Ramsey Numbers<br>for<br>Non-trivial Berge Cycles<br>Jiaxi Nie* Jacques Verstraëte ${ }^{\dagger}$<br>Department of Mathematics<br>University of California, San Diego<br>9500 Gilman Drive<br>La Jolla CA 92093-0112.

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#### Abstract

In this paper, we consider an extension of cycle-complete graph Ramsey numbers to Berge cycles in hypergraphs: for $k \geq 2$, a non-trivial Berge $k$-cycle is a family of sets $e_{1}, e_{2}, \ldots, e_{k}$ such that $e_{1} \cap e_{2}, e_{2} \cap e_{3}, \ldots, e_{k} \cap e_{1}$ has a system of distinct representatives and $e_{1} \cap e_{2} \cap \cdots \cap e_{k}=\emptyset$. In the case that all the sets $e_{i}$ have size three, let $\mathcal{B}_{k}$ denotes the family of all non-trivial Berge $k$-cycles. The Ramsey numbers $R\left(t, \mathcal{B}_{k}\right)$ denote the minimum $n$ such that every $n$-vertex 3 -uniform hypergraph contains either a non-trivial Berge $k$-cycle or an independent set of size $t$. We prove


$$
R\left(t, \mathcal{B}_{2 k}\right) \leq t^{1+\frac{1}{2 k-1}+\frac{4}{\sqrt{\operatorname{Tog} t}}}
$$

and moreover, we show that if a conjecture of Erdős and Simonovits [12] on girth in graphs is true, then this is tight up to a factor $t^{o(1)}$ as $t \rightarrow \infty$.

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## 1 Introduction

Let $\mathcal{F}$ be a family of $r$-graphs and $t \geq 1$. The Ramsey numbers $R(t, \mathcal{F})$ denote the minimum $n$ such that every $n$-vertex $r$-graph contains either a hypergraph in $\mathcal{F}$ or an independent set of size $t$. For $k \geq 2$, a Berge $k$-cycle is a family of sets $e_{1}, e_{2}, \ldots, e_{k}$ such that $e_{1} \cap e_{2}, e_{2} \cap$ $e_{3}, \ldots, e_{k} \cap e_{1}$ has a system of distinct representatives, and a Berge cycle is non-trivial if $e_{1} \cap e_{2} \cap \cdots \cap e_{k}=\emptyset$. Let $\mathcal{B}_{k}^{r}$ denote the family of non-trivial Berge $k$-cycles all of whose sets have size $r$. When $r=2, \mathcal{B}_{k}^{2}=\left\{C_{k}\right\}$, where $C_{k}$ denotes the graph cycle of length $k$. In this paper, we let $\mathcal{B}_{k}=\mathcal{B}_{k}^{3}$.

It is a notoriously difficult problem to determine even the order of magnitude of $R\left(t, C_{k}\right)$ the cycle-complete graph Ramsey numbers. Kim [18] proved $R\left(t, C_{3}\right)=\Omega\left(t^{2} / \log t\right)$, which gives the order of magnitude of $R\left(t, C_{3}\right)$ when combined with the results of Ajtai, Komlós and Szemerédi [2] and Shearer [30]. The current state-of-the-art results on $R\left(t, C_{3}\right)$ are due to Fiz Pontiveros, Griffiths and Morris [13] and Bohman and Keevash [6], using the random triangle-free process, which determines $R\left(t, C_{3}\right)$ up to a small constant factor.

$$
\left(\frac{1}{4}-o(1)\right) \frac{t^{2}}{\log t} \leq R\left(t, C_{3}\right) \leq(1+o(1)) \frac{t^{2}}{\log t}
$$

The case $R\left(t, C_{4}\right)$ is the subject of a notorious conjecture of Erdős [7], where he conjectured that $R\left(t, C_{4}\right)=o\left(t^{2-\epsilon}\right)$ for some $\epsilon>0$. The current best upper bounds on $R\left(t, C_{2 k}\right)$ is

$$
O\left(\left(\frac{t}{\log t}\right)^{k /(k-1)}\right)
$$

which come from the work of Caro, Li, Rousseau and Zhang [9]. For $R\left(t, C_{2 k+1}\right)$, the best upper bound is

$$
O\left(\frac{t^{(k+1) / k}}{\log ^{1 / k} t}\right)
$$

due to Sudakov [31. Recent results using pseudorandom graphs by Mubayi and the second author [26] give the best lower bounds on cycle-complete graph Ramsey numbers:

$$
R\left(C_{k}, n\right)=\Omega\left(\frac{t^{(k-1) /(k-2)}}{\log ^{2 /(k-2)} t}\right) .
$$

In particular, via random block constructions, they show that

$$
R\left(C_{5}, t\right) \geq(1+o(1)) t^{11 / 8}, \quad R\left(C_{7}, t\right) \geq(1+o(1)) t^{11 / 9}
$$

For $k \geq 3$, a loose $k$-cycle is a non-trivial Berge $k$-cycle, denoted $C_{k}^{r}$, with sets $e_{1}, e_{2}, \ldots, e_{k}$ of size $r$ such that $\left|e_{1} \cap e_{2}\right|=1,\left|e_{2} \cap e_{3}\right|=1, \ldots,\left|e_{k} \cap e_{1}\right|=1$, and for any other pairs of edges $e_{i}, e_{j}, e_{i} \cap e_{j}=\emptyset$. Ramsey type problems for loose cycles in $r$-graphs have been studied extensively [4, 10, 11, 14, 16 $20,24,26]$. For $r$-uniform hypergraphs with $r \geq 3$, Kostochka, Mubayi and the second author [19] proved for all $r \geq 3$, there exist constants $a, b>0$ such that

$$
\begin{equation*}
\frac{a t^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \leq R\left(t, C_{3}^{r}\right) \leq b t^{\frac{3}{2}} \tag{1}
\end{equation*}
$$

The following conjecture was proposed in [19]:
Conjecture I. For $r, k \geq 3$,

$$
\begin{equation*}
R\left(t, C_{k}^{r}\right)=t^{\frac{k}{k-1}+o(1)} \tag{2}
\end{equation*}
$$

The conjecture is true for $k=3$ due to (11). It is shown in [28] that $R\left(t, C_{4}^{3}\right) \leq t^{4 / 3+o(1)}$. Méroueh [24] showed $R\left(t, C_{k}^{3}\right)=O\left(t^{1+1 /\lfloor(k+1) / 2\rfloor}\right)$ for $k \geq 3$ and $R\left(t, C_{k}^{r}\right)=O\left(t^{1+1 /\lfloor k / 2\rfloor}\right)$ for $r \geq 4$ and every odd integers $k \geq 5$, improving earlier results of Collier-Cartaino, Graber and Jiang [10]. Conjecture $\square$ motivates our current study of non-trivial Berge $k$-cycles. In support of the above conjecture, we prove the following result for non-trivial Berge cycles of even length:

Theorem 1. For $k \geq 3$, and $t$ large enough,

$$
R\left(t, \mathcal{B}_{2 k}\right) \leq t^{\frac{2 k}{2 k-1}+\frac{4}{\sqrt{\log t}}} .
$$

Erdős and Simonovits [12] conjectured that there exists an $n$-vertex graph of girth more than $2 k$ with $\Theta\left(n^{1+1 / k}\right)$ edges. This notoriously difficult conjecture remains open, except when $k \in\{2,3,5\}$, largely due to the existence of generalized polygons [3, 32, 33]. Towards this conjecture, Lazebnik, Ustimenko and Woldar [22] gave the densest known construction, which has $\Omega\left(n^{1+2 /(3 k-2)}\right)$ edges. We prove the following theorem relating this conjecture to lower bounds on Ramsey numbers for non-trivial Berge cycles:

Theorem 2. Let $k \geq 2, r \geq 3$. Suppose there exists an $n$-vertex graph of girth more than $2 k$ with $c n^{1+1 / k}$ edges for any integer $n$ large enough and some positive constant $c$. Then for $t$ large enough and some positive constant $c_{k, r}$ dependent on $k$ and $r$,

$$
\begin{equation*}
R\left(t, \mathcal{B}_{k}^{r}\right) \geq c_{k, r}\left(\frac{t}{\log t}\right)^{\frac{k}{k-1}} . \tag{3}
\end{equation*}
$$

This shows that if the Erdős-Simonovits Conjecture is true, then Theorem 1 is tight up to a
$t^{o(1)}$ factor. Indeed, following the proof of Theorem 2, the known construction of Lazebnik, Ustimenko and Woldar [22] would give a weaker lower bound of $\Omega\left((t / \log t)^{(3 k-2) /(3 k-4)}\right)$.

Let $B_{k}$ be the family of 3 -uniform Berge $k$-cycles without non-triviality. Random graphs together with the Lovász local lemma give $R\left(t, B_{k}\right) \geq t^{(2 k-2) /(2 k-3)-o(1)}$, see [1] for similar computation. We prove the following theorem, which gives a substantially better lower bound for $B_{4}$ if the Erdős-Simonovits Conjecture is true.

Theorem 3. Suppose there exists an n-vertex graph of girth more than 8 with $c_{1} n^{5 / 4}$ edges for any integer $n$ large enough and some positive constant $c_{1}$. Then for $t$ large enough and some positive constant $c_{2}$,

$$
R\left(t, B_{4}\right) \geq\left(\frac{c_{2} t}{\sqrt{\log t}}\right)^{16 / 13}
$$

In fact, this is also a lower bound for $R\left(t,\left\{B_{2}, B_{3}, B_{4}\right\}\right)$. A natural 3-uniform analog of the Erdős-Simovits conjecture is that there exist $n$-vertex $\left\{B_{2}, B_{3}, \ldots, B_{k}\right\}$-free 3 -graphs with $n^{1+1 /\lfloor k / 2\rfloor-o(1)}$ edges. This is true for $k=3$ due to Ruzsa and Szemeredi [29]. The proof of Theorem 3 makes use of the fact that there exist $n$-vertex $\left\{B_{2}, B_{3}, B_{4}\right\}$-free 3 -graphs with $\Omega\left(n^{3 / 2}\right)$ edges, that is, the conjecture is true for $k=4$, which is due to Lazebnik and the second author [23]. More generally, following the proof of Theorem 3, if the 3-uniform analog of the Erdős-Simonovits Conjecture is true, then we have $R\left(t,\left\{B_{2}, B_{3}, \ldots, B_{2 k}\right\}\right) \geq$ $t^{2 k^{2} /\left(2 k^{2}-k-2\right)-o(1)}$ and $R\left(t,\left\{B_{2}, B_{3}, \ldots, B_{2 k+1}\right\}\right) \geq t^{2 k(k-1) /\left(2 k^{2}-3 k-1\right)-o(1)}$, which are substantially better than the lower bounds obtained by random graphs.

We prove Theorem 11 in Section 5. Theorem 2 in Section 2 and Theorem 3 in Section 3 , Theorem 2 is valid for all values of $k \geq 2$ and $r \geq 3$, while Theorem 1 only works for even values of $k$ and $r=3$. We believe that Theorem 1 should extend to odd values of $k$ and all $r \geq 3$ :

Conjecture II. For all $r, k \geq 3$,

$$
\begin{equation*}
R\left(t, \mathcal{B}_{k}^{r}\right) \leq t^{\frac{k}{k-1}+o(1)} . \tag{4}
\end{equation*}
$$

Notation and terminology. For a hypergraph $H$, let $V(H)$ denote the vertex set of $H, v(H)=|V(H)|$ and let $|H|$ be the number of edges in $H$. If all edges of $H$ have size $r$, we say $H$ is an $r$-uniform hypergraph, or an $r$-graph for short. For $v \in V(H)$, let $d_{H}(v)=|\{e \in H: v \in e\}|$ be the degree of $v$ in $H$. We denote the average degree of $H$ by $d(H)$, denote the minimum degree of $H$ by $\delta(H)$, and the maximum degree of $H$ by $\Delta(H)$. For $u, v \in V(H)$, let $d_{H}(u, v)=|\{w: u v w \in H\}|$ denote the codegree of the pair $\{u, v\}$. An
independent set in a hypergraph is a set of vertices containing no edge of the hypergraph. Let $\alpha(H)$ denote the largest size of an independent set in a hypergraph $H$.

## 2 Proof of Theorem 2

We will use the following lemma to get a large bipartite subgraph with large minimum degree and small maximum degree:

Lemma 4. Let $k \geq 3, c>0$, and let $G$ be an n-vertex graph of girth more than $2 k$ with more than $2 c n^{1+1 / k}$ edges. Then there exists a bipartite subgraph $G^{\prime}$ of $G$ such that $\delta\left(G^{\prime}\right) \geq c n^{1 / k}$, $\Delta\left(G^{\prime}\right) \leq n^{1 / k} / c^{k-1}$, and $v\left(G^{\prime}\right) \geq c^{k} n$.

Proof. A maximum cut of $G$ gives a bipartite subgraph with at least $c n^{1+1 / k}$ edges. A subgraph $G^{\prime}$ of this bipartite subgraph of minimum degree at least $\mathrm{cn}{ }^{1 / k}+1$ may be obtained by repeatedly removing vertices of degree at most $c n^{1 / k}$. Let $\Delta:=\Delta\left(G^{\prime}\right)$ be the maximum degree of $G^{\prime}$, and let $v$ be a vertex of maximum degree, then the number of vertices at distance $k$ from $v$ is at least $\Delta c^{k-1} n^{(k-1) / k}$, since $G$ has girth larger than $2 k$. In particular, $\Delta c^{k-1} n^{(k-1) / k} \leq n$ and so $\Delta \leq n^{1 / k} / c^{k-1}$. The number of vertices in $G^{\prime}$ is at least $c^{k} n$, since $G^{\prime}$ has minimum degree at least $c n^{1 / k}+1$ and girth larger than $2 k$.

Let $r \geq 2$, a star with vertex set $V$ is an $r$-graph on $V$ consisting of all edges containing a fixed vertex of $V$, i.e., the edge set of a star is $\{e \subset V:|e|=r, v \in e\}$ for some vertex $v \in V$. Let integers $d \geq m$ and let $S_{d, m}$ be a $d$-vertex $r$-graph consisting of $m$ vertex-disjoint stars of size $\lfloor d / m\rfloor$ or $\lceil d / m\rceil$.

Lemma 5. Let integer $r \geq 2$, and let integers $d \geq m$. The probability that a uniformly chosen set of $s$ vertices of $S_{d, m}$ is independent is at most

$$
\exp \left(-\frac{m(s-r m)}{2 d}\right)
$$

Proof. Let the vertex sets of these stars be $V_{1}, V_{2}, \ldots, V_{m}$. The probability that a uniformly chosen set of $s_{i}$ vertices in $V_{i}$ is independent in $S_{d, m}$ is at most $1-s_{i} /\lceil d / m\rceil \leq 1-m s_{i} / 2 d$ if $s_{i} \geq r$, and is 1 if $s_{i}<r$. Hence, this probability is at most $1-m\left(s_{i}-r\right) / 2 d$ for $0 \leq s_{i} \leq d$. Therefore a uniformly chosen set $I \subset S_{d, m}$ of $s$ vertices with $\left|I \cap V_{i}\right|=s_{i}$ is independent with probability at most

$$
\prod_{i=1}^{m}\left(1-\frac{m\left(s_{i}-r\right)}{2 d}\right) \leq \exp \left(-\sum_{i=1}^{m} \frac{m\left(s_{i}-r\right)}{2 d}\right)=\exp \left(-\frac{m(s-r m)}{2 d}\right)
$$

Now we are ready to prove Theorem 2,

Proof of Theorem 园. It suffices to show that for $n$ large enough, there exists an $n$-vertex $\mathcal{B}_{k}^{r}$-free $r$-graph with independence number $O\left(n^{1-\frac{1}{k}} \log n\right)$. Let $G$ be an $n$-vertex graph of girth more than $2 k$ with $2 c n^{1+1 / k}$ edges for some positive constant $c$. By Lemma 4, there exists a bipartite subgraph $G^{\prime}$ of $G$ with at least $N=c^{k} n$ vertices, minimum degree at least $c n^{1 / k}$ and maximum degree at most $n^{1 / k} / c^{k-1}$. Let $X, Y$ be the parts of this bipartite graph where $|Y| \geq|X|$. Let $m=8 \log n / c^{k}$. We form an $r$-graph $H$ with vertex set $Y$ by placing a random copy of $S_{d(x), m}$ on the vertex set $N_{G^{\prime}}(x)$, the neighborhood of $x$ in $G^{\prime}$, independently for each $x \in X$. Since $G^{\prime}$ has girth more than $2 k$, it is straightforward to check that $H$ does not contain any non-trivial Berge $k$-cycle. We now compute the expected number of independent sets of size $t=r m n^{1-1 / k} / c^{k+1}$ in $H$. Clearly, $\log t \geq(1-1 / k) \log n$. If $H$ has no independent set of size $t$ with positive probability, then since $v(H) \geq N / 2$, we find that

$$
R\left(t, \mathcal{B}_{k}^{r}\right) \geq N / 2 \geq \frac{c^{k}}{2}\left(\frac{c^{2 k+1} t}{8 r \log n}\right)^{\frac{k}{k-1}} \geq c_{k, r}\left(\frac{t}{\log t}\right)^{\frac{k}{k-1}}
$$

for some positive constant $c_{k, r}$. This is enough to prove Theorem 2.

For an independent $t$-set $I$ in $H, I \cap N_{G^{\prime}}(x)$ is an independent set in $S_{d(x), m}$ for all $x \in X$. Since these events are independent, setting $s(x)=\left|I \cap N_{G^{\prime}}(x)\right|$, and applying Lemma 5 gives: $\mathbb{P}(I$ independent in $H) \leq \prod_{x \in X} \exp \left(-\frac{m(s(x)-r m)}{2 d(x)}\right)=\exp \left(-\sum_{x \in X} \frac{m s(x)}{2 d(x)}+\sum_{x \in X} \frac{r m^{2}}{2 d(x)}\right)$.

For every $x \in X, c n^{1 / k} \leq d(x) \leq n^{1 / k} / c^{k-1}$ and therefore

$$
\mathbb{P}(I \text { independent in } H) \leq \exp \left(-\frac{c^{k-1} m \sum_{x \in X} s(x)}{2 n^{1 / k}}+\frac{|X| r m^{2}}{2 c n^{1 / k}}\right)
$$

Now $\sum_{x \in X} s(x)$ is precisely the number of edges of $G^{\prime}$ between $X$ and $I$. Since every vertex in $I$ has degree at least $c n^{1 / k}$, this number of edges is at least $c n^{1 / k} t=r m n / c^{k}$. Consequently, using $|X|<n / 2$,

$$
\mathbb{P}(I \text { independent in } H) \leq \exp \left(-\frac{c^{k} m t}{2}+\frac{c^{k} m t}{4}\right)=\exp \left(-\frac{c^{k} m t}{4}\right)
$$

The expected number of independent sets of size $t$ is at most

$$
\binom{n}{t} \exp \left(-\frac{c^{k} m t}{4}\right)<\exp \left(t \log n-\frac{c^{k} m t}{4}\right)=\exp (-t \log n) .
$$

This is vanishing as $n \rightarrow \infty$, and the proof of Theorem 2 is complete.

## 3 Proof of Theorem 3

Lazebnik and the second author [23] showed that there exist $n$-vertex $B_{4}$-free 3 -graphs with $(1 / 6+o(1)) n^{3 / 2}$ triples. More specifically, for $n$ large enough, there exists a linear $n$-vertex $B_{4}$-free 3 -graphs $J_{n}$ with $n^{3 / 2} / 10$ triples and maximum degree at most $n^{1 / 2}$. We want to find an upper bound for the probability that a random $s$-set is independent in $J_{n}$. We make use of the following lemma, where we make no effort to optimize the constants.

Lemma 6. Let $n, s$ be integers such that $s<\sqrt{n} / 2$. For $n$ large enough, the probability that a uniformly chosen set of $s$ vertices of $J_{n}$ is independent is at most

$$
\exp \left(-\frac{s^{3}-216}{80 n^{3 / 2}}\right)
$$

When $s \geq \sqrt{n} / 2$, the probability is at most 639/640.

Proof. This is trivial when $s<6$. When $6<s<\sqrt{n} / 2$, let $X$ be the uniformly chosen $s$-set. For any edge $e \in E\left(J_{n}\right)$, let $A_{e}$ be the event that $e \in X$. Then by inclusion-exclusion principle, for $n$ large enough, the probability that $X$ is not independent is at least

$$
\begin{aligned}
& \sum_{e \in E\left(J_{n}\right)} \mathbb{P}\left(A_{e}\right)-\sum_{\{e, f\} \subset E\left(J_{n}\right)} \mathbb{P}\left(A_{e} \wedge A_{f}\right) \\
\geq & \frac{1}{\binom{n}{s}}\left(\frac{n^{3 / 2}}{10}\binom{n-3}{s-3}-n\binom{n^{1 / 2}}{2}\binom{n-5}{s-5}-\binom{n^{3 / 2} / 10}{2}\binom{n-6}{s-6}\right) \\
\geq & \frac{s^{3}}{40 n^{3 / 2}}\left(1-\frac{4 s^{3}}{n^{3 / 2}}\right) \\
\geq & \frac{s^{3}}{80 n^{3 / 2}} .
\end{aligned}
$$

Therefore, for $s>6$ and $n$ large enough, the probability that $X$ is independent is at most

$$
1-\frac{s^{3}}{80 n^{3 / 2}} \leq \exp \left(-\frac{s^{3}}{80 n^{3 / 2}}\right)<\exp \left(-\frac{s^{3}-216}{80 n^{3 / 2}}\right)
$$

When $s \geq \sqrt{n} / 2$, the probability is at most

$$
1-\frac{(\sqrt{n} / 2)^{3}}{80 n^{3 / 2}}=\frac{639}{640}
$$

Now we are ready to prove Theorem 3,

Proof of Theorem 3. Let $G$ be an $n$-vertex graph of girth more than 8 with $2 c_{1} n^{5 / 4}$ edges for some positive constant $c_{1}$. By Lemma 4, there exists a bipartite subgraph $G^{\prime}$ of $G$ with at least $N=c_{1}^{4} n$ vertices, minimum degree at least $c_{1} n^{1 / 4}$ and maximum degree at most $n^{1 / 4} / c_{1}^{3}$. Let $X, Y$ be the parts of this bipartite graph where $|Y| \geq|X|$. We form a 3graph $H$ with vertex set $Y$ by placing a random copy of $J_{d(x)}$ on the vertex set $N_{G^{\prime}}(x)$, the neighborhood of $x$ in $G$, independently for each $x \in X$. Since $G$ has girth more than $2 k$, it is straightforward to check that $H$ does not contain any Berge 4-cycle. Let $m=8 c_{1}^{1 / 4} \sqrt{\log n}$, and let $t=m n^{13 / 16}$. Clearly, $\log t>13 \log n / 16$. If $H$ has no independent sets of size $t$ with positive probability, then since $v(H) \geq N / 2$, we conclude that

$$
R\left(t, B_{4}\right) \geq N / 2 \geq \frac{c_{1}^{4}}{2}\left(\frac{t}{8 c_{1}^{1 / 4} \sqrt{\log n}}\right)^{16 / 13} \geq c_{2}\left(\frac{t}{\sqrt{\log t}}\right)^{16 / 13}
$$

for some positive constant $c_{2}$. This is enough to prove Theorem 3.
Let $A$ be a $t$-set in $Y$, and let $X_{A}=\left\{x \in X| | N_{G^{\prime}}(x) \cap A \mid \geq \sqrt{t} / 2\right\}, \bar{X}_{A}=X \backslash A$. We now evaluate the probability that $A$ is independent in $H$ in two cases.
Case 1: When $\left|X_{A}\right|<n^{5 / 6}$. Since the induced bipartite subgraph of $G^{\prime}$ on $X_{A} \cup A$ has girth 8 , the number of edges of $G^{\prime}$ between $X_{A}$ and $A$ is less than $\left(n^{5 / 6}\right)^{5 / 4}=n^{25 / 24}$. If $A$ is independent in $H$, then $N_{G^{\prime}}(x) \cap A$ is also independent in $J_{d(x)}$ for all $x \in X$. Since these events are independent, setting $s(x)=\left|N_{G^{\prime}}(x) \cap A\right|$, and applying Lemma 6 gives

$$
\begin{aligned}
\mathbb{P}(A \text { independent in } H) & \leq \prod_{x \in \bar{X}_{A}} \exp \left(-\frac{s(x)^{3}-216}{80 d(x)^{3 / 2}}\right) \\
& =\exp \left(-\sum_{x \in \bar{X}_{A}} \frac{s(x)^{3}}{80 d(x)^{3 / 2}}+\sum_{x \in \bar{X}_{A}} \frac{27}{10 d(x)^{3 / 2}}\right)
\end{aligned}
$$

For every $x \in X, c_{1} n^{1 / 4} \leq d(x) \leq n^{1 / 4} / c_{1}^{3}$ and hence together with Jenson's inequality we
have

$$
\begin{aligned}
\mathbb{P}(A \text { independent in } H) & \leq \exp \left(-\frac{c_{1}^{9 / 2} \sum_{x \in \bar{X}_{A}} s(x)^{3}}{80 n^{3 / 8}}+\frac{27\left|\bar{X}_{A}\right|}{10 c_{1}^{3 / 2} n^{3 / 8}}\right) \\
& \leq \exp \left(-\frac{c_{1}^{9 / 2}\left(\sum_{x \in \bar{X}_{A}} s(x)\right)^{3}}{80 n^{3 / 8}\left|\bar{X}_{A}\right|^{2}}+\frac{27\left|\bar{X}_{A}\right|}{10 c_{1}^{3 / 2} n^{3 / 8}}\right)
\end{aligned}
$$

Note that $\sum_{x \in \bar{X}_{A}} s(x)$ is exactly the number of edges of $G^{\prime}$ between $\bar{X}_{A}$ and $A$, which is at least $t c_{1} n^{1 / 4}-n^{25 / 24}=(1-o(1)) c_{1} m n^{17 / 16}$. Also note that $\left|\bar{X}_{A}\right|<N / 2=c_{1}^{4} n / 2$. Consequently,

$$
\begin{aligned}
\mathbb{P}(A \text { independent in } H) & \leq \exp \left(-\frac{(1-o(1)) m^{3} n^{13 / 16}}{20 c_{1}^{1 / 2}}+\frac{27 c_{1}^{5 / 2} n^{5 / 8}}{20}\right) \\
& <\exp \left(-\frac{m^{3} n^{13 / 16}}{32 c_{1}^{1 / 2}}\right)
\end{aligned}
$$

Case 2: When $\left|X_{A}\right| \geq n^{5 / 6}$. Applying Lemma 6 gives

$$
\mathbb{P}(A \text { independent in } H) \leq(639 / 640)^{\left|X_{A}\right|} \leq \exp \left(-n^{5 / 6} / 640\right)<\exp \left(-\frac{m^{3} n^{13 / 16}}{32 c_{1}^{1 / 2}}\right)
$$

In both cases we have $\mathbb{P}(A$ independent in $H)<\exp \left(-\frac{m^{3} n^{13 / 16}}{32 c_{1}^{1 / 2}}\right)$. Therefore the expected number of independent sets of size $t$ in $H$ is at most

$$
\binom{n}{t} \exp \left(-\frac{m^{3} n^{13 / 16}}{32 c_{1}^{1 / 2}}\right)<\exp \left(m n^{13 / 16} \log n-\frac{m^{3} n^{13 / 16}}{32 c_{1}^{1 / 2}}\right)=\exp \left(-m n^{13 / 16} \log n\right)
$$

This is vanishing as $n \rightarrow \infty$, which completes the proof of Theorem 3.

## 4 Degrees, codegrees and independent sets

We make use of the following elementary lemma, whose proof is a standard probabilistic argument, included for completeness:

Lemma 7. Let $d \geq 1$, and let $H$ be a 3-graph of average degree at most $d$. Then

$$
\alpha(H) \geq \frac{2 v(H)}{3 d^{\frac{1}{2}}}
$$

Proof. Let $X$ be a subset of $V(H)$ whose elements are chosen independently with probability $p=d^{-1 / 2}$. We can get an independent set by deleting a vertex for each edge of $H$ contained in $X$. Then the expected size of such independent set is at least

$$
p v(H)-p^{3}|H|=p v(H)-\frac{p^{3} d v(H)}{3}=\frac{2 v(H)}{3 d^{\frac{1}{2}}} .
$$

Hence, there must exist an independent set of size at least the desired lower bound, which completes the proof.

Lemma 8. Let $H$ be a 3 -graph on $n$ vertices, and $0<\epsilon<1 / 2$. Then there exists an induced subgraph $G$ of $H$ satisfying the following properties:

1. $v(G) \geq n^{1-\frac{2}{\log _{2}\left(\frac{1}{\epsilon}\right)}}$,
2. $\Delta(G) \leq \frac{d(G)}{\epsilon}$.

Proof. Let $H=G^{(0)}$. We do the following for $i \geq 0$. If $\Delta\left(G^{(i)}\right) \leq d\left(G^{(i)}\right) / \epsilon$, we let $G=G^{(i)}$. Otherwise, iteratively delete vertices of $G^{(i)}$ with degree at least $d\left(G^{(i)}\right)$. Each deleted vertex will result in the loss of at least $d\left(G^{(i)}\right)$ edges. So we can delete at most

$$
\frac{\left|G^{(i)}\right|}{d\left(G^{(i)}\right)}=\frac{v\left(G^{(i)}\right) \cdot d\left(G^{(i)}\right)}{3 \cdot d\left(G^{(i)}\right)}=\frac{v\left(G^{(i)}\right)}{3}<\frac{v\left(G^{(i)}\right)}{2}
$$

vertices in this step. Let $G^{(i+1)}$ be the subgraph induced by the remaining vertices. Then we have $v\left(G^{(i+1)}\right)>v\left(G^{(i)}\right) / 2$. If $\Delta\left(G^{(i+1)}\right) \leq d\left(G^{(i+1)}\right) / \epsilon$, then we let $G=G^{(i+1)}$. Otherwise, we have

$$
d\left(G^{(i+1)}\right) \leq \epsilon \Delta\left(G^{(i+1)}\right)<\epsilon d\left(G^{(i)}\right) .
$$

Let $K=2 \log _{1 / \epsilon} n$. We must obtain an induced subgraph $G$ with $\Delta(G) \leq d(G) / \epsilon$ after at most $K$ repetitions. Otherwise, after $K$ repetitions, since the average degree decreases by at least a factor of $\epsilon$ after each repetition, the remaining graph $G^{(K)}$ will have no edge, which satisfies the condition $\Delta\left(G^{(K)}\right) \leq d\left(G^{(K)}\right) / \epsilon$. Suppose after $m \leq K$ repetitions we have the desired induced subgraph $G$ with $\Delta(G)<d(G) / \epsilon$. Since the number of vertices decreases by at most a factor of 2 , we also have

$$
v(G)>\frac{n}{2^{m}} \geq n^{1-\frac{2}{\log _{2}\left(\frac{1}{\epsilon}\right)}} .
$$

This completes the proof.

We use the following slightly weaker version of a lemma due to Méroueh [24]; the lemma is in fact valid for 3 -graphs $H$ with no loose $k$-cycles:

Lemma 9. Let $H$ be a $\mathcal{B}_{k}$-free 3 -graph. Then there exists a subgraph $H^{*}$ of $H$ such that $\left|H^{*}\right|>|H| /\left(3 k^{2}\right)$ and each edge of $H^{*}$ contains a pair of codegree 1.

Proof. Given a 3-graph $G$ and a pair of vertices $x, y$, we say that $\{x, y\}$ is $G$-light if $d_{G}(x, y)<$ $k$. Let $G_{1}=H$, and let $H_{1}$ consist of all edges of $G_{1}$ containing a $G_{1}$-light pair, and let $G_{2}=G_{1} \backslash H_{1}$. For $i \geq 2$, let $H_{i}$ consist of all edges of $G_{i}$ containing a $G_{i}$-light pair, and let $G_{i+1}=G_{i} \backslash H_{i}$. Suppose for contradiction that $G_{k}$ is not empty. Let $e_{1}=\left\{v_{1}, v_{2}, v_{3}\right\}$ be an edge in $G_{k}$, then by definition, $\left\{v_{2}, v_{3}\right\}$ is not a $G_{k-1}$ light pair, and hence, there exists an edge $e_{2}=\left\{v_{2}, v_{3}, v_{4}\right\}$ such that $v_{4} \neq v_{1}$. For $2 \leq i \leq k-1$, let $e_{i}=\left\{v_{i}, v_{i+1}, v_{i+2}\right\}$ be an edge in $G_{k+1-i}$. By definition, $\left\{v_{i+1}, v_{i+2}\right\}$ is not a $G_{k-i}$-light pair, and hence, there exists an edge $e_{i+1}=\left\{v_{i+1}, v_{i+2}, v_{i+3}\right\}$ in $G_{k-i}$ such that $v_{i+3}$ is distinct from all $v_{j}, 1 \leq j \leq i$. Therefore, we have a tight path of length $k$ in $G_{1}=H$, that is, a hypergraph consisting of $k+2$ distinct vertices $v_{i}, 1 \leq i \leq k+2$, and $k$ edges $e_{i}=\left\{v_{i}, v_{i+1}, v_{i+2}\right\}, 1 \leq i \leq k$. This is also a non-trivial Berge k-cycle. Indeed, when $k$ is even, $\left\{v_{2}, v_{4}, \ldots, v_{k}, v_{k+1}, v_{k-1}, \ldots, v_{3}\right\}$ forms a system of distinct representatives of $\left\{e_{1} \cap e_{2}, e_{2} \cap e_{4}, e_{4} \cap e_{6}, \ldots, e_{k-2} \cap e_{k}, e_{k} \cap e_{k-1}, e_{k-1} \cap\right.$ $\left.e_{k-3}, \ldots, e_{3} \cap e_{1}\right\}$, and when $k$ is odd, $\left\{v_{2}, v_{4}, \ldots, v_{k+1}, v_{k}, v_{k-2}, \ldots, v_{3}\right\}$ forms a system of distinct representatives of $\left\{e_{1} \cap e_{2}, e_{2} \cap e_{4}, e_{4} \cap e_{6}, \ldots, e_{k-3} \cap e_{k-1}, e_{k-1} \cap e_{k}, e_{k} \cap e_{k-2}, \ldots, e_{3} \cap e_{1}\right\}$. This results in a contradiction, since $H$ is $\mathcal{B}_{k}$-free. Therefore, $G_{k}$ must be empty, and hence $H$ can be partitioned into $k-1$ subgraphs $H_{i}, 1 \leq i \leq k-1$, such that each $H_{i}$ consists of edges containing a $G_{i}$-light pair, which is also $H_{i}$-light. Let $H^{\prime}$ be a subgraph $H_{i}$ with the most edges, then by the pigeonhole principle,

$$
\left|H^{\prime}\right|>\frac{|H|}{k}
$$

Now consider a graph $J$ whose vertex set is the set of 3 -edges of $H^{\prime}$, and two 3 -edges of $H^{\prime}$ form an edge of $J$ if they share an $H^{\prime}$-light pair. It is easy to see that $J$ has maximum degree at most $3 k-6$. Then we can greedily take an independent set of $J$ of size at least $v(J) /(3 k-5)$, and this independent set correspond to a subgraph $H^{*}$ of $H^{\prime}$ such that

$$
\left|H^{*}\right|>\frac{\left|H^{\prime}\right|}{3 k-5}>\frac{|H|}{3 k^{2}}
$$

and each edge of $H^{*}$ contains a pair of codegree 1.

## 5 Proof of Theorem 1

A key ingredient of the proof of Theorem 1 is a supersaturation theorem for cycles in graphs: we make use of the following result proved by Simonovits [8] (see Morris and Saxton [25] for stronger supersaturation):

Lemma 10. For every $n, k \geq 2$, there exist constants $\gamma, b_{0}>0$ such that for every $b \geq b_{0}$, any n-vertex graph $G$ with at least bn $n^{1+1 / k}$ edges contains at least $\gamma b^{2 k} n^{2}$ copies of $C_{2 k}$.

We next give a simple lemma which says that if a graph has many cycles of length $2 k$ containing a fixed edge, then it has many edges.

Lemma 11. Let $G$ be a graph containing $m$ cycles of length $2 k$, each containing an edge $e \in G$. Then $|G| \geq m^{1 /(k-1)} / 2$.

Proof. For each cycle $C$ of length $2 k$ containing $e$, let $M(C)$ be the perfect matching of $C$ containing $e$. Fixing a matching $M \subset G$ of size $k$ containing $e$, at most $(k-1)!2^{k-1}$ cycles $C$ have $M(C)=M$. It follows that the number of distinct matchings $M \subset G$ of size $k$ containing $e$ is at least $m /(k-1)!2^{k-1}$, and therefore

$$
\binom{|G|-1}{k-1} \geq \frac{m}{(k-1)!2^{k-1}}
$$

We conclude $|G|^{k-1} \geq m / 2^{k-1}$ and therefore $|G| \geq m^{1 /(k-1)} / 2$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. It suffices to show that for every large enough integer $n$, an $n$-vertex $\mathcal{B}_{2 k}$-free 3 -graph $H$ contains an independent set of size at least $n^{(2 k-1) /(2 k)-5 /(2 \sqrt{\log n})}$. By Lemma 8 with $\epsilon=\exp \left(-\sqrt{\log _{2} n}\right)$, we find an induced subgraph $H_{0}$ of $H$ with $n_{0}$ vertices, average degree $d_{0}$ and maximum degree $D_{0}$ such that $n_{0} \geq n^{1-2 / \sqrt{\log _{2} n}}$ and $D_{0}<d_{0} / \epsilon$. By Lemma 9, there is a subgraph $H_{1}$ of $H_{0}$ with at least $\left|H_{0}\right| /\left(4 k^{2}\right)$ edges such that each edge of $H_{1}$ contains a pair of codegree 1 in $H_{1}$. Let $\chi: V\left(H_{1}\right) \rightarrow\{1,2,3\}$ be a random 3-coloring and let $H_{2}$ consist of all triples in $H_{1}$ such that the pair of vertices of colors 1 and 2 has codegree 1 in $H_{1}$ and the last vertex in the triple has color 3 . The probability that an edge in $H_{1}$ is also an edge in $H_{2}$ is at least $1 / 27$, and therefore the expected number of edges in $H_{2}$ is at least $\left|H_{1}\right| / 27 \geq\left|H_{0}\right| /\left(108 k^{2}\right)$. Fix a coloring so that $\left|H_{2}\right| \geq\left|H_{0}\right| /\left(108 k^{2}\right)$. Consider the bipartite graph $G$ comprising all pairs of vertices of colors 1 and 2 contained in an edge of $H_{2}$. Thus, $|G|=\left|H_{2}\right|$ and $G$ has average degree $d_{G} \geq d_{0} /\left(108 k^{2}\right)$. For convenience, let
$b>0$ be defined by $d_{G}=2 b n_{0}^{1 / k}$ so $|G|=b n_{0}^{1+1 / k}$. By Lemma 10, there exist constants $\gamma, b_{0}>0$ such that if $b>b_{0}$, then $G$ must contain at least $\gamma b^{2 k} n_{0}^{2}$ copies of $C_{2 k}$. Notice that we must have $1 / \epsilon>b_{0}$ when $n$ is large enough. The proof is split into two cases.

Case 1. $b \geq 1 / \epsilon$. By the pigeonhole principle, there exists an edge $e$ such that the number of $C_{2 k}$ containing $e$ in $G$ is at least

$$
\frac{2 k \gamma b^{2 k} n_{0}^{2}}{|G|}=2 k \gamma b^{2 k-1} n_{0}^{1-\frac{1}{k}}
$$

Let $G^{\prime}$ be the union of all $2 k$-cycles in $G$ containing $e$. Then by Lemma 11, for some constant c,

$$
\left|G^{\prime}\right| \geq c b^{2+\frac{1}{k-1}} n_{0}^{\frac{1}{k}}=\frac{1}{2} c b^{1+\frac{1}{k-1}} d_{G} \geq \frac{1}{216 k^{2}} c \epsilon^{-1-\frac{1}{k-1}} d_{0}>D_{0}
$$

provided $n$ is large enough. Let $C$ be a $2 k$-cycle in $G$ containing $e$. Then there exist edges $e_{1} \cup\left\{v_{1}\right\}, e_{2} \cup\left\{v_{2}\right\}, \ldots, e_{2 k} \cup\left\{v_{2 k}\right\}$ in $H_{2}$ where $e_{1}, e_{2}, \ldots, e_{2 k} \in C$ and $v_{1}, v_{2}, \ldots, v_{2 k}$ have color 3. Since $H_{2}$ is $\mathcal{B}_{2 k}$-free, for some vertex $z$ we have $v_{1}=v_{2}=\cdots=v_{2 k}=z$. Since each cycle $C$ in $G^{\prime}$ contain $e$, they must have the same $z$. Now the degree of $z$ in $H_{2}$ is at least $\left|G^{\prime}\right|>D_{0}$, which contradicts the fact that $H_{0}$ has maximum degree at most $D_{0}$.

Case 2. $b<1 / \epsilon$. In this case, $d_{G}<2 n_{0}^{1 / k} / \epsilon$ and so $d_{0}<\left(216 k^{2} / \epsilon\right) n_{0}^{1 / k}$. By Lemma 7 on $H_{0}$,

$$
\alpha(H) \geq \alpha\left(H_{0}\right) \geq \frac{2 n_{0}}{3 d_{0}^{\frac{1}{2}}} \geq \frac{2}{3}\left(\frac{216 k^{2}}{\epsilon}\right)^{-\frac{1}{2}} n_{0}^{\frac{2 k-1}{2 k}} \geq \frac{1}{9 \sqrt{6} k} n^{\frac{2 k-1}{2 k}-\frac{5 k-2}{2 k \sqrt{\log _{2} n}}}>n^{\frac{2 k-1}{2 k}-\frac{5}{2 \sqrt{\log n}}} .
$$

Now let $n=t^{\frac{2 k}{2 k-1}+\frac{4}{\sqrt{\log t}}}$. Clearly, $\log n>\frac{2 k}{2 k-1} \log t$. Hence, an $n$-vertex $\mathcal{B}_{2 k}$-free 3 -graph $H$ contains an independent set of size

$$
n^{\frac{2 k-1}{2 k}-\frac{5}{2 \sqrt{\log n}}}=t^{\left(\frac{2 k}{2 k-1}+\frac{4}{\sqrt{\log t}}\right)\left(\frac{2 k-1}{2 k}-\frac{5}{2 \sqrt{\log n}}\right)}>t
$$

provided $n$ is large enough. Therefore, we have $R\left(t, \mathcal{B}_{2 k}\right)<t^{\frac{2 k}{2 k-1}+\frac{4}{\sqrt{\log t}}}$.

In fact, by more careful computation, we can obtain a slightly better upper bound $R\left(t, \mathcal{B}_{2 k}\right)<$ $t^{\frac{2 k}{2 k-1}+\frac{c}{\sqrt{\log t}}}$, where $c>\frac{5 k-2}{2 k-1} \cdot \sqrt{\frac{(2 k) \log 2}{2 k-1}}$.

## 6 Concluding remarks

- Notice that Theorem 2 is valid for odd values of $k$, we believe that Theorem 1 should extend to odd values of $k$. An obstacle to applying the same idea as in the proof for
even values of $k$ is that we don't have "good" supersaturation for odd cycles. New ideas may be required to complete the proof for odd values.
- It seems likely that Theorem 1 can be extended to $r$-uniform hypergraphs with $r \geq 4$, however when following the proof of Theorem 1, two obstacles arise. The first is that one requires supersaturation for Berge cycles in $r$-uniform hypergraphs for $r \geq 3$ (in other words, an $r$-uniform version of Lemma 8). A second obstacle is that an $r$-uniform analog of Lemma 9 is not straightforward: for instance if an edge $e$ in an $r$-graph is contained in $m$ Berge cycles of length $2 k$, then the number of edges may be as low as $m^{1 /(2 k-1)}$ : take a graph $2 k$-cycle, and replace one edge with the hyperedge $e$, and each other edge with $m^{1 /(2 k-1)}$ hyperedges. We believe these technical obstacles may be overcome (some of the ideas in the recent paper of Mubayi and Yepremyan [27] may apply).


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[^0]:    *E-mail: jin019@ucsd.edu
    ${ }^{\dagger}$ Research supported by NSF Award DMS-1952786. E-mail: jacques@ucsd.edu

