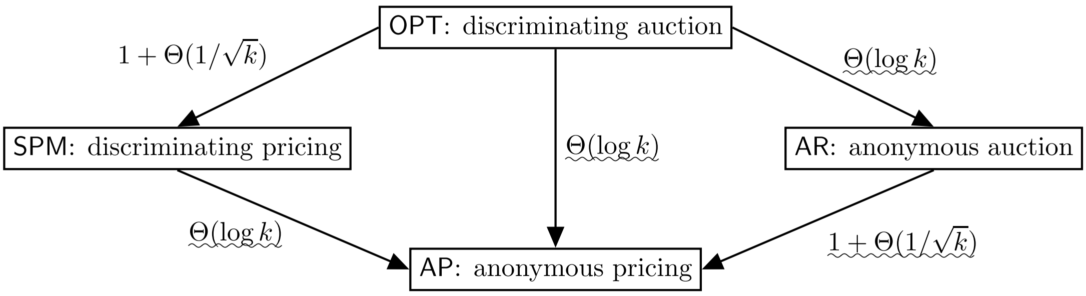


Figure 2: Demonstration for the revenue gaps among basic mechanisms in the k**-unit** setting, given that the value distributions are regular. Our new results are underwaved. The $1+\Theta(1/\sqrt{k})$ approximation result between AR and AP is given in Theorem 1, and the other three results are given in Theorem 2.

Theorem 1 (Anonymous Reserve vs. Anonymous Pricing)**.** *For the unit-demand buyers $j \in [n]$, in each of the following three settings,[4] the revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k)$ between* Anonymous Reserve *and* Anonymous Pricing *is $\mathfrak{R}_{\mathsf{AR/AP}}(k) = 1 + k \cdot \int_0^\infty \frac{T_k(x)\cdot(1-T_{k+1}(x))}{(k-\sum_{i\in[k]} T_i(x))^2} \cdot \mathrm{d}x$, where for each $i \in [k+1]$ the function $T_i(x) \stackrel{def}{=} e^{-x} \cdot \sum_{t\in[0:i-1]} \frac{1}{t!} \cdot x^t$.*

1. *The asymmetric general setting, where the buyers have independent but not necessarily identical value distributions.*

2. *The i.i.d. general setting, where the value distributions are identical.*

3. *The asymmetric regular setting, where the value distributions are regular but not necessarily identical.*

Asymptotically, this bound is of order $\mathfrak{R}_{\mathsf{AR/AP}}(k) = 1 + \Theta(1/\sqrt{k})$.

[2] In the k-unit setting, the k copies are *identical*. But in the multi-item setting, the items can be *heterogeneous*.

[3] Roughly speaking, this assumption means the seller extracts a higher expected revenue from a buyer via a moderate price, compared with an over-high price or an over-low price; see Section 2.1 for its definition.

[4] In the i.i.d. regular setting, the tight bound $1/(1-k^k/(e^k k!)) \approx 1/(1-1/\sqrt{2\pi k})$ follows from a combination of (upper bound) [Yan11, Section 4.2] and (lower bound) [DFK16, Section 4.3]. This "i.i.d. regular" bound and our bound $\mathfrak{R}_{\mathsf{AR/AP}}(k)$ are different for each $k \in \mathbb{N}_{\geq 1}$, but asymptotically are of the same order $= 1 + \Theta(1/\sqrt{k})$.

Theorem 2 (Discriminating Mechanisms vs. Anonymous Mechanisms)**.** *When the unit-demand buyers $j \in [n]$ have independent and regular value distributions, each of the next three revenue gaps is of order $\Theta(\log k)$:*

1. *The revenue gap $\mathfrak{R}_{\mathsf{OPT/AP}}(k)$ between* Myerson Auction *and* Anonymous Pricing*.*

2. *The revenue gap $\mathfrak{R}_{\mathsf{SPM/AP}}(k)$ between* Sequential Posted Pricing *and* Anonymous Pricing*.*

3. *The revenue gap $\mathfrak{R}_{\mathsf{OPT/AR}}(k)$ between* Myerson Auction *and* Anonymous Reserve*.*

Similar to the AR vs. AP revenue gap, the prior works [Yan11, Ala14] show that the OPT vs. SPM revenue gap is also of order $1 + \Theta(1/\sqrt{k})$. Consequently, regarding the discriminating mechanism group (OPT and SPM) and the anonymous mechanism group (AR and AP), each revenue gap across these two groups is $\Theta(\log k)$, but the revenue gap between the two mechanisms in one group tends to vanish (at the rate of $1/\sqrt{k}$) when the number of copies $k \in \mathbb{N}_{\geq 1}$ becomes large. These messages can be easily inferred from Figure 2.

As mentioned, the revenue gaps identify the power and the limit of "discrimination vs. anonymity" and "auction vs. pricing" in revenue maximization. Different from the single-item setting, where all the revenue gaps are universal constants (see Figure 1), our new results in the k-unit setting are more informative. When the number of copies $k \in \mathbb{N}_{\geq 1}$ is large:

- Auctions are not much more helpful than pricing schemes in extracting the revenue (i.e., just an $1 + \Theta(1/\sqrt{k})$ improvement), no matter whether discrimination is allowed or not.

- Discrimination is always very useful, and can even give an unbounded improvement (up to a $\Theta(\log k)$ factor) on the revenue.

These propositions meet what we observe in real business: auctions are rarely used in practice, whereas different kinds of price discrimination are rather common.

1.2.1 First Result: Anonymous Reserve vs. Anonymous Pricing

In this section, we sketch the proof of our $1 + \Theta(1/\sqrt{k})$ approximation result for the AR vs. AP revenue gap (Theorem 1). In fact, we can represent the exact ratio $\mathfrak{R}_{\mathsf{AR/AP}}$ as an explicit integration formula, (although this formula in general does not admit an elementary expression). We acquire this formula by solving a mathematical programming generalized from [JLTX20, Program (4)], which resolves the same problem for the single-item case $k = 1$.

However, many crucial properties of the single-item case do not preserve in the general case $k \geq 1$. In the single-item case, Anonymous Reserve relies on the first/second order statistics $b_{(1)}$ and $b_{(2)}$ (i.e., the biggest and second biggest sampled bids/values), and Anonymous Pricing relies on the $b_{(1)}$. Therefore, we only need to reason about these two random variables, $b_{(1)}$ and $b_{(2)}$, together with the correlation between them. In the k-unit case, however, up to $(k+1)$ random variables $b_{(1)}, \cdots, b_{(k+1)}$ must be taken into account, and the correlation among them becomes much more complicated.

For the above reasons, we cannot modify and re-adopt the approach of the work [JLTX20] in a naive way. Instead, with the purpose of handling the highly correlated order statistics $b_{(i)}$'s, we will develop a new structural lemma about the *Poisson binomial distributions* (PBDs). This new lemma mainly relies on the *log-concavity* of the PBDs.

Lemma (Bernoulli Sum Lemma)**.** *Given two arrays of Bernoulli random variables: $\{X_j\}_{j \in [n]}$ are i.i.d., while $\{Y_j\}_{j \in [n]}$ are independent yet not necessarily identically distributed. For the random sums $X = \sum_{j \in [n]} X_j$ and $Y = \sum_{j \in [n]} Y_j$, there exists some threshold $s \in \mathbb{R}$ such that:*

1. $\Pr[X \leq t] \geq \Pr[Y \leq t]$ *for any $t < s$.*

2. $\Pr[X \leq t] \leq \Pr[Y \leq t]$ *for any $t \geq s$.*

With the help of this lemma, we can characterize the worst-case instance of the mentioned mathematical programming, for $k \geq 1$ and $n \geq 1$. To this end, let us formulate the AR and AP revenues. Denote by F_j the cumulative distribution function (CDF) of buyer j's value, and D_i the CDF of the i-th order statistic $b_{(i)}$. The Anonymous Reserve revenue (Fact 3) is given by

$$\mathsf{AR}(r) \;=\; \mathsf{AP}(r) \;+\; k \cdot \int_r^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x, \qquad\qquad \forall r \geq 0.$$

where $\mathsf{AP}(r)$ is the revenue by posting the price $p = r$ in Anonymous Pricing. Further, the AP revenue (Fact 2) depends on the top-k CDF's $\{D_i(r)\}_{i \in [k]}$ at this reserve $r \geq 0$.

Now consider a Bernoulli sum $Y = \sum_{j \in [n]} Y_j$, for which the individual failure probabilities are $\Pr[Y_j = 0] = F_j(r)$. This choice of the failure probabilities ensures $\Pr[Y \leq i-1] = D_i(r)$ for every $i \geq 1$. Further, we can find another array of i.i.d. Bernoulli random variables $\{X_j\}_{j \in [n]}$ so that the sum $X = \sum_{j \in [n]} X_j$ satisfies

$$\Pr[X \leq k] \;=\; \Pr[Y \leq k] \;=\; D_{k+1}(r).$$

(The existence of such $\{X_j\}_{j \in [n]}$ is obvious.) Then our Bernoulli Sum Lemma shows that

$$\Pr[X \leq i-1] \;\geq\; \Pr[Y \leq i-1] \;=\; D_i(r)$$

for each $i \in [k]$, where the equality holds when the $\{Y_j\}_{j \in [n]}$ are also i.i.d.

Informally speaking, the above inequalities and the equality condition imply that, the ratio $\mathsf{AR}(r)/\mathsf{AP}(r)$ is maximized when the value CDF's are equal $F_1(r) = \cdots = F_n(r)$ at this reserve. Following this argument and with extra efforts, we have the next observation.

Observation. *For each $k \geq 1$ and $n \geq 1$, the worst case for the $\mathfrak{R}_{\mathsf{AR}/\mathsf{AP}}$ revenue gap happens when the value distributions are identical, i.e., $\mathbf{F}^* = \{F^*\}^n$, (although this worst-case common distribution F^* is given by an implicit equation and does not admit an elementary expression).*

Furthermore, it is noteworthy that the above approach enables a unified constructive proof for the upper-bound/lower-bound parts of the general case $k \geq 1$. In contrast, the former work [JLTX20] establishes these two parts of the single-item case separately, and their upper-bound proof is non-constructive.

Our Bernoulli Sum Lemma can find its applications in related directions. As mentioned, we leverage it mainly to handle the order statistics. Apart from the "simple vs. optimal mechanism design" paradigm, on other topics such as "learning simple mechanisms from samples" [CGL15, MM16, MR16, CD17, JLX19], the order statistics are also of fundamental interests. Conceivably, our new lemma would be helpful for those topics, in a similar manner as this paper.

1.2.2 Second Result: Discriminating Mechanisms vs. Anonymous Mechanisms

In this section we sketch the proof of Theorem 2, which claims that the revenue gaps $\mathfrak{R}_{\mathsf{OPT}/\mathsf{AP}}$, $\mathfrak{R}_{\mathsf{SPM}/\mathsf{AP}}$ and $\mathfrak{R}_{\mathsf{OPT}/\mathsf{AR}}$ are all of order $\Theta(\log k)$. In fact, any one bound implies the other two. This is because the revenue gaps within the discriminating/anonymous groups (OPT vs. SPM, and AR vs. AP) are both *constants* $1 + \Theta(1/\sqrt{k}) = \Theta(1)$, and these constants are dominated by the $\Theta(\log k)$ bound.

For these reasons, it suffices to only prove the OPT vs. AP revenue gap $\mathfrak{R}_{\mathsf{OPT}/\mathsf{AP}} = \Theta(\log k)$. Actually, an $\Omega(\log k)$ lower bound for this revenue gap is already shown in [HR09, Example 5.4], so we only need to prove the $O(\log k)$ upper bound.

We actually prove the $O(\log k)$ upper bound between Anonymous Pricing and a benchmark called Ex-Ante Relaxation (EAR in short). It is known that this benchmark always exceeds the Myerson Auction revenue [CHMS10]. To acquire the $O(\log k)$ upper bound, we will start with a mathematical programming generalized from [AHN+19, Equations (1) and (2)].

However, the general-case mathematical programming has a very different structure as it is in the single-item case. When $k = 1$, the worst-case instance (i.e., the optimal solution, see [AHN+19, Section 4.3]) turns out to be a *continuum of "small" buyers* – any single buyer has an infinitesimal contribution to the EAR benchmark, but there are infinitely many buyers $n \to \infty$ (in a sense of large markets [MSVV07, AGN14]). Accordingly, it is better to think about the "density" of different types of buyers, instead of the number of buyers.

But in the general case, the $\Omega(\log k)$ lower-bound instance [HR09, Example 5.4] essentially is constituted by "big" buyers – a certain amount of buyers contribute at least $1/k$ unit to the EAR benchmark each, while every other buyer contributes strictly 0 unit and can be omitted. More importantly (see Remark 5), if we insist on a continuum of "small" buyers in the general case $k \geq 1$, then the EAR vs. AP revenue gap turns out to be (at most) a universal constant for whatever $k \geq 1$.

For these reasons, the current approach must be very different from the single-item case. At a high level, to handle the general case $k \geq 1$, we will classify the buyers $j \in [n]$ into groups, and then bound the individual contributions from these groups to the EAR benchmark.

In more details, we can employ the technique developed in [AHN+19, Lemma 4.1], and thus transform the mentioned mathematical programming into the following one.

Variables:

- $\{v_j\}_{j \in [n]} \in \mathbb{R}^n_{\geq 0}$, where $v_j = \arg\max\{p \cdot (1 - F_j(p)) : p \geq 0\}$ for each $j \in [n]$, are the *monopoly prices* of the distributions $\mathbf{F} = \{F_j\}_{j \in [n]}$.
- $\{q_j\}_{j \in [n]} \in [0, 1]^n$, where $q_j = 1 - F_j(v_j)$ for each $j \in [n]$, are the *monopoly quantiles.*
- The resulting $\{v_j q_j\}_{j \in [n]} \in \mathbb{R}^n_{\geq 0}$ are the *monopoly revenues.*

Constraints:

- The capacity constraint, $\sum_{j \in [n]} q_j \leq k$.
- The feasibility constraint, $\mathsf{AP}(p, \mathbf{F}) \leq 1$ for all $p \in \mathbb{R}_{\geq 0}$.

Objective: Maximize the Ex-Ante Relaxation benchmark $\mathsf{EAR}(\mathbf{F}) = \sum_{j \in [n]} v_j q_j$.

Regarding the EAR benchmark, the monopoly revenues $\{v_j q_j\}_{j \in [n]}$ are precisely the individual contributions from the distributions $\{F_j\}_{j \in [n]}$. Given the capacity constraint (in a sense of the Knapsack Problem), the monopoly quantiles $\{q_j\}_{j \in [n]}$ can be viewed as the individual capacities. Therefore, the monopoly prices $\{v_j\}_{j \in [n]}$ can be viewed as the *bang-per-buck ratios* (i.e., the contribution to the EAR benchmark per unit of the capacity).

To find the optimal solution, clearly we prefer those distributions with higher bang-per-buck ratios $\{v_j\}_{j \in [n]}$, but also need to take the capacities $\{q_j\}_{j \in [n]}$ into account. Informally, we will classify the buyers into three groups $[n] = L \cup H_S \cup H_B$:

- $L = \{j \in [n] : v_j < 1/k\}$. Because these group-$L$ distributions have *lower* bang-per-buck ratios $v_j < 1/k$, conceivably the total contribution by this group to the EAR benchmark shall be small. Indeed, we will prove a *constant* upper bound $\sum_{j \in L} v_j q_j = O(1)$.
- $H_S = \{j \in [n] : v_j \geq 1/k \text{ and } v_j q_j < 1/(2k)\}$. In other words, the group-H_S distributions have *high* enough bang-per-buck ratios $v_j \geq 1/k$ but *small* capacities, i.e., $v_j q_j < 1/(2k)$. It turns out that the total contribution by this group is also small, and we also will prove a *constant* upper bound $\sum_{j \in H_S} v_j q_j = O(1)$.
- $H_B = \{j \in [n] : v_j \geq 1/k \text{ and } v_j q_j \geq 1/(2k)\}$. That is, these group-$H_B$ distributions have *high* enough bang-per-buck ratios and *big* enough capacities. Therefore, this group

should contribute the most to the EAR benchmark. Taking into account the feasibility constraint, $\mathsf{AP}(p, \mathbf{F}) \leq 1$ for all $p \in \mathbb{R}_{\geq 0}$, we will show $\sum_{j \in H_B} v_j q_j = O(\log k)$.

The actual grouping criteria in our proof are more complicated than the above ones, in order to handle other technical issues.

Finally, we notice that our grouping criteria borrow ideas from the "budget-feasible mechanism" literature [Sin10, CGL11, GJLZ20], where the target is to design approximately optimal mechanisms for the Knapsack Problem under the incentive concerns. We hope that these ideas can find more applications to the "simple vs. optimal mechanism design" research topic.

1.3 Further related works

The revenue gaps among the mentioned mechanisms, Myerson Auction, Sequential Posted Pricing, Anonymous Reserve, and Anonymous Pricing, are extensively studied in the literature. Below we provide an overview of the previous results (mainly in the single-item setting and in the k-unit settings). As a supplement, the reader can refer to the surveys [Luc17, CFH+18, JLQ+19c] and the textbook [Har13].

AR vs. AP. This revenue gap studies the relative power between the auction schemes and the pricing schemes, when the price discrimination is not allowed. The previously known results in the single-item case are shown in the next table.

i.i.d. regular	$e/(e-1) \approx 1.58$	[CHMS10, Thm 6] & [Har13, Thm 4.13]
i.i.d. general	$\pi^2/6 \approx 1.64$	[JLTX20, Thm 2]
asymmetric regular		
asymmetric general		

In the k-unit case, the tight bound $1/(1 - k^k/(e^k k!)) \approx 1/(1 - 1/\sqrt{2\pi k})$ for i.i.d. regular buyers is shown in [Yan11, Section 4.2] and [DFK16, Section 4.3]. Our new results settle the remaining pieces of the puzzle – even if the i.i.d. assumption and/or the regularity assumption are removed, this revenue gap is still of order $1 + \Theta(1/\sqrt{k})$.

SPM vs. AP. This revenue gap investigates the power of price discrimination in the pricing schemes. Below we summarize the known results and our new results, in both the single-item case and the k-unit case.

single-item case		SPM vs. AP	OPT vs. AP
i.i.d. regular	$e/(e-1) \approx 1.58$	[CHMS10, Thm 6] & [Har13, Thm 4.13]	
i.i.d. general	$2 - 1/n$	[DFK16, Thm 3]	[Har13, Thm 4.9]
asymmetric regular	constant $\mathcal{C}^* \approx 2.62$	[JLTX20, Thm 1]	[JLQ+19b, Thm 1]
asymmetric general	n	[AHN+19, Prop 6.1]	

k-unit case		SPM vs. AP	OPT vs. AP
i.i.d. regular	$1/(1 - k^k/(e^k k!))$	[DFK16, Thm 1]	[Yan11, Sec 4.2]
i.i.d. general	$2 - k/n$	[DFK16, Thm 3]	[Har13, Sec 4.5]
asymmetric regular	$\Theta(\log k)$	this work	
asymmetric general	n	[AHN+19, Prop 6.1]	

OPT vs. AP. This revenue gap is to illustrate that even the simplest mechanism, Anonymous Pricing, can approximate the optimal revenue in quite general settings. Actually, in each of the single-item/k-unit, i.i.d./asymmetric, regular/general settings, this ratio "coincedentally" is equal to the SPM vs. AP revenue gap, namely $\mathfrak{R}_{\mathsf{OPT/AP}} = \mathfrak{R}_{\mathsf{SPM/AP}}$.[5] (But the results respectively for $\mathfrak{R}_{\mathsf{OPT/AP}}$ and $\mathfrak{R}_{\mathsf{SPM/AP}}$ are credited to different works.) For brevity, we summarize the results on the both revenue gaps together in the above tables.

Instead of the regularity assumption, the stronger *monotone-hazard-rate* (MHR) distributional assumption is also very standard in the mechanism design literature. The previous works [GZ18, JLQ19a] study the OPT vs. AP revenue gap in the single-item i.i.d. MHR setting.

OPT vs. AR. This ratio studies the power of price discrimination in the auction schemes. When the value distributions are i.i.d. and regular, Myerson Auction and Anonymous Reserve turn out to be identical [Mye81]. The results beyond the i.i.d. regular case are given below.

single-item case		
i.i.d. general	$2 - 1/n$	[Har13, Thm 4.9]
asymmetric regular	LB ≈ 2.15	[HR09, Sec 5] & [JLTX20, Thm 3]
	UB $= \mathcal{C}^* \approx 2.62$	[HR09, Sec 5] & [JLQ+19b, Thm 1]
asymmetric general	n	[AHN+19, Prop 6.1]

k-unit case		
i.i.d. general	$2 - k/n$	[Har13, Sec 4.5]
asymmetric regular	$\Theta(\log k)$	this work
asymmetric general	n	[AHN+19, Prop 6.1]

Notably, the tight ratio in the single-item asymmetric regular setting is still unknown. Hartline and Roughgarden first prove that this ratio is between 2 and 4 [HR09, Section 5]. Afterwards, the lower bound is improved to ≈ 2.15 [JLTX20, Theorem 3]. But the best known upper bound just follows from the tight OPT vs. AP revenue gap $\mathcal{C}^* \approx 2.62$ by implication. We highly believe this factor-$\mathcal{C}^*$ barrier can be broken, for which new techniques tailored for Anonymous Reserve rather than Anonymous Pricing are required.

Beyond the Anonymous Reserve mechanism, other simple auctions with the more powerful *personalized reserves* are also extensively studied [HR09, BGL+18, MS20].

OPT vs. SPM. This revenue gap investigates the relative power between the auction schemes and the pricing schemes, when the price discrimination is allowed. Indeed, the previous works [HKS07, CFPV19] show that this problem is identical to the *ordered prophet inequality* problem in stopping theory. In each of the single-item/k-unit i.i.d./asymmetric settings, the tight revenue gaps under/without the regularity assumption turn out to be the same (see, e.g., [Yan11, Section 3.1]). The previous results in the single-item/k-unit cases are summarized below.

single-item case		
i.i.d.	constant $\beta \approx 1.34$	[CFH+17, Thm 1.3]
asymmetric	LB $= \beta \approx 1.34$	[CFH+17, Thm 1.3]
	UB $= 1/(1 - 1/e + 1/27) \approx 1.49$	[CSZ19, Thm 1.1]

k-unit case		
i.i.d./asymmetric	LB $= 1 + \Omega(1/\sqrt{k})$	[HKS07, Thm 7]
	UB $= 1/(1 - k^k/(e^k k!)) \approx 1/(1 - 1/\sqrt{2\pi k})$	[Yan11, Sec 4.2]

[5]The reader may wonder why the revenue gaps $\mathfrak{R}_{\mathsf{OPT/AP}}$ and $\mathfrak{R}_{\mathsf{SPM/AP}}$ are equal, in each of the single-item/k-unit, i.i.d./asymmetric, regular/general settings. This is because, in each of these settings, the worst-case instance $\{F_j^*\}_{j\in[n]}$ of the OPT vs. AP problem has a nice property: for each F_j^*, the corresponding *virtual-value distribution* is supported on the non-positive semiaxis $(-\infty, 0]$ plus a *single positive number* $v_j^* > 0$. When an instance satisfies this property, we can adopt the arguments in [JLTX20, Lemma 1] to show that OPT and SPM extract the same amount of revenue, which implies $\mathfrak{R}_{\mathsf{OPT/AP}} = \mathfrak{R}_{\mathsf{SPM/AP}}$.

Noticeably, the tight ratio in the single-item asymmetric setting is still unknown. The best known lower bound just follows from the tight "i.i.d." revenue gap $\beta \approx 1.34$ by implication. Recently, there is an outburst of activity on the upper bound [ACK18, BGL⁺18, CSZ19], and the best known result is $1/(1 - 1/e + 1/27) \approx 1.49$ [CSZ19, Theorem 1.1]. It remains an interesting open question to further refine the upper bound.

Beyond the k-unit setting, the OPT vs. SPM revenue gap is also studied in the more general *matroid* setting. For this, the work [CHMS10, Theorem 5] first shows an upper bound of 2, and then [Yan11, Section 4.1] improves it to $e/(e-1) \approx 1.58$.

The Sequential Posted Pricing mechanism crucially leverages the order in which the buyers participate in the mechanism. Instead, the *order-oblivious* counterpart mechanisms are extensively studied as well [CHMS10, Ala14, AW18, ACK18, BGL⁺18, EHKS18, CSZ19].

Organization. In Section 2 we introduce the notation and the requisite knowledge about the considered mechanisms. The Anonymous Reserve vs. Anonymous Pricing problem is investigated in Section 3 (with some technical details deferred to Appendix A). The Ex-Ante Relaxation vs. Anonymous Pricing problem is investigated in Section 4.

2 Notation and Preliminaries

This section includes the notation to be adopted in this paper, and the basic knowledge about probability (e.g. the regular/triangle distributions) and the concerning mechanisms.

Notation. Denote by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{N}_{\geq 1}$) the set of all non-negative real numbers (resp. positive integers). For any pair of integers $b \geq a \geq 0$, define the sets $[a] \stackrel{\text{def}}{=} \{1, 2, \cdots, a\}$ and $[a:b] \stackrel{\text{def}}{=} \{a, a+1, \cdots, b\}$. Denote by $\mathbb{1}\{\cdot\}$ the indicator function. The function $|\cdot|_+$ maps a real number $z \in \mathbb{R}$ to $\max\{0, z\}$.

2.1 Probability

We use the bold letter $\mathbf{F} = \{F_j\}_{j\in[n]}$ to denote an instance (namely an n-dimensional *product distribution*), where F_j is the bid distribution of the buyer $j \in [n]$. For ease of notation, F_j also represents the corresponding cumulative density function (CDF).

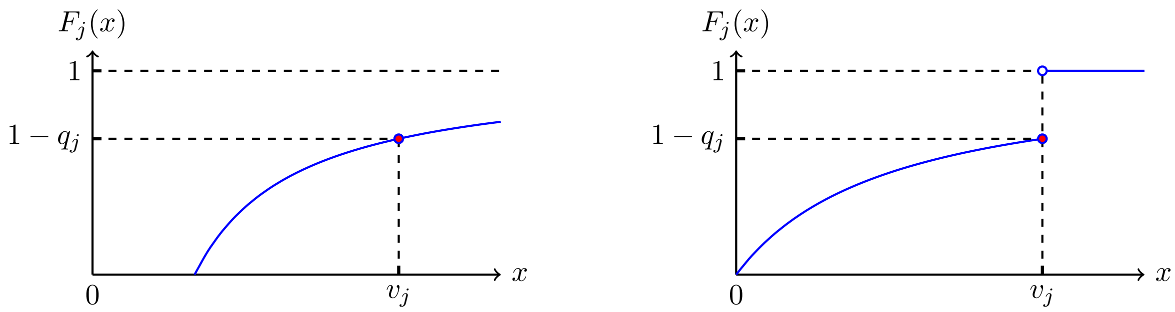
We assume the CDF's $\{F_j\}_{j\in[n]}$ to be *left-continuous*, in the sense that when the j-th buyer has a random bid $b_j \sim F_j$ for a price-p item, his willing-to-pay probability is $\Pr[b_j \geq p]$ rather than $\Pr[b_j > p]$. We also define the inverse CDF $F_j^{-1}(y) \stackrel{\text{def}}{=} \inf\{x \in \mathbb{R}_{\geq 0} : F_j(x) \geq y\}$ for any $y \in [0, 1]$; notice that possibly $F_j^{-1}(1) = \infty$. We say a distribution F_j *stochastically dominates* another $\overline{F}_j$, when $F_j(x) \leq \overline{F}_j(x)$ for all $x \in \mathbb{R}_{\geq 0}$. Further, an instance $\mathbf{F} = \{F_j\}_{j\in[n]}$ dominates another instance $\overline{\mathbf{F}} = \{\overline{F}_j\}_{j\in[n]}$, when F_j dominates $\overline{F}_j$ for each $j \in [n]$.

For a CDF F_j, we are also interested in two associated parameters (v_j, q_j). The *monopoly quantile* $q_j \in [0, 1]$ and the *monopoly price* $v_j \in \mathbb{R}_{\geq 0}$ are respectively given by

$$q_j \stackrel{\text{def}}{=} \arg\max_{q\in[0,1]}\{F_j^{-1}(1-q) \cdot q\} \qquad \text{and} \qquad v_j \stackrel{\text{def}}{=} F_j^{-1}(1 - q_j).$$

If there are multiple maximizers q_j, we would choose the smallest q_j among the alternatives; notice that possibly $q_j = 0$ and $v_j = \infty$.

Sampling a bid profile from the instance $\mathbf{b} = (b_j)_{j\in[n]} \sim \mathbf{F}$, the i-th highest bids (for $i \in [n]$) $b_{(1)} \geq \cdots \geq b_{(i)} \geq \cdots \geq b_{(n)}$ will be of particular interest. We denote by D_i the corresponding distributions/CDF's, namely $D_i(x) = \Pr[b_{(i)} < x]$ for all $x \in \mathbb{R}_{\geq 0}$. Again, we assume $\{D_i\}_{i\in[n]}$ to be left-continuous. The formulas for the i-th highest CDF's are given below.



(a) The CDF of a regular F_j

(b) The CDF of $\text{TRI}(v_j, q_j)$

Figure 3: Demonstration for the regular distribution and the triangle distribution.

Fact 1 (Order Statistics)**.** *For each $i \in [n+1]$, the i-th highest CDF is given by*

$$\begin{aligned} D_i(x) &= \sum_{t \in [0:i-1]} \sum_{|W|=t} \Big(\prod_{j \notin W} \Pr[b_j < x] \Big) \cdot \Big(\prod_{j \in W} \Pr[b_j \geq x] \Big) \\ &= \sum_{t \in [0:i-1]} \sum_{|W|=t} \Big(\prod_{j \notin W} F_j(x) \Big) \cdot \Big(\prod_{j \in W} (1 - F_j(x)) \Big), \qquad \forall x \geq 0. \end{aligned}$$

Regular distribution. Denote by REG this distribution family. According to [Mye81], a distribution is regular $F_j \in \text{REG}$ if and only if the *virtual value function* $\varphi_j(x) \stackrel{\text{def}}{=} x - \frac{1-F_j(x)}{f_j(x)}$ is non-decreasing on the support of F_j, where f_j is the probability density function (PDF). Such a regular CDF F_j is illustrated in Figure 3a.

Triangle distribution. This distribution family, denoted by TRI, is introduced in [AHN+19] and is a subset of the regular distribution family REG. Such a distribution $\text{TRI}(v_j, q_j)$ is determined by the monopoly price $v_j \in \mathbb{R}_{\geq 0}$ and the monopoly quantile $q_j \in [0, 1]$. In precise, the corresponding CDF is given below and is illustrated in Figure 3b.

$$F_j(x) \stackrel{\text{def}}{=} \begin{cases} \frac{(1-q_j) \cdot x}{(1-q_j) \cdot x + v_j q_j}, & \forall x \in [0, v_j] \\ 1, & \forall x \in (v_j, \infty) \end{cases}.$$

2.2 Mechanisms

We focus on such a revenue maximization scenario: the seller has $k \in \mathbb{N}_{\geq 1}$ *homogeneous* items and faces $n \geq k$ *unit-demand* buyers, and the buyers draw their bids $\mathbf{b} = \{b_j\}_{j \in [n]} \sim \mathbf{F}$ independently from a publicly known product distribution $\mathbf{F} = \{F_j\}_{j \in [n]}$. For convenience, we interchange buyer/bidder.

In the bulk of the work, we will concern three mechanisms: Anonymous Pricing, Anonymous Reserve, and Ex-Ante Relaxation. Below we briefly introduce these mechanisms; for more details, the reader can refer to [Har13, Chapter 4].

Anonymous Pricing. In such a mechanism, the seller posts an *a priori* price $p \in \mathbb{R}_{\geq 0}$ to any single item; then in an arbitrary coming order, each of the first k coming buyers that are willing to pay the price $p \in \mathbb{R}_{\geq 0}$, will get an item by paying this price. Given any bid profile $\mathbf{b} \sim \mathbf{F}$, let $b_{(n+1)} \stackrel{\text{def}}{=} 0$ and reorder the bids such that $b_{(1)} \geq \cdots \geq b_{(i)} \geq \cdots \geq b_{(n+1)}$.

Depending on how many bids exceed the posted price, the mechanism gives a revenue of

$$\text{REV}(\mathsf{AP}) = \sum_{i \in [k]} i \cdot p \cdot \mathbb{1}\{b_{(i)} \geq p > b_{(i+1)}\} + k \cdot p \cdot \mathbb{1}\{b_{(k+1)} \geq p\}$$

$$= \sum_{i \in [k]} p \cdot \mathbb{1}\{b_{(i)} \geq p\}.$$

Taking the randomness over $\mathbf{b} \sim \mathbf{F}$ into account results in the expected revenue.

Fact 2 (Revenue Formula for Anonymous Pricing)**.** *Under any posted price $p \in \mathbb{R}_{\geq 0}$, the* Anonymous Pricing *mechanism extracts an expected revenue of*

$$\mathsf{AP}(p, \mathbf{F}) \stackrel{def}{=} p \cdot \sum_{i \in [k]} (1 - D_i(p)).$$

Let $\mathsf{AP}(\mathbf{F}) \stackrel{def}{=} \max_{p \in \mathbb{R}_{\geq 0}} \{\mathsf{AP}(p, \mathbf{F})\}$ *denote the optimal* Anonymous Pricing *revenue.*

Anonymous Reserve. In such a mechanism, the seller sets an *a priori* reserve $r \in \mathbb{R}_{\geq 0}$ on any single item. When at most k bidders are willing to pay the reserve $r \in \mathbb{R}_{\geq 0}$, Anonymous Reserve has the same allocation/payment rule as Anonymous Pricing, thus the same revenue. But when at least $(k+1)$ bidders are willing to pay this reserve, each of the top-k bidders (with an arbitrary tie-breaking rule) wins an item by paying the $(k+1)$-th highest bid $b_{(k+1)} \geq r$.

Running on a specific bid profile $\mathbf{b} \sim \mathbf{F}$, the mechanism generates a revenue of

$$\begin{aligned} \textsc{Rev}(\mathsf{AR}) &= \sum_{i \in [k]} i \cdot r \cdot \mathbb{1}\{b_{(i)} \geq r > b_{(i+1)}\} \;+\; k \cdot b_{(k+1)} \cdot \mathbb{1}\{b_{(k+1)} \geq r\} \\ &= \sum_{i \in [k]} r \cdot \mathbb{1}\{b_{(i)} \geq r\} \;+\; k \cdot |b_{(k+1)} - r|_+. \end{aligned}$$

Taking the randomness over $\mathbf{b} \sim \mathbf{F}$ into account gives the expected revenue. (Note that [CGM15, Fact 1] get the revenue formula below in the single-item case $k = 1$.)

Fact 3 (Revenue Formula for Anonymous Reserve [CGM15, Fact 1])**.** *Under any reserve $r \in \mathbb{R}_{\geq 0}$, the* Anonymous Reserve *mechanism extracts an expected revenue of*

$$\mathsf{AR}(r, \mathbf{F}) \stackrel{def}{=} r \cdot \sum_{i \in [k]} (1 - D_i(r)) \;+\; k \cdot \int_r^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x.$$

Let $\mathsf{AR}(\mathbf{F}) \stackrel{def}{=} \max_{r \in \mathbb{R}_{\geq 0}} \{\mathsf{AR}(r, \mathbf{F})\}$ *denote the optimal* Anonymous Reserve *revenue.*

Sequential Posted Pricing. In such a mechanism, the seller sets an ordering $\sigma : [n] \mapsto [n]$ and *a priori* prices $\{p_j\}_{j \in [n]}$. The buyers come *sequentially* $\sigma(1), \cdots, \sigma(n)$, and each of the first k coming buyers that are willing to pay individual prices $b_{\sigma(j)} \geq p_j$ gets an item and pays p_j.

Myerson Auction. This mechanism ranks the buyers $\varphi_{(1)}(b_{(1)}) \geq \varphi_{(2)}(b_{(2)}) \geq \cdots \varphi_{(n)}(b_{(n)})$ in decreasing order of virtual values and allocates the items to the top-k buyers that have *nonnegative* virtual values $\varphi_{(j)}(b_{(j)}) \geq 0$.

Ex-Ante Relaxation. This notion is introduced by [CHMS10]. Although just being a "fake" mechanism,[6] Ex-Ante Relaxation is useful to upper bound the revenue from the optimal truthful mechanism, Myerson Auction.

For a regular instance, an Ex-Ante Relaxation mechanism is specified by an allocation rule $\mathbf{q}' = \{q'_j\}_{j \in [n]} \in [0, 1]^n$. Here, each $q_j \in [0, 1]$ represents the probability that the buyer $j \in [n]$ wins an item. This allocation rule is feasible iff $\sum_{j \in [n]} q'_j \leq k$, because we only have k items. The following fact characterizes the resulting "revenue".

[6] Namely, in the concerning Bayesian mechanism design setting, Ex-Ante Relaxation is unimplementable.

Fact 4 (Revenue Formula for Ex-Ante Relaxation [CHMS10, Lemma 2]). *Given a regular instance $\mathbf{F} = \{F_j\}_{j\in[n]}$, under any feasible allocation rule $\mathbf{q}' = \{q'_j\}_{j\in[n]} \in [0,1]^n$ that $\sum_{j\in[n]} q'_j \le k$, the* Ex-Ante Relaxation *mechanism extracts an expected revenue of*

$$\mathsf{EAR}(\mathbf{q}', \mathbf{F}) \stackrel{def}{=} \sum_{j\in[n]} F_j^{-1}(1 - q'_j) \cdot q'_j$$

Remark 1. We will study the Ex-Ante Relaxation mechanism just for the regular instances. The revenue formulas for the irregular instances are more complicated, for which the reader can refer to [CHMS10, Lemma 2].

Revenue monotonicity. Based on the revenue formulas given in Facts 2 to 4, one can easily check the following fact (a.k.a. the revenue monotonicity in the literature).

Fact 5 (Revenue Monotonicity). *Given that an instance $\mathbf{F} = \{F_j\}_{j\in[n]}$ stochastically dominates another instance $\overline{\mathbf{F}} = \{\overline{F}_j\}_{j\in[n]}$, the following hold:*

1. $\mathsf{AP}(p, \mathbf{F}) \ge \mathsf{AP}(p, \overline{\mathbf{F}})$ *for any posted price $p \in \mathbb{R}_{\ge 0}$, and thus* $\mathsf{AP}(\mathbf{F}) \ge \mathsf{AP}(\overline{\mathbf{F}})$.
2. $\mathsf{AR}(r, \mathbf{F}) \ge \mathsf{AR}(r, \overline{\mathbf{F}})$ *for any reserve $r \in \mathbb{R}_{\ge 0}$, and thus* $\mathsf{AR}(\mathbf{F}) \ge \mathsf{AR}(\overline{\mathbf{F}})$.
3. $\mathsf{EAR}(\mathbf{q}', \mathbf{F}) \ge \mathsf{EAR}(\mathbf{q}', \overline{\mathbf{F}})$ *for any allocation $\mathbf{q}' = \{q'_j\}_{j\in[n]} \in [0,1]^n$ with $\sum_{j\in[n]} q'_j \le k$.*

3 Anonymous Reserve vs. Anonymous Pricing

In this section, we investigate the Anonymous Reserve vs. Anonymous Pricing problem. Based on the revenue formulas (see Section 2.2), the revenue gap between both mechanisms is characterized by the following mathematical program.

$$\begin{aligned}
\sup \quad & \mathsf{AR}(r, \mathbf{F}) = r \cdot \sum_{i\in[k]} (1 - D_i(r)) + k \cdot \int_r^\infty (1 - D_{k+1}(x)) \cdot \mathrm{d}x, && \forall r \in \mathbb{R}_{\ge 0}, && \text{(P1)} \\
\text{s.t.} \quad & \mathsf{AP}(p, \mathbf{F}) = p \cdot \sum_{i\in[k]} (1 - D_i(p)) \le 1, && \forall p \in \mathbb{R}_{\ge 0}, && \text{(C1)} \\
& \mathbf{F} = \{F_j\}_{j\in[n]}, && \forall n \in \mathbb{N}_{\ge 1}.
\end{aligned}$$

By finding the optimal solution to Program (P1), we will prove the next theorem.

Theorem 3 (AR vs. AP). *Given that the seller has $k \in \mathbb{N}_{\ge 1}$ homogeneous items and faces $n \ge k$ independent unit-demand buyers, the revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k, n)$ between* Anonymous Reserve *and* Anonymous Pricing *satisfies the following:*

1. *The revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k, n)$ is maximized when all the buyers have the same bid distribution $\{F^*\}^n$, and their common CDF F^* is an implicit function given by $F^*(x) = 0$ for all $x \in [0, \frac{1}{k}]$ and $\mathsf{AP}(x, \{F^*\}^n) = 1$ for all $x \in (\frac{1}{k}, \infty)$.*
2. *Over all $n \ge k$, the supremum revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k) \stackrel{def}{=} \sup_{n\ge k} \mathfrak{R}_{\mathsf{AR/AP}}(k, n)$ is achieved by*
$$\mathfrak{R}_{\mathsf{AR/AP}}(k, \infty) = 1 + k \cdot \int_0^\infty \frac{T_k(x) \cdot (1 - T_{k+1}(x))}{(k - \sum_{i\in[k]} T_i(x))^2} \cdot \mathrm{d}x,$$
where the functions $T_i(x) \stackrel{def}{=} e^{-x} \cdot \sum_{t\in[0:i-1]} \frac{1}{t!} \cdot x^t$ for all $i \in [k+1]$.
3. *For each $k \in \mathbb{N}_{\ge 1}$, the supremum revenue gap is bounded between*
$$1 + \frac{0.1}{\sqrt{k}} \le \mathfrak{R}_{\mathsf{AR/AP}}(k) \le 1 + \frac{2}{\sqrt{k}}.$$

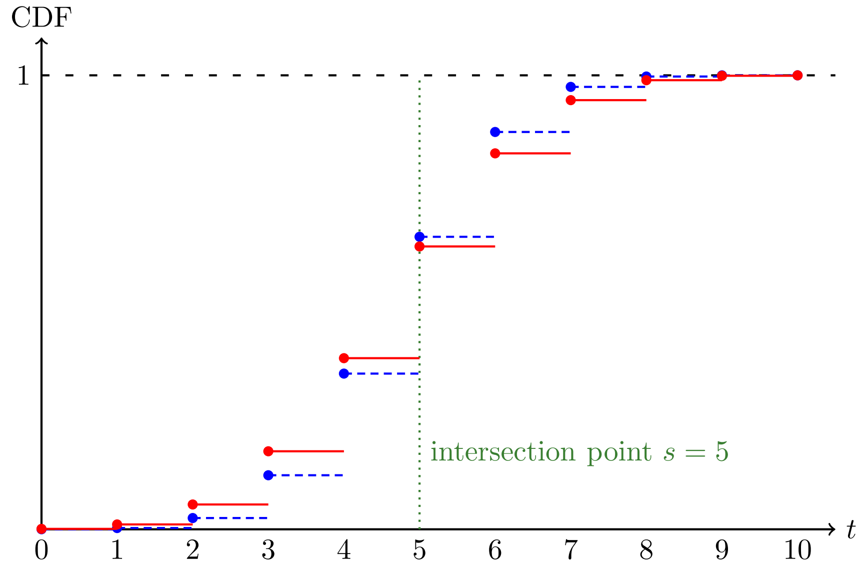


Figure 4: Demonstration for Theorem 4 when $n = 10$. Note that "full line $\geq$ dashed line" when $t < 5$ and "full line $\leq$ dashed line" when $t \geq 5$, where the full line refers to the sum of i.i.d. Bernoulli variables, and the dashed line refers to the sum of independent yet not necessarily identical Bernoulli variables.

4. *For each $k \in \mathbb{N}_{\geq 1}$, the ratio $\mathfrak{R}_{\mathsf{AR/AP}}(k)$ is tight not only in the asymmetric general setting, but (more restrictedly) also in the asymmetric regular setting and/or the i.i.d. general setting.*

We first outline our approach towards Theorem 3. Central to the upper-bound analysis is a basic result about the sum of independent Bernoulli random variables, which is formalized below as Theorem 4 and can be of independent interest (see Figure 4 for a demonstration). We will prove this theorem in Section 3.1.

Theorem 4 (Bernoulli Sum Lemma)**.** *For two arrays of Bernoulli random variables: $\{X_j\}_{j \in [n]}$ are independent and identically distributed (i.i.d. in short), while $\{Y_j\}_{j \in [n]}$ independent yet not necessarily identically distributed. For the random sums $X = \sum_{j \in [n]} X_j$ and $Y = \sum_{j \in [n]} Y_j$, there exists some threshold $s \in \mathbb{R}$ such that:*

1. $\Pr[X \leq t] \geq \Pr[Y \leq t]$ *for any $t < s$.*
2. $\Pr[X \leq t] \leq \Pr[Y \leq t]$ *for any $t \geq s$.*

Based on Theorem 4 together with further optimization arguments, we acquire Part 1 and Part 2 of Theorem 3 respectively in Sections 3.2 and 3.3. Part 3 of Theorem 3 requires some advanced tools from real analysis; its proof is technically involved and is deferred to Appendix A. (Notice that Parts 1 to 3 require no distributional assumption.) Eventually, we construct two matching lower-bound examples in Section 3.4, one in the i.i.d. general setting and one in the asymmetric regular setting, hence Part 4 of Theorem 3.

All the above results concern the Anonymous Reserve vs. Anonymous Pricing problem under a cardinality constraint, namely up to $k \in \mathbb{N}_{\geq 1}$ buyers can simultaneously win. In the literature on mechanism design, many works also consider the more general constraint that the winning buyers satisfy a *matroid* constraint (e.g. see [HR09]). For this setting, we will show in Section 3.5 an lower bound $\Omega(\log k)$ for the counterpart revenue gap.

3.1 Bernoulli sum lemma

The following well known result, that any independent Bernoulli sum is a *log-concave* random variable (e.g. see [JKM13]), is crucial to our proof of Theorem 4.

Fact 6 (Log-Concavity of Bernoulli Sum)**.** *Let $Z = \sum_{j\in[n]} Z_j$ be the sum of $n \in \mathbb{N}_{\geq 1}$ independent yet not necessarily identical Bernoulli random variables, then for any integer $t \in \mathbb{Z}$,*

$$\Pr[Z = t]^2 \;\geq\; \Pr[Z = t-1] \cdot \Pr[Z = t+1].$$

For ease of presentation, we will only justify Part 1 of Theorem 4, and Part 2 follows from similar arguments. Further, it suffices to consider the case that $\Pr[X \leq s] = \Pr[Y \leq s]$ (i.e. the other case that $\Pr[X \leq s] > \Pr[Y \leq s]$ can be accommodated by properly scaling the failure probability of the i.i.d. random variables $\{X_j\}_{j\in[n]}$). Further, since we concern about Bernoulli random variables and their sums, we safely assume $t \leq s$ to be integers between $[0 : n]$. We obtain Part 1 of Theorem 4 in two steps:

- Local transformation. In Lemma 1 and Corollary 1, we will pick a pair of non-identically distributed variables Y_{j_1} and Y_{j_2} (for some $j_1 \neq j_2 \in [n]$), and replace them by another pair of i.i.d. variables $\overline{Y}_{j_1}$ and $\overline{Y}_{j_2}$. The new pair is carefully constructed, so as to ensure certain properties.
- Global transformation. We conduct the local transformation on the variables $\{Y_j\}_{j\in[n]}$ round by round (in a nontrivial way), which preserves the mentioned properties by induction. Together with extra arguments from real analysis (see Claim 1), these properties will lead to Part 1 of Theorem 4.

Lemma 1 (Averaging Two Variables.)**.** *Assume w.l.o.g. that two variables Y_{j_1} and Y_{j_2} given in Theorem 4 (for some $j_1 \neq j_2 \in [n]$) are not identically distributed, then there exists another pair of i.i.d. Bernoulli random variables $\overline{Y}_{j_1}$ and $\overline{Y}_{j_2}$ such that:*

1. $\Pr[\overline{Y}_{j_1} + \overline{Y}_{j_2} + \sum_{j\notin\{j_1,j_2\}} Y_j \leq s] = \Pr[Y \leq s]$.
2. $\Pr[\overline{Y}_{j_1} + \overline{Y}_{j_2} + \sum_{j\notin\{j_1,j_2\}} Y_j \leq t] \geq \Pr[Y \leq t]$ *for any $t \in [0 : s-1]$.*

Proof of Lemma 1. For simplicity, we reindex the variables $\{Y_j\}_{j\in[n]}$ such that $j_1 = 1$ and $j_2 = 2$. We adopt the following notations:

- Let $q_j \stackrel{\text{def}}{=} \Pr[Y_j = 0] \in [0,1]$ for all $j \in [n]$ and $\overline{q} \stackrel{\text{def}}{=} \Pr[\overline{Y}_1 = 0] = \Pr[\overline{Y}_2 = 0] \in [0,1]$. W.l.o.g. we have $q_1 < q_2$, given that Y_1 and Y_2 are not identically distributed.
- Let $Y' \stackrel{\text{def}}{=} \sum_{j\in[3:n]} Y_j$ and $a_i \stackrel{\text{def}}{=} \Pr[Y' = i]$ for all $i \in \mathbb{Z}$. Because Y' is the sum of $(n-2)$ Bernoulli random variables, $a_i \neq 0$ only if $i \in [0 : n-2]$.

It follows from Fact 6 that $a_t^2 \geq a_{t-1} \cdot a_{t+1}$ for all $t \in \mathbb{Z}$. By induction, one can easily see that

$$a_t \cdot a_{s-1} \;\geq\; a_{t-1} \cdot a_s \tag{1}$$

for all integers $t \leq s \in \mathbb{Z}$, which is more convenient for our later use.

For any integer $i \in \mathbb{Z}$, by considering all of the four possibilities $(Y_1, Y_2) \in \{0,1\}^2$, we can reformulate the probability $\Pr[Y = i] \in [0,1]$ as follows:

$$\begin{aligned}
\Pr[Y = i] \;&=\; \Pr[Y_1 + Y_2 + Y' = i] \\
&=\; \Pr[Y' = i] \cdot \Pr[Y_1 = Y_2 = 0] \;+\; \Pr[Y' = i-2] \cdot \Pr[Y_1 = Y_2 = 1] \\
&\quad\;+\; \Pr[Y' = i-1] \cdot \Pr[Y_1 = 0, Y_2 = 1] \;+\; \Pr[Y' = i-1] \cdot \Pr[Y_1 = 1, = Y_2 = 0] \\
&=\; a_i \cdot q_1 \cdot q_2 \;+\; a_{i-2} \cdot (1-q_1) \cdot (1-q_2)
\end{aligned}$$

$$\begin{aligned}&+\ a_{i-1}\cdot q_1\cdot(1-q_2)\ +\ a_{i-1}\cdot(1-q_1)\cdot q_2\\ =\ &((a_i-a_{i-1})-(a_{i-1}-a_{i-2}))\cdot q_1\cdot q_2\ +\ (a_{i-1}-a_{i-2})\cdot(q_1+q_2)\ +\ a_{i-2}.\end{aligned}$$

Thus, we can rewrite the telescoping sum $\Pr[Y \le t] = \sum_{i\in[0:t]} \Pr[Y = i]$ as follows:

$$\begin{aligned}\Pr[Y\le t] &= ((a_t-a_{t-1})-(a_{-1}-a_{-2}))\cdot q_1\cdot q_2\ +\ (a_{t-1}-a_{-2})\cdot(q_1+q_2)\ +\ \sum_{i\in[0:t]} a_{i-2}\\ &= (a_t-a_{t-1})\cdot q_1\cdot q_2\ +\ a_{t-1}\cdot(q_1+q_2)\ +\ \mathrm{const}, \qquad (2)\end{aligned}$$

where the last step follows because $a_{-1} = a_{-2} = 0$ (recall that $a_i \ne 0$ only if $i \in [0 : n-2]$) and we denote const $\stackrel{\text{def}}{=} \sum_{i\in[0:t-2]} a_i = \Pr[Y' \le t-2]$.

We emphasize that Equation (2) is a *multilinear* function of $\{q_j\}_{j\in[n]} \in [0,1]^n$, and the last summand const is irrelevant to both $q_1 = \Pr[Y_1 = 0]$ and $q_2 = \Pr[Y_1 = 0]$. That is, suppose that q_2 and $\{q_j\}_{j=3}^n$ are held constant, we can regard Equation (2) as a linear function of $q_1 \in [0,1]$. Further, the corresponding slope

$$(a_t-a_{t-1})\cdot q_2+a_{t-1}\ =\ a_t\cdot q_2+a_{t-1}\cdot(1-q_2)$$

must be *non-negative*, because the probabilities $q_2, a_t, a_{t-1} \in [0,1]$. Similarly, when we regard Equation (2) as a univariate of $q_2 \in [0,1]$, this is also a non-decreasing linear function.

Following the above arguments but considering $\overline{q} \in [0,1]$ in place of q_1 and q_2, we also have

$$\Pr[\overline{Y}_1+\overline{Y}_2+Y'\le t]\ =\ (a_t-a_{t-1})\cdot\overline{q}^2\ +\ a_{t-1}\cdot 2\overline{q}\ +\ \mathrm{const}, \qquad (3)$$

Again, Equation (3) is a non-decreasing function in $\overline{q} \in [0,1]$. Given the monotonicity of Equations (2) and (3) and since $q_1 < q_2$, we can easily check that

$$\begin{aligned}\text{Equation (3)}|_{\overline{q}=q_1}\ &\le\ \text{Equation (2)},\\ \text{Equation (3)}|_{\overline{q}=q_2}\ &\ge\ \text{Equation (2)}.\end{aligned}$$

In the case that $t = s$, the equality $\Pr[\overline{Y}_1 + \overline{Y}_2 + Y' \le s] = \Pr[Y \le s]$ holds for at least one $\overline{q} \in [q_1, q_2] \subseteq [0,1]$, due to the intermediate value theorem. That is,

$$(a_s-a_{s-1})\cdot\overline{q}^2\ +\ a_{s-1}\cdot 2\overline{q}\ =\ (a_s-a_{s-1})\cdot q_1\cdot q_2\ +\ a_{s-1}\cdot(q_1+q_2). \qquad (4)$$

This accomplishes Part 1 of Lemma 1. We next show that the above particular $\overline{q} \in [q_1, q_2]$ (for which $\Pr[\overline{Y}_1 + \overline{Y}_2 + Y' \le s] = \Pr[Y \le s]$) guarantees Part 2:

$$\Pr[\overline{Y}_1+\overline{Y}_2+Y'\le t]\ \ge\ \Pr[Y\le t],$$

for all integers $t \in [0 : s-1]$. The proof is based on case analysis.

Case I ($a_s = a_{s-1}$). Based on Equation (1), i.e., $a_t \cdot a_{s-1} \ge a_{t-1} \cdot a_s$, we easily infer $a_t \ge a_{t-1}$ (note that $a_{t-1}, a_t, a_{s-1}, a_s \in [0,1]$ are probabilities). Moreover, Equation (4) degenerates into $a_{s-1} \cdot 2\overline{q} = a_{s-1} \cdot (q_1 + q_2)$, by which we can safely choose $\overline{q} = \frac{1}{2} \cdot (q_1 + q_2)$. (Particularly, when $a_{s-1} = 0$, the probabilities $\Pr[\overline{Y}_1 + \overline{Y}_2 + Y' \le s] = \Pr[Y \le s] =$ const depend not on $\overline{q} \in [0,1]$. That is, $\overline{q} \in [0,1]$ can be arbitrary, and we just choose $\overline{q} = \frac{1}{2} \cdot (q_1 + q_2)$.) As a consequence,

$$\begin{aligned}\Pr[\overline{Y}_1+\overline{Y}_2+Y'\le t]-\Pr[Y\le t]\ &=\ (a_t-a_{t-1})\cdot(\overline{q}^2-q_1\cdot q_2)\\ &=\ \frac{1}{4}\cdot(a_t-a_{t-1})\cdot(q_1-q_2)^2\\ &\ge\ 0,\end{aligned}$$

where the last step follows because $a_t \ge a_{t-1}$.

Case II ($a_s \neq a_{s-1}$). In this case, we can reformulate Equation (2) as follows:

$$\begin{aligned}
\Pr[Y \le t] &= \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot \left((a_s - a_{s-1}) \cdot q_1 \cdot q_2 + a_{s-1} \cdot (q_1 + q_2)\right) \\
&\quad + \left(a_{t-1} - \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}\right) \cdot (q_1 + q_2) \ + \ \text{const} \\
&= \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot \left((a_s - a_{s-1}) \cdot \overline{q}^2 + a_{s-1} \cdot 2\overline{q}\right) \\
&\quad + \left(a_{t-1} - \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}\right) \cdot (q_1 + q_2) \ + \ \text{const} \\
&= \Pr[\overline{Y}_1 + \overline{Y}_2 + Y' \le t] - \left(a_{t-1} - \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}\right) \cdot 2\overline{q} \\
&\quad + \left(a_{t-1} - \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}\right) \cdot (q_1 + q_2) \\
&= \Pr[\overline{Y}_1 + \overline{Y}_2 + Y' \le t] - \underbrace{\left(a_{t-1} - \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}\right) \cdot (2\overline{q} - q_1 - q_2)}_{\heartsuit},
\end{aligned}$$

where the first step rearranges Equation (2); the second step applies Equation (4) to the first summand; the third step applies Equation (3); and the last step is by elementary calculation.

To accomplish the lemma for a certain $t \in [0 : s-1]$, it remains to show $\heartsuit \ge 0$.

Case II.A ($a_s > a_{s-1}$). To see $\heartsuit \ge 0$ in this case, it suffices to show the following:

$$a_{t-1} \ \le \ \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}, \tag{5}$$

$$\overline{q} \ \le \ \frac{1}{2} \cdot (q_1 + q_2). \tag{6}$$

Because $a_s > a_{s-1}$, Equation (5) is equivalent to $a_{t-1} \cdot a_s \le a_t \cdot a_{s-1}$, i.e. what we have shown in Equation (1). And for Equation (6), assume on the opposite that $\overline{q} > \frac{1}{2} \cdot (q_1 + q_2)$, then

$$\begin{aligned}
\text{LHS of (4)} &= (a_s - a_{s-1}) \cdot \overline{q}^2 \ + \ a_{s-1} \cdot 2\overline{q} \\
&\ge (a_s - a_{s-1}) \cdot \overline{q}^2 \ + \ a_{s-1} \cdot (q_1 + q_2) \\
&> (a_s - a_{s-1}) \cdot q_1 \cdot q_2 \ + \ a_{s-1} \cdot (q_1 + q_2) \\
&= \text{RHS of (4)},
\end{aligned}$$

where the third step is *strict* because $a_s > a_{s-1}$ and $\overline{q} > \frac{1}{2} \cdot (q_1 + q_2) \ge \sqrt{q_1 \cdot q_2}$. This gives a contradiction. By refuting our assumption, we confirm Equation (6) and thus $\heartsuit \ge 0$.

Case II.B ($a_s < a_{s-1}$). Via similar arguments as in Case II.A, we have $a_{t-1} \ge \frac{a_t - a_{t-1}}{a_s - a_{s-1}} \cdot a_{s-1}$ and $\overline{q} \ge \frac{1}{2} \cdot (q_1 + q_2)$. Given these, we also have $\heartsuit \ge 0$.

Putting all the cases together accomplishes Lemma 1. $\square$

In the remainder of Section 3.1, we continue to adopt the notations given in the proof of Lemma 1, and introduce two more notations:

- Define Ξ_{j_1, j_2} as the operator specified by Lemma 1, i.e., replacing a pair of non-identical failure probabilities $q_{j_1} \neq q_{j_2}$ by another pair of identical probabilities $\{\overline{q}\}^2$.
- Define the operation composition $\Xi(\mathbf{q}) \stackrel{\text{def}}{=} \Xi_{1,n} \circ \cdots \circ \Xi_{1,3} \circ \Xi_{1,2}(\mathbf{q})$, i.e. modifying (q_1, q_2) first, then the new $(\overline{q}_1, \overline{q}_3)$ second, and so on. Note that both Ξ_{j_1, j_2} and Ξ are continuous mappings from $[0,1]^n$ to $[0,1]^n$.

Accessing the proof of Lemma 1, we can easily conclude the following corollary.

Input: failure probabilities $\mathbf{q} = \{q_j\}_{j\in[n]} \in [0,1]^n$ of $\{Y_j\}_{j\in[n]}$.

1. Initialize $\mathbf{q}^{(0)} \leftarrow \mathbf{q}$.
2. **for** $\tau \in \mathbb{N}_{\geq 1}$ **do**
3. Reindex $\mathbf{q}^{(\tau-1)}$ so that $q_1^{(\tau-1)} \leq q_2^{(\tau-1)} \leq \cdots \leq q_n^{(\tau-1)}$.
4. Update $\mathbf{q}^{(\tau)} \leftarrow \Xi(\mathbf{q}^{(\tau-1)})$.
5. **end for**

Figure 5: An algorithm for Part 1 of Theorem 4.

Corollary 1 (Averaging Two Variables.). *Given that $q_{j_1} \neq q_{j_2}$, the operation Ξ_{j_1,j_2} specified by Lemma 1 guarantees the strict inequalities* $\min\{q_{j_1}, q_{j_2}\} < \overline{q} < \max\{q_{j_1}, q_{j_2}\}$.

We are ready to prove Part 1 of Theorem 4, namely the existence of a desired array of i.i.d. Bernoulli random variables $\{X_j\}_{j\in[n]}$.

Claim 1 (Part 1 of Theorem 4)**.** $\Pr[X \leq t] \geq \Pr[Y \leq t]$ *for any* $t < s$.

Proof of Claim 1. Indeed, when not all the failure probabilities $\mathbf{q} = \{q_j\}_{j\in[n]} \in [0,1]^n$ of the given variables $\{Y_j\}_{j\in[n]}$ are the same, we can infer from Lemma 1 and Corollary 1 an iterative algorithm, that computes the common failure probability $q^* = \Pr[X_j = 0] \in [0,1]$ of the identically distributed variables $\{X_j\}_{j\in[n]}$. This algorithm is shown in Figure 5.

In a specific round $\tau \in \mathbb{N}_{\geq 1}$, because the interim probabilities $\mathbf{q}^{(\tau-1)} = \{q_j^{(\tau-1)}\}_{j\in[n]}$ are reindexed in increasing order, we can infer from Corollary 1 (together with the definition of the operation composition $\Xi : [0,1]^n \mapsto [0,1]^n$) that

$$\min_{j\in[n]} q_j^{(\tau)} \;\geq\; \min_{j\in[n]} q_j^{(\tau-1)} \qquad\qquad \text{and} \qquad\qquad \max_{j\in[n]} q_j^{(\tau)} \;\leq\; \max_{j\in[n]} q_j^{(\tau-1)}.$$

In particular, the second inequality above is strict, as long as not all the interim probabilities $\mathbf{q}^{(\tau-1)} = \{q_j^{(\tau-1)}\}_{j\in[n]}$ are identical.

We consider the distance $\ell(\mathbf{q}) \stackrel{\text{def}}{=} \max_{j_1,j_2\in[n]}(q_{j_2} - q_{j_1}) \geq 0$; notice that this is a continuous function from $[0,1]^n$ to $[0,1]$. The above arguments ensure that in each round $\tau \in \mathbb{N}_{\geq 1}$,

$$\ell(\mathbf{q}^{(\tau)}) \;=\; \ell(\Xi(\mathbf{q}^{(\tau-1)})) \;\leq\; \ell(\mathbf{q}^{(\tau-1)}), \tag{7}$$

where the inequality is strictly as long as not all $\{q_j^{(\tau-1)}\}_{j\in[n]}$ are identical. Due to the squeeze theorem, the sequence $\{\mathbf{q}^{(\tau)}\}_{\tau=1}^{\infty}$ converges to some limit $\mathbf{q}^* = \lim_{\tau\to\infty} \mathbf{q}^{(\tau)} \in [0,1]^n$. Further, since both ℓ and Ξ are continuous functions, we deduce that

$$\ell(\mathbf{q}^*) \;=\; \lim_{\tau\to\infty} \ell(\mathbf{q}^{(\tau)}) \;=\; \lim_{\tau\to\infty} \ell(\Xi(\mathbf{q}^{(\tau-1)})) \;=\; \ell(\Xi(\lim_{\tau\to\infty} \mathbf{q}^{(\tau-1)})) \;=\; \ell(\Xi(\mathbf{q}^*)).$$

As a result, it follows from Equation (7) that all coordinates of $\mathbf{q}^*$ must be the same, namely $\mathbf{q}^* = \{q^*\}^n$ for some common failure probability $q^* \in [0,1]$ of the i.i.d. $\{X_j\}_{j\in[n]}$.

We conclude with the existence of the desired i.i.d. Bernoulli random variables $\{X_j\}_{j\in[n]}$ and the sum $X = \sum_{j\in[n]} X_j$. In particular, applying Lemma 1 over all rounds $\tau \in \mathbb{N}_{\geq 1}$ gives

$$\Pr[X \leq t] \;\geq\; \Pr[Y \leq t], \qquad\qquad \forall t \in [0:s].$$

This completes the proof of Claim 1. $\square$

3.2 Worst-case instance

This section is to certify Part 1 of Theorem 3. Concretely, for any given $n \ge k \ge 1$, the worst-case instance $\mathbf{F} = \{F_j\}_{j\in[n]}$ of Program (P1) is achieved when the distributions $\{F_j\}_{j\in[n]}$ are identical and, for any posted price $p \ge 0$, make the most of the Anonymous Pricing revenue. We formalize this statement as the following claim.

Claim 2 (Part 1 of Theorem 3)**.** *The revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k,n)$ is maximized when all the buyers have the same bid distribution $\{F^*\}^n$, and their common CDF F^* is an implicit function given by $F^*(x) = 0$ for all $x \in [0, \frac{1}{k}]$ and $\mathsf{AP}(x, \{F^*\}^n) = 1$ for all $x \in (\frac{1}{k}, \infty)$.*

Proof of Claim 2. Recall Program (P1), an implicit constraint is that the input must be *CDF's*, i.e. each $F_j : \mathbb{R}_{\ge 0} \mapsto [0,1]$ is a non-decreasing mapping with $F_j(0) = 0$ and $F_j(\infty) = 1$. We relax this constraint and consider all mappings from the domain $\mathbb{R}_{\ge 0}$ to the codomain $[0,1]$. In fact, the lemma holds even under this relaxation.

For simplicity, we still denote by $\mathbf{F} = \{F_j\}_{j\in[n]}$ the given mappings. Even though $\{F_j\}_{j\in[n]}$ may not be CDF's, we can still write down the corresponding "i-th highest CDF" $D_i(x)$ and the "revenue" formulas $\mathsf{AP}(p,\mathbf{F})$ and $\mathsf{AR}(r,\mathbf{F})$:

$$\begin{aligned}
D_i(x) &= \sum_{t\in[0:i-1]} \sum_{|W|=t} \Big(\prod_{j\notin W} F_j(x)\Big)\cdot\Big(\prod_{j\in W}(1-F_j(x))\Big), && \forall x\ge 0,\ i\in[n+1]; \\
\mathsf{AP}(p,\mathbf{F}) &= p\cdot\sum_{i\in[k]}(1-D_i(p)), && \forall p\ge 0; \qquad (8)\\
\mathsf{AR}(r,\mathbf{F}) &= \mathsf{AP}(r,\mathbf{F}) \;+\; k\cdot\int_r^\infty (1-D_{k+1}(x))\cdot \mathrm{d}x, && \forall r\ge 0.
\end{aligned}$$

Notice that both "revenue" formulas satisfy the monotonicity given in Fact 5. For a bunch of mappings $\{F_j\}_{j\in[n]}$ that are feasible to Program (P1), we consider a two-step reduction:

(i) Pointwise convert $\mathbf{F} = \{F_j\}_{j\in[n]}$, according to Theorem 4, into a bunch of identical mappings $\overline{\mathbf{F}} = \{\overline{F}\}^n$.

(ii) Pointwise scale $\overline{\mathbf{F}} = \{\overline{F}\}^n$ into another bunch of identical mappings $\mathbf{F}^* = \{F^*\}^n$, for which $\mathsf{AP}(p,\mathbf{F}^*) = 1$ for any $p \ge \frac{1}{k}$ and $\mathsf{AP}(p,\mathbf{F}^*) = k\cdot p$ for any $p < \frac{1}{k}$.

Clearly, constraint (C1) holds for $\mathbf{F}^*$. Below we show that for any reserve $r \in \mathbb{R}_{\ge 0}$, the Anonymous Reserve revenue increases, namely $\mathsf{AR}(r,\mathbf{F}^*) \ge \mathsf{AR}(r,\mathbf{F})$.

Given any $x \ge 0$, let us consider the independent Bernoulli random variables $\{Y_j^{(x)}\}_{j\in[n]}$ with the failure probabilities $\Pr[Y_j^{(x)} = 0] \overset{\text{def}}{=} F_j(x)$. We denote their sum $Y^{(x)} \overset{\text{def}}{=} \sum_{j\in[n]} Y_j^{(x)}$. According to Part 1 of Theorem 4, there exists a particular bunch of i.i.d. variables $\{X_j^{(x)}\}_{j\in[n]}$, for which the sum $X^{(x)} \overset{\text{def}}{=} \sum_{j\in[n]} X_j^{(x)}$ satisfies

$$\begin{aligned}
\Pr[X^{(x)} \le k] &= \Pr[Y^{(x)} \le k]; \\
\Pr[X^{(x)} \le i] &\ge \Pr[Y^{(x)} \le i], && \forall i\in[0:k-1].
\end{aligned}$$

For each $i \in [n+1]$, one can easily see that $\Pr[Y^{(x)} \le i-1] = D_i(x)$, and we further denote $\overline{F}(x) \overset{\text{def}}{=} \Pr[X_j^{(x)} = 0]$ and $\overline{D}_i(x) \overset{\text{def}}{=} \Pr[X^{(x)} \le i-1]$. Take all $x \ge 0$ into account, it follows that

$$\begin{aligned}
\overline{D}_{k+1}(x) &= D_{k+1}(x), && \forall x\in\mathbb{R}_{\ge 0}; \\
\overline{D}_i(x) &\ge D_i(x), && \forall x\in\mathbb{R}_{\ge 0}, i\in[k].
\end{aligned}$$

In view of Equation (8), for any price $p \in \mathbb{R}_{\ge 0}$ we have

$$\mathsf{AP}(p,\overline{\mathbf{F}}) \;\le\; \mathsf{AP}(p,\mathbf{F}) \;\le\; \mathsf{AP}(p,\mathbf{F}^*), \qquad (9)$$

where the last inequality holds by the construction of $\mathbf{F}^* = \{F^*\}^n$ given in Step (ii).

Given Equation (9) and since either $\mathbf{F}^* = \{F^*\}^n$ or $\overline{\mathbf{F}} = \{\overline{F}\}^n$ involves *identical* mappings, we can infer from Fact 5 that the scaled common mapping F^* pointwise dominates $\overline{F}$. In terms of the "$(k+1)$-th highest CDF", we have $D^*_{k+1}(x) \leq \overline{D}_{k+1}(x) = D_{k+1}(x)$ for all $x \geq 0$. We thus deduce that for any reserve $r \in \mathbb{R}_{\geq 0}$,

$$k \cdot \int_r^\infty (1 - D^*_{k+1}(x)) \cdot \mathrm{d}x \;\geq\; k \cdot \int_r^\infty (1 - D_{k+1}(x)) \cdot \mathrm{d}x. \tag{10}$$

Combining Equations (9) and (10) together, we conclude that $\mathbf{F}^*$ gives a better Anonymous Reserve revenue than $\mathbf{F}$: for any reserve $r \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \mathsf{AR}(p, \mathbf{F}^*) &= \mathsf{AP}(p, \mathbf{F}^*) + k \cdot \int_r^\infty \big(1 - D^*_{k+1}(x)\big) \cdot \mathrm{d}x \\ &\geq \mathsf{AP}(p, \mathbf{F}) + k \cdot \int_r^\infty \big(1 - D_{k+1}(x)\big) \cdot \mathrm{d}x \\ &= \mathsf{AR}(p, \mathbf{F}). \end{aligned}$$

To complete the proof, it remains to show that the mapping F^* is indeed a CDF, namely that F^* is non-decreasing, $F^*(0) = 0$ and $F^*(\infty) = 1$. Under the construction given in Step (ii), we know from Equation (8) that $\sum_{i \in [k]} D^*_i(x) = |k - \frac{1}{x}|_+$ is an increasing function. Particularly, $D^*_i(x) = 0$ for any $x \leq \frac{1}{k}$ and each $i \in [k]$, and $\sum_{i \in [k]} D_i(\infty) = k$.

Indeed, suppose we regard $q \overset{\text{def}}{=} F^*(x)$ as a single variable, then each summand

$$D^*_i(q) \;:=\; \sum_{t=0}^{i-1} \binom{n}{t} \cdot q^{n-t} \cdot (1-q)^t$$

is an increasing function on $q \in [0, 1]$, with the minimum $D^*_i(q)|_{q=0} = 0$ and the maximum $D^*_i(q)|_{q=1} = 1$. To meet all the promised properties of $\sum_{i \in [k]} D^*_i$ (as a function of $x \geq 0$), the given F^* must be a CDF, namely an increasing function supported on $x \in (\frac{1}{k}, \infty)$ so that $F^*(\frac{1}{k}) = 0$ and $F^*(\infty) = 1$.

This completes the proof of Claim 2. ∎

3.3 Supremum revenue gap

In Section 3.2, we characterize the worst-case instance for any given population $n \geq k$. To avoid ambiguity, below we denote that instance by $\mathbf{F}^*_{(n)}$. In the next claim, we study the worst-case population and the resulting supremum revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k) = \sup_{n \geq k} \mathfrak{R}_{\mathsf{AR/AP}}(k, n)$.

Claim 3 (Part 2 of Theorem 3)**.** *Over all $n \geq k$, the supremum revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k) = \sup_{n \geq k} \mathfrak{R}_{\mathsf{AR/AP}}(k, n)$ is achieved by*

$$\mathfrak{R}_{\mathsf{AR/AP}}(k, \infty) \;=\; \lim_{n \to \infty} \mathsf{AR}(\mathbf{F}^*_{(n)}) \;=\; 1 + k \cdot \int_0^\infty \frac{T_k(x) \cdot (1 - T_{k+1}(x))}{(k - \sum_{i \in [k]} T_i(x))^2} \cdot \mathrm{d}x,$$

where the functions $T_i(x) \overset{\text{def}}{=} e^{-x} \cdot \sum_{t \in [0: i-1]} \frac{1}{t!} \cdot x^t$ for all $i \in [k+1]$.

Proof of Claim 3. We first show that $\{\mathfrak{R}_{\mathsf{AR/AP}}(k, n)\}_{n \geq k}$ is an *increasing* sequence, which by induction guarantees that $\mathfrak{R}_{\mathsf{AR/AP}}(k) = \mathfrak{R}_{\mathsf{AR/AP}}(k, \infty)$.

Indeed, the worst-case n-buyer instance $\mathbf{F}^*_{(n)} = \{F^*_{(n)}\}^n$ with the common CDF $F^*_{(n)}$ (specified by Claim 2) can be regarded as such a $(n+1)$-buyer instance: the index-$(n+1)$ buyer has a deterministic bid of zero, while every other buyer $i \in [n]$ still has the bid CDF $F^*_{(n)}$. This $(n+1)$-buyer instance is feasible to Program (P1), and gives a less Anonymous Reserve revenue than the worst-case $(n+1)$-buyer instance (due to Claim 2). That is, the n-buyer and $(n+1)$-buyer revenue gaps satisfy that $\mathfrak{R}_{\mathsf{AR/AP}}(k,n) \le \mathfrak{R}_{\mathsf{AR/AP}}(k,n+1)$, as desired.

It remains to prove the promised revenue formula for the limit instance $\mathbf{F}^*_{(\infty)}$. To this end, we first show the optimal Anonymous Reserve revenue from a specific n-buyer instance:

$$\mathsf{AR}(\mathbf{F}^*_{(n)}) \;=\; 1 \;+\; k \cdot \int_{1/k}^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x. \tag{11}$$

Indeed, this optimal revenue can be achieved by any reserve $r \in [0, \frac{1}{k}]$:

$$\begin{aligned}\mathsf{AR}(r, \mathbf{F}^*_{(n)}) \;&=\; r \cdot \sum_{i \in [k]} (1 - D_i(r)) \;+\; k \cdot \int_r^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x \\ &=\; k \cdot r \;+\; k \cdot \int_r^{1/k} \mathrm{d}x \;+\; k \cdot \int_{1/k}^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x \\ &=\; \text{RHS of (11)},\end{aligned}$$

where the second step follows because every i-th highest CDF $D_i(x) = 0$ for all $x \in [0, \frac{1}{k}]$, due to Claim 2 that the common CDF $F^*_{(n)}$ is supported on $x \in (\frac{1}{k}, \infty)$.

Moreover, any reserve $r \in (\frac{1}{k}, \infty)$ cannot generate a higher Anonymous Reserve revenue:

$$\begin{aligned}\mathsf{AR}(r, \mathbf{F}^*_{(n)}) \;&=\; \mathsf{AP}(r, \mathbf{F}^*_{(n)}) \;+\; k \cdot \int_r^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x \\ &=\; 1 + k \cdot \int_r^{\infty} (1 - D_{k+1}(x)) \cdot \mathrm{d}x \\ &\le\; \text{RHS of (11)},\end{aligned}$$

where the second step holds since $\mathsf{AP}(r, \mathbf{F}^*_{(n)}) = 1$ (see Claim 2); and the last step holds since the $(k+1)$-th highest CDF D_{k+1} is pointwise bounded within $[0, 1]$.

Given Equation (11), it remains to reason about the $(k+1)$-th highest CDF D_{k+1}. Below, we consider a specific bid $x \in (\frac{1}{k}, \infty)$ and, for each $n \ge k$ and all $i \in [k+1]$, use the shorthand $F^*_{(n)} = F^*_{(n)}(x)$ and $D_i = D_i(x)$. In addition, we denote by $\widehat{D}_i \stackrel{\text{def}}{=} \lim_{n \to \infty} D_i$ the i-th highest CDF resulted from the limit instance $\mathbf{F}^*_{(\infty)} = \lim_{n \to \infty} \mathbf{F}^*_{(n)}$.

It turns out that $F^*_{(\infty)} = 1$. Otherwise, any individual buyer is willing to pay with a *constant* probability $(1 - F^*_{(\infty)}) > 0$. This means the limit i-th highest CDF is $\widehat{D}_i = \lim_{n \to \infty} D_i = 0$ for all $i \in [k+1]$, since there are infinite buyers $n \to \infty$. This incurs a contradiction to constraint (C2), namely that the Anonymous Pricing revenue exceeds one (note that $x > \frac{1}{k}$ is given):

$$\mathsf{AP}(x, \mathbf{F}^*_{(\infty)}) \;=\; x \cdot \sum_{i \in [k]} (1 - \widehat{D}_i) \;=\; x \cdot k \;>\; 1.$$

Given that $F^*_{(\infty)} = \lim_{n \to \infty} F^*_{(n)} = 1$, for a sufficiently large $n \ge k$ we have

$$\begin{aligned} n \cdot \Big(\frac{1}{F^*_{(n)}} - 1\Big) \;&=\; (1 + o_n(1)) \cdot n \cdot \ln\Big(1 + \frac{1}{F^*_{(n)}} - 1\Big) \\ &=\; -\,(1 + o_n(1)) \cdot n \cdot \ln F^*_{(n)} \\ &=\; -\,(1 + o_n(1)) \cdot \ln D_1, \end{aligned} \tag{12}$$

where the first step uses the Maclaurin series of $\ln(1 + w)$ in the neighborhood of $w = 0$; and the last step follows because the highest CDF $D_1 = (F^*_{(n)})^n$.

Based on Equation (12), for any given $i \in [k+1]$ and a sufficiently large $n \geq k$, we can reformulate the i-th highest CDF D_i as follows:

$$\begin{aligned} D_i &= \sum_{t \in [0:i-1]} \binom{n}{t} \cdot (F^*_{(n)})^{n-t} \cdot (1 - F^*_{(n)})^t \\ &= \sum_{t \in [0:i-1]} \binom{n}{t} \cdot D_1 \cdot \Big(\frac{1}{F^*_{(n)}} - 1\Big)^t \\ &= (1 + o_n(1)) \cdot \sum_{t \in [0:i-1]} \binom{n}{t} \cdot \frac{1}{n^t} \cdot D_1 \cdot (-\ln D_1)^t \\ &= (1 + o_n(1)) \cdot \sum_{t \in [0:i-1]} \frac{1}{t!} \cdot D_1 \cdot (-\ln D_1)^t, \end{aligned}$$

where the first step applies Fact 1 (note that $\{F^*_{(n)}\}^n$ are i.i.d.); the second step follows since the highest CDF $D_1 = (F^*_{(n)})^n$; the third step applies Equation (12); and the last step uses the fact that $\binom{n}{t} \cdot \frac{1}{n^t} = (1 + o_n(1)) \cdot \frac{1}{t!}$.

Following the above equation, the limit i-th highest CDF $\widehat{D}_i = \lim_{n \to \infty} D_i$ satisfies that

$$\begin{aligned} \widehat{D}_i &= \sum_{t \in [0:i-1]} \frac{1}{t!} \cdot (\lim_{n \to \infty} D_1) \cdot (-\ln(\lim_{n \to \infty} D_1))^t \\ &= \sum_{t \in [0:i-1]} \frac{1}{t!} \cdot \widehat{D}_1 \cdot (-\ln \widehat{D}_1)^t. \end{aligned} \tag{13}$$

Note that this is an identity in the range $x \in (\frac{1}{k}, \infty)$. By taking the derivative, we also have

$$\begin{aligned} \frac{\mathrm{d}\widehat{D}_i}{\mathrm{d}\widehat{D}_1} &= \sum_{t \in [0:i-1]} \frac{1}{t!} \cdot (-\ln \widehat{D}_1)^t \; - \sum_{t \in [1:i-1]} \frac{1}{(t-1)!} \cdot (-\ln \widehat{D}_1)^{t-1} \\ &= \frac{1}{(i-1)!} \cdot (-\ln \widehat{D}_1)^{i-1}. \end{aligned} \tag{14}$$

We actually have one more identity $1 = \mathsf{AP}(x, \mathbf{F}^*_{(\infty)}) = x \cdot (k - \sum_{i \in [k]} \widehat{D}_i)$ for $x \in (\frac{1}{k}, \infty)$, due to Claim 2 (in the case that $n \to \infty$). Rearrange this identity and take the derivative:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\widehat{D}_1} &= \frac{\mathrm{d}}{\mathrm{d}\widehat{D}_1}\Big(\frac{1}{k - \sum_{i \in [k]} \widehat{D}_i}\Big) \\ &= \frac{1}{(k - \sum_{i \in [k]} \widehat{D}_i)^2} \cdot \sum_{i \in [k]} \frac{\mathrm{d}\widehat{D}_i}{\mathrm{d}\widehat{D}_1} \\ &= \frac{1}{(k - \sum_{i \in [k]} \widehat{D}_i)^2} \cdot \sum_{i \in [0:k-1]} \frac{1}{i!} \cdot (-\ln \widehat{D}_1)^i \\ &= \frac{1}{(k - \sum_{i \in [k]} \widehat{D}_i)^2} \cdot \frac{\widehat{D}_k}{\widehat{D}_1}, \end{aligned} \tag{15}$$

where the third step applies Equation (14); and the last step applies Equation (13).

Combining everything together, we deduce that

$$\mathsf{AR}(\mathbf{F}^*_{(\infty)}) \;=\; 1 \;+\; k \cdot \int_{1/k}^{\infty} (1 - \widehat{D}_{k+1}(x)) \cdot \mathrm{d}x$$

$$= 1 + k \cdot \int_{1/k}^{\infty} \frac{1 - \widehat{D}_{k+1}(x)}{(k - \sum_{i \in [k]} \widehat{D}_i(x))^2} \cdot \frac{\widehat{D}_k(x)}{\widehat{D}_1(x)} \cdot \mathrm{d}\widehat{D}_1(x), \tag{16}$$

where the first step applies Equation (11) for the limit instance $\mathbf{F}^*_{(\infty)} = \lim_{n\to\infty} \mathbf{F}^*_{(n)}$; and the last step follows from Equation (15).

For the above revenue formula $\mathsf{AR}(\mathbf{F}^*_{(\infty)})$, note that when the bid x ranges from $\frac{1}{k}$ to ∞, the highest CDF $\widehat{D}_1(x)$ ranges from 0 to 1. Moreover, Equation (13) characterizes, as a formula of $\widehat{D}_1(x)$, the i-th highest CDF $\widehat{D}_i(x)$. Thus, if we instead regard $\widehat{D}_1 \in (0, 1)$ as the variable,

$$\mathsf{AR}(\mathbf{F}^*_{(\infty)}) = 1 + k \cdot \int_0^1 \frac{1 - \widehat{D}_{k+1}}{(k - \sum_{i \in [k]} \widehat{D}_i)^2} \cdot \frac{\widehat{D}_k}{\widehat{D}_1} \cdot \mathrm{d}\widehat{D}_1,$$

where $\{\widehat{D}_i\}_{i \in [2:k+1]}$ once again are given by Equation (13).

Under the substitution $z \overset{\text{def}}{=} -\ln \widehat{D}_1 \in (0, \infty)$, we can check via elementary calculation that

$$\mathsf{AR}(\mathbf{F}^*_{(\infty)}) = 1 + k \cdot \int_0^{\infty} T_k(z) \cdot (1 - T_{k+1}(z)) \cdot (k - \sum_{i \in [k]} T_i(z))^{-2} \cdot \mathrm{d}z,$$

for the functions $\{T_i\}_{i \in [k+1]}$ defined in the statement of the claim.

This completes the proof of Claim 3. $\square$

Remark 2. In the single-item case $k = 1$, we can deduce from Claim 3 that

$$\mathfrak{R}_{\mathsf{AR/AP}}(1) = 1 + \int_0^{\infty} \frac{e^x - (1 + x)}{(e^x - 1)^2} \cdot \mathrm{d}x = \frac{\pi^2}{6} \approx 1.6449,$$

which recovers the known result [JLTX20, Theorem 2]. In the multi-unit case $k \geq 2$, however, the supremum revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}(k)$ does not have an elementary expression. We will show in Appendix A that $\mathfrak{R}_{\mathsf{AR/AP}}(k) = 1 + \Theta(1/\sqrt{k})$. Associated with numeric calculation, it turns out that the worst case $\arg\max\{\mathfrak{R}_{\mathsf{AR/AP}}(k) : k \in \mathbb{N}_{\geq 1}\}$ happens when $k = 1$.

3.4 Lower bound

We emphasize that all upper-bound results given in Sections 3.2 and 3.3 just require the input distributions $\{F_j\}_{j \in [n]}$ to be independent. In this section, we construct matching lower-bound instances respectively in the i.i.d. general setting and the asymmetric regular setting. For convenience, we reuse the notations introduced before.

I.I.D. general setting. Let us revisit the i.i.d. instance $\mathbf{F}^*_{(n)} = \{F^*_{(n)}\}^n$ specified by Claim 2. As mentioned, the revenue gap $\{\mathfrak{R}_{\mathsf{AR/AP}}(k, n)\}_{n \geq k}$ is an increasing sequence in the population $n \geq k$, and the limit/supremum revenue gap

$$\mathfrak{R}_{\mathsf{AR/AP}}(k) = \lim_{n \to \infty} \mathfrak{R}_{\mathsf{AR/AP}}(k, n)$$

is *finite* for any $k \in \mathbb{N}_{\geq 1}$ (see Appendix A). Accordingly, for a given $\varepsilon > 0$, there is a threshold population $N_1(\varepsilon) \geq k$ so that $\mathfrak{R}_{\mathsf{AR/AP}}(k, n) \geq \mathfrak{R}_{\mathsf{AR/AP}}(k) - \varepsilon$, for any $n \geq N_1(\varepsilon)$. Clearly, such instances $\mathbf{F}^*_{(n)} = \{F^*_{(n)}\}^n$ give the matching lower bound.

The common CDF $F^*_{(n)}$ specified in Claim 2 turns out to be the *equal-revenue* distribution (i.e. a "boundary-case" regular distribution) when $k = n = 1$, but is an irregular distribution otherwise. For example, when $k = 1$ and $n \geq 2$, we have

$$F^*_{(n)}(x) = \sqrt[n]{\left|1 - \frac{1}{x}\right|_+},$$

and the irregularity is shown in [JLTX20, Lemma 12]. In the other cases $n \geq k \geq 2$, the irregularity can be seen via similar but more technical arguments. For ease of presentation, here we omit the formal proof.

Asymmetric regular setting. We next use the triangle distributions to construct an instance whose revenue gap matches the bound $\mathfrak{R}_{\mathsf{AR}/\mathsf{AP}}(k)$ given in Claim 3. (Recall Section 2.1 that a triangle distribution must be regular.) To this end, we would reuse the notations introduced in the proof of Claim 3. The following claim is useful.

Claim 4 (Threshold for Lower Bound)**.** *Consider the limit instance $\mathbf{F}^*_{(\infty)}$ as well as its i-th highest CDF's $\{\widehat{D}_i\}_{i\in[k+1]}$ given in Claim 3. For any $\varepsilon > 0$, there exists a large enough $N_2(\varepsilon) \in \mathbb{N}_{\geq 1}$ so that*

$$k \cdot \int_a^b (1 - \widehat{D}_{k+1}(x)) \cdot \mathrm{d}x \ \geq \ \mathfrak{R}_{\mathsf{AR}/\mathsf{AP}}(k) - 1 - \varepsilon,$$

where $a \overset{def}{=} \frac{1}{k} + \frac{1}{N_2(\varepsilon)}$ and $b \overset{def}{=} \frac{1}{k} + N_2(\varepsilon)$; note that $a \leq b$.

Proof. According to Equation (16), in the limit case $N_2(\varepsilon) \to \infty$ we have

$$k \cdot \int_{1/k}^{\infty} (1 - \widehat{D}_{k+1}(x)) \cdot \mathrm{d}x \ = \ \mathsf{AR}(\mathbf{F}^*_{(\infty)}) - 1 \ = \ \mathfrak{R}_{\mathsf{AR}/\mathsf{AP}}(k) - 1.$$

We know from Claim 6 (see Appendix A) that the above improper integral $= \frac{\Theta(1)}{\sqrt{k}}$ is finite. In addition, the integrand is a non-negative function. Given these, we can easily see Claim 4. $\square$

Based on the above parameters $b \geq a$, we now construct a desired lower-bound instance.

Example 1 (Lower-Bound Instance in Asymmetric Regular Setting)**.** Denote $\delta \overset{\text{def}}{=} \frac{b-a}{n} > 0$, where the integer $n \geq k$ will be determined later. As Figure 6 shows, consider such an $(n+nk)$-buyer triangle instance $\mathbf{F} \overset{\text{def}}{=} \{\textsc{Tri}(v_{0,l}, q_{0,l})\}_{l\in[n]} \cup \{\textsc{Tri}(v_{j,l}, q_{j,l})\}_{j\in[n],l\in[k]}$:

- In the 0-th group, the involved monopoly prices $v_{0,l} \overset{\text{def}}{=} b$ for $l \in [n]$ are identical. In each group $j \in [n]$, the involved monopoly prices $v_{j,l} \overset{\text{def}}{=} b - j \cdot \delta$ for $l \in [k]$ are identical.
- In the 0-th group, the involved monopoly quantiles $\{q_{0,l}\}_{l\in[n]}$ are identical, which together give a *unit* Anonymous Pricing revenue

  $$\mathsf{AP}(p, \mathbf{F}) = 1 \text{ under the posted price } p = v_{0,l} = b.$$

  The remaining monopoly quantiles $\{q_{j,l}\}_{j\in[n],l\in[k]}$ are defined recursively. In each group $j \in [n]$, the involved $\{q_{j,l}\}_{l\in[k]}$ are identical, which give a *unit* Anonymous Pricing revenue

  $$\mathsf{AP}(p, \mathbf{F}) = 1 \text{ under the posted price } p = v_{j,l} = b - j \cdot \delta.$$

Claim 5 (Part 4 of Theorem 3 in Asymmetric Regular Setting)**.** *The $(n+nk)$-buyer triangle instance $\mathbf{F}$ in Example 1 is well defined, and satisfies the following:*

1. $\mathsf{AP}(p, \mathbf{F}) \leq 1$ *for any posted price $p \in \mathbb{R}_{\geq 0}$.*
2. *There exists a threshold $N_3(\varepsilon) \in \mathbb{N}_{\geq 1}$ such that for any $n \geq N_3(\varepsilon)$,*

$$\mathsf{AR}(\mathbf{F}) \ \geq \ \mathfrak{R}_{\mathsf{AR}/\mathsf{AP}}(k) - 2 \cdot \varepsilon.$$

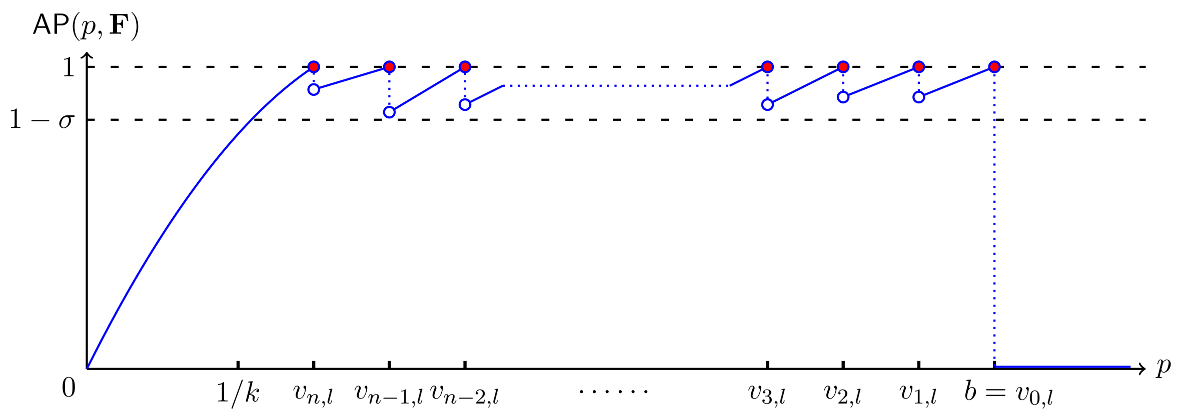


Figure 6: Demonstration for the triangle instance $\mathbf{F}$ given in Example 1, where $\sigma > 0$ can be arbitrarily small when $\delta > 0$ is small enough (i.e. when $n \in \mathbb{N}_{\geq 1}$ is large enough).

Proof of Claim 5. We first show that $\mathbf{F}$ is well defined or more precisely, the monopoly quantiles are well defined. Because the monopoly prices/quantiles in an individual group are identical, without ambiguity we denote $v_0 = v_{0,l} = b$ and $q_0 = q_{0,l}$ for $l \in [n]$ and $v_j = v_{j,l} = b - j \cdot \delta$ and $q_j = q_{j,l}$ for $l \in [k]$. Recall Section 2.1 that a triangle distribution $\textsc{Tri}(v_j, q_j)$ has the CDF

$$F_j(x) \;=\; \begin{cases} \frac{(1-q_j)\cdot x}{(1-q_j)\cdot x + v_j q_j}, & \text{when } x \in [0, v_j] \\ 1, & \text{when } x \in (v_j, \infty) \end{cases}.$$

Under the posted price $p = v_0 = b$, we have $F_0(p) = 1 - q_0$ for the 0-th group and $F_j(p) = 1$ for any other group $j \in [n]$. Thus, only the group-0 buyers contribute to the Anonymous Pricing revenue $\mathsf{AP}(v_0, \mathbf{F})$. This revenue formula $\mathsf{AP}(v_0, \mathbf{F})$ can be regarded as a continuous function in $q_0 \in [0, 1]$. Further, we observe that

(i) If a group-0 buyer is willing to pay $p = v_0 = b$ with probability $q_0 = 0$, then $\mathsf{AP}(v_0, \mathbf{F}) = 0$.

(ii) If a group-0 buyer is willing to pay $p = v_0 = b$ with probability $q_0 = 1$, then $\mathsf{AP}(v_0, \mathbf{F}) = v_0 \cdot \min\{n, k\} = b \cdot k > \frac{1}{k} \cdot k = 1$ (given that $n \geq k$ and $b > \frac{1}{k}$).

Given these and due to the intermediate value theorem, $\mathsf{AP}(v_0, \mathbf{F}) = 1$ for some $q_0 \in [0, 1]$. We conclude that the group-0 monopoly quantiles are well defined.

For some $m \in [0 : n-1]$, suppose that all the monopoly quantiles $q_j \in [0, 1]$ in the groups $j \in [0 : m]$ are well defined, below we justify the existence of the group-$(m+1)$ monopoly quantiles $q_{m+1} \in [0, 1]$.

By construction, under any posted price $p \in (v_{m+1}, v_m]$, the revenue $\mathsf{AP}(p, \mathbf{F})$ is contributed only by the buyers in the groups $j \in [0 : m]$. In particular, when $p = v_m$, by construction we have $\mathsf{AP}(v_m, \mathbf{F}) = 1$. Within the support $x \in [0, v_j]$, a triangle distribution $\textsc{Tri}(v_j, q_j)$ has the virtual value function

$$\varphi_j(x) \;=\; x - \frac{1 - F_j(x)}{f_j(x)} \;=\; \begin{cases} -\frac{v_j q_j}{1-q_j}, & \text{when } x \in [0, v_j) \\ v_j, & \text{when } x = v_j \end{cases}.$$

Hence, any allocation under any posted price $p \in (v_{m+1}, v_m)$ gives a *negative* virtual welfare. Due to the revenue-equivalence theorem [Mye81], the revenue formula $\mathsf{AP}(p, \mathbf{F})$ is a strictly increasing function in $p \in (v_{m+1}, v_m]$.

When $p = v_{m+1}$, we shall incorporate the contribution from the group-$(m+1)$ buyers into the revenue $\mathsf{AP}(v_{m+1}, \mathbf{F})$ as well. Once again, this revenue formula $\mathsf{AP}(v_{m+1}, \mathbf{F})$ can be regarded as a continuous function in $q_{m+1} \in [0, 1]$. And we have

(i) If a group-$(m+1)$ buyer is willing to pay $p = v_{m+1}$ with probability $q_{m+1} = 0$, then we have $\mathsf{AP}(v_{m+1}, \mathbf{F}) = \lim_{p \to v_{m+1}^+} \mathsf{AP}(p, \mathbf{F}) < 1$, where the inequality holds because $\mathsf{AP}(p, \mathbf{F})$ is a strictly increasing function when $p \in (v_{m+1}, v_m]$.

(ii) If a group-$(m+1)$ buyer is willing to pay $p = v_{m+1}$ with probability $q_0 = 1$, since there are k such buyers, we have $\mathsf{AP}(v_{m+1}, \mathbf{F}) = v_{m+1} \cdot k > \frac{1}{k} \cdot k = 1$, where the inequality holds because $v_{m+1} \geq a > \frac{1}{k}$ (by construction).

Once again, we deduce from the intermediate value theorem that $\mathsf{AP}(v_{m+1}, \mathbf{F}) = 1$ for some $q_{m+1} \in [0, 1]$, namely the group-$(m+1)$ monopoly quantiles are well defined. By induction, the triangle instance $\mathbf{F}$ is well defined.

From the above arguments, we also conclude Part 1 that $\mathsf{AP}(p, \mathbf{F}) \leq 1$ for all $p \in \mathbb{R}_{\geq 0}$.

We next justify Part 2 that the optimal Anonymous Reserve revenue $\mathsf{AR}(\mathbf{F}) \geq \mathfrak{R}_{\mathsf{AR/AP}}(k) - 2 \cdot \varepsilon$ when the $n \in \mathbb{N}_{\geq 1}$ is large enough. To this end, let us consider the specific reserve $r = v_n = a$. Indeed, when n is large enough, the $(k+1)$-th highest D_{k+1} resulted from $\mathbf{F}$ satisfies that

$$k \cdot \int_a^b (1 - D_{k+1}(x)) \cdot \mathrm{d}x \ \geq \ k \cdot \int_a^b (1 - \widehat{D}_{k+1}(x)) \cdot \mathrm{d}x - \varepsilon, \tag{17}$$

Assume Equation (17) to be true, then Part 2 follows immediately:

$$\begin{aligned}
\mathsf{AR}(a, \mathbf{F}) \ &= \ \mathsf{AP}(a, \mathbf{F}) \ + \ k \cdot \int_a^b (1 - D_{k+1}(x)) \cdot \mathrm{d}x \\
&\geq \ 1 \ + \ k \cdot \int_a^b (1 - D_{k+1}(x)) \cdot \mathrm{d}x \\
&\geq \ 1 \ + \ k \cdot \int_a^b (1 - \widehat{D}_{k+1}(x)) \cdot \mathrm{d}x - \varepsilon \\
&= \ \mathfrak{R}_{\mathsf{AR/AP}}(k) - 2 \cdot \varepsilon,
\end{aligned}$$

where the second step follows by construction, i.e. $\mathsf{AP}(a, \mathbf{F}) = \mathsf{AP}(v_n, \mathbf{F}) = 1$; the third step applies Equation (17); and the last step applies Claim 4.

We are left to prove Equation (17). By construction, one can easily see that in the limit case $n \to \infty$, every individual monopoly quantile q_j involved in $\mathbf{F}$ approaches to 0^+. Namely, the CDF $\lim_{n \to \infty} F_j(x) \to 1^-$ for any $x \in \mathbb{R}_{\geq 0}$. Given this, reusing the arguments for Equation (13), it can be seen that for each $i \in [k+1]$, the following holds for the limit i-th highest CDF:

$$\lim_{n \to \infty} D_i(x) \ = \ \sum_{t \in [0 : i-1]} \frac{1}{t!} \cdot (\lim_{n \to \infty} D_1(x)) \cdot (-\ln(\lim_{n \to \infty} D_1(x)))^t,$$

for all $x \in (\frac{1}{k}, \infty)$. Accessing the proof of Claim 3, for the limit instance $\mathbf{F}^*_{(\infty)}$ therein, we have the counterpart identities $\widehat{D}_i(x) = \sum_{t \in [0:i-1]} \frac{1}{t!} \cdot \widehat{D}_1(x) \cdot (-\ln \widehat{D}_1(x))^t$ for all $x \in (\frac{1}{k}, \infty)$.

By construction (as Figure 6 suggests), in the limit case $n \to \infty$, we have another identity[7]

$$\lim_{n \to \infty} \mathsf{AP}(x, \mathbf{F}) \ = \ 1,$$

[7]More precisely, by construction we have $\mathsf{AP}(v_j, \mathbf{F}) = 1$ for every $j \in [0 : n]$. Concerning the revenue formula $\mathsf{AP}(p, \mathbf{F}) = p \cdot \sum_{i \in [k]} (1 - D_i(p))$, we notice that the i-th highest CDF's $\{D_i\}_{i \in [k]}$ are increasing functions. Given these, for any $j \in [0 : n-1]$ and any posted price $p \in (v_{j+1}, v_j]$ we have $\mathsf{AP}(p, \mathbf{F}) \geq \frac{p}{v_j} \cdot \mathsf{AP}(v_j, \mathbf{F}) = \frac{p}{v_j} \geq \frac{v_{j+1}}{v_j}$. Under our construction that $v_j = b - j \cdot \delta$ for all $j \in [0 : n]$, where $\delta = \frac{b-a}{n}$, the minimum $\frac{v_{j+1}}{v_j}$ is equal to $\frac{v_n}{v_{n-1}} = \frac{a}{a + (b-a)/n} \geq \frac{1/k}{1/k + (b-a)/n} \geq 1 - (b-a) \cdot \frac{k}{n}$. Thus, for any $p \in [a, b]$ we have $\lim_{n \to \infty} \mathsf{AP}(p, \mathbf{F}) \geq \lim_{n \to \infty} (1 - (b-a) \cdot \frac{k}{n}) = 1$. On the other hand, we have shown that $\mathsf{AP}(p, \mathbf{F}) \leq 1$ for all $p \in [a, b]$ (see Part 1 of the claim).

for all $x \in [a, b]$. Accessing the proof of Claim 3, for the limit instance $\mathbf{F}^*_{(\infty)}$ therein, we have the counterpart identity $\mathsf{AP}(x, \mathbf{F}^*_{(\infty)}) = 1$ for all $x \in (\frac{1}{k}, \infty)$. Recall that $[a, b] \subseteq (\frac{1}{k}, \infty)$.

Based on the above identities, we can reapply the arguments for Claim 3 and deduce that

$$\lim_{n \to \infty} D_{k+1}(x) \;=\; \widehat{D}_{k+1}(x),$$

for all $x \in [a, b]$. Given this, and since both integrals $\int_a^b (1 - D_{k+1}(x)) \cdot \mathrm{d}x$ and $\int_a^b (1 - \widehat{D}_{k+1}(x)) \cdot \mathrm{d}x$ in Equation (17) are *definite* integrals, and both integrands $y = 1 - D_{k+1}(x)$ and $\widehat{y} = 1 - \widehat{D}_{k+1}(x)$ are bounded between $[0, 1]$, Equation (17) must hold for any sufficiently large $n \in \mathbb{N}_{\geq 1}$.

This completes the proof of Claim 5. $\square$

3.5 Matroid feasibility constraints

In Sections 3.2 to 3.4 we assume that any subset of up to $k \in \mathbb{N}_{\geq 1}$ willing-to-pay buyers can win simultaneously, i.e. the winners meet a rank-k *uniform matroid* constraint. To model some particular markets, many past works on Bayesian mechanism design also consider the general matroid constraints.

In this new scenario, the revenue gap between Anonymous Reserve and Anonymous Pricing is no longer a constant, or precisely, $\mathfrak{R}_{\mathsf{AR/AP}} = \Omega(\log k)$. In the rest of this section, we assume basic knowledge about matroid, for which the reader can turn to [Oxl06].

Regarding a general rank-k matroid constraint, Anonymous Pricing runs almost in the same way: a certain buyer $i \in [n]$, upon arriving, gets a copy of the item iff (i) he together with the past winning buyers form an independent set of the matroid; and (ii) he is willing to pay the posted price $p \geq 0$. No ambiguity would arise throughout the conduct of Anonymous Pricing, due to the greedy structure of matroids.

To implement Anonymous Reserve, the seller should use VCG Auction instead of $(k+1)$-th Price Auction: (i) the seller runs VCG Auction only on the buyers whose bids $\{b_j\}_{j \in [n]}$ are at least the reserve $r \geq 0$, by taking the matroid constraint into account; and (ii) each winner pays the threshold bid for him to keep winning.

Our lower-bound example with the $\Omega(\log k)$ revenue gap is constructed below.

Theorem 5 (AR vs. AP under a Matroid Constraint)**.** *When the seller faces $n \geq 1$ independent unit-demand buyers and the winners satisfy a rank-k matroid constraint, the revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}$ between* Anonymous Reserve *and* Anonymous Pricing *is lower bounded by $\Omega(\log k)$.*

Proof of Theorem 5. For simplicity, we assume that $n = 2m$ is an even integer and that $k \leq m$; the lower-bound instance for the general case is very similar. The buyers are divided into m pairs, and each pair $i \in [m]$ involves the $(2i-1)$-th and $2i$-th buyers. We consider a specific rank-k matroid $\mathcal{M}$ in terms of the collection $\mathcal{B}$ of its bases:

> Any base $B \in \mathcal{B}$ contains exactly one buyer from each chosen pair, for some choice of k pairs. In total, there are $|\mathcal{B}| = \binom{m}{k} \cdot 2^k$ bases.

One can easily justify the augmentation property, thus showing $\mathcal{M}$ to be a matroid (or more precisely, a laminar matroid with the laminar family $\{[1 : m], [m+1 : 2m], [1 : 2m]\}$ and the capacity function $c([1 : m]) = c([m+1 : 2m]) = c([1 : 2m]) = k$). Further, both buyers of each i-th pair have a deterministic bid $\max\{\frac{1}{i}, \frac{1}{k+1}\}$.

In Anonymous Pricing, when the seller posts a price $p > \frac{1}{k}$, exactly $\lfloor \frac{1}{p} \rfloor$ pairs would pay this price, hence a revenue $p \cdot \lfloor \frac{1}{p} \rfloor \leq 1$. When the price $p \leq \frac{1}{k}$, although all the k copies will be sold out, the revenue $p \cdot k$ is still at most 1. But when the seller instead employs VCG Auction (even without a reserve), either buyer in each of the top-k pairs will get an item by paying $\frac{1}{i}$, thus a revenue of $\sum_{j \in [k]} \frac{1}{i} = \Omega(\log k)$.

This completes the proof of Theorem 5. $\square$

Remark 3. We notice that a deterministic bid satisfies the regularity distributional assumption, as well as the stronger *monotone-hazard-rate* assumption.[8] Thus, the $\Omega(\log k)$ lower bound still holds for the revenue gap $\mathfrak{R}_{\mathsf{AR/AP}}$ in these restricted settings.

4 Ex-Ante Relaxation vs. Anonymous Pricing

In this section, we investigate the Ex-Ante Relaxation (EAR) vs. Anonymous Pricing (AP) problem, under the regularity assumption that $\mathbf{F} = \{F_j\}_{j\in[n]} \subseteq \textsc{Reg}$. Based on the revenue formulas (see Section 2.2), the revenue gap between both mechanisms is given by the optimal solution to the following mathematical program. Recall that D_i is the i-th highest bid distribution, and $\textsc{Reg}$ is the family of all regular distributions.

$$\begin{aligned}
\sup \qquad & \mathsf{EAR}(\mathbf{q}', \mathbf{F}) \;=\; \sum_{j\in[n]} F_j^{-1}(1-q'_j)\cdot q'_j && \text{(P2)}\\
\text{s.t.} \qquad & \mathsf{AP}(p, \mathbf{F}) \;=\; p\cdot \sum_{i\in[k]}(1-D_i(p)) \;\le\; 1, && \forall p\in\mathbb{R}_{\ge 0},\\
& \sum_{j\in[n]} q'_j \;\le\; k, &&\\
& \mathbf{q}' = \{q'_j\}_{j\in[n]}\in[0,1]^n,\;\; \mathbf{F} = \{F_j\}_{j\in[n]}\subseteq \textsc{Reg}, && \forall n\in\mathbb{N}_{\ge 1}.
\end{aligned}$$

We will establish an $O(\log k)$ upper bound for the optimal solution to Program (P2), which is formalized as Theorem 6. Combine this result with the matching lower bound by [HR09, Example 5.4], then the revenue gap gets understood.

Theorem 6 (EAR vs. AP)**.** *Given that the seller has $k\in\mathbb{N}_{\ge 1}$ homogeneous items and faces $n\ge k$ independent unit-demand buyers, who have regular value distributions $\mathbf{F} = \{F_j\}_{j\in[n]}\subseteq \textsc{Reg}$, the revenue gap between* Ex-Ante Relaxation *and* Anonymous Pricing *is $\mathfrak{R}_{\mathsf{EAR/AP}}(k) = \Theta(\log k)$.*

We establish Theorem 6 in three steps. First, we give a reduction from a regular instance to a triangle instance, which preserves the feasibility; then we just need to optimize n pairs of monopoly price and quantile $\{(v_j, q_j)\}_{j\in[n]}$ instead of n regular distributions $\{F_j\}_{j\in[n]}$. Second, we relax the constraint $\mathsf{AP}(p,\mathbf{F})\le 1$ to a more tractable constraint, which avoids the correlation among the order statistics $\{D_i\}_{i\in[k]}$. Afterwards, we divide all buyers into three careful groups under certain criteria for $\{(v_j, q_j)\}_{j\in[n]}$, and separately bound the contribution from each group to the EAR revenue. The total EAR revenue turns out to be $O(\log k)$.

Reduction to triangle instances. For the single-item case $k=1$, [AHN+19] show that the worst case of Program (P2) w.l.o.g. is achieved by a triangle instance. Indeed, their arguments work in the general case $k\in\mathbb{N}_{\ge 1}$ as well. Formally, we have the following lemma (see Figure 7 for a demonstration).

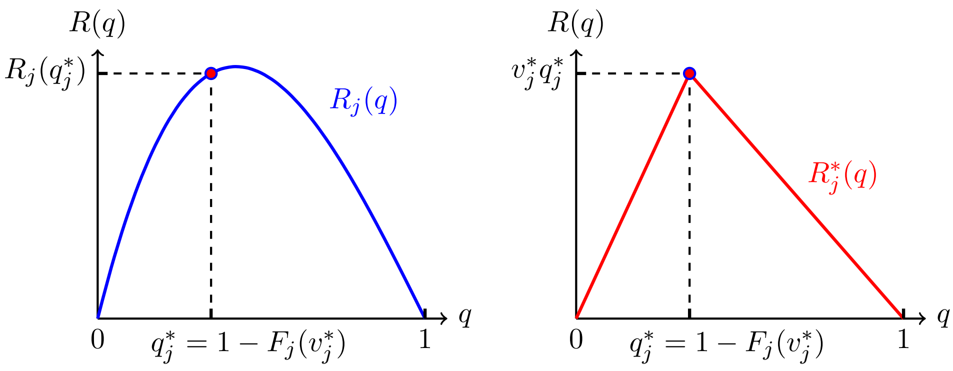
Lemma 2 (Reduction for EAR vs. AP [AHN+19, Lemma 4.1])**.** *Given a feasible solution $(\mathbf{q}', \mathbf{F})$ to Program (P2), there exists another n-buyer feasible instance $(\mathbf{q}^*, \mathbf{F}^*)$ such that:*

1. *The distributions $\mathbf{F}^* = \{F_j^*\}_{j\in[n]}\subseteq \textsc{Tri}$ are triangle distributions, and $\mathbf{q}^* = \{q_j^*\}_{j\in[n]}\in[0,1]^n$ (such that $\sum_{j\in[n]} q_j^*\le k$) are the monopoly quantiles thereof.*

2. *The* Ex-Ante Relaxation *revenue keeps the same, i.e.* $\mathsf{EAR}(\mathbf{q}^*, \mathbf{F}^*) = \mathsf{EAR}(\mathbf{q}', \mathbf{F})$.

3. *The distributions $\mathbf{F}^* = \{F_j^*\}_{j\in[n]}$ are stochastically dominated by $\mathbf{F} = \{F_j\}_{j\in[n]}$ and thus, for any price $p\in\mathbb{R}_{\ge 0}$, the* Anonymous Pricing *revenue drops, i.e.* $\mathsf{AP}(p,\mathbf{F}^*)\le \mathsf{AP}(p,\mathbf{F})$.

[8] A distribution F_j has *monotone hazard rate* if $y = \ln(1-F_j(x))$ is a concave function, e.g., see [JLX19].



(a) A concave revenue-quantile curve (b) A triangular revenue-quantile curve

Figure 7: Demonstration for the reduction in Lemma 2, in terms of the *revenue-quantile curves*. For a distribution F_j, its revenue-quantile curve is given by $R_j(q) = q \cdot F_j^{-1}(1-q)$ for $q \in [0, 1]$. This distribution F_j is regular iff the R_j is a concave function (as Figure 7a suggests). And the revenue-quantile curve of a triangle distribution is basically a triangle (i.e., a 2-piecewise linear function, as Figure 7b suggests); in particular, the two base angles have the tangent values v_j^* and $v_j^* q_j^* / (1 - q_j^*)$, respectively.

In view of Lemma 2, to establish Theorem 6 we can focus on Program (P3) in place of the previous Program (P2). For a triangle distribution $\text{Tri}(v_j, q_j)$, where $v_j = F_j^{-1}(1 - q_j) \geq 0$ is the monopoly price, we reuse F_j to denote its CDF. Recall Section 2.1 that $F_j(x) = \frac{(1-q_j)\cdot x}{(1-q_j)\cdot x + v_j q_j}$ for all $x \leq v_j$ and $F_j(x) = 1$ for all $x > v_j$.

$$\sup \qquad \mathsf{EAR}(\mathbf{F}) \;=\; \sum_{j\in[n]} v_j q_j \tag{P3}$$

$$\text{s.t.} \qquad \mathsf{AP}(p, \mathbf{F}) \;=\; p \cdot \sum_{i\in[k]} (1 - D_i(p)) \;\leq\; 1, \qquad \forall p \in \mathbb{R}_{\geq 0}, \tag{C2}$$

$$\sum_{j\in[n]} q_j \;\leq\; k, \tag{C3}$$

$$\mathbf{F} = \{\text{Tri}(v_j, q_j)\}_{j\in[n]} \subseteq \text{Reg}, \qquad \forall n \in \mathbb{N}_{\geq 1}.$$

For a single triangle distribution $\text{Tri}(v_j, q_j)$, the optimal Anonymous Pricing revenue from it equals $\mathsf{AP}(\text{Tri}(v_j, q_j)) = v_j q_j$, which $\leq \mathsf{AP}(\mathbf{F}) \leq 1$ due to constraint (C2). We thus add one more constraint

$$v_j q_j \;\leq\; 1, \qquad \forall j \in [n]. \tag{C4}$$

Relaxing constraint (C2). Given Program (P3), both the objective function $\mathsf{EAR}(\mathbf{F})$ and constraint (C3) are easy to deal with. However, constraint (C2) is rather complicated, because it involves the correlated top-k bids $\{b_{(i)}\}_{i\in[k]}$ and the corresponding order CDF's $\{D_i\}_{i\in[k]}$ (as formulas of the individual CDF's $\{F_j\}_{j\in[n]}$) are cumbersome.

The following Lemma 3 relaxes constraint (C2) to another constraint. The resulting constraint is much easier to reason about. Namely, it avoids the correlation among the top-k bids $\{b_{(i)}\}_{i\in[k]}$ and admits a clean formula of the individual CDF's $\{F_j\}_{j\in[n]}$. Later we will see that after this relaxation, the optimal objective value of Program (P3) blows up just by a constant multiplicative factor. Denote $m \stackrel{\text{def}}{=} \lfloor \frac{k}{2} \rfloor \geq 2$ for convenience.

Lemma 3 (Relaxed Constraint)**.** *The following is a necessary condition for constraint (C2):*

$$\sum_{j\in[n]} (1 - F_j(p)) \;\leq\; \frac{4}{p}, \qquad \forall p \in \left[\frac{1}{m}, \frac{1}{2}\right].$$

Proof of Lemma 3. Let us consider a specific price $p \in [\frac{1}{m}, \frac{1}{2}]$ for constraint (C2). For any $j \in [n]$, let the independent Bernoulli random variable $X_j \in \{0, 1\}$ denote whether the j-th buyer is willing to pay the price p, with the failure probability $\Pr[X_j = 0] = F_j(p)$. Then $X \stackrel{\text{def}}{=} \sum_{j\in[n]} X_j$ denotes how many buyers are willing to pay, and $Y \stackrel{\text{def}}{=} \min\{k, X\}$ denotes how many items are sold out in Anonymous Pricing.

We have the revenue $\mathsf{AP}(p, \mathbf{F}) = p \cdot \mathbf{E}[Y]$, and constraint (C2) is identical to $\mathbf{E}[Y] \le \frac{1}{p}$. For the equation given in Lemma 3, the LHS $= \sum_{j\in[n]} \Pr[X_j = 1] = \sum_{j\in[n]} \mathbf{E}[X_j] = \mathbf{E}[X]$.

On the opposite of Lemma 3, suppose that $\mathbf{E}[X] > \frac{4}{p}$. We have $\mathbf{E}[X] > 8$, given that the price $p \le \frac{1}{2}$. Since X is the sum of independent Bernoulli random variables, due to Chernoff bound, $\Pr[X < (1-\delta) \cdot \mathbf{E}[X]] < \frac{e^{-\delta\cdot\mathbf{E}[X]}}{(1-\delta)^{(1-\delta)\cdot\mathbf{E}[X]}}$ for any $\delta \in (0, 1)$. In particular,

$$\Pr\Big[X < \frac{1}{2} \cdot \mathbf{E}[X]\Big] \;<\; \Big(\frac{2}{e}\Big)^{\frac{1}{2}\cdot\mathbf{E}[X]} \;<\; \Big(\frac{2}{e}\Big)^{4} \;\approx\; 0.2931 \;<\; \frac{1}{2}, \tag{18}$$

where the first step follows by setting $\delta = \frac{1}{2}$; and the second step follows since $\mathbf{E}[X] > 8$.

And because $Y = \min\{k, X\}$, we further deduce that

$$\begin{aligned} \Pr\Big[Y \ge \min\Big\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\Big\}\Big] &= 1 - \Pr\Big[Y < \min\Big\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\Big\}\Big] \\ &= 1 - \Pr\Big[X < \min\Big\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\Big\}\Big] \\ &\ge 1 - \Pr\Big[X < \frac{1}{2} \cdot \mathbf{E}[X]\Big] \\ &> \frac{1}{2}, \end{aligned} \tag{19}$$

where the second step follows since $Y < \min\{k, \frac{1}{2} \cdot \mathbf{E}[X]\}$ holds only if $Y < k$, and thus only if $Y = X$; and the last step follows from Equation (18).

Based on the above arguments, we conclude a contradiction $\mathbf{E}[Y] > \frac{1}{p}$ as follows:

$$\begin{aligned} \mathbf{E}[Y] &\ge \Pr\Big[Y \ge \min\Big\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\Big\}\Big] \cdot \min\Big\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\Big\} \\ &> \frac{1}{2} \cdot \min\Big\{k,\ \frac{1}{2} \cdot \mathbf{E}[X]\Big\} \\ &\ge \frac{1}{2} \cdot \min\Big\{k,\ \frac{2}{p}\Big\} \\ &\ge \frac{1}{p}, \end{aligned}$$

where the second step applies Equation (19); the third step applies our assumption $\mathbf{E}[X] > \frac{4}{p}$; and the last step follows as $\frac{2}{p} \le 2m \le k$, given that $p \in [\frac{1}{m}, \frac{1}{2}]$ and $m = \lfloor \frac{k}{2} \rfloor$.

By refuting the assumption, we get $\mathbf{E}[X] \le \frac{4}{p}$ for any price $p \in [\frac{1}{m}, \frac{1}{2}]$. This completes the proof of Lemma 3. $\square$

Given a triangle instance $\{\textsc{Tri}(v_j, q_j)\}_{j\in[n]}$, by plugging the CDF formulas $\{F_j\}_{j\in[n]}$, we can reformulate Lemma 3 as follows:

$$\sum_{j\in[n]:v_j\ge p} \frac{v_j q_j}{(1-q_j)\cdot p + v_j q_j} \;\le\; \frac{4}{p}, \qquad\qquad \forall p \in \Big[\frac{1}{m},\ \frac{1}{2}\Big]. \tag{C2$'$}$$

Grouping the buyers. To upper bound the objective function $\mathsf{EAR}(\mathbf{F}) = \sum_{j\in[n]} v_j q_j$, let us partition all the buyers into three groups $[n] = A \sqcup B \sqcup C$, where

$$A \;\stackrel{\text{def}}{=}\; \Big\{j \in [n] : v_j \ge \frac{1}{m} \;\text{ and }\; \frac{v_j q_j}{1-q_j} \ge \frac{1}{m}\Big\},$$

$$B \stackrel{\text{def}}{=} \Big\{j \in [n] : v_j \geq \frac{1}{m} \ \text{ and } \ \frac{v_j q_j}{1 - q_j} < \frac{1}{m}\Big\},$$
$$C \stackrel{\text{def}}{=} \Big\{j \in [n] : v_j < \frac{1}{m}\Big\}.$$

Regarding the groups A, B and C given above, their individual contributions to the benchmark $\mathsf{EAR}(\mathbf{F})$ actually admit the following bounds:

$$\sum_{j \in A} v_j q_j \ = \ O(\log k), \qquad \sum_{j \in B} v_j q_j \ \leq \ 8, \qquad \sum_{j \in C} v_j q_j \ \leq \ 3.$$

Suppose these bounds to be true, then combining them together immediately gives Theorem 6. Below we explain the intuitions of our grouping criteria (Remark 4), give an interesting observation for the instances that are constituted by "small" distributions (Remark 5), and then verify the above three bounds in the reverse order.

Remark 4 (Grouping Criteria). Recall the objective function of Program (P3), i.e., $\mathsf{EAR}(\mathbf{F}) = \sum_{j \in [n]} v_j q_j$, and constraint (C3), i.e., $\sum_{j \in [n]} q_j \leq k$. Here the monopoly revenues $\{v_j q_j\}_{j \in [n]}$ are the individual contributions by the triangle distributions $\{\textsc{Tri}(v_j, q_j)\}_{j \in [n]}$, and (in the sense of the *Knapsack Problem*) the monopoly quantiles $\{q_j\}_{j \in [n]}$ can be regarded as the individual capacities. Thereby, the monopoly prices $\{v_j\}_{j \in [n]}$ somehow are the bang-per-buck ratios (i.e., the contribution to the EAR benchmark per unit of the capacity).

Of course we prefer those distributions with higher bang-per-buck ratios $\{v_j\}_{j \in [n]}$, but also need to take the capacities $\{q_j\}_{j \in [n]}$ into account. In particular:

- The group-C distributions have lower bang-per-buck ratios $v_j \leq 1/m$. So conceivably, the total contribution $\sum_{j \in C} v_j q_j$ by this group to the EAR benchmark shall be small, and we will prove an upper bound of 3.
- The group-B distributions have high enough bang-per-buck ratios $v_j \geq 1/m$ but small capacities, namely $v_j q_j / (1 - q_j) < 1/m$. It turns out that the total contribution $\sum_{j \in B} v_j q_j$ by this group is also small, and we will prove an upper bound of 8.
- The group-A distributions have high enough bang-per-buck ratios as well as big enough capacities. Thus, this group should contribute the most to the EAR benchmark, for which we will show $\sum_{j \in A} v_j q_j = O(\log k)$.

Indeed, our grouping criteria borrow ideas from the "budget-feasible mechanism design" literature [Sin10, CGL11, GJLZ20], where the primary goal is to design approximately optimal mechanisms for the Knapsack Problem under the incentive concerns.

Remark 5 ("Small" Distributions). As argued in Section 1.2, regarding a *continuum of "small" buyers* (i.e., any single buyer has an infinitesimal contribution to the EAR benchmark, but there are infinitely many buyers $n \to \infty$), the EAR vs. AP revenue gap would be (at most) a universal constant for whatever $k \geq 1$. This is because every "small" buyer belongs to either group B or group C, and thus the EAR benchmark is at most $\sum_{j \in B \cup C} v_j q_j \leq 8 + 3 = 11$.

Revenue from group C. Since such a buyer $j \in C$ has a monopoly price $v_j < \frac{1}{m}$, we have

$$\begin{aligned}
\sum_{j \in C} v_j q_j \ &\leq \ \frac{1}{m} \cdot \sum_{j \in C} q_j \\
&\leq \ \frac{1}{m} \cdot \sum_{j \in [n]} q_j \\
&\leq \ \frac{1}{m} \cdot k
\end{aligned}$$

$$\leq 3,$$

where the second step follows since $C \subseteq [n]$; the third step follows from constraint (C3); and the last step holds for $m = \lfloor \frac{k}{2} \rfloor$ and $k \geq 4$. (We will deal with the cases $k \in \{1, 2, 3\}$ separately, at the end of this section.)

Revenue from group B. Setting $p = \frac{1}{m}$ for constraint (C2′), we deduce that

$$\begin{aligned} 4m \;=\; \text{RHS of (C2}') \;\geq\; \text{LHS of (C2}') \;&=\; \sum_{j \in [n] : v_j \geq \frac{1}{m}} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{m} + v_j q_j} \\ &\geq\; \sum_{j \in B} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{m} + v_j q_j} \\ &\geq\; \sum_{j \in B} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{m} + (1 - q_j) \cdot \frac{1}{m}} \\ &\geq\; \frac{m}{2} \cdot \sum_{j \in B} v_j q_j, \end{aligned}$$

where the second line follows since $\{j \in [n] : v_j \geq \frac{1}{m}\} \supseteq B$ (see the definition of B); the third line follows since $\frac{v_j q_j}{1 - q_j} < \frac{1}{m}$ for any $j \in B$; and the last line drops the $(1 - q_j)$ terms and then rearranges the formula.

Rearranging the above equation immediately gives $\sum_{j \in B} v_j q_j \leq 8$, as desired.

Revenue from group A. To verify the upper bound about this group, we shall generalize the definition of A, and get a chain of subgroups $A = A_m \supseteq A_{m-1} \supseteq \cdots \supseteq A_2$:

$$A_t \stackrel{\text{def}}{=} \left\{ j \in [n] : v_j \geq \frac{1}{t} \text{ and } \frac{v_j q_j}{1 - q_j} \geq \frac{1}{t} \right\}, \qquad \forall t \in [2 : m].$$

Given an index $t \in [2 : m]$, by setting $p = \frac{1}{t} \in [\frac{1}{m}, \frac{1}{2}]$ for constraint (C2′), we deduce that

$$\begin{aligned} 4t \;=\; \text{RHS of (C2}') \;\geq\; \text{LHS of (C2}') \;&=\; \sum_{j \in [n] : v_j \geq \frac{1}{t}} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{t} + v_j q_j} \\ &\geq\; \sum_{j \in A_t} \frac{v_j q_j}{(1 - q_j) \cdot \frac{1}{t} + v_j q_j} \\ &\geq\; \sum_{j \in A_t} \frac{v_j q_j}{v_j q_j + v_j q_j} \\ &=\; \frac{1}{2} \cdot |A_t|, \end{aligned}$$

where the second step follows because $\{j \in [n] : v_j \geq \frac{1}{m}\} \supseteq A_t$ (see the definition of A_t); and the third step follows because $(1 - q_j) \cdot \frac{1}{t} \leq v_j q_j$ for each $j \in A_t$.

Based on the above equation, we easily bound the cardinality $|A_t| \leq 8t$ for each $t \in [2 : m]$. Combining the above arguments together gives

$$\begin{aligned} \sum_{j \in A} v_j q_j \;=\; \sum_{j \in A_m} v_j q_j \;&=\; \sum_{j \in A_2} v_j q_j + \sum_{t \in [3:m]} \sum_{j \in A_t \setminus A_{t-1}} v_j q_j \\ &\leq\; \sum_{j \in A_2} v_j q_j + \sum_{t \in [3:m]} \sum_{j \in A_t \setminus A_{t-1}} \frac{1}{t-1} \cdot 1 \\ &=\; \sum_{j \in A_2} v_j q_j + \sum_{t \in [3:m]} \frac{|A_t| - |A_{t-1}|}{t-1} \end{aligned}$$

$$
\begin{aligned}
&= \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + \frac{|A_m|}{m-1} + \sum_{t\in[3:m]} |A_t|\cdot\Big(\frac{1}{t-1}-\frac{1}{t}\Big)\\
&\le \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + \frac{8m}{m-1} + \sum_{t\in[3:m]} 8t\cdot\Big(\frac{1}{t-1}-\frac{1}{t}\Big)\\
&\le \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + 16 + \sum_{t\in[3:m]} 8t\cdot\Big(\frac{1}{t-1}-\frac{1}{t}\Big)\\
&= \Big(\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2}\Big) + 8 + \sum_{t\in[m-1]} \frac{8}{t}, \qquad (20)
\end{aligned}
$$

where the second line follows because the monopoly price $v_j \in (\frac{1}{t-1}, \frac{1}{t}]$ for each $j \in A_t \setminus A_{t-1}$ (see the definitions of A_t and A_{t-1}), and the monopoly quantiles $q_j \in [0,1]$ are bounded; the fifth line applies the bounds $|A_t| \le 8t$ for each $t \in [2:m]$; the sixth line holds for $m = \lfloor \frac{k}{2} \rfloor$ and $k \ge 4$; and the last line is by elementary calculation.

Because $v_j q_j \le 1$ for all $j \in A_2$ (see constraint (C4)) and $|A_2| \le 16$, we can bound the first term in Equation (20): $\sum_{j\in A_2} v_j q_j - \frac{|A_2|}{2} \le |A_2| - \frac{|A_2|}{2} \le 8$. Plug this into Equation (20):

$$
\sum_{j\in A} v_j q_j \;\le\; 16 + \sum_{t\in[m-1]} \frac{8}{t} \;=\; O(\log k),
$$

where the last step holds for $m = \lfloor \frac{k}{2} \rfloor$.

Upper bound when $k \in \{1,2,3\}$. Clearly, the optimal value $\mathfrak{R}_{\mathsf{EAR/AP}}(k)$ of Program (P2), which involves $k \in \mathbb{N}_{\ge 1}$ items in both mechanisms, is at most the revenue gap between the k-item Ex-Ante Relaxation and the 1-item Anonymous Pricing. The later revenue gap is given by the next mathematical program.

$$
\begin{aligned}
\sup \quad & \sum_{j\in[n]} F_j^{-1}(1-q'_j)\cdot q'_j && && \text{(P4)}\\
\text{s.t.} \quad & p\cdot(1-D_1(p)) \;\le\; 1, && \forall p \in \mathbb{R}_{\ge 0},\\
& \sum_{j\in[n]} q'_j \le k,\\
& \mathbf{q}' = \{q'_j\}_{j\in[n]} \in [0,1]^n,\ \mathbf{F} = \{F_j\}_{j\in[n]} \subseteq \textsc{Reg}, && \forall n \in \mathbb{N}_{\ge 1}.
\end{aligned}
$$

The only difference between Program (P4) and the one in [AHN+19, Section 4] is the constraint $\sum_{j\in[n]} q'_j \le k$ (rather than ≤ 1). We can resolve Program (P4) by following the exactly same steps as in [AHN+19, Section 4]. By doing so, we will get

$$
\mathfrak{R}_{\mathsf{EAR/AP}}(k) \;\le\; \text{optimal value of (P4)} \;=\; 1 + \mathcal{V}(\mathcal{Q}^{-1}(k)),
$$

where the functions $\mathcal{V}(p) \overset{\text{def}}{=} p\cdot\ln(\frac{p^2}{p^2-1})$ and $\mathcal{Q}(p) \overset{\text{def}}{=} \ln(\frac{p^2}{p^2-1}) - \frac{1}{2}\cdot\sum_{t=1}^{\infty} t^{-2}\cdot p^{-2t}$. Then we can derive Theorem 6 in the case $k \in \{1,2,3\}$ via numeric calculation, as the next table shows.

k	1	2	3
$1+\mathcal{V}(\mathcal{Q}^{-1}(k))$	≈ 2.7184	≈ 3.7897	≈ 4.8111

Acknowledgements. We would like to thank Xi Chen, Eric Neyman, Tim Roughgarden, and Rocco Servedio for helpful comments on an earlier version of this work.

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A Asymptotic Formulas for the Revenue Gap $\mathfrak{R}_{\mathsf{AR/AP}}(k)$

Claim 6 (Part 3 of Theorem 3)**.** *For each $k \in \mathbb{N}_{\geq 1}$, the supremum revenue gap*

$$\mathfrak{R}_{\mathsf{AR/AP}}(k) = 1 + k \cdot \int_0^\infty \frac{T_k(x) \cdot (1 - T_{k+1}(x))}{(k - \sum_{i \in [k]} T_i(x))^2} \cdot \mathrm{d}x,$$

where the functions $T_i(x) \stackrel{\text{def}}{=} e^{-x} \cdot \sum_{t \in [0:i-1]} \frac{1}{t!} \cdot x^t$ for all $i \in [k+1]$, is bounded between

$$1 + \frac{0.1}{\sqrt{k}} \ \leq \ \mathfrak{R}_{\mathsf{AR/AP}}(k) \ \leq \ 1 + \frac{2}{\sqrt{k}}.$$

The value of $\mathfrak{R}_{\mathsf{AR/AP}}(k)$ when $k \leq 24$ is listed in Table 6.

Proof of Claim 6 (Lower Bound). We define $lb(k) \stackrel{\text{def}}{=} 1 + k \cdot \int_0^\infty T_k(x) \cdot (1 - T_{k+1}(x)) \cdot k^{-2} \cdot \mathrm{d}x$. Apparently $lb(k) \leq \mathfrak{R}_{\mathsf{AR/AP}}(k)$ for all $k \geq 1$. Then we have

$$lb(k) \quad = \quad 1 + \frac{1}{k} \int_0^\infty T_k(x) \cdot (1 - T_{k+1}(x)) \cdot \mathrm{d}x$$

k	1	2	3	4	5	6	7	8
$\Re_{\mathsf{AR/AP}}(k)$	$\pi^2/6$	1.4445	1.3575	1.3065	1.2721	1.2470	1.2276	1.2121
c_k	0.6449	0.6287	0.6192	0.6130	0.6085	0.6050	0.6023	0.6000
k	9	10	11	12	13	14	15	16
$\Re_{\mathsf{AR/AP}}(k)$	1.1994	1.1886	1.1794	1.1714	1.1644	1.1581	1.1525	1.1475
c_k	0.5982	0.5965	0.5951	0.5939	0.5928	0.5918	0.5909	0.5901
k	17	18	19	20	21	22	23	24
$\Re_{\mathsf{AR/AP}}(k)$	1.1429	1.1387	1.1349	1.1313	1.1281	1.1250	1.1221	1.1195
c_k	0.5894	0.5887	0.5881	0.5875	0.5878	0.5865	0.5860	0.5855

Table 6: List of $\Re_{\mathsf{AR/AP}}(k)$ for $k \le 24$, where c_k means that $\Re_{\mathsf{AR/AP}}(k) = 1 + c_k/\sqrt{k}$.

$$
\begin{aligned}
&= 1 + \frac{1}{k}\int_0^\infty \sum_{i=0}^{k-1} e^{-x}x^i/i! \cdot (1 - \sum_{i=0}^{k} e^{-x}x^i/i!) \cdot \mathrm{d}x \\
&= 1 + \Big(\frac{1}{k}\int_0^\infty \sum_{i=0}^{k-1} e^{-x}x^i/i! \cdot \mathrm{d}x\Big) - \Big(\frac{1}{k}\int_0^\infty \sum_{i=0}^{k-1}\sum_{j=0}^{k} e^{-2x}x^{i+j}/i!/j!\mathrm{d}x\Big) \\
&= 2 - \frac{1}{k}\int_0^\infty \sum_{i=0}^{k-1}\sum_{j=0}^{k} e^{-2x}x^{i+j}/i!/j!\mathrm{d}x \\
&= 2 - \frac{1}{2k}\int_0^\infty \sum_{i=0}^{k-1}\sum_{j=0}^{k} e^{-x}x^{i+j}/i!/j!/2^{i+j}\mathrm{d}x \\
&= 2 - \frac{1}{2k}\sum_{i=0}^{k-1}\sum_{j=0}^{k}\binom{i+j}{i}/2^{i+j} \\
&= 2 - \frac{1}{2k}\left(\sum_{i=0}^{2k-1}\sum_{j=0}^{i}\binom{i}{j}/2^i - \sum_{\substack{i+j\le 2k-1 \\ i\ge k \text{ or } j>k}}\binom{i+j}{i}/2^{i+j}\right) \\
&= 1 + \frac{2}{2k}\cdot\sum_{i+j\le k}\binom{i+j+k}{j}/2^{i+j+k} \\
&= 1 + \frac{1}{k}\cdot\sum_{m=k}^{2k} g(m)/2^m,
\end{aligned}
$$

where the second step is by definition of $T_k(x)$, the third step is by Fact 8 that $\int_0^\infty e^{-x}x^n\mathrm{d}x = n!$, the fifth step is by substitution, the sixth step is by Fact 8, and in the last step we define $g(m) \stackrel{\text{def}}{=} \sum_{i=0}^{m-k}\binom{m}{i}$.

For all $m \in \{\lceil 2k - \sqrt{k}/2\rceil, \lceil 2k - \sqrt{k}/2\rceil + 1, \cdots, 2k\}$,

$$
\begin{aligned}
g(m) &= \sum_{i=0}^{m-k}\binom{m}{i} \\
&= \sum_{i=0}^{\lceil m/2\rceil}\binom{m}{i} - \sum_{i=m-k+1}^{\lceil m/2\rceil}\binom{m}{i} \\
&\ge 2^m/2 - (\lceil m/2\rceil - (m-k))\cdot\binom{m}{\lceil m/2\rceil}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{2^m}{2} - \frac{\sqrt{k}}{4}\cdot\frac{2\cdot 2^m}{\sqrt{\pi m}} \\
&\geq \frac{2^m}{2} - \frac{2^m}{2\sqrt{\pi}} \\
&\geq 2^m/5,
\end{aligned}
$$

where the fourth step is by Fact 7.

Therefore,

$$
\begin{aligned}
\Re_{\mathsf{AR/AP}}(k) &\geq lb(k) \\
&= 1 + \frac{1}{k}\cdot\sum_{m=k}^{2k} g(m)/2^m \\
&\geq 1 + \frac{1}{k}\sum_{m=\lceil 2k-\sqrt{k}/2\rceil}^{2k} g(m)/2^m \\
&\geq 1/(10\sqrt{k}).
\end{aligned}
$$

This accomplishes the lower-bound part of Claim 6. □

Proof of Claim 6 (Upper Bound). We define

$$
a(x) \stackrel{\text{def}}{=} \sum_{i=0}^{k-1}\frac{x^i}{i!}; \qquad b(x) \stackrel{\text{def}}{=} \sum_{i=k+1}^{\infty}\frac{x^i}{i!}.
$$

Then the integral part of $\Re_{\mathsf{AR/AP}}(k)$ can be written as

$$
\begin{aligned}
h(x) &\stackrel{\text{def}}{=} T_k(x)\cdot(1-T_{k+1}(x))\cdot\Big(k-\sum_{i\in[k]}T_i(x)\Big)^{-2} \\
&= a(x)b(x)\cdot\Big(\sum_{i\in[k]}(1-T_i(x))\Big)^{-2} \\
&= a(x)b(x)\cdot\Big(\sum_{i\in[k]}\sum_{j=i}^{\infty}\frac{x^j}{j!}\Big)^{-2} \\
&= a(x)b(x)\cdot\Big(\sum_{i\in[k]} i\cdot\frac{x^i}{i!} + k\cdot\sum_{i=k+1}^{\infty}\frac{x^i}{i!}\Big)^{-2} \\
&= a(x)b(x)\cdot(x\cdot a(x)+k\cdot b(x))^{-2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Re_{\mathsf{AR/AP}}(k) &= 1 + k\cdot\int_0^{\infty} h(x)\mathrm{d}x \\
&= 1 + k\cdot\Big(\underbrace{\int_0^{k-\sqrt{6k}} h(x)\mathrm{d}x}_{h_1} + \underbrace{\int_{k-\sqrt{6k}}^{k} h(x)\mathrm{d}x}_{h_2} + \underbrace{\int_k^{\infty} h(x)\mathrm{d}x}_{h_3}\Big).
\end{aligned}
$$

We are going to upper bound h_1, h_2, h_3 separately.

Case 1: Bound h_1

$$
h_1 = \int_0^{k-\sqrt{6k}} h(x)\mathrm{d}x
$$

$$
\begin{aligned}
&\le \int_0^{k-\sqrt{6k}} \frac{b(x)}{x^2 \cdot e^x} \mathrm{d}x \\
&= \int_0^{k-\sqrt{6k}} \sum_{i=k+1}^{\infty} \frac{x^{i-2}}{i!} e^{-x} \mathrm{d}x \\
&= \int_{\sqrt{6k}}^{k} \underbrace{\sum_{i=k+1}^{\infty} \frac{(k-x)^{i-2}}{i!} e^{-(k-x)}}_{G(x)} \mathrm{d}x, \qquad (21)
\end{aligned}
$$

where the second step is by Part 3 of Fact 9, the third step is by definition of $b(x)$.

We define $G(x) \stackrel{\text{def}}{=} \sum_{i=k+1}^{\infty} \frac{(k-x)^{i-2}}{i!} e^{-(k-x)}$. So

$$
\begin{aligned}
G(0) &= \sum_{i=k+1}^{\infty} \frac{k^{i-2}}{i!} e^{-k} \\
&= \frac{1}{k^2} \sum_{i=k+1}^{\infty} \frac{k^i}{i!} e^{-k} \\
&= \frac{1}{k^2} (1 - \Gamma(1+k,k)/\Gamma(1+k)) \\
&\le \frac{1}{2k^2},
\end{aligned}
$$

where the last step is by Lemma 4 that $\Gamma(1+k,k)/k! > 1/2$.

For the ease of the proof, we define $d(i,x) = \frac{(k-x)^{i-2}}{i!} e^{-(k-x)}$. So that $G(x) = \sum_{i=k+1}^{\infty} d(i,x)$. we have $\forall i \ge k+1$,

$$
\begin{aligned}
\ln(d(i,x)/d(i,0)) &= \ln\left(\frac{(k-x)^{i-2} e^{-(k-x)}/i!}{k^{i-2} e^{-k}/i!}\right) \\
&= \ln((1-x/k)^{i-2} e^x) \\
&= x + (i-2)\ln(1-x/k) \\
&= x - (i-2) \sum_{j=1}^{\infty} \frac{1}{j} (\frac{x}{k})^j \\
&= x - i \sum_{j=1}^{\infty} \frac{1}{j} (\frac{x}{k})^j + 2 \sum_{j=1}^{\infty} \frac{1}{j} (\frac{x}{k})^j \\
&\le -i \sum_{j=2}^{\infty} \frac{1}{j} (\frac{x}{k})^j + 2 \sum_{j=1}^{\infty} \frac{1}{j} (\frac{x}{k})^j \\
&= -\frac{i}{2} (\frac{x}{k})^2 - \sum_{j=1}^{\infty} \left(\frac{i}{j+2} \frac{x^2}{k^2} - \frac{2}{j} \right) (\frac{x}{k})^j \\
&\le -\frac{i}{2} (\frac{x}{k})^2 \\
&\le -\frac{1}{2} \frac{x^2}{k},
\end{aligned}
$$

where the sixth step follows from $i \ge k$, and the eighth step follows from $x \ge \sqrt{6k}$. So $d(i,x) \le e^{-\frac{x^2}{2k}} d(i,0)$.

Therefore,

$$
G(x) = \sum_{i=k+1}^{\infty} d(i,x)
$$

$$\leq \sum_{i=k+1}^{\infty} e^{-\frac{x^2}{2k}} d(i,0)$$
$$\leq e^{-\frac{x^2}{2k}} G(0).$$

Thus, we have

$$\begin{aligned}
h_1 &\leq \int_{\sqrt{6k}}^{k} G(x)\mathrm{d}x \\
&\leq \int_{\sqrt{6k}}^{k} e^{-\frac{x^2}{2k}} G(0)\mathrm{d}x \\
&\leq \sqrt{k}G(0) \int_{\sqrt{6}}^{\sqrt{k}} e^{-\frac{x^2}{2}} \mathrm{d}x \\
&\leq \sqrt{k}G(0) \int_{\sqrt{6}}^{\infty} e^{-\frac{x^2}{2}} \mathrm{d}x \\
&\leq 0.018\sqrt{k}G(0) \\
&\leq 0.009k^{-1.5},
\end{aligned}$$

where the first step is by Eq.(21), the fifth step is by $\int_{\sqrt{6}}^{\infty} e^{-\frac{x^2}{2}}\mathrm{d}x \leq 0.018$, the last step is by $G(0) \leq k^{-2}/2$.

Case 2, Bound h_2

$$\begin{aligned}
h_2 &= \int_{k-\sqrt{6k}}^{k} \frac{a(x)b(x)}{(x \cdot a(x) + k \cdot b(x))^2}\mathrm{d}x \\
&\leq \int_{k-\sqrt{6k}}^{k} \frac{1}{4kx}\mathrm{d}x \\
&= \ln(k/(k-\sqrt{6k}))/(4k) \\
&\leq (k/(k-\sqrt{6k})-1)/(4k) \\
&= (\sqrt{6}/4) \cdot k^{-0.5}/(k-\sqrt{6k}) \\
&\leq (2\sqrt{6}/4) \cdot k^{-1.5},
\end{aligned}$$

where the first step follows from $(a+b)^2 \geq 4ab$, the last step is by $k - \sqrt{6k} \geq 0.5k$ for $k \geq 24$.

Case 3, Bound h_3.

We have

$$\begin{aligned}
h_3 &= \int_{k}^{\infty} \frac{a(x)b(x)}{(xa(x)+kb(x))^2}\mathrm{d}x \\
&\leq k^{-2} \cdot \int_{k}^{\infty} a(x)e^{-x}\mathrm{d}x \\
&\leq a(k) \cdot k^{-2} \int_{k}^{\infty} \frac{x^{k-1}e^{-x}}{k^{k-1}}\mathrm{d}x \\
&\leq \frac{e^k k!}{2k^2 \cdot k^k} \int_{k}^{\infty} x^{k-1}e^{-x}/(k-1)!\mathrm{d}x \\
&= \frac{e^k k!}{2k^2 \cdot k^k} \big(\Gamma(k,k)/(k-1)!\big) \\
&\leq \frac{e^k k!}{2k^2 \cdot k^k} \cdot (1/2) \\
&\leq \frac{e^k}{4k^2 \cdot k^k} \cdot (ek^{k+1/2}e^{-k})
\end{aligned}$$

$$= \quad e/4 \cdot k^{-1.5},$$

where the second step is by Part 4 of Fact 9 that for all $x \geq k + c_2\sqrt{k}$, $b(x)/(xa(x)+kb(x))^2 \leq e^{-x}/k^2$, the third step is by Part 5 of Fact 9, the fourth step is by Part 2 of Fact 9 and that $\forall x,\ a(x)+b(x) \leq e^x$, the fifth step is by definition of incomplete gamma function, the sixth step is by Lemma 4. the last step is by Lemma 5.

Therefore, we can upper bound

$$\begin{aligned}
\Re_{\mathsf{AR/AP}}(k) &= 1 + k \cdot \int_0^\infty h(x)\mathrm{d}x \\
&= 1 + k \cdot (h_1 + h_2 + h_3) \\
&\leq 1 + (0.009 + 2\sqrt{6}/4 + e/4)/\sqrt{k} \\
&\leq 1 + 2/\sqrt{k}.
\end{aligned}$$

This accomplishes the upper-bound part of Claim 6. □

B Mathematical Tools

Lemma 4 (Incomplete gamma function [OLBC10, Chapter 8])**.** *Define the incomplete gamma function* $\Gamma(n,x) \stackrel{def}{=} \int_x^\infty t^{n-1}e^{-t}\mathrm{d}t$*. Then for all positive integer* n*, we have*

1. $\frac{\Gamma(n,n)}{(n-1)!} < \frac{1}{2} < \frac{\Gamma(n,n-1)}{(n-1)!}$.
2. $\frac{\Gamma(n,x)}{(n-1)!} = e^{-x}\sum_{i=0}^{n-1}\frac{x^i}{i!}$.

Lemma 5 (Stirling's approximation [Rob55])**.** *For all positive* n*, the following holds:*

$$\sqrt{2\pi}n^{n+1/2}e^{-n} \leq n! \leq en^{n+1/2}e^{-n}.$$

It's easy to see the following facts:

Fact 7. *For all positive integer* n*, we have*

$$\frac{1}{2}\frac{4^n}{\sqrt{\pi n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}}$$

Fact 8. *For all positive integer* n*, we have*

$$\int_0^\infty e^{-x}x^n\mathrm{d}x = \Gamma(n+1) = n!.$$

Fact 9. $a(x)$ *and* $b(x)$ *satisfies the following facts:*

1. *For all* $x \in [0,k]$, $b(x) \cdot e^{-x} \leq 1/2$,
2. *For all* $x \in [k,\infty)$, $a(x) \cdot e^{-x} \leq 1/2$,
3. *For all* $x \in [0,k-1]$,
$$\frac{a(x)}{(xa(x)+kb(x))^2} \leq \frac{1}{x^2e^x},$$
4. *For all* $x \in [k,\infty)$,
$$\frac{b(x)}{(xa(x)+kb(x))^2} \leq \frac{1}{k^2e^x},$$
5. *For all* $x \in [k,\infty)$, $a(k) \cdot x^{k-1}/k^{k-1} \geq a(x)$.

Proof. **Part 1.** For all $x \in [0, k]$,

$$\begin{aligned} b(x) \cdot e^{-x} &= 1 - \Gamma(k+1, x)/k! \\ &\leq 1 - \Gamma(k+1, k)/k! \\ &\leq 1/2, \end{aligned}$$

where the first step is by Part 2 of Lemma 4, the second step is because $\Gamma(k+1, x)$ is a decreasing function on $x \in [0, \infty)$, the third step is by Part 1 of Lemma 4.

Part 2. By the same reason, for all $x \in [k, \infty)$, we have that

$$\begin{aligned} a(x) \cdot e^{-x} &= \Gamma(k, x)/(k-1)! \\ &\leq \Gamma(k, k)/(k-1)! \\ &\leq 1/2. \end{aligned}$$

Part 3. For all $x \in [0, k-1]$, we have

$$\begin{aligned} & (x \cdot a(x) + k \cdot b(x))^2 - x^2 a(x) e^x \\ = {} & x^2 a(x)^2 + 2xk \cdot a(x) b(x) + k^2 b(x)^2 - x^2 a(x) \big(a(x) + b(x) + x^k/k!\big) \\ \geq {} & 2xk \cdot a(x) b(x) - x^2 a(x) \big(b(x) + x^k/k!\big) \\ \geq {} & (2k - (k-1)) b(x) - x^{k+1}/k! \\ \geq {} & (k+1) \cdot x^{k+1}/(k+1)! - x^{k+1}/k! \\ = {} & 0, \end{aligned}$$

where the fourth step is by $b(x) \geq x^{k+1}/(k+1)!$. Therefore, $\frac{a(x)}{(xa(x)+kb(x))^2} \leq \frac{1}{x^2 e^x}$ follows directly.

Part 4. For all $x \in [k, \infty)$, we have

$$\begin{aligned} & (x \cdot a(x) + k \cdot b(x))^2 - k^2 b(x) e^x \\ = {} & x^2 a(x)^2 + 2xk \cdot a(x) b(x) + k^2 b(x)^2 - k^2 b(x) \big(a(x) + b(x) + x^k/k!\big) \\ \geq {} & 2xk \cdot a(x) b(x) - k^2 b(x) \big(a(x) + x^k/k!\big) \\ \geq {} & (2x - k) \cdot a(x) - x^k/(k-1)! \\ \geq {} & k \cdot x^k/k! - x^k/(k-1)! \\ = {} & 0, \end{aligned}$$

where the fourth step is by $a(x) \geq x^k/k!$. Therefore, $\frac{b(x)}{(xa(x)+kb(x))^2} \leq \frac{1}{k^2 e^x}$ follows directly.

Part 5. For all $x \geq k$, we have

$$\begin{aligned} a(k) \cdot x^{k-1}/k^{k-1} &\geq \sum_{i=0}^{k-1} \frac{k^i}{i!} \frac{x^{k-1}}{k^{k-1}} \\ &\geq \sum_{i=0}^{k-1} \frac{k^i}{i!} \frac{x^i}{k^i} \\ &\geq \sum_{i=0}^{k-1} \frac{x^i}{i!} \\ &= a(x). \end{aligned}$$

$\square$