On Ray Shooting for Triangles in 3-Space and Related Problems^{*}

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Abstract

We consider several problems that involve lines in three dimensions, and present improved algorithms for solving them. The problems include (i) ray shooting amid triangles in \mathbb{R}^3 , (ii) reporting intersections between query lines (segments, or rays) and input triangles, as well as approximately counting the number of such intersections, (iii) computing the intersection of two nonconvex polyhedra, (iv) detecting, counting, or reporting intersections in a set of lines in \mathbb{R}^3 , and (v) output-sensitive construction of an arrangement of triangles in three dimensions.

Our approach is based on the polynomial partitioning technique.

For example, our ray-shooting algorithm processes a set of n triangles in \mathbb{R}^3 into a data structure for answering ray shooting queries amid the given triangles, which uses $O(n^{3/2+\varepsilon})$ storage and preprocessing, and answers a query in $O(n^{1/2+\varepsilon})$ time, for any $\varepsilon > 0$. This is a significant improvement over known results, obtained more than 25 years ago, in which, with this amount of storage, the query time bound is roughly $n^{5/8}$. The algorithms for the other problems have similar performance bounds, with similar improvements over previous results.

We also derive a nontrivial improved tradeoff between storage and query time. Using it, we obtain algorithms that answer m queries on n objects in

$$\max\left\{O(m^{2/3}n^{5/6+\varepsilon}+n^{1+\varepsilon}),\ O(m^{5/6+\varepsilon}n^{2/3}+m^{1+\varepsilon})\right\}$$

time, for any $\varepsilon > 0$, again an improvement over the earlier bounds.

Keywords: Ray shooting, Three dimensions, Polynomial partitioning, Tradeoff

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1 Introduction

In this paper we consider several algorithmic problems that involve, explicitly or implicitly, a finite set of lines in three dimensions. The main problems that we consider are:

- (i) Ray shooting amid triangles in three dimensions. We have a set \mathcal{T} of n triangles in \mathbb{R}^3 , and our goal is to preprocess \mathcal{T} into a data structure that supports efficient rayshooting queries, each of which specifies a ray ρ and asks for the first triangle of \mathcal{T} that is hit by ρ , if such a triangle exists.
- (ii) Intersection reporting, emptiness, and approximate counting queries amid triangles in three dimensions. For a set \mathcal{T} of n triangles in \mathbb{R}^3 , we want to preprocess \mathcal{T} into a data structure that supports efficient intersection reporting (resp., emptiness) queries, each of which specifies a line, ray, or segment ρ and asks for reporting the triangles of \mathcal{T} that ρ intersects (resp., determining whether such a triangle exists). We want the queries to be output-sensitive, so that their cost is a small (sublinear) overhead plus a term that is nearly linear in the output size k. In the related problem of approximate counting queries, we want to preprocess \mathcal{T} into a data structure, such that given a query ρ as above, it efficiently computes the number of triangles of \mathcal{T} that ρ intersects, up to some prescribed small relative error.
- (iii) Compute the intersection of two nonconvex polyhedra. The complexity of the intersection can be quadratic in the complexities of the input polyhedra, and we therefore seek an output-sensitive solution, where the running time is a small (subquadratic) overhead plus a term that is nearly linear in k, where k is the complexity of the intersection.
- (iv) Detect, count, or report intersections in a set of lines in 3-space. Again, in the reporting version we seek an output-sensitive solution, as above.
- (v) Output-sensitive construction of an arrangement of triangles in three dimensions.

All these problems, or variants thereof, have been considered in several works during the 1990s; see [5, 7, 11, 12, 16, 29, 32] for a sample of these works. See also Pellegrini [33] for a recent comprehensive survey of the state of the art in this area.

Pellegrini [32] presents solutions to some of these problems, including efficient data structures (albeit less efficient than ours) for the ray-shooting problem, and also (a) an output-sensitive algorithm for computing the intersection of two nonconvex polyhedra in time $O(n^{8/5+\varepsilon} + k \log k)$, for any $\varepsilon > 0$, where *n* is the number of vertices, edges, and facets of the two polyhedra and *k* is the (similarly defined) complexity of their intersection; (b) an output-sensitive algorithm for constructing an arrangement of *n* triangles in 3-space in $O(n^{8/5+\varepsilon} + k \log k)$ time, where *k* is the output size; and (c) an algorithm that, in $O(n^{8/5+\varepsilon})$ expected time, counts all pairs of intersecting lines, in a set of *n* lines in 3-space.

Background. Algorithmic problems that involve lines in three dimensions have been studied for more than 30 years, covering the problems mentioned above and several others. An early study by McKenna and O'Rourke [30] has developed some of the tools and techniques for tackling these problems. Various techniques for ray shooting, and for the

related problems of computing and verifying depth orders and hidden surface removal have been studied in de Berg's dissertation [11], and later by de Berg et al. [12]. Another work that developed some of the infrastructure for these problems is by Chazelle et al. [16], who presented several combinatorial and algorithmic results for problems involving lines in 3space. Agarwal and Matoušek [5] reduced ray shooting problems, via parametric search, to segment emptiness problems (where the query is a segment and we want to determine whether it intersects any input object), and obtained efficient solutions via this reduction. See also [29] and [7] for studies of some additional and special cases of the ray shooting problem.

Most of the works cited above suffer from the 'curse' of the four-dimensionality of (the parametric representation of) lines in space, which leads to algorithms whose complexity is inferior to those obtained in our work. Nevertheless, there are a few instances where better solutions can be obtained, such as in [15, 16] and some other works.

Our results. Using the polynomial partitioning technique of [21, 22], we derive more efficient algorithms for the problems listed above. In our first main result, presented in Section 2, we tackle the ray-shooting problem, and construct a data structure on an input set of n triangles, which requires $O(n^{3/2+\varepsilon})$ storage and preprocessing, so that a ray shooting query can be answered in $O(n^{1/2+\varepsilon})$ time, for any $\varepsilon > 0$. We then extend the technique, in Section 3, to obtain an equally-efficient data structure for the segment-triangle intersection reporting, emptiness, and approximate counting problems, where in the case of approximate counting the query time bound has an additional term that is nearly linear in the output size.

These are significant improvements over previous results, which, as already noted, have treated the lines supporting the edges of the input triangles and the line supporting the query ray (or segment) as points or surfaces in a suitable four-dimensional parametric space (in many of the earlier works, lines were actually represented as points on the *Klein quadric* in five-dimensional projective space; see [13, 26, 33, 36]). As a result, the algorithms obtained by these techniques were less efficient.

A weakness, or rather an intriguing peculiarity, of our analysis is that it does not provide a desirably sharp tradeoff between storage and query time. To make this statement more precise, the tradeoff that the earlier solutions provide, say for the ray shooting problem for specificity, is that, for n input triangles and with s storage, for s varying between n and n^4 , a ray-shooting query takes $O(n^{1+\varepsilon}/s^{1/4})$ time; see, e.g., [33] (the '4' in the exponent comes from the fact that lines in 3-space are represented as objects in four-dimensional parametric space). Thus, with storage $O(n^{3/2+\varepsilon})$, which is what our solution uses, the query time becomes about $O(n^{5/8})$, considerably weaker than our bound.

An ambitious, and maybe unrealistic goal would be to improve the tradeoff so that the query time is only $O(n^{1+\varepsilon}/s^{1/3})$. (This does indeed coincide with the bound that our main result gives, as the storage that it uses is $O(n^{3/2+\varepsilon})$, but this coincidence only holds for this amount of storage.) Although not achieving this goal, still, combining our technique with the known, aforementioned '4-dimensional' tradeoff, we are able to obtain an 'in between' tradeoff, which we present in Section 4. Concretely, the tradeoff is that, with s storage, the

cost of a query is

$$Q(n,s) = \begin{cases} O\left(\frac{n^{5/4+\varepsilon}}{s^{1/2}}\right), & s = O(n^{3/2+\varepsilon}), \\ O\left(\frac{n^{4/5+\varepsilon}}{s^{1/5}}\right), & s = \Omega(n^{3/2+\varepsilon}). \end{cases}$$
(1)

Note that this tradeoff contains our bounds $(s, Q) = (O(n^{3/2+\varepsilon}), O(n^{1/2+\varepsilon}))$, as a special case, that at the extreme ends $s = \Theta(n)$, $s = \Theta(n^4)$, of the range of s we get $Q = O(n^{3/4+\varepsilon})$, $Q = O(n^{\varepsilon})$, respectively,¹ as in the older tradeoff, and that the new tradeoff is better for any in-between value of s. A comparison between the two tradeoffs is illustrated in Figure 1. Our improved tradeoff applies to all the problems studied in this paper. In particular, it implies that, in all these problems, the overall cost of processing m queries with n input objects, including preprocessing cost, is

$$\max\Big\{O(m^{2/3}n^{5/6+\varepsilon} + n^{1+\varepsilon}), \ O(n^{2/3}m^{5/6+\varepsilon} + m^{1+\varepsilon})\Big\},$$
(2)

for any $\varepsilon > 0$; for the output-sensitive problems, this bounds the total overhead cost. The first (resp., second) bound dominates when $n \ge m$ (resp., $n \le m$).



Figure 1: The old tradeoff (green) and the new tradeoff (red). The x-axis is the storage as a function of n, and the y-axis is the query cost. Both axes are drawn in a logarithmic scale.

We then present, in Section 5, extensions of our technique for solving the other problems (iii), (iv) and (v) listed above. In all these applications, our algorithms are output-sensitive for the reporting versions, so that the query time bound, or the full processing cost bound, contains an additional term that is nearly linear in the output size. See Section 5 for the concrete bounds that we obtain.

2 Ray shooting amid triangles

Let \mathcal{T} be a collection of n triangles in \mathbb{R}^3 . We fix some sufficiently large constant parameter D, and construct a partitioning polynomial f of degree O(D) for \mathcal{T} , so that each of the $O(D^3)$ connected components τ of $\mathbb{R}^3 \setminus Z(f)$ (the cells of the partition) is crossed by at most n/D^2 triangle edges. We refer to triangles whose edge crosses τ as narrow triangles (with respect to τ), and refer to the remaining triangles that cross τ (but none of their edges do) as wide triangles. We denote the set of narrow (resp., wide) triangles in τ by \mathcal{N}_{τ}

¹The actual query time in the older tradeoff, with maximum storage, is $Q = O(\log n)$.

(resp., W_{τ}). The existence of such a partitioning polynomial is implied, as a special case, by the general machinery developed in Guth [21]. An algorithm for constructing f is given in a recent work of Agarwal et al. [2]. It runs in O(n) time, for any constant value of D, where the constant of proportionality depends (polynomially) on D.

For technical reasons, we want to turn any query ray into a bounded segment, and we do it by enclosing all the triangles of \mathcal{T} by a sufficiently large bounding box B_0 , and by clipping any query ray to its portion within B_0 .

For each (bounded) cell $\tau \subseteq B_0$ of the partition, we take the set \mathcal{W}_{τ} of wide triangles in τ , and prepare a data structure for efficient segment-shooting queries into the triangles of \mathcal{W}_{τ} , by segments that are fully contained in τ . The nontrivial details of this procedure are given in Section 2.1. As we show there, we can construct such a structure with storage and preprocessing $O(|\mathcal{W}_{\tau}|^{3/2+\varepsilon}) = O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$ (where the choice of D depends on ε), and each segment-shooting query takes $O(|\mathcal{W}_{\tau}|^{1/2+\varepsilon}) = O(n^{1/2+\varepsilon})$ time.

The preprocessing then recurses within each such cell τ of the partition, with the set \mathcal{N}_{τ} of the narrow triangles in τ . The recursion terminates when the number of input triangles becomes smaller than the constant threshold $n_0 := O(D^2)$, in which case we simply output the list of triangles in the subproblem.

A query, with a ray (now turned into a segment) ρ , emanating from a point q, is answered as follows. We first consider the case where ρ (that is, the line containing ρ) is not fully contained in Z(f), and discuss the (simpler, albeit still involved) case where $\rho \subset Z(f)$, later.

The case where $\rho \not\subset Z(f)$. We assume a standard model of algebraic computation, in which a variety of computations involving polynomials of constant degree, such as computing (some discrete representation of) the roots of such polynomials, performing comparisons and algebraic computations (of constant degree) with these roots, and so on, can be done exactly in time $C(\delta)$, where δ is the degree of the polynomial, and $C(\delta)$ is a constant that depends on δ ; see, e.g., [9, 10].

Using this model, we first locate the cell of the partition that contains the starting endpoint q of the segment ρ , in constant time (recalling that D is a constant). One way of doing this is to construct the cylindrical algebraic decomposition (CAD) of Z(f) (see [18, 34]), associate with each cell σ of the CAD the cell of $\mathbb{R}^3 \setminus Z(f)$ that contains it (or an indication that σ is contained in Z(f)), and then search with q in the CAD, coordinate by coordinate (see, e.g., [2] for more details concerning such an operation). We then find, in constant time, the t = O(D) points of intersection of ρ with Z(f), and sort them into a sequence $P := (p_1, \ldots, p_t)$ in their order along ρ ; we assume that $p_t \in \partial B_0$, and ignore the suffix of ρ from p_t onwards. The points in P partition ρ into a sequence of segments, each of which is a connected component of the intersection of ρ with some cell. The first segment is $e_1 = qp_1$, the subsequent segments are $e_2 = p_1p_2$, $e_3 = p_2p_3, \ldots, e_t = p_{t-1}p_t$. We denote by τ_i the cell containing the *i*-th segment, for $i = 1, \ldots, t$ (a cell can appear several times in this sequence). See Figure 2.

We now process the segments e_i in order. For each segment e_i , let τ_i denote the partition cell that contains e_i . We first perform a ray-shooting (or rather a segment-shooting) query in the structure for \mathcal{W}_{τ_i} with the segment e_i . As already mentioned (and will be described in Section 2.1), this step can be performed in $O(n^{1/2+\varepsilon})$ time, with $O(n^{3/2+\varepsilon})$ storage



Figure 2: A two-dimensional rendering of the the general structure of the ray-shooting mechanism.

and preprocessing, for any $\varepsilon > 0$. We then query with e_i in the substructure recursively constructed for \mathcal{N}_{τ_i} . If at least one of the two queries succeeds, i.e., outputs a point that lies on e_i , we report the point nearest to the starting point of e_i , and terminate the whole query. If both queries fail, we proceed to the next segment e_{i+1} and repeat this step. If the mechanism fails for all the segments, we report that ρ does not hit any triangle of \mathcal{T} .

The case where $\rho \subset Z(f)$. We use the cylindrical algebraic decomposition (CAD) of Z(f) (see [18, 34]), which has already been constructed for the earlier case. One of its byproducts is a stratification of Z(f), which is a decomposition of Z(f) into pairwise disjoint relatively open patches of dimensions 0, 1, and 2, called *strata* (each stratum is a cell of the CAD), so that each of the two-dimensional strata is xy-monotone and its relative interior is free of any singularities of Z(f), and Z(f) is the union of the closures of these two-dimensional strata, excluding possible components of Z(f) of dimension at most 1, which we may ignore. We compute the intersection arcs $\gamma_{\Delta} := Z(f) \cap \Delta$, for $\Delta \in \mathcal{T}$, and distribute each arc amid the closures of the two-dimensional strata that it traverses. We then project the closure of each two-dimensional stratum σ onto the xy-plane, including the portions of the arcs γ_{Δ} that the closure contains, and preprocess the resulting collection of O(n) algebraic arcs, each of degree O(D) = O(1), into a planar ray-shooting data structure, whose details are spelled out in Section $2.2.^2$ As we show there, we can answer a ray-shooting query in $O(n^{1/2+\varepsilon})$ time, using $O(n^{3/2+\varepsilon})$ storage, for any $\varepsilon > 0$, where the constants of proportionality depend on ε , as does the choice of D. The overall storage complexity, over all the (projected) strata of Z(f), is thus $O(n^{3/2+\varepsilon})$, and the overall query time, over all strata met by the query ray ρ , is $O(n^{1/2+\varepsilon})$, for a larger constant of proportionality (that depends on ε).

²This specific planar ray-shooting problem, amid constant-degree algebraic arcs, has not received full attention in the past, although several algorithms have been proposed, mostly with suboptimal solutions. Consult, e.g., Table 2 in Agarwal [1]; see also [4, 27].

Note that the recursion on D when the query ray comes to lie on the zero set of the current partitioning polynomial. When this happens, we solve the problem in this recursive instance using the (nonrecursive) procedure in Section 2.2 and terminate the (current branch of the) recursion. Another way of saying this is that the leaves of the D-recursion tree represent either constant-size subproblems or subproblems on the zero set of the current partitioning polynomial, and the inner nodes represent subproblems of shooting within the partition cells.

Analysis. The correctness of the procedure is fairly easy to establish. Denote by S(n) the maximum storage used by the structure for a set of at most n triangles, and denote by $S_0(n)$ (resp., $S_1(n)$) the maximum storage used by the auxiliary structure for a set of at most n wide triangles in a cell of the partition, as analyzed in Section 2.1 (resp., for a set of at most n intersection arcs on Z(f), which we process for planar ray-shooting in Section 2.2). Then S(n) obeys the recurrence

$$S(n) = O(D^3)S_0(n) + S_1(n) + O(D^3)S(n/D^2),$$
(3)

for $n > n_0$, and S(n) = O(n) for $n \le n_0$, where $n_0 := cD^2$, for a suitable constant $c \ge 1$. We show, in the respective Sections 2.1 and 2.2, that $S_0(n) = O(n^{3/2+\varepsilon})$ and $S_1(n) = O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$, where both constants of proportionality depend on D and ε , from which one can easily show that the solution of (3) is $S(n) = O(n^{3/2+\varepsilon})$, for a slightly larger, but still arbitrarily small $\varepsilon > 0$; to achieve this bound, we need to take D to be $2^{\Theta(1/\varepsilon)}$, as will follow from our analysis. Regarding the bound on the preprocessing time T(n), we obtain a similar recurrence as in (3), namely,

$$T(n) = O(n) + O(D^3)T_0(n) + T_1(n) + O(D^3)T(n/D^2),$$

where the non-recursive linear term is the time to compute the polynomial f, and $T_0(n)$, $T_1(n)$ are defined in an analogous manner as above, and have similar upper bounds as $S_0(n)$, $S_1(n)$ (see Sections 2.1 and 2.2).

Similarly, denote by Q(n) the maximum query time for a set of at most n triangles, and denote by $Q_0(n)$ (resp., $Q_1(n)$) the maximum query time in the auxiliary structure for a set of at most n wide triangles in a cell of the partition (resp., for a set of at most n intersection arcs within Z(f), when the query ray lies on Z(f)). Then Q(n) obeys the recurrence

$$Q(n) = \max\left\{O(D)Q_0(n) + O(D)Q(n/D^2), \ Q_1(n)\right\},\tag{4}$$

for $n > n_0$, and Q(n) = O(n) = O(1) for $n \le n_0$. (This reflects the observation, made above, that the current branch of the recursion terminates when the query ray lies on the zero set of the current partitioning polynomial.) Again, the analysis in Sections 2.1 and 2.2 shows that $Q_0(n) = Q_1(n) = O(n^{1/2+\varepsilon})$, for any $\varepsilon > 0$ (where the choice of D depends on ε , as above), from which one can easily show, using induction on n, that the solution of (4) is $Q(n) = O(n^{1/2+\varepsilon})$, for a slightly larger but still arbitrarily small $\varepsilon > 0$.

The main result of this section is therefore:

Theorem 2.1 Given a collection of n triangles in three dimensions, and a prescribed parameter $\varepsilon > 0$, we can process the triangles into a data structure of size $O(n^{3/2+\varepsilon})$, in time $O(n^{3/2+\varepsilon})$, so that a ray shooting query amid these triangles can be answered in $O(n^{1/2+\varepsilon})$ time.

2.1 Ray shooting into wide triangles

Preliminaries. In this subsection we present and analyze our procedure for ray shooting in the set W_{τ} of the wide triangles in a cell τ of the partition. We then present, in Section 2.2, a different approach that yields a procedure for ray shooting within Z(f). Both procedures have the performance bounds stated in Theorem 2.1. The efficiency of our structures depends on D being a constant, since the constants of proportionality depend polynomially (and rather poorly) on D.

We thus focus now on ray shooting in a set of wide triangles within a three-dimensional cell τ of the partition. To appreciate the difficulty in solving this subproblem, we make the following observation. A simple-minded approach might be to replace each wide triangle $\Delta \in W_{\tau}$ by the plane h_{Δ} supporting it. Denoting the set of these planes as \mathcal{H}_{τ} , we could then preprocess \mathcal{H}_{τ} for ray-shooting queries, each of which specifies a query ray ζ and asks for the first intersection of ζ with the planes of \mathcal{H}_{τ} . Using standard machinery (see, e.g. [1]), this would result in an algorithm with the performance bounds that we want. However, this approach is problematic, since, even though Δ is wide in τ , h_{Δ} could intersect τ in several connected components, some of which lie outside Δ . See Figure 3 for an illustration. In such cases, ray shooting amid the planes in \mathcal{H}_{τ} is not equivalent to ray shooting amid the triangles of \mathcal{W}_{τ} , even for rays, or rather portions thereof, that are contained in τ .



Figure 3: Wide triangles cannot be replaced by their supporting planes for ray shooting within τ .

Our solution is therefore more involved, and proceeds as follows.

Canonical sets of wide triangles. Consider first, for exposition sake, the case where the starting point of the shooting segment lies on $\partial \tau$ (the terminal point always lies on $\partial \tau$). As we will show, for each such segment query, the set of wide triangles in W_{τ} that it intersects can be decomposed into a small collection of precomputed "canonical" subsets, where in each canonical set the wide triangles can be treated as planes (for that particular query segment). We show below that the overall size of these sets, over all possible segment queries, is $O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$. Actually, to prepare for the complementary case, where the starting point of the query segment lies inside τ , we calibrate our algorithm, so that we control the storage that it uses, and consequently also the query time bound. To do so, we introduce a *storage parameter* s, which can range between n and n^2 , as a second input to our procedure, and then require that the actual storage and preprocessing cost be both $O(s^{1+\varepsilon})$, for any $\varepsilon > 0$. This relaxed notion of storage offers some simplification in the analysis. (We will also allow larger values of s when we discuss tradeoff between storage and query time, in Section 4.)

For each $\Delta \in \mathcal{W}_{\tau}$, let γ_{Δ} denote the intersection curve of Δ with $\partial \tau$. Note that γ_{Δ} does not have to be connected—it can have up to $O(D^2)$ connected components, by Harnack's curve theorem [24] (applied on the plane containing Δ). Note also that $\partial \tau$ does not have to be connected, so γ_{Δ} can have nonempty components on different connected components of $\partial \tau$, as well as several components on the same connected component of $\partial \tau$.

We construct the locus S_{τ} of points on $\partial \tau$ that are either singular points of Z(f) or points with z-vertical tangency. Since D is constant, S_{τ} is a curve of constant degree (by Bézout's theorem, its degree is $O(D^2)$). We take a random sample \mathcal{R} of r_0 triangles of \mathcal{W}_{τ} , where the analysis dictates that we choose $r_0 = D^{\Theta(1/\varepsilon)}$, for the arbitrarily small prescribed $\varepsilon > 0$. Since we have chosen D to be $2^{\Theta(1/\varepsilon)}$, the actual choice of r_0 is $2^{\Theta(1/\varepsilon^2)}$.

Let $\Gamma_{\mathcal{R}} = \{\gamma_{\Delta} \mid \Delta \in \mathcal{R}\}$, and let $\mathcal{A}_0 = \mathcal{A}(\Gamma_{\mathcal{R}} \cup \{S_{\tau}\})$ denote the arrangement of these curves within $\partial \tau$, together with S_{τ} . By construction, each face of \mathcal{A}_0 is *xy*-monotone and does not cross any other branch of Z(f) (at a singular point). We partition each face φ of \mathcal{A}_0 into pseudo-trapezoids (called trapezoids for short), using a suitably adapted version of a two-dimensional vertical decomposition scheme. Let \mathcal{A}_0^* denote the collection of these trapezoids on $\partial \tau$. The number of trapezoids in \mathcal{A}_0^* is proportional to the complexity of \mathcal{A}_0 , which is $O_D(r_0^2) = O(1)$ (we use the notation $O_D(\cdot)$ to indicate that the constant of proportionality depends on D, and recall that r_0 also depends on D).

We assume that the trapezoids are relatively open. To cover all possible cases, we also include in the collection of trapezoids the relatively open subarcs of arcs in $\Gamma_{\mathcal{R}}$ that the partition generates, the vertical edges of the trapezoids, and the vertices of the partition, but, for exposition sake, we will only handle here the case of two-dimensional trapezoids. (The inclusion of lower-dimensional 'trapezoids' is simpler to handle; it does not affect the essence of the forthcoming analysis, nor does it affect the asymptotic performance bounds.)

Let ψ_1, ψ_2 be two distinct trapezoids of \mathcal{A}_0^* . Let $S(\psi_1, \psi_2)$ denote the collection of all segments e such that one endpoint of e lies in ψ_1 , the other endpoint lies in ψ_2 , and the relative interior of e is fully contained in the open cell τ . We can parameterize such a segment e by four real parameters, so that two parameters specify the starting endpoint of e (as a point in ψ_1 , using, e.g., the *xy*-parameterization of the *xy*-monotone face containing ψ_1), and the other two parameters similarly specify the other endpoint. (Fewer parameters are needed when lower-dimensional trapezoids are involved.) Denote by \mathcal{F} the corresponding (at most) four-dimensional parametric space. Since each of τ, ψ_1, ψ_2 is of constant complexity, $S(\psi_1, \psi_2)$ is a semi-algebraic set in \mathcal{F} of constant complexity. More specifically, we can write $S(\psi_1, \psi_2)$ as an (implicitly) quantified formula of the form

$$S(\psi_1, \psi_2) = \{ (p_1, p_2) \mid p_1 \in \psi_1, p_2 \in \psi_2, \text{ and } p_1 p_2 \subset \tau \},\$$

where p_1p_2 denotes the line-segment connecting p_1 to p_2 . Using the singly exponential quantifier-elimination algorithm in [10, Theorem 14.16], we can construct, in $O_D(1)$ time,

a quantifier-free semi-algebraic representation of $S(\psi_1, \psi_2)$ of $O_D(1)$ complexity. Moreover, we can decompose $S(\psi_1, \psi_2)$ into its connected components, in $O_D(1)$ time as well.

For each segment $e \in S(\psi_1, \psi_2)$, let $\mathcal{T}(e)$ denote the set of all wide triangles of \mathcal{W}_{τ} that e crosses. We have the following technical lemma.

Lemma 2.2 In the above notations, each connected component C of $S(\psi_1, \psi_2)$ can be associated with a fixed set \mathcal{T}_C of wide triangles of \mathcal{W}_{τ} , none of which crosses $\psi_1 \cup \psi_2$, so that, for each segment $e \in C$, $\mathcal{T}_C \subseteq \mathcal{T}(e)$, and each triangle in $\mathcal{T}(e) \setminus \mathcal{T}_C$ crosses either ψ_1 or ψ_2 .

Proof. Pick an arbitrary but fixed segment e_0 in C, and define \mathcal{T}_C to consist of all the triangles in $\mathcal{T}(e_0)$ that do not cross $\psi_1 \cup \psi_2$. See Figure 4 for an illustration.



Figure 4: The set \mathcal{T}_C (consisting of the triangles depicted as black segments), and an illustration of the proof of Lemma 2.2.

Let e be another segment in C. Since C is connected, as a set in \mathcal{F} (recall that this is a four-dimensional parametric space representing the segments), there exists a continuous path π in C that connects e_0 and e (recall that each point on π represents a segment with one endpoint on ψ_1 and the other on ψ_2 , and π represents a continuous variation of such a segment from e_0 to e). Let Δ be a triangle in $\mathcal{T}(e_0)$ that does not cross $\psi_1 \cup \psi_2$ (that is, $\Delta \in \mathcal{T}_C$), and let h_Δ denote its supporting plane. As a segment e' traverses π from e_0 to e, the point $q_\Delta(e') := e' \cap h_\Delta$ is well defined and varies continuously in τ , unless e' comes to be contained in, or parallel to h_Δ , a situation that, as we now argue, cannot arise.

In what follows, we are going to argue that the segment e' is detached from Δ when either (i) the relative interior of e' touches the boundary of $\Delta \cap \tau$, which cannot happen since then e' would have to (partially) exit τ and meet its boundary, contrary to the assumption that e' is fully contained in τ (recall that τ is open), or (ii) $\Delta \cap \tau$ touches an endpoint of e', which again cannot happen because the endpoints of e' lie on $\psi_1 \cup \psi_2$, and Δ is assumed not to intersect $\psi_1 \cup \psi_2$. More formally, we argue as follows. By assumption, $q_{\Delta}(e_0)$ lies in Δ , and, as long as $q_{\Delta}(e')$ is defined (i.e., e' intersects Δ at a unique point), $q_{\Delta}(e')$ cannot reach $\partial \Delta$ because the corresponding segment e' is fully contained in the open cell τ and Δ is wide in τ . (We may assume that e' does not overlap Δ , because, since Δ is wide, that would mean that both endpoints of e' lie on Δ , but then Δ crosses both ψ_1 and ψ_2 , which we have assumed not to be the case.) We claim that $q_{\Delta}(e')$ must be nonempty throughout the motion of e' along π , for otherwise $q_{\Delta}(e')$ would have to reach an endpoint of e', which, by definition of $S(\psi_1, \psi_2)$, must lie on ψ_1 or on ψ_2 . But then Δ would have to intersect either ψ_1 or ψ_2 , contrary to assumption. It follows that $q_{\Delta}(e)$ also lies in Δ , so $\Delta \in \mathcal{T}(e)$. This establishes the first assertion of the lemma.

We next need to show that each triangle in $\mathcal{T}(e) \setminus \mathcal{T}_C$ must cross either ψ_1 or ψ_2 , which is our second assertion. Let Δ be a triangle in $\mathcal{T}(e) \setminus \mathcal{T}_C$, and assume to the contrary that Δ does not cross $\psi_1 \cup \psi_2$. We run the preceding argument in reverse (moving from e to e_0), and observe that, by assumption and by the same argument (and notations) as above, $q_{\Delta}(e')$ remains inside e', for all intermediate segments e' along the connecting path π , and does not reach $\partial \Delta \cap \tau$, so $\Delta \in \mathcal{T}(e_0)$ and thus also $\Delta \in \mathcal{T}_C$ (by definition of \mathcal{T}_C), contradicting our assumption. This establishes the second assertion, and thereby completes the proof. \Box

Lemma 2.2 and its proof show that, for each connected component C of $S(\psi_1, \psi_2)$, the canonical set \mathcal{T}_C , of wide triangles that are crossed by all segments in C and do not cross $\psi_1 \cup \psi_2$, assigned to C, is unique and is independent of the choice of e_0 . (This is because the sets $\mathcal{T}(e_0)$, for $e_0 \in C$, differ from each other only in triangles that cross either ψ_1 or ψ_2 .) The collection of all these sets \mathcal{T}_C , over all connected components C, and all pairs of trapezoids (ψ_1, ψ_2) , is part of the whole output collection of canonical sets over τ ; the rest of this collection is constructed recursively over the trapezoids ψ of \mathcal{A}_0^* .

The algorithm. For each trapezoid ψ of \mathcal{A}_0^* , the *conflict list* K_{ψ} of ψ is the set of all wide triangles that cross ψ . By standard random sampling arguments [14], with high probability, the size of each conflict list is $O\left(\frac{n}{r_0}\log r_0\right)$, where the constant of proportionality depends on D.

Two extreme situations that require special treatment are (i) 0-dimensional trapezoids (vertices), where there is no bound on the number of triangles of W_{τ} that can contain a vertex ψ (in which case we do not recurse at ψ), but it suffices just to maintain one of them in the structure, because if the starting (or the other) endpoint of a query segment lies at ψ , it does not matter which of these incident triangles we use. Also, we do not recurse at ψ (technically, it has no conflict list, as no triangle crosses it). (ii) triangles that fully contain a two-dimensional trapezoid ψ , where these triangles are contained in some planar component of Z(f) (where only the triangles do not overlap one another. Since we are handling here rays that are not contained in Z(f), such a ray ρ can cross Z(f) at only O(D) = O(1) points, and it is easy to find, in O(1) time, these crossing points, and then check, in $O(\log n)$ time (with linear storage), whether any of these points belongs to a triangle contained in Z(f). (The case where the triangles are overlapping is also easy to handle. The performance bounds deteriorate, but are still within the overall bounds that we derive.)

For every pair of trapezoids ψ_1 , ψ_2 , we compute $S(\psi_1, \psi_2)$ and decompose it into its connected components. We pick some arbitrary but fixed segment e_0 from each component C, compute the set $\mathcal{T}(e_0)$ of the wide triangles that cross e_0 , and remove from it any triangle that crosses $\psi_1 \cup \psi_2$, thereby obtaining the set \mathcal{T}_C . All this takes $O_D(r_0^4 n) = O_D(n)$ time, and the overall size of the produced canonical sets is also $O_D(n)$. Let s be the storage parameter associated with the problem, as defined earlier, and recall that we require that $n \leq s \leq n^2$. Each canonical set \mathcal{T}_C is preprocessed into a data structure that supports ray shooting in the set of planes $\mathcal{H}_C = \{h_\Delta \mid \Delta \in \mathcal{T}_c\}$, where h_Δ is the plane supporting Δ . We construct these structures so that they use $O(s^{1+\varepsilon})$ storage (and preprocessing), for any $\varepsilon > 0$, and a query takes $O(n \text{ polylog}(n)/s^{1/3})$ time (see, e.g., [1]).

We now process recursively each conflict list K_{ψ} , over all trapezoids ψ of \mathcal{A}_0^* . Each recursive subproblem uses the same parameter r_0 , but now the storage parameter that we allocate to each subproblem is only s/r_0^2 . We keep recursing until we reach conflict lists of size close to n^2/s . More precisely, after j levels of recursion, we get a total of at most $(c_0r_0^2)^j = c_0^j r_0^{2j}$ subproblems, each involving at most $\left(\frac{c_1 \log r_0}{r_0}\right)^j n$ wide triangles, for some constants c_0, c_1 that depend on D, and thus on ε (specifically, c_1 depends on c_0 and $c_0 = O(D^2)$ by Bézout's theorem), but are considerably smaller than r_0 , which, as already mentioned, we take to be $D^{\Theta(1/\varepsilon)}$.

We stop the recursion at the first level j^* at which $(c_1r_0 \log r_0)^{j^*} \ge s/n$. As a result, we have $r_0^{j^*} \le s/n$, and we get $c_0^{j^*} r_0^{2j^*} = O(s^2/n^{2-\varepsilon})$ subproblems, for any $\varepsilon > 0$, where the choice of D (and therefore also of c_0 , c_1 and r_0) depends, as above, on ε . Each of these subproblems involves at most

$$\left(\frac{c_1\log r_0}{r_0}\right)^{j^*} n = \left(\frac{(c_1\log r_0)^2}{c_1r_0\log r_0}\right)^{j^*} n \le (c_1\log r_0)^{2j^*} \cdot \frac{n^2}{s} = \frac{n^{2+\varepsilon}}{s}$$

triangles, for any $\varepsilon > 0$. For this estimate to hold, we choose $D = 2^{\Theta(1/\varepsilon)}$. Hence the overall size of the inputs, as well as of the canonical sets, at all the subproblems throughout the recursion, is $O\left(\frac{s^2}{n^{2-\varepsilon}}\right) \cdot \frac{n^{2+\varepsilon}}{s} = O(sn^{2\varepsilon}) = O(s^{1+\varepsilon})$, for a slightly larger $\varepsilon > 0$.

Note that the canonical sets that we encounter when querying with a fixed segment e are not necessarily pairwise disjoint. This is because the sets K_{ψ_1} and K_{ψ_2} are not necessarily disjoint (they are disjoint of \mathcal{T}_C , though). This does not pose a problem for ray shooting queries, but will be problematic for *counting queries*; see Section 3.

At the bottom of the recursion, each subproblem contains at most $n^{2+\varepsilon}/s$ wide triangles, which we merely store in the structure. As just calculated, the overall storage that this requires is $O(s^{1+\varepsilon})$, for a slightly larger ε , as above. We obtain the following recurrence for the overall storage $S(N_W, s_W)$ for the structure constructed on N_W wide triangles, where s_W denotes the storage parameter allocated to the structure (at the root $N_W = n, s_W = s$).

$$S(N_W, s_W) = \begin{cases} O_D(r_0^4 s_W^{1+\varepsilon}) + c_0 r_0^2 S\left(\frac{c_1 N_W \log r_0}{r_0}, \frac{s_W}{r_0^2}\right) & \text{for } N_W \ge n^{2+\varepsilon}/s, \\ O(N_W) & \text{for } N_W < n^{2+\varepsilon}/s. \end{cases}$$

(The overhead term is actually $O_D(r_0^4 N_W + r_0^4 s_W^{1+\varepsilon})$, but the second term dominates.) Throughout the recursion we have $N_W \leq s_W \leq N_W^2$. Indeed, starting with n and s, after j recursive levels we have $N_W \leq \left(\frac{c_1 \log r_0}{r_0}\right)^j n$ and $s_W = s/r_0^{2j}$. Hence the right inequality continues to hold (for $s \leq n^2$), and the left inequality holds as long as $\left(\frac{c_1 \log r_0}{r_0}\right)^j n \leq s/r_0^{2j}$, or $(c_1r_0 \log r_0)^j \leq s/n$, which indeed holds up to the terminal level j^* , by construction. Unfolding the first recurrence up to the terminal level j^* , where $N_W < n^{2+\varepsilon}/s$, the sum of the nonrecursive overhead terms $O_D(r_0^4 s_W^{1+\varepsilon})$, over all nodes at a fixed level j, is

$$c_0^j r_0^{2j} \cdot O\left(\frac{s_W^{1+\varepsilon}}{r_0^{2j(1+\varepsilon)}}\right) = O\left(\frac{c_0^j}{r_0^{2j\varepsilon}} s_W^{1+\varepsilon}\right) = O\left(s_W^{1+\varepsilon}\right),$$

by the choice of r_0 . Hence, starting the recurrence at $(N_W, s_w) = (n, s)$, the overall contribution of the overhead terms (over the logarithmically many levels) is $O(s^{1+\varepsilon})$, for a slightly larger ε . At the bottom of recurrence, we have, as already noted, $O(s^2/n^{2-\varepsilon})$ subproblems, each with at most $O(n^{2+\varepsilon}/s)$ triangles, so the sum of the terms N_W at the bottom of recurrence is also $O(s^{1+\varepsilon})$. In other words, the overall storage used by the data structure is $O(s^{1+\varepsilon})$. Using similar considerations, one can show that the overall preprocessing time is $O(s^{1+\varepsilon})$ as well, since the time obeys essentially the same recurrence.

Answering a query. To perform a query with a segment e that starts at a point a (that lies anywhere inside τ), we extend e from a backwards, find the first intersection point a' of the resulting backward ray with $\partial \tau$, and denote by e' the segment that starts at a' and contains e. See Figure 5 for an illustration. This takes $O_D(1)$ time. This step is vacuous when e starts on $\partial \tau$, in which case we have e' = e.



Figure 5: Segment shooting from inside the cell τ : Extending the segment backwards and the resulting canonical set of triangles.

We find the pair of trapezoids ψ_1 , ψ_2 that contain the endpoints of e', find the connected component $C \subseteq S(\psi_1, \psi_2)$ that contains e', and retrieve the canonical set \mathcal{T}_C . We then perform segment shooting along e from a in the structure constructed for \mathcal{H}_C , and then continue recursively in the subproblems for K_{ψ_1} and K_{ψ_2} . We output the triangle that ehits at a point nearest to a, or, if no such point is produced, report that e does not hit any wide triangle inside τ . In case both endpoints of e' lie in the same trapezoid ψ (that is, $\psi_1 = \psi_2$), we set \mathcal{T}_C to be empty at this step (it is easy to verify that this indeed must be the case), and then continue processing e' (and thus e) in the recursion on K_{ψ} .

The correctness of the procedure follows from the fact that e' intersects all the triangles of \mathcal{T}_C , and thus replacing these triangles by their supporting planes cannot produce any new (false) intersection of any of these triangles with e, and any other wide triangle that e hits must belong to $K_{\psi_1} \cup K_{\psi_2}$.

The query time $Q(N_W, s_W)$ satisfies the recurrence

$$Q(N_W, s_W) = \begin{cases} O_D(1) + O\left(\frac{N_W \text{polylog}(N_W)}{s_W^{1/3}}\right) + 2Q\left(\frac{c_1 N_W \log r_0}{r_0}, \frac{s_W}{r_0^2}\right) & \text{for } N_W \ge n^{2+\varepsilon}/s, \\ O(N_W) & \text{for } N_W < n^{2+\varepsilon}/s. \end{cases}$$

Unfolding the first recurrence, we see that when we pass from some recursive level to the next one, we get two descendant subproblems from each recursive instance, and the term N_W polylog $(N_W)/s_W^{1/3}$ is replaced in each of them by the (upper bound) term

$$\frac{\frac{c_1 N_W \log r_0}{r_0}}{\left(\frac{s_W}{r_0^2}\right)^{1/3}} \cdot \text{polylog}(N_W) = \frac{c_1 \log r_0}{r_0^{1/3}} \cdot \frac{N_W \text{polylog}(N_W)}{s_W^{1/3}}.$$

Hence the overall bound for the nonrecursive overhead terms in the unfolding, starting from $(N_W, s_W) = (n, s)$, is at most

$$O\left(\sum_{j\geq 0} \left(\frac{2c_1\log r_0}{r_0^{1/3}}\right)^j\right) \cdot \frac{n \operatorname{polylog}(n)}{s^{1/3}} = O\left(\frac{n \operatorname{polylog}(n)}{s^{1/3}}\right),$$

provided that r_0 is sufficiently large. Adding the cost at the 2^{j^*} subproblems at the bottom level j^* of the recursion, where the cost of each subproblem is at most $n^{2+\varepsilon}/s$, gives an overall bound for the query time of

$$Q(n,s) = O\left(\frac{n \text{ polylog}(n)}{s^{1/3}} + \frac{n^{2+\varepsilon}}{s}\right).$$
(5)

Starting with $s = n^{3/2}$, the query time is $O(n^{1/2+\varepsilon})$. We thus obtain

Proposition 2.3 For a (bounded) cell τ of the polynomial partition, and a set W of n wide triangles in τ , one can construct a data structure of size and preprocessing cost $O(n^{3/2+\varepsilon})$, so that a segment-shooting query within τ , from any starting point, can be answered in $O(n^{1/2+\varepsilon})$ time, for any $\varepsilon > 0$.

The case where the query ray is contained in Z(f) is discussed in detail in the following Section 2.2, culminating in Proposition 2.7, with the same performance bounds. Thus, combined with the results in this subsection, Theorem 2.1 follows.

2.2 Ray shooting within Z(f)

We now consider the case where the (line supporting the) query ray is contained in the zero set Z(f) of f. We present our result in a more general form, in which we are given a collection Γ of n constant-degree algebraic arcs in the plane,³ and preprocess it into a data structure of $O(n^{3/2+\varepsilon})$ size, which can be constructed in $O(n^{3/2+\varepsilon})$ time, that supports ray-shooting queries in time $O(n^{1/2+\varepsilon})$ per query, for any $\varepsilon > 0$.

³Recall that we project each stratum of Z(f) onto the xy-plane.

Ray shooting amid arcs in the plane. Let Γ be a set of *n* algebraic arcs of constant degree in the plane. We may assume, after breaking each arc into a constant number of subarcs, if needed, that each arc is *x*-monotone, has a smooth relative interior, and is either convex or concave. For concreteness, and with no loss of generality, we assume in what follows that all the arcs of Γ are convex. That is, the tangent directions turn counterclockwise along each arc as we traverse it from left to right.

We present the solution in four steps. We first discuss the problem of detecting an intersection between a query line and the input arcs. We then extend this machinery to detecting intersection with a query ray, and finally to detecting intersection with a query segment. Once we have such a procedure, we can use the parametric-search technique of Agarwal and Matoušek [5] (this is our fourth step) to perform ray shooting, with similar performance bounds. The reason for this gradual presentation of the technique is that each step uses the structure from the previous step as a substructure.

We remind the reader, again, that the problem of ray shooting in the plane amid a general collection of constant-degree algebraic arcs, which is the problem considered in this section, does not seem to have a solution with sharp performance bounds; see Table 2 in [1] and [4, 27].

2.2.1 Detecting line intersection with the arcs

Our approach is to transform the line-intersection problem to a planar point-location problem, by mapping the lines to points and the arcs $\gamma \in \Gamma$ to semi-algebraic sets (whose complexity depends on the complexity of γ). Our mapping is based on *quantifier elimination*, and proceeds as follows.⁴

Fix an arc $\gamma \in \Gamma$, and recall that it is a constant-degree algebraic arc in the plane, and, by assumption, γ is convex, smooth and x-monotone. Consider the smallest affine variety (curve) V_{γ} that contains γ , known as the Zariski closure of γ [19], and let F(x, y) be the bivariate polynomial whose zero set is V. F is a polynomial of constant degree, which we denote by d. Consider a line ℓ , given by the equation y := ax + b, where a, b are real coefficients. (Vertical lines are easier to handle, and we ignore this special case in what follows.) Then ℓ intersects γ if and only if there exists $x \in \mathbb{R}$ such that $(x, ax + b) \in \gamma$. This can be expressed as a quantified Boolean algebraic predicate of constant complexity (i.e., involving a constant number of variables, and a constant number of polynomial equalities and inequalities of constant degrees); one of the clauses of the predicate is F(x, ax + b) = 0and the others restrict (x, ax + b) to lie in γ . Using the singly exponential quantifierelimination algorithm in [10, Theorem 14.16] (also used earlier in Section 2.1), we can construct, in $O_d(1)$ time, a quantifier-free semi-algebraic set $G := G_{\gamma}$ in the *ab*-parametric plane, whose overall complexity is $O_d(1)$ as well, such that the quantified predicate is true if and only if $(a, b) \in G$; see, e.g., [6] for a concrete construction of such a set for the problem of intersection detection between lines and spheres in \mathbb{R}^3 .

We have thus mapped the setting of our problem to a planar point location problem amid a collection \mathcal{G} of *n* semi-algebraic regions of constant complexity. Using standard tech-

⁴There is another, more direct approach to solving this problem, which is easier to visualize but involves several levels of range searching structures. The running time of this approach, which we do not detail here, is asymptotically the same as the bound that we get here.

niques based on ε -cuttings (see, e.g., [17] for such constructions), one can construct, using overall storage and preprocessing time of $O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$, a data structure that supports point-location queries in the arrangement $\mathcal{A}(\mathcal{G})$ of these regions in time $O(n^{1/2})$ per query;⁵ note that the factor n^{ε} does not appear in the query time bound, but only in the preprocessing time bound.

Briefly, to do so, we construct a $(1/\sqrt{n})$ -cutting of $\mathcal{A}(\mathcal{G})$ in $O(n^{3/2})$ time. This is a partition of the plane into O(n) pseudo-trapezoids, each crossed by $O(n^{1/2})$ boundaries of the regions in \mathcal{G} (see [17]). Each pseudo-trapezoid (trapezoid for short) has a conflict list $\mathcal{G}_t au$ of the set of regions whose boundaries cross τ , and another list $\mathcal{G}_{\tau}^{(0)}$ of regions that fully contain τ . The lists \mathcal{G}_{τ} are stored explcitly at the respective regions τ , as their overall size is $O(n^{3/2})$. The overall size of the lists $\mathcal{G}_{\tau}^{(0)}$ is $O(n^2)$, so we store them implicitly in a persistent data structure, based on some tour of the trapezoids of the cutting, using the fact that $\mathcal{G}_{\tau}^{(0)}$ changes by only $O(n^{1/2})$ regions as we pass from τ to an adjacent trapezoid τ' (we gloss here over certain technical issues involved in the construction of such a tour). explcitly at the respective regions τ , as their overall size is $O(n^{3/2})$.

For the problem of detecting line intersection, it suffices to test whether the query point (a, b) (representing a query line ℓ) is contained in any of the input semi-algebraic regions. To do so, we locate (a, b) in the cutting. If its containing trapezoid τ has a nonempty list $\mathcal{G}_{\tau}^{(0)}$, we report that ℓ intersects an arc of Γ and stop. Otherwise we go over the conflict list \mathcal{G}_{τ} and test explicitly whether ℓ intersects any of the associated arcs of Γ , in $O(n^{1/2})$ time. Moreover, this point-location machinery can also return a compact representation of the set of the arcs from Γ that intersect ℓ , as a disjoint union of $O(n^{1/2})$ precomputed canonical subsets of Γ (namely, the set $\mathcal{G}_{\tau}^{(0)}$ of the trapezoid τ containing (a, b), and the $O(n^{1/2})$ singleton sets corresponding to those regions in \mathcal{G}_{τ} that contain (a, b)). This latter property is useful for the extensions of this procedure for detecting intersections of rays or segments with the given arcs, described below.

2.2.2 Detecting ray and segment intersections with the arcs

We next augment the data structure so that it can test whether a query ray ρ intersects any arc in Γ . A similar, somewhat more involved approach, which is spelled out later in this section, allows us also to test whether a query segment *s* intersects any arc in Γ . Using the parametric-search machinery of Agarwal and Matoušek [5], this latter structure allows us to answer ray shooting queries (finding the first arc of Γ hit by a query ray ρ) with similar performance bounds.

We comment that in principle we could have simply used an extended version of the quantifier-elimination technique used in the previous subsection. However, such an extension requires more parameters to represent a ray (three parameters) or a segment (four parameters). As a consequence, the space in which we need to perform the search becomes three- or four-dimensional, and the performance of the solution deteriorates. We therefore use a different, more explicit approach to these extended versions.

We also comment that the analysis presented next only applies to nonvertical rays and segments. Handling vertical rays is much simpler, and amounts, with some careful

 $^{{}^{5}}$ We comment that we need to exploit our model of computation in which root extraction, and manipulations of these roots, can be performed in constant time, for univariate polynomials of constant degree.

modifications, to point location of the apex of the ray in the arrangement of the given arcs, which can be implemented with standard techniques, with performance bounds that match the ones that we obtain for the general problem. We therefore assume in what follows that the query rays and segments are nonvertical.

So let ρ be a query ray, let q be the apex of ρ , and let ℓ be the line supporting ρ . We assume, without loss of generality, that ρ is directed to the right (for rays directed to the left, a symmetric set of conditions apply). We have:

Lemma 2.4 Let ρ , $q = (q_x, q_y)$ and ℓ be as above. Then ρ intersects a convex x-monotone arc γ , oriented from left to right, if and only if ℓ intersects γ , and one of the following conditions holds, where u and v are the left and right endpoints of γ , and where a is the slope of ℓ .

- (a) q lies to the left of u. See Figure 6(a).
- (b) q lies between u and v and below γ, and the tangent direction to γ at q_x is smaller than
 a. See Figure 6(b).
- (c) q lies between u and v and above γ , and v lies above ℓ . See Figure 6(c).



Figure 6: Scenarios where a ray ρ intersects a convex x-monotone arc γ : (a) q lies to the left of u. (b) q lies between u and v and below γ , and the tangent direction to γ at q_x is smaller than a. (c) q lies between u and v and above γ , and v lies above ℓ .

Proof. The 'only if' part of the lemma is simple, and we only consider the 'if' part. We are given that ℓ intersects γ . If q lies to the left of u then clearly ρ also intersects γ (this is

Case (a), where we actually have $\ell \cap \gamma = \rho \cap \gamma$), and if q lies to the right of v then clearly ρ does not intersect γ . Assume then that q lies between u and v. If q lies above γ , the ray intersects γ if and only if v lies above ℓ , as is easily checked, which is Case (c). If q lies below γ then, given that ℓ intersects γ , ρ intersects γ if and only if q lies to the left of the left intersection point in $\ell \cap \gamma$, and this happens if and only if the slope of the tangent to γ at q_x is smaller than the slope of ℓ . This is Case (b), and thus the proof is completed. \Box

Our data structure is constructed by taking the structure of Section 2.2.1 and augmenting it with additional levels, in three different ways, each testing for one of the conditions (a), (b), (c) in Lemma 2.4.

Testing for Condition (a) is easily done with a single additional level based on a onedimensional search tree on the left endpoints of the arcs.

Testing for Condition (b) requires three more levels. The first level is a segment tree on the x-spans of the arcs, which we search with q_x , to retrieve all the arcs whose x-span contains q_x , as the disjoint union of $O(\log n)$ canonical sets. The second level filters out those arcs that lie above q. As in the line-intersection structure given in Section 2.2.1 (except that the plane in which we perform the search is the actual xy-plane and not the parametric ab-plane), this level requires $O(n^{3/2+\varepsilon})$ storage and preprocessing (where n is the size of the present canonical set), and answers a query in $O(n^{1/2})$ time. In the third level we consider the tangent directions of the arcs of γ as partial functions of x, and construct their lower envelope (see [35, Corollary 6.2]). We can then test whether (q_x, a) lies above the envelope in logarithmic time.

Finally, testing for Condition (c) also requires three more levels, where the first two levels are as in case (b), and the third level tests whether there is any right endpoint v of an arc in the present canonical set that lies above ℓ , by constructing, in nearly linear time, the upper convex hull of the right endpoints and by testing, in logarithmic time, whether ℓ does not pass fully above the hull (see, e.g., [20]).

It is easily verified that the overall data structure has the desired performance bounds, namely, $O(n^{3/2+\varepsilon})$ storage and preprocessing cost, and $O(n^{1/2+\varepsilon})$ query time, for any $\varepsilon > 0$.

Detecting segment intersection. The same mechanism works when ρ is a segment, rather than a ray, except that the conditions for intersection with an arc of Γ are more involved. To simplify the presentation, we reduce the problem to the ray-intersection detection problem just treated, thereby avoiding explicit enumeration of quite a few subcases that need to be tested.

We associate with each arc $\gamma \in \Gamma$ the semi-unbounded region

 $\kappa = \kappa(\gamma) = \{(x, y) \mid u_x \le x \le v_x \text{ and } (x, y) \text{ lies strictly above } \gamma\}.$

That is, κ is bounded on the left by the upward vertical ray emanating from u, bounded on the right by the upward vertical ray emanating from v, and bounded from below by γ ; see Figure 7. Then we have the following extension of Lemma 2.4:

Lemma 2.5 Let γ , u, v, and κ be as above. Let s be a segment, with left endpoint $p = (p_x, p_y)$, right endpoint $q = (q_x, q_y)$, and slope a, and let ℓ be the line supporting s. Let ρ_p be the ray that starts at p and contains s (so ρ_p is rightward directed), and let ρ_q be the ray

that starts at q and contains s (so ρ_q is leftward directed). Then s intersects γ if and only if all the following conditions hold:

- (a) ℓ intersects γ .
- (b) At least one of p, q lies outside κ .
- (c) $\rho_p \cap \gamma \neq \emptyset$ and $\rho_q \cap \gamma \neq \emptyset$.

See Figure 7 for an illustration.



Figure 7: A segment s and a convex x-monotone arc γ , where the line ℓ containing s intersects γ : (a) Both endpoints of s lie inside κ and therefore s is disjoint from γ , although in this case ρ_p , ρ_q (depicted by the dash arrows in the figure) both intersect γ . (b) Several different positions of s, in each of which there is an endpoint of s outside κ . In this case s intersects γ if and only if both ρ_p and ρ_q intersect γ (these rays are drawn for an illustration for only two of the segments in the subfigure).

Proof. Here too, the 'only if' part of the lemma is simple, and we only consider the 'if' part. Condition (a) and the convexity of γ imply that $\ell \cap \gamma$ consists of one or two points.

Assume first that $\ell \cap \gamma$ consists of two points ξ , η . By Condition (b) at least one of p, q lies outside the (open) interval $\xi\eta$. Assume, without loss of generality, that p lies outside $\xi\eta$. If p lies to the right of that interval, ρ_p misses γ , contradicting Condition (c). Thus p lies to the left of $\xi\eta$. Then the only way in which $s \cap \gamma$ is empty is when q also lies to the left of $\xi\eta$, but then $\rho_q \cap \gamma$ would be empty, again contradicting Condition (c).

Assume next that $\ell \cap \gamma$ consists of one point ξ . By Condition (c), p must lie to the left of ξ and q must lie to the right of ξ , implying that s meets γ . See Figure 7(b) for an illustration. \Box

The description of the data structure is fairly straightforward given the criteria for intersection in Lemma 2.5. That is, we construct a multi-level data structure, where in the first level we test Condition (a), obtaining the set of arcs that ℓ intersects, as the disjoint union of canonical sets of arcs. At the next levels we test Condition (b). For an arc γ , with endpoints u and v, a point p lies outside $\kappa = \kappa(\gamma)$ when p lies either below γ , or to the left of u, or to the right of v. Collecting the arcs γ that satisfy this property is easily done using similar data structures as those described for the case of ray-intersection queries, where we first extract those arcs that lie above p and then the arcs that lie above q. We then use a one-dimensional search tree on the left (resp., right) endpoints of the arcs, to collect those arcs that lie to the right of p (resp., to the left of q). Overall this requires $O(n^{3/2+\varepsilon})$ storage and preprocessing and the query time is $O(n^{1/2+\varepsilon})$. To test for Condition (c), we build a multi-level data structure, over each final canonical set, that tests whether Conditions (a)–(c) of Lemma 2.4 are satisfied for the rightward-directed ray ρ_p , and that their symmetric counterparts are satisfied for the leftward-directed ray ρ_q .

The overall performance bounds remain the same: $O(n^{3/2+\varepsilon})$ storage and preprocessing cost, and $O(n^{1/2+\varepsilon})$ query time.

As noted above, the parametric-search approach in [5] yields a procedure for answering ray-shooting queries, given a procedure for answering segment-intersection queries, as well as a parallel procedure for the same task. The preprocessing that constructs the structure is performed only sequentially, as above. The query procedure, for detecting segment intersection, is easy to parallelize, since it is essentially a multi-level tree traversal. By allocating a processor to each node that the query visits, we can perform the traversal in parallel, in polylogarithmic parallel time. In other words, the parallel time to answer a segment-intersection detection query is O(polylog n), using at most $O(n^{3/2+\varepsilon})$ processors. Integrating these bounds with [5, Theorem 2.1], we obtain that ray-shooting queries in a planar collection of arcs can be answered using the same data structure for segment-intersection queries, where the query time for the former problem is only within a polylogarithmic factor of the time for the latter one, a factor that is hidden in our ε -notation, by slightly increasing ε .

A simple modification of the segment-intersection query procedure allows us to report an arc γ intersecting (i.e., containing) the endpoint of the query segment, when the segment is otherwise empty. The easy details are omitted.

In conclusion we have shown the following general result, which we believe to be of independent interest.

Proposition 2.6 A collection Γ of n constant-degree algebraic arcs in the plane can be preprocessed, in time and storage $O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$, into a data structure that supports ray shooting queries in Γ , in $O(n^{1/2+\varepsilon})$ time per query.

As a corollary, we thus obtain:

Proposition 2.7 For a partitioning polynomial f of sufficiently large constant degree, and a set W of n triangles, one can construct a data structure of size and preprocessing cost $O(n^{3/2+\varepsilon})$, so that a segment-shooting query with a segment that lies in Z(f), can be answered in $O(n^{1/2+\varepsilon})$ time, for any $\varepsilon > 0$.

As already concluded, Proposition 2.7, combined with Proposition 2.3 of the previous subsection, complete the proof of Theorem 2.1.

3 Segment-triangle intersection reporting, emptiness, and approximate counting queries

3.1 Segment-triangle intersection reporting and emptiness

We extend the technique presented in Section 2 to answer intersection reporting queries amid triangles in \mathbb{R}^3 . Here too we have a set \mathcal{T} of n triangles in \mathbb{R}^3 , and our goal is to preprocess \mathcal{T} into a data structure that supports efficient intersection queries, each of which specifies a line, ray or segment ρ and asks for reporting the triangles of \mathcal{T} that ρ intersects. In particular, this data structure also supports *segment emptiness* queries, in which we want to determine whether the query segment meets any input triangle. Unfortunately, for technical reasons, the method does not extend to segment-triangle intersection (exact) *counting* queries, in which we want to find the (exact) number of triangles that intersect a query segment (or a line or a ray). This issue will be discussed later on in this section, and a partial solution, which supports queries that approximately count the number of intersections, up to any prescribed relative error $\varepsilon > 0$, will be presented in Section 3.2.

Theorem 3.1 Given a collection of n triangles in three dimensions, and a prescribed parameter $\varepsilon > 0$, we can process the triangles into a data structure of size $O(n^{3/2+\varepsilon})$, in time $O(n^{3/2+\varepsilon})$, so that a segment-intersection reporting (resp., emptiness) query amid these triangles can be answered in $O(n^{1/2+\varepsilon}+k\log n)$ (resp., $O(n^{1/2+\varepsilon})$) time, where k is the number of triangles that the query segment crosses.

Proof. The algorithm that we develop here is a fairly easy adaptation of those given in the previous section, which is fairly straightforward, except for one significant issue, noted below. The preprocessing is almost identical, except that now we preprocess the sets \mathcal{T}_C of wide triangles for line- (ray-, or segment-)intersection reporting queries in a set of planes in \mathbb{R}^3 (namely, the corresponding planes in \mathcal{H}_C); in this case the reporting query time is $O\left(\frac{n \operatorname{polylog} n}{s^{1/3}} + k\right)$, where *n* is the number of wide triangles, *s* is the amount of storage allocated, and *k* is the output size; see [5, Theorem 3.3].

To answer a query with a segment (ray, or line) ρ , we trace ρ through the cells and subcells that it crosses, as before, but do not abort the search at cells where an intersection has been found, and instead follow the search to completion. We take each canonical set \mathcal{T}_C that the query collects, and access it via the intersection-reporting mechanism that we have constructed for it. At the bottom level we examine all the triangles in the subproblem, and report those that are crossed by ρ .

Recall that the canonical sets that we construct are not necessarily pairwise disjoint, even those that are encountered when querying with a fixed segment e. Thus, the triangles that we report may be reported multiple times, which we want to avoid. To this end, at each level of the recursion (on the trapezoidal cells ψ) within a single cell τ of the polynomial partition, we take the outputs of the two recursive calls (recall the details in Section 2), and keep only one copy of each triangle that is reported twice. (Recall that the sets of triangles in the recursive calls are disjoint from the canonical set \mathcal{T}_C but may share triangles between themselves.) Repeating this step at all levels of recursion guarantees that the reported triangles are all distinct. The overall cost of this overhead is $O(k \log n)$, where k is the output size. Note that this non-disjointness of the outputs makes it difficult to convert this procedure to one that counts the number of triangles that the query object crosses, and at the moment we do not know how to perform intersection-counting queries with the same performance bounds. (It is also conceivable that the upper bound on the cost of a counting query is larger; see, e.g., [5] for similar phenomena.)

A similar adaptation is applied for the substructure that handles queries that are contained in the zero set of the partitioning polynomial, and its easy details are omitted. (In fact, in this case, due to the nature of our range searching mechanism, the conflict lists comprising the answer to a single query are pairwise disjoint. Therefore in this case, we do obtain a range counting mechanism with similar asymptotic performance bounds).

This completes the description of the required adaptations, and establishes Theorem 3.1. \Box

3.2 Approximate segment-intersection counting

Let T be as above, and let $\delta > 0$ be a prescribed parameter. We want to preprocess T into a data structure that supports *approximate counting* queries of the following form: Given a query segment e, count how many triangles of T are crossed by e, up to a relative error of $1 \pm \delta$.

To do so, we use the notion of a relative (p, δ) -approximation, as developed and analyzed in Har-Peled and Sharir [25]. We recall this notion and its basic properties. Consider the range space (T, \mathcal{R}) , where \mathcal{R} is the collection of all subsets of T of the form $T(e) = \{\Delta \in T \mid \Delta \cap e \neq \emptyset\}$, where e is a segment. It is easily shown that (T, \mathcal{R}) has finite VC-dimension η .

(A brief argument for this latter property follows by bounding the *primal shatter function* of the range space, as a function of |T|. This is done by representing the lines supporting the triangle edges as surfaces in 4-space, and by observing that, for each of the $O(|T|^4)$ cells C of their arrangement, all the lines whose Plücker images lie in C meet the same subset T_C of triangles of T. For a segment e, we take the cell C containing the image of the line supporting e, and argue that there are only polynomially many subsets of T_C that can be crossed by such a segment e.)

For a segment e and a subset $X \subseteq T$, write $\overline{X}(e) := \frac{|X \cap e|}{|X|}$; this is the "relative size" of the range T(e) induced by e. Let 0 < p, $\delta < 1$ be given parameters. A subset $Z \subseteq T$ is called a *relative* (p, δ) -approximation if, for every segment e, we have:

$$\begin{aligned} (1-\delta)\overline{T}(e) &\leq \overline{Z}(e) \leq (1+\delta)\overline{T}(e), \quad \text{if } \overline{T}(e) \geq p\\ \overline{T}(e) &-\delta p \leq \overline{Z}(e) \leq \overline{T}(e) + \delta p, \quad \text{if } \overline{T}(e) \leq p. \end{aligned}$$

As shown in [25], a random sample Z of T of size $\frac{c}{\delta^2 p} \left(\eta \log \frac{1}{p} + \log \frac{1}{q} \right)$ is a relative (p, δ) -approximation with probability at least 1 - q, where c is some absolute constant.

We use this notion as follows. Using Theorem 3.1 we construct our data structure for exact segment intersection *reporting* queries, with $O(n^{3/2+\varepsilon})$ storage and preprocessing and query time $O(n^{1/2+\varepsilon} + k \log n)$, for any $\varepsilon > 0$, where k is the output size. We also construct

a relative (p, δ) -approximation Z for T, by an appropriate random sampling mechanism, where the value of p will be determined shortly.

An approximate counting query with a segment e is answered as follows. We first query with e in the data structure for exact segment intersection reporting, but stop the procedure as soon as we collect more than np triangles. If the output size k does not exceed this bound, we output k (as an exact count) and are done. The cost of the query so far is $O(n^{1/2+\varepsilon} + np\log n)$.

If we detect that k > np, we compute $\overline{Z}(e)$ by brute force, in O(|Z|) time, and output the value $k_{apx} := n\overline{Z}(e)$. By the properties of relative approximations, we have, since $\overline{T}(e) \ge p$,

$$(1-\delta)k \le k_{\text{apx}} \le (1+\delta)k,$$

so k_{apx} satisfies the desired approximation property.

The overall (deterministic) cost of the query is

$$O\left(n^{1/2+\varepsilon} + np\log n + |Z|\right) = O\left(n^{1/2+\varepsilon} + np\log n + \frac{1}{\delta^2 p}\left(\eta\log\frac{1}{p} + \log\frac{1}{q}\right)\right).$$

We ignore the effect of q, and balance the terms by choosing $p := \frac{1}{\delta n^{1/2}}$, making the query cost

$$O\left(n^{1/2+\varepsilon} + \frac{n^{1/2}}{\delta}\log n\right).$$

As long as δ is not too small (but we can still choose δ to be $1/n^{\varepsilon'}$, for some $\varepsilon' < \varepsilon$), the first term dominates the bound, which is thus asymptotically the same as the bound for reporting queries.

The storage is $O\left(n^{3/2+\varepsilon}+|Z|\right)=O\left(n^{3/2+\varepsilon}\right)$, as long as δ is not chosen too small.

We thus conclude:

Theorem 3.2 Given a collection of n triangles in three dimensions, and prescribed parameters $\varepsilon, \delta > 0$, where $\delta = \omega(1/n^{\varepsilon})$, we can process the triangles, using random sampling, into a data structure of size $O(n^{3/2+\varepsilon})$, in time $O(n^{3/2+\varepsilon})$, so that, for a query segment e, the number of intersections of e with the input triangles can be approximately computed, up to a relative error of $1 \pm \delta$, with very high probability, in $O(n^{1/2+\varepsilon})$ time.

4 Tradeoff between storage and query time

In this section we extend the technique in Sections 2 and 3 to obtain a tradeoff between storage (and preprocessing) and query time. A similar tradeoff holds for the other problems studied in Section 5.

For a quick overview of our approach, consider the ray-shooting structure of Section 2, and let s be the storage parameter that we allocate to the structure, which now satisfies $n \leq s \leq n^4$. We modify the procedure for ray shooting inside a cell τ by (i) stopping potentially the r_0 -recursion at some earlier 'premature' level, and (ii) modifying the structure at the bottom of recursion so that it uses the (weaker) ray-shooting technique of Pellegrini [32] instead of a brute-force scanning of the triangles (the current cost of $O(n^2/s)$, a consequence of this brute-force approach, is too expensive when s is small). A similar adaptation is applied to the procedure of ray shooting on the zero set of the partitioning polynomial. With additional care we obtain the performance bounds (1) and (2) announced in the introduction.

We now present the technique in detail. Consider the ray-shooting structure of Section 2, and let s be the storage parameter that we allocate to the structure, which satisfies $n \leq s \leq$ n^4 . As before, we use this notation to indicate that the actual storage (and preprocessing) that the structure uses may be $O(s^{1+\varepsilon})$, for any $\varepsilon > 0$. We comment that in the preceding sections s is assumed to be at most n^2 . Handling larger values of s require some care, detailed below. For the time being, we continue to assume that $s \leq n^2$, and will later show how to extend the analysis for larger values.

Consider first the subprocedure for handling ray shooting for rays that are not contained in the zero set of the partitioning polynomial. When $s = O(n^{3/2})$, we run the recursive preprocessing described in Section 2 up to some 'premature' level k, and when $s = \Omega(n^{3/2})$, we run it all the way down. With a suitable choice of parameters, we obtain $O(D^{3k+\varepsilon})$ subproblems at the bottom level of recursion, each involving at most n/D^{2k} (narrow) triangles.

Except for the bottom level, we build, at each node τ of the recursion, the same structure on the set \mathcal{W}_{τ} of wide triangles in τ , with one (significant) difference. First, since we start the recursion with storage parameter s, we allocate to each subproblem, at any level j, the storage parameter s/D^{3j} , thus ensuring that the storage used by the structure is $O(s^{1+\varepsilon})$. However, the cost of a query, even at the first level of recursion, given in (5), has the term $O(n^{2+\varepsilon}/s)$, which is the cost of a naive, brute-force processing of the conflict lists at the bottom instances of the r_0 -recursion within the partition cells. This is fine for $s = \Omega(n^{3/2+\varepsilon})$ but kills the efficiency of the procedure when s is smaller. For example, for s = n we get (near) linear query time, much more than what we aim to have. We therefore improve the performance at the bottom-level nodes of the r_0 -recurrence (within a cell), by constructing, for each respective conflict list, the ray-shooting data structure of Pellegrini [32], which, for N triangles and with storage parameter s, answers a query in time $O(N^{1+\varepsilon}/s^{1/4})$. Since at the bottom of the r_0 -recursion, both the number of triangles and the storage parameter are $n^{2+\varepsilon}/s$, the cost of a query at the bottom of the recursion is $O((n^2/s)^{3/4+\varepsilon})$. That is, the modified (improved) cost of a query at such a node is

$$Q(n,s) = O\left(\frac{n \operatorname{polylog}(n)}{s^{1/3}} + \frac{n^{3/2+\varepsilon}}{s^{3/4}}\right).$$
(6)

At each of the $O(D^{3k+\varepsilon})$ bottom-level cells τ , we take the set \mathcal{N}_{τ} of (narrow) triangles that have reached τ , whose size is now at most n/D^{2k} , allocate to it the storage parameter s/D^{3k} , and preprocess \mathcal{N}_{τ} using the aforementioned technique of Pellegrini [32], which results in a data structure, with storage parameter s/D^{3k} , which supports ray shooting queries in time

$$O\left(\frac{|\mathcal{N}_{\tau}|^{1+\varepsilon}}{(s/D^{3k})^{1/4}}\right) = O\left(\frac{(n/D^{2k})^{1+\varepsilon}}{(s/D^{3k})^{1/4}}\right) = O\left(\frac{n^{1+\varepsilon}}{s^{1/4}D^{(5/4+2\varepsilon)k}}\right)$$

Multiplying this bound by the number $O(D^{k+\varepsilon})$ of cells that the query ray crosses, the cost

of the query at the bottom-level cells is

$$Q_{\text{bot}}(n,s) = O\left(\frac{n^{1+\varepsilon}}{s^{1/4}D^{(1/4+\varepsilon)k}}\right).$$

The cost of a query at the inner recursive nodes of some depth j < k is the number, $O(D^{j+\varepsilon})$, of *j*-level cells that the ray crosses, times the cost of accessing the data structure for the wide triangles at each visited cell. Since we have allocated to each of the $O(D^{3j+\varepsilon})$ cells at level *j* the storage parameter $O(s/D^{3j})$, the cost of accessing the structure for wide triangles in a *j*-level cell is, according to (6), at most

$$Q_{\text{inner}}(n,s) = O\left(\frac{(n/D^{2j})\text{polylog}(n)}{(s/D^{3j})^{1/3}} + \frac{(n/D^{2j})^{3/2+\varepsilon}}{(s/D^{3j})^{3/4}}\right) = O\left(\frac{n \text{ polylog}(n)}{D^j s^{1/3}} + \frac{n^{3/2+\varepsilon}}{D^{(3/4+2\varepsilon)j} s^{3/4}}\right)$$

Summing over all *j*-level cells, for all *j*, and then adding the bottom-level cost, and the cost of traversing the structure with the query segment, the overall cost of a query is (we remind the reader that so far we only consider the case where $s \leq n^2$)

$$O\left(D^{k+\varepsilon} + \frac{n^{3/2+\varepsilon}D^{k/4}}{s^{3/4}} + \frac{n^{1+\varepsilon}}{s^{1/3}} + \frac{n^{1+\varepsilon}}{s^{1/4}D^{(1/4+\varepsilon)k}}\right).$$
 (7)

We choose k to (roughly) balance the second and the last terms; specifically, we choose

$$D^k = \frac{s}{n}.$$

Since $D^{k+\varepsilon}$ should not exceed $O(n^{1/2+\varepsilon})$, we require for this choice of k that $s = O(n^{3/2+\varepsilon})$. In this case it is easily verified that the second and last terms, which are $O(n^{5/4+\varepsilon}/s^{1/2})$, dominate both the first and third terms (recall that we assume $s \ge n$), and the query time is therefore

$$O(n^{5/4+\varepsilon}/s^{1/2}).$$

For larger values of s, that is, when $s = \Omega(n^{3/2+\varepsilon})$ (but we still assume $s \le n^2$), we balance the first term with the last term, so we choose

$$D^k = \frac{n^{4/5}}{s^{1/5}}.$$

Note that in this range we indeed have that $D^{k+\varepsilon} = O(n^{1/2+\varepsilon})$. Moreover, in this case the first and last terms dominate the second and third terms, as is easily verified. Therefore the query time is

$$O(n^{4/5+\varepsilon}/s^{1/5}).$$

As already promised, the case where the query ray lies on the current zero set will be presented later.

It remains to handle the range $n^2 < s \leq n^4$. Informally, at each cell τ of the polynomial partition, at any level j of the D-recursion, we have $n_{\tau} \leq n/D^{2j}$ wide triangles and storage parameter $s_{\tau} = s/D^{3j}$. Since $s \geq n^2$, we also have $s_{\tau} \geq n_{\tau}^2$. With such 'abundance' of storage, we run the r_0 -recursion until we reach subproblems of constant size, in which case we simply store the list of wide triangles at each bottom-level node, and the query simply inspects all of them, at a constant cost per subproblem. Hence the cost of a query at τ is $O(n_{\tau}^{1+\varepsilon}/s_{\tau}^{1/3})$. To be precise, this is the case as long as $s_{\tau} \leq n_{\tau}^3$. If $n^2 \leq s \leq n^3$ there will be some level j at whose cells $\tau \ s_{\tau} = s/D^{3j}$ becomes larger than $(n/D^{2j})^3 \geq n_{\tau}^3$, and then the cost becomes $O(n_{\tau}^{\varepsilon})$. When $n^3 < s \leq n^4$ the cost becomes $O(n_{\tau}^{\varepsilon})$ right away (and stays so). That is, the cost of a query in the structure for wide triangles at a cell τ at level j is

$$O\left(\frac{(n/D^{2j})^{1+\varepsilon}}{(s/D^{3j})^{1/3}}\right) = O\left(\frac{n^{1+\varepsilon}}{s^{1/3}D^{j(1+2\varepsilon)}}\right) \quad \text{for } s \le \frac{n^3}{D^{3j}}$$
$$O\left(n^{\varepsilon}\right) \quad \text{for } s > \frac{n^3}{D^{3j}}.$$

Since a query visits $O(D^{j+\varepsilon})$ cells τ at level j, the overall cost of searching amid the wide triangles, over all levels, is easily seen to be

$$O\left(\frac{n^{1+\varepsilon}}{s^{1/3}}\right) \quad \text{for } n^2 \le s \le n^3$$
$$O\left(D^k n^{\varepsilon}\right) \quad \text{for } n^3 < s \le n^4,$$

where k is the depth of the *D*-recursion.

Querying amid the narrow triangles is done again as in Section 2 (once again, recall that we now consider the case where $s > n^2$, whereas earlier in this section we assumed $s \le n^2$). At each node τ at the bottom level k of the D-recursion we use Pellegrini's data structure [32], which, with at most n/D^{2k} narrow triangles and storage parameter s/D^{3k} , answers a query in time

$$O\left(\frac{(n/D^{2k})^{1+\varepsilon}}{(s/D^{3k})^{1/4}}\right) = O\left(\frac{n^{1+\varepsilon}}{D^{(5/4+2\varepsilon)k}s^{1/4}}\right).$$

We multiply by the number of cells that the query visits, namely $O(D^{k+\varepsilon})$, and add the cost $O(D^{k+\varepsilon})$ of traversing these cells, for a total of

$$O\left(D^{k+\varepsilon} + \frac{n^{1+\varepsilon}}{s^{1/3}} + \frac{n^{1+\varepsilon}}{D^{(1/4+2\varepsilon)k}s^{1/4}}\right).$$

In other words, we get the same bound as in (7), except for the second term which is missing now (this term corresponds to querying at the bottom-level nodes of the r_0 -recursion on the wide triangles, which is not needed when $s > n^2$, since these bottom-level subproblems now have constant size). Repeating the same analysis as above, we get the same bound for the query cost.

Handling the zero set. We next analyze the case where the query ray lies on the zero set. In order to obtain the trade-off bounds for ray shooting within Z(f), we recall the multi-level data structure presented in Section 2.2. Each level in this data structure is either a one- or a two-dimensional search tree, where the dominating levels are those where we need to apply a planar decomposition over a set of planar regions (or in an arrangement of algebraic arcs) and preprocess it into a structure that supports point-location queries. A standard property of multi-level range searching data structures is that the overall complexity of their storage (resp., query time) is governed by the level with dominating storage (resp., query time) bound, up to a polylogarithmic factor [3]. Recall that in each level of our data structure we form a collection of canonical sets of the arcs in Γ , which are passed on to the next level for further processing. Our approach is to keep forming these canonical sets, where at the very last level we apply the ray-shooting data structure of Pellegrini [32], as described above. Therefore the overall query cost (resp., storage and preprocessing complexity) is the sum of the query (resp., storage and preprocessing time) bounds over all canonical sets of arcs that the query reaches (resp., all the sets) at the last level.

We now sketch the analysis in more detail. In order to simplify the presentation, we consider one of the dominating levels, and describe the ray-shooting data structure at that level. As stated above, we build this data structure only at the very last level, but the analysis for the dominating level subsumes the bounds for the last level, and thus for the entire multi-level data structure, up to a polylogarithmic factor. In such a scenario we have a set of algebraic arcs (or graphs of functions, or semi-algebraic regions represented by their bounding arcs), which we need to preprocess for planar point location. This is done using the technique of (1/r)-cuttings (see [17]), which forms a decomposition of the plane into $O(r^2)$ pseudo-trapezoidal cells, each meeting at most n/r arcs (the "conflict list" of the cell). The overall storage complexity is thus O(nr). More precisely, to achieve preprocessing time close to O(nr), one needs to use so-called *hierarchical-cuttings* (see [28] and also [8]), in which we construct a hierarchy of cuttings using a constant value r_0 as the cutting parameter, instead of the nonconstant r. Using this approach, both storage and preprocessing cost are $O(nr^{1+\varepsilon})$ for any $\varepsilon > 0$. Let s be our storage parameter as above, so we want to choose r such that s = rn. Thus we obtain that each cell of the cutting meets at most n^2/s arcs. Following our approach above, for each cell of the cutting, the amount of allocated storage is $s/r^2 = n^2/s$. We are now ready to apply Pellegrini's data structure, leading to a query time of $O\left(\frac{n^{3/2+\varepsilon}}{s^{3/4}}\right)$. Integrating this bound into the query time in (7), we recall that at each level $0 \le j \le k$ the actual storage parameter is $O(s/D^{3j})$, and the number of triangles at hand is $O(n/D^{2j})$. We now need to sum the query bound over all $O(D^j)$ cells reached by the query at the *j*th level, and over all *j*. We thus obtain an overall bound of

$$O\left(D^k \frac{(n/D^{2k})^{3/2+\varepsilon}}{(s/D^{3k})^{3/4}}\right) = O\left(\frac{n^{3/2+\varepsilon}D^{k/4}}{s^{3/4}}\right)$$

This is exactly the second term in (7). Therefore adding the query time for ray shooting on Z(f) does not increase the asymptotic bound in (7).

We comment that the overall storage and preprocessing time is $O(s^{1+\varepsilon})$ (see our discussion below). We also comment that the query bound we obtained applies when $n \leq s \leq n^2$. When s exceeds n^2 , every cell of the cutting has a conflict list of O(1) elements, which the query can handle in brute-force. This immediately brings the query time, for queries on the zero set, to $O(n^{\varepsilon})$.

Wrapping up. In summary, our analysis implies that the query bound Q(n,s) satisfies:

$$Q(n,s) = \begin{cases} O\left(\frac{n^{5/4+\varepsilon}}{s^{1/2}}\right), & s = O(n^{3/2+\varepsilon}), \\ O\left(\frac{n^{4/5+\varepsilon}}{s^{1/5}}\right), & s = \Omega(n^{3/2+\varepsilon}). \end{cases}$$
(8)

We recall that the overall storage (and preprocessing) is $O(s^{1+\varepsilon})$, since we allocate to each subproblem, at any level j, the storage parameter s/D^{3j} , thus at each fixed level the total

storage (and preprocessing) complexity is $O(s^{1+\varepsilon})$, and since there are only logarithmically many levels, the overall storage (and preprocessing) is $O(s^{1+\varepsilon})$ as well, for a slightly large ε .

Note that for the threshold $s \approx n^{3/2}$, both bounds yield a query cost of $O(n^{1/2+\varepsilon})$. Note also that in the extreme cases $s = n^4$, s = n (extreme for the older 'four-dimensional' tradeoff), we get the respective older bounds $O(n^{\varepsilon})$ and $O(n^{3/4+\varepsilon})$ for the query time. In this case, when either s = n and $s = n^4$ we have $D^k = O(1)$, implying that we handle all the narrow triangles at the root of the recursion tree, that is, we use the technique of Pellegrini [32] once altogether. Informally, the bound in (8) 'pinches' the tradeoff curve and pushes it down. The closer s is to $\Theta(n^{3/2+\varepsilon})$, the more significant is the improvement. See Figure 1.

Processing m queries. The improved tradeoff in (8) implies that the overall cost of processing m queries with n input triangles, including preprocessing cost, is

$$O(s^{1+\varepsilon} + mQ(n,s)) = \begin{cases} O\left(s^{1+\varepsilon} + \frac{mn^{5/4}}{s^{1/2}}\right), & s = O(n^{3/2+\varepsilon}), \\ O\left(s^{1+\varepsilon} + \frac{mn^{4/5}}{s^{1/5}}\right), & s = \Omega(n^{3/2+\varepsilon}). \end{cases}$$

To balance the terms in the first case we choose $s = m^{2/3}n^{5/6}$; this choice satisfies $s = O(n^{3/2+\varepsilon})$ when $m \le n$. To balance the terms in the second case we choose $s = m^{5/6}n^{2/3}$; this choice satisfies $s = \Omega(n^{3/2+\varepsilon})$ when $m \ge n$. Recall also that s has to be in the range between n and n^4 . So in the first case we must have $m^{2/3}n^{5/6} \ge n$, or $m \ge n^{1/4}$. Similarly, in the second case we must have $m^{5/6}n^{2/3} \le n^4$, or $m \le n^4$. We adjust the bounds, allowing also values of m outside this range, by adding the near-linear terms, which dominate the bound for such off-range values of m. We thus get

Corollary 4.1 We can process m ray-shooting queries on n triangles so that the total cost is

$$\max\left\{O(m^{2/3+\varepsilon}n^{5/6+\varepsilon}+n^{1+\varepsilon}),\ O(n^{2/3+\varepsilon}m^{5/6+\varepsilon}+m^{1+\varepsilon})\right\}.$$
(9)

5 Other applications

5.1 Detecting, counting or reporting line intersections in \mathbb{R}^3

It is more convenient, albeit not necessary, to consider the bichromatic version of the problem, in which we are given a set R of n red lines and a set B of n blue lines in \mathbb{R}^3 , and the detection problem asks whether there exists a pair of intersecting lines in $R \times B$. Similarly, the counting problem asks for the number of such intersecting pairs, and the reporting problem asks for reporting all these pairs.

An algorithm that solves the detection problem in $O(n^{3/2+\varepsilon})$ time is easily obtained by regarding the problem as a special degenerate (and much simpler) instance of the ray shooting problem, in which we regard the, say red lines as degenerate triangles (unbounded and of zero area), construct the data structure of Section 2 and query it with each of the blue lines. There exists a red-blue pair of intersecting lines if and only if at least one query has a positive outcome—the corresponding blue query line hits a red line. Since there are no wide triangles in this special variant, there is no need to construct the auxiliary data structure for wide triangles, as in Section 2.1, and we simply construct the recursive hierarchy of polynomial partitions, where each cell in each subproblem is associated with the set of red lines that cross it. A blue query line ℓ is propagated through the cells that it crosses until it reaches bottom-level cells, and we check, in each such cell, whether ℓ intersects any of the O(1) red lines associated with the cell.

Handling lines that lie fully in the zero set Z(f) is also an easy task (which can be performed using planar segment-intersection range searching, which also supports counting queries); further details are omitted.

Both correctness and runtime analysis follow easily, as special and simpler instances of the analysis in Section 2. Note that here we do not face the issue of non-disjointness of canonical sets of wide triangles, which has prevented us from extending the technique to segment-triangle intersection counting problems; see Section 3.

5.2 Computing the intersection of two polyhedra

Let K_1 and K_2 be two polyhedra in 3-space, not necessarily convex, each with n edges (so the number of vertices and faces of each of them is O(n)). The goal is to compute their intersection $K := K_1 \cap K_2$ in an output-sensitive manner. We note that computing the union $K_1 \cup K_2$ can be done using a very similar approach, within the same time bound.

While there are additional steps in the algorithm that construct a representation of K as a three-dimensional body, we will restrict here the presentation to the part that computes ∂K from ∂K_1 and ∂K_2 . Each face of ∂K is a connected portion of a face of ∂K_1 or of ∂K_2 , each edge is either a connected portion of an edge of ∂K_1 or of ∂K_2 , or a connected portion of the intersection of a face of ∂K_1 and a face of ∂K_2 . Finally, each vertex of ∂K is either a vertex of ∂K_1 or of ∂K_2 , or an intersection of an edge of one of these polyhedra with a face of the other. Note that not every vertex of K_1 or of K_2 is necessarily a vertex of K, but every edge-face intersection is a vertex of K.

The main step of the algorithm is to compute the vertices of ∂K , from which the other features of ∂K are fairly standard to construct, see, e.g., [32] where the graph of the edges of K is constructed by a tracing procedure [31], given the vertices of ∂K . We iterate over the edges of ∂K_1 , and compute the intersections of each such edge with the faces of ∂K_2 , using the algorithm in Theorem 3.1. We apply a symmetric procedure to compute the intersections of each edge of ∂K_2 with the faces of ∂K_1 . Collectively, these intersections are the vertices of K of the second type (edge-face intersection vertices). The cost of this step is $O(n^{3/2+\varepsilon} + k \log n)$, where k is the number of edge-face intersections: we preprocess the O(n) faces of, say K_1 , and query with the O(n) edges of K_2 , which overall takes $O(n \cdot n^{1/2+\varepsilon} + k \log n) = O(n^{3/2+\varepsilon} + k \log n)$ time. Then, applying the tracing procedure in [31] takes an additional cost of $O(k \log k)$.

This gives us all the edge-face intersection vertices. The other vertices of K are vertices of K_1 or of K_2 , and finding these vertices is done as follows. If such a vertex v, say of K_1 , is incident to an edge e of K_1 that intersects some face of K_2 , then it is easy to determine whether $v \in K_2$ (and thus in K). Otherwise, we collect, by a simple graph traversal, a maximal cluster of vertices of K_1 that are connected by edges that have no intersection with ∂K_2 . The vertices in such a cluster are either all inside K_2 (and in K) or all outside K_2 (and thus not in K). If the cluster consists of all vertices of K_1 then either K_1 , K_2 are disjoint, or one contains the other. In such a case, we only need to test, in O(n) time, if there is a vertex from one polyhedron that is contained in the other. Otherwise, we determine the status of the cluster (inside / outside K) by examining the edges that connect vertices from the cluster to vertices not in the cluster. By iteratively repeating this step, we construct all such clusters, from which we obtain all the vertices of K_1 and of K_2 the lie in K.

In summary we obtain:

Corollary 5.1 Given two arbitrary polyhedra K_1 and K_2 each of complexity O(n), the intersection $K_1 \cap K_2$ can be computed in time $O(n^{3/2+\varepsilon} + k \log n)$, where k is the size of the intersection.

As discussed in the introduction, the overhead term in Pellegrini's algorithm [32] is $O(n^{8/5+\varepsilon})$.

5.3 Output-sensitive construction of an arrangement of triangles

Let \mathcal{T} be a set of n possibly intersecting triangles in \mathbb{R}^3 , let $\mathcal{A} = \mathcal{A}(\mathcal{T})$ denote their arrangement, and let k denote its complexity, which, as in Section 5.2, we measure by the number of its vertices, as the number of its other features (edges, faces, and cells) is proportional to k. The goal is to construct \mathcal{A} in an output-sensitive manner with a small, subquadratic overhead. Pellegrini [32] gave such an algorithm that runs in $O(n^{8/5+\varepsilon} + k \log k)$, and the algorithm that we present here reduces the overhead to $O(n^{3/2+\varepsilon})$ time.

As in the previous subsection, we focus on the main step of the algorithm that constructs the features of \mathcal{A} (vertices, edges, and faces) on each triangle of \mathcal{T} . We will only briefly discuss the complementary part, which constructs the three-dimensional cells of \mathcal{A} and establishes the connections between the various features on the boundary of each cell. Albeit not trivial, this latter step uses standard techniques, follows the approach in [32] and in other works, and does not increase the overhead cost of the algorithm.

Fix a triangle $\Delta \in \mathcal{T}$. We first construct the set of intersection segments $\Delta \cap \Delta'$, for $\Delta' \in \mathcal{T} \setminus \{\Delta\}$. We observe that, for any such segment $e = \Delta \cap \Delta'$, each endpoint of e is either a vertex of Δ , or an intersection of an edge of one triangle with the other triangle.

We therefore take the collection of the 3n edges of the triangles of \mathcal{T} , and, for each such edge e, apply Theorem 3.1, which reports all k_e triangles that e meets. This identifies all the intersection segments $\Delta \cap \Delta'$. We then take all the intersection segments within a fixed triangle Δ , and run a sweepline procedure within Δ to obtain the portion of \mathcal{A} on Δ . Gluing these portions to each other, and some additional steps, complete the construction of \mathcal{A} .

6 Conclusion

In this paper we have managed to improve the performance of ray shooting amid triangles in three dimensions, as well as of several related problems. The improvement is based on the polynomial partitioning technique of Guth. The improvement is most significant when the storage is about $n^{3/2}$ and the query takes about $n^{1/2}$ time, but one gets an improvement for all values of the storage between n and n^4 , except at the very ends of this range. This is a significant improvement, the first in nearly 30 years, in this basic problem.

There are several open questions that our work raises. First, the improvement for the special values of $O(n^{3/2+\varepsilon})$ storage and $O(n^{1/2+\varepsilon})$ query time seems too specialized, and one would like to get similar improvements for all possible values of the storage, ideally obtaining query time of $O(n^{1+\varepsilon}/s^{1/3})$, where s is the storage allocated to the structure, as in the case of ray shooting amid planes. Alternatively, can one establish a lower-bound argument that shows the limitations of our technique?

Another open issue follows from our current inability to extend the technique to counting queries, due to the fact that the canonical sets that we collect during a query are not necessarily pairwise disjoint. It would be interesting to obtain such an extension, or, alternatively, to establish a gap between the performances of the counting and reporting versions of the segment intersection query problem.

Finally, could one obtain similar bounds for non-flat input objects? for shooting along non-straight curves? It would also be interesting to find additional applications of the general technique developed in this paper.

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