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THE BOUSSINESQ SYSTEM WITH NON-SMOOTH BOUNDARY CONDITIONS : EXISTENCE, RELAXATION AND TOPOLOGY OPTIMIZATION. *

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Abstract. In this paper, we tackle a topology optimization problem which consists in finding the optimal shape of a solid located inside a fluid that minimizes a given cost function. The motion of the fluid is modeled thanks to the Boussinesq system which involves the unsteady Navier-Stokes equation coupled to a heat equation. In order to cover several models presented in the literature, we choose a non-smooth formulation for the outlet boundary conditions. This paper aims at proving existence of solutions to the resulting equations, along with the study of a relaxation scheme of the non-smooth conditions. A second part covers the topology optimization problem itself for which we proved the existence of optimal solutions and provides the definition of first order necessary optimality conditions.

Key words. Non-smooth boundary conditions, topology optimization, relaxation scheme, directional do-nothing boundary conditions

AMS subject classifications. 49K20, 49Q10, 76D03, 76D55

1. Introduction.

Directional do-nothing conditions. For many engineering applications, simulations of flows coupled with the temperature are useful for predicting the behaviour of physical designs before their manufacture, reducing the cost of the development of new products. The relevance of the model and the adequacy with the experiment therefore become important [20, 47, 53]. In this paper, we choose to model the flow with the Boussinesq system which involves the Navier-Stokes equations coupled with an energy equation. In most mathematical papers analyzing this model [12, 32, 54], homogeneous Dirichlet boundary conditions are considered on the whole boundary. This simplifies the mathematical analysis of the incompressible Navier-Stokes equation since the non-linear term vanishes after integrating by part hence simplifying the derivation of a priori estimates [11, 25, 31, 54].

However, several applications use different boundary conditions that model inlet, no-slip and outlet conditions [1]. Unlike the inlet and the no-slip conditions, the outlet conditions are more subject to modelling choices. A popular one consists in using a do-nothing outlet condition (see e.g. [8, 29, 30, 40, 52, 55]) which naturally comes from integration by parts when defining a weak formulation of the Navier-Stokes equations. However, since this outlet condition does not deal with re-entering flows, several papers use a non-smooth outlet boundary conditions for their numerical simulations (see e.g. [5, 27]). A focus on non-smooth outflow conditions when the temperature appears can be found in [16, 27, 48, 50].

We emphasize that such non-smooth boundary conditions can be used to solve problems involving reversal flows (or re-entering flows) which appear when modeling heat

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transfer and fluid flows driven by natural or mixed convection in open channels [27]. Among the potential applications, we can find the so-called *mur Trombe* [56], the wall solar chimney [7] or even the cooling of electronic equipment [39].

In particular, directional do-nothing (DDN) boundary conditions are non-smooth conditions that become popular. The idea is originally described in [17], and several other mathematical studies followed [5, 13, 15]. These conditions were considered especially for turbulent flows. In this situation, the flow may alternatively exit and re-enter the domain. These directional boundary conditions tries to capture this phenomenon, while limiting the reflection. It is worth noting that other boundary conditions can be used, namely the so-called local/global Bernouilli boundary conditions [16, 27, 50]. The latter implies the do-nothing boundary condition is satisfied for exiting fluid and that both the normal velocity gradient and the total pressure vanish for re-entering fluid. Nevertheless, in this paper, we are going to use non-smooth DDN boundary condition since they are easier to impose though a variational formulation.

Concerning the mathematical study of Boussinesq system with directional do-nothing conditions, the literature is rather scarce. To the best of our knowledge, we only found [6, 19], where the steady case is studied in depth, but the unsteady case only presents limited results. Indeed, while [19, p. 16, Theorem 3.2] gives existence and uniqueness of a weak solution with additional regularity to the steady-state Boussinesq system involving non-smooth boundary conditions at the inlet, it requires the source terms and the physical constants like for example the Reynolds number to be small enough. We emphasize that these limitations comes from the proof which relies on a fixed-point strategy. The first aim of this paper will then be to fill that gap by proving existence and, in a two-dimensional setting, uniqueness of solutions for the unsteady Boussinesq system with non-smooth DDN boundary condition at the outlet.

Topology optimization. On top of the previous considerations, this paper aims at using these equations in a topology optimization (TO) framework. In fluid mechanics, the term *topology optimization* refers to the problem of finding the shape of a solid located inside a fluid that optimizes a given physical effect. There exist various mathematical methods to deal with such problems that fall into the class of PDE-constrained optimization, such as the topological asymptotic expansion [3, 18, 46] or the shape optimization method [28, 44, 45]. In this paper, we choose to locate the solid thanks to a penalization term added in the unsteady Navier-Stokes equations, as exposed in [4]. However, the binary function introduced in [4] is usually replaced by a smooth approximation, referred as *interpolation function* [50], in order to be used in gradient-based optimization algorithms. We refer to the review papers [1, 26] for many references that deal with numerical resolution of TO problems applied to several different physical settings. However, as noted in [1, Section 4.7], most problems tackling topology optimization for flows only focus on steady flows, and time-dependant approaches are still rare. Furthermore, to the best of our knowledge, no paper is dedicated to the mathematical study of unsteady TO problems involving DDN boundary conditions, even though they are already used in numerical studies [16, 27, 48, 50]. Therefore, the second goal of this paper will be to prove existence of optimal solution to a TO problem involving Boussinesq system with non-smooth DDN boundary conditions at the outlet.

First order optimality conditions. As hinted above, a gradient based method is often used in order to compute an optimal solution of a TO problem. However, the introduction of the non-smooth DDN boundary conditions implies that the control-to-state mapping is no longer differentiable. The literature presents several ways to deal with such PDE-constrained optimization problems. Most focus on elliptic equations, using subdifferential calculus [21, 35, 23] or as the limit of relaxation schemes [9, 22, 41, 51]. We may also cite [43] for a semilinear parabolic case, [57] which involves the Maxwell equations, and [10] which analyzes the optimal control of an optical flow model. In the last reference, it should be noted that the relaxation is made on the nonsmooth initial condition, which is different from the nonsmoothness we have in our problem. We emphasize that using directly a subdifferential approach presents several drawbacks: the subdifferential of composite functions may be hardly computed, and the result may be hardly enlightening nor used [21]. We will therefore use a differentiable relaxation approach, as studied in [51]. First, we will be able to use standard first order necessary optimality conditions since the relaxed control-to-state mapping will be smooth. A convergence analysis will let us design necessary optimality condition for the non-smooth problem. Secondly, we find this approach more advantageous, as the approximated problem may be used as a numerical scheme for solving the TO problem.

1.1. Problem settings. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a bounded open set with Lipschitz boundary whose outward unitary normal is $\mathbf{n}$. We assume the fluid occupies a region $\Omega_f \subset \Omega$ and that a solid fills a region Ω_s such that $\Omega = \Omega_f \cup \Omega_s$. The penalized Boussinesq approximation (see e.g. [50] for the steady case) of the Navier-Stokes equations coupled to convective heat transfer reads:

$$
\begin{aligned}
&\nabla \cdot \mathbf{u} = 0, \\
&\partial_t \theta + \nabla \cdot (\mathbf{u}\theta) - \nabla \cdot (Ck(\alpha)\nabla\theta) = 0, \qquad \text{a.e. in } \Omega \\
&\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - A\Delta\mathbf{u} + \nabla p - B\theta\mathbf{e}_y + h(\alpha)\mathbf{u} = f, \\
&\mathbf{u}(0) = u_0(\alpha), \ \theta(0) = \theta_0(\alpha),
\end{aligned} \tag{1.1}
$$

where $\mathbf{u}$ denotes the velocity of the fluid, p the pressure and θ the temperature (all dimensionless), $u_0(\alpha), \theta_0(\alpha)$ are initial conditions. In (1.1), $A = \mathrm{Re}^{-1}$ with Re being the Reynolds number, $B = \mathrm{Ri}$ is the Richardson number and $C = (\mathrm{Re}\,\mathrm{Pr})^{-1}$ where Pr is the Prandtl number, $-\mathbf{e}_y$ is the direction in which the gravity acts on the flow. In a topology optimization problem, it is classical to introduce a function $\alpha : x \in \Omega \mapsto \alpha(x) \in \mathbb{R}^+$ as optimization parameter (see e.g. [1, 26]). The function $h(\alpha)$ then penalizes the flow in order to mimic the presence of a solid:

- if $h \equiv 0$, then one retrieves the classical Boussinesq approximation.
- if, for some large enough $\alpha_{\max} > 0$, $h : s \in [0, \alpha_{\max}] \mapsto h(s) \in [0, \alpha_{\max}]$ is a smooth function such that $h(0) = 0$ and $h(\alpha_{\max}) = \alpha_{\max}$, one retrieves the formulations used in topology optimization [1, 12, 50]. In the sequel, we work in this setting since we wish to study a TO problem.

Since the classical Boussinesq problem is retrieved when $h(\alpha) = 0$, the fluid zones $\Omega_f \subset \Omega$ and the solid ones $\Omega_s \subset \Omega$ can be defined as $\Omega_s := \{x \in \Omega \mid \alpha(x) < s_0\}$, $\Omega_f := \{x \in \Omega \mid \alpha(x) > s_0\}$, for some $s_0 \in (0, \alpha_{\max})$ and where $\alpha_{\max}$ is large enough to ensure the velocity $\mathbf{u}$ is small enough for the Ω_s above to be considered as a solid (see [4, Corollary 4.1]). Several examples of the function h can be found in the literature (see e.g. [1, 49]) such as

$$
h(s) = \alpha_{\max}\left(\frac{s}{\alpha_{\max}}\right)^p \text{ or } h(s) = \alpha_{\max}\left(\frac{1}{1 + \mathrm{e}^{-p(s-s_0)}} - \frac{1}{1 + \mathrm{e}^{ps_0}}\right)
$$

for some parameter $p \geq 1$ which is usually chosen large enough. The function $k(\alpha) : x \in \Omega \mapsto k(\alpha(x))$ is the dimensionless diffusivity defined as $k(\alpha)|_{\Omega_f} = 1$ and $k(\alpha)|_{\Omega_s} = k_s/k_f$ with k_s and k_f are respectively the diffusivities of the solid and the fluid. We also assume that k is a smooth regularization of $(k_s/k_f)\mathbf{1}_{\Omega_s} + \mathbf{1}_{\Omega_f}$. In this framework, α is thus defined as a parameter function, which will let us control the distribution of the solid in Ω.

Let us now specify the boundary conditions. Assume $\partial\Omega = \Gamma$ is Lipschitz and is split into three disjoint parts: $\Gamma = \Gamma_{\mathrm{w}} \cup \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$. Here, Γ_{w} are the walls, Γ_{in} the inlet/entrance and Γ_{out} is the exit/outlet of the computational domain.

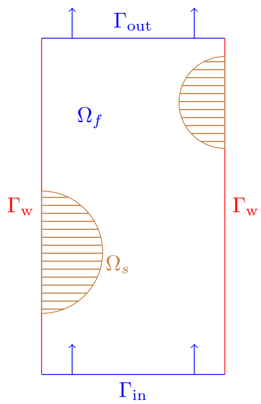

Fig. 1: Sketch of Ω

Let β be a function defined on Γ_{out} and define: $\forall x \in \mathbb{R}\ :\ x^+ = \mathrm{pos}(x) = \max(0,x), x^- = \mathrm{neg}(x) = \max(0,-x), x = x^+ - x^-$. Inspired by [17], we supplement (1.1) with the following boundary conditions:

(1.2)
$$\begin{aligned}
&\underline{\text{On } \Gamma_{\mathrm{in}}}: && \mathbf{u} = \mathbf{u}_{\mathrm{in}},\ \theta = 0,\\
&\underline{\text{On } \Gamma_{\mathrm{w}}}: && \mathbf{u} = 0,\ Ck\partial_n\theta = \phi,\\
&\underline{\text{On } \Gamma_{\mathrm{out}}}: && A\partial_n\mathbf{u} - \mathbf{n}p = A\partial_n\mathbf{u}^{\mathrm{ref}} - \mathbf{n}p^{\mathrm{ref}}\\
& && \qquad\qquad - \frac{1}{2}(\mathbf{u}\cdot\mathbf{n})^-(\mathbf{u} - \mathbf{u}^{\mathrm{ref}}),\\
& && Ck\partial_n\theta + \beta(\mathbf{u}\cdot\mathbf{n})^-\theta = 0,
\end{aligned}$$

with $\phi \in L^2(0,T;L^2(\Gamma_{\mathrm{w}}))$, $f \in L^2(0,T;(H^1(\Omega))')$, $\mathbf{u}_{\mathrm{in}} \in L^2(0,T;H^{1/2}_{00}(\Gamma_{\mathrm{in}}))$, $\partial_n = \mathbf{n}\cdot\nabla$ and $(\mathbf{u}^{\mathrm{ref}}, p^{\mathrm{ref}})$ denotes a reference solution.

As stated in [34], this nonlinear condition is physically meaningful: if the flow is outward, we impose the constraint coming from the selected reference flow ; if it is inward, we need to control the increase of energy, so, according to Bernoulli's principle, we add a term that is quadratic with respect to velocity.

Weak formulation. To define a weak formulation of (1.1)-(1.2), we introduce $V^u = \{\mathbf{u} \in H^1(\Omega)^d;\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}_{|\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{w}}} = 0\}$, and define H^u as the closure of V^u in $(L^2(\Omega))^d$. Similarly, we define $V^\theta = \{\theta \in H^1(\Omega);\ \theta_{|\Gamma_{\mathrm{in}}} = 0\}$, and $H^\theta = L^2(\Omega)$. We identify H^u and H^θ with their dual, and denote by $(V^u)'$ (resp. $(V^\theta)'$) the dual of V^u (resp. V^θ). Multiplying (1.1)-(1.2) with $\varphi \in V^\theta$ and integrating by parts, the result reads as:

$$\int_\Omega \partial_t\theta\varphi - \int_\Omega \theta\mathbf{u}\cdot\nabla\varphi + \int_\Omega Ck\nabla\theta\cdot\nabla\varphi + \int_\Gamma(\theta(\mathbf{u}\cdot\mathbf{n}) - Ck\partial_n\theta)\varphi = 0,$$

for all $\varphi \in V^\theta$. From (1.2), the boundary term reduces to:

$$\begin{aligned}
\int_\Gamma(\theta(\mathbf{u}\cdot\mathbf{n}) - Ck\partial_n\theta)\varphi &= -\int_{\Gamma_{\mathrm{w}}}\phi\varphi + \int_{\Gamma_{\mathrm{out}}}\left((\mathbf{u}\cdot\mathbf{n}) + \beta(\mathbf{u}\cdot\mathbf{n})^-\right)\theta\varphi\\
&\qquad - \int_{\Gamma_{\mathrm{out}}}\left(\beta\theta(\mathbf{u}\cdot\mathbf{n})^- + Ck\partial_n\theta\right)\varphi\\
&= -\int_{\Gamma_{\mathrm{w}}}\phi\varphi + \int_{\Gamma_{\mathrm{out}}}\left((\mathbf{u}\cdot\mathbf{n}) + \beta(\mathbf{u}\cdot\mathbf{n})^-\right)\theta\varphi,
\end{aligned}$$

and the weak form of the heat transfer equation is then

$$\text{(WF.1)}\quad \begin{aligned}
&\int_\Omega \partial_t\theta\varphi - \int_\Omega \theta\mathbf{u}\cdot\nabla\varphi + \int_\Omega Ck\nabla\theta\cdot\nabla\varphi + \int_{\Gamma_{\mathrm{out}}}\left((\mathbf{u}\cdot\mathbf{n}) + \beta(\mathbf{u}\cdot\mathbf{n})^-\right)\theta\varphi\\
&= \int_{\Gamma_{\mathrm{w}}}\phi\varphi,\ \forall\varphi \in V^\theta.
\end{aligned}$$

For the Navier-Stokes equations, we are going to use the next formula to replace the inertial term $(\mathbf{u}\cdot\nabla)\mathbf{u})$ by a symmetric one which helps to get a priori estimates (see also [14, 17]). For all $\mathbf{\Psi} \in V^u$, the latter is given as

$$\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{\Psi} = \frac{1}{2}\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{\Psi} - ((\mathbf{u}\cdot\nabla)\mathbf{\Psi})\cdot\mathbf{u} + \frac{1}{2}\int_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{\Psi}).$$

Multiplying (1.1) by $\mathbf{\Psi} \in V^u$, integrating by parts and using the boundary conditions, the weak formulation of the Navier-Stokes system is then defined as

$$\begin{aligned}
&\int_\Omega \partial_t\mathbf{u}\cdot\mathbf{\Psi} + \frac{1}{2}\{((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{\Psi} - ((\mathbf{u}\cdot\nabla)\mathbf{\Psi})\cdot\mathbf{u}\} + A\nabla\mathbf{u} : \nabla\mathbf{\Psi} + h(\alpha)\mathbf{u}\cdot\mathbf{\Psi} \\
\text{(WF.2)}\quad & - \int_\Omega B\theta\cdot\mathbf{e}_y\cdot\mathbf{\Psi} + \frac{1}{2}\int_{\Gamma_{\text{out}}}(\mathbf{u}\cdot\mathbf{n})^+(\mathbf{u}\cdot\mathbf{\Psi}) \\
&= \int_\Omega f\cdot\mathbf{\Psi} + \int_{\Gamma_{\text{out}}}(A\partial_n\mathbf{u}^{\text{ref}} - \mathbf{n}p^{\text{ref}})\cdot\mathbf{\Psi} + \frac{1}{2}\int_{\Gamma_{\text{out}}}(\mathbf{u}\cdot\mathbf{n})^-(\mathbf{u}^{\text{ref}}\cdot\mathbf{\Psi})
\end{aligned}$$

for all $\mathbf{\Psi} \in V^u$. A weak solution to (1.1)-(1.2) is then defined as $(\mathbf{u},\theta) \in L^2(0,T;V^u)\times L^2(0,T;V^\theta)$ such that $(\partial_t\mathbf{u},\partial_t\theta) \in (V^u)'\times(V^\theta)'$ and satisfying the weak formulations (WF) in the sense of $\mathcal{D}'(]0,T[)$.

1.2. The topology optimization problem. A goal of this paper is to analyze the next topology optimization problem

$$\text{(OPT)}\qquad \begin{aligned} &\min\ \mathcal{J}(\alpha,\mathbf{u},\theta) \\ &\text{s.t.}\ \begin{cases} (\mathbf{u},\theta) \text{ solution of (WF) parametrized by } \alpha, \\ \alpha \in \mathcal{U}_{ad}, \end{cases} \end{aligned}$$

where $\mathcal{J}$ is a given cost function. For some $\kappa > 0$, we set $\mathcal{U}_{\text{ad}} = \{\alpha \in \text{BV}(\Omega) : 0 \le \alpha(x) \le \alpha_{\max} \text{ a.e. on } \Omega,\ |D\alpha|(\Omega) \le \kappa\}$ where BV(Ω) stands for functions of bounded variations, and $|D\alpha|$ is the total variation of $D\alpha$, the distributional derivative of α which is a finite Radon measure in Ω. As shown in [2], the weak-* convergence in BV(Ω) is defined as follows: $(\alpha_\varepsilon)_\varepsilon \subset \text{BV}(\Omega)$ weakly-* converges to $\alpha \in \text{BV}(\Omega)$ if (α_ε) strongly converges to α in $L^1(\Omega)$ and $(D\alpha_\varepsilon)$ weakly-* converges to $D\alpha$ in Ω, meaning:

$$\lim_{\varepsilon\to 0}\int_\Omega \nu\, dD\alpha_\varepsilon = \int_\Omega \nu\, dD\alpha,\ \forall \nu \in C_0(\Omega),$$

where $C_0(\Omega)$ denotes the closure, in the sup norm, of the set of real continuous functions with compact support over Ω. We choose $\mathcal{U}_{\text{ad}}$ as a subset of BV(Ω) since it is a nice way to approximate piecewise constant functions, which is close to the desired solid distribution.

Remark 1.1. *The set $\mathcal{U}_{ad}$ has been used for instance in [24, 55] and have the property that any sequence $(\alpha_n)_n \subset \mathcal{U}_{ad}$ is bounded in $BV(\Omega)$ and thus have a subsequence that converges strongly in $L^1(\Omega)$ toward some $\alpha \in \mathcal{U}_{ad}$. It then has a further subsequence that converges almost everywhere in Ω toward α and thus $h(\alpha_n)$ and $k(\alpha_n)$ converge almost everywhere respectively toward $h(\alpha)$ and $k(\alpha)$. The last statement is going to be useful to prove some smoothness result on the control-to-state mapping $\alpha \mapsto (\mathbf{u}(\alpha),\theta(\alpha))$. In addition, we emphasize we may actually replace the above $\mathcal{U}_{ad}$ by any Banach space $\mathcal{B}_{ad}$ for which any $(\alpha_n)_n \subset \mathcal{B}_{ad}$ has a subsequence that converges toward some $\alpha \in \mathcal{B}_{ad}$ strongly in $L^p(\Omega)$ for $p \ge 1$.*

It is classical for these problems to compute first order optimality conditions (see e.g. [38, 49]). This approach needs smoothness of the control-to-state mapping. However, the presence of the non-differentiable function $\text{neg}(x) = x^-$ makes this approach hardly used in practice. Therefore, we adopt a smoothing approach, as studied in [41, 51], and we approximate the neg function with a C^1 positive approximation, denoted neg_ε. We suppose this approximation satisfies the following assumptions:

(**A1**) $\forall s \in \mathbb{R}$, $\text{neg}_\varepsilon(s) \geq \text{neg}(s)$.
(**A2**) $\forall s \in \mathbb{R}$, $-1 \leq \text{neg}'_\varepsilon(s) \leq 0$.
(**A3**) neg_ε converges to neg uniformly over $\mathbb{R}$.
(**A4**) for every $\delta > 0$, the sequence $(\text{neg}'_\varepsilon)_{\varepsilon>0}$ converges uniformly to 0 on $[\delta, +\infty)$ and uniformly to -1 on $(-\infty, -\delta]$ as $\varepsilon \to 0$.

As presented in [51], we may choose:

$$\text{neg}_\varepsilon(s) = \begin{cases} s^- & \text{if } |s| \geq \frac{\varepsilon}{2}, \\ \frac{1}{2}\left(\frac{1}{2} - \frac{s}{\varepsilon}\right)^3 \left(\frac{3\varepsilon}{2} + s\right) & \text{if } |s| < \frac{\varepsilon}{2}. \end{cases} \tag{1.3}$$

We also introduce the notation $\text{pos}_\varepsilon(s) = s + \text{neg}_\varepsilon(s)$. Note that the function pos_ε is non-negative, since for any $s \in \mathbb{R}$, $\text{pos}_\varepsilon(s) = s + \text{neg}_\varepsilon(s) \geq s + \text{neg}(s) = \text{pos}(s) \geq 0$.

Remark that, owing to the mean value theorem, (**A2**)-(**A3**) imply that, for all $x \in \mathbb{R}$ and for ε small enough

$$|\text{neg}_\varepsilon(x)| \leq |x| + O(\varepsilon). \tag{1.4}$$

We redefine (WF) with an approximation of s^- and s^+, which gives:

$$\begin{aligned} &\int_\Omega \partial_t \theta_\varepsilon \varphi - \int_\Omega \theta_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\varphi + \int_{\Gamma_{\text{out}}} \left((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})\right) \theta_\varepsilon \varphi \\ &+ \int_\Omega Ck\nabla\theta_\varepsilon \cdot \nabla\varphi = \int_{\Gamma_{\text{w}}} \phi\varphi. \end{aligned} \tag{WFe.1}$$

$$\begin{aligned} &\int_\Omega \partial_t \mathbf{u}_\varepsilon \cdot \mathbf{\Psi} + \frac{1}{2}\left\{((\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon) \cdot \mathbf{\Psi} - ((\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{\Psi}) \cdot \mathbf{u}_\varepsilon\right\} + A\nabla\mathbf{u}_\varepsilon : \nabla\mathbf{\Psi} \\ &+ \int_\Omega h(\alpha)\mathbf{u}_\varepsilon \cdot \mathbf{\Psi} - B\theta_\varepsilon \cdot \mathbf{e}_y \cdot \mathbf{\Psi} + \frac{1}{2}\int_{\Gamma_{\text{out}}} \text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \mathbf{\Psi}) \\ &= \int_\Omega f \cdot \mathbf{\Psi} + \int_{\Gamma_{\text{out}}} (A\partial_n \mathbf{u}^{\text{ref}} - \mathbf{n}p^{\text{ref}}) \cdot \mathbf{\Psi} + \frac{1}{2}\int_{\Gamma_{\text{out}}} \text{neg}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}^{\text{ref}} \cdot \mathbf{\Psi}) \end{aligned} \tag{WFe.2}$$

for all $(\mathbf{\Psi}, \varphi) \in V^u \times V^\theta$.

We then define the approximate optimal control problem:

$$\begin{aligned} &\min \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon) \\ &\text{s.t.} \begin{cases} (\mathbf{u}_\varepsilon, \theta_\varepsilon) \text{ solution of (WFe.1)} - \text{(WFe.2) parametrized by } \alpha_\varepsilon, \\ \alpha_\varepsilon \in \mathcal{U}_{ad}. \end{cases} \end{aligned} \tag{OPTe}$$

As it will be made clear later, the control-to-state mapping in (WFe.1)-(WFe.2) is smooth, which will let us derive first order conditions.

1.3. Summary of the paper. The rest of this introduction is dedicated to the presentation of some notations used in this article and some important results of the literature. The core of this paper is organized in two sections. First, we will prove, in Theorem 2.4, the existence of solutions to (WFe), which will let us prove, with a compactness argument, the existence of solutions to (WF). This latter result is proved in Theorem 2.5. We then focus on the two dimensional case, where we prove uniqueness of the solutions in Proposition 2.7. This will let us prove stronger convergence results in the corollaries 2.9 and 2.10, which will be useful for the analysis of the optimization problem. This is an extension of the work done by [17], where only the pressure and the velocity were considered, and to [6, 19], where the steady case was studied in depth, but the results concerning the unsteady case were obtained using restrictive assumptions. We then study the approximate optimal control problem (OPTe), for which we will derive first order conditions in the Theorem 3.7. We conclude this paper with the convergence of the optimality conditions of (OPTe) in Lemma 3.9, which let us design first order conditions of (OPT).

Notations. We set $a \lesssim b$ if there exists a constant $C(\Omega) > 0$ depending only on Ω such that $a \leq C(\Omega)b$. Denote:

- $\mathcal{A} : V^u \to (V^u)'$ defined by $\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle_{(V^u)',V^u} = A\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}$,
- $\mathcal{B} : V^u \times V^u \to (V^u)'$ defined by $\langle \mathcal{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{(V^u)',V^u} = \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}$,
- $\mathcal{T} : V^\theta \to (V^u)'$ defined by $\langle \mathcal{T}\theta, \mathbf{v}\rangle_{(V^u)',V^u} = \int_\Omega B\theta\mathbf{e}_y\cdot\mathbf{v}$,
- $\mathcal{P} : V^u \times V^u \to (V^u)'$ defined by $\langle \mathcal{P}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} \text{pos}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})$,
- $\mathcal{P}_\varepsilon : V^u \times V^u \to (V^u)'$ given by $\langle \mathcal{P}_\varepsilon(\mathbf{u}, \mathbf{v})), \mathbf{w}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} \text{pos}_\varepsilon\,(\mathbf{u}\cdot\mathbf{n})\,(\mathbf{v}\cdot\mathbf{w})$.
- $\mathcal{N} : V^u \times V^u \to (V^u)'$ defined by $\langle \mathcal{N}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} \text{neg}(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{w})$,
- $\mathcal{N}_\varepsilon : V^u \times V^u \to (V^u)'$ given by $\langle \mathcal{N}_\varepsilon(\mathbf{u}, \mathbf{v})), \mathbf{w}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} \text{neg}_\varepsilon\,(\mathbf{u}\cdot\mathbf{n})\,(\mathbf{v}\cdot\mathbf{w})$.
- $\mathcal{C}(\alpha) : V^\theta \to (V^\theta)'$ defined by $\langle \mathcal{C}(\alpha)\theta, \varphi\rangle_{(V^\theta)',V^\theta} = \int_\Omega Ck(\alpha)\nabla\theta\cdot\nabla\varphi$,
- $\mathcal{D} : V^u \times V^\theta \to (V^\theta)'$ defined by $\langle \mathcal{D}(\mathbf{u}, \theta), \varphi\rangle_{(V^\theta)',V^\theta} = \int_\Omega \theta\mathbf{u}\cdot\nabla\varphi$,
- $\mathcal{M} : V^u \times V^\theta \to (V^\theta)'$ defined by $\langle \mathcal{M}(\mathbf{u}, \theta), \varphi\rangle_{(V^\theta)',V^\theta} = \int_{\Gamma_{\text{out}}} ((\mathbf{u}\cdot\mathbf{n}) + \beta\text{neg}(\mathbf{u}\cdot\mathbf{n}))\theta\varphi$,
- $\mathcal{M}_\varepsilon : V^u \times V^\theta \to (V^\theta)'$ defined by $\langle \mathcal{M}(\mathbf{u}, \theta), \varphi\rangle_{(V^\theta)',V^\theta} = \int_{\Gamma_{\text{out}}} ((\mathbf{u}\cdot\mathbf{n}) + \beta\text{neg}_\varepsilon\,(\mathbf{u}\cdot\mathbf{n}))\theta\varphi$.

We will also denote by σ^{ref} the element of $(V^u)'$ defined by $\langle \sigma^{\text{ref}}, \mathbf{w}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} (A\partial_n\mathbf{u}^{\text{ref}} - p^{\text{ref}}\mathbf{n})\cdot\mathbf{w}$, $h(\alpha) : V^u \to (V^u)'$ the function defined by $\langle h(\alpha)\mathbf{u}, \mathbf{v}\rangle_{(V^u)',V^u} = \int_\Omega h(\alpha)\mathbf{u}\cdot\mathbf{v}$, and ϕ the element of $(V^\theta)'$ defined by $\langle \phi, \varphi\rangle_{(V^\theta)',V^\theta} = \int_{\Gamma_{\text{out}}} \phi\varphi$.

Results from the literature. We now recall two results that will be heavily used throughout this paper.

PROPOSITION 1.2. *([14, Proposition III.2.35]) Let Ω be a Lipschitz domain of $\mathbb{R}^d$ with compact boundary. Let $p \in [1, +\infty]$ and $q \in [p, p^*]$, where p^* is the critical exponent associated with p, defined as:*

$$\begin{cases} \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d} & \textit{for } p < d, \\ p^* \in [1, +\infty[& \textit{for } p = d, \\ p^* = +\infty & \textit{for } p > d. \end{cases}$$

Then, there exists a positive constant C such that, for any $u \in W^{1,p}(\Omega)$:

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{L^p(\Omega)}^{1+\frac{d}{q}-\frac{d}{p}}\|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p}-\frac{d}{q}}.$$

PROPOSITION 1.3. *([14, Theorem III.2.36]) Let Ω be a Lipschitz domain of $\mathbb{R}^d$ with compact boundary, and $1 < p < d$. Then for any $r \in \left[p, \frac{p(d-1)}{d-p}\right]$, there exists a positive constant C such that, for any $u \in W^{1,p}(\Omega)$:*

$$\|u_{|\partial\Omega}\|_{L^r(\partial\Omega)} \leq C\|u\|_{L^p(\Omega)}^{1-\frac{d}{p}+\frac{d-1}{r}} \|u\|_{W^{1,p}(\Omega)}^{\frac{d}{p}-\frac{d-1}{r}}.$$

In the case $p = d$, the previous result holds true for any $r \in [p, +\infty[$.

2. Existence of solutions. In this section, we will focus on proving the existence of solutions to (WFe) and prove their convergence toward the ones of (WF).

We make the following assumptions throughout this paper:

ASSUMPTIONS 2.1.

- *The source term $f \in L^2(0, T; (H^1(\Omega))')$.*
- *$(\mathbf{u}^{ref}, p^{ref})$ are such that:*

$$\begin{cases} \mathbf{u}^{ref} \in L^r(0, T; (H^1(\Omega))^d) \cap L^\infty(0, T; (L^2(\Omega))^d) \\ \text{with } r = 2 \text{ if } d = 2 \text{ and } r = 4 \text{ if } d = 3, \\ \nabla \cdot \mathbf{u}^{ref} = 0, \\ \partial_t \mathbf{u}^{ref} \in L^2(0, T; (L^2(\Omega))^d), \\ \mathbf{u}^{ref} = 0 \text{ on } \Gamma_w \\ \mathbf{u}^{ref} = \mathbf{u}_{in} \text{ on } \Gamma_{in}. \end{cases}$$

  and $A\partial_n \mathbf{u}^{ref} - p^{ref}\mathbf{n} \in L^2(0, T; H^{-\frac{1}{2}}(\partial\Omega))$.
- *There exists k_{min} such that $k(x) \geq k_{min} > 0$ and $h(x) \geq 0$ for a.e. $x \in \Omega$.*
- *The initial condition $\mathbf{u}_0$ (resp. θ_0) is a Fréchet-differentiable function from $\mathcal{U}_{ad}$ to V^u (resp. V^θ). Furthermore, for all $\alpha \in \mathcal{U}_{ad}$, $\mathbf{u}_0(\alpha)|_{\Gamma_{in}} = \mathbf{u}_{in}(0)$, $\mathbf{u}_0(\alpha)|_{\Gamma_w} = 0$, and $\theta_0(\alpha)|_{\Gamma_{in}} = 0$.*
- *$\beta \in L^\infty(0, T; L^\infty(\Gamma_{out}))$ such that $\beta(t, x) \geq \frac{1}{2}$, for a.e. $(t, x) \in [0, T] \times \Gamma_{out}$.*

2.1. Existence in dimension 2 or 3. In this part, we work with a fixed $\varepsilon > 0$ and a given α_ε in $\mathcal{U}_{\text{ad}}$.

To prove the existence of solutions to (WFe), we follow the classical Fadeo-Galerkin method as used in [17, 42, 54]. By construction, V^u and V^θ are separable. Therefore, both admit a countable Hilbert basis $(w_k^u)_k$ and $(w_k^\theta)_k$. Let us construct an approximate problem, which will converge to a solution of the original problem (WFe). Denote by V_n^u (resp. V_n^θ) the space spanned by $(w_k^u)_{k\leq n}$ (resp. $(w_k^\theta)_{k\leq n}$). We consider the following Galerkin approximated problem:
find $t \mapsto \mathbf{v}_n(t) \in V_n^u$ and $t \mapsto \theta_n(t) \in V_n^\theta$ such that, defining $\mathbf{u}_n = \mathbf{v}_n + \mathbf{u}^{ref}$, $(\mathbf{u}_n, \theta_n)$ satisfy (WFe) for all $t \in [0, T]$ and for all $(\mathbf{\Psi}, \varphi) \in V_n^u \times V_n^\theta$.

With a similar pattern of proof as in [54, p.283], such $(\mathbf{u}_n, \theta_n)$ exist. We now prove that these solutions are bounded uniformly with respect to n and ε:

PROPOSITION 2.2. *There exist positive constants c_1^θ, c_2^θ, $c_1^{\mathbf{v}}$ and $c_2^{\mathbf{v}}$, independent of ε and n, such that:*

$$\sup_{[0,T]} \|\theta_n\|_{L^2(\Omega)} \leq c_1^\theta, \tag{2.1}$$

$$\int_0^T \|\nabla\theta_n\|_{L^2(\Omega)}^2 \leq c_2^\theta, \tag{2.2}$$

$$\text{(2.3)} \qquad \sup_{[0,T]} \|\mathbf{v}_n\|_{L^2(\Omega)} \leq c_1^{\mathbf{v}}, \qquad\qquad \text{(2.4)} \qquad \int_0^T \|\nabla \mathbf{v}_n\|^2_{L^2(\Omega)} \leq c_2^{\mathbf{v}}.$$

Proof. Taking $\varphi_n = \theta_n$ in (WFe.1) and integrating by part give:

$$\frac{d}{dt}\|\theta_n\|^2_{L^2(\Omega)} - \frac{1}{2}\int_{\Gamma_{\text{out}}} \theta_n^2(\mathbf{u}_n \cdot \mathbf{n}) + \int_\Omega Ck|\nabla\theta_n|^2 + \int_{\Gamma_{\text{out}}} \left((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon\left(\mathbf{u}_n \cdot \mathbf{n}\right)\right)\theta_n^2 = \int_{\Gamma_{\text{w}}} \phi\theta_n.$$

Since $\beta \geq \frac{1}{2}$ and using assumption (**A1**), one has on Γ_{out}:

$$\begin{aligned}\left((\mathbf{u}_n \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon\left(\mathbf{u}_n \cdot \mathbf{n}\right)\right)\theta_n^2 - \frac{1}{2}(\mathbf{u}_n \cdot \mathbf{n})\theta_n^2 &\geq \frac{1}{2}\left((\mathbf{u}_n \cdot \mathbf{n}) + \text{neg}_\varepsilon\left(\mathbf{u}_n \cdot \mathbf{n}\right)\right)\theta_n^2 \\ &\geq \frac{1}{2}\text{pos}_\varepsilon\left(\mathbf{u}_n \cdot \mathbf{n}\right)\theta_n^2 \geq 0.\end{aligned}$$

Therefore: $\frac{d}{dt}\|\theta_n\|^2_{L^2(\Omega)} + Ck_{\min}\|\nabla\theta_n\|^2_{L^2(\Omega)} \leq \|\phi\|_{L^2(\Gamma_{\text{w}})}\|\theta_n\|_{L^2(\Gamma_{\text{w}})}$. Using the continuity of the trace operator and Young's inequality, one proves that there exists a positive constant $C(\Omega)$ such that, for any $\nu > 0$:

$$\frac{d}{dt}\|\theta_n\|^2_{L^2(\Omega)} + Ck_{\min}\|\nabla\theta_n\|^2_{L^2(\Omega)} \leq \frac{1}{2\nu}\|\phi\|^2_{L^2(\Gamma_{\text{w}})} + \frac{C(\Omega)\nu}{2}(\|\theta_n\|^2_{L^2(\Omega)} + \|\nabla\theta_n\|^2_{L^2(\Omega)}).$$

Taking ν small enough, we are left with:

$$\frac{d}{dt}\|\theta_n\|^2_{L^2(\Omega)} \leq \frac{1}{2\nu}\|\phi\|^2_{L^2(\Gamma_{\text{w}})} + \frac{C(\Omega)\nu}{2}\|\theta_n\|^2_{L^2(\Omega)}.$$

Integrating this equation and using Gronwall's lemma then give (2.1) and (2.2).

Now, take $\mathbf{\Psi}_n = \mathbf{v}_n$ in (WFe.2). After some calculations, one gets:

$$\begin{aligned}&\frac{d}{dt}|\mathbf{v}_n|^2 + A|\nabla\mathbf{v}_n|^2 + \frac{1}{2}\int_{\Gamma_{\text{out}}} \text{pos}_\varepsilon\left(\mathbf{u}_n \cdot \mathbf{n}\right)|\mathbf{v}_n|^2 + \int_\Omega h|\mathbf{v}_n|^2 \\ &= \int_\Omega f_\theta \cdot \mathbf{v}_n - \int_\Omega \partial_t\mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n - A\int_\Omega \nabla\mathbf{u}^{\text{ref}} : \nabla\mathbf{v}_n - \int_\Omega h\mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n \\ &\quad - \int_\Omega (\mathbf{u}_n \cdot \nabla)\mathbf{u}^{\text{ref}} \cdot \mathbf{v}_n + \int_{\Gamma_{\text{out}}} (A\partial_n\mathbf{u}^{\text{ref}} - \mathbf{n}p^{\text{ref}})\mathbf{v}_n\end{aligned}$$

where $f_\theta = f + B\theta_n\mathbf{e}_y$. First, using (2.2), one has $\|f_\theta\|_{(H^u)'} \leq \|f\|_{(H^u)'} + Bc_1^\theta$. Secondly, (**A1**) gives that $\int_{\Gamma_{\text{out}}} \text{neg}_\varepsilon\left(\mathbf{u}_n \cdot \mathbf{n}\right)|\mathbf{v}_n|^2 \geq 0$. Following then the same pattern of proof as in [17, Proposition 2], one proves (2.3) and (2.4). ∎

Following [14, 54], we need to bound the fractional derivatives of the solution in order to prove some convergence results. For any real-valued function f defined on $[0,T]$, define by $\tilde{f}$ the extension by 0 of f to the whole real line $\mathbb{R}$, and by $\mathscr{F}(\tilde{f})$ the Fourier transform of $\tilde{f}$, which we define as: $\mathscr{F}(\tilde{f})(\tau) = \int_{\mathbb{R}} \tilde{f}(t)e^{-it\tau}dt$. Using the Hausdorff-Young inequality [14, Theorem II.5.20] we can prove the

Proposition 2.3. *For all $\sigma \in [0, \frac{1}{6})$, there exists a constant $C(\sigma) > 0$ independent of ε and n such that:*

$$\int_{\mathbb{R}} |\tau|^{2\sigma} \left\|\mathscr{F}\left(\widetilde{\theta_n}\right)\right\|^2_{(L^2(\Omega))^d} \leq C(\sigma), \tag{2.5}$$

$$\int_{\mathbb{R}} |\tau|^{2\sigma} \|\mathscr{F}(\widetilde{\mathbf{u}_n})\|^2_{L^2(\Omega)} \leq C(\sigma). \tag{2.6}$$

Proof. We emphasize that (2.6) is proved if (2.5) holds by using [14, Proposition VII.1.3] by replacing f by $f_\theta = f + B\theta\mathbf{e}_y$. The proof of (2.5) consists in adapting the one of [14, Proposition VII.1.3] and is thus omitted. ∎

Combining the two previous results, we now have the following existence theorem for (WFe).

THEOREM 2.4. *For all $(\mathbf{v}_0, \theta_0) \in H^u \times H^\theta$ and all $T > 0$, there exists $\mathbf{v}_\varepsilon \in L^\infty(0,T;H^u) \cap L^2(0,T;V^u)$, $\theta_\varepsilon \in L^\infty(0,T,H^\theta) \cap L^2(0,T;V^\theta)$ solution of (WFe) such that, defining $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{u}^{ref}(0)$ and $\mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{u}^{ref}$, one has for all $(\mathbf{\Psi}, \varphi) \in V^u \times V^\theta$: $\left(\int_\Omega \mathbf{u}_\varepsilon \cdot \mathbf{\Psi}\right)(0) = \int_\Omega \mathbf{u}_0 \cdot \mathbf{\Psi}$, $\left(\int_\Omega \theta_\varepsilon \varphi\right)(0) = \int_\Omega \theta_0 \varphi$. Moreover, one has $\mathbf{v}'_\varepsilon = \frac{d\mathbf{v}_\varepsilon}{dt} \in L^{\frac{4}{3}}(0,T;(V^u)')$ and $\theta'_\varepsilon \in L^{\frac{4}{3}}(0,T;(V^\theta)')$.*

Proof. The proof of existence is similar to part (iv) of the proof of [54, Theorem 3.1] and the proof of [14, Proposition VII.1.4], where estimates (2.1)-(2.4) and (2.5)-(2.6) are used in a compactness argument.

We only add the proof that $(\mathbf{u}_n, \theta_n)$ converges to a solution of (WFe.1). Using (2.1), (2.2), (2.5) and [54, Theorem 2.2], one shows that, up to a subsequence, θ_n strongly converges to an element θ_ε of $L^2(0,T;H^\theta)$, weakly converges in $L^2(0,T;V^\theta)$, and weak-$\star$ converges in $L^\infty(0,T;L^2(\Omega))$. These results imply that θ_n strongly converges to θ_ε in $L^2(0,T;L^2(\Gamma))$ thanks to Proposition 1.3. The only technical points which need more details are the non-linear terms in (WFe.1). Using the strong convergence of $\mathbf{u}_n$ to $\mathbf{u}_\varepsilon$ in $L^2(0,T;H^u)$ proved in [54, Eq (3.41)], one proves that $(\theta_n \mathbf{u}_n)$ strongly converges to $\theta_\varepsilon \mathbf{u}_\varepsilon$ in $L^1(0,T;L^2(\Omega))$. Furthermore, notice that:

$$\begin{aligned}
\int_0^T \|(\mathbf{u}_n \cdot \mathbf{n})\theta_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Gamma)} \leq &\int_0^T \|\mathbf{u}_n\|^{\frac{4}{3}}_{L^{\frac{8}{3}}(\Gamma)} \|\theta_n\|^{\frac{4}{3}}_{L^{\frac{8}{3}}(\Gamma)} \\
\leq & C \int_0^T \|\mathbf{u}_n\|^{\frac{1}{3}}_{L^2(\Omega)} \|\theta_n\|^{\frac{1}{3}}_{L^2(\Omega)} \|\mathbf{u}_n\|_{H^1(\Omega)} \|\theta_n\|_{H^1(\Omega)} \\
\leq & C \|\mathbf{u}_n\|^{\frac{1}{3}}_{L^\infty(0,T;L^2(\Omega))} \|\theta_n\|^{\frac{1}{3}}_{L^\infty(0,T;L^2(\Omega))} \\
& \quad \|\mathbf{u}_n\|_{L^2(0,T;H^1(\Omega))} \|\theta_n\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

This inequality together with (2.1)-(2.4) proves that $((\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$ is bounded in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Gamma))$, which is reflexive. Therefore, it proves that, up to a subsequence, there exists a weak limit κ_1 in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Gamma))$ of $((\mathbf{u}_n \cdot \mathbf{n})\theta_n)_n$. A simple adaptation of the above reasoning proves that $(\mathrm{neg}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n})\, \theta_n)_n$ weakly converges to some κ_2 in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Gamma))$. Using the strong convergence of θ_n in $L^2(0,T;L^2(\Gamma))$, [14, Proposition II.2.12] implies that:

$$((\mathbf{u}_n \cdot \mathbf{n}) + \beta \mathrm{neg}_\varepsilon (\mathbf{u}_n \cdot \mathbf{n}))\theta_n \rightharpoonup ((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \mathrm{neg}_\varepsilon (\mathbf{u}_\varepsilon \cdot \mathbf{n}))\theta_\varepsilon \text{ in } L^{\frac{4}{3}}(0,T;L^1(\Gamma))$$

obtained using the uniform Lipschitz continuity with respect to ε of $s \in \mathbb{R} \mapsto \mathrm{neg}_\varepsilon(s)$. By uniqueness of the limit in the sense of distribution, we can identify $\kappa_1 + \beta\kappa_2$ with $((\mathbf{u}_\varepsilon \cdot \mathbf{n}) + \beta \mathrm{neg}_\varepsilon (\mathbf{u}_\varepsilon \cdot \mathbf{n}))\theta_\varepsilon$. Therefore, $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ is a solution of (WF.1).

The convergence of the weak derivative with respect to time of $\mathbf{v}_\varepsilon$ in $L^{\frac{4}{3}}(0,T;(V^u)')$ is proved in [14, Proposition V.1.3]. Concerning the weak derivative with respect to time of θ_ε, remark that, for all $\varphi \in V^\theta$ with $\varphi \neq 0$:

$$\begin{aligned}
\frac{\langle \partial_t \theta_n, \varphi \rangle_{(V^\theta)', V^\theta}}{\|\varphi\|_{V^\theta}} = & \frac{1}{\|\varphi\|_{V^\theta}} (\langle \mathcal{D}(\mathbf{u}_n, \theta_n), \varphi \rangle_{(V^\theta)', V^\theta} - \langle \mathcal{C}(\alpha_\varepsilon)\theta_n, \varphi \rangle_{(V^\theta)', V^\theta} \\
& - \langle \mathcal{M}_\varepsilon(\mathbf{u}_n, \theta_n), \varphi \rangle_{(V^\theta)', V^\theta} + \langle \phi, \varphi \rangle_{(V^\theta)', V^\theta}).
\end{aligned}$$

Using Proposition 1.2 we prove the following inequalities:

$$\begin{aligned}\frac{\langle \mathcal{D}(\mathbf{u}_n,\theta_n),\varphi\rangle_{(V^\theta)',V^\theta}}{\|\varphi\|_{V^\theta}} &= \frac{\int_\Omega \theta\mathbf{u}\cdot\nabla\varphi}{\|\varphi\|_{V^\theta}}\\ &\lesssim \|\theta_n\|_{L^4(\Omega)}\|\mathbf{u}_n\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\|\varphi\|_{V^\theta}^{-1}\\ &\lesssim \left(\|\theta_n\|_{L^2(\Omega)}\|\mathbf{u}_n\|_{L^2(\Omega)}\right)^{\frac{1}{4}}\left(\|\theta_n\|_{H^1(\Omega)}\|\mathbf{u}_n\|_{H^1(\Omega)}\right)^{\frac{3}{4}},\end{aligned}$$

$$\frac{\langle \mathcal{C}(\alpha_\varepsilon)\theta_n,\varphi\rangle_{(V^\theta)',V^\theta}}{\|\varphi\|_{V^\theta}} = \frac{\int_\Omega Ck(\alpha_\varepsilon)\nabla\theta\cdot\nabla\varphi}{\|\varphi\|_{V^\theta}}\lesssim \|\nabla\theta_n\|_{L^2(\Omega)}\lesssim\|\theta_n\|_{H^1(\Omega)}.$$

Using now Proposition 1.3, we obtain

$$\begin{aligned}\int_{\Gamma_{\text{out}}}\operatorname{neg}_\varepsilon(\mathbf{u}_n\cdot\mathbf{n})\,\theta_n\varphi &\lesssim \|\mathbf{u}_n\|_{L^2(\Gamma)}\|\theta_n\|_{L^4(\Gamma)}\|\varphi\|_{L^4(\Gamma)}\\ &\lesssim \|\mathbf{u}_n\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{u}_n\|_{H^1(\Omega)}^{\frac{1}{2}}\|\theta_n\|_{H^1(\Omega)}\|\varphi\|_{H^1(\Omega)},\end{aligned}$$

which helps to get the inequality:

$$\frac{\langle \mathcal{M}_\varepsilon(\mathbf{u}_n,\theta_n),\varphi\rangle_{(V^\theta)',V^\theta}}{\|\varphi\|_{V^\theta}}\lesssim \|\mathbf{u}_n\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{u}_n\|_{H^1(\Omega)}^{\frac{1}{2}}\|\theta_n\|_{H^1(\Omega)}.$$

Since $(\mathbf{u}_n)$ is bounded in $L^2(0,T;H^1(\Omega)^d)\cap L^\infty(0,T;L^2(\Omega)^d)$ (the same goes for (θ_n)), these inequalities prove that $(\partial_t\theta_n)_n$ is bounded in $L^{\frac{4}{3}}(0,T;(V^\theta)')$. Therefore, $(\partial_t\theta_n)_n$ weakly converges in $L^{\frac{4}{3}}(0,T;(V^\theta)')$. By continuity of the weak derivative with respect to time, this weak limit needs to be $\partial_t\theta_\varepsilon$. ∎

We now use the existence of solutions to the approximate problem (WFe) to prove existence of solutions to the limit problem (WF), along with the convergence of the approximate solutions to those of (WF).

THEOREM 2.5. *Let $(\alpha_\varepsilon)\subset\mathcal{U}_{ad}$ and $\alpha\in\mathcal{U}_{ad}$ such that $\alpha_\varepsilon\overset{*}{\rightharpoonup}\alpha$ in $BV(\Omega)$. Define by $(\mathbf{v}_\varepsilon,\theta_\varepsilon)$ a solution of (WFe) parametrized by α_ε, and define $\mathbf{u}_\varepsilon=\mathbf{v}_\varepsilon+\mathbf{u}^{ref}$. Then, there exists $(\mathbf{v},\theta)\in L^\infty(0,T;H^u)\cap L^2(0,T;V^u)\times L^\infty(0,T,H^\theta)\cap L^2(0,T;V^\theta)$ such that, defining $\mathbf{u}=\mathbf{v}+\mathbf{u}^{ref}$, up to a subsequence, we have*

- $\mathbf{u}_\varepsilon\overset{*}{\rightharpoonup}\mathbf{u}$ *in* $L^\infty(0,T;H^u)$
- $\theta_\varepsilon\overset{*}{\rightharpoonup}\theta$ *in* $L^\infty(0,T;H^\theta)$,
- $\mathbf{u}_\varepsilon\rightharpoonup\mathbf{u}$ *in* $L^2(0,T;V^u)$ *and in* $L^2(0,T;(L^6(\Omega))^d)$,
- $\theta_\varepsilon\rightharpoonup\theta$ *in* $L^2(0,T;V^\theta)$ *and in* $L^2(0,T;L^6(\Omega))$,
- $\mathbf{u}_\varepsilon\rightharpoonup\mathbf{u}$ *in* $L^4(0,T;(L^2(\Gamma))^d)$
- $\theta_\varepsilon\rightharpoonup\theta$ *in* $L^4(0,T;L^2(\Gamma))$,
- $\mathbf{u}_\varepsilon\xrightarrow[\varepsilon\to0]{}\mathbf{u}$ *in* $L^2(0,T;(L^2(\Omega))^d)$
- $\theta_\varepsilon\xrightarrow[\varepsilon\to0]{}\theta$ *in* $L^2(0,T;L^2(\Omega))$,
- $\mathbf{u}_\varepsilon\xrightarrow[\varepsilon\to0]{}\mathbf{u}$ *in* $L^2(0,T;(L^2(\Gamma))^d)$
- $\theta_\varepsilon\xrightarrow[\varepsilon\to0]{}\theta$ *in* $L^2(0,T;L^2(\Gamma))$,
- $\partial_t\mathbf{u}_\varepsilon\rightharpoonup\partial_t\mathbf{u}$ *in* $L^{\frac{4}{3}}(0,T;(V^u)')$
- $\partial_t\theta_\varepsilon\rightharpoonup\partial_t\theta$ *in* $L^{\frac{4}{3}}(0,T;(V^\theta)')$.

Furthermore, $(\mathbf{v},\theta)$ is a solution to (WF) parametrized by α.

Proof. Using (2.1)-(2.4) and (2.5)-(2.6), we prove that there exists $\mathbf{u}$ and θ such that all the convergences above are verified in the same manner as in [14, Proposition VII.1.4].

Let us prove first that $\mathbf{u}$ is a solution of (WF.2) parametrized by α and θ.

- With the same pattern of proof as in Theorem 2.4, one proves immediately that $(\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \rightharpoonup (\mathbf{u} \cdot \nabla)\mathbf{u}$ in $L^1(0,T;(L^1(\Omega))^d)$, and $(\mathbf{u}_\varepsilon \cdot \mathbf{n})\mathbf{u}^{\text{ref}} \rightharpoonup (\mathbf{u} \cdot \mathbf{n})\mathbf{u}^{\text{ref}}$ in $L^4(0,T;(L^{\frac{4}{3}}(\Gamma))^d)$.
- Regarding the penalization term:

$$\|h(\alpha_\varepsilon)\mathbf{u}_\varepsilon - h(\alpha)\mathbf{u}\|^2_{L^2(0,T;L^2(\Omega)^d)} \lesssim \|h\|^2_\infty \|\mathbf{u}_\varepsilon - \mathbf{u}\|^2_{L^2(0,T;L^2(\Omega)^d)} + \int_0^T \int_\Omega (h(\alpha_\varepsilon) - h(\alpha))^2 |\mathbf{u}|^2.$$

  Since $\alpha_\varepsilon \to \alpha$ strongly in $L^1(\Omega)$, $h(\alpha_\varepsilon) \to h(\alpha)$ pointwise in Ω up to a subsequence (not relabeled). Lebesgue dominated convergence theorem then implies: $h(\alpha_\varepsilon)\mathbf{u}_\varepsilon \xrightarrow[\varepsilon \to 0]{} h(\alpha)\mathbf{u}$ in $L^2(0,T;(L^2(\Omega))^d)$.
- Concerning the boundary terms, we only consider the term with the approximation of the pos function. First, we claim that there exists γ such that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})\,\mathbf{u}_\varepsilon \rightharpoonup \gamma$ in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Gamma)^d)$. Notice that, for ε large enough and using (1.4), we have:

$$\begin{aligned}\int_0^T \|\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})\,\mathbf{u}_\varepsilon\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Gamma)} &\lesssim \int_0^T \left(\|\mathbf{u}_\varepsilon\|^{\frac{4}{3}}_{L^{\frac{8}{3}}(\Gamma)} + C\right)\|\mathbf{u}_\varepsilon\|^{\frac{4}{3}}_{L^{\frac{8}{3}}(\Gamma)} \\ &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\Gamma)} + \left(\int_0^T \|\mathbf{u}_\varepsilon\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\Gamma)}\right)^{\frac{1}{2}}.\end{aligned} \tag{2.7}$$

  In addition, from Proposition 1.3, we have

$$\|\mathbf{u}_\varepsilon\|^{\frac{8}{3}}_{L^{\frac{8}{3}}(\Gamma)} \lesssim \|\mathbf{u}_\varepsilon\|^{\frac{2}{3}}_{L^2(\Omega)} \|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)}.$$

  Since $\mathbf{u}_\varepsilon$ is bounded in $L^\infty(0,T;(L^2(\Omega))^d)$ and $L^2(0,T;(H^1(\Omega))^d)$ as proved in Proposition 2.2, we see that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n})\,\mathbf{u}_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Gamma)^d)$ uniformly in ε. Since this Banach space is reflexive, it proves the claimed weak convergence.
- Let us now prove that γ can be identified with $(\mathbf{u}\cdot\mathbf{n})^+\mathbf{u}$. First, since $\mathbf{u}_\varepsilon \to \mathbf{u}$ strongly in $L^2(0,T;L^2(\Gamma)^d)$, $\text{pos}_\varepsilon(\cdot) \to (\cdot)^+$ uniformly and $|\text{neg}'_\varepsilon(\cdot)| \le 1$, one proves that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon \cdot \mathbf{n}) - \text{pos}_\varepsilon(\mathbf{u} \cdot \mathbf{n}) \to 0$ and $\text{pos}_\varepsilon(\mathbf{u}\cdot\mathbf{n}) \to (\mathbf{u}\cdot\mathbf{n})^+$ in $L^2(0,T;L^2(\Gamma))$. Therefore, $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n}) \to (\mathbf{u}\cdot\mathbf{n})^+$ in $L^2(0,T;L^2(\Gamma))$. Then, the weak convergence of $\mathbf{u}_\varepsilon$ in $L^4(0,T;L^2(\Gamma)^d)$ and [14, Proposition II.2.12] implies that $\text{pos}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\,\mathbf{u}_\varepsilon \rightharpoonup (\mathbf{u}\cdot\mathbf{n})^+\mathbf{u}$ weakly in $L^{\frac{4}{3}}(0,T;L^1(\Gamma)^d)$. Using [14, Proposition II.2.9], we argue that $\gamma = (\mathbf{u}\cdot\mathbf{n})^+\mathbf{u}$.
- Regarding $\partial_t \mathbf{u}_\varepsilon$, remark that:

$$\begin{aligned}\|\partial_t \mathbf{u}_\varepsilon\|_{(V^u)'} \lesssim\ & \|\mathcal{B}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)\|_{(V^u)'} + \|\mathcal{A}\mathbf{u}_\varepsilon\|_{(V^u)'} + \|h(\alpha)\mathbf{u}_\varepsilon\|_{(V^u)'} + \|\mathcal{T}\theta_\varepsilon\|_{(V^u)'} \\ &+ \|\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)\|_{(V^u)'} + \|\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\text{ref}})\|_{(V^u)'} + \|f + \sigma^{\text{ref}}\|_{(V^u)'}.\end{aligned}$$

  We now bound each term depending on ε:
  - Since the Stokes operator is continuous, $\|\mathcal{A}\mathbf{u}_\varepsilon\|_{(V^u)'} \lesssim \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ and therefore, $\mathcal{A}\mathbf{u}_\varepsilon$ is bounded in $L^2(0,T;(V^u)')$.

– Obviously, $\|h(\alpha)\mathbf{u}_\varepsilon\|_{(V^u)'} \leq \|h\|_\infty\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}$ and therefore, $h(\alpha)\mathbf{u}_\varepsilon$ is bounded in $L^\infty(0,T;(V^u)')$.
– $\|\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\text{ref}})\|_{(V^u)'} \lesssim \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ and therefore, $\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\text{ref}})$ is bounded in $L^2(0,T;(V^u)')$.

We are left with the boundary term $\mathcal{P}_\varepsilon$ and the non linear term $\mathcal{B}$. Concerning $\mathcal{B}$, remark that :

$$\forall\mathbf{\Psi}\in V^u, \langle\mathcal{B}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon),\mathbf{\Psi}\rangle_{(V^u)',V^u} = -\int_\Omega(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{\Psi}\cdot\mathbf{u}_\varepsilon + \frac{1}{2}\int_\Gamma(\mathbf{u}_\varepsilon\cdot\mathbf{n})(\mathbf{u}_\varepsilon\cdot\mathbf{\Psi}).$$

The first term can be treated as in [54, Lemma 3.1] while the second one on the boundary needs more details.

Let $0\neq\mathbf{\Psi}\in V^u$. Since the proof is similar in dimension 2, we will only focus on the dimension $d=3$. Using Hölder's inequality and Proposition 1.3, we obtain:

$$\frac{\int_{\Gamma_{\text{out}}}|(\mathbf{u}_\varepsilon\cdot\mathbf{n})(\mathbf{u}_\varepsilon\cdot\mathbf{\Psi})|}{\|\mathbf{\Psi}\|_{V^u}} \lesssim \|\mathbf{u}_\varepsilon\|^{\frac{1}{2}}_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|^{\frac{3}{2}}_{H^1(\Omega)}.$$

Therefore:

$$\int_0^T\left(\sup_{\mathbf{\Psi}\in V^u\setminus\{0\}}\frac{\int_{\Gamma_{\text{out}}}|(\mathbf{u}_\varepsilon\cdot\mathbf{n})(\mathbf{u}_\varepsilon)\cdot\mathbf{\Psi}|}{\|\mathbf{\Psi}\|_{V^u}}\right)^{\frac{4}{3}} \lesssim \|\mathbf{u}_\varepsilon\|^{\frac{2}{3}}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}.$$

This proves that $(\mathcal{B}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon))_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0,T;(V^u)')$. We prove analogously that $(\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon))_\varepsilon$ is bounded in $L^{\frac{4}{3}}(0,T;(V^u)')$. These bounds prove that $(\partial_t\mathbf{u}_\varepsilon)$ is bounded in $L^{\frac{4}{3}}(0,T;(V^u)')$, and by continuity of the time derivative, we argue that $(\partial_t\mathbf{u}_\varepsilon)$ weakly converges to $\partial_t\mathbf{u}$ in $L^{\frac{4}{3}}(0,T;(V^u)')$.

Concerning θ, the convergence is largely proved in the same way as in Theorem 2.4. The only difference concerns the convergence of $\text{neg}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\,\theta_\varepsilon$ to $(\mathbf{u}\cdot\mathbf{n})^-\theta$, which is proved in the same manner as (2.7). All these convergence results let us say that $(\mathbf{u},\theta)$ is a solution to (WF) in the distribution sense. □

2.2. Further results in dimension 2. It is notably known that the solution of the Navier-Stokes equations with homogeneous Dirichlet boundary conditions are unique in dimension 2. We prove here that uniqueness still holds with the boundary conditions (1.2). Denote $\mathbb{X}^u = L^2(0,T;V^u)\cap L^\infty(0,T;H^u)$ and $\mathbb{X}^\theta = L^2(0,T;V^\theta)\cap L^\infty(0,T;H^\theta)$. These space are endowed with the norm:

$$\|\mathbf{u}\|_{\mathbb{X}^u} = \max\{\|\mathbf{u}\|_{L^2(0,T;V^u)},\|\mathbf{u}\|_{L^\infty(0,T;H^u)}\},$$

and the same definition follows for $\|\cdot\|_{\mathbb{X}^\theta}$.

LEMMA 2.6. *Assume $d=2$. Then the solution $(\mathbf{v}_\varepsilon,\theta_\varepsilon)$ of (WFe) is such that:*

$$\partial_t\mathbf{v}_\varepsilon\in(\mathbb{X}^u)',\ \ \partial_t\theta_\varepsilon\in(\mathbb{X}^\theta)'.$$

Proof. The proof being similar, we will only focus on $\partial_t\mathbf{u}_\varepsilon$. First, remark that:

$$\partial_t\mathbf{u}_\varepsilon = -\mathcal{B}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon) - \mathcal{A}\mathbf{u}_\varepsilon - h(\alpha)\mathbf{u}_\varepsilon + \mathcal{T}\theta_\varepsilon - \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon) + \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\text{ref}}) + f + \sigma^{\text{ref}}.$$

Due to the fact that $\mathbf{u}_\varepsilon\in\mathbb{X}^u$ and $\theta_\varepsilon\in\mathbb{X}^\theta$, it is straightforward to prove that $\mathcal{A}\mathbf{u}_\varepsilon$, $h(\alpha)\mathbf{u}_\varepsilon$, $\mathcal{T}\theta_\varepsilon$, $\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\text{ref}})$ and $f+\sigma^{\text{ref}}$ are in $(\mathbb{X}^u)'$. Concerning $\mathcal{B}$, we use once again the identity:

$$\forall\mathbf{\Psi}\in V^u, \langle\mathcal{B}(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon),\mathbf{\Psi}\rangle_{(V^u)',V^u} = -\int_\Omega(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{\Psi}\cdot\mathbf{u}_\varepsilon + \frac{1}{2}\int_\Gamma(\mathbf{u}_\varepsilon\cdot\mathbf{n})(\mathbf{u}_\varepsilon\cdot\mathbf{\Psi}),$$

and only focus on the boundary part.

Let $\mathbf{\Psi} \in \mathbb{X}^u$. Notice that, using Proposition 1.3:

$$\begin{aligned}\int_0^T \int_\Gamma (\mathbf{u}_\varepsilon \cdot \mathbf{n})(\mathbf{u}_\varepsilon \cdot \mathbf{\Psi}) &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|_{L^2(\Gamma)} \|\mathbf{u}_\varepsilon\|_{L^4(\Gamma)} \|\mathbf{\Psi}\|_{L^4(\Gamma)} \\ &\lesssim \int_0^T \|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^{\frac{3}{4}} \|\mathbf{\Psi}\|_{L^2(\Omega)}^{\frac{1}{4}} \|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^{\frac{5}{4}} \|\mathbf{\Psi}\|_{H^1(\Omega)}^{\frac{3}{4}} \\ &\lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{3}{4}} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}^{\frac{5}{4}} \|\mathbf{\Psi}\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{4}} \|\mathbf{\Psi}\|_{L^2(0,T;H^1(\Omega))}^{\frac{3}{4}} \\ &\lesssim \|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{3}{4}} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;H^1(\Omega))}^{\frac{5}{4}} \|\mathbf{\Psi}\|_{\mathbb{X}^u}\end{aligned}$$

This proves that $\mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)$ is in $(\mathbb{X}^u)'$. Similar computations for $\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)$ show that $\partial_t \mathbf{u}_\varepsilon \in (\mathbb{X}^u)'$. ∎

We may now prove uniqueness of the solution. We only sketch the proof.

Proposition 2.7. *Let $d = 2$. Then, the solution $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ of (WFe) is unique.*

Sketch of proof Let $(\mathbf{u}_{\varepsilon 1}, \theta_{\varepsilon 1})$ and $(\mathbf{u}_{\varepsilon 2}, \theta_{\varepsilon 2})$ be two solutions of (WF.1)-(WF.2). Define $\mathbf{u} = \mathbf{v} = \mathbf{u}_{\varepsilon 1} - \mathbf{u}_{\varepsilon 2}$ and $\theta = \theta_{\varepsilon 1} - \theta_{\varepsilon 2}$. Slightly adapting the proof in [14, Section VII.1.2.5], one proves that:

$$\frac{d}{dt}|\mathbf{v}|_{L^2(\Omega)}^2 + A|\nabla \mathbf{v}|_{L^2(\Omega)}^2 \lesssim g^v(t)|\mathbf{v}|_{L^2(\Omega)}^2 + B|\theta|_{L^2(\Omega)}^2 + \nu^v |\nabla \mathbf{v}|_{L^2(\Omega)}^2 \tag{2.8}$$

where ν^v is a positive constant and g^v is a function in $L^1([0,T])$.

Testing the differential equation (WFe.1) with θ and using Lemma A.2, it proves that:

$$\begin{aligned}&\frac{d}{dt}|\theta|_{L^2(\Omega)}^2 + 2C\int_\Omega k|\nabla\theta|^2 + \int_{\Gamma_{\text{out}}} \theta^2 \left(\frac{1}{2}(\mathbf{u}_{\varepsilon 1} \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon (\mathbf{u}_{\varepsilon 1} \cdot \mathbf{n})\right) \\ &\qquad = -\int_{\Gamma_{\text{out}}} \left(\beta \left(\text{neg}_\varepsilon (\mathbf{u}_{\varepsilon 1} \cdot \mathbf{n}) - \text{neg}_\varepsilon (\mathbf{u}_{\varepsilon 2} \cdot \mathbf{n})\right) + \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})\right) \theta_{\varepsilon 2}\theta.\end{aligned}$$

With a similar proof as the one of Proposition 2.2, we can prove that, on Γ_{out}, $\theta^2 \left(\frac{1}{2}(\mathbf{u}_1 \cdot \mathbf{n}) + \beta \text{neg}_\varepsilon (\mathbf{u}_1 \cdot \mathbf{n})\right) \geq 0$. Therefore, using (**A3**), one has:

$$\frac{d}{dt}|\theta|_{L^2(\Omega)}^2 + 2C\int_\Omega k|\nabla\theta|^2 \lesssim \left(|\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2}\right) |\mathbf{u} \cdot \mathbf{n}|_{L^3(\Gamma_{\text{out}})} |\theta_{\varepsilon 2}|_{L^3(\Gamma_{\text{out}})} |\theta|_{L^3(\Gamma_{\text{out}})}. \tag{2.9}$$

Using Sobolev embeddings and Young inequality, we prove:

$$\begin{aligned}&\left(|\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2}\right) |\mathbf{u} \cdot \mathbf{n}|_{L^3(\Gamma_{\text{out}})} |\theta_{\varepsilon 2}|_{L^3(\Gamma_{\text{out}})} |\theta|_{L^3(\Gamma_{\text{out}})} \\ &\qquad \lesssim \left(|\beta|_{L^\infty(\Gamma_{\text{out}})} + \frac{1}{2}\right)^3 \frac{|\theta_{\varepsilon 2}|_{L^2(\Omega)} |\nabla\theta_{\varepsilon 2}|_{L^2(\Omega)}^2}{2(\nu^\theta)^3} (|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2) \\ &\qquad\quad + \frac{(\nu^\theta)^{\frac{3}{2}}}{2} \left(|\nabla \mathbf{u}|_{L^2(\Omega)}^2 + |\nabla\theta|_{L^2(\Omega)}^2\right),\end{aligned}$$

where ν^θ is a positive constant. Therefore, summing (2.8) and (2.9) gives $\frac{d}{dt}(|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2) \lesssim \max(g_1^v, g^\theta)(|\mathbf{u}|_{L^2(\Omega)}^2 + |\theta|_{L^2(\Omega)}^2)$, with g_1^v and g^θ integrable. Therefore, applying Gronwall's lemma and noticing that $|\mathbf{u}(0)|^2_{L^2(\Omega)} + |\theta(0)|^2_{L^2(\Omega)} = 0$, one shows that $\mathbf{u} = 0$ and $\theta = 0$. □

Note that we may also prove that, for $d = 2$, the solution $(\mathbf{u}, \theta)$ of (WF) is unique, and that $\partial_t \mathbf{u} \in (\mathbb{X}^u)'$, $\partial_t\theta \in (\mathbb{X}^\theta)'$. We can also state stronger convergence (compared to the ones stated in Theorem 2.5) in dimension 2. These results will be useful in the analysis of the optimisation problems.

Denote $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\varepsilon$ and $\bar\theta = \theta - \theta_\varepsilon$. The variational formulation verified by $(\bar{\mathbf{u}}, \bar\theta)$ reads as: for all $(\mathbf{\Psi}, \varphi) \in V^u \times V^\theta$:

(2.10a)

$$\begin{aligned} 0 =&\langle \partial_t \bar{\mathbf{u}} + \mathcal{A}\bar{\mathbf{u}} + h(\alpha)\bar{\mathbf{u}}, \mathbf{\Psi}\rangle_{(V^u)',V^u} + \langle (h(\alpha) - h(\alpha_\varepsilon))\mathbf{u}_\varepsilon, \mathbf{\Psi}\rangle_{(V^u)',V^u} + \\ &\frac{1}{2}\langle \mathcal{P}(\mathbf{u}, \bar{\mathbf{u}}) + \mathcal{P}(\mathbf{u}, \mathbf{u}_\varepsilon) - \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon), \mathbf{\Psi}\rangle_{(V^u)',V^u} + \langle \mathcal{T}\bar\theta, \mathbf{\Psi}\rangle_{(V^u)',V^u} \\ &- \frac{1}{2}\langle \mathcal{N}(\mathbf{u}, \mathbf{u}^{\mathrm{ref}}) - \mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\mathrm{ref}}), \mathbf{\Psi}\rangle_{(V^u)',V^u} + \langle \mathcal{B}(\mathbf{u}, \bar{\mathbf{u}}) + \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}_\varepsilon), \mathbf{\Psi}\rangle_{(V^u)',V^u}, \end{aligned}$$

$$\begin{aligned} 0 =&\langle \partial_t\bar\theta, \varphi\rangle_{(V^\theta)',V^\theta} - \langle \mathcal{D}(\mathbf{u}, \bar\theta) + \mathcal{D}(\bar{\mathbf{u}}, \theta_\varepsilon), \varphi\rangle_{(V^\theta)',V^\theta} \\ &+ \langle (\mathcal{C}(\alpha) - \mathcal{C}(\alpha_\varepsilon))\theta + \mathcal{C}(\alpha_\varepsilon)\bar\theta, \varphi\rangle_{(V^\theta)',V^\theta} \\ &+ \langle \mathcal{M}(\mathbf{u}, \theta) + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon), \varphi\rangle_{(V^\theta)',V^\theta}. \end{aligned} \tag{2.10b}$$

We now bound some of the terms above in the following lemma. The proof is omitted since it mainly relies on Proposition 1.2, Theorem 1.3 and Hölder's inequality.

Lemma 2.8. *Suppose $d = 2$. Denote $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_\varepsilon$ and $\bar\theta = \theta - \theta_\varepsilon$. Let $C_\varepsilon = \sup_{s\in\mathbb{R}} |\mathrm{neg}_\varepsilon(s) - s^-|$. Owning to **(A1)**, one has $C_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$. The following inequalities are then valid:*

1.
(2.11)

$$\begin{aligned} \langle \mathcal{B}(\bar{\mathbf{u}}, \mathbf{u}_\varepsilon), \bar{\mathbf{u}}\rangle_{(V^u)',V^u} \lesssim &\|\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega)} \\ &+ \left(\|\bar{\mathbf{u}}\|_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^3_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega)}\right)^{\frac{1}{2}}. \end{aligned}$$

2.

$$\mathrm{neg}(\mathbf{u}\cdot\mathbf{n}) - \mathrm{neg}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n}) \le |\bar{\mathbf{u}}\cdot\mathbf{n}| + C_\varepsilon \tag{2.12}$$

$$\begin{aligned} &\int_{\Gamma_{out}} (\mathrm{pos}(\mathbf{u}\cdot\mathbf{n}) - \mathrm{pos}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n}))\mathbf{u}_\varepsilon\cdot\bar{\mathbf{u}} \\ &\lesssim \left(\|\bar{\mathbf{u}}\|^{\frac14}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{\frac34}_{L^2(\Omega)} + C_\varepsilon\right)\|\mathbf{u}_\varepsilon\|^{\frac12}_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|^{\frac12}_{H^1(\Omega)} \\ &\qquad \times \|\bar{\mathbf{u}}\|^{\frac14}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{\frac34}_{L^2(\Omega)}. \end{aligned} \tag{2.13a}$$

$$\begin{aligned} &\int_{\Gamma_{out}} (\mathrm{neg}(\mathbf{u}\cdot\mathbf{n}) - \mathrm{neg}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n}))\mathbf{u}^{ref}\cdot\bar{\mathbf{u}} \\ &\lesssim \left(\|\bar{\mathbf{u}}\|^{\frac14}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{\frac34}_{L^2(\Omega)} + C_\varepsilon\right)\|\mathbf{u}^{ref}\|^{\frac12}_{L^2(\Omega)}\|\mathbf{u}^{ref}\|^{\frac12}_{H^1(\Omega)} \\ &\qquad \times \|\bar{\mathbf{u}}\|^{\frac14}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{\frac34}_{L^2(\Omega)}. \end{aligned} \tag{2.13b}$$

3.

$$\int_\Omega \theta_\varepsilon \bar{\mathbf{u}} \cdot \nabla \bar\theta \lesssim \|\theta_\varepsilon\|^{\frac12}_{L^2(\Omega)} \|\theta_\varepsilon\|^{\frac12}_{H^1(\Omega)} \|\bar{\mathbf{u}}\|^{\frac12}_{L^2(\Omega)} \|\nabla \bar{\mathbf{u}}\|^{\frac12}_{L^2(\Omega)} \|\nabla\bar\theta\|_{L^2(\Omega)}. \tag{2.14}$$

4.

(2.15a)

$$\int_{\Gamma_{out}} (\bar{\mathbf{u}}\cdot\mathbf{n})\theta_\varepsilon\bar\theta \lesssim \|\bar{\mathbf{u}}\|^{\frac14}_{L^2(\Omega)} \|\nabla\bar{\mathbf{u}}\|^{\frac34}_{L^2(\Omega)} \|\theta_\varepsilon\|^{\frac12}_{L^2(\Omega)} \|\nabla\theta_\varepsilon\|^{\frac12}_{L^2(\Omega)} \|\bar\theta\|^{\frac14}_{L^2(\Omega)} \|\nabla\bar\theta\|^{\frac34}_{L^2(\Omega)}.$$

(2.15b)

$$\begin{aligned}\int_{\Gamma_{out}} \left(\operatorname{neg}(\mathbf{u}\cdot\mathbf{n}) - \operatorname{neg}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\right)\theta_\varepsilon\bar\theta \lesssim \left(\|\bar{\mathbf{u}}\|^{\frac14}_{L^2(\Omega)}\|\nabla\bar{\mathbf{u}}\|^{\frac34}_{L^2(\Omega)} + C_\varepsilon\right)\|\theta_\varepsilon\|^{\frac12}_{L^2(\Omega)}&\\ \|\nabla\theta_\varepsilon\|^{\frac12}_{L^2(\Omega)}\|\bar\theta\|^{\frac14}_{L^2(\Omega)}\|\nabla\bar\theta\|^{\frac34}_{L^2(\Omega)}.&\end{aligned}$$

Corollary 2.9. *Suppose $d = 2$. Under the assumptions of Theorem 2.5, $\mathbf{u}_\varepsilon \to \mathbf{u}$ strongly in $L^\infty(0,T;L^2(\Omega)^2)$ and $\theta_\varepsilon \to \theta$ strongly in $L^\infty(0,T;L^2(\Omega))$.*

Proof. Since $d = 2$, one has $\partial_t\bar{\mathbf{u}} \in (\mathbb{X}^u)'$ and we may choose $\boldsymbol{\Psi} = \bar{\mathbf{u}}(t)$ for fixed t in (2.10a). After rearranging the terms, and using Lemma A.2, we obtain:

$$\begin{aligned}&\frac{d}{dt}\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + 2A\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + 2\int_\Omega h(\alpha)|\bar{\mathbf{u}}|^2 + \int_{\Gamma_{\text{out}}} \operatorname{pos}(\mathbf{u}\cdot\mathbf{n})|\bar{\mathbf{u}}|^2 =\\ &-2\langle (h(\alpha)-h(\alpha_\varepsilon))\mathbf{u}_\varepsilon, \bar{\mathbf{u}}\rangle_{(V^u)',V^u} - \int_\Omega B\bar\theta\mathbf{e}_y\cdot\bar{\mathbf{u}}\\ &-\langle \mathcal{B}(\bar{\mathbf{u}},\mathbf{u}_\varepsilon),\bar{\mathbf{u}}\rangle_{(V^u)',V^u} + \int_{\Gamma_{\text{out}}}\left(\operatorname{neg}(\mathbf{u}\cdot\mathbf{n}) - \operatorname{neg}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\right)\mathbf{u}^{\text{ref}}\cdot\bar{\mathbf{u}}\\ &-\int_{\Gamma_{\text{out}}}\left(\operatorname{pos}(\mathbf{u}\cdot\mathbf{n}) - \operatorname{pos}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\right)\mathbf{u}_\varepsilon\cdot\bar{\mathbf{u}}.\end{aligned}$$

Therefore, (2.11), (2.13), Proposition 1.3 and Young's inequality imply there exists $C_1 > 0$ independent of ε such that:

$$\begin{aligned}\frac{d}{dt}\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + C_1\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} \lesssim \|\bar\theta\|^2_{L^2(\Omega)} + 2\int_\Omega |h(\alpha)-h(\alpha_\varepsilon)|^2|\mathbf{u}_\varepsilon|^2&\\ + g_1^u\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + (g_2^u)^{\frac45}\|\bar{\mathbf{u}}\|^{\frac25}_{L^2(\Omega)},&\end{aligned}$$

where $g_1^u = \|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)} + \|\mathbf{u}_\varepsilon\|^2_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|^2_{H^1(\Omega)} + \|\mathbf{u}^{\text{ref}}\|^2_{L^2(\Omega)}\|\mathbf{u}^{\text{ref}}\|^2_{H^1(\Omega)}$ and $g_2^u = C_\varepsilon^2\left(\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|_{H^1(\Omega)} + \|\mathbf{u}^{\text{ref}}\|_{L^2(\Omega)}\|\mathbf{u}^{\text{ref}}\|_{H^1(\Omega)}\right)$. Using once again Young's inequality, one has:

$$\begin{aligned}\frac{d}{dt}\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + C_1\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} \lesssim &\|\bar\theta\|^2_{L^2(\Omega)} + (1+g_1^u)\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)}\\ &+ 2\int_\Omega |h(\alpha)-h(\alpha_\varepsilon)|^2|\mathbf{u}_\varepsilon|^2 + g_2^u.\end{aligned} \tag{2.16}$$

We now move back to (2.10b) and choose $\varphi = \bar\theta$, which gives, after some manipulation:

$$\frac{1}{2}\frac{d}{dt}\|\bar\theta\|^2_{L^2(\Omega)} + C\int_\Omega k(\alpha_\varepsilon)|\nabla\bar\theta|^2 + \int_{\Gamma_{\text{out}}}\left(\frac{1}{2}(\mathbf{u}\cdot\mathbf{n}) + \beta\text{neg}(\mathbf{u}\cdot\mathbf{n})\right)\bar\theta^2$$
$$= \int_\Omega \theta_\varepsilon\bar{\mathbf{u}}\cdot\nabla\bar\theta - C\int_\Omega (k(\alpha)-k(\alpha_\varepsilon))\nabla\theta\cdot\nabla\bar\theta$$
$$- \int_{\Gamma_{\text{out}}} \left[\left((\bar{\mathbf{u}}\cdot\mathbf{n}) + \beta\left(\text{neg}(\mathbf{u}\cdot\mathbf{n}) - \text{neg}_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\right)\right]\theta_\varepsilon\bar\theta.$$

Since $\beta \geq \frac{1}{2}$, $\frac{1}{2}(\mathbf{u}\cdot\mathbf{n}) + \beta\text{neg}(\mathbf{u}\cdot\mathbf{n}) \geq \frac{1}{2}\text{pos}(\mathbf{u}\cdot\mathbf{n}) \geq 0$. Thus, $\int_{\Gamma_{\text{out}}}(\frac{1}{2}(\mathbf{u}\cdot\mathbf{n})+\beta\text{neg}(\mathbf{u}\cdot\mathbf{n}))\bar\theta^2$ is positive. Therefore, using (2.15), Proposition 1.3 and Young's inequality, one proves that there exist $C_3 > 0, C_4 > 0$, such that:

$$\begin{aligned}\frac{d}{dt}\|\bar\theta\|^2_{L^2(\Omega)} + C_3\|\nabla\bar\theta\|^2_{L^2(\Omega)} &\lesssim \|\theta_\varepsilon\|^2_{L^2(\Omega)}\|\nabla\theta_\varepsilon\|^2_{L^2(\Omega)}\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + C_4\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} \\ &+ \left(C\int_\Omega (k(\alpha)-k(\alpha_\varepsilon))^2|\nabla\theta|^2\right) + g_1^\theta\|\bar\theta\|^2_{L^2(\Omega)} + g_2^\theta,\end{aligned} \tag{2.17}$$

where $g_1^\theta = 1 + \|\theta_\varepsilon\|^2_{L^2(\Omega)}\|\theta_\varepsilon\|^2_{H^1(\Omega)}$, $g_2^\theta = C_\varepsilon^2\|\theta_\varepsilon\|_{L^2(\Omega)}\|\theta_\varepsilon\|_{H^1(\Omega)}$.

Summing (2.16) and (2.17) and choosing C_4 small enough, there exists $C^* > 0$ such that:

$$\begin{aligned}&\frac{d}{dt}(\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + \|\bar\theta\|^2_{L^2(\Omega)}) + C^*(\|\nabla\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + \|\nabla\bar\theta\|^2_{L^2(\Omega)}) \lesssim g_2^u + g_2^\theta \\ &+ (1 + \|\theta_\varepsilon\|^2_{L^2(\Omega)}\|\nabla\theta_\varepsilon\|^2_{L^2(\Omega)} + g_1^u)\|\bar{\mathbf{u}}\|^2_{L^2(\Omega)} + (g_1^\theta + 1)\|\bar\theta\|^2_{L^2(\Omega)} \\ &+ \int_\Omega (k(\alpha)-k(\alpha_\varepsilon))^2|\nabla\theta|^2 + \int_\Omega |h(\alpha)-h(\alpha_\varepsilon)|^2|\mathbf{u}_\varepsilon|^2.\end{aligned} \tag{2.18}$$

We now introduce the following functions

$$a_\varepsilon^u = (1 + \|\theta_\varepsilon\|^2_{L^2(\Omega)}\|\nabla\theta_\varepsilon\|^2_{L^2(\Omega)} + g_1^u), \qquad b_\varepsilon^u = \int_\Omega |h(\alpha)-h(\alpha_\varepsilon)|^2|\mathbf{u}_\varepsilon|^2 + g_2^u,$$
$$a_\varepsilon^\theta = (1 + g_1^\theta), \qquad b_\varepsilon^\theta = \int_\Omega (k(\alpha)-k(\alpha_\varepsilon))^2|\nabla\theta|^2 + g_2^\theta.$$

Since $\mathbf{u}$ and $\mathbf{u}_\varepsilon$ both belong to $L^2(0,T;H^1(\Omega)^2)\cap L^\infty(0,T;L^2(\Omega)^2)$ (the same holds for θ and θ_ε), a_ε^u, b_ε^u, a_ε^θ and b_ε^θ are integrable, and so are $a_\varepsilon = \max(a_\varepsilon^u, a_\varepsilon^\theta)$ and $b_\varepsilon = b_\varepsilon^u + b_\varepsilon^\theta$. Grönwall's lemma proves that for all $t \in [0,T]$, $\|\bar{\mathbf{u}}(t)\|^2_{L^2(\Omega)} + \|\bar\theta(t)\|^2_{L^2(\Omega)} \leq \left(\int_0^t b_\varepsilon(s)ds\right)\exp\left(\int_0^t a_\varepsilon(s)ds\right)$. Since $a_\varepsilon \geq 0$ and $b_\varepsilon \geq 0$, $t \mapsto \left(\int_0^t b_\varepsilon(s)ds\right)$ and $t \mapsto \exp\left(\int_0^t a_\varepsilon(s)ds\right)$ are non-decreasing and we have

$$\sup_{t\in[0,T]}\left(\|\bar{\mathbf{u}}(t)\|_{L^2(\Omega)} + \|\bar\theta(t)\|_{L^2(\Omega)}\right) \leq \left(\int_0^T b_\varepsilon(s)ds\right)^{\frac{1}{2}}\exp\left(\frac{1}{2}\int_0^T a_\varepsilon(s)ds\right). \tag{2.19}$$

Since, on one hand, $\alpha_\varepsilon \to \alpha$ in $L^1(\Omega)$ and α_ε is independent of time, and on the other hand, $\mathbf{u}_\varepsilon \to \mathbf{u}$ strongly in $L^2(0,T;L^2(\Omega))$, Lebesgue's dominated convergence gives a subsequence (ε_k) such that:

(2.20)

$$\int_0^T\int_\Omega |h(\alpha)-h(\alpha_{\varepsilon_k})|^2|\mathbf{u}_{\varepsilon_k}|^2 \xrightarrow[k\to+\infty]{} 0, \qquad \int_0^T\int_\Omega |k(\alpha)-k(\alpha_{\varepsilon_k})|^2|\nabla\theta|^2 \xrightarrow[k\to+\infty]{} 0.$$

Notice that, owning to the convergence of $\mathbf{u}_\varepsilon$ and θ_ε, $\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}$ and $\|\theta_\varepsilon\|_{L^2(\Omega)}\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega)}$ are bounded w.r.t ε in $L^1([0,T])$. Therefore, since $C_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$, it proves that $\int_0^T (g_2^u + g_\theta^2) \xrightarrow[\varepsilon_k\to+\infty]{} 0$. Gathering the previous convergence results then ensure that $\int_0^T b_{\varepsilon_k}(s)ds \xrightarrow[k\to+\infty]{} 0$. In addition, thanks to Theorem 2.5, we show that $\int_0^T a_\varepsilon(s)ds$ is bounded w.r.t. ε. Therefore, it proves that $\|\mathbf{u}-\mathbf{u}_{\varepsilon_k}\|_{L^\infty(0,T,L^2(\Omega))} + \|\theta-\theta_{\varepsilon_k}\|_{L^\infty(0,T,L^2(\Omega))} \xrightarrow[k\to+\infty]{} 0$. ∎

Corollary 2.10. *Suppose $d=2$. Under the assumptions of Theorem 2.5, $\nabla\mathbf{u}_\varepsilon \to \nabla\mathbf{u}$ strongly in $L^2(0,T;L^2(\Omega)^2)$ and $\nabla\theta_\varepsilon \to \nabla\theta$ strongly in $L^2(0,T;L^2(\Omega))$.*

Proof. Move back to (2.18). We integrate each side of the inequality:

$$\int_0^T \|\nabla\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \|\nabla\bar{\theta}\|_{L^2(\Omega)}^2 \lesssim F_\varepsilon^{u,\theta} + \int_0^T (g_1^u + \|\theta_\varepsilon\|_{L^2(\Omega)}^2\|\nabla\theta_\varepsilon\|_{L^2(\Omega)}^2 + 1)\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2 + \int_0^T (g_1^\theta + 1)\|\bar{\theta}\|_{L^2(\Omega)}^2,$$

with

$$F_\varepsilon^{u,\theta} = \|\mathbf{u}_0(\alpha_\varepsilon) - \mathbf{u}_0(\alpha)\|_{L^2(\Omega)}^2 + \|\theta_0(\alpha_\varepsilon) - \theta_0(\alpha)\|_{L^2(\Omega)}^2 + \int_0^T (g_2^u + g_2^\theta) + \int_0^T\int_\Omega |k(\alpha)-k(\alpha_\varepsilon)|^2\,|\nabla\theta|^2 + \int_0^T\int_\Omega |h(\alpha)-h(\alpha_\varepsilon)|^2\,|\mathbf{u}_\varepsilon|^2.$$

- From Assumptions 2.1, the initial conditions are continuous with respect to α and thus the two first terms in $F_\varepsilon^{u,\theta}$ goes to 0 as $\varepsilon\to 0$.
- The third, forth and fifth terms in $F_\varepsilon^{u,\theta}$ have been already treated (see (2.20)).
- We now prove convergence for the term $g_1^u\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$. The main problem concerns the term $\int_0^T (1+\|\mathbf{u}_\varepsilon\|_{L^2(\Omega)}^2)\|\mathbf{u}_\varepsilon\|_{H^1(\Omega)}^2\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$. First, remark that $(\mathbf{u}_\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;L^2(\Omega)^2)$ Secondly, as proved in Theorem 2.5, up to a subsequence, $\mathbf{u}_\varepsilon$ weakly converges to $\mathbf{u}$ in $L^2(0,T,H^1(\Omega))$ and $\bar{\mathbf{u}}\to 0$ in $L^\infty(0,T;L^2(\Omega))$. Therefore, the whole term converges to 0.
- Concerning the other terms in g_1^u, they are all independent of ε, and we mainly use the fact that $\|\bar{\mathbf{u}}\|_{L^2(\Omega)}\to 0$ in $L^\infty([0,T])$.
- We may do the same proof concerning $\int_0^T \|\theta_\varepsilon\|_{L^2(\Omega)}\|\nabla\theta_\varepsilon\|_{L^2(\Omega)}^2\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2$ and $\int_0^T \|\theta_\varepsilon\|_{L^2(\Omega)}\|\theta_\varepsilon\|_{H^1(\Omega)}^2\|\bar{\theta}\|_{L^2(\Omega)}^2$.

Therefore, $\int_0^T (1+\|\theta_{\varepsilon k}\|_{L^2(\Omega)}\|\nabla\theta_{\varepsilon k}\|_{L^2(\Omega)}^2 + g_1)\|\bar{\mathbf{u}}\|_{L^2(\Omega)}^2) \xrightarrow[\varepsilon_k\to 0]{} 0$ and $\int_0^T (g_1^\theta+1)\|\bar{\theta}\|_{L^2(\Omega)}^2 \xrightarrow[\varepsilon_k\to 0]{} 0$. It eventually proves that $\|\nabla(\mathbf{u}-\mathbf{u}_{\varepsilon_k})\|_{L^2(0,T;L^2(\Omega))} + \|\nabla(\theta-\theta_{\varepsilon_k})\|_{L^2(0,T;L^2(\Omega))} \xrightarrow[k\to+\infty]{} 0$.

Owing to Urysohn's subsequence principle and the uniqueness of the solution to (WF), we actually obtain that the whole sequence $(\mathbf{u}_\varepsilon,\theta_\varepsilon)$ strongly converges toward $(\mathbf{u},\theta)$. ∎

REMARK 2.11. *If $\alpha_\varepsilon = \alpha$, then the next estimate holds*

$$\sup_{t\in[0,T]} \Big(\|\mathbf{u}(t) - \mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)} + \|\theta(t) - \theta_\varepsilon(t)\|_{L^2(\Omega)}\Big)$$
$$+ \left(\int_0^T \|\nabla\bar{\mathbf{u}}(t) - \nabla\mathbf{u}_\varepsilon(t)\|^2_{L^2(\Omega)} + \|\nabla\theta(t) - \nabla\theta_\varepsilon(t)\|^2_{L^2(\Omega)}\right)^{1/2} = O(C_\varepsilon).$$

The convergence of $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ toward $(\mathbf{u}, \theta)$ as $\varepsilon \to 0$ thus has the same rate as the one of neg_ε *toward* neg.

3. First order necessary conditions for the non-smooth optimization problem. We now begin the analysis of the optimization problems (OPT) and (OPTe). Let us detail first some assumptions made on the objective functional:

ASSUMPTIONS 3.1.
- *We assume that there are no terminal costs, i.e. there is no term in the cost functional concentrated on the terminal time T.*
- *For $d = 2$, $\mathcal{J}$ is lower semi-continuous with respect to the (weak-*, strong, strong) topology of $\mathcal{U}_{ad} \times L^2(0,T;V^u) \times L^2(0,T;V^\theta)$.*
- *In dimension 3, $\mathcal{J}$ is either lower semi-continuous with respect to the (weak-*, strong, strong) topology of $\mathcal{U}_{ad} \times L^2(0,T;H^u) \times L^2(0,T;H^\theta)$, or lower semi-continuous with respect to the (weak-*, weak, weak) topology of $\mathcal{U}_{ad} \times L^2(0,T;V^u) \times L^2(0,T;V^\theta)$.*

The existence of solutions to (OPTe) and (OPT) is rather classical and we refer for instance to [24, 36, 38]. We state a first result that let us see that a solution of (OPT) can be approximated by (OPTe).

THEOREM 3.2. *Assume Assumptions 3.1 are verified. Let $(\alpha^*_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ be a globally optimal solution of (OPTe). Then $(\alpha^*_\varepsilon) \subset \mathcal{U}_{ad}$ is a bounded sequence. Furthermore, there exists $(\alpha^*, \mathbf{u}^*, \theta^*) \in \mathcal{U}_{ad} \times L^2(0,T;V^u) \times L^2(0,T;V^\theta)$ such that a subsequence of $(\alpha^*_\varepsilon, \mathbf{u}_\varepsilon,, \theta_\varepsilon)$ converges to $(\alpha^*, \mathbf{u}^*, \theta^*)$ in the topology of Assumptions 3.1, and for all $(\alpha, \mathbf{u}, \theta)$ in $\mathcal{U}_{ad} \times L^2(0,T;V^u) \times L^2(0,T;V^\theta)$: $\mathcal{J}(\alpha^*, \mathbf{u}^*, \theta^*) \le \mathcal{J}(\alpha, \mathbf{u}, \theta)$. Hence, any accumulation point of $(\alpha^*_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon)$ is a globally optimal solution of (OPT).*

Proof. The proof can be adapted from [24, Theorem 15] or [36, Theorem 3]. ☐

However, the fact that this only concerns global solutions may appear restrictive. Under an additional assumption, we can state a slightly stronger result.

COROLLARY 3.3. *Assume Assumptions 3.1 hold. Let α^* be a local strict solution of (OPT), meaning that there exists $\rho > 0$ such that $\mathcal{J}(\alpha^*, \mathbf{u}^*, \theta^*) < \mathcal{J}(\alpha, \mathbf{u}, \theta)$ for all $\alpha \neq \alpha^*$ such that $\|\alpha^* - \alpha\|_{BV} < \rho$. Then, there exists a family of local solution (α^*_ε) of (OPTe) such that (α^*_ε) converges weak-* to α^*.*

Proof. Similar to [41, Theorem 3.14]. ☐

3.1. First order necessary conditions for (OPTe). From now on, we set $d = 2$ in order to have uniqueness of solution of (WFe). We make the following assumption on the cost function:

ASSUMPTIONS 3.4. *Assume $d = 2$ and $\mathcal{J}$ is Fréchet-differentiable.*

We define the sets $W^u(0,T) = \{\mathbf{u} \in \mathbb{X}^u;\ \partial_t\mathbf{u} \in (\mathbb{X}^u)'\}$, and $W^\theta(0,T) = \{\theta \in \mathbb{X}^\theta;\ \partial_t\theta \in (\mathbb{X}^\theta)'\}$. Write, in $(\mathbb{X}^u)' \times (\mathbb{X}^\theta)'$, the equation (WFe) as $e(\mathbf{u}_\varepsilon, \theta_\varepsilon, \alpha_\varepsilon) = 0$, where $e : W^u(0,T) \times W^\theta(0,T) \times \mathcal{U}_{\text{ad}} \to (\mathbb{X}^u)' \times (\mathbb{X}^\theta)' \times H^u \times H^\theta$ is defined as:

$$e(\mathbf{u}_\varepsilon, \theta_\varepsilon, \alpha_\varepsilon) = \begin{pmatrix} \partial_t \mathbf{u}_\varepsilon + \mathcal{A}\mathbf{u}_\varepsilon + \mathcal{B}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) + h(\alpha_\varepsilon)\mathbf{u}_\varepsilon \\ +\frac{1}{2}\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) - \frac{1}{2}\mathcal{N}_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}^{\text{ref}}) - f - \sigma^{\text{ref}} \\ \partial_t \theta_\varepsilon - \mathcal{D}(\mathbf{u}_\varepsilon, \theta_\varepsilon) + \mathcal{C}(\alpha_\varepsilon)\theta_\varepsilon + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon) - \phi \\ \mathbf{u}_\varepsilon(0,\cdot) - \mathbf{u}_0(\alpha_\varepsilon) \\ \theta_\varepsilon(0,\cdot) - \theta_0(\alpha_\varepsilon) \end{pmatrix}.$$

The operators $\mathcal{P}_\varepsilon$, $\mathcal{N}_\varepsilon$ and $\mathcal{M}_\varepsilon$ are Fréchet differentiable with the same smoothness as the approximation neg_ε. Their derivatives with respect to $\mathbf{u}_\varepsilon$ are denoted by $d_u\mathcal{P}_\varepsilon : W^u(0,T)^2 \to \mathcal{L}(W^u(0,T), (\mathbb{X}^u)')$, $\mathcal{N}'_\varepsilon : W^u(0,T)^2 \to \mathcal{L}(W^u(0,T), (\mathbb{X}^u)')$, $d_u\mathcal{M}_\varepsilon : W^u(0,T) \times W^\theta(0,T) \to \mathcal{L}(W^u(0,T), (\mathbb{X}^\theta)')$, defined by:

$$d_u\mathcal{P}_\varepsilon(\mathbf{u},\mathbf{u})\mathbf{v} = \mathcal{P}_\varepsilon(\mathbf{u},\mathbf{v}) + \mathcal{P}'_\varepsilon(\mathbf{u},\mathbf{u})\mathbf{v},$$

$$\langle \mathcal{N}'_\varepsilon(\mathbf{u},\mathbf{w})\mathbf{v}, \mathbf{\Psi}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} \text{neg}'_\varepsilon\,(\mathbf{u}\cdot\mathbf{n})\,(\mathbf{v}\cdot\mathbf{n})\mathbf{w}\cdot\mathbf{\Psi}.$$

$$\langle d_u\mathcal{M}_\varepsilon(\mathbf{u},\theta)\mathbf{v}, \varphi\rangle_{(V^\theta)',V^\theta} = \int_{\Gamma_{\text{out}}} (1 + \beta\text{neg}'_\varepsilon\,(\mathbf{u}\cdot\mathbf{n}))\,(\mathbf{v}\cdot\mathbf{n})\theta\varphi,$$

where $\mathcal{P}'_\varepsilon(\mathbf{u},\mathbf{w})$ is defined by:

$$\langle \mathcal{P}'_\varepsilon(\mathbf{u},\mathbf{w})\mathbf{v}, \mathbf{\Psi}\rangle_{(V^u)',V^u} = \int_{\Gamma_{\text{out}}} \text{pos}'_\varepsilon\,(\mathbf{u}\cdot\mathbf{n})\,(\mathbf{v}\cdot\mathbf{n})\mathbf{w}\cdot\mathbf{\Psi}.$$

Furthermore, these operators are bounded, as proved in the following lemma:

Lemma 3.5. *Given $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$ solution of (WFe):*

$$\begin{aligned}
\|d_u\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)\mathbf{v}\|_{(\mathbb{X}^u)'} \lesssim& (\|\mathbf{u}_\varepsilon\|^{\frac14}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}_\varepsilon\|^{\frac34}_{L^2(0,T;H^1(\Omega))} + C_\varepsilon) \\
&\|\mathbf{v}\|^{\frac12}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^2(0,T;H^1(\Omega))}, \\
\|\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{ref})\mathbf{v}\|_{(\mathbb{X}^u)'} \lesssim& \|\mathbf{u}_\varepsilon\|^{\frac14}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^\infty(0,T;L^2(\Omega))} \\
&\|\mathbf{u}_\varepsilon\|^{\frac34}_{L^2(0,T;H^1(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^2(0,T;H^1(\Omega))}, \\
\|d_u\mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon,\theta_\varepsilon)\mathbf{v}\|_{(\mathbb{X}^\theta)'} \lesssim& \|\theta_\varepsilon\|^{\frac14}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^\infty(0,T;L^2(\Omega))} \\
&\|\theta_\varepsilon\|^{\frac34}_{L^2(0,T;H^1(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Proof. The proof is similar to the proof of Lemma 2.6. Thanks to (**A2**), we obtain also:

$$\begin{aligned}
\langle \mathcal{P}'_\varepsilon(\mathbf{u},\mathbf{u})\mathbf{v}, \mathbf{\Psi}\rangle_{(V^u)',V^u} \lesssim& \|\mathbf{u}_\varepsilon\|^{\frac14}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^\infty(0,T;L^2(\Omega))} \\
&\|\mathbf{u}_\varepsilon\|^{\frac34}_{L^2(0,T;H^1(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^2(0,T;H^1(\Omega))}\|\mathbf{\Psi}\|_{\mathbb{X}^u}.
\end{aligned}$$

Analogously, using (**A4**), ones proves that there exists $C_\varepsilon > 0$ such that:

$$\begin{aligned}
\int_0^T \langle \mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{v}), \mathbf{\Psi}\rangle_{(V^u)',V^u} \lesssim& (\|\mathbf{u}_\varepsilon\|^{\frac14}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{u}_\varepsilon\|^{\frac34}_{L^2(0,T;H^1(\Omega))} + C_\varepsilon) \\
&\|\mathbf{v}\|^{\frac12}_{L^\infty(0,T;L^2(\Omega))}\|\mathbf{v}\|^{\frac12}_{L^2(0,T;H^1(\Omega))}\|\mathbf{\Psi}\|_{\mathbb{X}^u}.
\end{aligned}$$

Adding the two inequalities and dividing by $\|\mathbf{\Psi}\|_{H^1(\Omega)}$ concludes the proof. The proof of the second and third inequalities being similar, they are thus omitted. ∎

Using the results of [38, Section 1.8.2], one shows easily that e is Fréchet differentiable w.r.t. $(\mathbf{u}_\varepsilon, \theta_\varepsilon)$, with derivative given by:

$$e'_{\mathbf{u}_\varepsilon,\theta_\varepsilon}(\alpha_\varepsilon)\begin{pmatrix}\mathbf{v}\\ \ell\end{pmatrix} = \begin{pmatrix} \partial_t \mathbf{v} + \mathcal{A}\mathbf{v} + \mathcal{B}(\mathbf{v},\mathbf{u}_\varepsilon) + \mathcal{B}(\mathbf{u}_\varepsilon,\mathbf{v}) + h(\alpha_\varepsilon)\mathbf{v} + \mathcal{T}\ell \\ \hfill +\frac{1}{2} d_u\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)\mathbf{v} - \frac{1}{2}\mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\mathrm{ref}})\mathbf{v} \\ \partial_t\ell - \mathcal{D}(\mathbf{u}_\varepsilon,\ell) - \mathcal{D}(\mathbf{v},\theta_\varepsilon) + \mathcal{C}(\alpha_\varepsilon)\ell + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon,\ell) \\ \hfill + d_u\mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon,\theta_\varepsilon)\mathbf{v} \\ \mathbf{v}(0,\cdot) \\ \ell(0,\cdot)\end{pmatrix}.$$

For defining first order conditions (see [38]), a question of interest is to determine if, for all $g = (g^u, g^\theta, \mathbf{v}_0, \ell_0) \in (\mathbb{X}^u)' \times (\mathbb{X}^\theta)' \times H^u \times H^\theta$, the following linearized equation

$$e'_{\mathbf{u}_\varepsilon,\theta_\varepsilon}(\alpha_\varepsilon)\begin{pmatrix}\mathbf{v}\\ \ell\end{pmatrix} = g \tag{3.1}$$

admits a solution $(\mathbf{v},\ell) \in W^u(0,T) \times W^\theta(0,T)$.

Theorem 3.6. *For all $\alpha_\varepsilon \in \mathcal{U}_{ad}$, Eq. (3.1) admits a unique solution. Therefore, $e'_{\mathbf{u}_\varepsilon,\theta_\varepsilon}(\alpha_\varepsilon)$ is invertible.*

Sketch of proof. Using Lemma 3.5, the proof can be adapted from Theorem 2.5 and [37, Appendix A2]. Uniqueness is proved as for Proposition 2.7 (see also [37, Appendix A2]). ∎

A consequence of Theorem 3.6 is that for all $G = (g_1, g_2) \in W^u(0,T)' \times W^\theta(0,T)'$, the following adjoint equation admits a unique solution $\Lambda_\varepsilon = (\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta, \lambda_\varepsilon^{\mathbf{u}_0}, \lambda_\varepsilon^{\theta_0}) \in \mathbb{X}^u \times \mathbb{X}^\theta \times H^u \times H^\theta$:

$$(e'_{\mathbf{u}_\varepsilon,\theta_\varepsilon}(\alpha_\varepsilon))^*\Lambda_\varepsilon = G, \tag{3.2}$$

where $(e'_{\mathbf{u}_\varepsilon,\theta_\varepsilon}(\alpha_\varepsilon))^*$ denotes the adjoint operator of $e'_{\mathbf{u}_\varepsilon,\theta_\varepsilon}(\alpha_\varepsilon)$.

After some calculations, equation (3.2) is equivalent to solve, for all $(\mathbf{v},\ell) \in W^u(0,T) \times W^\theta(0,T)$, the following variational problem:

(3.3)
$$\begin{aligned}
&\langle -\partial_t\lambda_\varepsilon^{\mathbf{u}} + \mathcal{A}\lambda_\varepsilon^{\mathbf{u}} + \frac{1}{2}\left((\nabla\mathbf{u}_\varepsilon)^\intercal\lambda_\varepsilon^{\mathbf{u}} - (\nabla\lambda_\varepsilon^{\mathbf{u}})^\intercal\mathbf{u}_\varepsilon\right) - \mathcal{B}(\mathbf{u}_\varepsilon,\lambda_\varepsilon^{\mathbf{u}}) + h(\alpha_\varepsilon)\lambda_\varepsilon^{\mathbf{u}} - \mathcal{D}_1(\theta_\varepsilon)\lambda_\varepsilon^\theta \\
&\quad + \frac{1}{2}\mathcal{P}_\varepsilon(\mathbf{u}_\varepsilon,\lambda_\varepsilon^{\mathbf{u}}) + \frac{1}{2}(\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon) - \mathcal{N}'_\varepsilon(\mathbf{u}_\varepsilon,\mathbf{u}^{\mathrm{ref}}))^*\lambda_\varepsilon^{\mathbf{u}} \\
&\quad + \left(d_u\mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon,\theta_\varepsilon)\right)^*\lambda_\varepsilon^\theta, \mathbf{v}\rangle_{W^u(0,T)',W^u(0,T)} \\
&\qquad + \langle \mathbf{v}(0,\cdot), \lambda_\varepsilon^{u_0}\rangle_H \\
&\qquad = \langle g_1, \mathbf{v}\rangle_{W^u(0,T)',W^u(0,T)}, \\
&\langle -\partial_t\lambda_\varepsilon^\theta + \mathcal{T}^*\lambda_\varepsilon^{\mathbf{u}} + \mathcal{C}(\alpha_\varepsilon)\lambda_\varepsilon^\theta - \mathcal{D}_2(\mathbf{u}_\varepsilon)\lambda_\varepsilon^\theta + \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon)^*\lambda_\varepsilon^\theta, \ell\rangle_{W^\theta(0,T)',W^\theta(0,T)} \\
&\qquad = \langle g_2, \ell\rangle_{W^\theta(0,T)',W^\theta(0,T)}
\end{aligned}$$

where $\langle\mathcal{D}(\theta,\mathbf{u}),\varphi\rangle = \langle\mathcal{D}_1(\theta)\varphi,\mathbf{u}\rangle = \langle\mathcal{D}_2(\mathbf{u})\varphi,\theta\rangle$, $\langle\mathcal{M}_\varepsilon(\mathbf{u})\theta,\varphi\rangle = \langle\mathcal{M}_\varepsilon(\mathbf{u})\varphi,\theta\rangle = \int_{\Gamma_{\mathrm{out}}}\left((\mathbf{u}\cdot\mathbf{n}) + \beta\,\mathrm{neg}_\varepsilon(\mathbf{u}\cdot\mathbf{n})\right)\theta\varphi$. This equation, in turn, is the weak formulation of:

$$
\begin{aligned}
& -\partial_t\lambda_\varepsilon^{\mathbf{u}} - A\Delta\lambda_\varepsilon^{\mathbf{u}} + h(\alpha_\varepsilon)\lambda_\varepsilon^{\mathbf{u}} + (\nabla\mathbf{u}_\varepsilon)^\intercal\lambda_\varepsilon^{\mathbf{u}} - (\mathbf{u}_\varepsilon\cdot\nabla)\lambda_\varepsilon^{\mathbf{u}} - \theta_\varepsilon\nabla\lambda_\varepsilon^{\theta} = g_1\\
& \nabla\cdot\lambda_\varepsilon^{\mathbf{u}} = 0,\\
& -\partial_t\lambda_\varepsilon^{\theta} + B\lambda_\varepsilon^{\mathbf{u}}\cdot\mathbf{e}_y - \nabla\cdot(Ck(\alpha_\varepsilon)\nabla\lambda_\varepsilon^{\theta}) - \nabla\cdot(\mathbf{u}_\varepsilon\lambda_\varepsilon^{\theta}) = g_2\\
& \lambda_\varepsilon^{\mathbf{u}}|_{\Gamma_{\rm w}\cup\Gamma_{\rm in}} = 0,\\
& \lambda_\varepsilon^{\theta}|_{\Gamma_{\rm in}} = 0,\\
& \partial_n\lambda_\varepsilon^{\theta}|_{\Gamma_{\rm w}} = 0,\\
& A\partial_n\lambda_\varepsilon^{\mathbf{u}}|_{\Gamma_{\rm out}} = \frac{1}{2}\left(\mathrm{pos}_\varepsilon\left(\mathbf{u}_\varepsilon\cdot\mathbf{n}\right) + (\mathbf{u}_\varepsilon\cdot\mathbf{n})\right)\lambda_\varepsilon^{\mathbf{u}} + (1+\beta\mu_\varepsilon)\theta_\varepsilon\lambda_\varepsilon^{\theta}\mathbf{n}\\
& \qquad\qquad\qquad + \frac{1}{2}\mu_\varepsilon\left((\mathbf{u}_\varepsilon - \mathbf{u}^{\rm ref})\cdot\lambda_\varepsilon^{\mathbf{u}}\right)\mathbf{n},\\
& Ck(\alpha_\varepsilon)\partial_n\lambda_\varepsilon^{\theta} + \beta\lambda_\varepsilon^{\theta}\mathrm{neg}_\varepsilon\left(\mathbf{u}_\varepsilon\cdot\mathbf{n}\right)|_{\Gamma_{\rm out}} = 0\\
& \lambda_\varepsilon^{\mathbf{u}}(T) = 0, \lambda_\varepsilon^{\theta}(T) = 0,
\end{aligned}
\tag{3.4a}
$$

$$
\mu_\varepsilon = \mathrm{neg}_\varepsilon'\left(\mathbf{u}_\varepsilon\cdot\mathbf{n}\right)
\tag{3.4b}
$$

and, as shown in a similar fashion in [37], $\lambda_\varepsilon^{u_0} = \lambda_\varepsilon^{\mathbf{u}}(0,\cdot)$, $\lambda_\varepsilon^{\theta_0} = \lambda_\varepsilon^{\theta}(0,\cdot)$. Furthermore, we can argue that the weak solution $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^{\theta})$ of (3.4) are in $L^\infty(0,T;L^2(\Omega)^2)\times L^\infty(0,T;L^2(\Omega))$, as done in Theorem 2.4.

An other consequence of Theorem 3.6 is that we can apply [38, Corollary 1.3] which states that at any local solution $(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon^*, \theta_\varepsilon^*)$ of (OPTe), the following optimality conditions hold:

THEOREM 3.7. *Let α_ε^* be an optimal solution of* (OPTe) *with associated states $(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*)$. Then there exist adjoint states $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^{\theta}) \in \mathbb{X}^u\times\mathbb{X}^\theta$ such that, denoting $(\lambda_\varepsilon^{u_0}, \lambda_\varepsilon^{\theta_0}) = (\lambda_\varepsilon^{\mathbf{u}}(0,\cdot), \lambda_\varepsilon^{\theta}(0,\cdot))$ and $\Lambda_\varepsilon = (\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^{\theta}, \lambda_\varepsilon^{\mathbf{u}_0}, \lambda_\varepsilon^{\theta_0})$:*

$$
\begin{aligned}
& e(\alpha_\varepsilon^*, \mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) = 0,\\
& \mathcal{J}'_{\mathbf{u}_\varepsilon^*,\theta_\varepsilon^*}(\alpha_\varepsilon^*) + (e_{\mathbf{u}_\varepsilon^*,\theta_\varepsilon^*}(\alpha_\varepsilon^*)')^*\Lambda_\varepsilon = 0,\\
& \left\langle \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*)')^*\Lambda_\varepsilon, \alpha - \alpha_\varepsilon^*\right\rangle_{\mathcal{U}_{ad}',\mathcal{U}_{ad}} \geq 0,\ \forall\alpha\in\mathcal{U}_{ad},\\
& \alpha_\varepsilon \in \mathcal{U}_{ad}.
\end{aligned}
\tag{3.5}
$$

REMARK 3.8. *As stated in [38, Eq. (1.89)], since e and $\mathcal{J}$ are Fréchet differentiable, the mapping $\alpha_\varepsilon \mapsto \hat{\mathcal{J}}(\alpha_\varepsilon) = \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon)$ is Fréchet differentiable, and $\hat{\mathcal{J}}'(\alpha_\varepsilon) = \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*))^*\Lambda_\varepsilon$, which reads as:*

$$
\begin{aligned}
(e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*))^*\Lambda_\varepsilon = \int_0^T & \left(h'(\alpha_\varepsilon)\mathbf{u}_\varepsilon\cdot\lambda_\varepsilon^{\mathbf{u}} + Ck'(\alpha_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\lambda_\varepsilon^{\theta}\right)\\
& + \mathbf{u}_0'(\alpha_\varepsilon)\cdot\lambda_\varepsilon^{u_0} + \theta_0'(\alpha_\varepsilon)\lambda_\varepsilon^{\theta_0}.
\end{aligned}
$$

3.2. Limit adjoint system. To conclude this paper, we will now study the convergence, as $\varepsilon\to 0$, of the adjoint states $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^{\theta})$ to functions $(\lambda^{\mathbf{u}}, \lambda^{\theta})$. The main difficulty concerns the multiplier μ_ε defined in (3.4b). We will prove that at the limit, μ is defined thanks to the convex-hull of the Heaviside function $H:\mathbb{R}\multimap[0,1]$, given by:

$$
H(u) = \begin{cases} \{0\} & \text{if } u < 0, \\ \{1\} & \text{if } u > 0, \\ [0,1] & \text{if } u = 0. \end{cases} \tag{3.6}
$$

As we will prove in this section, these limit adjoint states $(\lambda^{\mathbf{u}}, \lambda^{\theta})$ let us define necessary conditions of optimality for the unrelaxed problem (OPT).

LEMMA 3.9. *Let $(\alpha_\varepsilon) \subset \mathcal{U}_{ad}$ and $\alpha \in \mathcal{U}_{ad}$ such that $\alpha_\varepsilon \overset{*}{\rightharpoonup} \alpha$. Define by $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^{\theta})$ a weak solution of (3.4) parametrized by α_ε. Then, there exists $\lambda^{\mathbf{u}} \in L^\infty(0,T;\, H^u) \cap L^2(0,T;V^u)$, $\lambda^\theta \in L^\infty(0,T,H^\theta) \cap L^2(0,T;V^\theta)$ such that, up to a subsequence:*
- *$\lambda_\varepsilon^{\mathbf{u}} \to \lambda^{\mathbf{u}}$ in $L^\infty(0,T;(L^2(\Omega))^2)$ and $\lambda_\varepsilon^\theta \to \lambda^\theta$ in $L^\infty(0,T;L^2(\Omega))$,*
- *$\lambda_\varepsilon^{\mathbf{u}} \xrightarrow[\varepsilon\to 0]{} \lambda^{\mathbf{u}}$ in $L^2(0,T;(H^1(\Omega))^2)$ and $\lambda_\varepsilon^\theta \xrightarrow[\varepsilon\to 0]{} \lambda^\theta$ in $L^2(0,T;(H^1(\Omega)))$,*
- *$\lambda_\varepsilon^{\mathbf{u}} \xrightarrow[\varepsilon\to 0]{} \lambda^{\mathbf{u}}$ in $L^2(0,T;(L^2(\Gamma))^2)$ and $\lambda_\varepsilon^\theta \xrightarrow[\varepsilon\to 0]{} \lambda^\theta$ in $L^2(0,T;(L^2(\Gamma)))$.*

Furthermore, there exists $\mu \in L^\infty([0,T]\times\Gamma_{out})$ defined by $-\mu \in H(-\mathbf{u}\cdot\mathbf{n})$ a.e. in Γ_{out} such that $(\lambda^{\mathbf{u}}, \lambda^\theta)$ is a weak solution to (3.4a) parametrized by α and μ, replacing $\operatorname{neg}_\varepsilon(\cdot)$ (resp. $\operatorname{pos}_\varepsilon(\cdot)$) by $\operatorname{neg}(\cdot)$ (resp. $\operatorname{pos}(\cdot)$).

Proof. The proof is very similar to the ones presented in section 2.
- In a similar manner as for Proposition 2.2 and Proposition 2.3, one shows that, for all $\sigma \in [0, \frac{1}{6})$, there exist constants $c_\lambda^\theta(\sigma)$ and $c_\lambda^u(\sigma)$, independent of ε, such that:

$$
\sup_{[0,T]} \|\lambda_\varepsilon^{\mathbf{u}}\|_{L^2(\Omega)} + \int_0^T \|\nabla\lambda_\varepsilon^{\mathbf{u}}\|_{L^2(\Omega)} + \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathscr{F}\left(\widetilde{\lambda_\varepsilon^{\mathbf{u}}}\right)\right\|_{L^2(\Omega)} d\tau \leq c_\lambda^u(\sigma),
$$

$$
\sup_{[0,T]} \|\lambda_\varepsilon^{\theta}\|_{L^2(\Omega)} + \int_0^T \|\nabla\lambda_\varepsilon^{\theta}\|_{L^2(\Omega)} + \int_{\mathbb{R}} |\tau|^{2\sigma} \left\| \mathscr{F}\left(\widetilde{\lambda_\varepsilon^{\theta}}\right)\right\|_{L^2(\Omega)} d\tau \leq c_\lambda^\theta(\sigma).
$$

- These bounds prove a weaker set of convergence in the same manner as in Theorem 2.5. Since once again, we set $d = 2$, one proves the strong convergence stated above as in Corollary 2.9.

We only need to prove that $(\lambda^{\mathbf{u}}, \lambda^\theta)$ is a weak solution to (3.4a). The terms $\langle(\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))^* \lambda_\varepsilon^{\mathbf{u}}\rangle_{W^u(0,T)',W^u(0,T)}$ and $\langle (d_u\mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \,\lambda_\varepsilon^\theta, \mathbf{v}\rangle_{W^u(0,T)',W^u(0,T)}$ need a more thorough examination. We start with the first term for which we have

$$
\langle(\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon))^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v}\rangle_{W^u} = \int_0^T \int_{\Gamma_{\text{out}}} \operatorname{pos}'_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n})\,(\mathbf{u}_\varepsilon\cdot\lambda_\varepsilon^{\mathbf{u}})\,\mathbf{n}\cdot\mathbf{v}.
$$

In the same spirit as in [21, Proof of Lemma 4.3], we prove that up to a subsequence (not relabeled) one has $\operatorname{neg}'_\varepsilon(\mathbf{u}_\varepsilon\cdot\mathbf{n}) \overset{*}{\rightharpoonup} \mu$ in $L^\infty([0,T]\times\Gamma_{\text{out}})$, and such that $-1 \leq \mu \leq 0$ a.e. in Γ_{out} and

$$
\mu = -1 \text{ a.e. in } \{\mathbf{u}\cdot\mathbf{n} < 0\},\ \mu = 0 \text{ a.e. in } \{\mathbf{u}\cdot\mathbf{n} > 0\}.
$$

Furthermore, due to the convergence presented above, $\mathbf{u}_\varepsilon\cdot\lambda_\varepsilon^{\mathbf{u}} \to \mathbf{u}\cdot\lambda^{\mathbf{u}}$ in $L^1(0,T;\, L^1(\Gamma_{\text{out}}))$. Therefore, it proves that:

$$
\langle(\mathcal{P}'_\varepsilon(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon)^* \lambda_\varepsilon^{\mathbf{u}}, \mathbf{v}\rangle_{W^u(0,T)',W^u(0,T)} \to \int_0^T \int_{\Gamma_{\text{out}}} (1+\mu)\,(\mathbf{u}\cdot\lambda^{\mathbf{u}})\,\mathbf{n}\cdot\mathbf{v}.
$$

Similarly, we have that:

$$\langle (d_u \mathcal{M}_\varepsilon(\mathbf{u}_\varepsilon, \theta_\varepsilon))^* \lambda^\theta_\varepsilon, \mathbf{v}\rangle_{W^u(0,T)', W^u(0,T)} \to \int_0^T \int_{\Gamma_{\text{out}}} (1+\beta\mu)\,(\mathbf{v}\cdot\mathbf{n})\theta\lambda^\theta.$$

All other terms in (3.3) can be dealt with as in the proof of Theorem 2.5. Therefore, $(\lambda^{\mathbf{u}}, \lambda^\theta)$ is a weak solution to (3.4a) parametrized by α and μ. □

We may now prove the final result of this paper ; namely the necessary optimality conditions of (OPT).

THEOREM 3.10. *Let α^* be an optimal solution of* (OPT) *with associated state $\mathbf{u}^*, \theta^*$. Then there exist a multiplier $\mu \in L^\infty([0,T]\times\Gamma_{out})$ and adjoint states $(\lambda^{\mathbf{u}}, \lambda^\theta) \in \mathbb{X}^u \times \mathbb{X}^\theta$ solution of* (3.4a) *such that, denoting $(\lambda^{u_0}, \lambda^{\theta_0}) = (\lambda^{\mathbf{u}}(0,\cdot), \lambda^\theta(0,\cdot))$ and $\Lambda = (\lambda^{\mathbf{u}}, \lambda^\theta, \lambda^{\mathbf{u}_0}, \lambda^{\theta_0})$:*

$$\langle \mathcal{J}'_{\alpha^*}(\mathbf{u}^*, \theta^*) + (e_{\alpha^*}(\mathbf{u}^*, \theta^*)')^*\Lambda, \alpha - \alpha^*\rangle_{\mathcal{U}'_{ad}, \mathcal{U}_{ad}} \geq 0,\ \forall \alpha \in \mathcal{U}_{ad}.$$

Proof. The proof follows the lines of [21, Theorem 4.4]. Denote by S_ε the solution operator which associates to α the solution of the relaxed equations (WFe) and by S the solution operator which to α associates the solution of (WF). For some $\rho > 0$, consider the auxiliary optimal control problem:

$$\text{(3.7)}\qquad \begin{aligned} &\min\ F_\varepsilon(\alpha_\varepsilon) = \mathcal{J}(\alpha_\varepsilon, \mathbf{u}_\varepsilon, \theta_\varepsilon) + \frac{1}{2}\|\alpha^* - \alpha_\varepsilon\|^2_{L^2(\Omega)} \\ &\text{s.t.}\ \begin{cases} (\mathbf{u}_\varepsilon, \theta_\varepsilon) = S_\varepsilon(\alpha_\varepsilon), \\ \alpha_\varepsilon \in \mathcal{U}_{ad}, \\ \|\alpha_\varepsilon - \alpha^*\|_{L^2(\Omega)} \leq \rho. \end{cases} \end{aligned}$$

Since α_ε and α^* are both in $\mathcal{U}_{\text{ad}}$, they are both bounded in $L^\infty(\Omega)$ and therefore, $\|\alpha^* - \alpha_\varepsilon\|_{L^2(\Omega)}$ is well defined. It is classical to show that (3.7) admits a global minimizer $\alpha^*_\varepsilon \in \mathcal{U}_{\text{ad}}$.

Using (2.19) (but with $\alpha_\varepsilon \equiv \alpha$), one proves that (in the norm of the topology given in Assumptions 3.1 with $d = 2$):

$$\text{(3.8)}\qquad \|S(\alpha) - S_\varepsilon(\alpha)\| \lesssim C_\varepsilon,\ \forall \alpha \in \mathcal{U}_{\text{ad}},$$

where C_ε has been defined in (2.12).

Note that due to the Fréchet-differentiability of $\mathcal{J}$ supposed in Assumptions 3.4 and (3.8), it holds, for ε small enough:

$$|\mathcal{J}(\alpha, S(\alpha)) - \mathcal{J}(\alpha, S_\varepsilon(\alpha))| \lesssim C_\varepsilon,\ \forall \alpha \in \mathcal{U}_{\text{ad}},\ \|\alpha - \alpha^*\| \leq \rho.$$

We obtain as a consequence that $F_\varepsilon(\alpha^*) \lesssim C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*))$, and:

$$F_\varepsilon(\alpha) \gtrsim -C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*)) + \frac{1}{2}\|\alpha - \alpha^*\|^2_{L^2(\Omega)},\ \forall \alpha \in \mathcal{U}_{\text{ad}},\ \|\alpha - \alpha^*\|_{L^2(\Omega)} \leq \rho.$$

Therefore, for all $\alpha \in \mathcal{U}_{\text{ad}}$ such that $\|\alpha - \alpha^*\|_{L^2(\Omega)} \leq \rho$:

$$F_\varepsilon(\alpha^*) \lesssim C_\varepsilon + \mathcal{J}(\alpha^*, S(\alpha^*)) \lesssim C_\varepsilon + \mathcal{J}(\alpha, S(\alpha)) \lesssim 2C_\varepsilon + F_\varepsilon(\alpha).$$

Hence, for some constant C', and denoting $C'_\varepsilon = C'C_\varepsilon$, one has the implication:

$$\forall \alpha \in \mathcal{U}_{\text{ad}},\ 2C'_\varepsilon < \frac{1}{2}\|\alpha - \alpha^*\|^2_{L^2(\Omega)} \leq \frac{1}{2}\rho^2 \implies F_\varepsilon(\alpha^*) < F_\varepsilon(\alpha).$$

One has therefore the following necessary condition of optimality:

$$\|\alpha_\varepsilon^* - \alpha^*\|_{L^2(\Omega)} \leq \sqrt{4C'_\varepsilon}. \tag{3.9}$$

Hence, for ε small enough, α_ε^* is in the ρ-ball around α^* ; therefore, α_ε^* is a local solution of (OPTe). Using Theorem 3.7, one then proves that there exists adjoint states $(\lambda_\varepsilon^{\mathbf{u}}, \lambda_\varepsilon^\theta)$ solution of (3.4a) such that, for all $\alpha \in \mathcal{U}_{\mathrm{ad}}$:

$$\left\langle \mathcal{J}'_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*) + (e_{\alpha_\varepsilon^*}(\mathbf{u}_\varepsilon^*, \theta_\varepsilon^*)')^* \Lambda_\varepsilon, \alpha - \alpha_\varepsilon^* \right\rangle_{\mathcal{U}'_{\mathrm{ad}}, \mathcal{U}_{\mathrm{ad}}} + \langle \alpha_\varepsilon^* - \alpha^*, \alpha - \alpha_\varepsilon^* \rangle_{L^2(\Omega)} \geq 0. \tag{3.10}$$

From (3.9), one has $\alpha_\varepsilon^* \to \alpha^*$ strongly in $L^2(\Omega)$, and therefore, in $L^1(\Omega)$. Since $(\alpha_\varepsilon^* - \alpha^*)_\varepsilon \subset \mathcal{U}_{\mathrm{ad}}$, one has also $(\alpha_\varepsilon^* - \alpha^*)_\varepsilon$ bounded in $BV(\Omega)$. Hence, $\alpha_\varepsilon^* \xrightarrow{*} \alpha^*$ in $\mathcal{U}_{\mathrm{ad}}$. Using then Corollary 2.9, Assumptions 3.1 and Lemma 3.9, we can pass to the limit in (3.10), which concludes this proof. ∎

4. Conclusions and perspectives. In this paper, we obtained a set of theoretical results (existence, uniqueness in 2d, relaxation and definition of first order necessary optimality conditions) for a topology optimization problem involving Boussinesq system with non-smooth boundary conditions.

As a perspective, we must now consider how these results can help design a numerical method. It should be noted that these non-smooth outlet conditions have already been studied outside of an optimization context [5, 13, 15]. Also, the use of smooth first order conditions for a topology optimization problem is not new [16, 48], and the smooth approximation found in Theorem 3.7 can straightforwardly be used in this context for a fixed ε. Finally, we emphasize that the numerical use of the nonsmooth first order conditions as given in Theorem 3.10 needs more research. This could be inspired by the approaches used in nonsmooth optimization with subdifferentials [33]. A continuity approach, as experimented in [51] together with Remark 2.11 may also provide a good basis to get error estimates between the optimized solution of the relaxed optimization problem and the non-smooth ones.

Appendix A. Technical lemma. Let $\mathbb{X} = L^2(0,T;H^1(\Omega)) \cap L^4(0,T;L^2(\Omega))$, and denote by $\mathbb{X}'$ the dual of $\mathbb{X}$ with the following dual pairing: $\langle f, g\rangle_{\mathbb{X}',\mathbb{X}} = \int_0^T \langle f(t), g(t)\rangle_{L^2(\Omega)}$. Denote $E_{\mathbb{X}} = \{\mathbf{u} \in \mathbb{X} | \mathbf{u}' = \frac{d\mathbf{u}}{dt} \in \mathbb{X}'\}$. We endow $E_{\mathbb{X}}$ with the norm: $\|\mathbf{u}\|_{E_{\mathbb{X}}} = \|\mathbf{u}\|_{\mathbb{X}} + \|\mathbf{u}'\|_{\mathbb{X}'}$, where $\|\mathbf{u}\|_{\mathbb{X}} = \max\{\|\mathbf{u}\|_{L^2(0,T;H^1(\Omega))}, \|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))}\}$. Finally, denote $\mathcal{D}(0,T;X)$ the set of infinitely differentiable functions from $[0,T]$ to X with compact support in $[0,T]$.

LEMMA A.1. *Let $\mathbf{u} \in E_{\mathbb{X}}$. There exists $(\mathbf{u}_n)_n \subset \mathcal{D}(0,T;H^1(\Omega))$ such that:*

$$\mathbf{u}_n \to \mathbf{u} \text{ in } L^2(0,T;H^1(\Omega)), \quad \mathbf{u}'_n \rightharpoonup \mathbf{u}' \text{ in } \mathbb{X}'.$$

Proof. From [14, Theorem II.2.26], one proves directly that there exists $(\mathbf{u}_n)_n \subset \mathcal{D}(0,T;H^1(\Omega))$ such that $\mathbf{u}_n \to \mathbf{u}$ strongly in $L^2(0,T;H^1(\Omega))$.

For all $\varphi \in \mathcal{D}(0,T;H^1(\Omega))$, one has:

$$\langle \mathbf{u}'_n, \varphi\rangle_{\mathbb{X}',\mathbb{X}} = -\langle \mathbf{u}_n, \varphi'\rangle_{\mathbb{X}',\mathbb{X}} \xrightarrow[n\to+\infty]{} -\langle \mathbf{u}, \varphi'\rangle_{\mathbb{X}',\mathbb{X}} = \langle \mathbf{u}', \varphi\rangle_{\mathbb{X}',\mathbb{X}}.$$

By the density result [14, Theorem II.2.26], we prove that:

$$\forall \varphi \in \mathbb{X}, \langle \mathbf{u}'_n, \varphi\rangle_{\mathbb{X}',\mathbb{X}} \xrightarrow[n\to+\infty]{} \langle \mathbf{u}', \varphi\rangle_{\mathbb{X}',\mathbb{X}}.$$

∎

LEMMA A.2. *Let $\mathbf{u}, \mathbf{v} \in E_{\mathbb{X}}$. Then, $t \mapsto \langle \mathbf{u}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)}$ is in $W^{1,1}([0,T])$ and for all $t \in [0,T]$:*

$$\frac{d}{dt} \langle \mathbf{u}(t), \mathbf{v}(t) \rangle_{L^2(\Omega)} = \left\langle \frac{d\mathbf{u}}{dt}(t), \mathbf{v}(t) \right\rangle_{L^2(\Omega)} + \left\langle \frac{d\mathbf{v}}{dt}(t), \mathbf{u}(t) \right\rangle_{L^2(\Omega)}.$$

Proof. Using Lemma A.1, the proof is a simple adaptation of [14, Theorem II.5.12]. ◻

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