Ramsey Numbers for Non-trivial Berge Cycles

Jiaxi Nie^{*}

Jacques Verstraëte[†]

Department of Mathematics University of California, San Diego 9500 Gilman Drive La Jolla CA 92093-0112.

September 21, 2021

Abstract

In this paper, we consider an extension of cycle-complete graph Ramsey numbers to Berge cycles in hypergraphs: for $k \geq 2$, a non-trivial Berge k-cycle is a family of sets e_1, e_2, \ldots, e_k such that $e_1 \cap e_2, e_2 \cap e_3, \ldots, e_k \cap e_1$ has a system of distinct representatives and $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$. In the case that all the sets e_i have size three, let \mathcal{B}_k denotes the family of all non-trivial Berge k-cycles. The Ramsey numbers $R(t, \mathcal{B}_k)$ denote the minimum n such that every n-vertex 3-uniform hypergraph contains either a non-trivial Berge k-cycle or an independent set of size t. We prove

$$R(t, \mathcal{B}_{2k}) \le t^{1 + \frac{1}{2k-1} + \frac{4}{\sqrt{\log t}}}$$

and moreover, we show that if a conjecture of Erdős and Simonovits [12] on girth in graphs is true, then this is tight up to a factor $t^{o(1)}$ as $t \to \infty$.

^{*}E-mail: jin019@ucsd.edu

[†]Research supported by NSF Award DMS-1952786. E-mail: jacques@ucsd.edu

1 Introduction

Let \mathcal{F} be a family of r-graphs and $t \geq 1$. The Ramsey numbers $R(t, \mathcal{F})$ denote the minimum n such that every n-vertex r-graph contains either a hypergraph in \mathcal{F} or an independent set of size t. For $k \geq 2$, a *Berge k-cycle* is a family of sets e_1, e_2, \ldots, e_k such that $e_1 \cap e_2, e_2 \cap e_3, \ldots, e_k \cap e_1$ has a system of distinct representatives, and a Berge cycle is *non-trivial* if $e_1 \cap e_2 \cap \cdots \cap e_k = \emptyset$. Let \mathcal{B}_k^r denote the family of non-trivial Berge k-cycles all of whose sets have size r. When r = 2, $\mathcal{B}_k^2 = \{C_k\}$, where C_k denotes the graph cycle of length k. In this paper, we let $\mathcal{B}_k = \mathcal{B}_k^3$.

It is a notoriously difficult problem to determine even the order of magnitude of $R(t, C_k)$ – the cycle-complete graph Ramsey numbers. Kim [18] proved $R(t, C_3) = \Omega(t^2/\log t)$, which gives the order of magnitude of $R(t, C_3)$ when combined with the results of Ajtai, Komlós and Szemerédi [2] and Shearer [30]. The current state-of-the-art results on $R(t, C_3)$ are due to Fiz Pontiveros, Griffiths and Morris [13] and Bohman and Keevash [6], using the random triangle-free process, which determines $R(t, C_3)$ up to a small constant factor.

$$(\frac{1}{4} - o(1))\frac{t^2}{\log t} \le R(t, C_3) \le (1 + o(1))\frac{t^2}{\log t}.$$

The case $R(t, C_4)$ is the subject of a notorious conjecture of Erdős [7], where he conjectured that $R(t, C_4) = o(t^{2-\epsilon})$ for some $\epsilon > 0$. The current best upper bounds on $R(t, C_{2k})$ is

$$O\left(\left(\frac{t}{\log t}\right)^{k/(k-1)}\right),$$

which come from the work of Caro, Li, Rousseau and Zhang [9]. For $R(t, C_{2k+1})$, the best upper bound is

$$O\left(\frac{t^{(k+1)/k}}{\log^{1/k}t}\right)$$

due to Sudakov [31]. Recent results using pseudorandom graphs by Mubayi and the second author [26] give the best lower bounds on cycle-complete graph Ramsey numbers:

$$R(C_k, n) = \Omega\left(\frac{t^{(k-1)/(k-2)}}{\log^{2/(k-2)} t}\right)$$

In particular, via random block constructions, they show that

$$R(C_5,t) \ge (1+o(1))t^{11/8}, \quad R(C_7,t) \ge (1+o(1))t^{11/9}.$$

For $k \geq 3$, a loose k-cycle is a non-trivial Berge k-cycle, denoted C_k^r , with sets e_1, e_2, \ldots, e_k of size r such that $|e_1 \cap e_2| = 1$, $|e_2 \cap e_3| = 1, \ldots, |e_k \cap e_1| = 1$, and for any other pairs of edges $e_i, e_j, e_i \cap e_j = \emptyset$. Ramsey type problems for loose cycles in r-graphs have been studied extensively [4, 10, 11, 14, 16–20, 24, 26]. For r-uniform hypergraphs with $r \geq 3$, Kostochka, Mubayi and the second author [19] proved for all $r \geq 3$, there exist constants a, b > 0 such that

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \le R(t, C_3^r) \le bt^{\frac{3}{2}},\tag{1}$$

The following conjecture was proposed in [19]:

Conjecture I. For $r, k \geq 3$,

$$R(t, C_k^r) = t^{\frac{k}{k-1} + o(1)}.$$
(2)

The conjecture is true for k = 3 due to (1). It is shown in [28] that $R(t, C_4^3) \leq t^{4/3+o(1)}$. Méroueh [24] showed $R(t, C_k^3) = O(t^{1+1/\lfloor (k+1)/2 \rfloor})$ for $k \geq 3$ and $R(t, C_k^r) = O(t^{1+1/\lfloor k/2 \rfloor})$ for $r \geq 4$ and every odd integers $k \geq 5$, improving earlier results of Collier-Cartaino, Graber and Jiang [10]. Conjecture I motivates our current study of non-trivial Berge k-cycles. In support of the above conjecture, we prove the following result for non-trivial Berge cycles of even length:

Theorem 1. For $k \geq 3$, and t large enough,

$$R(t, \mathcal{B}_{2k}) \le t^{\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}}}.$$

Erdős and Simonovits [12] conjectured that there exists an *n*-vertex graph of girth more than 2k with $\Theta(n^{1+1/k})$ edges. This notoriously difficult conjecture remains open, except when $k \in \{2, 3, 5\}$, largely due to the existence of generalized polygons [3, 32, 33]. Towards this conjecture, Lazebnik, Ustimenko and Woldar [22] gave the densest known construction, which has $\Omega(n^{1+2/(3k-2)})$ edges. We prove the following theorem relating this conjecture to lower bounds on Ramsey numbers for non-trivial Berge cycles:

Theorem 2. Let $k \ge 2$, $r \ge 3$. Suppose there exists an n-vertex graph of girth more than 2k with $cn^{1+1/k}$ edges for any integer n large enough and some positive constant c. Then for t large enough and some positive constant $c_{k,r}$ dependent on k and r,

$$R(t, \mathcal{B}_k^r) \ge c_{k,r} \left(\frac{t}{\log t}\right)^{\frac{k}{k-1}}.$$
(3)

This shows that if the Erdős-Simonovits Conjecture is true, then Theorem 1 is tight up to a

 $t^{o(1)}$ factor. Indeed, following the proof of Theorem 2, the known construction of Lazebnik, Ustimenko and Woldar [22] would give a weaker lower bound of $\Omega((t/\log t)^{(3k-2)/(3k-4)})$.

Let B_k be the family of 3-uniform Berge k-cycles without non-triviality. Random graphs together with the Lovász local lemma give $R(t, B_k) \ge t^{(2k-2)/(2k-3)-o(1)}$, see [1] for similar computation. We prove the following theorem, which gives a substantially better lower bound for B_4 if the Erdős-Simonovits Conjecture is true.

Theorem 3. Suppose there exists an n-vertex graph of girth more than 8 with $c_1 n^{5/4}$ edges for any integer n large enough and some positive constant c_1 . Then for t large enough and some positive constant c_2 ,

$$R(t, B_4) \ge \left(\frac{c_2 t}{\sqrt{\log t}}\right)^{16/13}$$

•

In fact, this is also a lower bound for $R(t, \{B_2, B_3, B_4\})$. A natural 3-uniform analog of the Erdős-Simovits conjecture is that there exist *n*-vertex $\{B_2, B_3, \ldots, B_k\}$ -free 3-graphs with $n^{1+1/\lfloor k/2 \rfloor - o(1)}$ edges. This is true for k = 3 due to Ruzsa and Szemeredi [29]. The proof of Theorem 3 makes use of the fact that there exist *n*-vertex $\{B_2, B_3, B_4\}$ -free 3-graphs with $\Omega(n^{3/2})$ edges, that is, the conjecture is true for k = 4, which is due to Lazebnik and the second author [23]. More generally, following the proof of Theorem 3, if the 3-uniform analog of the Erdős-Simonovits Conjecture is true, then we have $R(t, \{B_2, B_3, \ldots, B_{2k}\}) \geq t^{2k^2/(2k^2-k-2)-o(1)}$ and $R(t, \{B_2, B_3, \ldots, B_{2k+1}\}) \geq t^{2k(k-1)/(2k^2-3k-1)-o(1)}$, which are substantially better than the lower bounds obtained by random graphs.

We prove Theorem 1 in Section 5, Theorem 2 in Section 2 and Theorem 3 in Section 3. Theorem 2 is valid for all values of $k \ge 2$ and $r \ge 3$, while Theorem 1 only works for even values of k and r = 3. We believe that Theorem 1 should extend to odd values of k and all $r \ge 3$:

Conjecture II. For all $r, k \geq 3$,

$$R(t, \mathcal{B}_k^r) \le t^{\frac{k}{k-1} + o(1)}.$$
(4)

Notation and terminology. For a hypergraph H, let V(H) denote the vertex set of H, v(H) = |V(H)| and let |H| be the number of edges in H. If all edges of H have size r, we say H is an r-uniform hypergraph, or an r-graph for short. For $v \in V(H)$, let $d_H(v) = |\{e \in H : v \in e\}|$ be the degree of v in H. We denote the average degree of H by d(H), denote the minimum degree of H by $\delta(H)$, and the maximum degree of H by $\Delta(H)$. For $u, v \in V(H)$, let $d_H(u, v) = |\{w : uvw \in H\}|$ denote the codegree of the pair $\{u, v\}$. An

independent set in a hypergraph is a set of vertices containing no edge of the hypergraph. Let $\alpha(H)$ denote the largest size of an independent set in a hypergraph H.

2 Proof of Theorem 2

We will use the following lemma to get a large bipartite subgraph with large minimum degree and small maximum degree:

Lemma 4. Let $k \ge 3$, c > 0, and let G be an n-vertex graph of girth more than 2k with more than $2cn^{1+1/k}$ edges. Then there exists a bipartite subgraph G' of G such that $\delta(G') \ge cn^{1/k}$, $\Delta(G') \le n^{1/k}/c^{k-1}$, and $v(G') \ge c^k n$.

Proof. A maximum cut of G gives a bipartite subgraph with at least $cn^{1+1/k}$ edges. A subgraph G' of this bipartite subgraph of minimum degree at least $cn^{1/k} + 1$ may be obtained by repeatedly removing vertices of degree at most $cn^{1/k}$. Let $\Delta := \Delta(G')$ be the maximum degree of G', and let v be a vertex of maximum degree, then the number of vertices at distance k from v is at least $\Delta c^{k-1}n^{(k-1)/k}$, since G has girth larger than 2k. In particular, $\Delta c^{k-1}n^{(k-1)/k} \leq n$ and so $\Delta \leq n^{1/k}/c^{k-1}$. The number of vertices in G' is at least $c^k n$, since G' has minimum degree at least $cn^{1/k} + 1$ and girth larger than 2k.

Let $r \ge 2$, a *star* with vertex set V is an r-graph on V consisting of all edges containing a fixed vertex of V, i.e., the edge set of a star is $\{e \subset V : |e| = r, v \in e\}$ for some vertex $v \in V$. Let integers $d \ge m$ and let $S_{d,m}$ be a d-vertex r-graph consisting of m vertex-disjoint stars of size $\lfloor d/m \rfloor$ or $\lceil d/m \rceil$.

Lemma 5. Let integer $r \ge 2$, and let integers $d \ge m$. The probability that a uniformly chosen set of s vertices of $S_{d,m}$ is independent is at most

$$\exp\left(-\frac{m(s-rm)}{2d}\right).$$

Proof. Let the vertex sets of these stars be V_1, V_2, \ldots, V_m . The probability that a uniformly chosen set of s_i vertices in V_i is independent in $S_{d,m}$ is at most $1 - \frac{s_i}{\lceil d/m \rceil} \le 1 - \frac{ms_i}{2d}$ if $s_i \ge r$, and is 1 if $s_i < r$. Hence, this probability is at most $1 - \frac{m(s_i - r)}{2d}$ for $0 \le s_i \le d$. Therefore a uniformly chosen set $I \subset S_{d,m}$ of s vertices with $|I \cap V_i| = s_i$ is independent with probability at most

$$\prod_{i=1}^{m} \left(1 - \frac{m(s_i - r)}{2d}\right) \le \exp\left(-\sum_{i=1}^{m} \frac{m(s_i - r)}{2d}\right) = \exp\left(-\frac{m(s - rm)}{2d}\right).$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. It suffices to show that for n large enough, there exists an n-vertex \mathcal{B}_k^r -free r-graph with independence number $O(n^{1-\frac{1}{k}}\log n)$. Let G be an n-vertex graph of girth more than 2k with $2cn^{1+1/k}$ edges for some positive constant c. By Lemma 4, there exists a bipartite subgraph G' of G with at least $N = c^k n$ vertices, minimum degree at least $cn^{1/k}$ and maximum degree at most $n^{1/k}/c^{k-1}$. Let X, Y be the parts of this bipartite graph where $|Y| \geq |X|$. Let $m = 8 \log n/c^k$. We form an r-graph H with vertex set Y by placing a random copy of $S_{d(x),m}$ on the vertex set $N_{G'}(x)$, the neighborhood of x in G', independently for each $x \in X$. Since G' has girth more than 2k, it is straightforward to check that H does not contain any non-trivial Berge k-cycle. We now compute the expected number of independent sets of size $t = rmn^{1-1/k}/c^{k+1}$ in H. Clearly, $\log t \geq (1 - 1/k) \log n$. If H has no independent set of size t with positive probability, then since $v(H) \geq N/2$, we find that

$$R(t, \mathcal{B}_k^r) \ge N/2 \ge \frac{c^k}{2} \left(\frac{c^{2k+1}t}{8r\log n}\right)^{\frac{k}{k-1}} \ge c_{k,r} \left(\frac{t}{\log t}\right)^{\frac{k}{k-1}},$$

for some positive constant $c_{k,r}$. This is enough to prove Theorem 2.

For an independent t-set I in H, $I \cap N_{G'}(x)$ is an independent set in $S_{d(x),m}$ for all $x \in X$. Since these events are independent, setting $s(x) = |I \cap N_{G'}(x)|$, and applying Lemma 5 gives:

$$\mathbb{P}(I \text{ independent in } H) \le \prod_{x \in X} \exp\left(-\frac{m(s(x) - rm)}{2d(x)}\right) = \exp\left(-\sum_{x \in X} \frac{ms(x)}{2d(x)} + \sum_{x \in X} \frac{rm^2}{2d(x)}\right).$$

For every $x \in X$, $cn^{1/k} \le d(x) \le n^{1/k}/c^{k-1}$ and therefore

$$\mathbb{P}(I \text{ independent in } H) \le \exp\left(-\frac{c^{k-1}m\sum_{x\in X}s(x)}{2n^{1/k}} + \frac{|X|rm^2}{2cn^{1/k}}\right).$$

Now $\sum_{x \in X} s(x)$ is precisely the number of edges of G' between X and I. Since every vertex in I has degree at least $cn^{1/k}$, this number of edges is at least $cn^{1/k}t = rmn/c^k$. Consequently, using |X| < n/2,

$$\mathbb{P}(I \text{ independent in } H) \leq \exp\left(-\frac{c^k m t}{2} + \frac{c^k m t}{4}\right) = \exp\left(-\frac{c^k m t}{4}\right).$$

The expected number of independent sets of size t is at most

$$\binom{n}{t} \exp\left(-\frac{c^k m t}{4}\right) < \exp\left(t \log n - \frac{c^k m t}{4}\right) = \exp\left(-t \log n\right).$$

This is vanishing as $n \to \infty$, and the proof of Theorem 2 is complete.

3 Proof of Theorem 3

Lazebnik and the second author [23] showed that there exist *n*-vertex B_4 -free 3-graphs with $(1/6 + o(1))n^{3/2}$ triples. More specifically, for *n* large enough, there exists a linear *n*-vertex B_4 -free 3-graphs J_n with $n^{3/2}/10$ triples and maximum degree at most $n^{1/2}$. We want to find an upper bound for the probability that a random *s*-set is independent in J_n . We make use of the following lemma, where we make no effort to optimize the constants.

Lemma 6. Let n, s be integers such that $s < \sqrt{n}/2$. For n large enough, the probability that a uniformly chosen set of s vertices of J_n is independent is at most

$$\exp\left(-\frac{s^3-216}{80n^{3/2}}\right).$$

When $s \ge \sqrt{n}/2$, the probability is at most 639/640.

Proof. This is trivial when s < 6. When $6 < s < \sqrt{n/2}$, let X be the uniformly chosen s-set. For any edge $e \in E(J_n)$, let A_e be the event that $e \in X$. Then by inclusion-exclusion principle, for n large enough, the probability that X is not independent is at least

$$\sum_{e \in E(J_n)} \mathbb{P}(A_e) - \sum_{\{e,f\} \subset E(J_n)} \mathbb{P}(A_e \wedge A_f)$$

$$\geq \frac{1}{\binom{n}{s}} \left(\frac{n^{3/2}}{10} \binom{n-3}{s-3} - n\binom{n^{1/2}}{2} \binom{n-5}{s-5} - \binom{n^{3/2}/10}{2} \binom{n-6}{s-6}\right)$$

$$\geq \frac{s^3}{40n^{3/2}} \left(1 - \frac{4s^3}{n^{3/2}}\right)$$

$$\geq \frac{s^3}{80n^{3/2}}.$$

Therefore, for s > 6 and n large enough, the probability that X is independent is at most

$$1 - \frac{s^3}{80n^{3/2}} \le \exp\left(-\frac{s^3}{80n^{3/2}}\right) < \exp\left(-\frac{s^3 - 216}{80n^{3/2}}\right).$$

When $s \ge \sqrt{n}/2$, the probability is at most

$$1 - \frac{(\sqrt{n}/2)^3}{80n^{3/2}} = \frac{639}{640}.$$

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let G be an n-vertex graph of girth more than 8 with $2c_1n^{5/4}$ edges for some positive constant c_1 . By Lemma 4, there exists a bipartite subgraph G' of G with at least $N = c_1^4 n$ vertices, minimum degree at least $c_1n^{1/4}$ and maximum degree at most $n^{1/4}/c_1^3$. Let X, Y be the parts of this bipartite graph where $|Y| \ge |X|$. We form a 3graph H with vertex set Y by placing a random copy of $J_{d(x)}$ on the vertex set $N_{G'}(x)$, the neighborhood of x in G, independently for each $x \in X$. Since G has girth more than 2k, it is straightforward to check that H does not contain any Berge 4-cycle. Let $m = 8c_1^{1/4}\sqrt{\log n}$, and let $t = mn^{13/16}$. Clearly, $\log t > 13 \log n/16$. If H has no independent sets of size t with positive probability, then since $v(H) \ge N/2$, we conclude that

$$R(t, B_4) \ge N/2 \ge \frac{c_1^4}{2} \left(\frac{t}{8c_1^{1/4}\sqrt{\log n}}\right)^{16/13} \ge c_2 \left(\frac{t}{\sqrt{\log t}}\right)^{16/13},$$

for some positive constant c_2 . This is enough to prove Theorem 3.

Let A be a t-set in Y, and let $X_A = \{x \in X | |N_{G'}(x) \cap A| \ge \sqrt{t}/2\}, \overline{X}_A = X \setminus A$. We now evaluate the probability that A is independent in H in two cases.

Case 1: When $|X_A| < n^{5/6}$. Since the induced bipartite subgraph of G' on $X_A \cup A$ has girth 8, the number of edges of G' between X_A and A is less than $(n^{5/6})^{5/4} = n^{25/24}$. If A is independent in H, then $N_{G'}(x) \cap A$ is also independent in $J_{d(x)}$ for all $x \in X$. Since these events are independent, setting $s(x) = |N_{G'}(x) \cap A|$, and applying Lemma 6 gives

$$\mathbb{P}(A \text{ independent in } H) \leq \prod_{x \in \overline{X}_A} \exp\left(-\frac{s(x)^3 - 216}{80d(x)^{3/2}}\right)$$
$$= \exp\left(-\sum_{x \in \overline{X}_A} \frac{s(x)^3}{80d(x)^{3/2}} + \sum_{x \in \overline{X}_A} \frac{27}{10d(x)^{3/2}}\right)$$

For every $x \in X$, $c_1 n^{1/4} \leq d(x) \leq n^{1/4}/c_1^3$ and hence together with Jenson's inequality we

have

$$\mathbb{P}(A \text{ independent in } H) \le \exp\left(-\frac{c_1^{9/2} \sum_{x \in \overline{X}_A} s(x)^3}{80n^{3/8}} + \frac{27|\overline{X}_A|}{10c_1^{3/2}n^{3/8}}\right)$$
$$\le \exp\left(-\frac{c_1^{9/2} (\sum_{x \in \overline{X}_A} s(x))^3}{80n^{3/8}|\overline{X}_A|^2} + \frac{27|\overline{X}_A|}{10c_1^{3/2}n^{3/8}}\right)$$

Note that $\sum_{x \in \overline{X}_A} s(x)$ is exactly the number of edges of G' between \overline{X}_A and A, which is at least $tc_1 n^{1/4} - n^{25/24} = (1 - o(1))c_1 m n^{17/16}$. Also note that $|\overline{X}_A| < N/2 = c_1^4 n/2$. Consequently,

$$\begin{split} \mathbb{P}(A \text{ independent in } H) &\leq \exp\left(-\frac{(1-o(1))m^3n^{13/16}}{20c_1^{1/2}} + \frac{27c_1^{5/2}n^{5/8}}{20}\right) \\ &< \exp\left(-\frac{m^3n^{13/16}}{32c_1^{1/2}}\right). \end{split}$$

Case 2: When $|X_A| \ge n^{5/6}$. Applying Lemma 6 gives

$$\mathbb{P}(A \text{ independent in } H) \le (639/640)^{|X_A|} \le \exp(-n^{5/6}/640) < \exp\left(-\frac{m^3 n^{13/16}}{32c_1^{1/2}}\right).$$

In both cases we have $\mathbb{P}(A \text{ independent in } H) < \exp\left(-\frac{m^3 n^{13/16}}{32c_1^{1/2}}\right)$. Therefore the expected number of independent sets of size t in H is at most

$$\binom{n}{t} \exp\left(-\frac{m^3 n^{13/16}}{32c_1^{1/2}}\right) < \exp\left(mn^{13/16}\log n - \frac{m^3 n^{13/16}}{32c_1^{1/2}}\right) = \exp\left(-mn^{13/16}\log n\right).$$

This is vanishing as $n \to \infty$, which completes the proof of Theorem 3.

4 Degrees, codegrees and independent sets

We make use of the following elementary lemma, whose proof is a standard probabilistic argument, included for completeness:

Lemma 7. Let $d \ge 1$, and let H be a 3-graph of average degree at most d. Then

$$\alpha(H) \ge \frac{2v(H)}{3d^{\frac{1}{2}}}.$$

Proof. Let X be a subset of V(H) whose elements are chosen independently with probability $p = d^{-1/2}$. We can get an independent set by deleting a vertex for each edge of H contained in X. Then the expected size of such independent set is at least

$$pv(H) - p^{3}|H| = pv(H) - \frac{p^{3}dv(H)}{3} = \frac{2v(H)}{3d^{\frac{1}{2}}}.$$

Hence, there must exist an independent set of size at least the desired lower bound, which completes the proof. $\hfill \Box$

Lemma 8. Let H be a 3-graph on n vertices, and $0 < \epsilon < 1/2$. Then there exists an induced subgraph G of H satisfying the following properties:

1.
$$v(G) \ge n^{1 - \frac{2}{\log_2(\frac{1}{\epsilon})}},$$

2. $\Delta(G) \le \frac{d(G)}{\epsilon}.$

Proof. Let $H = G^{(0)}$. We do the following for $i \ge 0$. If $\Delta(G^{(i)}) \le d(G^{(i)})/\epsilon$, we let $G = G^{(i)}$. Otherwise, iteratively delete vertices of $G^{(i)}$ with degree at least $d(G^{(i)})$. Each deleted vertex will result in the loss of at least $d(G^{(i)})$ edges. So we can delete at most

$$\frac{|G^{(i)}|}{d(G^{(i)})} = \frac{v(G^{(i)}) \cdot d(G^{(i)})}{3 \cdot d(G^{(i)})} = \frac{v(G^{(i)})}{3} < \frac{v(G^{(i)})}{2}$$

vertices in this step. Let $G^{(i+1)}$ be the subgraph induced by the remaining vertices. Then we have $v(G^{(i+1)}) > v(G^{(i)})/2$. If $\Delta(G^{(i+1)}) \le d(G^{(i+1)})/\epsilon$, then we let $G = G^{(i+1)}$. Otherwise, we have

$$d(G^{(i+1)}) \le \epsilon \Delta(G^{(i+1)}) < \epsilon d(G^{(i)}).$$

Let $K = 2 \log_{1/\epsilon} n$. We must obtain an induced subgraph G with $\Delta(G) \leq d(G)/\epsilon$ after at most K repetitions. Otherwise, after K repetitions, since the average degree decreases by at least a factor of ϵ after each repetition, the remaining graph $G^{(K)}$ will have no edge, which satisfies the condition $\Delta(G^{(K)}) \leq d(G^{(K)})/\epsilon$. Suppose after $m \leq K$ repetitions we have the desired induced subgraph G with $\Delta(G) < d(G)/\epsilon$. Since the number of vertices decreases by at most a factor of 2, we also have

$$v(G) > \frac{n}{2^m} \ge n^{1 - \frac{2}{\log_2(\frac{1}{\epsilon})}}$$

This completes the proof.

We use the following slightly weaker version of a lemma due to Méroueh [24]; the lemma is in fact valid for 3-graphs H with no loose k-cycles:

Lemma 9. Let H be a \mathcal{B}_k -free 3-graph. Then there exists a subgraph H^* of H such that $|H^*| > |H|/(3k^2)$ and each edge of H^* contains a pair of codegree 1.

Proof. Given a 3-graph G and a pair of vertices x, y, we say that $\{x, y\}$ is G-light if $d_G(x, y) < d_G(x, y)$ k. Let $G_1 = H$, and let H_1 consist of all edges of G_1 containing a G_1 -light pair, and let $G_2 = G_1 \setminus H_1$. For $i \ge 2$, let H_i consist of all edges of G_i containing a G_i -light pair, and let $G_{i+1} = G_i \setminus H_i$. Suppose for contradiction that G_k is not empty. Let $e_1 = \{v_1, v_2, v_3\}$ be an edge in G_k , then by definition, $\{v_2, v_3\}$ is not a G_{k-1} -light pair, and hence, there exists an edge $e_2 = \{v_2, v_3, v_4\}$ such that $v_4 \neq v_1$. For $2 \leq i \leq k-1$, let $e_i = \{v_i, v_{i+1}, v_{i+2}\}$ be an edge in G_{k+1-i} . By definition, $\{v_{i+1}, v_{i+2}\}$ is not a G_{k-i} -light pair, and hence, there exists an edge $e_{i+1} = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ in G_{k-i} such that v_{i+3} is distinct from all $v_j, 1 \le j \le i$. Therefore, we have a tight path of length k in $G_1 = H$, that is, a hypergraph consisting of k+2 distinct vertices v_i , $1 \le i \le k+2$, and k edges $e_i = \{v_i, v_{i+1}, v_{i+2}\}, 1 \le i \le k$. This is also a non-trivial Berge k-cycle. Indeed, when k is even, $\{v_2, v_4, \ldots, v_k, v_{k+1}, v_{k-1}, \ldots, v_3\}$ forms a system of distinct representatives of $\{e_1 \cap e_2, e_2 \cap e_4, e_4 \cap e_6, \dots, e_{k-2} \cap e_k, e_k \cap e_{k-1}, e_{k-1} \cap e_k, e_k \cap e_k, e_k \cap e_{k-1}, e_{k-1} \cap e_k, e_k \cap e_k, e_$ $e_{k-3},\ldots,e_3\cap e_1$, and when k is odd, $\{v_2,v_4,\ldots,v_{k+1},v_k,v_{k-2},\ldots,v_3\}$ forms a system of distinct representatives of $\{e_1 \cap e_2, e_2 \cap e_4, e_4 \cap e_6, \dots, e_{k-3} \cap e_{k-1}, e_{k-1} \cap e_k, e_k \cap e_{k-2}, \dots, e_3 \cap e_1\}$. This results in a contradiction, since H is \mathcal{B}_k -free. Therefore, G_k must be empty, and hence H can be partitioned into k-1 subgraphs H_i , $1 \le i \le k-1$, such that each H_i consists of edges containing a G_i -light pair, which is also H_i -light. Let H' be a subgraph H_i with the most edges, then by the pigeonhole principle,

$$|H'| > \frac{|H|}{k}.$$

Now consider a graph J whose vertex set is the set of 3-edges of H', and two 3-edges of H' form an edge of J if they share an H'-light pair. It is easy to see that J has maximum degree at most 3k - 6. Then we can greedily take an independent set of J of size at least v(J)/(3k-5), and this independent set correspond to a subgraph H^* of H' such that

$$|H^*| > \frac{|H'|}{3k-5} > \frac{|H|}{3k^2},$$

and each edge of H^* contains a pair of codegree 1.

11

5 Proof of Theorem 1

A key ingredient of the proof of Theorem 1 is a supersaturation theorem for cycles in graphs: we make use of the following result proved by Simonovits [8] (see Morris and Saxton [25] for stronger supersaturation):

Lemma 10. For every $n, k \ge 2$, there exist constants $\gamma, b_0 > 0$ such that for every $b \ge b_0$, any n-vertex graph G with at least $bn^{1+1/k}$ edges contains at least $\gamma b^{2k}n^2$ copies of C_{2k} .

We next give a simple lemma which says that if a graph has many cycles of length 2k containing a fixed edge, then it has many edges.

Lemma 11. Let G be a graph containing m cycles of length 2k, each containing an edge $e \in G$. Then $|G| \ge m^{1/(k-1)}/2$.

Proof. For each cycle C of length 2k containing e, let M(C) be the perfect matching of C containing e. Fixing a matching $M \subset G$ of size k containing e, at most $(k-1)!2^{k-1}$ cycles C have M(C) = M. It follows that the number of distinct matchings $M \subset G$ of size k containing e is at least $m/(k-1)!2^{k-1}$, and therefore

$$\binom{|G|-1}{k-1} \ge \frac{m}{(k-1)!2^{k-1}}$$

We conclude $|G|^{k-1} \ge m/2^{k-1}$ and therefore $|G| \ge m^{1/(k-1)}/2$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. It suffices to show that for every large enough integer n, an n-vertex \mathcal{B}_{2k} -free 3-graph H contains an independent set of size at least $n^{(2k-1)/(2k)-5/(2\sqrt{\log n})}$. By Lemma 8 with $\epsilon = \exp(-\sqrt{\log_2 n})$, we find an induced subgraph H_0 of H with n_0 vertices, average degree d_0 and maximum degree D_0 such that $n_0 \geq n^{1-2/\sqrt{\log_2 n}}$ and $D_0 < d_0/\epsilon$. By Lemma 9, there is a subgraph H_1 of H_0 with at least $|H_0|/(4k^2)$ edges such that each edge of H_1 contains a pair of codegree 1 in H_1 . Let $\chi : V(H_1) \to \{1, 2, 3\}$ be a random 3-coloring and let H_2 consist of all triples in H_1 such that the pair of vertices of colors 1 and 2 has codegree 1 in H_1 and the last vertex in the triple has color 3. The probability that an edge in H_1 is also an edge in H_2 is at least 1/27, and therefore the expected number of edges in H_2 is at least $|H_1|/27 \geq |H_0|/(108k^2)$. Fix a coloring so that $|H_2| \geq |H_0|/(108k^2)$. Consider the bipartite graph G comprising all pairs of vertices of colors 1 and 2 contained in an edge of H_2 . Thus, $|G| = |H_2|$ and G has average degree $d_G \geq d_0/(108k^2)$. For convenience, let

b > 0 be defined by $d_G = 2bn_0^{1/k}$ so $|G| = bn_0^{1+1/k}$. By Lemma 10, there exist constants $\gamma, b_0 > 0$ such that if $b > b_0$, then G must contain at least $\gamma b^{2k} n_0^2$ copies of C_{2k} . Notice that we must have $1/\epsilon > b_0$ when n is large enough. The proof is split into two cases.

Case 1. $b \ge 1/\epsilon$. By the pigeonhole principle, there exists an edge e such that the number of C_{2k} containing e in G is at least

$$\frac{2k\gamma b^{2k}n_0^2}{|G|} = 2k\gamma b^{2k-1}n_0^{1-\frac{1}{k}}$$

Let G' be the union of all 2k-cycles in G containing e. Then by Lemma 11, for some constant c,

$$|G'| \ge cb^{2+\frac{1}{k-1}}n_0^{\frac{1}{k}} = \frac{1}{2}cb^{1+\frac{1}{k-1}}d_G \ge \frac{1}{216k^2}c\epsilon^{-1-\frac{1}{k-1}}d_0 > D_0$$

provided n is large enough. Let C be a 2k-cycle in G containing e. Then there exist edges $e_1 \cup \{v_1\}, e_2 \cup \{v_2\}, \ldots, e_{2k} \cup \{v_{2k}\}$ in H_2 where $e_1, e_2, \ldots, e_{2k} \in C$ and v_1, v_2, \ldots, v_{2k} have color 3. Since H_2 is \mathcal{B}_{2k} -free, for some vertex z we have $v_1 = v_2 = \cdots = v_{2k} = z$. Since each cycle C in G' contain e, they must have the same z. Now the degree of z in H_2 is at least $|G'| > D_0$, which contradicts the fact that H_0 has maximum degree at most D_0 .

Case 2. $b < 1/\epsilon$. In this case, $d_G < 2n_0^{1/k}/\epsilon$ and so $d_0 < (216k^2/\epsilon)n_0^{1/k}$. By Lemma 7 on H_0 ,

$$\alpha(H) \ge \alpha(H_0) \ge \frac{2n_0}{3d_0^{\frac{1}{2}}} \ge \frac{2}{3} \left(\frac{216k^2}{\epsilon}\right)^{-\frac{1}{2}} n_0^{\frac{2k-1}{2k}} \ge \frac{1}{9\sqrt{6}k} n^{\frac{2k-1}{2k} - \frac{5k-2}{2k\sqrt{\log_2 n}}} > n^{\frac{2k-1}{2k} - \frac{5}{2\sqrt{\log n}}}.$$

Now let $n = t^{\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}}}$. Clearly, $\log n > \frac{2k}{2k-1} \log t$. Hence, an *n*-vertex \mathcal{B}_{2k} -free 3-graph H contains an independent set of size

$$n^{\frac{2k-1}{2k} - \frac{5}{2\sqrt{\log n}}} = t^{\left(\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}}\right)\left(\frac{2k-1}{2k} - \frac{5}{2\sqrt{\log n}}\right)} > t$$

provided *n* is large enough. Therefore, we have $R(t, \mathcal{B}_{2k}) < t^{\frac{2k}{2k-1} + \frac{4}{\sqrt{\log t}}}$.

In fact, by more careful computation, we can obtain a slightly better upper bound $R(t, \mathcal{B}_{2k}) < t^{\frac{2k}{2k-1} + \frac{c}{\sqrt{\log t}}}$, where $c > \frac{5k-2}{2k-1} \cdot \sqrt{\frac{(2k)\log 2}{2k-1}}$.

6 Concluding remarks

• Notice that Theorem 2 is valid for odd values of k, we believe that Theorem 1 should extend to odd values of k. An obstacle to applying the same idea as in the proof for

even values of k is that we don't have "good" supersaturation for odd cycles. New ideas may be required to complete the proof for odd values.

• It seems likely that Theorem 1 can be extended to r-uniform hypergraphs with $r \ge 4$, however when following the proof of Theorem 1, two obstacles arise. The first is that one requires supersaturation for Berge cycles in r-uniform hypergraphs for $r \ge 3$ (in other words, an r-uniform version of Lemma 8). A second obstacle is that an r-uniform analog of Lemma 9 is not straightforward: for instance if an edge e in an r-graph is contained in m Berge cycles of length 2k, then the number of edges may be as low as $m^{1/(2k-1)}$: take a graph 2k-cycle, and replace one edge with the hyperedge e, and each other edge with $m^{1/(2k-1)}$ hyperedges. We believe these technical obstacles may be overcome (some of the ideas in the recent paper of Mubayi and Yepremyan [27] may apply).

7 Acknowledgments

We would like to thank the anonymous referees for their careful reading of the paper and helpful suggestions. In particular, one of the referee's comments on Berge cycles without non-triviality leads to Theorem 3.

References

- [1] N. Alon and J. H. Spencer, The probabilistic method. John Wiley & Sons, (2016)
- [2] M. Ajtai, J. Komlós and E. Szemerédi. A note on Ramsey numbers. J. Combin. Theory Ser. A, 29, 354–360, (1980).
- [3] C. T. Benson. Minimal regular graphs of girths eight and twelve. Canad. J. Math., 18 (1966), 1091–1094.
- [4] B. Bollobás, E. Győri. Pentagons vs. triangles. Discrete Math., 308 (2008), no. 19, 4332–4336.
- [5] J. A. Bondy, M. Simonovits. Cycles of even length in graphs. J. Combin. Theory Ser. B, 16 (1974), 97–105.

- [6] T. Bohman P. Keevash. Dynamic concentration of the triangle-free process. In: Nešetřil J, Pellegrini M, eds. The Seventh European Conference on Combinatorics, Graph Theory and Applications. Pisa: Edizioni della Normale; 2013:pp. 489–495.
- [7] F. Chung. Open problems of Paul Erdős in graph theory. Journal of Graph Theory, 25 (1997), 3–36.
- [8] R. Faudree and M. Simonovits. Cycle-supersaturated graphs. in preparation.
- [9] Y. Caro, Y. Li, C. Rousseau, Y. Zhang. Asymptotic bounds for some bipartite graphcomplete graph Ramsey numbers. Discrete Math., 220 (2000), 51–56.
- [10] C. Coller-Cartaino, N. Graber, T. Jiang. Linear Turán Numbers of Linear Cycles and Cycle-Complete Ramsey Numbers. Combinatorics, Probability and Computing, 27(3), 358–386, (2018)
- [11] S. Das, C. Lee and B. Sudakov. Rainbow Turán Problem for Even Cycles. European J. Combin. 34 (2013), 905–915.
- [12] P. Erdős and M. Simonovits. Compactness results in extremal graph theory. Combinatorica 2 (1982) no. 3, 275–288.
- [13] G. Fiz Pontiveros, S. Griffiths, and R. Morris. The triangle-free process and the Ramsey number R(3, k). Mem. Amer. Math. Soc. 263 (2020), no. 1274, v+125 pp.
- [14] Z. Füredi, T. Jiang. Hypergraph Turán numbers of linear cycles. J. Combin. Theory Ser. A, 123, 252–270, 2014.
- [15] A. Frieze, D. Mubayi. On the chromatic number of simple triangle-free 3-graphs. Electronic Journal of Combinatorics 15 (2008), no. 1, Research Paper 121, 27 pp.
- [16] E. Győri, N. Lemons. 3-uniform hypergraphs avoiding a given odd cycle. Combinatorica 32, 2 (March 2012), 187–203.
- [17] E. Győri, N. Lemons. Hypergraphs with no odd cycle of given length. Electron. Notes Discrete Math. 34(2009), 359–362.
- [18] J.H. Kim. The Ramsey number R(3,t) has order of magnitude $t^2/\log t$. Random Structures and Algorithms, 7 (1995) 173–207.
- [19] A. Kostochka, D. Mubayi, J. Verstraëte. Hypergraph Ramsey Numbers: Triangles versus Cliques. J. Combin. Theory. Ser. A, 120 (2013), no. 7, 1491–1507.

- [20] A. Kostochka, D. Mubayi and J. Verstraëte. Turán problems and shadows I: Paths and cycles. J. Combin. Theory. Ser. A, 129 (2015), 57–79.
- [21] J. Louck, C. Timmons. Triangle-free induced subgraphs of polarity graphs. Preprint: https://arxiv.org/abs/1703.06347
- [22] F. Lazebnik, V.A. Ustimenko, A.J. Woldar. New upper bounds on the order of cages. Electron. J. Combin. 14(R13), 1–11 (1997)
- [23] F. Lazebnik, J. Verstraëte. On hypergraphs of girth five. Electron. J. Combin. 10 (2003): R25.
- [24] A. Méroueh. The Ramsey number of loose cycles versus cliques. J. Graph Theory, 2019, 90: 172–188.
- [25] R. Morris, D. Saxton. The number of C_{2l} -free graphs. Advances in Mathematics, Vol. 298, 2016, 534–580, ISSN 0001-8708.
- [26] D. Mubayi, J. Verstraëte. A survey of Turán problems for expansions, Recent trends in combinatorics. 117–143, IMA Vol. Math. Appl., 159, Springer, New York, 2016. Vii 706. ISBN: 978-3-319-24296-5.
- [27] D. Mubayi, L. Yepremyan. Random Turán theorem for hypergraph cycles. Preprint: https://arxiv.org/abs/2007.10320
- [28] J. Nie, J. Verstraëte. Loose Cycle-Complete Hypergraph Ramsey Numbers. Preprint (2021+).
- [29] I.Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles. Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai, 18, 939–945 (1978).
- [30] J. B. Shearer. A note on the independence number of triangle-free graphs. Discrete Math., 46, 83–87, (1983).
- [31] B. Sudakov. A note on odd cycle-complete graph Ramsey numbers. Electron. J. Combin. 9 (2002), no. 1, Note 1, 4 pp.
- [32] H. van Maldeghem. Generalized polygons, Monographs in Mathematics. Birkhäuser 1998. http://cage.ugent.be/~fdc/contactforum/vanmaldeghem.pdf
- [33] J. Verstraëte. Extremal problems for cycles in graphs. Recent trends in combinatorics, 83–116, IMA Vol. Math. Appl., 159, Springer, [Cham], 2016.