# Finite-Element Domain Approximation for Maxwell Variational Problems on Curved Domains

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January 5, 2022

#### 1 Introduction

We consider the problem of domain approximation in finite element methods for Maxwell's equations on curved domains, i.e., when affine or polynomial meshes fail to cover the domain of interest exactly, forcing to approximate the domain by a sequence of (potentially curved) polyhedra arising from inexact meshes. In particular, we aim at finding conditions on the quality of these approximations that ensure convergence rates of the discrete solutions—in the approximate domains—to the continuous one in the original domain. This analysis is classical in the context of the Laplace equation [15] but has not been studied in the Maxwell case. In [4], we showed the effects of numerical integration on the convergence of the curl-conforming finite element method (FEM) for Maxwell variational problems and found necessary conditions on quadrature rules to ensure error convergence rates both in affine and curved meshes. However, we discarded the error terms associated with domain mesh approximation.

When approximating solutions to variational problems on a given *original domain* by solutions to analogous problems on *approximate* (computational) domains, the choice of error measure is not straightforward as approximate and exact solutions do not share the same domain. Indeed, several choices for error measures can be considered: comparisons between extensions of continuous or discrete solutions to a *hold-all* domain [9, 10, 15, 41]; mismatch measured at the intersection between the original and computational domains [27, 38, 23, 40, 42]; mapping of solutions from computational to the original domains or vice-versa [3, 19, 28]; and finally, in [14, 17] the error is measured in a Hilbert space common to solutions on the approximate and original domains.

In the present note, our main results are condensed in theorems 4.18, 4.20, 5.8, and 5.9 in which we estimate the convergence of Maxwell solutions in a series of approximate domains  $\{\widetilde{D}_i\}_{i\in\mathbb{N}}$  to the continuous solution in a given original domain—denoted D, and approximated by the sequence  $\{\widetilde{D}_i\}_{i\in\mathbb{N}}$ —in two different ways. Following [28], theorem 4.18 estimates the error through curl-conforming pull-backs mapping fields in approximate domains  $\{\widetilde{D}_i\}_{i\in\mathbb{N}}$  to fields in the original one D (cf. [25, Sec. 2.5]). Alternatively, and in the spirit of [15], theorem 4.20 bounds the error of approximate solutions to an extension of the solution in the original domain D, allowing for its evaluation in each approximate domain in the sequence even though one can not ensure that  $\widetilde{D}_i \subseteq D$  for any  $i \in \mathbb{N}$  without additional assumptions. Then, theorems 5.8 and 5.9 correspond to discrete analogues to theorems 4.18 and 4.20, respectively. Moreover, our findings allow for a straightforward combination with our earlier results in [4], and so theorems 5.12 and 5.13 correspond to fully discrete versions of theorems 5.8 and 5.9, respectively, by incorporating the effects of numerical integration on error convergence rates.

The structure of the manuscript is as follows. In section 2 we set notation and introduce the Maxwell variational problems considered throughout, as well as basic parameter and overarching assumptions. In section 3, we introduce finite elements on curved meshes as in [15] and introduce

elementary results concerning the continuity and approximation properties of the classical curl-conforming interpolation operator (see [30, Sec. 5.5]). Section 4 introduces the issue of solving Maxwell variational problems on approximate domains at the continuous level; a viewpoint which is then directly applied to the discrete level in section 5. Then, section 6 displays a simple numerical example confirming our findings followed by concluding remarks in Section 7. Appendices provide proofs of various technical lemmas and results.

## 2 General definitions and Maxwell problem statement

#### 2.1 General notation

Set  $i = \sqrt{-1}$ . For  $d \in \mathbb{N}$ , we denote the canonical vectors in  $\mathbb{R}^d$  as  $\{e_i\}_{i=1}^d$  and the inner product between two elements  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^d$  is written  $\mathbf{x} \cdot \mathbf{y}$ . Let  $\Omega$  be an open bounded Lipschitz domain in  $\mathbb{R}^d$  with boundary  $\partial \Omega$ . For  $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $C^m(\Omega)$  denotes the set of complex-valued functions with m-continuous derivatives on  $\Omega$ , while  $C_0^m(\Omega)$  is the subset of elements in  $C^m(\Omega)$  with compact support in  $\Omega$ . Infinitely smooth functions with compact support in  $\Omega$  belong to  $\mathcal{D}(\Omega) := \bigcap_{m=0}^{\infty} C_0^m(\Omega)$ . For  $k \in \mathbb{N}_0$  and  $q \in \mathbb{N}$ ,  $\mathbb{P}_k(\Omega; \mathbb{C}^q)$  is the space of polynomials of degree less than or equal to k from  $\Omega$  to  $\mathbb{C}^q$ .  $\widetilde{\mathbb{P}}_k(\Omega; \mathbb{C}^q)$  denotes the space of homogeneous polynomials of degree k from  $\Omega$  to  $\mathbb{C}^q$ . For  $p \geq 1$  and  $s \in \mathbb{R}$ ,  $L^p(\Omega)$  and  $W^{p,s}(\Omega)$  are the class of p-integrable functions on  $\Omega$  and the standard Sobolev spaces of order s, respectively. If p = 2, we employ the standard notation  $H^s(\Omega) := W^{2,s}(\Omega)$ .

Norms and semi-norms over a general Banach space Y are indicated by subscripts. However, the norm and semi-norm of  $H^s(\Omega)$  will be written as  $\|\cdot\|_{s,\Omega}$  and  $|\cdot|_{s,\Omega}$ , respectively. The topological dual of the Banach space Y will be denoted as Y'. For a Hilbert space X, we write its inner product as  $(\cdot,\cdot)_X$ , and its duality pairing as  $\langle\cdot,\cdot\rangle_{X'\times X}$ . Again, we make an exception for  $H^s(\Omega)$  and write its inner and duality products as  $(\cdot,\cdot)_{s,\Omega}$ , and  $\langle\cdot,\cdot\rangle_{s,\Omega}$ , respectively. These are understood in the sesquilinear sense.

General scalar-valued functions and function spaces are differentiated from their vector-valued counterparts by the use of boldface symbols for the latter. Components of vector-valued functions are identified by subscript, e.g.,  $V_2 = \mathbf{V} \cdot \mathbf{e}_2$ . For a square matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$ , with  $n \in \mathbb{N}$ , we denote its induced matrix norm by  $\|\mathbf{A}\|_{\mathbb{C}^{n \times n}}$ , its determinant by  $\det(\mathbf{A})$ , its transpose by  $\mathbf{A}^{\top}$ , its cofactor matrix by  $\mathbf{A}^{\text{co}}$  and its inverse by  $\mathbf{A}^{-1} = \det(\mathbf{A})^{-1}\mathbf{A}^{\text{co}}$ , when invertible. The Jacobian matrix of a differentiable function  $\mathbf{U} : \mathbb{R}^n \to \mathbb{C}^n$  is  $d\mathbf{U} : \mathbb{R}^n \to \mathbb{C}^{n \times n}$ . Moreover,  $\mathbf{I} : \mathbb{C}^n \to \mathbb{C}^n$  is the identity map while  $\mathbf{I} \in \mathbb{R}^{n \times n}$  denotes the identity matrix, so that  $d\mathbf{I} = \mathbf{I}$ .

Finally, norms of vector-valued functions in  $\mathbf{W}^{p,s}(\Omega)$ , for  $p \geq 1$  and  $s \in \mathbb{R}$ , are computed as the p-sum of the  $W^{p,s}(\Omega)$ -norms of their components, e.g.,  $\|\mathbf{U}\|_{\mathbf{W}^{p,s}(\Omega)}^p = \sum_{i=1}^d \|U_i\|_{W^{p,s}(\Omega)}^p$  for  $p \in [1,\infty)$  and the customary modification when  $p = \infty$ . Norms for matrix-valued functions are computed analogously. For a multi-index  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)^{\top} \in \mathbb{N}_0^d$ , we write  $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$  and  $\mathbf{x}^{\boldsymbol{\alpha}} = \prod_{i=1}^n \mathbf{x}_i^{\alpha_i}$ . For  $n \in \mathbb{N}$ , we write the set of integers  $\{1, 2, 3, \dots, n\}$  as  $\{1: n\}$ 

#### 2.2 Functional spaces

Let  $\Omega$  be an open and bounded Lipschitz domain in  $\mathbb{R}^3$ . We introduce the following functional spaces of vector-valued functions:

$$\begin{split} \boldsymbol{H}(\boldsymbol{\operatorname{curl}};\Omega) := \left\{ \mathbf{U} \in \boldsymbol{L}^2(\Omega) \ : \ \boldsymbol{\operatorname{curl}} \, \mathbf{U} \in \boldsymbol{L}^2(\Omega) \right\}, \\ \boldsymbol{H}(\boldsymbol{\operatorname{curl}} \, \boldsymbol{\operatorname{curl}};\Omega) := \left\{ \mathbf{U} \in \boldsymbol{H}(\boldsymbol{\operatorname{curl}};\Omega) \ : \ \boldsymbol{\operatorname{curl}} \, \boldsymbol{\operatorname{curl}} \, \mathbf{U} \in \boldsymbol{L}^2(\Omega) \right\}, \end{split}$$

together with the inner product on  $H(\mathbf{curl}; \Omega)$ :

$$(\mathbf{U}, \mathbf{V})_{H(\mathbf{curl}:\Omega)} := (\mathbf{U}, \mathbf{V})_{0,\Omega} + (\mathbf{curl}\,\mathbf{U}, \mathbf{curl}\,\mathbf{V})_{0,\Omega}$$

so that  $H(\mathbf{curl}; \Omega)$  is Hilbert [30, Sec. 3.5.3]. For s > 0, let us define the scale of smooth spaces [30, Sec. 3.5.3] useful to characterize the regularity of Maxwell solutions:

$$H^s(\mathbf{curl};\Omega) := \{ \mathbf{U} \in H^s(\Omega) : \mathbf{curl} \, \mathbf{U} \in H^s(\Omega) \},$$

with norm and semi-norm given by

$$\|\mathbf{U}\|_{\boldsymbol{H}^{s}(\mathbf{curl};\Omega)} := \left(\|\mathbf{curl}\,\mathbf{U}\|_{s,\Omega}^{2} + \|\mathbf{U}\|_{s,\Omega}^{2}\right)^{\frac{1}{2}}, \quad |\mathbf{U}|_{\boldsymbol{H}^{s}(\mathbf{curl};\Omega)} := \left(|\mathbf{curl}\,\mathbf{U}|_{s,\Omega}^{2} + |\mathbf{U}|_{s,\Omega}^{2}\right)^{\frac{1}{2}}.$$

We also require appropriate trace spaces [11, 13, 30]. As in [11], we introduce two Hilbert spaces of tangential vector fields on  $\partial\Omega$  and their duals:

$$\begin{split} \boldsymbol{H}_{\parallel}^{\frac{1}{2}}(\partial\Omega) &:= \{\mathbf{n} \times (\mathbf{U} \times \mathbf{n}) \ : \ \mathbf{U} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \}, \quad \boldsymbol{H}_{\perp}^{\frac{1}{2}}(\partial\Omega) := \{\mathbf{U} \times \mathbf{n} \ : \ \mathbf{U} \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega) \}, \\ \boldsymbol{H}_{\parallel}^{-\frac{1}{2}}(\partial\Omega) &:= \left(\boldsymbol{H}_{\parallel}^{\frac{1}{2}}(\partial\Omega)\right)' \quad \text{and} \quad \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\partial\Omega) := \left(\boldsymbol{H}_{\perp}^{\frac{1}{2}}(\partial\Omega)\right)'. \end{split}$$

where **n** is the outward unit normal vector on  $\partial\Omega$ . Trace spaces on  $\boldsymbol{H}(\boldsymbol{\operatorname{curl}};\Omega)$  are then defined through first order differential operators on  $\partial\Omega$ :

$$\begin{aligned} \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial\Omega) &:= \{ \mathbf{U} \in \boldsymbol{H}_{\parallel}^{-\frac{1}{2}}(\partial\Omega) : \ \mathrm{div}_{\partial\Omega} \, \mathbf{U} \in H^{-\frac{1}{2}}(\partial\Omega) \}, \\ \boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial\Omega) &:= \{ \mathbf{U} \in \boldsymbol{H}_{\perp}^{-\frac{1}{2}}(\partial\Omega) : \ \mathrm{curl}_{\partial\Omega} \, \mathbf{U} \in H^{-\frac{1}{2}}(\partial\Omega) \}, \end{aligned}$$

where  $\operatorname{div}_{\partial\Omega}$  and  $\operatorname{curl}_{\partial\Omega}$  are the divergence and scalar curl surface operators, respectively (*cf.* [12] and [33, Sec. 2.5.6] for detailed definitions). Moreover, it holds that (*cf.* [12, Thm. 2])

$$\boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial\Omega) = \left(\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial\Omega)\right)'$$
.

We define the following trace operators

$$\begin{split} \gamma_{\mathrm{D}}: \boldsymbol{H}(\boldsymbol{\mathrm{curl}};\Omega) \!\to\! \boldsymbol{H}^{-\frac{1}{2}}(\partial\Omega), \quad \gamma_{\mathrm{D}}^{\times}: \boldsymbol{H}(\boldsymbol{\mathrm{curl}};\Omega) \!\to\! \boldsymbol{H}^{-\frac{1}{2}}(\partial\Omega), \\ \gamma_{\mathrm{N}}: \boldsymbol{H}(\boldsymbol{\mathrm{curl}}\,\boldsymbol{\mathrm{curl}};\Omega) \!\to\! \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial\Omega), \end{split}$$

as the unique continuous extensions of their actions on  $\mathbf{U} \in \mathcal{C}^{\infty}(\overline{\Omega})$  given by

$$\gamma_D \mathbf{U} := \mathbf{n} \times (\left. \mathbf{U} \right|_{\partial\Omega} \times \mathbf{n}), \quad \gamma_D^\times \mathbf{U} := \mathbf{n} \times \left. \mathbf{U} \right|_{\partial\Omega} \quad \text{and} \quad \gamma_N \mathbf{U} := \mathbf{n} \times \left. \mathbf{curl} \left. \mathbf{U} \right|_{\partial\Omega},$$

dubbed the Dirichlet, flipped Dirichlet trace and Neumann traces, respectively. Range spaces are characterized as

$$\operatorname{Im}(\gamma_{\mathrm{D}}) = \boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial\Omega), \qquad \operatorname{Im}(\gamma_{\mathrm{D}}^{\times}) = \boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial\Omega).$$

Moreover, for U and  $\mathbf{V} \in H(\mathbf{curl}, \Omega)$ , the following Green identity holds

$$(\mathbf{U}, \mathbf{curl} \, \mathbf{V})_{\Omega} - (\mathbf{curl} \, \mathbf{U}, \mathbf{V})_{\Omega} = -\langle \gamma_{\mathrm{D}}^{\times} \mathbf{U}, \gamma_{\mathrm{D}} \mathbf{V} \rangle_{\partial \Omega},$$

where  $\langle \cdot, \cdot \rangle_{\partial\Omega}$  denotes the duality between  $\boldsymbol{H}_{\mathrm{div}}^{-\frac{1}{2}}(\partial\Omega)$  and  $\boldsymbol{H}_{\mathrm{curl}}^{-\frac{1}{2}}(\partial\Omega)$  (cf. [30, Sec. 3] and [11]). The subset of  $\boldsymbol{H}(\mathbf{curl};\Omega)$ -elements satisfying zero boundary conditions is defined through the flipped Dirichlet trace as

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) := \{ \mathbf{U} \in \boldsymbol{H}(\mathbf{curl};\Omega) : \gamma_{\mathrm{D}}^{\times} \mathbf{U} = \mathbf{0} \text{ on } \partial\Omega \}.$$

By continuity of the flipped Dirichlet trace,  $H_0(\mathbf{curl};\Omega)$  is a closed subspace of  $H(\mathbf{curl};\Omega)$ .

#### 2.3 Maxwell variational problems

Let  $D \subset \mathbb{R}^3$  be an open, bounded domain with boundary  $\Gamma := \partial D$  of class  $\mathcal{C}^{\mathfrak{M}}$  for some  $\mathfrak{M} \in \mathbb{N}$ . For a circular frequency  $\omega > 0$  and time dependency  $e^{i\omega t}$ , the time-harmonic Maxwell equations on D read

$$\mathbf{curl} \mathbf{E} + \imath \omega \mu \mathbf{H} = \mathbf{0},$$

$$\imath \omega \varepsilon \mathbf{E} - \mathbf{curl} \mathbf{H} = -\mathbf{J}.$$
(2.1)

where **E** and **H** belong to  $H(\mathbf{curl}; \mathbf{D})$  and represent the electric and magnetic fields, respectively. The magnetic permeability  $\mu$  and electric permittivity  $\varepsilon$  are assumed to be symmetric matrix-valued functions with coefficients in  $L^{\infty}(\mathbf{D})$ , and  $\mathbf{J} \in L^{2}(\mathbf{D})$  is an imposed current in  $\mathbf{D}$ .

The system eq. (2.1) is converted into a second order system for **E** or **H** by eliminating the remaining field, requiring pointwise invertibility assumptions on either  $\varepsilon$  or  $\mu$  depending on the specific choice. Without loss of generality, we follow [4] and consider the system for the electric field only, assuming the existence of a pointwise inverse of  $\mu$ . Thus,

$$\mathbf{H} = i \frac{1}{\omega} \mu^{-1} \mathbf{curl} \mathbf{E},$$

from where

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E} - \omega^2 \varepsilon \mathbf{E} = -\imath \omega \mathbf{J}.\tag{2.2}$$

The system is completed by imposing boundary conditions on traces of E, e.g.,

$$\gamma_{\mathrm{D}}^{\times} \mathbf{E} = \mathbf{g}_{D}, \quad \text{or} \quad \mathbf{n} \times (\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{E}) = \mathbf{g}_{N},$$

for 
$$\boldsymbol{g}_D \in \boldsymbol{H}_{\operatorname{div}}^{-\frac{1}{2}}(\partial\Omega)$$
 or  $\boldsymbol{g}_N \in \boldsymbol{H}_{\operatorname{\mathbf{curl}}}^{-\frac{1}{2}}(\partial\Omega)$ .

We proceed by considering the system eq. (2.2) with perfect electric conductor (PEC) boundary conditions, i.e., homogeneous (flipped) Dirichlet boundary conditions given by  $\gamma_{\rm D}^{\times} \mathbf{E} = \mathbf{0}$ . The associated sesquilinear and antilinear forms on  $H_0(\mathbf{curl}; \mathbf{D})$  for the Maxwell PEC cavity problem, respectively, are

$$\Phi(\mathbf{U}, \mathbf{V}) := \int_{D} \mu^{-1} \operatorname{\mathbf{curl}} \mathbf{U} \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{V}} - \omega^{2} \epsilon \mathbf{U} \cdot \overline{\mathbf{V}} \, \mathrm{d} \mathbf{x} \quad \text{and} \quad \mathbf{F}(\mathbf{V}) := -\imath \omega \int_{D} \mathbf{J} \cdot \overline{\mathbf{V}} \, \mathrm{d} \mathbf{x}, \qquad (2.3)$$

which are continuous on  $\mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  if we assume  $\mu^{-1}$  has coefficients in  $L^{\infty}(\mathbf{D})$ . Then, the problem under consideration reads:

**Problem 2.1** (Continuous variational problem). Find  $\mathbf{E} \in H_0(\mathbf{curl}; \mathbf{D})$  such that

$$\Phi(\mathbf{E}, \mathbf{V}) = \mathbf{F}(\mathbf{V}),$$

for all  $\mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$ .

In this work, we are concerned with the approximation of D, the *original domain*, by computational domains  $\{\widetilde{D}_i\}_{i\in\mathbb{N}}$  and its consequences on the FEM error convergence rates. Hence, we take for granted the necessary conditions for the unique solvability of Problem 2.1.

**Assumption 2.2** (Wellposedness). We assume the sesquilinear form  $\Phi$  in (2.3) satisfies the following conditions:

$$\begin{aligned} |\Phi(\mathbf{U}, \mathbf{V})| &< C_1 \|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}; D)} \|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; D)} & \forall \mathbf{U}, \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; D), \\ & \sup_{\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; D) \setminus \{\mathbf{0}\}} |\Phi(\mathbf{U}, \mathbf{V})| > 0 & \forall \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; D) \setminus \{\mathbf{0}\}, \end{aligned}$$

and

$$\inf_{\mathbf{U}\in \boldsymbol{H}_0(\mathbf{curl}; \mathbf{D})\backslash \{\boldsymbol{0}\}} \left(\sup_{\mathbf{V}\in \boldsymbol{H}_0(\mathbf{curl}; \mathbf{D})\backslash \{\boldsymbol{0}\}} \frac{|\Phi(\mathbf{U}, \mathbf{V})|}{\|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})}\|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})}}\right) \geq C_2,$$

for positive constants  $C_1$  and  $C_2$ .

We denote the unique solution of Problem 2.1 as  $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$ . Lastly, for examples of problems satisfying Assumption 2.2 we refer to [5, 21, 29, 30].

## 3 Curl-conforming finite elements

We begin by introducing the reference tetrahedron from which all meshes will be constructed.

**Definition 3.1** (Reference element). We define  $\check{K}$  as the tetrahedron with vertices  $\mathbf{0}$ ,  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  and  $\mathbf{e}_3$ , and refer to it as the reference element or reference tetrahedron.

We also recall our smoothness assumptions on our original domain—stated in section 1—, requiring D to be of class  $\mathcal{C}^{\mathfrak{M}}$  for  $\mathfrak{M} \in \mathbb{N}$ .

**Assumption 3.2.** The bounded domain D is of class  $\mathcal{C}^{\mathfrak{M}}$  for  $\mathfrak{M} \in \mathbb{N}$ .

Assumption 3.2 is required to ensure convergence rates of approximate domains to D built by polynomial interpolation [28]. We point out that one could easily adjust the following analysis to piecewise smooth domains.

#### 3.1 Curl-conforming finite element spaces on straight and curved meshes

As in [4], we introduce<sup>1</sup>  $\mathfrak{T}^{\mathsf{p}}$  a family of quasi-uniform straight meshes of D, written  $\tau_{h_i}^{\mathsf{p}}$ , with  $h_i > 0$  for all  $i \in \mathbb{N}$   $h_i \to 0$  as i grows to infinity, constructed by straight tetrahedrons and indexed by their mesh-sizes, i.e.,  $\mathfrak{T}^{\mathsf{p}} := \{\tau_{h_i}^{\mathsf{p}}\}_{i \in \mathbb{N}}$ . Throughout,  $\tau_h^{\mathsf{p}}$  denotes an arbitrary mesh in  $\mathfrak{T}^{\mathsf{p}}$ . An arbitrary tetrahedron in any of the meshes of  $\mathfrak{T}^{\mathsf{p}}$  is denoted  $K^{\mathsf{p}}$ , and we assume each tetrahedron  $K^{\mathsf{p}}$  to be constructed from K by an affine mapping, denoted  $K^{\mathsf{p}} : K \to K^{\mathsf{p}}$ . The polyhedral domain covered by  $\tau_h^{\mathsf{p}}$  is denoted  $K^{\mathsf{p}}$  with boundary  $\Gamma_h^{\mathsf{p}} := \partial D_h^{\mathsf{p}}$ .

domain covered by  $\tau_h^{\mathsf{p}}$  is denoted  $\mathsf{D}_h^{\mathsf{p}}$  with boundary  $\Gamma_h^{\mathsf{p}} := \partial \mathsf{D}_h^{\mathsf{p}}$ .

Now, for each polyhedral mesh  $\tau_h^{\mathsf{p}} \in \mathfrak{T}^{\mathsf{p}}$ , we introduce  $\tau_h$  as the approximated curved mesh constructed from  $\tau_h^{\mathsf{p}}$ , in the sense that it shares its nodes with  $\tau_h^{\mathsf{p}}$  but is composed of curved tetrahedrons. As before, we introduce the family of curved meshes as  $\mathfrak{T} := \{\tau_{h_i}\}_{i \in \mathbb{N}}$ .

For a given  $K^{\mathsf{p}} \in \tau_h^{\mathsf{p}}$  we refer to the element of  $\tau_h$  that shares its nodes with  $K^{\mathsf{p}}$  as K and consider bijective mappings  $T_K : \check{K} \mapsto K$  to be polynomial of degree  $\mathfrak{K} \in \mathbb{N}$ , with  $\mathfrak{K} < \mathfrak{M}$  and fixed throughout. Also, we refer to an arbitrary mesh in  $\mathfrak{T}$  by  $\tau_h$  and the domain covered by  $\tau_h$  by  $D_h$  with boundary  $\Gamma_h := \partial D_h$ .

**Assumption 3.3** (Assumptions on  $\mathfrak{T}^p$  and  $\mathfrak{T}$ .). The meshes in  $\mathfrak{T}^p$  are assumed to be affine, quasiuniform and such that their boundary nodes are located on  $\Gamma$  and the polyhedral domains  $\{D_{h_i}^p\}_{i\in\mathbb{N}}$ approximate D. The family of approximate meshes  $\mathfrak{T}$  is assumed to be  $\mathfrak{K}$ -regular, i.e., for each  $K \in \tau_h$ , the mappings  $T_K$  are  $C^{\mathfrak{K}+1}$ -diffeomorphisms that belong to  $\mathbb{P}_{\mathfrak{K}}(\check{K};\mathbb{R}^3)$  for some integer  $\mathfrak{K} < \mathfrak{M}$ , with  $\mathfrak{M}$  as in Assumption 3.2. Moreover, they satisfy

$$\sup_{\mathbf{x}\in\check{K}}\|\mathbf{d}^{n}\mathbf{T}_{K}(\mathbf{x})\| \leq C_{n}h^{n} \quad and \quad \sup_{\mathbf{x}\in\check{K}}\|\mathbf{d}^{n}\left(\mathbf{T}_{K}^{-1}\right)(\mathbf{x})\| \leq C_{-n}h^{-n} \quad \forall n\in\{1:\mathfrak{K}+1\}, \tag{3.1}$$

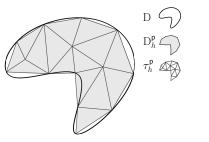
where  $C_n$  and  $C_{-n}$  are positive constants independent from the mesh-size for all  $n \in \{1 : \Re + 1\}$ . Therein,  $d^n T_K$  is the Fréchet derivative of order n of  $T_K$  and  $\|d^n \widetilde{T}_K(\mathbf{x})\|$  is the induced norm, with functional spaces omitted for brevity, and the curved domains  $\{D_{h_i}\}_{i\in\mathbb{N}}$  approximate D. Furthermore, we assume that  $\det(dT_K(\mathbf{x})) > 0$  for all  $\mathbf{x} \in \check{K}$  and that there exists some positive  $\theta \in \mathbb{R}$ , independent of h > 0, such that for all  $K \in \tau_h$ , it holds that

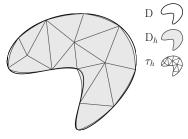
$$\frac{1}{\theta} \le \frac{\det(\mathrm{d} T_K(\mathbf{x}))}{\det(\mathrm{d} T_K(\mathbf{y}))} \le \theta \quad \forall \ \mathbf{x}, \mathbf{y} \in \breve{K}.$$

Our assumptions on  $\mathfrak{T}^{\mathsf{p}}$  follow from [15] and are satisfied by constructions of curved meshes by polynomial approximations of the domain D (*cf.* [28]). Figure 1 displays a 2D example of our setting.

The sense in which we assume the approximate domains to converge to D will be made clear in the following section. Note that we have limited the polynomial degree of our approximate domains

<sup>&</sup>lt;sup>1</sup>The superindex **p** stands for polyhedral.





(a) Polygonal mesh of D.

(b) Curved mesh of D.

Figure 1: Two-dimensional example of a smooth original domain D, with associated straight (polygonal) and curved meshes,  $\tau_h^{\mathsf{p}} \in \mathfrak{T}_h^{\mathsf{p}}$  of D (fig. 1a) and  $\tau_h \in \mathfrak{T}$  of D (fig. 1b), together with respective approximated domains  $D_h^{\mathsf{p}}$  and  $D_h$ . Note that  $\tau_h^{\mathsf{p}}$  and  $\tau_h$  share same nodes and that curved edges do not necessarily match the boundary of D.

by the smoothness of D (by imposing  $\mathfrak{K} < \mathfrak{M}$  where D is of class  $\mathcal{C}^{\mathfrak{M}}$ ) since no gain is derived from additional orders of approximation. Moreover, for a multi-index  $\alpha \in \mathbb{N}_0^3$ , Assumption 3.3 implies the following estimates:

$$\sup_{\mathbf{x} \in \check{K}} \left| \frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}} T_{K,i}(\mathbf{x}) \right| \le C h^{|\alpha|} \quad \text{and} \quad \sup_{\mathbf{x} \in K} \left| \frac{\partial^{\alpha}}{\partial \mathbf{x}^{\alpha}} T_{K,i}^{-1}(\mathbf{x}) \right| \le C h^{-|\alpha|} \qquad \forall i \in \{1:3\}, \tag{3.2}$$

$$ch^3 \le |\det(\mathrm{d}T_K(\mathbf{x}))| \le Ch^3 \quad \forall \ \mathbf{x} \in \check{K},$$
 (3.3)

where c and C are positive generic constants—not necessarily equal in each appearance—independent of K and the mesh-size. The estimates in (3.2) follow from norm equivalence over finite-dimensional spaces and (3.1), while (3.3) follows from (3.2) by straightforward computation (cf. Lemma 8 in [4]).

With the above definitions, we consider finite elements as triples  $(K, P_K, \Sigma_K)$ , with  $K \in \tau_h$ ,  $P_K$  a space of polynomials over K and  $\Sigma_K := \{\sigma_i^K\}_{i=1}^{n_{\Sigma}}, n_{\Sigma} \in \mathbb{N}$ , a set of linear functionals acting on  $P_K$  (cf. [30]). Let  $k \in \mathbb{N}$  refer to the polynomial degree of the curl-conforming (Nédélec) finite element space on the reference tetrahedron  $\check{K}$  defined as

$$\mathbf{P}_{\check{K}}^{c} := \mathbb{P}_{k-1}(\check{K}; \mathbb{C}^{3}) \oplus \{ \mathbf{p} \in \widetilde{\mathbb{P}}_{k}(\check{K}, \mathbb{C}^{3}) : \mathbf{x} \cdot \mathbf{p}(\mathbf{x}) = 0 \}.$$
(3.4)

Finite element spaces on arbitrary tetrahedrons K (straight or curved) are defined via a curl-conforming pull-back as follows

$$\mathbf{P}_{K}^{c} := \{ \mathbf{p} : \psi_{K}^{c}(\mathbf{p}) \in \mathbf{P}_{K}^{c} \} \quad \text{where} \quad \psi_{K}^{c}(\mathbf{V}) := d\mathbf{T}_{K}^{\top}(\mathbf{V} \circ \mathbf{T}_{K}). \tag{3.5}$$

The pull-back in eq. (3.5) defines an isomorphism between  $\boldsymbol{H}(\boldsymbol{\text{curl}};K)$  and  $\boldsymbol{H}(\boldsymbol{\text{curl}};\check{K})$  and satisfies (cf. [25, Lem. 2.2] and [30])

$$\operatorname{\mathbf{curl}} \psi_K^c(\mathbf{V}) = \mathrm{d} T_K^{\mathsf{co}} \operatorname{\mathbf{curl}} \mathbf{V} \circ T_K.$$

We refer to [30, Sec. 5.5] for the definition of degrees of freedom for the reference finite element space in eq. (3.4). The curl-conforming discrete spaces on  $\tau_h \in \mathfrak{T}$  are then constructed as

$$\mathbf{P}^{c}(\tau_{h}) := \left\{ \mathbf{V}_{h} \in \mathbf{H}_{0}(\mathbf{curl}; \mathbf{D}_{h}) : \mathbf{V}_{h}|_{K} \in \mathbf{P}_{K}^{c} \quad \forall K \in \tau_{h} \right\}, \tag{3.6}$$

where  $D_h$  is the domain covered by  $\tau_h$ .

#### 3.2 Curl-conforming interpolation on curved meshes

We now focus on proving continuity and approximation properties for the classical curl-conforming interpolation operator on curved meshes. We shall denote the reference curl-conforming finite

element by  $(\check{K}, \boldsymbol{P}^c_{\check{K}}, \Sigma^c_{\check{K}})$  and let us introduce  $\{\phi_\sigma\}_{\sigma \in \Sigma^c_{\check{K}}}$  as the basis of  $\boldsymbol{P}^c_{\check{K}}$  associated with the degrees of freedom  $\Sigma^c_{\check{K}}$  so that for any pair of degrees of freedom  $\sigma$ ,  $\sigma' \in \Sigma^c_{\check{K}}$ , it holds that

$$\sigma(\phi_{\sigma'}) = \begin{cases} 1, & \text{if } \sigma = \sigma', \\ 0, & \text{if } \sigma \neq \sigma'. \end{cases}$$

For further details, we refer to [30, Sec. 5.5].

**Definition 3.4** (Local interpolation operator). Let  $s \in \mathbb{N}$ . We define the canonical interpolation operator on  $\check{K}$ 

$$m{reve{r}}:m{H}^s(\mathbf{curl};reve{K}) om{P}^c_{reve{K}}$$

as the operator mapping  $U \in H^s(\mathbf{curl}; \check{K})$  to the unique element in  $P^c_{\check{K}}$  having the same degrees of freedom as U, i.e.,

$$reve{r}(\mathbf{U}) := \sum_{\sigma \in \Sigma_{reve{K}}^c} \sigma(\mathbf{U}) oldsymbol{\phi}_{\sigma}.$$

For any  $\tau_h \in \mathfrak{T}$  and any  $K \in \tau_h$  we denote the canonical interpolation operator on K as  $\mathbf{r}_K$ , mapping  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; K)$  to  $\mathbf{P}_K^c$  as follows

$$m{r}_K: egin{cases} m{H}^s(\mathbf{curl};K) 
ightarrow m{P}^c_K, \ \mathbf{U} \mapsto (\psi^c_K)^{-1}(m{r}(\psi^c_K(\mathbf{U}))). \end{cases}$$

**Assumption 3.5.** From here onwards, we assume that  $k \leq \Re$ .

The next results are proven in appendix B.

**Proposition 3.6.** Let Assumptions 3.3 and 3.5 hold. For  $K \in \tau_h$  and  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; K)$  for some  $s \in \{1 : k\}$ , one has that

$$\|\mathbf{U} - \mathbf{r}_K(\mathbf{U})\|_{\mathbf{H}(\mathbf{curl};K)} \le Ch^s \|\mathbf{U}\|_{\mathbf{H}^s(\mathbf{curl};K)},$$

where C > 0 is independent of K, U and  $\tau_h \in \mathfrak{T}$ .

**Proposition 3.7.** Let Assumptions 3.3 and 3.5 hold. For  $K \in \tau_h$  and  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; K)$  with  $s \in \{1 : k\}$ , one can show that

$$\|\mathbf{r}_K(\mathbf{U})\|_{\mathbf{H}^s(\mathbf{curl}:K)} \le c \|\mathbf{U}\|_{\mathbf{H}^s(\mathbf{curl}:K)},$$

where c > 0 is independent of K, U and  $\tau_h \in \mathfrak{T}$ .

**Definition 3.8** (Global interpolation operator). Let  $s \in \mathbb{N}$ . For all  $\tau_h \in \mathfrak{T}$  define the canonical interpolation operator on the entire mesh  $\tau_h$  as

$$\Pi_h: \mathbf{H}^s(\mathbf{curl}; \mathcal{D}_h) \to \mathbf{P}^c(\tau_h),$$

i.e., the operator mapping  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; D_h)$  to the unique element in  $\mathbf{P}^c(\tau_h)$  having the same degrees of freedom as  $\mathbf{U}$  element by element:

$$\Pi_h(\mathbf{U})|_K = \mathbf{r}_K(\mathbf{U}) \quad \forall K \in \tau_h.$$

The global interpolation operator follows [30, Sec. 5.5] and enjoys results analogous to propositions 3.6 and 3.7, which we omit since we will require more specialized versions later on.

## 4 Variational problems on approximate domains: continuous problem

We now focus on the solution of Problem 2.1 on a countable family of domains  $\mathfrak{D} := \{\widetilde{D}_i\}_{i \in \mathbb{N}}$  that approximate the original domain D. Specifically, we are interested in computing the rate of convergence of solutions on each domain  $\widetilde{D} \in \mathfrak{D}$  to  $\mathbf{E}$  in D. Rather than immediately considering the approximate domains defined by meshes in  $\mathfrak{T}$ , we study the problem in a more general setting, so as to derive conditions on  $\mathfrak{D}$  transferable to our meshes in  $\mathfrak{T}$ . We will return to our original discrete problem—identifying  $\widetilde{D}_i$  with  $D_{h_i}$ —in Section 5. Moreover, since we are to consider Problem 2.1 for domains in  $\mathfrak{D}$ , which need not be contained in D, we require the data  $\mu$ ,  $\varepsilon$  and  $\mathbf{J}$  of Problem 2.1 to have extensions to a hold-all domain, denoted  $D_H$ , containing D and each domain in  $\mathfrak{D}$ .

Assumption 4.1 (Extension of parameters). There exists an open and bounded Lipschitz domain  $D_H$ , referred to as the hold-all domain, such that  $\overline{D} \subset D_H$  and  $\overline{\widetilde{D}} \subset D_H$  for all  $\widetilde{D} \in \mathfrak{D}$ . Both  $\mu$  and  $\epsilon$  are complex symmetric matrix-valued functions with coefficients in  $L^{\infty}(D_H)$  and  $\mu$  has a pointwise inverse  $(\mu^{-1})$  almost everywhere on  $D_H$ , with coefficients in  $L^{\infty}(D_H)$  as well. The imposed current J may be extended to  $D_H$  so that F in (2.3) may be extended to  $H_0(\operatorname{\mathbf{curl}}; D_H)'$ .

We consider  $H_0(\mathbf{curl}; D)$  and  $H_0(\mathbf{curl}; \widetilde{D})$  to be closed subspaces of  $H(\mathbf{curl}; D_H)$  by identifying elements in  $H_0(\mathbf{curl}; D)$  and  $H_0(\mathbf{curl}; \widetilde{D})$  with their extension by  $\mathbf{0}$  to  $D_H$ . We may then continuously extend the sesquilinear form in eq. (2.3) as follows:

$$\Phi(\mathbf{U}, \mathbf{V}) := \int_{\mathbf{D}_H} \mu^{-1} \operatorname{\mathbf{curl}} \mathbf{U} \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{V}} - \omega^2 \epsilon \mathbf{U} \cdot \overline{\mathbf{V}} \, \mathrm{d} \mathbf{x} \qquad \forall \, \mathbf{U}, \mathbf{V} \in \mathbf{H}_0(\operatorname{\mathbf{curl}}; \mathbf{D}_H)$$
(4.1)

while the right-hand side  $\mathbf{F}$  in eq. (2.3) is extended to  $\mathbf{H}_0(\mathbf{curl}; D_H)'$  by Assumption 4.1, e.g., by taking an  $\mathbf{L}^2(D_H)$ -extension of  $\mathbf{J}$ .

**Problem 4.2** (Continuous variational problem on inexact domains). Find  $\widetilde{\mathbf{E}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  such that

$$\Phi(\widetilde{\mathbf{E}}, \mathbf{V}) = \mathbf{F}(\mathbf{V}),$$

for all  $\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$ .

As before, we assume Problem 4.2 is well posed on each  $\widetilde{D} \in \mathfrak{D}$ , with uniform constants.

**Assumption 4.3** (Wellposedness on  $\mathfrak{D}$ ). We assume the sesquilinear form in eq. (4.1) to satisfy the following conditions:

$$|\Phi(\mathbf{U}, \mathbf{V})| < C_1 \|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}})} \|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}})} \quad \forall \ \mathbf{U}, \ \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}),$$

$$\sup_{\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}) \setminus \{\mathbf{0}\}} |\Phi(\mathbf{U}, \mathbf{V})| > 0 \quad \forall \ \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}) \setminus \{\mathbf{0}\},$$

and

$$\inf_{\mathbf{U}\in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}})\backslash\{\mathbf{0}\}} \left(\sup_{\mathbf{V}\in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}})\backslash\{\mathbf{0}\}} \frac{|\Phi(\mathbf{U},\mathbf{V})|}{\|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}})}\|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}})}}\right) \geq C_2,$$

for all  $\widetilde{D} \in \mathfrak{D}$ , with positive constants  $C_1$  and  $C_2$  independent of  $\widetilde{D} \in \mathfrak{D}$ .

**Example 4.4.** Taking  $\mu^{-1}$  and  $\epsilon$  in  $L^{\infty}(D_H; \mathbb{C}^{3\times 3})$  and such that

$$\inf_{\mathbf{x}\in D_H} \operatorname{Re}\left(e^{i\theta}\mu(\mathbf{x})^{-1}\right), \ \inf_{\mathbf{x}\in D_H} \operatorname{Re}\left(-e^{i\theta}\epsilon(\mathbf{x})\right) \geq \alpha > 0$$

for some  $\theta \in [0, 2\pi)$  and  $\alpha > 0$  is enough to ensure the conditions in Assumption 4.3 (cf. [5, 21]).

For general  $\widetilde{\mathbf{D}} \in \mathfrak{D}$ , we denote the unique solution of Problem 4.2 as  $\widetilde{\mathbf{E}} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$ , respectively,  $\widetilde{\mathbf{E}}_i \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}_i)$  for each  $i \in \mathbb{N}$ .

#### 4.1 On the convergence of domains

We introduce several different notions of convergence of a sequence of domains to a limit, so that our conditions on the sequence  $\mathfrak{D}$  are clearly defined.

**Definition 4.5** (Mosco convergence). We say  $\mathfrak{D}$  approximates D in the sense of Mosco if the following conditions hold:

- (a) For every  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  there exists a sequence  $\{\mathbf{U}_i\}_{i \in \mathbb{N}}$ , with  $\mathbf{U}_i \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}_i)$  for all  $i \in \mathbb{N}$ , such that  $\mathbf{U}_i$  converges to  $\mathbf{U}$  strongly in  $\mathbf{H}(\mathbf{curl}; \mathbf{D}_H)$ .
- (b) Weak limits in  $H_0(\mathbf{curl}; D_H)$  of every sequence  $\{\mathbf{U}_i\}_{i \in \mathbb{N}}$  satisfying  $\mathbf{U}_i \in H_0(\mathbf{curl}; \widetilde{D}_i)$  for all  $i \in \mathbb{N}$ , belong to  $H_0(\mathbf{curl}; D)$ .

Note that we have identified each  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  with its extension by zero to  $\mathbf{H}_0(\mathbf{curl}; \mathbf{D}_H)$ .

The notion of Mosco convergence originated in the study of variational inequalities [31, 32] with applications to partial differential equations found in [14, 17, 34]. The original definition of Mosco convergence corresponds to the convergence of the spaces  $\mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}_i)$  to  $\mathbf{H}(\mathbf{curl}; \mathbf{D})$  rather than to the convergence of the domains  $\widetilde{\mathbf{D}}_i$  to  $\mathbf{D}$ , but we choose the latter convention since both are equivalent in our context.

**Lemma 4.6.** Let Assumptions 2.2, 4.1, and 4.3 hold and let  $\mathbf{E}$  and  $\widetilde{\mathbf{E}}_i$  denote the unique solutions of Problems 2.1 and 4.2, respectively. Assume  $\mathfrak{D}$  approximates  $\mathbf{D}$  in the sense of Mosco. Then,  $\{\widetilde{\mathbf{E}}_i\}_{i\in\mathbb{N}}$  converges to  $\mathbf{E}$  in  $\mathbf{H}(\mathbf{curl}; D_H)$ .

*Proof.* Let  $\{\mathbf{E}_i\}_{i\in\mathbb{N}}$  be a sequence as in item (a) in definition 4.5, strongly converging to  $\mathbf{E}$  in  $\mathbf{H}(\mathbf{curl}; \mathbf{D}_H)$ . Then, by Assumption 4.3, for each  $i \in \mathbb{N}$  there exists  $\mathbf{V}_i \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}_i)$ , with  $\|\mathbf{V}_i\|_{\mathbf{H}(\mathbf{curl}; \widetilde{\mathbf{D}}_H)} = 1$ , such that

$$\frac{C_2}{2} \|\widetilde{\mathbf{E}}_i - \mathbf{E}_i\|_{\mathbf{H}(\mathbf{curl}; \mathbf{D}_H)} \le \left| \Phi(\widetilde{\mathbf{E}}_i - \mathbf{E}_i, \mathbf{V}_i) \right| = \left| \mathbf{F}(\mathbf{V}_i) - \Phi(\mathbf{E}_i, \mathbf{V}_i) \right|, \tag{4.2}$$

where the positive constant  $C_2$  is as in Assumption 4.3. Moreover, since the sequence  $\{\mathbf{V}_i\}_{i\in\mathbb{N}}$  is bounded it has a weakly convergent subsequence—still denoted  $\{\mathbf{V}_i\}_{i\in\mathbb{N}}$ —to a limit point  $\mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  due to item (b) in definition 4.5, so that

$$\lim_{i \to \infty} \mathbf{F}(\mathbf{V}_i) = \mathbf{F}(\mathbf{V}) \quad \text{and} \quad \lim_{i \to \infty} \Phi(\mathbf{E}_i, \mathbf{V}_i) = \Phi(\mathbf{E}, \mathbf{V}),$$

and the result follows by taking the limit as i grows to infinity in (4.2).

Notice that it is not straightforward to derive convergence rates of approximate solutions  $\widetilde{\mathbf{E}}_i$  to  $\mathbf{E}$ , since we cannot estimate  $\|\mathbf{E} - \mathbf{E}_i\|_{H(\mathbf{curl}; \mathbf{D}_H)}$  in lemma 4.6 without further assumptions on  $\mathfrak{D}$ . However, the notion of Mosco convergence gives minimum conditions to ensure strong convergence of the approximate solutions.<sup>2</sup>

**Definition 4.7** (Hausdorff convergence). We say  $\mathfrak D$  approximates D in the sense of Hausdorff if

$$\lim_{i \to \infty} d_{\mathcal{H}}(\overline{\mathbf{D}}_H \setminus \widetilde{\mathbf{D}}_i, \overline{\mathbf{D}}_H \setminus \mathbf{D}) = 0,$$

where  $d_{\mathcal{H}}(\cdot,\cdot)$  denotes the Hausdorff metric between closed subsets of  $\mathbb{R}^3$ , defined as

$$d_{\mathcal{H}}(\Omega_1, \Omega_2) := \max \left\{ \sup_{\mathbf{x} \in \Omega_1} \operatorname{dist}(\mathbf{x}, \Omega_2), \sup_{\mathbf{y} \in \Omega_2} \operatorname{dist}(\mathbf{y}, \Omega_1) \right\},\,$$

for two closed subsets  $\Omega_1$  and  $\Omega_2$  of  $\mathbb{R}^3$ .

<sup>&</sup>lt;sup>2</sup>The conditions in definition 4.5 are further studied in [17] and Lemma 2.7 in [14], for example.

**Lemma 4.8** (Lemmas 3 and 4 in [34]). Suppose  $\mathfrak{D}$  approximates D in the sense of Hausdorff and that D and all  $\widetilde{D} \in \mathfrak{D}$  are Lipschitz continuous domains with uniform Lipschitz constant. Then,  $\mathfrak{D}$  approximates D in the sense of Mosco.

lemma 4.8 shows that uniform point-wise approximation—together with mild assumptions on the regularity of D and  $\mathfrak{D}$ —implies Mosco convergence, and so it provides us with sufficient—geometric—conditions that ensure the strong convergence of  $\{\widetilde{\mathbf{E}}_i\}_{i\in\mathbb{N}}$  to  $\mathbf{E}$ . Still, the notion is too weak for us to compute meaningful estimates as it gives almost no information on the domains  $\widetilde{\mathbf{D}}$  in  $\mathfrak{D}$ . Instead of the previous definitions of convergence of domains, we shall consider the following (stronger) notion, which appears in [18, 37] in the context of shape optimization.

**Definition 4.9** (Convergence in the sense of transformations). We say  $\mathfrak{D}$  approximates D in the sense of transformations of order  $\mathfrak{n} \in \mathbb{N}_0$  if there exist bijective transformations  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$  such that:

$$\mathbf{T}_{i}: \mathbf{D}_{H} \to \mathbf{D}_{H}, \quad \mathbf{T}_{i}|_{\widetilde{\mathbf{D}}_{i}}: \widetilde{\mathbf{D}}_{i} \to \mathbf{D}, \quad \mathbf{T}_{i}, \mathbf{T}_{i}^{-1} \in \boldsymbol{W}^{\mathfrak{n}, \infty}(\mathbf{D}_{H})$$

$$\lim_{i \to \infty} \|\mathbf{T}_{i} - \mathbf{I}\|_{\boldsymbol{W}^{\mathfrak{n}, \infty}(\mathbf{D}_{H})} + \|\mathbf{T}_{i}^{-1} - \mathbf{I}\|_{\boldsymbol{W}^{\mathfrak{n}, \infty}(\mathbf{D}_{H})} = 0.$$

For any transformation  $\mathbf{T}$  satisfying the previous conditions for a domain  $\widetilde{D} \in \mathfrak{D}$ , we denote the associated discrepancy between D and  $\widetilde{D}$ , subject to the transformation  $\mathbf{T}$ , as

$$d_{\mathfrak{n}}(\mathrm{D}_{\mathit{H}},\mathbf{T}):=\|\mathbf{T}-\mathsf{I}\|_{\boldsymbol{W}^{\mathfrak{n},\infty}(\mathrm{D}_{\mathit{H}})}+\|\mathbf{T}^{-1}-\mathsf{I}\|_{\boldsymbol{W}^{\mathfrak{n},\infty}(\mathrm{D}_{\mathit{H}})}.$$

It is straightforward to see that convergence in the sense of transformations of order zero implies Hausdorff convergence and that convergence of order one implies, together with the Lipschitz continuity of D, the results of lemma 4.8. We continue our analysis under the following assumption.

**Assumption 4.10** (Assumptions on  $\mathfrak{D}$ ). We assume that the countable family  $\mathfrak{D}$  approximates D in the sense of transformations of order  $\mathfrak{n}=1$ . Moreover, we assume the respective family of transformations  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$  is such that

$$d_1(D_H, \mathbf{T}_i) < 1, \quad d_1(D_H, \mathbf{T}_{i+1}) < d_1(D_H, \mathbf{T}_i),$$
 (4.3)

$$\vartheta^{-1} \leq \|\det(\mathbf{d}\mathbf{T}_{i})\|_{L^{\infty}(\mathbf{D}_{H})}, \ \|\mathbf{d}\mathbf{T}_{i}\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})}, \ \|\mathbf{d}\mathbf{T}_{i}^{\mathsf{co}}\|_{L^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})}, \leq \vartheta,$$
$$\vartheta^{-1} \leq \|\det(\mathbf{d}(\mathbf{T}_{i}^{-1}))\|_{L^{\infty}(\mathbf{D}_{H})}, \ \|\mathbf{d}(\mathbf{T}_{i}^{-1})\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})}, \ \|\mathbf{d}(\mathbf{T}_{i}^{-1})^{\mathsf{co}}\|_{L^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})}, \leq \vartheta,$$

$$(4.4)$$

for some  $\vartheta > 1$  and for all  $i \in \mathbb{N}$ .

We will further assume, for simplicity, that all determinants  $\det(d\mathbf{T})$  are positive almost everywhere on  $D_H$ .

**Remark 4.11.** The conditions in eqs. (4.3) and (4.4) only restrict the quality of "bad" approximations of D, as convergence in the sense of transformations of order one implies

$$\lim_{i \to \infty} \|\det(\mathbf{d}\mathbf{T}_i)\|_{L^{\infty}(\mathbf{D}_H)}, \ \|\mathbf{d}\mathbf{T}_i\|_{\mathbf{L}^{\infty}(\mathbf{D}_H;\mathbb{C}^{3\times3})}, \ \|\mathbf{d}\mathbf{T}_i^{\mathsf{co}}\|_{L^{\infty}(\mathbf{D}_H;\mathbb{C}^{3\times3})} = 1,$$

$$\lim_{i \to \infty} \|\det(\mathbf{d}(\mathbf{T}_i^{-1}))\|_{L^{\infty}(\mathbf{D}_H)}, \ \|\mathbf{d}(\mathbf{T}_i^{-1})\|_{\mathbf{L}^{\infty}(\mathbf{D}_H;\mathbb{C}^{3\times3})}, \ \|\mathbf{d}(\mathbf{T}_i^{-1})^{\mathsf{co}}\|_{L^{\infty}(\mathbf{D}_H;\mathbb{C}^{3\times3})} = 1.$$

Moreover, by the norm equivalence over finite-dimensional spaces—and some algebra in the last case—it holds that

$$\|\mathbf{d}\mathbf{T}_{i} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H})}, \ \|\mathbf{d}\mathbf{T}_{i}^{-1} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})}, \ \|\mathbf{d}\mathbf{T}_{i}^{\mathsf{co}} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})} \leq Cd_{1}(\mathbf{D}_{H}, \mathbf{T}_{i}),$$

$$where \ C > 0 \ is \ independent \ of \ i \in \mathbb{N}.$$

$$(4.5)$$

As aforementioned, we will assess the quality of the solutions of Problem 4.2 as approximations to the solution of Problem 2.1 in two different ways:

(a) through isomorphisms  $\Psi_i : H_0(\mathbf{curl}; D) \to H_0(\mathbf{curl}; \widetilde{D}_i)$  to measure  $\|\Psi_i \mathbf{E} - \widetilde{\mathbf{E}}_i\|_{H(\mathbf{curl}; \widetilde{D}_i)}$  as i grows towards infinity—equivalently,  $\|\mathbf{E} - \Psi_i^{-1} \widetilde{\mathbf{E}}_i\|_{H(\mathbf{curl}; D)}$ —; and,

(b) through an appropriate extension to  $D_H$  of  $\mathbf{E}$ , allowing us to measure  $\|\mathbf{E} - \widetilde{\mathbf{E}}_i\|_{H(\mathbf{curl};\widetilde{D}_i)}$  as i grows towards infinity.

We now introduce a curl-conforming pull-back that will act as the mentioned isomorphism between  $H_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  and  $H_0(\mathbf{curl}; \mathbf{D})$ .

**Lemma 4.12** (Lemma 2.2 in [25]). For  $\widetilde{D} \in \mathfrak{D}$ , let  $\mathbf{T} : \widetilde{D} \to D$  be a continuous, bijective and bi-Lipschitz mapping from  $\widetilde{D}$  to D, so that  $\mathbf{T} \in \mathbf{W}^{1,\infty}(\widetilde{D})$  and  $\mathbf{T}^{-1} \in \mathbf{W}^{1,\infty}(D)$ . Then,  $\mathbf{T}$  induces an isomorphism between  $\mathbf{H}_0(\mathbf{curl}; D)$  and  $\mathbf{H}_0(\mathbf{curl}; \widetilde{D})$ , given by

$$\Psi: \left\{ \begin{aligned} \boldsymbol{H}_0(\boldsymbol{\operatorname{curl}}; D) &\to \boldsymbol{H}_0(\boldsymbol{\operatorname{curl}}; \widetilde{D}) \\ \boldsymbol{U} &\mapsto d\boldsymbol{\mathrm{T}}^\top (\boldsymbol{\mathrm{U}} \circ \boldsymbol{\mathrm{T}}) \end{aligned} \right..$$

Moreover, it holds that

$$\operatorname{\mathbf{curl}} \Psi(\mathbf{U}) = d\mathbf{T}^{\operatorname{\mathsf{co}}}(\operatorname{\mathbf{curl}} \mathbf{U} \circ \mathbf{T}) \in \boldsymbol{L}^2(\widetilde{\mathbf{D}}).$$

Since **T** and **T**<sup>-1</sup> possess analogous properties, the results of lemma 4.12 hold for **T**<sup>-1</sup> as well. Hence, we shall denote the inverse of the mapping  $\Psi : \mathbf{H}_0(\mathbf{curl}; D) \to \mathbf{H}_0(\mathbf{curl}; \widetilde{D})$  by  $\Psi^{-1}$ , for which one has

$$\Psi^{-1}: \begin{cases} \boldsymbol{H}_0(\boldsymbol{\operatorname{curl}}; \widetilde{\boldsymbol{\mathrm{D}}}) \to \boldsymbol{H}_0(\boldsymbol{\operatorname{curl}}; \boldsymbol{\mathrm{D}}) \\ \boldsymbol{\mathrm{U}} \mapsto \mathrm{d}(\boldsymbol{\mathrm{T}}^{-1})^\top (\boldsymbol{\mathrm{U}} \circ \boldsymbol{\mathrm{T}}^{-1}) \end{cases},$$
  
$$\boldsymbol{\operatorname{curl}} \Psi^{-1}(\boldsymbol{\mathrm{U}}) = \mathrm{d}(\boldsymbol{\mathrm{T}}^{-1})^{\mathsf{co}}(\boldsymbol{\operatorname{curl}} \boldsymbol{\mathrm{U}} \circ \boldsymbol{\mathrm{T}}^{-1}) \in \boldsymbol{L}^2(\widetilde{\boldsymbol{\mathrm{D}}}).$$

From here onwards, and for general  $\widetilde{D} \in \mathfrak{D}$ , we refer to the isomorphism introduced in lemma 4.12 as  $\Psi : \mathcal{H}_0(\mathbf{curl}; D) \to \mathcal{H}_0(\mathbf{curl}; \widetilde{D})$ —respectively,  $\Psi_i : \mathcal{H}_0(\mathbf{curl}; D) \to \mathcal{H}_0(\mathbf{curl}; \widetilde{D}_i)$  for each  $i \in \mathbb{N}$ .

Lemma 4.13. Let Assumption 4.10 hold. Then, 1the following bounds are satisfied

$$\|\Psi_i\mathbf{U}\|_{H(\mathbf{curl};\widetilde{\mathbf{D}}_i)} \le C\|\mathbf{U}\|_{H(\mathbf{curl};\mathbf{D})}$$
 and  $\|\Psi_i^{-1}\mathbf{V}\|_{H(\mathbf{curl};\mathbf{D})} \le C\|\mathbf{V}\|_{H(\mathbf{curl};\widetilde{\mathbf{D}}_i)}$ ,

for all  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; D)$  and all  $\mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{D}_i)$ , where the positive constant C depends on  $\vartheta > 1$  introduced in Assumption 4.10, but not on  $i \in \mathbb{N}$ .

*Proof.* Fix  $i \in \mathbb{N}$  and let  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$ . Then, by Assumption 4.10, one has

$$\begin{split} \|\Psi_{i}\mathbf{U}\|_{0,\widetilde{\mathbf{D}}_{i}}^{2} &= \int_{\widetilde{\mathbf{D}}_{i}} \|\Psi_{i}\mathbf{U}(\mathbf{x})\|_{\mathbb{C}^{3}}^{2} \, \mathrm{d}\mathbf{x} = \int_{\widetilde{\mathbf{D}}_{i}} \|\mathrm{d}\mathbf{T}_{i}^{\top}\mathbf{U} \circ \mathbf{T}(\mathbf{x})\|_{\mathbb{C}^{3}}^{2} \, \mathrm{d}\mathbf{x} \\ &\leq \int_{\widetilde{\mathbf{D}}_{i}} \|\mathrm{d}\mathbf{T}_{i}^{\top}(\mathbf{x})\|_{\mathbb{C}^{3\times3}}^{2} \|\mathbf{U} \circ \mathbf{T}(\mathbf{x})\|_{\mathbb{C}^{3}}^{2} \, \mathrm{d}\mathbf{x} \leq C\vartheta^{2} \int_{\widetilde{\mathbf{D}}_{i}} \|\mathbf{U} \circ \mathbf{T}(\mathbf{x})\|_{\mathbb{C}^{3}}^{2} \, \mathrm{d}\mathbf{x} \\ &= C\vartheta^{2} \int_{\mathbf{D}} \|\mathbf{U}(\mathbf{x})\|_{\mathbb{C}^{3}}^{2} \, \det(\mathrm{d}(\mathbf{T}_{i}^{-1}(\mathbf{x}))) \, \mathrm{d}\mathbf{x} \leq C\vartheta^{3} \|\mathbf{U}\|_{0,\mathbf{D}}^{2}, \end{split}$$

where the positive constant C follows from the norm equivalence over finite dimensional spaces. An analogous computation yields

$$\|\mathbf{curl}\,\Psi_i\mathbf{U}\|_{0,\widetilde{\mathbf{D}}_i}^2 \leq C\vartheta^3\|\mathbf{curl}\,\mathbf{U}\|_{0,\mathbf{D}}^2.$$

From where the estimate for  $\Psi_i$  follows straightforwardly. The estimate for  $\Psi_i^{-1}$  is retrieved by repeating the arguments exposed above.

The next results follow from arguments similar to [37, Prop. 2.32] and will be of use throughout (cf. [7, Lem. 5.1]), and whose proofs are provided in appendix C.

**Lemma 4.14.** Let  $\Upsilon$  and  $\Omega$  be open Lipschitz domains in  $\mathbb{R}^3$  such that  $\Upsilon$  is convex and  $\Omega \subset \Upsilon$ . Let  $\mathbf{T}$  be a continuous, bijective and bi-Lipschitz transformation—so that  $\mathbf{T}$  and  $\mathbf{T}^{-1}$  belong to  $\mathbf{W}^{1,\infty}(\Upsilon)$ —mapping  $\Upsilon$  onto itself. Then, it holds that

$$||U \circ \mathbf{T} - U||_{\mathbf{L}^{\infty}(\Omega)} \le ||\mathbf{T} - \mathsf{I}||_{L^{\infty}(\Upsilon)} ||U||_{W^{1,\infty}(\Upsilon)},$$

for all  $U \in W^{1,\infty}(\Upsilon)$ .

**Lemma 4.15.** Let  $\Upsilon$  and  $\Omega$  be open Lipschitz domains in  $\mathbb{R}^3$  such that  $\Upsilon$  is convex and  $\Omega \subset \Upsilon$ . Let  $\mathbf{T}$  be a continuous, bijective and bi-Lipschitz transformation— $\mathbf{T}$  and  $\mathbf{T}^{-1}$  belong to  $\mathbf{W}^{1,\infty}(\Upsilon)$ —mapping  $\Upsilon$  onto itself and such that

$$\sup_{\substack{\mathbf{x},\mathbf{y}\in\Upsilon\\\mathbf{x}\neq\mathbf{y}}} \frac{\|(\mathbf{T}(\mathbf{x})-\mathbf{x})-(\mathbf{T}(\mathbf{y})-\mathbf{y})\|_{\mathbb{R}^{3}}}{\|\mathbf{x}-\mathbf{y}\|_{\mathbb{R}^{3}}} \leq \kappa < 1 \quad and \quad \vartheta^{-1} \leq \|\det(\mathrm{d}\mathbf{T})\|_{L^{\infty}(\Upsilon)} \leq \vartheta, \tag{4.6}$$

for some  $\kappa \in (0,1)$  and  $\vartheta > 1$ . Then, one has

$$\|\mathbf{U} \circ \mathbf{T} - \mathbf{U}\|_{0,\Omega} \le (\vartheta^{\frac{1}{2}} + 1) \|\mathbf{T} - \mathbf{I}\|_{L^{\infty}(\Upsilon)}^{s} \|\mathbf{U}\|_{s,\Upsilon},$$

for all  $\mathbf{U} \in \mathbf{H}^s(\Upsilon)$ , with  $0 \le s \le 1$ .

#### 4.2 Convergence of solution pull-backs in approximate domains

We begin by estimating the convergence to zero of the following approximation error:

$$\|\Psi_i \mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl}:\widetilde{\mathbf{D}}_i)},$$

through an application of Strang's lemma [35, Thm. 4.2.11]. As in [5, 25], we note that if  $\mathbf{E} \in \mathcal{H}_0(\mathbf{curl}; \mathbf{D})$  is the unique solution of Problem 2.1, then  $\Psi \mathbf{E} \in \mathcal{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  is the unique solution of a modified Maxwell problem on  $\widetilde{\mathbf{D}}$  arising from transferring the sesquilinear and antilinear forms  $\Phi(\cdot, \cdot)$  and  $\mathbf{F}(\cdot)$  from  $\mathbf{D}$  to  $\widetilde{\mathbf{D}}$  by a change of variables (cf. in [5, Sec. 2.5.2]). Specifically, for  $\widetilde{\mathbf{D}} \in \mathfrak{D}$ , we introduce the modified sesqulinear and antilinear forms as

$$\widehat{\Phi}(\mathbf{U}, \mathbf{V}) := \Phi(\Psi^{-1}\mathbf{U}, \Psi^{-1}\mathbf{V}) \quad \text{and} \quad \widehat{\mathbf{F}}(\mathbf{V}) := \mathbf{F}(\Psi^{-1}\mathbf{V}), \tag{4.7}$$

for all  $\mathbf{U}, \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$ .

**Problem 4.16.** (Modified variational problem on  $\widetilde{D}$ ) Find  $\widehat{\mathbf{E}} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{D})$  such that

$$\widehat{\Phi}(\widehat{\mathbf{E}}, \mathbf{V}) = \widehat{\mathbf{F}}(\mathbf{V}) \quad \forall \, \mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{D}).$$

**Proposition 4.17.** Let Assumptions 2.2 and 4.10 hold and let  $\mathbf{E}$  denote the solution of Problem 2.1. Then,  $\Psi \mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  is the unique solution of Problem 4.16.

*Proof.* Take  $\mathbf{U}, \mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$ . Then, with  $C_1 > 0$  as in Assumption 2.2 and C > 0 as in lemma 4.13, we have that

$$\begin{split} |\widehat{\Phi}(\mathbf{U}, \mathbf{V})| &= \left|\Phi(\Psi^{-1}\mathbf{U}, \Psi^{-1}\mathbf{V})\right| \leq C_1 \|\Psi^{-1}\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})} \|\Psi^{-1}\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})} \\ &\leq C_1 C^2 \|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}})} \|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}})}, \end{split}$$

and

$$|\widehat{\mathbf{F}}(\mathbf{V})| = \left|\mathbf{F}(\Psi^{-1}\mathbf{V})\right| \leq \|\mathbf{F}\|_{\boldsymbol{H}_0(\mathbf{curl}; \mathbf{D})'} \|\Psi^{-1}\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})} \leq C \|\mathbf{F}\|_{\boldsymbol{H}_0(\mathbf{curl}; \mathbf{D})'} \|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}})}.$$

Moreover, since  $\Psi : \mathbf{H}_0(\mathbf{curl}; D) \to \mathbf{H}_0(\mathbf{curl}; \widetilde{D})$  is an isomorphism, for every  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{D})$  it holds that

$$\begin{split} \sup_{\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}}) \backslash \{\mathbf{0}\}} & \frac{|\widehat{\Phi}(\mathbf{U},\mathbf{V})|}{\|\mathbf{U}\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}})} \|\mathbf{V}\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}})}} \\ \geq C^{-2} \sup_{\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}}) \backslash \{\mathbf{0}\}} & \frac{|\Phi(\Psi^{-1}\mathbf{U},\Psi^{-1}\mathbf{V})|}{\|\Psi^{-1}\mathbf{U}\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}})} \|\Psi^{-1}\mathbf{V}\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathbf{D}})}} \geq C^{-2}C_2, \end{split}$$

where the positive constant  $C_2$  is as in Assumption 2.2 and C > 0 comes from lemma 4.13. Moreover, for every  $\mathbf{V} \in H_0(\mathbf{curl}; \widetilde{\mathbf{D}}) \setminus \{\mathbf{0}\}$  we have that

$$\sup_{\mathbf{U}\in \boldsymbol{H}_0(\mathbf{curl};\widetilde{D})\backslash\{\mathbf{0}\}}|\widehat{\Phi}(\mathbf{U},\mathbf{V})|=\sup_{\mathbf{U}\in \boldsymbol{H}_0(\mathbf{curl};D)\backslash\{\mathbf{0}\}}|\Phi(\mathbf{U},\Psi^{-1}\mathbf{V})|>0.$$

Hence, since  $\widehat{\Phi}(\cdot,\cdot)$  satisfies the inf-sup conditions, we can conclude that Problem 4.16 is well posed and has a unique solution in  $\mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  [35, Sec. 2.1.6]. Moreover, since  $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  solves Problem 2.1 there holds that

$$\widehat{\Phi}(\Psi \widetilde{\mathbf{E}}, \mathbf{V}) = \Phi(\widetilde{\mathbf{E}}, \Psi^{-1} \mathbf{V}) = \mathbf{F}(\Psi^{-1} \mathbf{V}) = \widehat{\mathbf{F}}(\mathbf{V}),$$

for all  $\mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$ , and so  $\Psi \mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}})$  is the unique solution of Problem 4.16.

**Theorem 4.18.** Let Assumptions 2.2, 4.1, 4.3, and 4.10 hold and let  $\mathbf{E}$  and  $\mathbf{E}_i$  denote the unique solutions of Problems 2.1 and 4.2 on  $\widetilde{\mathbf{D}}_i$  for each  $i \in \mathbb{N}$ . Moreover, assume that  $\mu^{-1}$ ,  $\epsilon$  and  $\mathbf{J}$  have coefficients in  $W^{1,\infty}(\mathbf{D}_H)$ . Then, it holds that

$$\|\Psi_i \mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}}_i)} \le Cd_1(\mathbf{D}_H, \mathbf{T}_i)(\|\mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})} + \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)}),$$

where C depends on  $\omega$ ,  $\mu$ ,  $\epsilon$  and **J** but is independent of  $i \in \mathbb{N}$ .

*Proof.* Fix  $i \in \mathbb{N}$ , recall the sesquilinear and antilinear forms in eq. (4.7) and let  $\mathbf{U}, \mathbf{V} \in \mathbf{H}_0(\mathbf{curl}; \widetilde{\mathbf{D}}_i)$ . We begin by noticing that the sesquilinear and antilinear forms in (4.7) may be written as

$$\widehat{\Phi}_i(\mathbf{U}, \mathbf{V}) = \int_{\widetilde{\mathbf{D}}_i} \mu_{\mathbf{T}_i}^{-1} \operatorname{\mathbf{curl}} \mathbf{U} \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{V}} - \omega^2 \epsilon_{\mathbf{T}_i} \mathbf{U} \cdot \overline{\mathbf{V}} \, \mathrm{d} \mathbf{x} \quad \text{and} \quad \widehat{\mathbf{F}}_i(\mathbf{V}) = -\imath \omega \int_{\widetilde{\mathbf{D}}_i} \mathbf{J}_{\mathbf{T}_i} \cdot \overline{\mathbf{V}} \, \mathrm{d} \mathbf{x},$$

where

$$\mu_{\mathbf{T}_i} := \det(\mathrm{d}\mathbf{T}_i) \, \mathrm{d}\mathbf{T}_i^{-1} (\mu \circ \mathbf{T}_i) \, \mathrm{d}\mathbf{T}_i^{-\top}, \quad \epsilon_{\mathbf{T}_i} := \det(\mathrm{d}\mathbf{T}_i) \, \mathrm{d}\mathbf{T}_i^{-1} (\epsilon \circ \mathbf{T}_i) \, \mathrm{d}\mathbf{T}_i^{-\top},$$
$$\mathbf{J}_{\mathbf{T}_i} := \det(\mathrm{d}\mathbf{T}_i) \, \mathrm{d}\mathbf{T}_i^{-1} (\mathbf{J} \circ \mathbf{T}_i).$$

Then, one has that

$$\left| \Phi(\mathbf{U}, \mathbf{V}) - \widehat{\Phi}_{i}(\mathbf{U}, \mathbf{V}) \right| \leq \left| \int_{\widetilde{\mathbf{D}}_{i}} (\mu^{-1} - \mu_{\mathbf{T}_{i}}^{-1}) \operatorname{\mathbf{curl}} \mathbf{U} \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{V}} \, d\mathbf{x} \right| + \omega^{2} \left| \int_{\widetilde{\mathbf{D}}_{i}} (\epsilon - \epsilon_{\mathbf{T}_{i}}) \mathbf{U} \cdot \overline{\mathbf{V}} \, d\mathbf{x} \right| \\
\leq C \left( \|\mu^{-1} - \mu_{\mathbf{T}}^{-1}\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})} + \omega^{2} \|\epsilon - \epsilon_{\mathbf{T}}\|_{\mathbf{L}^{\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})} \right) \|\mathbf{U}\|_{\mathbf{H}(\operatorname{\mathbf{curl}};\widetilde{\mathbf{D}}_{i})} \|\mathbf{V}\|_{\mathbf{H}(\operatorname{\mathbf{curl}};\widetilde{\mathbf{D}}_{i})} \\
\leq C d_{1}(\mathbf{D}_{H}, \mathbf{T}_{i}) (\|\epsilon\|_{\mathbf{W}^{1,\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})} + \|\mu^{-1}\|_{\mathbf{W}^{1,\infty}(\mathbf{D};\mathbb{C}^{3\times3})}) \|\mathbf{U}\|_{\mathbf{H}(\operatorname{\mathbf{curl}};\widetilde{\mathbf{D}}_{i})} \|\mathbf{V}\|_{\mathbf{H}(\operatorname{\mathbf{curl}};\widetilde{\mathbf{D}}_{i})}, \tag{4.8}$$

where C > 0 in the first inequality follows from the norm equivalence on finite-dimensional spaces and is independent of  $i \in \mathbb{N}$ . The last bound is derived by applying lemma 4.14 and Assumption 4.10, whereby the positive constant C depends on  $\vartheta > 1$  in Assumption 4.10.

Analogously, we have that

$$|\mathbf{F}(\mathbf{V}) - \widehat{\mathbf{F}}_i(\mathbf{V})| \leq C d_1(\mathbf{D}_H, \mathbf{T}_i) \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)} \|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \widetilde{\mathbf{D}}_i)},$$

where C > 0 is independent of  $i \in \mathbb{N}$ . Finally, by Assumption 4.3 and proposition 4.17, it holds that

$$\begin{split} &\widetilde{C}_{s}\|\Psi_{i}\mathbf{E}-\widetilde{\mathbf{E}}_{i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_{i})} \leq \sup_{\mathbf{V}\in\boldsymbol{H}_{0}(\mathbf{curl};\widetilde{\mathbf{D}}_{i})\backslash\{\mathbf{0}\}} \frac{|\Phi(\Psi_{i}\mathbf{E}-\widetilde{\mathbf{E}}_{i},\mathbf{V})|}{\|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_{i})}} \\ &\leq \sup_{\mathbf{V}\in\boldsymbol{H}_{0}(\mathbf{curl};\mathbf{D})\backslash\{\mathbf{0}\}} \frac{|\Phi(\Psi_{i}\mathbf{E},\mathbf{V})-\widehat{\Phi}_{i}(\Psi_{i}\mathbf{E},\mathbf{V})|+|\widehat{\mathbf{F}}_{i}(\mathbf{V})-\mathbf{F}(\mathbf{V})|}{\|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_{i})}}. \\ &\leq Cd_{1}(\mathbf{D}_{H};\mathbf{T}_{i})\Big[(\omega^{2}\|\epsilon\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_{H};\mathbb{C}^{3\times3})}+\|\mu^{-1}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D};\mathbb{C}^{3\times3})})\|\Psi_{i}\mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_{i})}+\omega\|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_{H})})\Big], \\ &\text{and the claimed estimate then follows by lemma 4.13.} \quad \Box \end{split}$$

#### 4.3 Convergence to an extended solution over $D_H$

Throughout this section, we aim at deriving convergence rate estimates for the approximation error

$$\|\mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}_i)}$$

To this end, we will require **E** to be extended to  $D_H$  with some higher regularity  $H^r(\text{curl}; D_H)$ , for some  $r \in (0, 1]$ —see Assumption 4.19 below. Our results in section 4.2 require no such assumption.

**Assumption 4.19** (Extension to  $D_H$ ).  $\mathbf{E} \in H_0(\mathbf{curl}; D)$ —the solution of Problem 2.1—may be extended to  $H^r(\mathbf{curl}; D_H)$ , with  $r \in (0, 1]$ . We slightly abuse notation by referring to the extension of  $\mathbf{E}$  as  $\mathbf{E}$  as well. Furthermore, we assume the hold-all domain  $D_H$  to be convex.

For an example of an extension operator from  $\mathbf{H}^1(\mathbf{curl}; D)$  to  $\mathbf{H}^1(\mathbf{curl}; D_H)$  under the condition that D be a domain of class  $C^2$  (cf. [24, Thm. 2]).

**Theorem 4.20.** Let Assumptions 2.2, 4.1, 4.3, 4.10, and 4.19 hold and let  $\mathbf{E}$  and  $\widetilde{\mathbf{E}}_i$  denote respectively the unique solutions of Problems 2.1 and 4.2 on  $\widetilde{D}_i$  for all  $i \in \mathbb{N}$ . Then, it holds that

$$\|\mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_i)} \le C \left[ (d_1(\mathbf{D}_H, \mathbf{T}_i) + d_0(\mathbf{D}_H, \mathbf{T}_i)^r) \|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl};\mathbf{D}_H)} + d_1(\mathbf{D}_H, \mathbf{T}_i) \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)} \right],$$
(4.9)

for all  $i \in \mathbb{N}$ , where the positive constant C depends on  $\omega$ ,  $\mu$ ,  $\epsilon$  and **J** but is independent of  $i \in \mathbb{N}$ .

*Proof.* Fix  $i \in \mathbb{N}$  and recall  $\Psi_i : H_0(\mathbf{curl}; D) \to H_0(\mathbf{curl}; \widetilde{D}_i)$  as in lemma 4.12. Then, by the triangle inequality, one has

$$\|\mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_i)} \le \|\mathbf{E} - \Psi_i \mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_i)} + \|\Psi_i \mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_i)}, \tag{4.10}$$

where  $\mathbf{E}$  is chosen to be extended by Assumption 4.19. We start by bounding the first term in the right-hand side of eq. (4.10) thanks to lemma 4.15:

$$\begin{split} \|\mathbf{E} - \Psi_i \mathbf{E}\|_{0,\widetilde{\mathbf{D}}_i} &= \|\mathbf{E} - \mathbf{d} \mathbf{T}_i^{\top} \mathbf{E} \circ \mathbf{T}_i\|_{0,\widetilde{\mathbf{D}}_i} \\ &\leq \|\mathbf{d} \mathbf{T}_i^{\top} - \mathsf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_H)} \|\mathbf{E}\|_{0,\mathbf{D}_H} + (\vartheta^{\frac{1}{2}} + 1) \|\mathbf{d} \mathbf{T}_i^{\top}\|_{\mathbf{L}^{\infty}(\mathbf{D}_H)} \|\mathbf{T}_i - \mathsf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_H)}^r \|\mathbf{E}\|_{r,\mathbf{D}_H} \\ &\leq \left( \|\mathbf{d} \mathbf{T}_i^{\top} - \mathsf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_H)} + (\vartheta^{\frac{1}{2}} + 1)\vartheta \|\mathbf{T}_i - \mathsf{I}\|_{\mathbf{L}^{\infty}(\mathbf{D}_H)}^r \right) \|\mathbf{E}\|_{r,\mathbf{D}_H}. \end{split}$$

Simarly, we have that

$$\|\mathbf{curl}(\mathbf{E} - \Psi_i \mathbf{E})\|_{0,\widetilde{\mathbf{D}}_i} \leq \left(\|\mathbf{d}\mathbf{T}_i^{\mathsf{co}} - \mathsf{I}\|_{\boldsymbol{L}^{\infty}(\mathbf{D}_H)} + (\vartheta^{\frac{1}{2}} + 1)\vartheta\|\mathbf{T}_i - \mathsf{I}\|_{\boldsymbol{L}^{\infty}(\mathbf{D}_H)}^r\right)\|\mathbf{curl}\,\mathbf{E}\|_{r,\mathbf{D}_H},$$

so that, by Assumption 4.10, we have that

$$\begin{split} \|\mathbf{E} - \Psi_i(\mathbf{E})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_i)}^2 &= \|\mathbf{E} - \Psi_i(\mathbf{E})\|_{0,\widetilde{\mathbf{D}}_i}^2 + \|\mathbf{curl}\,\mathbf{E} - \mathbf{curl}\,\Psi_i(\mathbf{E})\|_{0,\widetilde{\mathbf{D}}_i}^2 \\ &\leq C^2 \left(d_1(\mathbf{D}_H,\mathbf{T}_i) + (\vartheta^{\frac{1}{2}} + 1)\vartheta d_0(\mathbf{D}_H,\mathbf{T}_i)^r\right)^2 \left(\|\mathbf{E}\|_{r,\mathbf{D}_H}^2 + \|\mathbf{curl}\,\mathbf{E}\|_{r,\mathbf{D}_H}^2\right) \\ &\leq C^2 (\vartheta^{\frac{1}{2}} + 1)^2 \vartheta^2 \left(d_1(\mathbf{D}_H,\mathbf{T}_i) + d_0(\mathbf{D}_H,\mathbf{T}_i)^r\right)^2 \|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl};\mathbf{D}_H)}^2, \end{split}$$

where C arises from eq. (4.5) and is independent of  $i \in \mathbb{N}$ . Furthermore, the second term in the right-hand side of eq. (4.10) may be bounded by direct application of theorem 4.18, yielding

$$\begin{aligned} \|\mathbf{E} - \widetilde{\mathbf{E}}_i\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathbf{D}}_i)} &\leq C\left(d_1(\mathbf{D}_H, \mathbf{T}_i) + d_0(\mathbf{D}_H, \mathbf{T}_i)^r\right) \\ &\cdot \|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl};\mathbf{D}_H)} + Cd_1(\mathbf{D}_H, \mathbf{T}_i)(\|\mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl};\mathbf{D})} + \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)}). \end{aligned}$$

The stated result follows by reordering the terms in the last equation.

## 5 Variational problems on approximate domains: discrete problem

We now analyze the discrete version of Problem 4.2, and consider the family of approximate domains  $\mathfrak{D}$  corresponding to curved domains  $\{D_{h_i}\}_{i\in\mathbb{N}}$  introduced in section 3, i.e.,  $D_{h_i} \equiv \widetilde{D}_i$ . This discrete setting is signaled by denoting an arbitrary element of  $\mathfrak{D}$  by  $D_h$  instead of  $\widetilde{D}$ . We also recall the discrete space  $P_0^c(\tau_h)$  in eq. (3.6) as the space of curl-conforming piecewise polynomials of degree  $k \in \mathbb{N}$  with null flipped-Dirichlet trace on  $\Gamma$ .

**Assumption 5.1** (Discrete inf-sup conditions). Assume the sesquilinear form in eq. (4.1) to satisfy the following conditions:

$$\inf_{\mathbf{U} \in \mathbf{P}_{0}^{c}(\tau_{h}) \setminus \{\mathbf{0}\}} \left( \sup_{\mathbf{V} \in \mathbf{P}_{0}^{c}(\tau_{h}) \setminus \{\mathbf{0}\}} \frac{|\Phi(\mathbf{U}, \mathbf{V})|}{\|\mathbf{U}\|_{\mathbf{H}(\mathbf{curl}; \mathbf{D}_{h})} \|\mathbf{V}\|_{\mathbf{H}(\mathbf{curl}; \mathbf{D}_{h})}} \right) \ge C_{2},$$

$$\inf_{\mathbf{V} \in \mathbf{P}_{0}^{c}(\tau_{h}) \setminus \{\mathbf{0}\}} \left( \sup_{\mathbf{U} \in \mathbf{P}_{0}^{c}(\tau_{h}) \setminus \{\mathbf{0}\}} \frac{|\Phi(\mathbf{U}, \mathbf{V})|}{\|\mathbf{U}\|_{\mathbf{H}(\mathbf{curl}; \mathbf{D}_{h})} \|\mathbf{V}\|_{\mathbf{H}(\mathbf{curl}; \mathbf{D}_{h})}} \right) \ge C_{3}, \tag{5.1}$$

for all  $\tau_h \in \mathfrak{T}$  ( $D_h \in \mathfrak{D}$ ), where the positive constants  $C_2$  and  $C_3$  are independent of the mesh-size h.

**Problem 5.2** (Discrete variational problem on inexact domains). Find  $\mathbf{E}_h \in P_0^c(\tau_h)$  such that

$$\Phi(\mathbf{E}_h, \mathbf{V}) = \mathbf{F}(\mathbf{V}),$$

for all  $\mathbf{V} \in \mathbf{P}_0^c(\tau_h)$ .

Assumption 5.1 ensures uniqueness and existence of solutions of Problem 5.2, whose solutions are denoted  $\mathbf{E}_h \in P_0^c(\tau_h)$  for general  $\tau_h \in \mathfrak{T}$  and  $\mathbf{E}_{h_i} \in P_0^c(\tau_{h_i})$  for each  $i \in \mathbb{N}$ , respectively. Note that the condition in eq. (5.1) is stronger than required for the purposes of proving the unique solvability of Problem 5.2. This stronger condition is necessary to prove a discrete analogue of proposition 4.17 (see proposition 5.6) via a perturbation argument.

#### 5.1 Convergence of domains in a discrete setting

Let us start with the following result, regarding the approximation of functions in  $H_0(\mathbf{curl}; D)$  by discrete functions in  $P_0^c(\tau_h)$ . Here, once again we identify elements of  $H_0(\mathbf{curl}; D)$  and  $H_0(\mathbf{curl}; D_h)$  with their zero-extensions to  $D_H$ .

**Lemma 5.3.** Suppose  $\mathfrak{D}$  approximates D in the sense of Hausdorff and that D and all  $D_h \in \mathfrak{D}$  are Lipschitz continuous domains with uniform Lipschitz constant in  $\mathfrak{D}$ . Then, it holds that

- (a) For every  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  there exists a sequence  $\{\mathbf{U}_i\}_{i \in \mathbb{N}}$  with  $\mathbf{U}_i \in \mathbf{P}_0^c(\tau_{h_i})$  for all  $i \in \mathbb{N}$ , such that  $\mathbf{U}_i$  converges to  $\mathbf{U}$  strongly in  $\mathbf{H}_0(\mathbf{curl}; \mathbf{D}_H)$ .
- (b) Weak limits of every sequence  $\{\mathbf{U}_i\}_{i\in\mathbb{N}}$ , with  $\mathbf{U}_i\in P_0^c(\tau_{h_i})$ , belong to  $H_0(\mathbf{curl}; \mathbf{D})$ .

Proof. item (a): Take  $\mathbf{U} \in H_0(\mathbf{curl}; \mathbf{D})$  and set an arbitrary  $\epsilon > 0$ . By density of  $\mathcal{C}_0^{\infty}(\mathbf{D})$  in  $H_0(\mathbf{curl}; \mathbf{D})$ , there exists some  $\widetilde{\mathbf{U}}_{\epsilon} \in \mathcal{C}_0^{\infty}(\mathbf{D})$  such that  $\|\widetilde{\mathbf{U}}_{\epsilon} - \mathbf{U}\|_{H_0(\mathbf{curl}; \mathbf{D}_H)} \leq \frac{\epsilon}{2}$ . Moreover, there exists some  $i_{\epsilon} \in \mathbb{N}$  such that  $\sup_{\epsilon} (\widetilde{\mathbf{U}}_{\epsilon}) \subset \mathbf{D}_{h_i}$  for all  $i > i_{\epsilon}$  (cf. proof of Lemma 3 in [34, Sec. 3]), and another  $i'_{\epsilon} \in \mathbb{N}$  such that  $\|\mathbf{\Pi}_{h_i}\widetilde{\mathbf{U}}_{\epsilon} - \widetilde{\mathbf{U}}_{\epsilon}\|_{H(\mathbf{curl}; \mathbf{D}_H)} \leq \frac{\epsilon}{2}$  for all  $i > i'_{\epsilon}$ . Then, it holds that

$$\|\mathbf{\Pi}_{h_i}\widetilde{\mathbf{U}}_{\epsilon} - \mathbf{U}\|_{\mathbf{H}(\mathbf{curl}; \mathbf{D}_H)} \le \epsilon,$$

for all  $i > i'_{\epsilon}$ . Repeating the procedure above for a decreasing sequence of  $\epsilon > 0$  allows the construction of a strongly convergent sequence to  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  (cf. [14, Lem. 2.4]).

item (b): Follows from lemma 4.8 by the inclusion 
$$P_0^c(\tau_{h_i}) \subset H_0(\operatorname{curl}; D_{h_i})$$
.

In order to obtain convergence rates of finite element solutions to  $\mathbf{E}$ —thereby proving discrete analogues to theorems 4.18 and 4.20—we need one further assumption on the transformations  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$ .

**Assumption 5.4.** For each element of the family  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$  mapping  $D_{h_i}$  to D, as given in Assumption 4.10, we assume that the transformations and their inverses belong to  $\mathbf{W}^{\Re+1,\infty}(K)$  for each tetrahedron K in their corresponding mesh, i.e., for each  $i\in\mathbb{N}$ , we assume that

$$\mathbf{T}_i|_K \in \boldsymbol{W}^{\mathfrak{K}+1,\infty}(K)$$

for all  $K \in \tau_{h_i}$ , and that

$$\lim_{i \to \infty} \max_{K \in \tau_{h_i}} \|\mathbf{T}_i|_K - \mathbf{I}\|_{\mathbf{W}^{\mathfrak{K}, \infty}(K)} + \max_{K \in \tau_{h_i}} \|\mathbf{T}_i^{-1}|_K - \mathbf{I}\|_{\mathbf{W}^{\mathfrak{K}, \infty}(K)} = 0,$$

$$\max_{K \in \tau_{h_i}} \|\mathbf{T}_i|_K \|_{\mathbf{W}^{\mathfrak{K}+1, \infty}(K)} + \max_{K \in \tau_{h_i}} \|\mathbf{T}_i^{-1}|_K \|_{\mathbf{W}^{\mathfrak{K}+1, \infty}(K)} \le c_{\mathfrak{T}},$$

for all  $i \in \mathbb{N}$ , where  $c_{\mathfrak{T}}$  is a positive constant independent of  $i \in \mathbb{N}$ . Moreover, the family  $\{\mathbf{T}_i\}_{i \in \mathbb{N}}$  satisfies the following bound:

$$d_0(D_H, \mathbf{T}_i) \le Ch_i^{\mathfrak{K}+1} \quad and \quad d_1(D_H, \mathbf{T}_i) \le Ch_i^{\mathfrak{K}},$$
 (5.2)

for  $\mathfrak{K} \in \{1 : \mathfrak{M} - 1\}$  as in Assumption 3.3, with C > 0 independent of  $i \in \mathbb{N}$ .

Assumption 5.4 is justified by constructions of mappings  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$  in the context of curved meshes in finite elements (cf. [28, Prop. 2 and 3] for properties of these mappings and Sections 3 and 5 therein for their construction satisfying Assumptions 3.3 and 5.4). Also, Assumption 3.2 related to smoothness requirements on D to be of class  $\mathcal{C}^{\mathfrak{M}}$ , has no direct relevance on the coming proofs of our discrete approximation results. Indeed, the smoothness of the domain only plays a role in proving decay rates such as eq. (5.2) in practical constructions of the transformations  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$ , which we have assumed through Assumption 5.4. Nonetheless, we opt to enforce Assumption 3.2 in our coming results to emphasize its necessity for the construction of the transformations  $\{\mathbf{T}_i\}_{i\in\mathbb{N}}$  and, therefore, our main results.

#### 5.2 Discrete convergence of solution pull-backs in approximate domains

We now extend the results in section 4.2 to our discrete setting, i.e. we estimate the error

$$\|\Psi_i \mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})},$$

as i grows towards infinity, where  $\Psi_i: H_0(\mathbf{curl}; D) \to H_0(\mathbf{curl}; D_{h_i})$  was introduced in lemma 4.12.

**Problem 5.5** (Modified discrete variational problem on  $D_h$ ). Find  $\hat{\mathbf{E}}_h \in P_0^c(\tau_h)$  such that

$$\widehat{\Phi}(\widehat{\mathbf{E}}_h, \mathbf{V}) = \widehat{\mathbf{F}}(\mathbf{V}) \quad \forall \, \mathbf{V} \in \mathbf{P}_0^c(\tau_h).$$

**Proposition 5.6.** Let Assumptions 2.2, 3.3, 4.10, and 5.1 hold. Moreover, assume that  $\mu^{-1}$ ,  $\epsilon$  and **J** have coefficients in  $W^{1,\infty}(D_H)$ . Then, Problem 5.5 is well posed for all sufficiently small h > 0.

Proof. The continuity of  $\widehat{\Phi}(\cdot,\cdot)$  and  $\widehat{\mathbf{F}}(\cdot)$  on  $P_0^c(\tau_h)$  follows from the proof of proposition 4.17 and the conformity of the finite element space, i.e.  $P_0^c(\tau_h) \subset H_0(\mathbf{curl}; D_h)$ . The inf-sup conditions on  $\widehat{\Phi}(\cdot,\cdot)$  follow by a perturbation argument through eq. (4.8) and our assumptions<sup>3</sup>.

In order to estimate the convergence rates for approximations of  $\mathbf{E}_{h_i}$  to  $\Psi_i \mathbf{E}$  as  $i \in \mathbb{N}$  grows to infinity, one needs to show that  $\Psi_i \mathbf{E}$  preserves the smoothness of  $\mathbf{E}$  to some degree.

**Lemma 5.7.** Let Assumptions 3.3 and 5.4 hold. Let  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D}) \cap \mathbf{H}^s(\mathbf{curl}; \mathbf{D})$  for some  $s \in \{1 : \mathfrak{K}\}$  and let  $K \in \tau_h$  be an arbitrary tetrahedron of the mesh  $\tau_h \in \mathfrak{T}$ . Then, for  $\Psi : \mathbf{H}_0(\mathbf{curl}; \mathbf{D}) \to \mathbf{H}_0(\mathbf{curl}; \mathbf{D}_h)$  as introduced in lemma 4.12, there holds that  $\Psi \mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; K)$  and that

$$\|\Psi_i(\mathbf{U})\|_{\mathbf{H}^s(\mathbf{curl};K)} \le \vartheta^{\frac{1}{2}} C \|\mathbf{U}\|_{\mathbf{H}^s(\mathbf{curl};\mathbf{T}_i(K))},$$

where the positive constant C depends on  $c_{\mathfrak{T}}$  in Assumption 5.4 but not on the mesh-size or  $\mathbf{U} \in \mathbf{H}_0(\mathbf{curl}; D) \cap \mathbf{H}^s(\mathbf{curl}; D)$ .

*Proof.* The result is a direct consequence of [15, Lem. 1] or [16, Lem. 3] together with our assumptions.  $\Box$ 

**Theorem 5.8.** Let Assumptions 2.2, 3.2, 3.3, 4.1, 4.3, 4.10, 5.1, and 5.4 hold, let  $\mathbf{E}$  and  $\mathbf{E}_{h_i}$  denote the unique solutions of Problems 2.1 and 5.2, respectively, and assume that  $\mathbf{E} \in \mathbf{H}^s(\mathbf{curl}; \mathbf{D})$  for some  $s \in \{1:k\}$ . Furthermore, let  $\mu^{-1}$ ,  $\epsilon$  and  $\mathbf{J}$  have coefficients in  $W^{1,\infty}(\mathbf{D})$ . Then, there exists some  $i \in \mathbb{N}$  such that, for all i > i, it holds that

$$\|\Psi_i \mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D})} \le C(h_i^s \|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl}; \mathbf{D})} + h_i^{\mathfrak{K}} \|\mathbf{J}\|_{\boldsymbol{W}^{1, \infty}(\mathbf{D}_H)}), \tag{5.3}$$

where C > 0 depends on  $\omega$ ,  $\mu$ ,  $\epsilon$  and  $\mathbf{J}$  but is independent of  $i \in \mathbb{N}$ .

*Proof.* Fix  $i \in \mathbb{N}$  large enough so that the results of proposition 5.6 hold true and let  $\widehat{\mathbf{E}}_i \in P_0^c(\tau_{h_i})$  denote the solution of Problem 5.5. Then, one can write

$$\|\Psi_i\mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \leq \|\Psi_i\mathbf{E} - \widehat{\mathbf{E}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} + \|\widehat{\mathbf{E}}_{h_i} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})}.$$

Lemma 5.7 implies that  $\Psi_i \mathbf{E} \in \mathbf{H}^s(\mathbf{curl}; K)$  for all  $K \in \tau_{h_i}$ , with

$$\|\Psi_i(\mathbf{E})\|_{\mathbf{H}^s(\mathbf{curl};K)} \le C \|\mathbf{E}\|_{\mathbf{H}^s(\mathbf{curl};\mathbf{T}_i(K))},$$

for C positive independent of  $i \in \mathbb{N}$ . Since  $\Psi_i \mathbf{E} \in \mathcal{H}_0(\mathbf{curl}; D_{h_i})$  is the solution of Problem 4.16 (see proposition 4.17), Theorem 5.41 in [30] yields

$$\begin{split} \|\Psi_{i}\mathbf{E} - \widehat{\mathbf{E}}_{h_{i}}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_{i}})} &\leq Ch_{i}{}^{s} \left(\sum_{K \in \tau_{h_{i}}} \|\Psi_{i}\mathbf{E}\|_{\boldsymbol{H}^{s}(\mathbf{curl}; K)}^{2}\right)^{\frac{1}{2}} \\ &\leq Ch_{i}{}^{s} \left(\sum_{K \in \tau_{h_{i}}} \|\mathbf{E}\|_{\boldsymbol{H}^{s}(\mathbf{curl}; \mathbf{T}_{i}(K))}^{2}\right)^{\frac{1}{2}} \leq Ch_{i}{}^{s} \|\mathbf{E}\|_{\boldsymbol{H}^{s}(\mathbf{curl}; \mathbf{D})}, \end{split}$$

<sup>&</sup>lt;sup>3</sup>See, for example, the proof of [35, Thm. 4.2.11]

where the positive constant C is not necessarily equal in each appearance. Furthermore, by a reasoning analogous to that in theorem 4.18, followed by an application of lemma 4.13 together with Assumption 5.4 and the wellposedness of Problem 5.5 (also, see proposition 4.17), we have that

$$\|\widehat{\mathbf{E}}_{h_i} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \le Cd_1(\mathbf{D}_H, \mathbf{T}_i)(\|\widehat{\mathbf{E}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} + \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)}) \le Ch_i^{\mathfrak{K}} \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)}$$

where again C may vary, but remains  $h_i$ -independent. The result in eq. (5.3) is then deduced by a simple combination of the above estimates.

#### 5.3 Discrete convergence to an extended solution over $D_H$

We now transfer our results in section 4.3 to our discrete setting.

**Theorem 5.9.** Let Assumptions 2.2, 3.2, 3.3, 4.1, 4.3, 4.10, 4.19, 5.1, and 5.4 hold, let  $\mathbf{E}$  and  $\mathbf{E}_{h_i}$  denote the unique solutions of Problems 2.1 and 5.2 and assume that  $\mathbf{E} \in \mathbf{H}^s(\mathbf{curl}; \mathbf{D})$  for some  $s \in \{1:k\}$ . Furthermore, let  $\mu^{-1}$ ,  $\epsilon$  and  $\mathbf{J}$  have coefficients in  $W^{1,\infty}(\mathbf{D})$ . By Assumption 4.19  $\mathbf{E} \in \mathbf{H}^r(\mathbf{curl}; \mathbf{D}_H)$  for some  $r \in (0,1]$ . Then, there exists some  $i \in \mathbb{N}$  such that, for all i > i, it holds that

$$\|\mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\mathcal{D}_{h_i})} \leq C \bigg[ (h_i^{\mathfrak{K}} + h_i^{r(\mathfrak{K}+1)}) \|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl};\mathcal{D}_H)} + h_i^s \|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl};\mathcal{D})} + h_i^{\mathfrak{K}} \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathcal{D}_H)} \bigg],$$

where C > 0 depends on  $\omega$ ,  $\mu$ ,  $\epsilon$  and  $\mathbf{J}$ , but is independent of  $i \in \mathbb{N}$ .

*Proof.* Fix  $i \in \mathbb{N}$  large enough so that the results of proposition 5.6 hold true. The triangle inequality yields

$$\|\mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \leq \|\mathbf{E} - \Psi_i \mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} + \|\Psi_i \mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})}.$$

From the proof of theorem 4.20, it follows that

$$\|\mathbf{E} - \Psi_i \mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h, \cdot})} \le C \left( d_1(\mathbf{D}_H, \mathbf{T}_i) + d_0(\mathbf{D}_H, \mathbf{T}_i)^r \right) \|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl}; \mathbf{D}_H)},$$

which, together with Assumption 5.4, gives

$$\|\mathbf{E} - \Psi_i \mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \le C \left( h_i^{\Re} + h_i^{r(\Re+1)} \right) \|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl}; \mathbf{D}_H)},$$

where C is a positive constant depending only on  $\vartheta$  in Assumption 4.10. Moreover, a direct application of theorem 5.8 leads to

$$\|\Psi_i\mathbf{E}-\mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\mathcal{D}_{h_i})}\leq C(h_i^s\|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl};\mathcal{D})}+h_i^{\mathfrak{K}}\|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathcal{D}_H)}),$$

where the positive constant C follows from theorem 5.8. The result then follows by a straightforward combination of previous estimates.

#### 5.4 A fully discrete estimate

We now deduce convergence estimates for the fully discrete Maxwell variational problem under consideration by incorporating our findings in [4]. To this end, let us introduce quadrature rules for the numerical approximation of Problem 5.2.

**Definition 5.10** (Quadratures). For  $L \in \mathbb{N}$ , let  $\{\check{w}_l\}_{l=1}^L \subset \mathbb{R}$  be a set of quadrature weights and let  $\{\check{b}_l\}_{l=1}^L \subset \check{K}$  be a set of corresponding quadrature points. Then, we introduce the following linear operator over  $\mathcal{C}(\check{K})$ ,

$$Q_{reve{K}}: egin{cases} \mathcal{C}(reve{K}) 
ightarrow \mathbb{C} \ \phi \mapsto \sum_{l=1}^L reve{w}_l \phi(reve{b}_l) \end{cases}$$

Moreover, we say  $Q_{\breve{K}}$  is exact on polynomials of degree  $\mathfrak{n} \in \mathbb{N}_0$  if and only if

$$Q_{\breve{K}}(\phi) = \int_{\breve{K}} \phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} \quad \forall \ \phi \in \mathbb{P}_{\mathfrak{n}}(\breve{K}; \mathbb{C}).$$

Quadratures on arbitrary mesh elements  $K \in \tau_h$  are defined from a quadrature on  $\check{K}$  through the mappings in Assumption 3.3 as follows

$$Q_K(\phi) := Q_{\check{K}}(\det(\mathrm{d}T_K)\phi \circ T_K). \tag{5.4}$$

Then, for three distinct quadrature rules over  $\check{K}$ —denoted as  $Q^1_{\check{K}}$ ,  $Q^2_{\check{K}}$  and  $Q^3_{\check{K}}$  and which shall be specified later on—we introduce the numerical approximations of the sesquilinear and antilinear forms in (2.3) as

$$\Phi_h(\mathbf{U}_h, \mathbf{V}_h) := \sum_{K \in \tau_h} Q_K^1(\mu^{-1} \operatorname{\mathbf{curl}} \mathbf{U}_h \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{V}_h}) + Q_K^2(-\omega^2 \epsilon \mathbf{U}_h \cdot \overline{\mathbf{V}_h}), \tag{5.5}$$

$$\mathbf{F}_h(\mathbf{V}_h) := \sum_{K \in \tau_h} Q_K^3(-\imath \omega \mathbf{J} \cdot \overline{\mathbf{V}_h}), \tag{5.6}$$

for all  $\mathbf{U}_h$  and  $\mathbf{V}_h$  in  $P_0^c(\tau_h)$ , where  $Q_K^i$  is constructed from  $Q_{\check{K}}^i$  through (5.4) for all  $i \in \{1:3\}$ .

**Problem 5.11** (Fully discrete problem). Find  $\widetilde{\mathbf{E}}_h \in P_0^c(\tau_h)$  such that

$$\Phi_h(\widetilde{\mathbf{E}}_h, \mathbf{V}_h) = \mathbf{F}_h(\mathbf{V}_h),$$

for all  $\mathbf{V}_h \in \mathbf{P}_0^c(\tau_h)$ .

#### 5.4.1 Fully discrete convergence of solution pull-backs in approximate domains

We now present a fully discrete version of theorem 5.8, stating the approximation properties of the solution of Problem 5.11 to the pull-back of the solution of Problem 2.1. To limit the number of parameters with effects on the convergence rate, we restrict ourselves to the case of isoparametric finite elements ( $\Re = k$ ).

**Theorem 5.12.** Let all the assumptions in theorem 5.8 hold and take  $\mathfrak{K} = k$ . Additionally, let us assume that there holds that

$$\mathbf{J} \in \mathbf{W}^{s,q}(\mathbf{D}_H) \quad and \quad \epsilon_{i,j}, \ (\mu^{-1})_{i,j} \in W^{s,\infty}(\mathbf{D}_H) \quad \forall \ i, \ j \in \{1:3\},\$$

for some  $q > \min(2, s/3)$ , where  $s \in \{1 : k\}$  is as in theorem 5.8, as well as the following conditions on the quadrature rules defining  $\Phi_h(\cdot, \cdot)$  and  $\mathbf{F}_h(\cdot)$  in eqs. (5.5) and (5.6), respectively,

- $Q_{\check{K}}^1$  is exact for polynomials of degree 2k+s-3 and
- $Q_{\breve{K}}^2$  and  $Q_{\breve{K}}^3$  are exact for polynomials of degree 3k+s-3.

Then, there exists some  $i \in \mathbb{N}$  such that for all  $i \in \mathbb{N}$  with i > i, Problem 5.11 is uniquely solvable and its solution, denoted  $\widetilde{\mathbf{E}}_{h_i}$ , satisfies

$$\|\Psi_i \mathbf{E} - \widetilde{\mathbf{E}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \le Ch_i^s \left( \|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl}; \mathbf{D})} + \|\mathbf{J}\|_{\boldsymbol{W}^{s,q}(\mathbf{D}_H)} + \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)} \right),$$

where the positive constant C is independent of the mesh-size.

*Proof.* Fix  $i \in \mathbb{N}$  as in the proof of theorem 5.8 and let  $\mathbf{E}_{h_i} \in \mathbf{P}_0^c(\tau_{h_i})$  denote the solution of Problem 5.2. Performing small modifications of [4, Thm. 4] (*cf.* proof of [35, Thm. 4.2.11]), it follows that

$$\begin{split} \|\Psi_{i}\mathbf{E} - \widetilde{\mathbf{E}}_{h_{i}}\|_{\boldsymbol{H}(\mathbf{curl};\mathbf{D}_{h_{i}})} &\leq C \Bigg[ \|\Psi_{i}(\mathbf{E}) - \mathbf{E}_{h_{i}}\|_{\boldsymbol{H}(\mathbf{curl};\mathbf{D}_{h_{i}})} + \|\mathbf{\Pi}_{h_{i}}(\Psi_{i}\mathbf{E}) - \mathbf{E}_{h_{i}}\|_{\boldsymbol{H}(\mathbf{curl};\mathbf{D}_{h_{i}})} \\ &+ h_{i}^{s} \left( \sum_{K \in \tau_{h_{i}}} \|\mathbf{\Pi}_{h_{i}}(\Psi_{i}\mathbf{E})\|_{\boldsymbol{H}^{s}(\mathbf{curl};K)}^{2} \right)^{\frac{1}{2}} + h_{i}^{s} \|\mathbf{J}\|_{\boldsymbol{W}^{s,q}(\mathbf{D}_{H})} \Bigg], \end{split}$$

where C > 0 depends on the problem's parameters, including the constants in Assumptions 4.3 and 5.1, but it is independent of  $i \in \mathbb{N}$ . theorem 5.8 then yields

$$\|\Psi_i(\mathbf{E}) - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \le Ch_i^s(\|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl}; \mathbf{D})} + \|\mathbf{J}\|_{\boldsymbol{W}^{1,\infty}(\mathbf{D}_H)}),$$

with C from theorem 5.8 independently of  $i \in \mathbb{N}$ . The continuity of the global interpolation operator (proposition 3.7 and definition 3.8) leads to

$$\|\mathbf{\Pi}_{h_i}(\Psi_i(\mathbf{E})) - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} = \|\mathbf{\Pi}_{h_i}(\Psi_i(\mathbf{E}) - \mathbf{E}_{h_i})\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \le C\|\Psi_i(\mathbf{E}) - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})},$$

where the positive constant C is as in proposition 3.7 and is also independent of  $i \in \mathbb{N}$ . Moreover, by proposition 3.7 and lemma 5.7, for every  $K \in \tau_{h_i}$  it holds that

$$\|\mathbf{\Pi}_{h_i}(\Psi_i(\mathbf{E}))\|_{\mathbf{H}^s(\mathbf{curl};K)} = \|\mathbf{r}_K(\Psi_i(\mathbf{E}))\|_{\mathbf{H}^s(\mathbf{curl};K)} \le C\|\Psi_i(\mathbf{E})\|_{\mathbf{H}^s(\mathbf{curl};K)} \le C\|\mathbf{E}\|_{\mathbf{H}^s(\mathbf{curl};\mathbf{T}_i(K))},$$
 so that

$$\sum_{K \in \tau_{h_s}} \|\mathbf{\Pi}_{h_i}(\Psi_i(\mathbf{E}))\|_{\boldsymbol{H}^s(\mathbf{curl};K)}^2 \le C^2 \|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl};\mathrm{D})}^2,$$

where C > 0 follows from proposition 3.7 and lemma 5.7. The estimate follows by an application of lemma 4.13.

#### 5.4.2 Fully discrete convergence to an extended solution over $D_H$

We continue by presenting the corresponding fully discrete version of theorem 5.13, stating the convergence of the solution of Problem 5.11 to a smooth extension of the solution of Problem 2.1. For simplicity (as before) we consider only the case of isoparametric finite elements ( $\Re = k$ ).

**Theorem 5.13.** Let all the assumptions in theorem 5.9 hold and take  $\mathfrak{K} = k$ . Moreover, assume that there holds that

$$\mathbf{J} \in \boldsymbol{W}^{s,q}(\mathbf{D}^H) \quad and \quad \epsilon_{i,j}, \ (\mu^{-1})_{i,j} \in W^{s,\infty}(\mathbf{D}_H) \quad \forall \ i, \ j \in \{1:3\},$$

for some  $q > \min(2, s/3)$ , where  $s \in \{1 : k\}$  is as in theorem 5.13, as well as the following conditions on the quadrature rules defining  $\Phi_h(\cdot, \cdot)$  and  $\mathbf{F}_h(\cdot)$  in eqs. (5.5) and (5.6), respectively,

- $Q_{\breve{K}}^1$  is exact for polynomials of degree 2k+s-3 and
- $Q_{\check{K}}^2$  and  $Q_{\check{K}}^3$  are exact for polynomials of degree 3k+s-3.

Then, there exists some  $\mathfrak{i} \in \mathbb{N}$  such that for all  $i \in \mathbb{N}$  with  $i > \mathfrak{i}$ , Problem 5.11 is uniquely solvable and its solution, denoted  $\widetilde{\mathbf{E}}_{h_i}$ , satisfies

$$\|\mathbf{E} - \widetilde{\mathbf{E}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \le C(\vartheta)(h^{\mathfrak{K}} + h^{r(\mathfrak{K}+1)})\|\mathbf{E}\|_{\boldsymbol{H}^r(\mathbf{curl}; \mathbf{D}_{H})} + C(\vartheta, \mathfrak{T})h_i^s\|\mathbf{E}\|_{\boldsymbol{H}^s(\mathbf{curl}; \mathbf{D})} + Ch_i^s,$$

where positive constants  $C(\vartheta)$  and  $C(\mathfrak{T})$  depend only on  $\vartheta$  and  $\mathfrak{T}$ , respectively, and C is a positive constant independent of the mesh-size.

*Proof.* Fix  $i \in \mathbb{N}$  as in the proof of theorem 5.9. By the triangle inequality, we have that

$$\|\mathbf{E} - \widetilde{\mathbf{E}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} \leq \|\mathbf{E} - \Psi_i \mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})} + \|\Psi_i \mathbf{E} - \widetilde{\mathbf{E}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})},$$

and the result follows by an application of lemma 4.15 and theorem 5.12 (also see the proof of theorem 4.20).

#### 6 Numerical Results

We test our main results on a simple numerical example. For simplicity, and in order to study only the effects of domain approximation quality on the convergence rate of the finite element method, we consider the exact domain D to be the ball with radius 1 centered at the origin, i.e.

$$D := \{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_{\mathbb{R}^3} < 1 \}.$$

Since D is a convex domain, the approximate domains in  $\mathfrak{D}$  may be chosen to be contained in D, so that D  $\equiv$  D<sub>H</sub> in Assumption 4.19 and no extension of the solution  $\mathbf{E} \in \mathbf{H}_0(\mathbf{curl}; \mathbf{D})$  of Problem 2.1 outside of D is required. Hence, our following results consider the error measurement as in section 5.3 only.

We consider the coercive variational problem on  $H_0(\mathbf{curl}; \mathbf{D})$  given by the following sesquilinear and antilinear forms:

$$\Phi(\mathbf{U}, \mathbf{V}) := \int_{D} (\mu_0 \mathbf{I})^{-1} \operatorname{\mathbf{curl}} \mathbf{U} \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{V}} - \omega^2(\epsilon_0 \mathbf{I}) \mathbf{U} \cdot \overline{\mathbf{V}} \, d\mathbf{x} \quad \text{and} \quad \mathbf{F}(\mathbf{V}) := -\imath \omega \int_{D} \mathbf{J} \cdot \overline{\mathbf{V}} \, d\mathbf{x},$$

where  $\epsilon_0 := 1$ ,  $\mu_0 := 2$ ,  $\omega := 1$  and  $\mathbf{J} := i[J_1, J_2, J_3]^{\top}$ , with

$$J_{1}(\mathbf{x}) := x_{1} - \frac{\pi^{2}}{8} x_{1} x_{2} \cos\left(\frac{\pi}{2} \|\mathbf{x}\|_{\mathbb{R}^{3}}^{2}\right),$$

$$J_{2}(\mathbf{x}) := x_{2} + \left(\frac{1}{4} + \frac{\pi^{2}}{8} (x_{1}^{2} + x_{3}^{2})\right) \cos\left(\frac{\pi}{2} \|\mathbf{x}\|_{\mathbb{R}^{3}}^{2}\right) + \frac{\pi}{4} \sin\left(\frac{\pi}{2} \|\mathbf{x}\|_{\mathbb{R}^{3}}^{2}\right),$$

$$J_{3}(\mathbf{x}) := x_{3} - \frac{\pi^{2}}{8} x_{2} x_{3} \cos\left(\frac{\pi}{2} \|\mathbf{x}\|_{\mathbb{R}^{3}}^{2}\right).$$

Under the above choices, the exact solution to Problem 2.1 is

$$\mathbf{E}(\mathbf{x}) := \left[ x_1, x_2 + \frac{1}{4} \cos\left(\frac{\pi}{2} \|\mathbf{x}\|_{\mathbb{P}^3}^2\right), x_3 \right]^{\top}. \tag{6.1}$$

The various meshes used throughout our experiments were constructed using GMSH [22], while Problem 5.2 was solved using GETDP version 3.4.0 [20].

#### 6.1 Approximate domains

Let us consider two different sequences of meshes of different order. The first sequence considers meshes constructed from straight tetrahedrons only  $(\mathfrak{K}=1)$ , while the second sequence considers meshes consisting of second order elements curved tetrahedrons  $(\mathfrak{K}=2)$ . fig. 2 shows the first three meshes of first and second order. For more details on the conditions satisfied by the second order mesh elements, we refer to [26].

#### 6.2 Convergence results

We employ first and second order curl-conforming elements on both straight and curved (order 2) meshes in order to test the results exposed in Theorem 5.9. To that end, we measure the error

$$\|\mathbf{E} - \mathbf{E}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl}; \mathbf{D}_{h_i})}$$

as  $i \in \mathbb{N}$  grows towards infinity, where  $\mathbf{E} \in H_0(\mathbf{curl}; \mathbf{D})$  is as in (6.1) (the solution to Problem 2.1) and  $\mathbf{E}_{h_i} \in H_0(\mathbf{curl}; \mathbf{D}_{h_i})$  denotes the solution to Problem 5.2. fig. 3 displays the convergence of the solution to Problem 5.2 to the continuous one when using a first-order curl-conforming approximation (k = 1) together with first and second order mesh elements  $(\mathfrak{K} = 1)$  and  $\mathfrak{K} = 2$ , respectively). fig. 4, on the other hand, displays the convergence of the solution to Problem 5.2 to the continuous solution when using a second-order curl-conforming approximation (i.e. k = 1) on the same meshes as before.

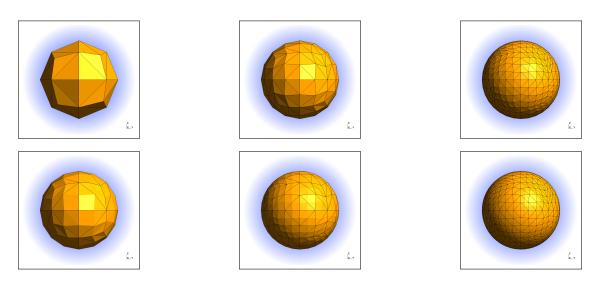


Figure 2: First three meshes of order  $\mathfrak{K}=1$ , followed by the first three meshes of order  $\mathfrak{K}=2$ .

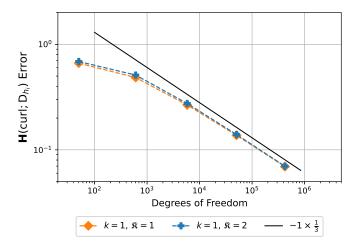


Figure 3: Error convergence in  $H(\operatorname{curl}; D_{h_i})$ -norm of solutions to Problem 5.2 with respect to that of Problem 2.1 using first order curl-conforming finite elements on straight  $(\mathfrak{K}=1)$  and curved  $(\mathfrak{K}=2)$  meshes. In both cases, we observe the expected linear behavior with respect to the mesh-size.

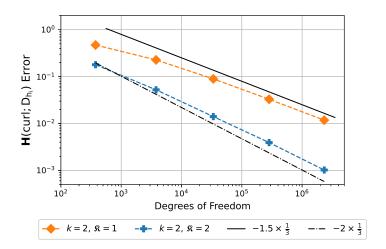


Figure 4: Error convergence in  $\mathbf{H}(\mathbf{curl}; \mathbf{D}_{h_i})$ -norm of solutions to Problem 5.2 with respect to that of Problem 2.1 using second-order curl-conforming finite elements on straight ( $\mathfrak{K}=1$ ) and curved ( $\mathfrak{K}=2$ ) meshes. A pre-asymptotic regime is observed in both cases after which the latter achieves the expected second order—with respect to  $h_i$ —convergence rate, while the former case only attains a degenerated rate of roughly order 1.5 with respect to the mesh-size due to the low-order approximation of the boundary of D.

#### 7 Conclusions

For the family of Maxwell variational problems here considered, theorems 4.18 and 4.20 provide sufficient conditions on the family of approximate domains  $\{\widetilde{D}_i\}_{i\in\mathbb{N}}$  to ensure convergence rates of: (i) pull-backs of continuous solutions in approximate domains to those in the original one; and, (ii) continuous solutions in approximate domains to smooth extensions of the exact solution. theorem 5.8 extend these results to their discrete counterparts to then include the effects of numerical integration for a fully discrete analysis in theorem 5.12, based on our previous work [4].

Our results on curved meshes established various properties of (local) interpolation on curved meshes and pull-backs  $\psi^c$  in (3.5). These correspond to lemmas A.2 and A.3 and propositions 3.6 and 3.7, which are of independent interest. Also the simple numerical examples in section 6 confirm our findings. Observe the failure of second-order polynomials to achieve second-order convergence rates to the solution on straight meshes in fig. 4.

We left out the issue of demonstrating the regularities of the electric field that are required in order to ensure different rates of convergence (cf. [1] for globally smooth boundaries). However, results ensuring arbitrary degrees of regularity in domains with corners and allowing for an application of the results as in [8], are, to the best of our knowledge, unavailable for Maxwell's equations.

Finally, we were able to consider integer degrees of regularity only in theorems 5.9 and 5.13 ( $s \in \{1:k\}$ ). This stems from the same deficiency in lemma A.2 and further improvements are left as future work as well as extensions to more specific and varied Maxwell variational problems such as problems in periodic media, FEM/BEM couplings and applications in uncertainty quantification [6, 5, 36].

## Acknowledgments

The authors would like to thank Dr. José Pinto and Dr. Fernando Henríquez for their invaluable comments on earlier versions of this manuscript.

## A Technical results concerning curl-conforming finite elements

**Lemma A.1.** Let Assumption 3.3 hold and take  $s \in \{0 : \Re + 1\}$  and  $K \in \tau_h$ . Then, for any  $\check{U} \in H^s(\check{K})$  and with  $U := \check{U} \circ T_K^{-1}$ , it holds that  $U \in H^s(K)$  and that,

$$|\breve{U}|_{s,\breve{K}} \le C \inf_{\mathbf{x} \in \breve{K}} |\det(\mathbf{d}T_K(\mathbf{x}))|^{-\frac{1}{2}} h^s ||U||_{s,K}, \tag{A.1}$$

where the constant C > 0 is independent of K and h. Analogously, for any  $U \in H^s(K)$  and with  $\check{U} := U \circ T_K$ , it holds that  $\check{U} \in H^s(\check{K})$  and

$$|U|_{s,K} \le C \sup_{\mathbf{x} \in \check{K}} |\det(\mathbf{d}T_K(\mathbf{x}))|^{\frac{1}{2}} h^{-s} ||\check{U}||_{s,\check{K}}, \tag{A.2}$$

with a positive constant C as before.

*Proof.* The statement in eq. (A.1) is nothing more than Lemma 1 in [15], while eq. (A.2) follows by replacing  $T_K^{-1}$  with  $T_K$  in Lemma 1 in [15], together—in both cases—with Assumption 3.3 and eq. (3.1). Note that Lemma 3 in [16] ensures that the constants in eqs. (A.1) and (A.2) depend only on s.

**Lemma A.2.** Let Assumption 3.3 hold. For all  $K \in \tau_h$  and all  $\mathbf{V} \in \mathbf{H}^s(\mathbf{curl}; K)$  with  $s \in \{0 : \mathfrak{K} + 1\}$ , it holds that

$$|\psi_K^c(\mathbf{V})|_{s,\check{K}} \le Ch^{s-\frac{1}{2}} \|\mathbf{V}\|_{s,K} \quad and \quad |\mathbf{curl}\,\psi_K^c(\mathbf{V})|_{s,\check{K}} \le Ch^{s+\frac{1}{2}} \|\mathbf{curl}\,\mathbf{V}\|_{s,K},$$
 (A.3)

for a positive constant C independent of  $K \in \tau_h$  and h. Also, for all  $\mathbf{V} \in \mathbf{H}^s(\mathbf{curl}; \check{K})$  it holds that

$$\left| (\psi_K^c)^{-1}(\mathbf{V}) \right|_{s,K} \leq C h^{\frac{1}{2}-s} \|\mathbf{V}\|_{s,\check{K}} \quad and \quad \left| \mathbf{curl} (\psi_K^c)^{-1}(\mathbf{V}) \right|_{s,K} \leq C h^{-(s+\frac{1}{2})} \|\mathbf{curl} \, \mathbf{V}\|_{s,\check{K}},$$

for a positive constant C as before.

*Proof.* We will prove only the estimates in eq. (A.3), as those for the inverse of the pull-back follow analogously by noticing that

$$(\psi_K^c)^{-1}(\mathbf{V}) = \mathrm{d} \boldsymbol{T}_K^{-\top}(\mathbf{V} \circ \boldsymbol{T}_K^{-1}) \quad \text{and} \quad \operatorname{\mathbf{curl}}(\psi_K^c)^{-1}(\mathbf{V}) = (\mathrm{d} \boldsymbol{T}_K^{-1})^{\operatorname{co}} \operatorname{\mathbf{curl}} \mathbf{V} \circ \boldsymbol{T}_K^{-1},$$

show casing the same structure and satisfying analogous properties, despite the different signs in the powers of h.

Take  $\mathbf{V} \in \mathbf{H}^s(K)$  for any  $K \in \tau_h$  and  $s \in \mathbb{N}_0$ . Let  $\mathbf{A}_K = (a_{ij})_{i,j=1}^3 \in \mathbf{W}^{\infty,\mathfrak{K}+1}(K;\mathbb{C}^{3\times 3})$ , be either  $d\mathbf{T}_K^{\mathsf{T}}$  or  $d\mathbf{T}_K^{\mathsf{co}}$ . By definition, it holds that

$$\left|\mathbf{A}_{K}\mathbf{V}\circ\mathbf{T}_{K}\right|_{s,\check{K}}=\left(\sum_{i=1}^{3}\left|\sum_{j=1}^{3}a_{ij}\mathbf{V}_{j}\circ\mathbf{T}_{K}\right|_{s,\check{K}}^{2}\right)^{rac{1}{2}}.$$

Moreover, for  $i \in \{1:3\}$ , by Titu's lemma [2, Sec. 1.2] we have that

$$\left| \sum_{j=1}^{3} a_{ij} \mathbf{V}_{j} \circ \mathbf{T}_{K} \right|_{s,\check{K}}^{2} \leq 3 \sum_{j=1}^{3} \left| a_{ij} \mathbf{V}_{j} \circ \mathbf{T}_{K} \right|_{s,\check{K}}^{2},$$

and, for any pair  $i, j \in \{1:3\}$ , it holds that

$$|a_{ij}\mathbf{V}_{j} \circ \mathbf{T}_{K}|_{s,\check{K}}^{2} = \sum_{|\boldsymbol{\alpha}|=s} \left\| \frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} (a_{ij}\mathbf{V}_{j} \circ \mathbf{T}_{K}) \right\|_{0,\check{K}}^{2} \leq C \sum_{\substack{|\boldsymbol{\alpha}_{1}| \leq s \\ |\boldsymbol{\alpha}_{2}|=s-|\boldsymbol{\alpha}_{1}|}} \left\| \frac{\partial^{\boldsymbol{\alpha}_{1}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{1}}} a_{ij} \frac{\partial^{\boldsymbol{\alpha}_{2}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{2}}} \mathbf{V}_{j} \circ \mathbf{T}_{K} \right\|_{0,\check{K}}^{2}$$

$$\leq C \sum_{\substack{|\boldsymbol{\alpha}_{1}| \leq s \\ |\boldsymbol{\alpha}_{2}|=s-|\boldsymbol{\alpha}_{1}|}} \left\| \frac{\partial^{\boldsymbol{\alpha}_{1}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{1}}} a_{ij} \right\|_{L^{\infty}(\check{K})}^{2} \left\| \frac{\partial^{\boldsymbol{\alpha}_{2}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{2}}} \mathbf{V}_{j} \circ \mathbf{T}_{K} \right\|_{0,\check{K}}^{2},$$

$$(A.4)$$

where the positive constant C in eq. (A.4) follows from the Cauchy-Schwarz inequality and depends only on s. For the particular choice  $\mathbf{A}_k = \mathrm{d} \mathbf{T}_K^{\top}$ , eq. (3.2) ensures the existence of a uniform constant C > 0, independent of K and the mesh-size, such that

$$\sum_{\substack{|\boldsymbol{\alpha}_{1}| \leq s \\ |\boldsymbol{\alpha}_{2}| = s - |\boldsymbol{\alpha}_{1}|}} \left\| \frac{\partial^{\boldsymbol{\alpha}_{1}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{1}}} a_{ij} \right\|_{L^{\infty}(\tilde{K})}^{2} \left\| \frac{\partial^{\boldsymbol{\alpha}_{2}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{2}}} \mathbf{V}_{j} \circ \mathbf{T}_{K} \right\|_{0,\tilde{K}}^{2} \leq C \sum_{\substack{|\boldsymbol{\alpha}_{1}| \leq s \\ |\boldsymbol{\alpha}_{2}| = s - |\boldsymbol{\alpha}_{1}|}} h^{2(|\boldsymbol{\alpha}_{1}|+1)} \left\| \frac{\partial^{\boldsymbol{\alpha}_{2}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{2}}} \mathbf{V}_{j} \circ \mathbf{T}_{K} \right\|_{0,\tilde{K}}^{2} \\
= C \sum_{m=0}^{s} h^{2m+2} \left| \mathbf{V}_{j} \circ \mathbf{T}_{K} \right|_{s-m,\tilde{K}}^{2} \leq C \left( \inf_{\mathbf{x} \in \tilde{K}} \det(\mathbf{d}\mathbf{T}_{K}(\mathbf{x})) \right)^{-1} h^{2s+2} \|\mathbf{V}_{j}\|_{s,K}^{2},$$

where C now includes the constant in lemma A.1. The estimate for  $\psi_K^c(\mathbf{V})$  is retrieved by the bound in eq. (3.3).

For  $\mathbf{A}_K = \mathrm{d} T_K^{\mathsf{co}}$ , we proceed analogously

$$\sum_{\substack{|\boldsymbol{\alpha}_{1}| \leq s \\ |\boldsymbol{\alpha}_{2}| = s - |\boldsymbol{\alpha}_{1}|}} \left\| \frac{\partial^{\boldsymbol{\alpha}_{1}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{1}}} a_{ij} \right\|_{L^{\infty}(\tilde{K})}^{2} \left\| \frac{\partial^{\boldsymbol{\alpha}_{2}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{2}}} \mathbf{V}_{j} \circ \boldsymbol{T}_{K} \right\|_{0,\tilde{K}}^{2} \leq C \sum_{\substack{|\boldsymbol{\alpha}_{1}| \leq s \\ |\boldsymbol{\alpha}_{2}| = s - |\boldsymbol{\alpha}_{1}|}} h^{2(|\boldsymbol{\alpha}_{1}|+2)} \left\| \frac{\partial^{\boldsymbol{\alpha}_{2}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}_{2}}} \mathbf{V}_{j} \circ \boldsymbol{T}_{K} \right\|_{0,\tilde{K}}^{2} \tag{A.5}$$

$$= C \sum_{m=0}^{s} h^{2m+4} \left| \mathbf{V}_{j} \circ \boldsymbol{T}_{K} \right|_{s-m,\tilde{K}}^{2} \leq C \left( \inf_{\mathbf{x} \in \tilde{K}} \det(\mathbf{d}\boldsymbol{T}_{K}) \right)^{-1} h^{2s+4} \|\mathbf{V}_{j}\|_{s,K}^{2},$$

where eq. (A.5) is deduced from eq. (3.2) and the cofactor matrix definition. The bound for  $\operatorname{\mathbf{curl}} \psi_K^c(\mathbf{V})$  then follows as before.

**Lemma A.3.** Let Assumption 3.3 hold and let  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; \check{K})$  for  $s \in \{1:k\}$ . Then, for any  $l \in \{1:s\}$  it holds that,

$$\|\mathbf{U} - \breve{r}(\mathbf{U})\|_{l,\breve{K}} \leq C \left( |\mathbf{U}|_{s,\breve{K}} + |\mathbf{curl}\,\mathbf{U}|_{s,\breve{K}} \right) \quad and \quad \|\mathbf{curl}\,\mathbf{U} - \mathbf{curl}\,\breve{r}(\mathbf{U})\|_{l,\breve{K}} \leq C \left| \mathbf{curl}\,\mathbf{U}|_{s,\breve{K}} ,$$

for a positive constant C > 0 independent of  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; \breve{K})$ .

*Proof.* Take  $s \in \{1:k\}$  and  $l \in \{1:s\}$ . We will prove only the estimate

$$\left\|\mathbf{U} - \check{r}(\mathbf{U})\right\|_{l,\check{K}} \le C\left(\left|\mathbf{U}\right|_{s,\check{K}} + \left|\mathbf{curl}\,\mathbf{U}\right|_{s,\check{K}}\right),$$

since the remaining estimate follows by analogous arguments. Let  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; \check{K})$  and take any  $\phi \in \mathbb{P}_{k-1}(\check{K}; \mathbb{C}^3)$ . Then, by the invariance of the canonical interpolation operator, we have that

$$\|\mathbf{U} - \check{r}(\mathbf{U})\|_{l,\check{K}} = \|(\mathbf{I} - \check{r})(\mathbf{U} + \phi)\|_{l,\check{K}} \le \|\mathbf{U} + \phi\|_{l,\check{K}} + \|\check{r}(\mathbf{U} + \phi)\|_{l,\check{K}}. \tag{A.6}$$

Lemma 5.38 in [30] then allows us to bound the degrees of freedom of  $(U + \phi)$  through the  $\mathbf{H}^s(\mathbf{curl}; \check{K})$ -norm (since  $s \geq 1$ ). Specifically, we have that

$$\|\check{r}(\mathbf{U}+\boldsymbol{\phi})\|_{l,\check{K}} \le C\left(\|\mathbf{U}+\boldsymbol{\phi}\|_{s,\check{K}} + \|\mathbf{curl}(\mathbf{U}+\boldsymbol{\phi})\|_{s,\check{K}}\right),$$
 (A.7)

where C may depend on  $l \in \{1:s\}$  and  $s \in \{1:k\}$ , but is independent of **U** and  $\phi$ . A combination of eqs. (A.6) and (A.7), together with the fact that  $l \leq s$ , yields

$$\|\mathbf{U} - \check{r}(\mathbf{U})\|_{l,\check{K}} \le C \left( \|\mathbf{U} + \boldsymbol{\phi}\|_{s,\check{K}} + \|\mathbf{curl}(\mathbf{U} + \boldsymbol{\phi})\|_{s,\check{K}} \right), \tag{A.8}$$

where C is not necessarily the same as before, but is still independent of  $\mathbf{U}$  and  $\phi$ . Since eq. (A.8) holds for any  $\phi \in \mathbb{P}_{k-1}(\check{K}; \mathbb{C}^3)$  we may take the infimum of its right-hand side over  $\mathbb{P}_{k-1}(\check{K}; \mathbb{C}^3)$ , which we may then bound by [30, Thm. 5.5], to obtain

$$\|\mathbf{U} - \breve{r}(\mathbf{U})\|_{l, \breve{K}} \le C \left( |\mathbf{U}|_{s, \breve{K}} + |\mathbf{curl} \, \mathbf{U}|_{s, \breve{K}} \right),$$

for a positive constant C as before.

The estimate

$$\|\operatorname{curl} \mathbf{U} - \operatorname{curl} \breve{r}(\mathbf{U})\|_{l,\breve{K}} \leq C |\operatorname{curl} \mathbf{U}|_{s,\breve{K}},$$

follows by analogous arguments employing Lemma 5.40 in [30] and [30, Lem. 5.15] in lieu of [30, Lem. 5.38].  $\Box$ 

## B Proofs of propositions 3.6 and 3.7

Proof of proposition 3.6. For  $U \in H^s(\mathbf{curl}; K)$ , we first estimate the  $L^2$ -portion of the norm:

$$\|\mathbf{U} - \mathbf{r}_K(\mathbf{U})\|_{0,K} \le \sup_{\mathbf{x} \in \check{K}} \det(d\mathbf{T}_K(\mathbf{x}))^{\frac{1}{2}} \sup_{\mathbf{x} \in \check{K}} \|d\mathbf{T}^{-1}(\mathbf{x})\| \|\psi_K^c(\mathbf{U}) - \check{\mathbf{r}}(\psi_K^c(\mathbf{U}))\|_{0,\check{K}}. \tag{B.1}$$

By Lemma A.3, one has

$$\left\|\psi_K^c(\mathbf{U}) - \check{r}(\psi_K^c(\mathbf{U}))\right\|_{0,\check{K}} \le c\left(\left|\psi_K^c(\mathbf{U})\right|_{s,\check{K}} + \left|\operatorname{\mathbf{curl}}\psi_K^c(\mathbf{U})\right|_{s,\check{K}}\right),$$

where c is a positive constant independent of  $K \in \tau_h$  and h. By Lemma A.2, it holds that

$$\left\|\psi_K^c(\mathbf{U}) - \check{r}(\psi_K^c(\mathbf{U}))\right\|_{0,\check{K}} \leq c \left(h^{s-\frac{1}{2}}\|\mathbf{U}\|_{s,K} + h^{s+\frac{1}{2}}\|\mathbf{curl}\,\mathbf{U}\|_{s,K}\right),$$

and combining the last equation with eqs. (3.1), (3.3), and (B.1) yields the estimate

$$\|\mathbf{U} - \mathbf{r}_K(\mathbf{U})\|_{0,K} \le c \left(h^s \|\mathbf{U}\|_{s,K} + h^{s+1} \|\mathbf{curl} \,\mathbf{U}\|_{s,K}\right) \le ch^s \|\mathbf{U}\|_{\mathbf{H}^s(\mathbf{curl};K)},$$

where the positive constant c is as before. We continue with the estimate for the curl. Proceeding as before, we have that

$$\begin{aligned} \|\mathbf{curl}\,\mathbf{U} - \mathbf{curl}\,\mathbf{r}_K(\mathbf{U})\|_{0,K} &\leq \sup_{\mathbf{x} \in \check{K}} \det(\mathrm{d}\mathbf{T}_K(\mathbf{x}))^{-\frac{1}{2}} \sup_{\mathbf{x} \in \check{K}} \|\mathrm{d}\mathbf{T}(\mathbf{x})\| \|\mathbf{curl}\,\psi_K^c(\mathbf{U}) - \mathbf{curl}\,\check{\mathbf{r}}(\psi_K^c(\mathbf{U}))\|_{0,K} \\ &\leq c \sup_{\mathbf{x} \in \check{K}} \det(\mathrm{d}\mathbf{T}_K(\mathbf{x}))^{-\frac{1}{2}} \sup_{\mathbf{x} \in \check{K}} \|\mathrm{d}\mathbf{T}(\mathbf{x})\| \, |\mathbf{curl}\,\psi_K^c(\mathbf{U})|_{s,\check{K}} \,, \end{aligned}$$

where the last inequality follows from Lemma A.3. Lemma A.2, together with eq. (3.1) and eq. (3.3), leads to

$$\|\mathbf{curl}\,\mathbf{U} - \mathbf{curl}\,\mathbf{r}_K(\mathbf{U})\|_{0,K} \le ch^s \|\mathbf{d}\mathbf{T}(\mathbf{x})\| \|\mathbf{curl}\,\mathbf{U}\|_{s,K}$$

The combination of the  $L^2$ - and curl-estimates yield the approximation result.

Proof of proposition 3.7. Take  $\mathbf{U} \in \mathbf{H}^s(\mathbf{curl}; K)$ . Then, it holds that

$$\|\boldsymbol{r}_K(\mathbf{U})\|_{\boldsymbol{H}^s(\mathbf{curl};K)} \leq \|\boldsymbol{r}_K(\mathbf{U}) - \mathbf{U}\|_{\boldsymbol{H}^s(\mathbf{curl};K)} + \|\mathbf{U}\|_{\boldsymbol{H}^s(\mathbf{curl};K)}.$$

Moreover,

$$\|\boldsymbol{r}_{K}(\mathbf{U}) - \mathbf{U}\|_{\boldsymbol{H}^{s}(\mathbf{curl};K)} = \|(\psi_{K}^{c})^{-1}(\check{\boldsymbol{r}}(\psi_{K}^{c}(\mathbf{U})) - \psi_{K}^{c}(\mathbf{U}))\|_{\boldsymbol{H}^{s}(\mathbf{curl};K)}. \tag{B.2}$$

From Lemma A.2, for  $l \in \{0: s\}$ , it follows that

$$\left| \left( \psi_K^c \right)^{-1} (\breve{\boldsymbol{r}}(\psi_K^c(\mathbf{U})) - \psi_K^c(\mathbf{U}) \right) \right|_{l,K} \le c h^{\frac{1}{2} - l} \| \breve{\boldsymbol{r}}(\psi_K^c(\mathbf{U})) - \psi_K^c(\mathbf{U}) \|_{l,\check{K}},$$

and a sequential application of Lemmas A.2 and A.3 yields

$$\| \check{r}(\psi_K^c(\mathbf{U})) - \psi_K^c(\mathbf{U}) \|_{l,\check{K}} \leq c \left( h^{s-\frac{1}{2}} \| \mathbf{U} \|_{s,K} + h^{s+\frac{1}{2}} \| \mathbf{curl} \, \mathbf{U} \|_{s,K} \right),$$

so that

$$\|\mathbf{r}_K(\mathbf{U}) - \mathbf{U}\|_{s,K} \le c \|\mathbf{U}\|_{\mathbf{H}^s(\mathbf{curl};K)}. \tag{B.3}$$

We may proceed analogously and bound the curl portion of the norm as

$$\|\operatorname{curl} \mathbf{r}_K(\mathbf{U}) - \operatorname{curl} \mathbf{U}\|_{\mathbf{H}^s(K)} \le c \|\operatorname{curl} \mathbf{U}\|_{s,K}.$$
 (B.4)

Then, combining the results in eqs. (B.2) to (B.4) completes the proof.

## C Proofs of lemmas 4.14 and 4.15

Proof of lemma 4.14. Take U as a smooth function in  $\Upsilon$ , i.e.  $u \in \mathcal{C}^{\infty}(\Upsilon)$ . Then, for all  $x \in \Omega$ , it holds that

$$U \circ \mathbf{T}(\mathbf{x}) - U(\mathbf{x}) = \int_0^1 \nabla U((1-t)\mathbf{x} + t\mathbf{T}(\mathbf{x})) \cdot (\mathbf{T}(\mathbf{x}) - \mathbf{x}) \, dt.$$

Observe that  $(1-t)\mathbf{x} + t\mathbf{T}(\mathbf{x}) \in \Upsilon$ , for all  $t \in [0,1]$ , and define  $\mathbf{T}_t(\mathbf{x}) := (1-t)\mathbf{x} + t\mathbf{T}(\mathbf{x})$ . Then, the convexity of  $\Upsilon$  implies  $\mathbf{T}_t(\mathbf{x}) \in D_H$  for all  $\mathbf{x} \in D_H$ . Moreover, one has

$$|U \circ \mathbf{T}(\mathbf{x}) - U(\mathbf{x})| = \left| \int_0^1 \nabla U(\mathbf{T}_t(\mathbf{x})) \cdot (\mathbf{T}(\mathbf{x}) - \mathbf{x}) \, \mathrm{d}t \right|$$

$$\leq \|\mathbf{T} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\Upsilon)} \int_0^1 \|\nabla U(\mathbf{T}_t(\mathbf{x}))\|_{\mathbf{L}^{\infty}(\Omega)} \, \mathrm{d}t \leq \|\mathbf{T} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\Upsilon)} \|U\|_{W^{1,\infty}(\Upsilon)}.$$

The statement follows by taking the supremum over  $\mathbf{x} \in \Omega$  and by density of  $\mathcal{C}^{\infty}(\Upsilon)$  in  $W^{1,\infty}(\Upsilon)$ .

Proof of lemma 4.15. We start by proving the statement for s=0. Set **T** as required, then for any  $\mathbf{U} \in L^2(\Upsilon)$  it holds that

$$\|\mathbf{U}\circ\mathbf{T}-\mathbf{U}\|_{0,\Omega}\leq\|\mathbf{U}\|_{0,\Omega}+\|\mathbf{U}\circ\mathbf{T}\|_{0,\Omega}\leq\|\mathbf{U}\|_{0,\Omega}+\vartheta^{\frac{1}{2}}\|\mathbf{U}\|_{0,\mathbf{T}(\Omega)}\leq(\vartheta^{\frac{1}{2}}+1)\|\mathbf{U}\|_{0,\Upsilon}.$$

Now we prove for s = 1. Take **U** as a smooth function in  $\Upsilon$ , i.e.  $\mathbf{U} \in \mathcal{C}^{\infty}(\Upsilon)$ . Then, for all  $\mathbf{x} \in \Omega$  one has

$$\mathbf{U} \circ \mathbf{T}(\mathbf{x}) - \mathbf{U}(\mathbf{x}) = \int_0^1 d\mathbf{U}((1-t)\mathbf{x} + t\mathbf{T}(\mathbf{x})) \cdot (\mathbf{T}(\mathbf{x}) - \mathbf{x}) dt.$$

Note that  $(1-t)\mathbf{x} + t\mathbf{T}(\mathbf{x}) \in \Upsilon$  for all  $t \in [0,1]$  and  $\mathbf{T}_t(\mathbf{x}) := (1-t)\mathbf{x} + t\mathbf{T}(\mathbf{x})$  satisfies all the conditions in eq. (4.6). In particular, we have that

$$\|\mathbf{x} - \mathbf{y}\|_{\mathbb{R}^3} \le \|\mathbf{T}_t(\mathbf{x}) - \mathbf{T}_t(\mathbf{y})\|_{\mathbb{R}^3} + t\|(\mathbf{T}(\mathbf{x}) - \mathbf{x}) - (\mathbf{T}(\mathbf{y}) - \mathbf{y})\|_{\mathbb{R}^3}$$
$$\le \|\mathbf{T}_t(\mathbf{x}) - \mathbf{T}_t(\mathbf{y})\|_{\mathbb{R}^3} + t\kappa \|\mathbf{x} - \mathbf{y}\|_{\mathbb{R}^3},$$

implying the invertibility of  $\mathbf{T}_t: \Omega \to \mathbf{T}_t(\Omega)$ . Moreover,

$$\begin{aligned} &\|\mathbf{U} \circ \mathbf{T} - \mathbf{U}\|_{0,\Omega}^{2} = \int_{\Omega} \|\mathbf{U} \circ \mathbf{T}(\mathbf{x}) - \mathbf{U}(\mathbf{x})\|_{\mathbb{R}^{3}}^{2} \, \mathrm{d}\mathbf{x} = \int_{\Omega} \left\| \int_{0}^{1} \mathrm{d}\mathbf{U}(\mathbf{T}_{t}(\mathbf{x})) \cdot (\mathbf{T}(\mathbf{x}) - \mathbf{x}) \, \mathrm{d}t \right\|_{\mathbb{R}^{3}}^{2} \, \mathrm{d}\mathbf{x} \\ &\leq \int_{\Omega} \int_{0}^{1} \|\mathrm{d}\mathbf{U}(\mathbf{T}_{t}(\mathbf{x})) \cdot (\mathbf{T}(\mathbf{x}) - \mathbf{x})\|_{\mathbb{R}^{3}}^{2} \, \mathrm{d}t \, \mathrm{d}\mathbf{x} \quad \text{(Jensen's inequality)} \\ &\leq \|\mathbf{T} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\Upsilon)}^{2} \int_{\Omega} \int_{0}^{1} \|\mathrm{d}\mathbf{U}(\mathbf{T}_{t}(\mathbf{x}))\|_{\mathbb{R}^{3\times 3}}^{2} \, \mathrm{d}t \, \mathrm{d}\mathbf{x} \\ &= \|\mathbf{T} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\Upsilon)}^{2} \int_{0}^{1} \int_{\mathbf{T}_{t}(\Omega)} \|\mathrm{d}\mathbf{U}(\mathbf{x})\|_{\mathbb{R}^{3\times 3}}^{2} \, \mathrm{det}(\mathrm{d}\mathbf{T}_{t}(\mathbf{x}))^{-1} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq \vartheta \|\mathbf{T} - \mathbf{I}\|_{\mathbf{L}^{\infty}(\Upsilon)}^{2} \|\mathbf{U}\|_{1,\Upsilon}^{2}, \end{aligned}$$

so that

$$\|\mathbf{U} \circ \mathbf{T} - \mathbf{U}\|_{0,\Omega} \le \vartheta^{\frac{1}{2}} \|\mathbf{T} - \mathbf{I}\|_{L^{\infty}(\Upsilon)} \|\mathbf{U}\|_{1,\Upsilon} \le (\vartheta^{\frac{1}{2}} + 1) \|\mathbf{T} - \mathbf{I}\|_{L^{\infty}(\Upsilon)} \|\mathbf{U}\|_{1,\Upsilon}.$$

The statement for s=1 follows by density of  $\mathcal{C}^{\infty}(\Upsilon)$  in  $H^1(\Upsilon)$ . The result for real  $s \in (0,1)$  follows by applying real interpolation in Sobolev spaces (cf. [39, Lem. 22.3]).

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