# EXISTENCE AND EMERGENT DYNAMICS OF QUADRATICALLY SEPARABLE STATES TO THE LOHE TENSOR MODEL 

SEUNG-YEAL HA, DOHYUN KIM, AND HANSOL PARK


#### Abstract

A tensor is a multi-dimensional array of complex numbers, and the Lohe tensor model is an aggregation model on the space of tensors with the same rank and size. It incorporates previously well-studied aggregation models on the space of low-rank tensors such as the Kuramoto model, Lohe sphere and matrix models as special cases. Due to its structural complexities in cubic interactions for the Lohe tensor model, explicit construction of solutions with specific structures looks daunting. Recently, we obtained completely separable states by associating rank-1 tensors. In this paper, we further investigate another type of solutions, namely "quadratically separable states" consisting of tensor products of matrices and their component rank-2 tensors are solutions to the double matrix model whose emergent dynamics can be studied using the same methodology of the Lohe matrix model.


## 1. Introduction

Collective behaviors often appear in large population systems for weakly coupled oscillators or interacting units [3, 6, 7] in diverse scientific disciplines including biology, social sciences, engineering with various space and time scales, for instance, colonies of bacteria [25], school of fish 4, 26], flock of starlings [29], pedestrian dynamics [18, opinion dynamics 17, power grid networks [22, etc. For a brief introduction to collective dynamics, we refer the reader to survey articles and book 1, 5, 10, 24. Mathematical approach toward the understanding of collective motions has been established in literature by Winfree 28] and Kuramoto [19] in the 1970s, and by Vicsek [27] in the 1990s. After their remarkable works, the aforementioned models have been extended in several directions, particularly in high-dimensional extension to Riemannian manifolds including the hypersurfaces [2, 12] and the matrix Lie group [11 which have attracted lots of interest thanks to its powerful application, for instance, nonconvex optimization. In this work, among high-dimensional models, we are concerned with the Lohe tensor model in [13.

Next, we briefly discuss tensors and an aggregation model on the space of tensors, namely "the Lohe tensor model". A rank- $m$ complex valued tensor can be represented as a multidimensional array of complex numbers with multi-indices. The rank of a tensor is the number of indices, say a rank- $m$ tensor with size $d_{1} \times \cdots \times d_{m}$ is an element of $\mathbb{C}^{d_{1} \times \cdots \times d_{m}}$. For example, scalars, vectors and matrices correspond to rank- 0,1 and 2 tensors, respectively.

[^0]Let $T$ be a rank- $m$ tensor with a size $d_{1} \times \cdots \times d_{m}$. Then, we denote $\left(\alpha_{1}, \cdots, \alpha_{m}\right)$ th component of the tensor $T$ by $[T]_{\alpha_{1} \cdots \alpha_{m}}$, and we set $\bar{T}$ by the rank- $m$ tensor whose components are the complex conjugate of the elements in $T$ :

$$
[\bar{T}]_{\alpha_{1} \cdots \alpha_{m}}:=\overline{[T]_{\alpha_{1} \cdots \alpha_{m}}}
$$

Let $\mathcal{T}_{m}\left(\mathbb{C} ; d_{1} \times \cdots \times d_{m}\right)$ be the collection of all rank- $m$ tensors with size $d_{1} \times \cdots \times d_{m}$. Then, it is a complex vector space. Several well-known first-order aggregation models, for instance, the Kuramoto model [19], the swarm sphere model [23] and matrix models [8, 20] can be regarded as aggregation models on $\mathcal{T}_{0}(\mathbb{C} ; 1), \mathcal{T}_{1}(\mathbb{R} ; d)$ and $\mathcal{T}_{2}(\mathbb{C} ; d \times d)$, respectively. Let $A_{j}$ be the skew-hermitian rank- $2 m$ tensor with size $\left(d_{1} \times \cdots \times d_{m}\right) \times\left(d_{1} \times \cdots \times d_{m}\right)$. For simplicity, we introduce handy notation as follows: for $T \in \mathcal{T}_{m}\left(\mathbb{C} ; d_{1} \times \cdots \times d_{m}\right)$ and $A \in \mathcal{T}_{2 m}\left(\mathbb{C} ; d_{1} \times \cdots \times d_{m} \times d_{1} \times \cdots \times d_{m}\right)$, we set

$$
\begin{aligned}
& {[T]_{\alpha_{*}}:=[T]_{\alpha_{1} \alpha_{2} \cdots \alpha_{m}}, \quad[T]_{\alpha_{* 0}}:=[T]_{\alpha_{10} \alpha_{20} \cdots \alpha_{m 0}}, \quad[T]_{\alpha_{* 1}}:=[T]_{\alpha_{11} \alpha_{21} \cdots \alpha_{m 1}},} \\
& {[T]_{\alpha_{* i_{*}}}:=[T]_{\alpha_{1 i_{1}} \alpha_{2 i_{2}} \cdots \alpha_{m i_{m}}}, \quad[T]_{\alpha_{*\left(1-i_{*}\right)}}:=[T]_{\alpha_{1\left(1-i_{1}\right)} \alpha_{2\left(1-i_{2}\right) \cdots \alpha_{m\left(1-i_{m}\right)}},}^{[A]_{\alpha_{*} \beta_{*}}:=[A]_{\alpha_{1} \alpha_{2} \cdots \alpha_{m} \beta_{1} \beta_{2} \cdots \beta_{m}} .}}
\end{aligned}
$$

Then, the Lohe tensor model in a component form reads as follows:

$$
\left\{\begin{align*}
{\left[\dot{T}_{j}\right]_{\alpha_{* 0}}=} & {\left[A_{j}\right]_{\alpha_{* 0} \alpha_{* 1}}\left[T_{j}\right]_{\alpha_{* 1}} }  \tag{1.1}\\
& +\sum_{i_{*} \in\{0,1\}^{m}} \kappa_{i_{*}}\left(\left[T_{c}\right]_{\alpha_{* i *}}\left[\bar{T}_{j}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*(1-i *)}}-\left[T_{j}\right]_{\alpha_{* i_{*}}}\left[\bar{T}_{c}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*\left(1-i_{*}\right)}}\right) \\
{\left[\bar{A}_{j}\right]_{\alpha_{* 0} \alpha_{* 1}}=} & -\left[A_{j}\right]_{\alpha_{* 1} \alpha_{* 0}}
\end{align*}\right.
$$

where $\kappa_{i_{*}}$ 's are nonnegative coupling strengths.
Before we discuss our main issues, we introduce a concept of a "quadratically separable state" for the Lohe tensor model (1.1).

Definition 1.1. Let $\left\{T_{i}\right\}$ be a quadratically separable state to (1.1), if it is decomposed as a tensor product of rank-2 tensors (or matrices):

$$
T_{i}=U_{i}^{1} \otimes U_{i}^{2} \otimes \cdots \otimes U_{i}^{m}, \quad U_{i}^{k} \in \mathbb{C}_{1}^{k} \times d_{2}^{k}, \quad\left\|U_{i}^{k}\right\|_{F}=1, \quad 1 \leq i \leq N, \quad 1 \leq k \leq m
$$

where $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm induced by Frobenius inner product: for matrices $A$ and B,

$$
\langle A, B\rangle_{\mathrm{F}}:=\operatorname{tr}\left(A^{\dagger} B\right), \quad\|A\|_{\mathrm{F}}:=\sqrt{\operatorname{tr}\left(A^{\dagger} A\right)}
$$

In this paper, we are interested in the following simple questions:

- (Q1): Are there quadratically separable states for the Lohe tensor model?
- (Q2): If so, do they exhibit collective behaviors under which circumstances?

Our main results deal with the raised two questions (Q1) and (Q2). More precisely, our main results of this paper can be summarized as follows.

First, we introduce the double matrix model induced from the Lohe tensor model whose elements have rank-4 with a specific condition on natural frequencies $B_{j}$ and $C_{j}$ (see (3.9)):

$$
\left\{\begin{align*}
\dot{U}_{j}=B_{j} U_{j} & +\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger} U_{j}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right)  \tag{1.2}\\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{j} U_{j}^{\dagger} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right), \\
\dot{V}_{j}=C_{j} V_{j} & +\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k} V_{j}^{\dagger} V_{j}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right) \\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{j} V_{j}^{\dagger} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right),
\end{align*}\right.
$$

where $B_{j} \in \mathbb{C}^{d_{1} \times d_{2} \times d_{1} \times d_{2}}$ and $C_{j} \in \mathbb{C}^{d_{3} \times d_{4} \times d_{3} \times d_{4}}$ are skew-symmetric rank-4 tensors. For a solution $\left\{\left(U_{i}, V_{i}\right)\right\}$ to the double matrix model (1.2), a special solution $T_{i}$ to (1.1) can be represented as follows:

$$
T_{i}(t) \equiv U_{i}(t) \otimes V_{i}(t), \quad t>0
$$

Precisely, if $T_{i}$ is initially decomposed into the tensor product of two matrices $U_{i}$ and $V_{i}$, then its separability is propagated along the flow for all time. For details, we refer the reader to Section 3

Second, we study emergent dynamics of the double matrix model by investigating several aggregation quantities:

$$
\begin{array}{ll}
\mathcal{D}(\mathcal{U}(t)):=\max _{1 \leq i, j \leq N}\left\|U_{i}(t)-U_{j}(t)\right\|_{\mathrm{F}}, & \mathcal{S}(\mathcal{U}(t)):=\max _{1 \leq i, j \leq N}\left|n-\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}}(t)\right|, \\
\mathcal{D}(\mathcal{V}(t)):=\max _{1 \leq i, j \leq N}\left\|V_{i}(t)-V_{j}(t)\right\|_{\mathrm{F}}, \quad \mathcal{S}(\mathcal{V}(t)):=\max _{1 \leq i, j \leq N}\left|m-\left\langle V_{i}, V_{j}\right\rangle_{\mathrm{F}}(t)\right| .
\end{array}
$$

For a homogeneous ensemble (i.e. $B_{i}=B_{j}$ and $C_{i}=C_{j}$ for all $i$ and $j$.) we show that system (1.2) exhibits complete aggregation in which all relative distances for $\left\{U_{i}\right\}$ and $\left\{V_{i}\right\}$ tend to zero respectively (see Theorem 4.1). On the other hand, for a heterogeneous ensemble ( $B_{i} \neq B_{j}$ and $C_{i} \neq C_{j}$ in general) complete aggregation (one-point collapse) would not be expected. Instead, our concern is dedicated to emergence of locked states in which relative distances converge to positive definite values (see Theorem 4.2). For our analytical results, we need to assume that the size of unitary matrices satisfy $\min (n, m)>4 \sqrt{\max (n, m)}$ that requires restriction on $n, m$. In fact, this technical assumption on the sizes is mainly due to the fact that elements are complex-valued. Thus, when the unitary groups $\mathbf{U}(n)$ and $\mathbf{U}(m)$ are replaced by the special orthogonal groups $\mathbf{S O}(n)$ and $\mathbf{S O}(m)$, such restriction on $n, m$ would be removed (see Theorem C.1 and Theorem C.2).

The rest of the paper is organized as follows. In Section 2, we begin with previous results on the relation between the Lohe tensor model and the swarm double sphere model presented in [9. As a natural extension, we construct the double matrix model in Section 3 and study existence and uniqueness of quadratically separable states. In Section 4, we study emergent dynamics of the double matrix model for both homogeneous and heterogeneous ensembles. Next, the double matrix model is further generalized to the multi matrix model in Section 5 Finally, Section 6 is devoted to a brief summary of the paper and future work. In Appendix

A and Appendix B, we provide proofs of Lemma 4.1 and Lemma 4.2, respectively. In Appendix Cl emergent dynamics of the double matrix model on $\mathbf{S O}(n) \times \mathbf{S O}(m)$ is provided.

For simplicity of presentation, we use the following abbreviated jargons:

- LT model: Lohe tensor model, LM model: Lohe matrix model,
- SDS model: swarm double sphere model, SMS model: swarm multi-sphere model,
- DM model: double matrix model, DUM model: double unitary matrix model,
- DSOM model: double special orthogonal matrix model, QSS: quadratically separable state,
- MM model: multiple matrix model, MUM model: multiple unitary matrix model.


## 2. Preliminaries

In this section, we review how the SDS model [9, 21] can be related to the LT model via completely separable states, and discuss extension of the SDS model to the SMS model leading to the DM model.
2.1. From the LT model to the SDS model. In this subsection, we briefly recall the relation between the LT model and the SDS model which was first observed in 9. In [21], Lohe introduced a first-order aggregation model on the product of two unit spheres $\left(u_{i}, v_{i}\right) \in \mathbb{S}^{d_{1}-1} \times \mathbb{S}^{d_{2}-1}:$

$$
\left\{\begin{array}{l}
\dot{u}_{i}=\Omega_{i} u_{i}+\frac{\kappa}{N} \sum_{j=1}^{N}\left\langle v_{i}, v_{j}\right\rangle\left(u_{j}-\left\langle u_{i}, u_{j}\right\rangle u_{i}\right), \quad t>0  \tag{2.1}\\
\dot{v}_{i}=\Lambda_{i} v_{i}+\frac{\kappa}{N} \sum_{j=1}^{N}\left\langle u_{i}, u_{j}\right\rangle\left(v_{j}-\left\langle v_{i}, v_{j}\right\rangle v_{i}\right), \\
\left(u_{i}, v_{i}\right)(0)=\left(u_{i}^{0}, v_{i}^{0}\right) \in \mathbb{S}^{d_{1}-1} \times \mathbb{S}^{d_{2}-1}, \quad 1 \leq i \leq N
\end{array}\right.
$$

where $\Omega_{i}$ and $\Lambda_{i}$ are skew-symmetric matrices of sizes $d_{1} \times d_{1}$ and $d_{2} \times d_{2}$, respectively:

$$
\Omega_{i}^{\top}=-\Omega_{i}, \quad \Lambda_{i}^{\top}=-\Lambda_{i}, \quad 1 \leq i \leq N,
$$

and $\kappa$ denotes the (uniform) coupling strength.
On the other hand, if we choose the following parameters:

$$
m=2, \quad \kappa_{00}=\kappa_{11}=0, \quad \kappa_{01}=\kappa_{10}=\kappa,
$$

system (1.1) reduces to the generalized Lohe matrix model in [15]:

$$
\left\{\begin{array}{l}
\dot{T}_{i}=A_{i} T_{i}+\kappa\left(T_{c} T_{i}^{\dagger} T_{i}-T_{i} T_{c}^{\dagger} T_{i}\right)+\kappa\left(T_{i} T_{i}^{\dagger} T_{c}-T_{i} T_{c}^{\dagger} T_{i}\right), \quad t>0  \tag{2.2}\\
T_{i}(0)=T_{i}^{0}, \quad\left\|T_{i}^{0}\right\|_{\mathrm{F}}=1, \quad T_{c}:=\frac{1}{N} \sum_{k=1}^{N} T_{k}, \quad i=1, \cdots, N
\end{array}\right.
$$

Next, we present how models (2.1) and (2.2) can be viewed as equivalent systems under well-prepared natural frequency tensors and initial data in the following proposition.

Proposition 2.1. 9 Systems (2.1) and (2.2) are equivalent in the following sense.
(1) Suppose $\left\{\left(u_{i}, v_{i}\right)\right\}$ is a solution to (2.1). Then, rank-2 real tensors $T_{i}$ defined by $T_{i}:=u_{i} \otimes v_{i}$ is a solution to (2.2) with initial data $T_{i}^{0}=u_{i}^{0} \otimes v_{i}^{0}$ and well-prepared free flow tensors $A_{i}$ :

$$
\begin{equation*}
A_{i} T_{i}:=\Omega_{i} T_{i}+T_{i} \Lambda_{i}^{\top} \tag{2.3}
\end{equation*}
$$

(2) Suppose $T_{i}$ is a solution to (2.2) $-(\sqrt{2.3})$ with completely factorized initial data:

$$
T_{i}^{0}=: u_{i}^{0} \otimes v_{i}^{0}, \quad 1 \leq i, j \leq N,
$$

for rank-1 real tensors $u_{i}^{0} \in \mathbb{S}^{d_{1} 1}$ and $v_{i}^{0} \in \mathbb{S}^{d_{2}-1}$. Then, there exists a pair of unit vectors $\left(u_{i}(t), v_{i}(t)\right)$ such that

$$
T_{i}(t)=u_{i}(t) \otimes v_{i}(t), \quad t>0,
$$

where $\left(u_{i}, v_{i}\right)$ is a solution to (2.1) with initial data $\left(u_{i}, v_{i}\right)(0)=\left(u_{i}^{0}, v_{i}^{0}\right)$.
By applying the completely separability stated in Proposition 2.1, emergent behaviors for (2.1) and those for (2.2) are exactly the same. Thus, it suffices to investigate the SDS model (2.1).

Proposition 2.2. 9 Suppose the initial data $\left\{\left(u_{i}^{0}, v_{i}^{0}\right)\right\}$ satisfy the following conditions:

$$
\min _{1 \leq i, j \leq N}\left\langle u_{i}^{0}, u_{j}^{0}\right\rangle>0, \quad \min _{1 \leq i, j \leq N}\left\langle v_{i}^{0}, v_{j}^{0}\right\rangle>0,
$$

and let $\{(U, V)\}$ be a solution to system (2.1). Then, we have

$$
\lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left|u_{i}(t)-u_{j}(t)\right|=0 \quad \text { and } \quad \lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left|v_{i}(t)-v_{j}(t)\right|=0
$$

Now, it is worthwhile mentioning that system (2.1) can be represented as a coupled gradient flow:

$$
\left\{\begin{array}{l}
\dot{u}_{i}=-\frac{N \kappa}{2} \mathbb{P}_{T_{u_{i}} \mathbb{S}^{d_{1}-1}}\left(\nabla_{u_{i}} \mathcal{E}(U, V)\right),  \tag{2.4}\\
\dot{v}_{i}=-\frac{N \kappa}{2} \mathbb{P}_{T_{v_{i}} \mathbb{S}^{\mathbb{S}_{2}-1}}\left(\nabla_{v_{i}} \mathcal{E}(U, V)\right),
\end{array}\right.
$$

where the projection operators $\mathbb{P}_{T_{u_{i}} \mathbb{S}^{d_{1}-1}}$ and $\mathbb{P}_{T_{v_{i}} \mathbb{S}_{2}-1}$ onto the tangent spaces of $\mathbb{S}^{d_{1}-1}$ and $\mathbb{S}^{d_{2}-1}$ at $u_{i}$ and $v_{i}$ are defined by the formulae, respectively: for $w_{1} \in \mathbb{R}^{d_{1}}$ and $w_{2} \in \mathbb{R}^{d_{2}}$,

$$
\left\{\begin{array}{l}
\mathbb{P}_{T_{u_{i}} \mathbb{S}^{d_{1}-1}}\left(w_{1}\right):=w_{1}-\left\langle w_{1}, u_{i}\right\rangle u_{i}, \\
\mathbb{P}_{T_{v_{i}}} \mathbb{S}^{d_{2}-1}\left(w_{2}\right):=w_{2}-\left\langle w_{2}, v_{i}\right\rangle v_{i},
\end{array}\right.
$$

and the potential function $\mathcal{E}(U, V)$ is defined as

$$
\begin{equation*}
\mathcal{E}(U, V):=1-\frac{1}{N^{2}} \sum_{i, j=1}^{N}\left\langle u_{i}, u_{j}\right\rangle\left\langle v_{i}, v_{j}\right\rangle . \tag{2.5}
\end{equation*}
$$

Thanks to the gradient flow formulation (2.4), any solution to system (2.1) converges to an equilibrium as $t \rightarrow \infty$.
2.2. From the SDS model to the SMS model. In this subsection, we extend the SDS model (2.1) on the product of two unit spheres to an aggregation model on the product of multiple unit spheres, namely, the SMS model. Note that the SDS model can be represented as a gradient flow with a potential function as can be seen in (2.4)-(2.5). Thus, we first generalize the potential function (2.5) as follows: for $u_{i}^{k} \in \mathbb{S}^{d_{k}-1}, \quad i=1, \cdots, N, \quad k=$ $1, \cdots, m$, we set

$$
\begin{equation*}
\mathcal{E}\left(U^{1}, U^{2}, \cdots, U^{m}\right):=1-\frac{1}{N^{2}} \sum_{i, j=1}^{N} \prod_{k=1}^{m}\left\langle u_{i}^{k}, u_{j}^{k}\right\rangle, \quad U^{k}:=\left\{u_{1}^{k}, u_{2}^{k}, \cdots, u_{N}^{k}\right\} . \tag{2.6}
\end{equation*}
$$

Using the same spirit for a gradient flow with the potential function (2.6), we propose the SMS model as follows.

$$
\left\{\begin{array}{l}
\dot{u}_{i}^{k}=\frac{\kappa}{N} \sum_{j=1}^{N}\left(\prod_{l \neq k}\left\langle u_{i}^{l}, u_{j}^{l}\right\rangle\right)\left(u_{j}^{k}-\left\langle u_{i}^{k}, u_{j}^{k}\right\rangle u_{i}^{k}\right), \quad t>0,  \tag{2.7}\\
u_{i}^{k}(0)=u_{i}^{k, 0} \in \mathbb{S}^{d_{k}-1}
\end{array} \quad i \in\{1,2, \cdots, N\}, \quad k \in\{1,2, \cdots, m\} .\right.
$$

As in Section 2.1, we set rank- $m$ real tensor $T_{i}$ :

$$
T_{i}:=u_{i}^{1} \otimes \cdots \otimes u_{i}^{m}, \quad i=1, \cdots, N
$$

Then, it is easy to check that $T_{i}$ satisfies

$$
\begin{equation*}
\left[\dot{T}_{p}\right]_{\alpha_{* 0}}=\frac{\kappa}{N} \sum_{k=1}^{m} \sum_{\ell=1}^{N}\left(\left[T_{\ell}\right]_{\alpha_{* i k}}\left[\bar{T}_{p}\right]_{\alpha_{* 1}}\left[T_{p}\right]_{\left.\alpha_{*(1-i * *)}^{k}\right)}-\left[T_{p}\right]_{\alpha_{* i k}}\left[\bar{T}_{\ell}\right]_{\alpha_{* 1}}\left[T_{p}\right]_{\left.\alpha_{*(1-i k)}^{k}\right)}\right) . \tag{2.8}
\end{equation*}
$$

It should be noted that (2.8) can be derivable from the Lohe tensor model (1.1) with the following conditions:

$$
\kappa_{i_{*}}=\{\begin{array}{ll}
\kappa & \text { when } i_{*}=i_{*}^{k}, \quad 1 \leq k \leq m, \\
0 & \text { otherwise, }
\end{array} \quad \text { where } \quad i_{*}^{k}:=\underbrace{(0, \cdots, 0,1,0, \cdots, 0)}_{\text {only } k^{t h} \text { index is } 1} .
$$

Hence, systems (2.7) and (2.8) can be related in view of a completely separable state, and since the emergent dynamics of the LT model has been discussed in literature [13, 14, 15], we conclude that system (2.7) exhibits complete aggregation under suitable circumstances.
Proposition 2.3. [9] Suppose that initial data $T^{0}=\left\{T_{i}^{0}\right\}$ are completely factorized as a tensor product of rank-1 real tensors:

$$
\begin{aligned}
& T_{i}^{0}=u_{i}^{1,0} \otimes u_{i}^{2,0} \otimes \cdots \otimes u_{i}^{m, 0}, \quad i=1, \cdots, N \\
& \mathcal{A}\left(U^{k, 0}\right):=\min _{1 \leq i, j \leq N}\left\langle u_{i}^{k, 0}, u_{j}^{k, 0}\right\rangle>0, \quad k=1, \cdots, m,
\end{aligned}
$$

and let $T=\left\{T_{i}\right\}$ be a solution to system (2.8) and $\left\{U^{1}, \cdots, U^{d}\right\}$ be a solution to system (2.7). Then, the following assertions hold.
(1) $T_{i}=T_{i}(t)$ is completely separable in the sense that

$$
T_{i}(t)=u_{i}^{1}(t) \otimes u_{i}^{2}(t) \otimes \cdots \otimes u_{i}^{m}(t), \quad t>0, \quad i=1, \cdots, N
$$

(2) The solution exhibits the complete aggregation:

$$
\lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left\|T_{i}(t)-T_{j}(t)\right\|_{\mathrm{F}}=0
$$

Proof. For a proof, we refer the reader to Theorem 6.2 and Proposition 7.1 in 9 .

## 3. Existence of Rank-4 quadratically separable states

In this section, we present existence of the QSS for the LT model with rank-4 tensors and introduce the DM model than can be induced from the LT model.
3.1. The DM model. In this subsection, we propose the DM model consisting of two generalized Lohe matrix model on the rectangular matrices with possibly different sizes:

$$
U_{j} \in \mathbb{C}^{d_{1} \times d_{2}} \quad \text { and } \quad V_{j} \in \mathbb{C}^{d_{3} \times d_{4}}, \quad j=1, \cdots, N .
$$

Below, we sketch our strategy how to derive the DM model from the LT model:

- Step A (A homogeneous ensemble): we present a DM model for a homogeneous ensemble (Section 3.1.1).
- Step B (A heterogeneous ensemble): by adding natural frequency tensors with suitable structure conditions, we derive the DM model from the LT model (Section (3.1.2).
3.1.1. A homogeneous ensemble. Let $T_{j} \in \mathbb{C}^{d_{1} \times d_{2} \times d_{3} \times d_{4}}$ be a rank- 4 tensor that is a solution to (1.1) with zero natural frequency tensors $A_{j} \equiv O$ :

$$
\begin{equation*}
\left[\dot{T}_{j}\right]_{\alpha_{* 0}}=\sum_{i_{*} \in\{0,1\}^{4}}\left[\frac{\kappa_{i_{*}}}{N} \sum_{k=1}^{N}\left(\left[T_{k}\right]_{\alpha_{* i *}}\left[\bar{T}_{j}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*(1-i *)}}-\left[T_{j}\right]_{\alpha_{* i *}}\left[\bar{T}_{k}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\left.\alpha_{*(1-i *)}\right)}\right)\right] . \tag{3.1}
\end{equation*}
$$

For a given solution $T_{j}$ to (3.1), we assume that there exist two matrices $U_{j} \in \mathbb{C}^{d_{1} \times d_{2}}$ and $V_{j} \in \mathbb{C}^{d_{3} \times d_{4}}$ such that

$$
T_{j}=U_{j} \otimes V_{j}, \quad\left[T_{j}(t)\right]_{\alpha \beta \gamma \delta}=\left[U_{j}(t)\right]_{\alpha \beta}\left[V_{j}(t)\right]_{\gamma \delta} \quad \text { in a component form. }
$$

Next, we rewrite cubic interaction terms in (3.1) in terms of $U_{j}$ and $V_{j}$. For this, we decompose the index vectors $i_{*}$ and $\alpha_{* i_{*}}$ as

$$
i_{*}:=\left(i_{1}, i_{2}, i_{3}, i_{4}\right), \quad \alpha_{* i_{*}}:=\left(\beta_{* j_{*}}, \gamma_{* k_{*}}\right), \quad j_{*}:=\left(i_{1}, i_{2}\right), \quad k_{*}:=\left(i_{3}, i_{4}\right),
$$

where $j_{*}$ and $k_{*}$ correspond to the index vectors for $U_{j}$ and $V_{j}$, respectively. We now observe

$$
\begin{aligned}
& {\left[T_{i}\right]_{\alpha_{* i *}}\left[\bar{T}_{j}\right]_{\alpha_{* 1}}\left[T_{k}\right]_{\alpha_{*\left(1-i_{*}\right)}}} \\
& \quad=\left[T_{i}\right]_{\left(\beta_{* j_{*}}, \gamma_{* k_{*}}\right.}\left[\bar{T}_{j}\right]_{\left(\beta_{* 1}, \gamma_{* 1}\right)}\left[T_{k}\right]_{\left(\beta_{*\left(1-j_{*}\right)}, \gamma_{*\left(1-k_{*}\right)}\right)} \\
& \quad=\left[U_{i}\right]_{\beta_{* j *}}\left[V_{i}\right]_{\gamma_{* k}}\left[\bar{U}_{j}\right]_{\beta_{* 1}}\left[\bar{V}_{j}\right]_{\gamma_{* 1}}\left[U_{k}\right]_{\beta_{*\left(1-j_{*}\right)}}\left[V_{k}\right]_{\gamma_{*\left(1-k_{*}\right)}} \\
& \quad=\left(\left[U_{i}\right]_{\beta_{* j *}}\left[\bar{U}_{j}\right]_{\beta_{* 1}}\left[U_{k}\right]_{\beta_{*\left(1-j_{*}\right)}}\right)\left(\left[V_{i}\right]_{\gamma_{* k_{*}}}\left[\bar{V}_{j}\right]_{\gamma_{* 1}}\left[V_{k}\right]_{\left.\gamma_{*\left(1-k_{*}\right)}\right)}\right) .
\end{aligned}
$$

By interchanging the roles of $j \leftrightarrow k$, the term inside of the summation in the right-hand side of (3.1) becomes

$$
\begin{aligned}
{\left[T_{k}\right]_{\alpha_{* i *}} } & {\left[\bar{T}_{j}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*\left(1-i_{*}\right)}}-\left[T_{j}\right]_{\alpha_{* i *}}\left[\bar{T}_{k}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*\left(1-i_{*}\right)}} } \\
= & \left(\left[U_{k}\right]_{\beta_{* j_{*}}}\left[\bar{U}_{j}\right]_{\beta_{* 1}}\left[U_{j}\right]_{\beta_{*\left(1-j_{*}\right)}}\right)\left(\left[V_{k}\right]_{\gamma_{* k *}}\left[\bar{V}_{j}\right]_{\gamma_{* 1}}\left[V_{j}\right]_{\gamma_{*\left(1-k_{*}\right)}}\right) \\
& -\left(\left[U_{j}\right]_{\beta_{* j}}\left[\bar{U}_{k}\right]_{\beta_{* 1}}\left[U_{j}\right]_{\left.\beta_{*\left(1-j_{*}\right)}\right)}\right)\left(\left[V_{j}\right]_{\gamma_{* k_{*}}}\left[\bar{V}_{k}\right]_{\gamma_{* 1}}\left[V_{j}\right]_{\gamma_{*\left(1-k_{*}\right)}}\right) .
\end{aligned}
$$

Since the left-hand side of (3.1) has the form of

$$
\begin{equation*}
\dot{T}_{j}=\dot{U}_{j} \otimes V_{j}+U_{j} \otimes \dot{V}_{j} \tag{3.2}
\end{equation*}
$$

one should impose either $\kappa_{*}=(1,1)$ or $j_{*}=(1,1)$ to derive the restriction on $\kappa_{i_{*}}$ :

$$
\kappa_{i_{*}}=0 \quad \text { for all } i_{*} \in\{0,1\}^{4} \text { with }\left(i_{1}, i_{2}\right) \neq(1,1) \text { and }\left(i_{3}, i_{4}\right) \neq(1,1)
$$

Then, the right-hand side of (3.1) further reduces to

$$
\begin{aligned}
& \sum_{i_{*} \in\{0,1\}^{4}}\left[\frac{\kappa_{i_{*}}}{N} \sum_{k=1}^{N}\left(\left[T_{k}\right]_{\alpha_{* i *}}\left[\bar{T}_{j}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*\left(1-i_{*}\right)}}-\left[T_{j}\right]_{\alpha_{* i *}}\left[\bar{T}_{k}\right]_{\alpha_{* 1}}\left[T_{j}\right]_{\alpha_{*\left(1-i_{*}\right)}}\right)\right] \\
& =\sum_{j_{*} \in\{0,1\}^{2}}\left[\frac { \kappa _ { ( j _ { * } , 1 , 1 ) } } { N } \sum _ { k = 1 } ^ { N } \left(\left[T_{k}\right]_{\beta_{* j *} \gamma_{* 1}}\left[\bar{T}_{j}\right]_{\beta_{* 1} \gamma_{* 1}}\left[T_{j}\right]_{\beta_{*\left(1-j_{*}\right)} \gamma_{* 0}}\right.\right. \\
& \left.\left.-\left[T_{j}\right]_{\beta_{* j *} \gamma_{* 1}}\left[\bar{T}_{k}\right]_{\beta_{* 1} \gamma_{* 1}}\left[T_{j}\right]_{\beta_{*\left(1-j_{*}\right)} \gamma_{* 0}}\right)\right] \\
& +\sum_{k_{*} \in\{0,1\}^{2}}\left[\frac { \kappa _ { ( 1 , 1 , k _ { * } ) } } { N } \sum _ { k = 1 } ^ { N } \left(\left[T_{k}\right]_{\beta_{* 1} \gamma_{* k_{*}}}\left[\bar{T}_{j}\right]_{\beta_{* 1} \gamma_{* 1}}\left[T_{j}\right]_{\beta_{* 0} \gamma_{*\left(1-k_{*}\right)}}\right.\right. \\
& \left.\left.-\left[T_{j}\right]_{\beta_{* 1} \gamma_{* k_{*}}}\left[\bar{T}_{k}\right]_{\beta_{* 1} \gamma_{* 1}}\left[T_{j}\right]_{\beta_{* 0} \gamma_{*(1-k *)}}\right)\right] \\
& =\left[V_{j}\right]_{\gamma_{* 0}} \sum_{j_{*} \in\{0,1\}^{2}}\left[\frac { \kappa _ { ( j _ { * } , 1 , 1 ) } } { N } \sum _ { k = 1 } ^ { N } \left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}}\left[U_{k}\right]_{\beta_{* j *}}\left[\bar{U}_{j}\right]_{\beta_{* 1}}\left[U_{j}\right]_{\beta_{*\left(1-j_{*}\right)}}\right.\right. \\
& \left.\left.-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}}\left[U_{j}\right]_{\beta_{* j}}\left[\bar{U}_{k}\right]_{\beta_{* 1}}\left[U_{j}\right]_{\beta_{*\left(1-j_{*}\right)}}\right)\right] \\
& +\left[U_{j}\right]_{\beta_{* 0}} \sum_{k_{*} \in\{0,1\}^{2}}\left[\frac { \kappa _ { ( 1 , 1 , k _ { * } ) } } { N } \sum _ { k = 1 } ^ { N } \left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}}\left[V_{k}\right]_{\gamma_{* k_{*}}}\left[\bar{V}_{j}\right]_{\gamma_{* 1}}\left[V_{j}\right]_{\gamma_{*\left(1-k_{*}\right)}}\right.\right. \\
& \left.\left.-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}}\left[V_{j}\right]_{\gamma_{* k_{*}}}\left[\bar{V}_{k}\right]_{\gamma_{* 1}}\left[V_{j}\right]_{\left.\gamma_{*\left(1-k_{*}\right)}\right)}\right)\right] .
\end{aligned}
$$

By comparing $\cdot \otimes V_{j}$ and $U_{j} \otimes \cdot$ in (3.2), one has

$$
\left\{\begin{aligned}
& {\left[\dot{U}_{j}\right]_{\beta_{* 0}}=} \sum_{j_{*} \in\{0,1\}^{2}}\left(\frac { \kappa _ { ( j _ { * } , 1 , 1 ) } } { N } \sum _ { k = 1 } ^ { N } \left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}}\left[U_{k}\right]_{\beta_{* j_{*}}}\left[\bar{U}_{j}\right]_{\beta_{* 1}}\left[U_{j}\right]_{\beta_{*\left(1-j_{*}\right)}}\right.\right. \\
&\left.\left.-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}}\left[U_{j}\right]_{\beta_{* j *}}\left[\bar{U}_{k}\right]_{\beta_{* 1}}\left[U_{j}\right]_{\beta_{*\left(1-j_{*}\right)}}\right)\right) \\
& {\left[\dot{V}_{j}\right]_{\gamma_{* 0}}=} \sum_{k_{*} \in\{0,1\}^{2}}\left(\frac { \kappa _ { ( 1 , 1 , k _ { * } ) } } { N } \sum _ { k = 1 } ^ { N } \left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}}\left[V_{k}\right]_{\gamma_{* k_{*}}}\left[\bar{V}_{j}\right]_{\gamma_{* 1}}\left[V_{j}\right]_{\gamma_{*\left(1-k_{*}\right)}}\right.\right. \\
&\left.\left.-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}}\left[V_{j}\right]_{\gamma_{* * *}}\left[\bar{V}_{k}\right]_{\gamma_{* 1}}\left[V_{j}\right]_{\gamma_{*\left(1-k_{*}\right)}}\right)\right)
\end{aligned}\right.
$$

If we choose the coupling strengths as for remaining $i_{*}$ :

$$
\begin{equation*}
\kappa_{(0,1,1,1)}=\kappa_{(1,1,0,1)}=\kappa_{1}, \quad \kappa_{(1,0,1,1)}=\kappa_{(1,1,1,0)}=\kappa_{2}, \quad \kappa_{i_{*}}=0, \tag{3.3}
\end{equation*}
$$

we obtain the desired system for $\left(U_{j}, V_{j}\right)$ :

$$
\left\{\begin{align*}
\dot{U}_{j}= & \frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger} U_{j}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right)  \tag{3.4}\\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{j} U_{j}^{\dagger} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right), \\
\dot{V}_{j}= & \frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k} V_{j}^{\dagger} V_{j}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right) \\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{j} V_{j}^{\dagger} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right)
\end{align*}\right.
$$

3.1.2. A heterogeneous ensemble. Similar to several aggregation models such as the Lohe matrix model [20] and the Lohe tensor model [13], a natural candidate for heterogeneous (or non-identical) extension of (3.4) would be the model (3.4) together with natural frequency tensors $B_{j}$ and $C_{j}$ whose ranks are 4 . Thus, the DM model for a heterogeneous ensemble reads as

$$
\left\{\begin{align*}
\dot{U}_{j}= & B_{j} U_{j}+\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger} U_{j}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right)  \tag{3.5}\\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{j} U_{j}^{\dagger} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right), \\
\dot{V}_{j}= & C_{j} V_{j}+\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k} V_{j}^{\dagger} V_{j}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right) \\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{j} V_{j}^{\dagger} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right),
\end{align*}\right.
$$

where $B_{j} \in \mathbb{C}^{d_{1} \times d_{2} \times d_{1} \times d_{2}}$ and $C_{j} \in \mathbb{C}^{d_{3} \times d_{4} \times d_{3} \times d_{4}}$ are rank- 4 tensors satisfying skewsymmetric properties: for suitable indices,

$$
\begin{equation*}
\left[B_{j}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}}=-\left[\bar{B}_{j}\right]_{\alpha_{2} \beta_{2} \alpha_{1} \beta_{1}}, \quad\left[C_{j}\right]_{\gamma_{1} \delta_{1} \gamma_{2} \delta_{2}}=-\left[\bar{C}_{j}\right]_{\gamma_{2} \delta_{2} \gamma_{1} \delta_{1}} . \tag{3.6}
\end{equation*}
$$

Moreover, $U_{j} B_{j}$ and $C_{j} V_{j}$ can be understood as tensor contractions between rank-4 and rank-2 tensors:

$$
\left[B_{j} U_{j}\right]_{\alpha \beta}=\left[B_{j}\right]_{\alpha \beta \gamma \delta}\left[U_{j}\right]_{\gamma \delta}, \quad\left[C_{j} V_{j}\right]_{\alpha \beta}=\left[C_{j}\right]_{\alpha \beta \gamma \delta}\left[V_{j}\right]_{\gamma \delta}
$$

Next, we find a condition for $A_{j}$ in (1.1) in terms of $B_{j}$ and $C_{j}$ in (3.5) so that models (1.1) and (3.5) are equivalent. For this, it suffices to focus on the free flows by setting $\kappa_{1}=\kappa_{2}=0$, i.e.,

$$
\dot{U}_{j}=B_{j} U_{j}, \quad \dot{V}_{j}=C_{j} V_{j}, \quad j=1, \cdots, N .
$$

If we use the relations above and

$$
\left(U_{j} \otimes V_{j}\right)^{\prime}=\left(B_{j} U_{j}\right) \otimes V_{j}+U_{j} \otimes\left(C_{j} V_{j}\right)
$$

then we can find

$$
\begin{equation*}
A_{j}\left(U_{j} \otimes V_{j}\right)=\left(B_{j} U_{j}\right) \otimes V_{j}+U_{j} \otimes\left(C_{j} V_{j}\right) \tag{3.7}
\end{equation*}
$$

In addition, if we rewrite (3.7) as a component form, then it becomes

$$
\begin{align*}
0= & {\left[A_{j}\right]_{\alpha_{1} \beta_{1} \gamma_{1} \delta_{1} \alpha_{2} \beta_{2} \gamma_{2} \delta_{2}}\left[U_{j}\right]_{\alpha_{2} \beta_{2}}\left[V_{j}\right]_{\gamma_{2} \delta_{2}}-\left[B_{j}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}}\left[U_{j}\right]_{\alpha_{2} \beta_{2}}\left[V_{j}\right]_{\gamma_{1} \delta_{1}} } \\
& -\left[U_{j}\right]_{\alpha_{1} \beta_{1}}\left[C_{j}\right]_{\gamma_{1} \delta_{1} \gamma_{2} \delta_{2}}\left[V_{j}\right]_{\gamma_{2} \delta_{2}} \\
= & \left(\left[A_{j}\right]_{\alpha_{1} \beta_{1} \gamma_{1} \delta_{1} \alpha_{2} \beta_{2} \gamma_{2} \delta_{2}}-\left[B_{j}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}} \delta_{\gamma_{1} \gamma_{2}} \delta_{\delta_{1} \delta_{2}}-\left[C_{j}\right]_{\gamma_{1} \delta_{1} \gamma_{2} \delta_{2}} \delta_{\alpha_{1} \alpha_{2}} \delta_{\beta_{1} \beta_{2}}\right)  \tag{3.8}\\
& \times\left[U_{j}\right]_{\alpha_{2} \beta_{2}}\left[V_{j}\right]_{\gamma_{2} \delta_{2}} .
\end{align*}
$$

Since (3.8) holds for arbitrary $U_{j}$ and $V_{j}$, we can find an explicit relation between $A_{j}$ and $B_{j}, C_{j}$ :

$$
\begin{equation*}
\left[A_{j}\right]_{\alpha_{1} \beta_{1} \gamma_{1} \delta_{1} \alpha_{2} \beta_{2} \gamma_{2} \delta_{2}}=\left[B_{j}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}} \delta_{\gamma_{1} \gamma_{2}} \delta_{\delta_{1} \delta_{2}}+\left[C_{j}\right]_{\gamma_{1} \delta_{1} \gamma_{2} \delta_{2}} \delta_{\alpha_{1} \alpha_{2}} \delta_{\beta_{1} \beta_{2}} \tag{3.9}
\end{equation*}
$$

So far, we have derived the DM model (3.5) with (3.9) from the LT model (1.1). However, if we use the same argument reversely and recall that system (3.5) admits a unique solution, then one can identify the LT model from the DM model. Thus, we can say that system (3.5)(3.9) and system (1.1) are equivalent in some sense. Below, we summarize the argument above in the following proposition.
Proposition 3.1. (1) Suppose $\left\{\left(U_{i}, V_{i}\right)\right\}$ is a solution to (3.5). Then, a rank-4 tensor $T_{i}$ defined by $T_{i}:=U_{i} \otimes V_{i}$ is a $Q S S$ to (1.1) with a well-prepared free flow tensor $A_{i}$ satisfying (3.9).
(2) Suppose a rank-4 tensor $T_{i}$ is a solution to (1.1) with (3.9) and quadratically separable initial data:

$$
T_{i}^{0}=: U_{i}^{0} \otimes V_{i}^{0}, \quad 1 \leq i \leq N,
$$

for rank-2 tensors $U_{i}^{0} \in \mathbb{C}^{d_{1} \times d_{2}}$ and $V_{i}^{0} \in \mathbb{C}^{d_{3} \times d_{4}}$ with unit norms. Then, there exist two matrices with unit norms $U_{i}=U_{i}(t)$ and $V=V_{i}(t)$ such that

$$
T_{i}(t)=U_{i}(t) \otimes V_{i}(t), \quad t>0
$$

where $\left(U_{i}, V_{i}\right)$ is a solution to (3.5) with $\left(U_{i}, V_{i}\right)(0)=\left(U_{i}^{0}, V_{i}^{0}\right)$.
3.2. Gradient flow formulation of the DM model. In this subsection, we show that system (3.5) can be formulated as a gradient flow with a suitable analytical potential induced from the Lohe tensor model. Recall from [14] that the following functional can be associated with the LT model:

$$
\begin{equation*}
\mathcal{V}(T)=1-\frac{1}{N^{2}} \sum_{i, j=1}^{N}\left\langle T_{i}, T_{j}\right\rangle_{\mathrm{F}} \tag{3.10}
\end{equation*}
$$

If we use the decomposition $T_{i}=U_{i} \otimes V_{i}$, then we find the corresponding functional for (3.5):

$$
\begin{equation*}
\mathcal{E}(U, V):=1-\frac{1}{N^{2}} \sum_{i, j=1}^{N}\left\langle U_{i} \otimes V_{i}, U_{j} \otimes V_{j}\right\rangle_{\mathrm{F}}=1-\frac{1}{N^{2}} \sum_{i, j=1}^{N}\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}}\left\langle V_{i}, V_{j}\right\rangle_{\mathrm{F}} \tag{3.11}
\end{equation*}
$$

In the following lemma, we show that $\mathcal{E}(U, V)$ is non-increasing along the flow (3.5).
Lemma 3.1. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (3.5) with $\left(B_{j}, C_{j}\right)=(O, O)$. Then, the functional $\mathcal{E}(U, V)$ is non-increasing in time:

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{E}(U, V)=-\frac{\kappa_{1}}{N} \sum_{j=1}^{N}\left\|\frac{1}{N} \sum_{i=1}^{N}\left(\left\langle V_{j}, V_{i}\right\rangle_{\mathrm{F}} U_{i} U_{j}^{\dagger}-\left\langle V_{i}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{i}^{\dagger}\right)\right\|_{\mathrm{F}}^{2}
$$

$$
\begin{aligned}
& -\frac{\kappa_{1}}{N} \sum_{j=1}^{N}\left\|\frac{1}{N} \sum_{i=1}^{N}\left(\left\langle U_{j}, U_{i}\right\rangle_{\mathrm{F}} V_{i} V_{j}^{\dagger}-\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{i}^{\dagger}\right)\right\|_{\mathrm{F}}^{2} \\
& -\frac{\kappa_{2}}{N} \sum_{j=1}^{N}\left\|\frac{1}{N} \sum_{i=1}^{N}\left(\left\langle V_{j}, V_{i}\right\rangle_{\mathrm{F}} U_{j}^{\dagger} U_{i}-\left\langle V_{i}, V_{j}\right\rangle_{\mathrm{F}} U_{i}^{\dagger} U_{j}\right)\right\|_{\mathrm{F}}^{2} \\
& -\frac{\kappa_{2}}{N} \sum_{j=1}^{N}\left\|\frac{1}{N} \sum_{i=1}^{N}\left(\left\langle U_{j}, U_{i}\right\rangle_{\mathrm{F}} V_{j}^{\dagger} V_{i}-\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}} V_{i}^{\dagger} V_{j}\right)\right\|_{\mathrm{F}}^{2} .
\end{aligned}
$$

Proof. It follows from Proposition 4.1 in [14] that $\mathcal{V}(T)$ in (3.10) for the LT model (1.1) satisfies

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{V}(T)=-\frac{1}{N} \sum_{j=1}^{N} \sum_{i_{*} \in\{0,1\}^{4}} \kappa_{i_{*}}\left\|M^{i_{*}}\left(T_{c}\right) M^{i_{*}}\left(T_{j}\right)^{\dagger}-M^{i_{*}}\left(T_{j}\right) M^{i_{*}}\left(T_{c}\right)^{\dagger}\right\|_{\mathrm{F}}^{2} \tag{3.12}
\end{equation*}
$$

Here, we recall from [14, Definition 3.1] that for a rank-4 tensor $T, \mathcal{M}^{i_{*}}(T)$ is defined as a rank- 2 tensor reshaped from $T$ or it can be understood as a bijective linear map which conserves the Frobenius norm. Since we only consider four types among $i_{*} \in\{0,1\}^{4}$ as in (3.3), we set

$$
\begin{equation*}
\kappa_{(0,1,1,1)}=\kappa_{(1,1,0,1)}=\kappa_{1}, \quad \kappa_{(1,0,1,1)}=\kappa_{(1,1,1,0)}=\kappa_{2}, \quad \kappa_{i_{*}}=0 \quad \text { for other } i_{*} . \tag{3.13}
\end{equation*}
$$

On the other hand, in (3.12), we are concerned with the term $M^{i_{*}}\left(T_{i}\right) M^{i_{*}}\left(T_{j}\right)^{\dagger}$ for $T_{i}=$ $U_{i} \otimes V_{i}$ and $i_{*}$ in (3.13), for instance, if $i_{*}=(0,1,1,1)$, then we have

$$
\begin{aligned}
{\left[M^{i_{*}}\left(T_{i}\right) M^{i_{*}}\left(T_{j}\right)^{\dagger}\right]_{\alpha_{20} \alpha_{21}} } & =\left[T_{i}\right]_{\alpha_{11} \alpha_{20} \alpha_{31} \alpha_{41}}\left[\bar{T}_{j}\right]_{\alpha_{11} \alpha_{21} \alpha_{31} \alpha_{41}} \\
& =\left[U_{i}\right]_{\alpha_{11} \alpha_{20}}\left[V_{i}\right]_{\alpha_{31} \alpha_{41}}\left[\bar{U}_{j}\right]_{\alpha_{11} \alpha_{21}}\left[\bar{V}_{j}\right]_{\alpha_{31} \alpha_{41}}=\left\langle V_{j}, V_{i}\right\rangle_{\mathrm{F}}\left[U_{j}^{\dagger} U_{i}\right]_{\alpha_{21} \alpha_{20}} .
\end{aligned}
$$

If we repeat similar calculations as above, we obtain the desired dissipative estimate.
3.3. The DM model. In this subsection, we further reduce the DM model (3.5) to the model defined on the product of two unitary matrices $\mathbf{U}(n) \times \mathbf{U}(m)$ which we call as the DUM model. By the construction of model (3.5), we know that the Frobenius norms of $U_{j}$ and $V_{j}$ are conserved. In addition to the conservation of the Frobenius norms, we show that unitarity is also preserved, when rectangular matrices are replaced by square matrices.

Lemma 3.2. Suppose that the system parameters and initial data satisfy

$$
d_{1}=d_{2}=n, \quad d_{3}=d_{4}=m, \quad U_{j}^{0} \in \mathbf{U}(n), \quad V_{j}^{0} \in \mathbf{U}(m),
$$

and let $\left\{\left(U_{j}, V_{j}\right)\right\}$ be a solution to (3.5). Then, one has

$$
U_{j}(t) \in \mathbf{U}(n), \quad V_{j}(t) \in \mathbf{U}(m), \quad t>0, \quad j=1, \cdots, N
$$

Proof. Since $U_{j}$ and $V_{j}$ have the same structure, we focus only on $U_{j}$. We rewrite the dynamics of $U_{j}$ as a simpler form:

$$
\dot{U}_{j}=B_{j} U_{j}+\left(P_{j}-P_{j}^{\dagger}\right) U_{j}+U_{j}\left(Q_{j}-Q_{j}^{\dagger}\right),
$$

where $P_{j}$ and $Q_{j}$ are defined as

$$
P_{j}:=\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger}, \quad Q_{j}:=\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{j}^{\dagger} U_{k} .
$$

Then, one has

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(U_{j} U_{j}^{\dagger}\right)= & B_{j} U_{j} U_{j}^{\dagger}-U_{j} U_{j}^{\dagger} B_{j}+\left(P_{j}-P_{j}^{\dagger}\right) U_{j} U_{j}^{\dagger}+U_{j}\left(Q_{j}-Q_{j}^{\dagger}\right) U_{j}^{\dagger} \\
& -U_{j} U_{j}^{\dagger}\left(P_{j}-P_{j}^{\dagger}\right)-U_{j}\left(Q_{j}-Q_{j}^{\dagger}\right) U_{j}^{\dagger} \\
= & \left(B_{j}+P_{j}-P_{j}^{\dagger}\right) U_{j} U_{j}^{\dagger}-U_{j} U_{j}^{\dagger}\left(B_{j}+P_{j}-P_{j}^{\dagger}\right)
\end{aligned}
$$

By straightforward calculation, we find

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left\|I_{n}-U_{j} U_{j}^{\dagger}\right\|_{\mathrm{F}}=0 \tag{3.14}
\end{equation*}
$$

Since we assume $U_{j}^{0} \in \mathbf{U}(n)$, i.e., $I_{n}-U_{j} U_{j}^{\dagger}=O$, the relation (3.14) yields the desired result.

Due to the unitarity, system (3.5) can further be reduced to the model on $\mathbf{U}(n) \times \mathbf{U}(m)$ by using the relations:

$$
U_{j} U_{j}^{\dagger}=I_{n}=U_{j}^{\dagger} U_{j}, \quad V_{j} V_{j}^{\dagger}=I_{m}=V_{j}^{\dagger} V_{j}, \quad j=1, \cdots, N
$$

Thus, the DM model reads as

$$
\left\{\begin{array}{l}
\dot{U}_{j}=B_{j} U_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right) \\
\dot{V}_{j}=C_{j} V_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right)
\end{array}\right.
$$

where $\kappa=\kappa_{1}+\kappa_{2}$.
Note that natural frequency tensors $B_{j}$ and $C_{j}$ have rank- 4 satisfying skew-symmetric properties as in (3.6). In order to give a meaning of Hamiltonian, we associate two Hermitian matrices, namely, $H_{j} \in \mathbb{C}^{n \times n}$ and $G_{j} \in \mathbb{C}^{m \times m}$ :

$$
\left[B_{j}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}}=:\left[-\mathrm{i} H_{j}\right]_{\alpha_{1} \alpha_{2}} \delta_{\beta_{1} \beta_{2}}, \quad\left[C_{j}\right]_{\gamma_{1} \delta_{1} \gamma_{2} \delta_{2}}=:\left[-\mathrm{i} G_{j}\right]_{\gamma_{1} \gamma_{2}} \delta_{\delta_{1} \delta_{2}}
$$

Then, system (3.10) reduces to the model on $\mathbf{U}(n) \times \mathbf{U}(m)$ :

$$
\left\{\begin{array}{l}
\dot{U}_{j}=-\mathrm{i} H_{j} U_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right)  \tag{3.15}\\
\dot{V}_{j}=-\mathrm{i} G_{j} V_{j}+\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right)
\end{array}\right.
$$

where $H_{j} U_{j}$ and $G_{j} V_{j}$ are now usual matrix products. Then as in Lemma 3.1, system (3.15) also satisfies the dissipative energy estimate. Since the proof can be directly obtained from Lemma 3.1, we omit it.

Corollary 3.1. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (3.15) with $H_{j}=G_{j} \equiv O$. Then the Lyapunov functional (5.1) satisfies

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{E}(U, V)=-\frac{\kappa}{N} \sum_{j=1}^{N}\left(\left\|\dot{U}_{j}\right\|_{\mathrm{F}}^{2}+\left\|\dot{V}_{j}\right\|_{\mathrm{F}}^{2}\right), \quad t>0
$$

## 4. Emergent dynamics of rank-4 quadratically separable states

In this section, we study the emergent behavior of the QSS for the Lohe tensor model by investigating the dynamics of the DUM model which reads as follows:

$$
\left\{\begin{array}{l}
\dot{U}_{j}=-\mathrm{i} H_{j} U_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right), \quad t>0  \tag{4.1}\\
\dot{V}_{j}=-\mathrm{i} G_{j} V_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right) \\
\left(U_{j}, V_{j}\right)(0)=\left(U_{j}^{0}, V_{j}^{0}\right) \in \mathbf{U}(n) \times \mathbf{U}(m), \quad 1 \leq j \leq N
\end{array}\right.
$$

where $H_{j} \in \mathbb{C}^{n \times n}$ and $G_{j} \in \mathbb{C}^{m \times m}$ are Hermitian matrices which play roles of natural frequencies for each oscillator.

First, we recall several definitions of emergent behaviors for (4.1).
Definition 4.1. [16] $\operatorname{Let}(\mathcal{U}, \mathcal{V}):=\left\{U_{j}, V_{j}\right\}_{j=1}^{N}$ be a solution to (4.1).
(1) System (4.1) exhibits complete aggregation if the following estimate holds:

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left(\left\|U_{i}(t)-U_{j}(t)\right\|_{\mathrm{F}}+\left\|V_{i}(t)-V_{j}(t)\right\|_{\mathrm{F}}\right)=0 \tag{4.2}
\end{equation*}
$$

(2) The state $(\mathcal{U}, \mathcal{V})$ tends to a phase-locked state if the following relation holds:

$$
\exists \lim _{t \rightarrow \infty} U_{i}(t) U_{j}(t)^{\dagger} \quad \text { and } \quad \exists \lim _{t \rightarrow \infty} V_{i}(t) V_{j}(t)^{\dagger}
$$

In order to investigate emergent behaviors for (4.1), we denote the following quantities for notational simplicity:

$$
\begin{align*}
& X_{i j}:=U_{i} U_{j}^{\dagger}, \quad S_{i j}:=I_{n}-U_{i} U_{j}^{\dagger}, \quad d_{i j}:=\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}},  \tag{4.3}\\
& Y_{i j}:=V_{i} V_{j}^{\dagger}, \quad T_{i j}:=I_{m}-V_{i} V_{j}^{\dagger} \quad c_{i j}:=\left\langle V_{i}, V_{j}\right\rangle_{\mathrm{F}} .
\end{align*}
$$

It follows from simple observations that

$$
\begin{array}{ll}
\left\|U_{j}\right\|_{\mathrm{F}}=n, \quad\left\|V_{j}\right\|_{\mathrm{F}}=m, \quad\left|d_{i j}\right| \leq n, \quad\left|c_{i j}\right| \leq m \\
\left\|S_{i j}\right\|_{\mathrm{F}}^{2}=\left\|U_{i}-U_{j}\right\|_{\mathrm{F}}^{2}=2 \operatorname{Re}\left(n-d_{i j}\right), \quad\left\|T_{i j}\right\|_{\mathrm{F}}^{2}=\left\|V_{i}-V_{j}\right\|_{\mathrm{F}}^{2}=2 \operatorname{Re}\left(m-c_{i j}\right)
\end{array}
$$

Then, it is easy to see that

$$
\left\|U_{i}-U_{j}\right\|_{\mathrm{F}} \rightarrow 0 \Longleftrightarrow\left|n-d_{i j}\right| \rightarrow 0, \quad\left\|V_{i}-V_{j}\right\|_{\mathrm{F}} \rightarrow 0 \Longleftrightarrow\left|m-c_{i j}\right| \rightarrow 0
$$

Thus, the complete aggregation in (4.2) can be represented in terms of the quantities in (4.3):
$\lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left(\left\|S_{i j}(t)\right\|_{\mathrm{F}}+\left\|T_{i j}(t)\right\|_{\mathrm{F}}\right)=0 \quad$ or $\quad \lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left(\left|n-d_{i j}(t)\right|+\left|m-c_{i j}(t)\right|\right)=0$.
In this regard, we define aggregation quantities: for $t>0$,

$$
\begin{array}{ll}
\mathcal{D}(\mathcal{U}(t)):=\max _{1 \leq i, j \leq N}\left\|U_{i}(t)-U_{j}(t)\right\|_{\mathrm{F}}, & \mathcal{S}(\mathcal{U}(t)):=\max _{1 \leq i, j \leq N}\left|n-d_{i j}(t)\right|, \\
\mathcal{D}(\mathcal{V}(t)):=\max _{1 \leq i, j \leq N}\left\|V_{i}(t)-V_{j}(t)\right\|_{\mathrm{F}}, \quad \mathcal{S}(\mathcal{V}(t)):=\max _{1 \leq i, j \leq N}\left|m-c_{i j}(t)\right| .
\end{array}
$$

Note that $\mathcal{S}(\mathcal{U})$ has a second-order with respect to the state $\mathcal{U}$, whereas $\mathcal{D}(\mathcal{U})$ has a firstorder.

In the following two subsections, we present emergent dynamics for homogeneous and heterogeneous ensembles respectively.
4.1. Homogeneous ensemble. In this subsection, we deal with the emergent dynamics of (4.1) with a homogeneous ensemble:

$$
H_{j} \equiv H, \quad G_{j} \equiv G, \quad j=1, \cdots, N
$$

and by the solution splitting property, we may assume

$$
H=O, \quad G=O
$$

In this setting, system (3.5) becomes

$$
\left\{\begin{array}{l}
\dot{U}_{j}=\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger} U_{j}\right), \quad t>0  \tag{4.4}\\
\dot{V}_{j}=\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}} V_{k}-\left\langle U_{k}, U_{j}\right\rangle_{\mathrm{F}} V_{j} V_{k}^{\dagger} V_{j}\right), \\
\left(U_{j}, V_{j}\right)(0)=\left(U_{j}^{0}, V_{j}^{0}\right) \in \mathbf{U}(n) \times \mathbf{U}(m)
\end{array}\right.
$$

Without loss of generality, we may assume

$$
\begin{equation*}
n \geq m \tag{4.5}
\end{equation*}
$$

Our goal of this subsection is to find a sufficient condition under which

$$
\lim _{t \rightarrow \infty} \mathcal{L}(t)=0, \quad \mathcal{L}(t):=\mathcal{D}(\mathcal{U}(t))+\mathcal{D}(\mathcal{V}(t))+\mathcal{S}(\mathcal{U}(t))+\mathcal{S}(\mathcal{V}(t))
$$

where $\mathcal{L}=\mathcal{L}(t)$ is called as a total aggregation functional.
In [16], it suffices to study the temporal evolutions of $\mathcal{D}(\mathcal{U})$. However in our case, time evolutions of $\mathcal{S}(\mathcal{U})$ as well as $\mathcal{S}(\mathcal{V})$ are needed to achieve complete aggregation estimates. Below, we derive a differential inequality for $\mathcal{L}$.
Lemma 4.1. Let $\left\{\left(U_{j}, V_{j}\right)\right\}$ be a solution to (4.4) with (4.5). Then, the total aggregation functional $\mathcal{L}$ satisfies

$$
\begin{equation*}
\dot{\mathcal{L}} \leq-2 \kappa(m-4 \sqrt{n}) \mathcal{L}+\kappa(4 n+9) \mathcal{L}^{2}+\kappa\left(2 n+\frac{8}{3}\right) \mathcal{L}^{3}, \quad t>0 \tag{4.6}
\end{equation*}
$$

Proof. Since a proof is lengthy, we provide it in Appendix A.
We are now ready to provide a sufficient condition leading to the complete aggregation for (4.4).

Theorem 4.1. Suppose that the system parameters and initial data satisfy
(i) $n \geq m>4 \sqrt{n}$.
(ii) $\mathcal{L}^{0}<\alpha_{n, m}:=\frac{-(12 n+27)+\sqrt{(12 n+27)^{2}+48(m-4 \sqrt{n})(3 n+4)}}{4(3 n+4)}$,
and let $\left\{\left(U_{j}, V_{j}\right)\right\}$ be a solution to (4.4). Then, we have

$$
\lim _{t \rightarrow \infty} \mathcal{L}(t)=0
$$

Moreover, the convergence rate is at least exponential. In other words, system (4.4) exhibits complete aggregation with an exponential convergence.

Proof. Consider an auxiliary quadratic polynomial:

$$
f(s):=\left(2 n+\frac{8}{3}\right) s^{2}+(4 n+9) s-2(m-4 \sqrt{n})
$$

Since $m>4 \sqrt{n}$, the algebraic relation $f=0$ admits a unique positive root $\alpha_{n, m}$ defined in (4.7) (ii). Then, the relation (4.6) is rewritten in terms of $f$ :

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{L} \leq \kappa \mathcal{L} f(\mathcal{L}), \quad t>0
$$

Since initial data satisfy (4.7)(ii), the desired result follows from dynamical systems theory.

Remark 4.1. (i) In (4.7) (i), we have assumed that $n \geq m>4 \sqrt{n}$ which imposes a restriction on the size of $U_{i}$ such as

$$
n>16 \text {. }
$$

It also should be mentioned that such restriction arises, for instance, when we estimate the terms $\mathcal{I}_{15}$ in A.7) and $\mathcal{I}_{25}$ in A.10). We indeed show in Appendix $\mathbb{0}$ that this technical assumption would be removed.
(ii) Since $m \leq n$, we have

$$
\limsup _{n \rightarrow \infty} \alpha_{n, m} \leq \lim _{n \rightarrow \infty} \frac{-(12 n+27)+\sqrt{(12 n+27)^{2}+48(n-4 \sqrt{n})(3 n+4)}}{4(3 n+4)}=\frac{1}{2} .
$$

4.2. Heterogeneous ensemble. In this subsection, we study the case of heterogeneous Hamiltonians in which $H_{j}$ and $G_{j}$ in (4.1) are given to be different in general. In order to establish the emergence of the phase-locked state, we will follow a strategy developed in [16]. For any two solutions $\left\{U_{j}, V_{j}\right\}$ and $\left\{\tilde{U}_{j}, \tilde{V}_{j}\right\}$ to (4.1), we define the diameters measuring the dissimilarity of two configurations:

$$
\begin{align*}
d(U, \tilde{U})(t) & :=\max _{1 \leq i, j \leq N}\left\|U_{i}(t) U_{j}^{\dagger}(t)-\tilde{U}_{i}(t) \tilde{U}_{j}^{\dagger}(t)\right\|_{\mathrm{F}} \\
d(V, \tilde{V})(t) & :=\max _{1 \leq i, j \leq N}\left\|V_{i}(t) V_{j}^{\dagger}(t)-\tilde{V}_{i}(t) \tilde{V}_{j}^{\dagger}(t)\right\|_{\mathrm{F}} \tag{4.8}
\end{align*}
$$

Then, we will show that the diameters above converge to zero:

$$
\begin{equation*}
\lim _{t \rightarrow \infty}(d(U, \tilde{U})(t)+d(V, \tilde{V})(t))=0 \tag{4.9}
\end{equation*}
$$

As a next step, since our system is autonomous, for any $T>0$, we choose $\tilde{U}_{j}$ and $\tilde{V}_{j}$ as

$$
\tilde{U}_{j}(t)=U_{j}(t+T), \quad \tilde{V}_{j}(t)=V_{j}(t+T) .
$$

By discretizing the time $t \in \mathbb{R}_{+}$as $n \in \mathbb{Z}_{+}$and setting $T=m \in \mathbb{Z}_{+}$, we conclude that $\left\{U_{i}(n) U_{j}^{\dagger}(n)\right\}_{n \in \mathbb{Z}_{+}}$and $\left\{V_{i}(n) V_{j}^{\dagger}(n)\right\}_{n \in \mathbb{Z}_{+}}$are indeed Cauchy sequences in the complete
spaces $\mathbf{U}(n)$ and $\mathbf{U}(m)$, respectively. Hence, there exist two constant unitary matrices $U_{i j}^{\infty} \in \mathbf{U}(n)$ and $V_{i j}^{\infty} \in \mathbf{U}(m)$ such that

$$
\lim _{t \rightarrow \infty}\left\|U_{i}(t) U_{j}^{\dagger}(t)-U_{i j}^{\infty}\right\|_{\mathrm{F}}=0, \quad \lim _{t \rightarrow \infty}\left\|V_{i}(t) V_{j}^{\dagger}(t)-V_{i j}^{\infty}\right\|_{\mathrm{F}}=0
$$

Hence, we aim to find a sufficient condition under which (4.9) holds. To show (4.9), we associate another diameters measuring the difference between two solution configurations $\left\{U_{j}, V_{j}\right\}$ and $\left\{\tilde{U}_{j}, \tilde{V}_{j}\right\}:$

$$
\begin{align*}
\mathcal{S}(U, \tilde{U})(t) & :=\max _{1 \leq i, j \leq N}\left|\left\langle U_{i}, U_{j}\right\rangle(t)-\left\langle\tilde{U}_{i}, \tilde{U}_{j}\right\rangle(t)\right|, \\
\mathcal{S}(V, \tilde{V})(t) & :=\max _{1 \leq i, j \leq N}\left|\left\langle V_{i}, V_{j}\right\rangle(t)-\left\langle\tilde{V}_{i}, \tilde{V}_{j}\right\rangle(t)\right| . \tag{4.10}
\end{align*}
$$

Note that

$$
\mathcal{S}(U, \tilde{U}) \leq \sqrt{n} d(U, \tilde{U}), \quad \mathcal{S}(V, \tilde{V}) \leq \sqrt{m} d(V, \tilde{V})
$$

To this end, our goal of this subsection is to find a sufficient framework leading to

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathcal{F}(t)=0, \quad \mathcal{F}(t):=d(U, \tilde{U})(t)+d(V, \tilde{V})(t)+\mathcal{S}(U, \tilde{U})(t)+\mathcal{S}(V, \tilde{V})(t) \tag{4.11}
\end{equation*}
$$

Below, we derive a differential inequality for $\mathcal{F}$ in (4.11).
Lemma 4.2. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ and $\left\{\left(\tilde{U}_{i}, \tilde{V}_{i}\right)\right\}$ be any two solutions to (4.1) with (4.5), respectively. Then, the aggregation functional $\mathcal{F}$ satisfies

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F} \leq-\kappa\left(2 m-8 \sqrt{n}-\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{\kappa}\right) \mathcal{F}+\kappa(4 n+22) \mathcal{L} \mathcal{F}+20 \kappa \mathcal{L}^{2} \mathcal{F} \tag{4.12}
\end{equation*}
$$

where $\mathcal{D}(\mathcal{H})$ and $\mathcal{D}(\mathcal{G})$ are defined as

$$
\mathcal{D}(\mathcal{H}):=\max _{1 \leq i, j \leq N}\left\|H_{i}-H_{j}\right\|_{\infty}, \quad \mathcal{D}(\mathcal{G}):=\max _{1 \leq i, j \leq N}\left\|G_{i}-G_{j}\right\|_{\infty}
$$

Proof. We postpone its proof in Appendix B,
In what follows, using the differential inequality (4.12), we provide a sufficient condition leading to the phase-locked state. First, in order to make a leading coefficient of (4.12) negative, we assume that a coupling strength $\kappa$ is sufficiently large:

$$
\kappa>\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{2(m-4 \sqrt{n})}
$$

For a handy notation, we denote

$$
\Lambda:=2(m-4 \sqrt{n})-\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{\kappa} .
$$

Next, we show that we can make the total aggregation functional $\mathcal{L}$ small as we wish by controlling the coupling strength $\kappa$.
Proposition 4.1. Suppose system parameters and initial data satisfy

$$
\begin{equation*}
\text { (i) } n \geq m, \quad \mathcal{D}(\mathcal{H}) \geq \mathcal{D}(\mathcal{G}), \quad \text { (ii) } \kappa \geq \kappa_{\mathrm{c}}, \quad \mathcal{L}^{0} \ll \nu_{2} \tag{4.13}
\end{equation*}
$$

where $\kappa_{c}$ and $\nu_{2}$ are specified later in (4.18) and (4.19), respectively, and let $\left\{\left(U_{j}, V_{j}\right)\right\}$ be a solution to (4.1). Then, we have

$$
\lim _{\kappa \rightarrow \infty} \limsup _{t \rightarrow \infty} \mathcal{L}(t)=0
$$

Proof. Since we assumed (4.13)(i), it follows from (4.6) that

$$
\begin{equation*}
\dot{\mathcal{L}} \leq-2 \kappa(m-4 \sqrt{n}) \mathcal{L}+\kappa(4 n+9) \mathcal{L}^{2}+\kappa\left(2 n+\frac{8}{3}\right) \mathcal{L}^{3}+2(1+3 \sqrt{n}) \mathcal{D}(\mathcal{H}) . \tag{4.14}
\end{equation*}
$$

Now, we introduce an auxiliary cubic polynomial:

$$
\begin{equation*}
g(s):=2(m-4 \sqrt{n}) s-(4 n+9) s^{2}-\left(2 n+\frac{8}{3}\right) s^{3} . \tag{4.15}
\end{equation*}
$$

Then, (4.14) can be rewritten as

$$
\dot{\mathcal{L}} \leq \kappa\left(\frac{2(1+3 \sqrt{n}) \mathcal{D}(\mathcal{H})}{\kappa}-g(\mathcal{L})\right) .
$$

By investigating roots of the polynomial $g$ in (4.15), we deduce that for a sufficient large $\kappa$, the polynomial $g$ admits one negative root, say, $\nu_{0}<0$, and two positive roots, say, $0<\nu_{1}<\nu_{2}$ with continuous dependence of $\kappa$, i.e.,

$$
\begin{equation*}
\lim _{\kappa \rightarrow \infty} \nu_{1}(\kappa)=0, \quad \lim _{\kappa \rightarrow \infty} \nu_{2}(\kappa)=\alpha_{*}, \quad \alpha_{*}: \text { a unique positive root of } g . \tag{4.16}
\end{equation*}
$$

Since we assume (4.13) (ii), there exists a finite entrance time $T_{*}>0$ such that

$$
\begin{equation*}
\mathcal{L}(t)<\nu_{1}, \quad t>T_{*} . \tag{4.17}
\end{equation*}
$$

Finally, we combine (4.16) and (4.17) to obtain the desired estimate:

$$
\lim _{\kappa \rightarrow \infty} \limsup _{t \rightarrow \infty} \mathcal{L}(t)=0
$$

Remark 4.2. For an explicit value for $\kappa_{\mathrm{c}}$ in (4.13)(ii), by simple calculus, we know that the polynomial $g=g(s)$ in (4.15) admits a global maximum in $\mathbb{R}_{+}$at $s=s_{*}$ :

$$
s_{*}:=\frac{-(4 n+9)+\sqrt{(4 n+9)^{2}+2(6 n+8)(m-4 \sqrt{n})}}{6 n+8} .
$$

Thus, in order to guarantee the existence of a positive root for $g$, one should impose

$$
\begin{equation*}
g\left(s_{*}\right)>\frac{2(1+3 \sqrt{n}) \mathcal{D}(\mathcal{H})}{\kappa}, \quad \text { i.e., } \quad \kappa>\frac{2(1+3 \sqrt{n}) \mathcal{D}(\mathcal{H})}{g\left(s_{*}\right)}=: \kappa_{\mathrm{c}} \text {. } \tag{4.18}
\end{equation*}
$$

For this $\kappa_{\mathrm{c}}, \nu_{2}$ in (4.13) (ii) can be explicitly determined as the largest positive root of

$$
\begin{equation*}
g(s)=\frac{2(1+3 \sqrt{n}) \mathcal{D}(\mathcal{H})}{\kappa} \quad \text { when } \kappa>\kappa_{\mathrm{c}} . \tag{4.19}
\end{equation*}
$$

It follows from Proposition 4.1 that under the assumption on the smallness of the initial data $\mathcal{L}^{0}$ and largeness of the coupling strength $\kappa$, we have

$$
\lim _{\kappa \rightarrow \infty} \limsup _{t \rightarrow \infty} \mathcal{L}(t)=0
$$

Hence, there exists a (large) coupling strength $\kappa_{\mathrm{p}}$ larger than $\kappa_{\mathrm{c}}$ such that for $t \gg 1$,

$$
(4 n+22) \mathcal{L}(t)+20 \mathcal{L}(t)^{2}<\frac{\Lambda}{2}, \quad \Lambda=2(m-4 \sqrt{n})-\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{\kappa}
$$

Then, (4.12) becomes

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F} \leq-\Lambda \kappa \mathcal{F}+\kappa(4 n+22) \mathcal{L \mathcal { F }}+20 \kappa \mathcal{L}^{2} \mathcal{F} \leq-\Lambda \kappa \mathcal{F}+\frac{\Lambda \kappa}{2} \mathcal{F}=-\frac{\Lambda \kappa}{2} \mathcal{F}
$$

This yields a desired exponential decay for $\mathcal{F}$. The argument above can be stated and shown as follows.

Theorem 4.2. Suppose system parameters and initial data satisfy
(i) $n \geq m, \quad \mathcal{D}(\mathcal{H}) \geq \mathcal{D}(\mathcal{G})$,
(ii) $\kappa>\kappa_{\mathrm{p}}>\kappa_{\mathrm{c}}, \quad \max \left\{\mathcal{L}^{0}, \tilde{\mathcal{L}}^{0}\right\} \leq \nu_{2}$,
and let $\left\{\left(U_{i}, V_{i}\right)\right\}$ and $\left\{\left(\tilde{U}_{i}, \tilde{V}_{i}\right)\right\}$ be any two solutions to (4.1), respectively. Then, the following assertions hold.
(1) The functional $\mathcal{F}$ converges to zero with an exponential rate.
(2) The normalized velocities $\mathrm{i} \dot{U}_{j} U_{j}^{\dagger}$ and $\mathrm{i} \dot{V}_{j} V_{j}^{\dagger}$ synchronize:

$$
\lim _{t \rightarrow \infty}\left\|i \dot{U}_{j} U_{j}^{\dagger}-\mathrm{i} \dot{\tilde{U}}_{j} \tilde{U}_{j}^{\dagger}\right\|_{\mathrm{F}}=0, \quad \lim _{t \rightarrow \infty}\left\|i \dot{V}_{j} V_{j}^{\dagger}-\mathrm{i} \dot{\tilde{V}_{j}} \tilde{V}_{j}^{\dagger}\right\|_{\mathrm{F}}=0
$$

(3) There exist unitary matrices $X_{\infty} \in \mathbf{U}(n)$ and $Y_{\infty} \in \mathbf{U}(m)$ such that

$$
\begin{array}{ll}
\lim _{t \rightarrow \infty} U_{i}^{\dagger}(t) \tilde{U}_{i}(t)=X_{\infty}, & \lim _{t \rightarrow \infty}\left\|\tilde{U}_{i}(t)-U_{i}(t) X_{\infty}\right\|_{\mathrm{F}}=0 \\
\lim _{t \rightarrow \infty} V_{i}^{\dagger}(t) \tilde{V}_{i}(t)=Y_{\infty}, & \lim _{t \rightarrow \infty}\left\|\tilde{V}_{i}(t)-V_{i}(t) Y_{\infty}\right\|_{\mathrm{F}}=0
\end{array}
$$

(4) Asymptotic phase-locking emerge: for any indices $i$ and $j$,

$$
\exists \lim _{t \rightarrow \infty} U_{i}(t) U_{j}^{\dagger}(t) \quad \text { and } \quad \exists \lim _{t \rightarrow \infty} U_{i}(t) U_{j}^{\dagger}(t)
$$

Moreover, there exist phase-locked state $\mathcal{X}^{\infty}:=\left\{X_{i}^{\infty}\right\}_{i=1}^{N}$ and $\mathcal{Y}^{\infty}:=\left\{Y_{i}^{\infty}\right\}_{i=1}^{N}$, and unitary matrices $P \in \mathbf{U}(n)$ and $Q \in \mathbf{U}(m)$ such that

$$
\lim _{t \rightarrow \infty}\left\|U_{i}(t)-X_{i}^{\infty} P\right\|_{\mathrm{F}}=0=\lim _{t \rightarrow \infty}\left\|V_{i}(t)-Y_{i}^{\infty} Q\right\|_{\mathrm{F}}=0, \quad \max \left\{\mathcal{D}\left(\mathcal{X}^{\infty}\right), \mathcal{D}\left(\mathcal{Y}^{\infty}\right)\right\}<\nu_{2}
$$

Proof. (i) It follows from (4.12) in Proposition 4.1 that $\mathcal{F}$ satisfies

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F} \leq-\kappa\left(2 m-8 \sqrt{n}-\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{\kappa}\right) \mathcal{F}+\kappa(4 n+22) \mathcal{L} \mathcal{F}+20 \kappa \mathcal{L}^{2} \mathcal{F} \tag{4.20}
\end{equation*}
$$

Since $\mathcal{L}$ can be sufficiently small under (4.13), i.e.,

$$
\mathcal{L}(t)<\nu_{1}, \quad t>T_{*},
$$

we choose $\kappa_{\mathrm{p}}>\kappa_{\mathrm{c}}$ sufficiently large such that

$$
(4 n+22) \mathcal{L}+20 \mathcal{L}^{2}<\frac{\Lambda}{2}, \quad \Lambda=2(m-4 \sqrt{n})-\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{\kappa}
$$

Thus, (4.20) becomes

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{F} \leq-\frac{\Lambda \kappa}{2} \mathcal{F}, \quad t>T_{*} \tag{4.21}
\end{equation*}
$$

and the relation (4.21) yields the desired exponential convergence of $\mathcal{F}$ toward zero.
(ii) For the second assertion, we claim:

$$
\begin{equation*}
\left\|i \dot{U}_{j} U_{j}^{\dagger}-\mathrm{i} \dot{\tilde{U}}_{j} \tilde{U}_{j}^{\dagger}\right\|_{\mathrm{F}} \leq 2 \kappa(m+n) \mathcal{F}, \quad\left\|\mathrm{i} \dot{V}_{j} V_{j}^{\dagger}-\mathrm{i} \dot{\tilde{V}}_{j} \tilde{V}_{j}^{\dagger}\right\|_{\mathrm{F}} \leq 2 \kappa(m+n) \mathcal{F} \tag{4.22}
\end{equation*}
$$

Once the relations (4.22) hold, it follows from the first assertion to derive the second assertion. Note that

$$
\begin{align*}
\left\|\mathrm{i} \dot{U}_{j} U_{j}^{\dagger}-\mathrm{i} \dot{\tilde{U}}_{j} \tilde{U}_{j}^{\dagger}\right\|_{\mathrm{F}}=\| \frac{\kappa}{N} \sum_{k=1}^{N} & \left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger}-\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{j} U_{k}^{\dagger}\right)  \tag{4.23}\\
& -\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle\tilde{V}_{j}, \tilde{V}_{k}\right\rangle_{\mathrm{F}} \tilde{U}_{k} \tilde{U}_{j}^{\dagger}-\left\langle\tilde{V}_{k}, \tilde{V}_{j}\right\rangle_{\mathrm{F}} \tilde{U}_{j} \tilde{U}_{k}^{\dagger}\right) \|_{\mathrm{F}}
\end{align*}
$$

By algebraic manipulations, one has

$$
\begin{aligned}
& \left\|\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger}-\left\langle\tilde{V}_{j}, \tilde{V}_{k}\right\rangle_{\mathrm{F}} \tilde{U}_{k} \tilde{U}_{j}^{\dagger}\right\|_{\mathrm{F}} \\
& \quad \leq\left\|\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}}\left(U_{k} U_{j}^{\dagger}-\tilde{U}_{k} \tilde{U}_{j}^{\dagger}\right)\right\|_{\mathrm{F}}+\left\|\left(\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}}-\left\langle\tilde{V}_{j}, \tilde{V}_{k}\right\rangle_{\mathrm{F}}\right) \tilde{U}_{k} \tilde{U}_{j}^{\dagger}\right\|_{\mathrm{F}} \leq(m+n) \mathcal{F} .
\end{aligned}
$$

Thus, the relation (4.23) yields

$$
\left\|i \dot{U}_{j} U_{j}^{\dagger}-\mathrm{i} \dot{\tilde{U}}_{j} \tilde{U}_{j}^{\dagger}\right\|_{\mathrm{F}} \leq 2 \kappa(m+n) \mathcal{F}
$$

This shows the desired synchronization of the normalized velocities and the exactly same argument is applied to $\mathrm{i} \dot{V}_{j} V_{j}^{\dagger}$.
(iii) Since system (4.1) is autonomous, we directly use Theorem 2(3) of 16. Hence, we briefly sketch a proof. We observe

$$
\left\|\frac{\mathrm{d}}{\mathrm{~d} s}\left(U_{i}^{\dagger} \tilde{U}_{i}\right)\right\|_{\mathrm{F}}=\left\|\dot{U}_{i} U_{i}^{\dagger}-\dot{\tilde{U}}_{i} \tilde{U}_{i}^{\dagger}\right\|_{\mathrm{F}} \leq 2 \kappa(m+n) \mathcal{F}
$$

Since $\mathcal{F}$ tends to zero exponentially,

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left(U_{i}^{\dagger}(t) \tilde{U}_{i}(t)\right)=U_{i}^{0, \dagger} \tilde{U}_{i}^{0}+\int_{0}^{\infty} \frac{\mathrm{d}}{\mathrm{~d} s}\left(U_{i}^{\dagger}(s) \tilde{U}_{i}(s)\right) \mathrm{d} s \tag{4.24}
\end{equation*}
$$

In addition, since $d(U, \tilde{U})$ converges to zero, one deduces that the right-hand side of (4.24) does not depend on the index $i$. This establishes the desired assertion.
(iv) We first show the existence of the asymptotic limit of $U_{i} U_{j}^{\dagger}$ and $V_{i} V_{j}^{\dagger}$. For any $T>0$, since system (4.1) is autonomous, we choose $\tilde{U}_{i}$ and $\tilde{V}_{i}$ as

$$
\tilde{U}_{i}(t)=U_{i}(t+T), \quad V_{i}(t)=V_{i}(t+T) .
$$

If we discretize the time $t \in \mathbb{R}_{+}$and $n \in \mathbb{Z}_{+}$and choose $T=m \in \mathbb{Z}_{+}$, we use the convergence of $d(U, \tilde{U})$ and $d(V, \tilde{V})$ to conclude that $\left\{U_{i}(n) U_{j}^{\dagger}(n)\right\}_{n \in \mathbb{Z}_{+}}$and $\left\{V_{i}(n) V_{j}^{\dagger}(n)\right\}_{n \in \mathbb{Z}_{+}}$become Cauchy sequences in the complete spaces $\mathbf{U}(n)$ and $\mathbf{U}(m)$, respectively. Thus, the limits of $U_{i} U_{j}^{\dagger}$ and $V_{i} V_{j}^{\dagger}$ exist. In particular, if we denote

$$
X_{i}^{\infty}:=\lim _{t \rightarrow \infty} U_{i}(t) U_{1}^{\dagger}(t)
$$

then one has

$$
\lim _{t \rightarrow \infty} U_{i}(t) U_{j}^{\dagger}(t)=\lim _{t \rightarrow \infty} U_{i} U_{1}^{\dagger}(t)\left(U_{j}(t) U_{1}^{\dagger}(t)\right)^{\dagger}=X_{i}^{\infty} X_{j}^{\infty, \dagger}, \quad \lim _{t \rightarrow \infty}\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}}=\left\langle X_{i}^{\infty}, X_{j}^{\infty}\right\rangle_{\mathrm{F}}
$$

On the other hand, we recall (A.2)

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left(U_{i} U_{j}^{\dagger}\right)=\mathrm{i}\left(U_{i} U_{j}^{\dagger} H_{j}-H_{i} U_{i} U_{j}^{\dagger}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle V_{i}, V_{k}\right\rangle_{\mathrm{F}} U_{k} U_{j}^{\dagger}-\left\langle V_{k}, V_{i}\right\rangle_{\mathrm{F}} U_{i} U_{k}^{\dagger} U_{i} U_{j}^{\dagger}+\left\langle V_{k}, V_{j}\right\rangle_{\mathrm{F}} U_{i} U_{k}^{\dagger}-\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}} U_{i} U_{j}^{\dagger} U_{k} U_{j}^{\dagger}\right) \tag{4.25}
\end{align*}
$$

In (4.25), if we let $t \rightarrow \infty$ and apply Barbalat's lemma, the left-hand side of (4.25) vanishes and consequently, the relation (4.25) becomes

$$
\begin{align*}
O= & \mathrm{i}\left(X_{i}^{\infty} X_{j}^{\infty, \dagger} H_{j}-H_{i} X_{i}^{\infty} X_{j}^{\infty, \dagger}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle Y_{i}^{\infty}, Y_{k}^{\infty}\right\rangle_{\mathrm{F}} X_{k}^{\infty} X_{j}^{\infty, \dagger}-\left\langle Y_{k}^{\infty}, Y_{i}^{\infty, \dagger}\right\rangle_{\mathrm{F}} X_{i}^{\infty} X_{k}^{\infty, \dagger} X_{i}^{\infty} X_{j}^{\infty, \dagger}\right.  \tag{4.26}\\
& \left.+\left\langle Y_{k}^{\infty}, Y_{j}^{\infty}\right\rangle_{\mathrm{F}} X_{i}^{\infty} X_{k}^{\infty, \dagger}-\left\langle Y_{j}^{\infty}, Y_{k}^{\infty, \dagger}\right\rangle_{\mathrm{F}} X_{i}^{\infty} X_{j}^{\infty, \dagger} X_{k}^{\infty} X_{j}^{\infty, \dagger}\right) .
\end{align*}
$$

By performing left-multiplication $X_{i}^{\infty, \dagger}$ and right-multiplication $X_{j}^{\infty}$ with (4.26), we obtain

$$
\begin{align*}
& -\mathrm{i} X_{j}^{\infty, \dagger} H_{j} X_{j}^{\infty}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle Y_{j}^{\infty}, Y_{k}^{\infty}\right\rangle_{\mathrm{F}} X_{j}^{\infty, \dagger} X_{k}^{\infty}-\left\langle Y_{k}^{\infty}, Y_{j}^{\infty}\right\rangle_{\mathrm{F}} X_{k}^{\infty, \dagger} X_{j}^{\infty}\right)  \tag{4.27}\\
& \quad=-\mathrm{i} X_{i}^{\infty, \dagger} H_{i} X_{i}^{\infty}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle Y_{i}^{\infty}, Y_{k}^{\infty}\right\rangle_{\mathrm{F}} X_{i}^{\infty, \dagger} X_{k}^{\infty}-\left\langle Y_{k}^{\infty}, Y_{i}^{\infty}\right\rangle_{\mathrm{F}} X_{k}^{\infty, \dagger} X_{i}^{\infty}\right)
\end{align*}
$$

Since the relation (4.27) does not depend on the index, we can set

$$
-\mathrm{i} \Theta:=-\mathrm{i} X_{j}^{\infty, \dagger} H_{j} X_{j}^{\infty}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle Y_{j}^{\infty}, Y_{k}^{\infty}\right\rangle_{\mathrm{F}} X_{j}^{\infty, \dagger} X_{k}^{\infty}-\left\langle Y_{k}^{\infty}, Y_{j}^{\infty}\right\rangle_{\mathrm{F}} X_{k}^{\infty, \dagger} X_{j}^{\infty}\right)
$$

This yields

$$
X_{j}^{\infty} \Theta X_{j}^{\infty, \dagger}=H_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle Y_{k}^{\infty}, Y_{j}^{\infty}\right\rangle_{\mathrm{F}} X_{k}^{\infty, \dagger} X_{j}^{\infty}-\left\langle Y_{j}^{\infty}, Y_{k}^{\infty}\right\rangle_{\mathrm{F}} X_{j}^{\infty, \dagger} X_{k}^{\infty}\right)
$$

Similarly for $\left\{V_{i}\right\}$, we can define a matrix $\Gamma$ independent of the index such that

$$
Y_{j}^{\infty} \Gamma Y_{j}^{\infty, \dagger}=G_{j}++\frac{\kappa}{N} \sum_{k=1}^{N}\left(\left\langle X_{k}^{\infty}, X_{j}^{\infty}\right\rangle_{\mathrm{F}} Y_{k}^{\infty, \dagger} Y_{j}^{\infty}-\left\langle X_{j}^{\infty}, X_{k}^{\infty}\right\rangle_{\mathrm{F}} Y_{j}^{\infty, \dagger} Y_{k}^{\infty}\right)
$$

Then, $\left\{\left\{X_{i}^{\infty}\right\}, \Theta\right\}$ and $\left\{\left\{Y_{i}^{\infty}\right\}, \Gamma\right\}$ indeed consist of the phase-locked state. Moreover, one has

$$
\mathcal{D}\left(\mathcal{X}^{\infty}\right)=\max _{1 \leq i, j \leq N}\left\|X_{i}^{\infty} X_{j}^{\infty, \dagger}-I_{n}\right\|=\lim _{t \rightarrow \infty} \max _{1 \leq i, j \leq N}\left\|U_{i}(t) U_{j}^{\dagger}(t)-I_{n}\right\|_{\mathrm{F}}=\lim _{t \rightarrow \infty} \mathcal{D}(\mathcal{U})<\nu_{2}
$$

Exactly the same estimate holds for $\mathcal{D}\left(\mathcal{Y}^{\infty}\right)$ as well.
Remark 4.3. In [20], the phase-locked state of (4.1) are defined to be of the following form:

$$
U_{i}(t)=U_{i}^{\infty} e^{-\mathrm{i} \Gamma_{U} t}, \quad V_{i}(t)=V_{i}^{\infty} e^{-\mathrm{i} \Gamma_{V} t}
$$

where $U_{i}^{\infty} \in \mathbf{U}(n)$ and $V_{i}^{\infty} \in \mathbf{U}(m)$ are unitary matrices, and $\Lambda_{U}$ and $\Lambda_{V}$ satisfy

$$
\begin{aligned}
U_{i}^{\infty} \Gamma_{U} U_{i}^{\infty, \dagger} & =H_{i}+\frac{\mathrm{i} \kappa}{N} \sum_{k=1}^{N}\left(\left\langle V_{k}^{\infty}, V_{j}^{\infty}\right\rangle_{\mathrm{F}} U_{k}^{\infty} U_{i}^{\infty, \dagger}-\left\langle V_{j}^{\infty}, V_{k}^{\infty}\right\rangle_{\mathrm{F}} U_{i}^{\infty} U_{k}^{\infty, \dagger}\right), \\
V_{i}^{\infty} \Gamma_{V} V_{i}^{\infty, \dagger} & =G_{i}+\frac{\mathrm{i} \kappa}{N} \sum_{k=1}^{N}\left(\left\langle U_{k}^{\infty}, U_{j}^{\infty}\right\rangle_{\mathrm{F}} V_{k}^{\infty} V_{i}^{\infty, \dagger}-\left\langle U_{j}^{\infty}, U_{k}^{\infty}\right\rangle_{\mathrm{F}} V_{i}^{\infty} V_{k}^{\infty, \dagger}\right) .
\end{aligned}
$$

## 5. Rank-2m Quadratically separable state

In this section, we study a quadratically separable state of the LT model for rank- $2 m$ tensors by introducing the MM model whose solution configuration is given as follows.

$$
\left\{\mathcal{U}^{1}, \mathcal{U}^{2}, \cdots, \mathcal{U}^{m}\right\}, \quad \mathcal{U}^{p}:=\left(U_{1}^{p}, \cdots, U_{N}^{p}\right), \quad p=1, \cdots, m
$$

In the following two subsections, we introduce extended models for the DM model (3.5) and the DUM model (3.15). Since the procedures are similar as those in Section 3, we omit details.
5.1. The MM model. In this subsection, we introduce the MM model:

$$
\left\{\begin{align*}
\dot{U}_{j}^{p}= & B_{j}^{p} U_{j}^{p}+\frac{\kappa_{1}}{N} \sum_{k=1}^{N}\left(\prod_{\substack{\ell=1 \\
\ell \neq p}}^{m}\left\langle U_{j}^{\ell}, U_{k}^{\ell}\right\rangle_{\mathrm{F}} U_{k}^{p}\left(U_{j}^{p}\right)^{\dagger} U_{j}^{p}-\prod_{\substack{\ell=1 \\
\ell \neq p}}^{m}\left\langle U_{k}^{l}, U_{j}^{l}\right\rangle_{\mathrm{F}} U_{j}^{p}\left(U_{k}^{p}\right)^{\dagger} U_{j}^{p}\right)  \tag{5.1}\\
& +\frac{\kappa_{2}}{N} \sum_{k=1}^{N}\left(\prod_{\substack{\ell=1 \\
\ell \neq p}}^{m}\left\langle U_{j}^{\ell}, U_{k}^{\ell}\right\rangle_{\mathrm{F}} U_{j}^{p}\left(U_{j}^{p}\right)^{\dagger} U_{k}^{p}-\prod_{\substack{\ell=1 \\
\ell \neq p}}^{m}\left\langle U_{k}^{\ell}, U_{j}^{\ell}\right\rangle_{\mathrm{F}} U_{j}^{p}\left(U_{k}^{p}\right)^{\dagger} U_{j}^{p}\right),
\end{align*}\right.
$$

where $B_{j}^{p}$ is a rank- $4 m$ tensor satisfying the skew-symmetric property.
For a solution $\left\{\mathcal{U}^{p}\right\}_{p=1}^{m}$ to (5.1), we set a rank- $2 m$ tensor denoted by $T_{i}$ :

$$
\begin{equation*}
T_{i}(t)=U_{i}^{1}(t) \otimes U_{i}^{2}(t) \otimes \cdots \otimes U_{i}^{m}(t), \quad U_{i}^{p} \in \mathbb{C}^{d_{1}^{p} \times d_{2}^{p}}, \quad i=1, \cdots, N, \quad p=1, \cdots, m \tag{5.2}
\end{equation*}
$$

which can be written in a component form:

$$
\left[T_{i}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2} \cdots \alpha_{m} \beta_{m}}=\left[U_{i}^{1}\right]_{\alpha_{1} \beta_{1}}\left[U_{i}^{2}\right]_{\alpha_{2} \beta_{2}} \cdots\left[U_{i}^{m}\right]_{\alpha_{m} \beta_{m}} .
$$

In order to relate the model (5.1) with the LT model (1.1), we consider the index vector $i_{*}$ in $\kappa_{i_{*}}$. Then, we introduce two subsets of $\{0,1\}^{2 d}$ : for $q=1, \cdots, m$,

$$
\begin{aligned}
& \Lambda_{1}=\left\{i_{*} \in\{0,1\}^{2 d}: 0 \text { appears once at } 2 q-1 \text {-th coordinate }\right\}, \\
& \Lambda_{2}=\left\{i_{*} \in\{0,1\}^{2 d}: 0 \text { appears once at } 2 q \text {-th coordinate }\right\} .
\end{aligned}
$$

Note that $\left|\Lambda_{1}\right|=\left|\Lambda_{2}\right|=m$ and for instance with $m=2$,

$$
(0,1,1,1),(1,1,0,1) \in \Lambda_{1}, \quad(1,0,1,1),(1,1,1,0) \in \Lambda_{2} .
$$

In this regard, we choose the index vector as

$$
\kappa_{i_{*}}= \begin{cases}\kappa_{1} & \text { if } i_{*} \in \Lambda_{1}  \tag{5.3}\\ \kappa_{2} & \text { if } i_{*} \in \Lambda_{2} \\ 0 & \text { otherwise }\end{cases}
$$

which generalizes (3.3) for the DM model.
Next, for the natural frequency tensors, we use $\left\{B_{j}^{p}\right\}_{p=1}^{m}$ to associate a rank- $4 m$ tensor $A_{j}$ as

$$
\begin{equation*}
\left[A_{j}\right]_{\alpha_{1} \alpha_{2} \cdots \alpha_{2 m} \beta_{1} \beta_{2} \cdots \beta_{2 m}}=\sum_{k=1}^{m}\left(\left[B_{j}^{k}\right]_{\alpha_{2 k-1} \alpha_{2 k} \beta_{2 k-1} \beta_{2 k}} \prod_{\substack{\ell=1 \\ \ell \neq k}}^{d}\left(\delta_{\alpha_{2 \ell-1} \beta_{2 \ell-1}} \delta_{\alpha_{2 \ell} \beta_{2 \ell}}\right)\right) \tag{5.4}
\end{equation*}
$$

which corresponds to (3.9) for the DM model. Then, it follows from straightforward calculation that the Lohe tensor model with (5.3) and (5.4) reduces to (5.1) whose solutions can be related by (5.2). The argument above is summarized in the following proposition analogous to Proposition 3.1.
Proposition 5.1. The following assertions hold.
(1) Suppose $\left\{\mathcal{U}^{p}\right\}_{p=1}^{m}$ is a solution to (5.1). Then, a rank-2m tensor defined by $T_{i}:=$ $U_{i}^{1} \otimes U_{i}^{2} \otimes \cdots \otimes U_{i}^{m}$ is the $Q S S$ to (1.1) with well-prepared initial data and free flow tensors $A_{i}$ satisfying (5.4).
(2) Suppose a rank-2m tensor $T_{i}$ is a solution to (1.1) with (5.4) and quadratically separable initial data:

$$
T_{i}^{0}:=U_{i}^{1,0} \otimes U_{i}^{2,0} \otimes \cdots \otimes U_{i}^{p, 0}, \quad 1 \leq i \leq N
$$

for rank-2 tensors $U_{i}^{p, 0} \times \mathbb{C}_{1}^{p} \times d_{2}^{p}$ with unit norms. Then, there exist matrices $\left\{U_{i}^{p}\right\}$ with unit norms such that

$$
T_{i}(t)=U_{i}^{1}(t) \otimes U_{i}^{2}(t) \otimes \cdots \otimes U_{i}^{p}(t), \quad t>0
$$

where $\left\{U_{i}^{p}\right\}$ is a solution to (5.1) with $U_{i}^{p, 0}=U_{i}^{p}(0)$.
5.2. The MUM model. In this subsection, we further reduce to the MM model to the model on the product of the unitary groups. Since the unitary group is concerned, we set

$$
\begin{equation*}
d_{1}^{p}=d_{2}^{p}=: d_{p}, \quad 1 \leq p \leq m \tag{5.5}
\end{equation*}
$$

For the modeling of natural frequencies, we also define Hermitian matrices with a size $d_{p} \times d_{p}$ :

$$
\left[-\mathrm{i} H_{j}^{p}\right]_{\alpha_{1} \alpha_{2}} \delta_{\beta_{1} \beta_{2}}:=\left[B_{j}^{p}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}}
$$

In addition, $\left\{H_{j}^{p}\right\}$ satisfy

$$
\begin{equation*}
\left[A_{j}\right]_{\alpha_{1} \alpha_{2} \cdots \alpha_{2 m} \beta_{1} \beta_{2} \cdots \beta_{2 m}}=\sum_{k=1}^{m}\left(\left[-\mathrm{i} H_{j}^{k}\right]_{\alpha_{2 k-1} \beta_{2 k-1}} \delta_{\alpha_{2 k} \beta_{2 k}} \prod_{\substack{\ell=1 \\ \ell \neq k}}^{m}\left(\delta_{\alpha_{2 \ell-1} \beta_{2 \ell-1}} \delta_{\alpha_{2 \ell} \beta_{2 \ell}}\right)\right) \tag{5.6}
\end{equation*}
$$

By Lemma 3.2, one can verify that system (5.1) with (5.5) conserves the unitarity of $U_{j}^{p}$. Thus, system (5.1) reduces to the following model on the unitary group:

$$
\left\{\begin{array}{l}
\dot{U}_{j}^{p}=-\mathrm{i} H_{j}^{p} U_{j}^{p}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\prod_{\substack{\ell=1 \\
\ell \neq p}}^{m}\left\langle U_{j}^{l}, U_{k}^{l}\right\rangle_{\mathrm{F}} U_{k}^{p}-\prod_{\substack{\ell=1 \\
\ell \neq p}}^{m}\left\langle U_{k}^{\ell}, U_{j}^{\ell}\right\rangle_{\mathrm{F}} U_{j}^{p}\left(U_{k}^{p}\right)^{\dagger} U_{j}^{p}\right)  \tag{5.7}\\
U_{j}^{p}(0)=U_{j}^{p, 0} \in \mathrm{U}\left(d_{p}\right)
\end{array}\right.
$$

where $\kappa=\kappa_{1}+\kappa_{2}$ and $H_{j}^{p} U_{j}^{p}$ is a usual matrix product. As in Proposition [5.1, existence and uniqueness of the QSS for (5.7) can be stated as follows.
Proposition 5.2. The following assertions hold.
(1) Suppose $\left\{\mathcal{U}^{p}\right\}_{p=1}^{m}$ is a solution to (5.7). Then, a rank-2m tensor defined by $T_{i}:=$ $U_{i}^{1} \otimes U_{i}^{2} \otimes \cdots \otimes U_{i}^{m}$ is a quadratically separable state to (1.1) with a well-prepared free flow tensor $A_{i}$ satisfying (5.6).
(2) Suppose a rank-2m tensor $T_{i}$ is a solution to (1.1) with (5.6) and quadratically separable initial data:

$$
T_{i}^{0}:=U_{i}^{1,0} \otimes U_{i}^{2,0} \otimes \cdots \otimes U_{i}^{p, 0}, \quad 1 \leq i \leq N
$$

for rank-2 tensors $U_{i}^{p, 0} \times \mathbb{C}_{1}^{n_{1}^{p} \times n_{2}^{p}}$ with unit norms. Then, there exist matrices $\left\{U_{i}^{p}\right\}$ with unit norms such that

$$
T_{i}(t)=U_{i}^{1}(t) \otimes U_{i}^{2}(t) \otimes \cdots \otimes U_{i}^{p}(t), \quad t>0,
$$

where $\left\{U_{i}^{p}\right\}$ is a solution to (5.7) with $U_{i}^{p, 0}=U_{i}^{p}(0)$.
Remark 5.1. In Section 4 , we provided the emergent dynamics of the double unitary matrix model (4.1). However for its generalized model (5.7), emergent dynamics will not be studied, since it can be straightforwardly obtained from the results in Section 4.

## 6. Conclusion

In this paper, we have studied the existence and emergent dynamics of quadratically separable states for the Lohe tensor model which incorporates several well-known lowrank aggregation models such as the Kuramoto model, the Lohe sphere model and the Lohe matrix model, etc. In our previous work [9, we obtained completely separable states as special solutions to the Lohe tensor model defined as tensor products of rank-1 real tensors (or vectors). In analogy with the previous work, we consider the states in which a solution can be decomposed as a tensor product of rank-2 tensors (or matrices), namely, a quadratically separable state. Precisely, if initial data are quadratically separable, then such separability is preserved along the Lohe tensor flow. Moreover, by introducing and analyzing double matrix and unitary matrix models, we are able to study asymptotic behavior of the quadratically separable states to the Lohe tensor model. Of course, there are several issues that are not discussed in this work. For example, one can naturally consider the state consisting of tensors with possibly different ranks and sizes. We explore this issue in a future work.

## Appendix A. Proof of Lemma 4.1

In this appendix, we provide a proof of Lemma 4.1 in which a differential inequality for the aggregation functional $\mathcal{L}=\mathcal{D}(\mathcal{U})+\mathcal{D}(\mathcal{V})+\mathcal{S}(\mathcal{U})+\mathcal{S}(\mathcal{V})$ is derived. We divide a proof into two steps:

- Step A: we derive differential inequalities for $\mathcal{D}(\mathcal{U})$ and $\mathcal{D}(\mathcal{V})$ (see Lemma A.1).
- Step B: we derive differential inequalities for $\mathcal{S}(\mathcal{U})$ and $\mathcal{S}(\mathcal{V})$ (see Lemma A.2).

Lemma A.1. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (4.4). Then, $\mathcal{D}(\mathcal{U})$ and $\mathcal{D}(\mathcal{V})$ satisfy

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{U}) \leq-2 m \kappa \mathcal{D}(\mathcal{U})+m \kappa \mathcal{D}(\mathcal{U})^{3}+6 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+4 \kappa \sqrt{n} S(\mathcal{V}),  \tag{A.1}\\
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{V}) \leq-2 n \kappa \mathcal{D}(\mathcal{V})+n \kappa \mathcal{D}(\mathcal{V})^{3}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}+4 \kappa \sqrt{m} S(\mathcal{U})
\end{align*}
$$

Proof. By straightforward calculations, one finds a differential equation for $G_{i j}=U_{i} U_{j}^{\dagger}$ :

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} G_{i j}=\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{i k} G_{k j}-c_{k i} G_{i k} G_{i j}+c_{k j} G_{i k}-c_{j k} G_{i j} G_{k j}\right) \tag{A.2}
\end{equation*}
$$

By using the relation $G_{i j}=I_{n}-S_{i j}$, we see that $S_{i j}$ satisfies

$$
\begin{gather*}
\frac{\mathrm{d}}{\mathrm{~d} t} S_{i j}=\frac{\kappa}{N} \sum_{k=1}^{N}\left[c_{i k} S_{k j}-c_{k i} S_{i k}-c_{k i} S_{i j}+c_{i} S_{k} S_{i j}+c_{k j} S_{i k}-c_{j k} S_{k j}-c_{j k} S_{i j}+c_{j k} S_{i j} S_{k j}\right. \\
\left.+\left(c_{k i}-c_{i k}+c_{j k}-c_{k j}\right) I_{n}\right] \tag{A.3}
\end{gather*}
$$

After algebraic manipulation, we rewrite (A.3) in terms of $S_{i j}$ and $c_{i j}-m$ that are expected to converge to zero:

$$
\begin{align*}
\frac{\mathrm{d} S_{i j}}{d t}= & -2 m \kappa S_{i j}+\frac{m \kappa}{N} \sum_{k=1}^{N}\left(S_{i k} S_{i j}+S_{i j} S_{k j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left[\left(c_{i k}-m\right) S_{k j}-\left(c_{k i}-m\right) S_{i k}-\left(c_{i k}-m\right) S_{i j}+\left(c_{k i}-m\right) S_{i k} S_{i j}\right] \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left[\left(c_{k j}-m\right) S_{i k}-\left(c_{j k}-m\right) S_{k j}-\left(c_{j k}-m\right) S_{i j}+\left(c_{j k}-m\right) S_{i j} S_{k j}\right]  \tag{A.4}\\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left[\left(c_{k i}-c_{i k}+c_{j k}-c_{k j}\right) I_{n}\right] .
\end{align*}
$$

On the other hand for an $n \times n$ matrix $A$, one has

$$
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\|A\|_{\mathrm{F}}^{2}=\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \operatorname{tr}\left(A A^{\dagger}\right)=\frac{1}{2} \operatorname{tr}\left(\dot{A} A^{\dagger}+A \dot{A}^{\dagger}\right)=\operatorname{Retr}\left(\dot{A} A^{\dagger}\right)
$$

We multiply (A.4) with $S_{i j}^{\dagger}$ to find

$$
\begin{align*}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|S_{i j}\right\|_{\mathrm{F}}^{2}=-2 m \kappa\left\|S_{i j}\right\|_{\mathrm{F}}^{2}+\frac{m \kappa}{N} \sum_{k=1}^{N} \operatorname{Retr}\left(S_{i k} S_{i j} S_{i j}^{\dagger}+S_{i j} S_{k j} S_{i j}^{\dagger}\right) \\
& \quad+\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{i k}-m\right) \operatorname{tr}\left(S_{k j} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left[\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left(c_{i k}-m\right)\left\|S_{i j}\right\|_{\mathrm{F}}^{2} \\
& \quad+\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{k j}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left[\left(c_{j k}-m\right) \operatorname{tr}\left(S_{k j} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left(c_{j k}-m\right)\left\|S_{i j}\right\|_{\mathrm{F}}^{2}  \tag{A.5}\\
& \quad+\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j} S_{i j}^{\dagger}\right)\right]+\operatorname{Re}\left[\left(c_{j k}-m\right) \operatorname{tr}\left(S_{i j} S_{k j} S_{i j}^{\dagger}\right)\right] \\
& \quad+\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{k i}-c_{i k}+c_{j k}-c_{k j}\right) \operatorname{tr}\left(S_{i j}^{\dagger}\right)\right] \\
& =:
\end{align*}-2 m \kappa\left\|S_{i j}\right\|_{\mathrm{F}}^{2}+\mathcal{I}_{11}+\mathcal{I}_{12}+\mathcal{I}_{13}+\mathcal{I}_{14}+\mathcal{I}_{15} .
$$

Below, we present estimates for $\mathcal{I}_{1 k}, k=1, \cdots, 5$, respectively.

- (Estimate of $\mathcal{I}_{11}$ ): We use (A.6) and

$$
\begin{equation*}
S_{i j}+S_{j i}=S_{i j} S_{j i}, \quad S_{i j}^{\dagger}=S_{j i} \tag{A.6}
\end{equation*}
$$

to derive

$$
\begin{aligned}
\operatorname{Retr}\left(S_{i k} S_{i j} S_{j i}\right) & =\frac{1}{2} \operatorname{tr}\left(S_{i k} S_{i j} S_{j i}+S_{i j} S_{j i} S_{k i}\right)=\frac{1}{2} \operatorname{tr}\left(S_{i j} S_{j i}\left(S_{k i}+S_{i k}\right)\right) \\
& =\frac{1}{2} \operatorname{tr}\left(S_{i j} S_{j i} S_{k i} S_{i k}\right)=\frac{1}{2}\left\|S_{k i} S_{i j}\right\|_{\mathrm{F}}^{2}
\end{aligned}
$$

Similarly, one has

$$
\operatorname{Retr}\left(S_{i j} S_{k j} S_{i j}^{\dagger}\right)=\frac{1}{2}\left\|S_{i j} S_{j k}\right\|_{\mathrm{F}}^{2}
$$

Hence, $\mathcal{I}_{11}$ satisfies

$$
\mathcal{I}_{11}=\frac{m \kappa}{N} \sum_{k=1}^{N} \operatorname{Retr}\left(S_{i k} S_{i j} S_{i j}^{\dagger}+S_{i j} S_{k j} S_{i j}^{\dagger}\right)=\frac{m \kappa}{2 N} \sum_{k=1}^{N}\left(\left\|S_{k i} S_{i j}\right\|_{\mathrm{F}}^{2}+\left\|S_{i j} S_{j k}\right\|_{\mathrm{F}}^{2}\right) \leq m \kappa \mathcal{D}(\mathcal{U})^{4}
$$

- (Estimates of $\mathcal{I}_{12}$ and $\left.\mathcal{I}_{13}\right)$ : By the maximality of $\mathcal{D}(\mathcal{U})$ and $\mathcal{S}(\mathcal{V})$, we have

$$
\begin{aligned}
\mathcal{I}_{12} & =\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{i k}-m\right) \operatorname{tr}\left(S_{k j} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left[\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left(c_{i k}-m\right)\left\|S_{i j}\right\|_{\mathrm{F}}^{2} \\
& \leq 3 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}, \\
\mathcal{I}_{13} & =\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{k j}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left[\left(c_{j k}-m\right) \operatorname{tr}\left(S_{k j} S_{i j}^{\dagger}\right)\right]-\operatorname{Re}\left(c_{j k}-m\right)\left\|S_{i j}\right\|_{\mathrm{F}}^{2} \\
& \leq 3 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2} .
\end{aligned}
$$

- (Estimate of $\mathcal{I}_{14}$ ): By straightforward calculation, one has

$$
\mathcal{I}_{14}=\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j} S_{i j}^{\dagger}\right)\right]+\operatorname{Re}\left[\left(c_{j k}-m\right) \operatorname{tr}\left(S_{i j} S_{k j} S_{i j}^{\dagger}\right)\right] \leq 2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{3}
$$

- (Estimate of $\left.\mathcal{I}_{15}\right):$ Note that for an $n \times n$ matrix $A$,

$$
\operatorname{tr}(A)=\operatorname{tr}\left(I_{n} A\right) \leq\left\|I_{n}\right\|_{\mathrm{F}}\|A\|_{\mathrm{F}}=\sqrt{n}\|A\|_{\mathrm{F}} .
$$

This yields

$$
\begin{equation*}
\mathcal{I}_{15}=\frac{\kappa}{N} \sum_{k=1}^{N} \operatorname{Re}\left[\left(c_{k i}-c_{i k}+c_{j k}-c_{k j}\right) \operatorname{tr}\left(S_{i j}^{\dagger}\right)\right] \leq 4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U}) \tag{A.7}
\end{equation*}
$$

In (A.5), we collect all the estimates for $\mathcal{I}_{1 k}, k=1, \cdots, 5$ to obtain
$\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\left\|S_{i j}\right\|_{\mathrm{F}}^{2} \leq-2 m \kappa\left\|S_{i j}\right\|_{\mathrm{F}}^{2}+m \kappa \mathcal{D}(\mathcal{U})^{4}+6 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{3}+4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})$.
Hence, $\mathcal{D}(\mathcal{U})$ satisfies

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{U}) \leq-2 m \kappa \mathcal{D}(\mathcal{U})+m \kappa \mathcal{D}(\mathcal{U})^{3}+6 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V})
$$

Similarly, one can find a differential inequality for $\mathcal{D}(\mathcal{V})$ by exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$ :

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{V}) \leq-2 n \kappa \mathcal{D}(\mathcal{V})+n \kappa \mathcal{D}(\mathcal{V})^{3}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}+4 \kappa \sqrt{m} \mathcal{S}(\mathcal{U})
$$

In (A.1), note that $\mathcal{S}(\mathcal{U})$ and $\mathcal{S}(\mathcal{V})$ appear in the differential inequalities for $\mathcal{D}(\mathcal{U})$ and $\mathcal{D}(\mathcal{V})$. Hence, we derive the differential inequalities for $\mathcal{S}(\mathcal{U})$ and $\mathcal{S}(\mathcal{V})$ below.

Lemma A.2. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (4.4). Then, $\mathcal{S}(\mathcal{U})$ and $\mathcal{S}(\mathcal{V})$ satisfy

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{U}) \leq-2 m \kappa \mathcal{S}(\mathcal{U})+2 m \kappa \mathcal{D}(\mathcal{U})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V}),  \tag{A.8}\\
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{V}) \leq-2 n \kappa \mathcal{S}(\mathcal{V})+2 n \kappa \mathcal{D}(\mathcal{V})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}+4 \kappa \sqrt{m} \mathcal{S}(\mathcal{U})
\end{align*}
$$

Proof. In (A.4), we take trace to obtain

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t}(n- & \left.d_{i j}\right)=-2 m \kappa\left(n-d_{i j}\right)+\frac{m \kappa}{N} \sum_{k=1}^{N} \operatorname{tr}\left(S_{i k} S_{i j}+S_{i j} S_{k j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{i k}-m\right)\left(n-d_{k j}\right)-\left(c_{k i}-m\right)\left(n-d_{i k}\right)-\left(c_{i k}-m\right)\left(n-d_{i j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k j}-m\right)\left(n-d_{i k}\right)-\left(c_{j k}-m\right)\left(n-d_{k j}\right)-\left(c_{j k}-m\right)\left(n-d_{i j}\right)  \tag{A.9}\\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}\right)+\left(c_{j k}-m\right) \operatorname{tr}\left(S_{i j} S_{k j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left[\left(c_{k i}-c_{i k}\right)+\left(c_{j k}-c_{k j}\right)\right] \sqrt{n} \\
& =:-2 m \kappa\left(n-d_{i j}\right)+\mathcal{I}_{21}+\mathcal{I}_{22}+\mathcal{I}_{23}+\mathcal{I}_{24}+\mathcal{I}_{25} .
\end{align*}
$$

Below, we present estimates of $\mathcal{I}_{2 k}, k=1, \cdots, 5$, separately.

- (Estimate of $\left.\mathcal{I}_{21}\right):$ We use the definition of $\mathcal{D}(\mathcal{U})$ to find

$$
\mathcal{I}_{21}=\frac{m \kappa}{N} \sum_{k=1}^{N} \operatorname{tr}\left(S_{i k} S_{i j}+S_{i j} S_{k j}\right) \leq 2 m \kappa \mathcal{D}(\mathcal{U})^{2}
$$

- (Estimates of $\mathcal{I}_{22}$ and $\mathcal{I}_{23}$ ): It is easy to see that

$$
\begin{aligned}
& \mathcal{I}_{22}=\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{i k}-m\right)\left(n-d_{k j}\right)-\left(c_{k i}-m\right)\left(n-d_{i k}\right)-\left(c_{i k}-m\right)\left(n-d_{i j}\right) \leq 3 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V}), \\
& \mathcal{I}_{23}=\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k j}-m\right)\left(n-d_{i k}\right)-\left(c_{j k}-m\right)\left(n-d_{k j}\right)-\left(c_{j k}-m\right)\left(n-d_{i j}\right) \leq 3 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V}) .
\end{aligned}
$$

- (Estimate of $\left.\mathcal{I}_{24}\right)$ : Similar to $\mathcal{I}_{21}$, one has

$$
\mathcal{I}_{24}=\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}\right)+\left(c_{j k}-m\right) \operatorname{tr}\left(S_{i j} S_{k j}\right) \leq 2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}
$$

- (Estimate of $\left.\mathcal{I}_{25}\right)$ : We find

$$
\begin{equation*}
\mathcal{I}_{25}=\frac{\kappa}{N} \sum_{k=1}^{N}\left[\left(c_{k i}-c_{i k}\right)+\left(c_{j k}-c_{k j}\right)\right] \sqrt{n} \leq 4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V}) \tag{A.10}
\end{equation*}
$$

In (A.9), we combine all estimates to obtain

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{U}) \leq-2 m \kappa \mathcal{S}(\mathcal{U})+2 m \kappa \mathcal{D}(\mathcal{U})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V})
$$

By exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$, we derive a differential inequality for $\mathcal{S}(\mathcal{V})$ :

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{V}) \leq-2 n \kappa \mathcal{S}(\mathcal{V})+2 n \kappa \mathcal{D}(\mathcal{V})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}+4 \kappa \sqrt{m} \mathcal{S}(\mathcal{U})
$$

Remark A.1. For homogeneous Hamiltonians, we add (|А.8) $)_{1}$ and (A.8) $)_{2}$ to find

$$
\frac{\mathrm{d}}{\mathrm{~d} t}(\mathcal{S}(\mathcal{U})+\mathcal{S}(\mathcal{V})) \leq-2 \kappa(m-2 \sqrt{m}) \mathcal{S}(\mathcal{U})-2 \kappa(n-2 \sqrt{n}) \mathcal{S}(\mathcal{V})+\mathcal{O}\left((\mathcal{S}(\mathcal{U})+\mathcal{S}(\mathcal{V}))^{2}\right)
$$

Hence, in order to obtain the desired convergence, we assume

$$
n>2 \sqrt{n}, \quad m>2 \sqrt{m}, \quad \text { i.e., } \quad n, m>4
$$

which requires the restriction on the size of $U_{i}$ and $V_{j}$. This technical assumption arises from the estimate of $\mathcal{I}_{25}$ in (A.10).

Now, we are ready to present a proof of Lemma 4.1 using Lemma A. 1 and Lemma A.2.
Proof of Lemma 4.1] Recall the inequalities in (A.1) and (A.8):

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{U}) \leq-2 m \kappa \mathcal{D}(\mathcal{U})+m \kappa \mathcal{D}(\mathcal{U})^{3}+6 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V}) \\
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{V}) \leq-2 n \kappa \mathcal{D}(\mathcal{V})+n \kappa \mathcal{D}(\mathcal{V})^{3}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}+4 \kappa \sqrt{m} \mathcal{S}(\mathcal{U})  \tag{A.11}\\
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{U}) \leq-2 m \kappa \mathcal{S}(\mathcal{U})+2 m \kappa \mathcal{D}(\mathcal{U})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}+4 \kappa \sqrt{n} \mathcal{S}(\mathcal{V}) \\
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{V}) \leq-2 n \kappa \mathcal{S}(\mathcal{V})+2 n \kappa \mathcal{D}(\mathcal{V})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}+4 \kappa \sqrt{m} \mathcal{S}(\mathcal{U})
\end{align*}
$$

Without loss of generality, we may assume $n \geq m$, and add all the inequalities in (A.11) to find

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{L} \leq & -2 \kappa(m-4 \sqrt{n}) \mathcal{L}+n \kappa\left(\mathcal{D}(\mathcal{U})^{3}+2 \mathcal{D}(\mathcal{U})^{2}+\mathcal{D}(\mathcal{V})^{3}+2 \mathcal{D}(\mathcal{V})^{2}\right) \\
& +6 \kappa(\mathcal{D}(\mathcal{U}) \mathcal{S}(\mathcal{V})+\mathcal{D}(\mathcal{V}) \mathcal{S}(\mathcal{U})+2 \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V}))+4 \kappa\left(\mathcal{D}(\mathcal{U})^{2} \mathcal{S}(\mathcal{V})+\mathcal{D}(\mathcal{V})^{2} \mathcal{S}(\mathcal{U})\right) \\
= & -2 \kappa(m-4 \sqrt{n}) \mathcal{L}+n \kappa \mathcal{I}_{31}+6 \kappa \mathcal{I}_{32}+4 \kappa \mathcal{I}_{33} . \tag{A.12}
\end{align*}
$$

In the sequel, we provide estimates for $\mathcal{I}_{3 k}, k=1,2,3$, respectively.

- (Estimate of $\mathcal{I}_{31}$ ): We use a rough estimate to find

$$
\mathcal{I}_{31}=\mathcal{D}(\mathcal{U})^{3}+2 \mathcal{D}(\mathcal{U})^{2}+\mathcal{D}(\mathcal{V})^{3}+2 \mathcal{D}(\mathcal{V})^{2} \leq \mathcal{L}^{3}+2 \mathcal{L}^{2} .
$$

- (Estimate of $\mathcal{I}_{32}$ ): By straightforward calculation, one has

$$
\begin{aligned}
\mathcal{I}_{32} & =\mathcal{D}(\mathcal{U}) \mathcal{S}(\mathcal{V})+\mathcal{D}(\mathcal{V}) \mathcal{S}(\mathcal{U})+2 \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V}) \\
& \leq \frac{1}{2} \mathcal{D}(\mathcal{U})^{2}+\frac{1}{2} \mathcal{D}(\mathcal{V})^{2}+\frac{3}{2} \mathcal{S}(\mathcal{U})^{2}+\frac{3}{2} \mathcal{S}(\mathcal{V})^{2} \leq \frac{3}{2} \mathcal{L}^{2} .
\end{aligned}
$$

- (Estimate of $\mathcal{I}_{33}$ ): We use Young's inequality that for $a, b>0$,

$$
a^{2} b \leq \frac{2}{3} a^{3}+\frac{1}{3} b^{3}
$$

to find

$$
\mathcal{I}_{33}=\mathcal{D}(\mathcal{U})^{2} \mathcal{S}(\mathcal{V})+\mathcal{D}(\mathcal{V})^{2} \mathcal{S}(\mathcal{U}) \leq \frac{2}{3} \mathcal{D}(\mathcal{U})^{3}+\frac{1}{3} \mathcal{S}(\mathcal{V})^{3}+\frac{2}{3} \mathcal{D}(\mathcal{V})^{3}+\frac{1}{3} \mathcal{S}(\mathcal{U})^{3} \leq \frac{2}{3} \mathcal{L}^{3} .
$$

In (A.12), we combine all the estimates to obtain the desired inequality for $\mathcal{L}$ :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{L} & \leq-2 \kappa(m-4 \sqrt{n}) \mathcal{L}+2 \kappa n\left(\mathcal{L}^{3}+2 \mathcal{L}^{2}\right)+9 \kappa \mathcal{L}^{2}+\frac{8 \kappa}{3} \mathcal{L}^{3} \\
& =-2 \kappa(m-4 \sqrt{n}) \mathcal{L}++\kappa(4 n+9) \mathcal{L}^{2}+\kappa\left(2 n+\frac{8}{3}\right) \mathcal{L}^{3} .
\end{aligned}
$$

## Appendix B. Proof of Lemma 4.2

In this appendix, we present a proof of Lemma 4.2 in which the differential inequality for the aggregation functional $\mathcal{F}$ in (4.11) will be derived.
Lemma B.1. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (4.1). Then, $d(U, \tilde{U})$ and $d(V, \tilde{V})$ satisfy

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} d(U, \tilde{U}) \leq & -2 m \kappa d(U, \tilde{U})+4 m \kappa \mathcal{L} d(U, \tilde{U})+6 \mathcal{L}(d(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) \\
& +2 \mathcal{L}^{2}(4 d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))+4 \sqrt{n} \mathcal{S}(V, \tilde{V})  \tag{B.1}\\
\frac{\mathrm{d}}{\mathrm{~d} t} d(V, \tilde{V}) \leq & -2 n \kappa d(V, \tilde{V})+4 n \kappa \mathcal{L} d(V, \tilde{V})+6 \mathcal{L}(d(V, \tilde{V})+\mathcal{S}(U, \tilde{U})) \\
& +2 \mathcal{L}^{2}(4 d(V, \tilde{V})+\mathcal{S}(U, \tilde{U}))+4 \sqrt{m} \mathcal{S}(U, \tilde{U}) .
\end{align*}
$$

Proof. First, we recall (A.4):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} S_{i j}= & -2 m \kappa S_{i j}+\frac{m \kappa}{N} \sum_{k=1}^{N} S_{i k} S_{i j}+S_{i j} S_{k j}+\mathrm{i}\left(H_{i}-H_{j}\right)+\mathrm{i}\left(S_{i j} H_{j}-H_{i} S_{i j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{i k}-m\right) S_{k j}-\left(c_{k i}-m\right) S_{i k}-\left(c_{i k}-m\right) S_{i j}+\left(c_{k i}-m\right) S_{i k} S_{i j} \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k j}-m\right) S_{i k}-\left(c_{j k}-m\right) S_{k j}-\left(c_{j k}-m\right) S_{i j}+\left(c_{j k}-m\right) S_{i j} S_{k j} \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k i}-c_{i k}+c_{j k}-c_{k j}\right) I_{n} .
\end{aligned}
$$

Thus, $S_{i j}-\tilde{S}_{i j}$ satisfies

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(S_{i j}\right. & \left.-\tilde{S}_{i j}\right)=-2 m \kappa\left(S_{i j}-\tilde{S}_{i j}\right)+\frac{m \kappa}{N} \sum_{k=1}^{N} \underbrace{\left(S_{i k} S_{i j}+S_{i j} S_{k j}-\tilde{S}_{i k} \tilde{S}_{i j}-\tilde{S}_{i j} \tilde{S}_{k j}\right)}_{=: \mathcal{I}_{41}} \\
& +\underbrace{\mathrm{i}\left(\left(S_{i j}-\tilde{S}_{i j}\right) H_{j}-H_{i}\left(S_{i j}-\tilde{S}_{i j}\right)\right)}_{=: \mathcal{I}_{42}}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\mathcal{I}_{43}-\tilde{\mathcal{I}}_{43}\right)+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\mathcal{I}_{44}-\tilde{\mathcal{I}}_{44}\right)  \tag{B.2}\\
& +\frac{\kappa}{N} \sum_{k=1}^{N} \underbrace{\left(\left(c_{k i}-c_{i k}+c_{j k}-c_{k j}\right)-\left(\tilde{c}_{k i}-\tilde{c}_{i k}+\tilde{c}_{j k}-\tilde{c}_{k j}\right)\right) I_{n}}_{=: \mathcal{I}_{45}},
\end{align*}
$$

where $\mathcal{I}_{43}$ and $\mathcal{I}_{44}$ are defined as

$$
\begin{aligned}
& \mathcal{I}_{43}:=\left(c_{i k}-m\right) S_{k j}-\left(c_{k i}-m\right) S_{i k}-\left(c_{i k}-m\right) S_{i j}+\left(c_{k i}-m\right) S_{i k} S_{i j}, \\
& \mathcal{I}_{44}:=\left(c_{k j}-m\right) S_{i k}-\left(c_{j k}-m\right) S_{k j}-\left(c_{j k}-m\right) S_{i j}+\left(c_{j k}-m\right) S_{i j} S_{k j} .
\end{aligned}
$$

Below, we provide the estimates for $\mathcal{I}_{4 k}, k=1, \cdots, 5$, respectively.

- (Estimate of $\mathcal{I}_{41}$ ): Note that

$$
\begin{aligned}
\mathcal{I}_{41} & =S_{i k} S_{i j}+S_{i j} S_{k j}-\tilde{S}_{i k} \tilde{S}_{i j}-\tilde{S}_{i j} \tilde{S}_{k j} \\
& =S_{i k}\left(S_{i j}-\tilde{S}_{i j}\right)+\left(S_{i k}-\tilde{S}_{i k}\right) \tilde{S}_{i j}+S_{i j}\left(S_{k j}-\tilde{S}_{k j}\right)+\left(S_{i j}-\tilde{S}_{i j}\right) \tilde{S}_{k j}
\end{aligned}
$$

This yields

$$
\begin{equation*}
\left\|\mathcal{I}_{41}\right\|_{\mathrm{F}} \leq 4 \mathcal{L} d(U, \tilde{U}) \tag{B.3}
\end{equation*}
$$

- (Estimate of $\left.\mathcal{I}_{42}\right)$ : Note that for a skew-Hermitian matrix $\Omega$ and a matrix $A$, one has

$$
\operatorname{tr}\left(\Omega A A^{\dagger}\right)=0
$$

This implies

$$
\operatorname{Retr}\left(\mathcal{I}_{42}\left(S_{i j}-\tilde{S}_{i j}\right)^{\dagger}\right)=\operatorname{Retr}\left(\mathrm{i} H_{j}\left(S_{i j}-\tilde{S}_{i j}\right)\left(S_{i j}-\tilde{S}_{i j}\right)^{\dagger}\right)=0
$$

- (Estimate of $\mathcal{I}_{43}$ ): We observe

$$
\begin{aligned}
& \left\|\left(c_{i k}-m\right) S_{k j}-\left(\tilde{c}_{i k}-m\right) \tilde{S}_{k j}\right\|_{\mathrm{F}} \\
& \quad=\left\|\left(c_{i k}-m\right)\left(S_{k j}-\tilde{S}_{k j}\right)+\left(c_{i k}-\tilde{c}_{i k}\right) \tilde{S}_{k j}\right\|_{F} \\
& \quad \leq \mathcal{L} d(U, \tilde{U})+\mathcal{L S}(V, \tilde{V})=\mathcal{L}(d(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) .
\end{aligned}
$$

Moreover, we use (B.3)

$$
\begin{aligned}
& \left\|\left(c_{k i}-m\right) S_{i k} S_{i j}-\left(\tilde{c}_{k i}-m\right) \tilde{S}_{i k} \tilde{S}_{i j}\right\|_{\mathrm{F}} \\
& \quad=\left\|\left(c_{k i}-m\right)\left(S_{i k} S_{i j}-\tilde{S}_{i k} \tilde{S}_{i j}\right)+\left(c_{i k}-\tilde{c}_{i k}\right) \tilde{S}_{i k} \tilde{S}_{i j}\right\|_{\mathrm{F}} \\
& \quad \leq \mathcal{L} \cdot 4 \mathcal{L} d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}) \mathcal{L}^{2}=\mathcal{L}^{2}(4 d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))
\end{aligned}
$$

to obtain

$$
\left\|\mathcal{I}_{43}-\tilde{\mathcal{I}}_{43}\right\|_{\mathrm{F}} \leq 3 \mathcal{L}(d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))+\mathcal{L}^{2}(4 d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))
$$

- (Estimate of $\mathcal{I}_{44}$ ): Similar to $\mathcal{I}_{43}$, one finds

$$
\left\|\mathcal{I}_{44}-\tilde{\mathcal{I}}_{44}\right\|_{\mathrm{F}} \leq 3 \mathcal{L}(d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))+\mathcal{L}^{2}(4 d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))
$$

- (Estimate of $\mathcal{I}_{45}$ ): We directly find

$$
\left\|\mathcal{I}_{45}\right\|_{\mathrm{F}} \leq 4 \sqrt{n} \mathcal{S}(V, \tilde{V})
$$

In (B.2), we combine all the estimates to obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} d(U, \tilde{U}) \leq & -2 m \kappa d(U, \tilde{U})+4 m \kappa \mathcal{L} d(U, \tilde{U})+6 \mathcal{L}(d(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) \\
& +2 \mathcal{L}^{2}(4 d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))+4 \sqrt{n} \mathcal{S}(V, \tilde{V})
\end{aligned}
$$

By exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$, we find the differential inequality for $d(V, \tilde{V})$ :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} d(V, \tilde{V}) \leq & -2 n \kappa d(V, \tilde{V})+4 n \kappa \mathcal{L} d(V, \tilde{V})+6 \mathcal{L}(d(V, \tilde{V})+\mathcal{S}(U, \tilde{U})) \\
& +2 \mathcal{L}^{2}(4 d(V, \tilde{V})+\mathcal{S}(U, \tilde{U}))+4 \sqrt{m} \mathcal{S}(U, \tilde{U})
\end{aligned}
$$

Below, we derive differential inequalities for $\mathcal{S}(U, \tilde{U})$ and $\mathcal{S}(V, \tilde{V})$.
Lemma B.2. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (4.1). Then, $\mathcal{S}(U, \tilde{U})$ and $\mathcal{S}(V, \tilde{V})$ satisfy

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(U, \tilde{U}) \leq & -2 m \kappa \mathcal{S}(U, \tilde{U})+4 \mathcal{L} d(U, \tilde{U})+\mathcal{D}(\mathcal{H}) d(U, \tilde{U})+6 \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) \\
& +8 \mathcal{L}^{2} d(U, \tilde{U})+2 \mathcal{L}^{2} \mathcal{S}(V, \tilde{V})+4 \sqrt{n} \mathcal{S}(V, \tilde{V}) \\
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(V, \tilde{V}) \leq & -2 n \kappa \mathcal{S}(V, \tilde{V})+4 \mathcal{L} d(V, \tilde{V})+\mathcal{D}(\mathcal{H}) d(V, \tilde{V})+6 \mathcal{L}(\mathcal{S}(V, \tilde{V})+\mathcal{S}(U, \tilde{U}))  \tag{B.4}\\
& +8 \mathcal{L}^{2} d(V, \tilde{V})+2 \mathcal{L}^{2} \mathcal{S}(U, \tilde{U})+4 \sqrt{m} \mathcal{S}(U, \tilde{U})
\end{align*}
$$

Proof. We recall (A.9):

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t}\left(n-d_{i j}\right)= & -2 m \kappa\left(n-d_{i j}\right)+\frac{m \kappa}{N} \sum_{k=1}^{N} \operatorname{tr}\left(S_{i k} S_{i j}+S_{i j} S_{k j}\right) \\
& +\operatorname{itr}\left(H_{i}-H_{j}\right)+\operatorname{itr}\left(S_{i j} H_{j}-H_{i} S_{i j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{i k}-m\right)\left(n-d_{k j}\right)-\left(c_{k i}-m\right)\left(n-d_{i k}\right)-\left(c_{i k}-m\right)\left(n-d_{i j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k j}-m\right)\left(n-d_{i k}\right)-\left(c_{j k}-m\right)\left(n-d_{k j}\right)-\left(c_{j k}-m\right)\left(n-d_{i j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left(c_{k i}-m\right) \operatorname{tr}\left(S_{i k} S_{i j}\right)+\left(c_{j k}-m\right) \operatorname{tr}\left(S_{i j} S_{k j}\right) \\
& +\frac{\kappa}{N} \sum_{k=1}^{N}\left[\left(c_{k i}-c_{i k}\right)+\left(c_{j k}-c_{k j}\right)\right] \sqrt{n}
\end{aligned}
$$

We denote

$$
p_{i j}:=n-d_{i j}, \quad q_{i j}:=m-c_{i j} .
$$

Then, $p_{i j}-\tilde{p}_{i j}$ satisfies

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t}\left(p_{i j}-\tilde{p}_{i j}\right)=-2 m \kappa\left(p_{i j}-\tilde{p}_{i j}\right)+\frac{m \kappa}{N} \sum_{k=1}^{N} \underbrace{\operatorname{tr}\left(S_{i k} S_{i j}+S_{i j} S_{k j}-\tilde{S}_{i k} \tilde{S}_{i j}-\tilde{S}_{i j} \tilde{S}_{k j}\right)}_{=: \mathcal{I}_{51}} \\
& \quad+\underbrace{\operatorname{itr}\left(\left(S_{i j}-\tilde{S}_{i j}\right)\left(H_{j}-H_{i}\right)\right)}_{=: \mathcal{I}_{56}}+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\mathcal{I}_{53}-\tilde{\mathcal{I}}_{53}\right)+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\mathcal{I}_{54}-\tilde{\mathcal{I}}_{54}\right) \\
& \quad+\frac{\kappa}{N} \sum_{k=1}^{N}\left(\mathcal{I}_{55}-\tilde{\mathcal{I}}_{55}\right)+\frac{\kappa}{N} \sum_{k=1}^{N} \underbrace{\left(p_{i k}-\tilde{p}_{i k}+p_{k j}-\tilde{p}_{k j}\right)+\left(\tilde{p}_{k i}-\tilde{p}_{k i}\right)+\left(\tilde{p}_{j k}-p_{j k}\right)} \sqrt{n} . \tag{B.5}
\end{align*}
$$

Below, we present the estimates for $\mathcal{I}_{5 k}, k=1, \cdots, 6$, separately.

- (Estimate of $\mathcal{I}_{51}$ ): We recall (B.3) to find

$$
\left|\mathcal{I}_{51}\right| \leq 4 \mathcal{L} d(U, \tilde{U})
$$

- (Estimate of $\mathcal{I}_{52}$ ): we set

$$
\mathcal{D}(\mathcal{H}):=\max _{1 \leq i, j \leq N}\left\|H_{i}-H_{j}\right\|_{\infty}
$$

Then, one has

$$
\left|\mathcal{I}_{52}\right| \leq \mathcal{D}(\mathcal{H}) d(U, \tilde{U})
$$

- (Estimate of $\left.\mathcal{I}_{53}\right)$ : Note that

$$
\begin{aligned}
& \left|\left(c_{i k}-m\right)\left(n-d_{k j}\right)-\left(\tilde{c}_{i k}-m\right)\left(n-\tilde{d}_{k j}\right)\right| \\
& \quad \leq\left|p_{i k}-\tilde{p}_{i k}\right|\left|q_{i k}\right|+\left|\tilde{p}_{i k}\right|\left|q_{k j}-\tilde{q}_{k j}\right| \leq \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))
\end{aligned}
$$

Thus, we find

$$
\left\|\mathcal{I}_{53}-\tilde{\mathcal{I}}_{53}\right\|_{\mathrm{F}} \leq 3 \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))
$$

- (Estimate of $\mathcal{I}_{54}$ ): We observe

$$
\left\|\mathcal{I}_{54}-\tilde{\mathcal{I}}_{54}\right\|_{\mathrm{F}} \leq 3 \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))
$$

- (Estimate of $\mathcal{I}_{55}$ ): Note that

$$
\begin{aligned}
& q_{k i} \operatorname{tr}\left(S_{i k} S_{i j}-\tilde{q}_{k i} \operatorname{tr}\left(\tilde{S}_{i k} \tilde{S}_{i j}\right)\right. \\
& \quad=q_{i j} \operatorname{tr}\left(S_{i k} S_{i j}-\tilde{S}_{i k} \tilde{S}_{i j}\right)+\left(q_{i j}-\tilde{q}_{i j}\right) \operatorname{tr}\left(\tilde{S}_{i k} \tilde{S}_{i j}\right) \leq 4 \mathcal{L}^{2} d(U, \tilde{U})+\mathcal{L}^{2} \mathcal{S}(V, \tilde{V})
\end{aligned}
$$

Thus, we have

$$
\left\|\mathcal{I}_{55}-\tilde{\mathcal{I}}_{55}\right\|_{\mathrm{F}} \leq 8 \mathcal{L}^{2} d(U, \tilde{U})+2 \mathcal{L}^{2} \mathcal{S}(V, \tilde{V})
$$

- (Estimate of $\mathcal{I}_{56}$ ): We directly find

$$
\left|\mathcal{I}_{56}\right| \leq 4 \sqrt{n} \mathcal{S}(V, \tilde{V})
$$

In (B.5), we collect all the estimates to find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(U, \tilde{U}) \leq & -2 m \kappa \mathcal{S}(U, \tilde{U})+4 \mathcal{L} d(U, \tilde{U})+\mathcal{D}(\mathcal{H}) d(U, \tilde{U})+6 \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) \\
& +8 \mathcal{L}^{2} d(U, \tilde{U})+2 \mathcal{L}^{2} \mathcal{S}(V, \tilde{V})+4 \sqrt{n} \mathcal{S}(V, \tilde{V})
\end{aligned}
$$

By exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$, we can derive a differential inequality for $\mathcal{S}(V, \tilde{V})$ :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(V, \tilde{V}) \leq & -2 n \kappa \mathcal{S}(V, \tilde{V})+4 \mathcal{L} d(V, \tilde{V})+\mathcal{D}(\mathcal{H}) d(V, \tilde{V})+6 \mathcal{L}(\mathcal{S}(V, \tilde{V})+\mathcal{S}(U, \tilde{U})) \\
& +8 \mathcal{L}^{2} d(V, \tilde{V})+2 \mathcal{L}^{2} \mathcal{S}(U, \tilde{U})+4 \sqrt{m} \mathcal{S}(U, \tilde{U})
\end{aligned}
$$

Now, we are ready to provide a proof of Lemma 4.2 with the help of Lemma B. 1 and Lemma B.2.

Proof of Lemma 4.2: recall the inequalities in (B.1) and (B.4):

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} d(U, \tilde{U}) \leq & -2 m \kappa d(U, \tilde{U})+4 m \kappa \mathcal{L} d(U, \tilde{U})+6 \mathcal{L}(d(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) \\
& +2 \mathcal{L}^{2}(4 d(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))+4 \sqrt{n} \mathcal{S}(V, \tilde{V}) \\
\frac{\mathrm{d}}{\mathrm{~d} t} d(V, \tilde{V}) \leq & -2 n \kappa d(V, \tilde{V})+4 n \kappa \mathcal{L} d(V, \tilde{V})+6 \mathcal{L}(d(V, \tilde{V})+\mathcal{S}(U, \tilde{U})) \\
& +2 \mathcal{L}^{2}(4 d(V, \tilde{V})+\mathcal{S}(U, \tilde{U}))+4 \sqrt{m} \mathcal{S}(U, \tilde{U}) \\
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(U, \tilde{U}) \leq & -2 m \kappa \mathcal{S}(U, \tilde{U})+4 \mathcal{L} d(U, \tilde{U})+\mathcal{D}(\mathcal{H}) d(U, \tilde{U})+6 \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))  \tag{B.6}\\
& +8 \mathcal{L}^{2} d(U, \tilde{U})+2 \mathcal{L}^{2} \mathcal{S}(V, \tilde{V})+4 \sqrt{n} \mathcal{S}(V, \tilde{V}) \\
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(V, \tilde{V}) \leq & -2 n \kappa \mathcal{S}(V, \tilde{V})+4 \mathcal{L} d(V, \tilde{V})+\mathcal{D}(\mathcal{G}) d(V, \tilde{V})+6 \mathcal{L}(\mathcal{S}(V, \tilde{V})+\mathcal{S}(U, \tilde{U})) \\
& +8 \mathcal{L}^{2} d(V, \tilde{V})+2 \mathcal{L}^{2} \mathcal{S}(U, \tilde{U})+4 \sqrt{m} \mathcal{S}(U, \tilde{U})
\end{align*}
$$

We add all the inequalities in (B.6) to obtain the desired inequality for $\mathcal{F}$ :

$$
\begin{aligned}
\frac{\mathrm{d} \mathcal{F}}{d t} \leq & -\kappa\left(2 m-8 \sqrt{n}-\frac{\mathcal{D}(\mathcal{H})}{\kappa}\right) \mathcal{F}+4(n+1) \kappa \mathcal{L}(d(U, \tilde{U})+d(V, \tilde{V}))+6 \kappa \mathcal{L} \mathcal{F} \\
& +12 \kappa \mathcal{L}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V}))+16 \kappa \mathcal{L}^{2}(d(U, \tilde{U})+d(V, \tilde{V}))+4 \kappa \mathcal{L}^{2}(\mathcal{S}(U, \tilde{U})+\mathcal{S}(V, \tilde{V})) \\
\leq & -\kappa\left(2 m-8 \sqrt{n}-\frac{\max \{\mathcal{D}(\mathcal{H}), \mathcal{D}(\mathcal{G})\}}{\kappa}\right) \mathcal{F}+\kappa(4 n+22) \mathcal{L} \mathcal{F}+20 \kappa \mathcal{L}^{2} \mathcal{F} .
\end{aligned}
$$

## Appendix C. Emergent dynamics of the DSOM model

In this section, we replace the product of two unitary groups $\mathbf{U}(n) \times \mathbf{U}(m)$ by the product of two special orthogonal groups $\mathbf{S O}(n) \times \mathbf{S O}(m)$. Since all elements of special orthogonal matrices are real-valued, one has for any (real-valued) square matrices $A$ and $B$,

$$
\begin{equation*}
\langle A, B\rangle_{\mathrm{F}}=\operatorname{tr}\left(A B^{\top}\right)=\operatorname{tr}\left(B A^{\top}\right)=\langle B, A\rangle_{\mathrm{F}} . \tag{C.1}
\end{equation*}
$$

Thus, model (3.5) reduces to

$$
\left\{\begin{array}{l}
\dot{U}_{j}=B_{j} U_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}}\left(U_{k}-U_{j} U_{k}^{\top} U_{j}\right), \quad t>0  \tag{C.2}\\
\dot{V}_{j}=C_{j} V_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}}\left(V_{k}-V_{j} V_{k}^{\top} V_{j}\right), \\
\left(U_{j}, V_{j}\right)(0)=\left(U_{j}^{0}, V_{j}^{0}\right) \in \mathbf{S O}(n) \times \mathbf{S O}(m), \quad 1 \leq j \leq N
\end{array}\right.
$$

where $B_{j} \in \mathbb{R}^{n \times n \times n \times n}$ and $C_{j} \in \mathbb{R}^{m \times m \times m \times m}$ are given rank-4 tensors satisfying skewsymmetric properties:

$$
\left[B_{j}\right]_{\alpha_{1} \beta_{1} \alpha_{2} \beta_{2}}=-\left[B_{j}\right]_{\alpha_{2} \beta_{2} \alpha_{1} \beta_{1}}, \quad\left[C_{j}\right]_{\gamma_{1} \delta_{1} \gamma_{2} \delta_{2}}=-\left[C_{j}\right]_{\gamma_{2} \delta_{2} \gamma_{1} \delta_{1}} .
$$

Remark C.1. Due to the symmetric property of the Frobenius inner product on real-valued matrices, one has

$$
\begin{equation*}
d_{i j}=\left\langle U_{i}, U_{j}\right\rangle_{\mathrm{F}}=\left\langle U_{j}, U_{i}\right\rangle_{\mathrm{F}}=d_{j i}, \quad c_{i j}=\left\langle V_{i}, V_{j}\right\rangle_{\mathrm{F}}=\left\langle V_{j}, V_{i}\right\rangle_{\mathrm{F}}=c_{j i} . \tag{C.3}
\end{equation*}
$$

Due to the relation (C.3), for example, the last term $c_{k i}-c_{i k}+c_{j k}-c_{k j}$ in (A.3) vanishes. Thus, it would be expected that the relation (C.3) relaxes the condition such as the dimension condition $n \geq m>4 \sqrt{n}$ in (4.7)(i) (see also Remark A.1).

The contents of this appendix are exactly the same as those of Section 4 and we only provide a sketch of the proof. In Section C.1, we are concerned with the identical (or homogeneous) Hamiltonian, and in Section C.2, the non-identical (or heterogeneous) Hamiltonian is considered to exhibit the phase-locked state.
C.1. Homogeneous ensemble. In this subsection, thanks to the solution splitting property, we assume that

$$
\begin{equation*}
B_{j} \equiv O, \quad C_{j} \equiv O, \quad j=1, \cdots, N \tag{C.4}
\end{equation*}
$$

Below, we state the main theorem for an identical ensemble under which the complete aggregation occurs. For this, we recall diameters and aggregation functional:

$$
\begin{aligned}
& \mathcal{D}(\mathcal{U}(t))=\max _{1 \leq i, j \leq N}\left\|U_{i}(t)-U_{j}(t)\right\|_{\mathrm{F}}, \quad \mathcal{S}(\mathcal{U}(t))=\max _{1 \leq i, j \leq N}\left|n-d_{i j}(t)\right|, \\
& \mathcal{D}(\mathcal{V}(t))=\max _{1 \leq i, j \leq N}\left\|V_{i}(t)-V_{j}(t)\right\|_{\mathrm{F}}, \quad \mathcal{S}(\mathcal{V}(t))=\max _{1 \leq i, j \leq N}\left|m-c_{i j}(t)\right|, \\
& \mathcal{L}=\mathcal{D}(\mathcal{U})+\mathcal{D}(\mathcal{V})+\mathcal{S}(\mathcal{U})+\mathcal{S}(\mathcal{V}) .
\end{aligned}
$$

Theorem C.1. Suppose initial data satisfy

$$
\begin{equation*}
\mathcal{L}^{0}<\alpha_{m, n}=\frac{-(12 n+27)+\sqrt{(12 n+27)^{2}+24 m(3 n+4)}}{4(4 n+3)}, \tag{C.5}
\end{equation*}
$$

and let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (C.2) with (C.4). Then, we have

$$
\lim _{t \rightarrow \infty} \mathcal{L}(t)=0
$$

Proof. By recalling the inequalities in (A.1) with the relation (C.1), we see

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{U}) \leq-2 m \kappa \mathcal{D}(\mathcal{U})+m \kappa \mathcal{D}(\mathcal{U})^{3}+6 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2}  \tag{C.6}\\
& \frac{\mathrm{~d}}{\mathrm{~d} t} \mathcal{D}(\mathcal{V}) \leq-2 n \kappa \mathcal{D}(\mathcal{V})+n \kappa \mathcal{D}(\mathcal{V})^{3}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2}
\end{align*}
$$

Similarly, we use (A.8) to find

$$
\begin{align*}
& \frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{U}) \leq-2 m \kappa \mathcal{S}(\mathcal{U})+2 m \kappa \mathcal{D}(\mathcal{U})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{V}) \mathcal{D}(\mathcal{U})^{2} \\
& \frac{\mathrm{~d}}{\mathrm{~d} t} \mathcal{S}(\mathcal{V}) \leq-2 n \kappa \mathcal{S}(\mathcal{V})+2 n \kappa \mathcal{D}(\mathcal{V})^{2}+6 \kappa \mathcal{S}(\mathcal{U}) \mathcal{S}(\mathcal{V})+2 \kappa \mathcal{S}(\mathcal{U}) \mathcal{D}(\mathcal{V})^{2} \tag{C.7}
\end{align*}
$$

Now, we add (C.6) and (C.7) to obtain a differential inequality for $\mathcal{L}$ :

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{L} \leq-2 \kappa m \mathcal{L}+\kappa(4 n+9) \mathcal{L}^{2}+\kappa\left(2 n+\frac{8}{3}\right) \mathcal{L}^{3} \tag{C.8}
\end{equation*}
$$

If we introduce an auxiliary polynomial:

$$
\begin{equation*}
f(s):=\left(2 n+\frac{8}{3}\right) s^{2}+(4 n+9) s-2 m, \tag{C.9}
\end{equation*}
$$

then (C.8) can be written as

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \mathcal{L} \leq \kappa \mathcal{L} f(\mathcal{L})
$$

Furthermore, we notice that $\alpha_{m, n}$ in (C.5) becomes a unique positive root of the quadratic polynomial $f$ in (C.9). Hence, the desired zero convergence of $\mathcal{L}$ directly follows from the dynamical systems theory.
C.2. Heterogeneous ensemble. In this subsection, we are concerned with the heterogeneous Hamiltonians where the natural frequency tensors are different in general. Similar to Section 4.2, there exist two skew-symmetric matrices $\Omega_{j} \in \mathbb{R}^{n \times n}$ and $\Psi_{j} \in \mathbb{R}^{m \times m}$ such that

$$
B_{j} U_{j}=\Omega_{j} U_{j}, \quad C_{j} V_{j}=\Psi_{j} V_{j}, \quad 1 \leq j \leq N .
$$

Thus, our model reads as

$$
\left\{\begin{array}{l}
\dot{U}_{j}=\Omega_{j} U_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left\langle V_{j}, V_{k}\right\rangle_{\mathrm{F}}\left(U_{k}-U_{j} U_{k}^{\top} U_{j}\right),  \tag{C.10}\\
\dot{V}_{j}=\Psi_{j} V_{j}+\frac{\kappa}{N} \sum_{k=1}^{N}\left\langle U_{j}, U_{k}\right\rangle_{\mathrm{F}}\left(V_{k}-V_{j} V_{k}^{\top} V_{j}\right), \\
\left(U_{j}, V_{j}\right)(0)=\left(U_{j}^{0}, V_{j}^{0}\right) \in \mathbf{S O}(n) \times \mathbf{S O}(m), \quad 1 \leq j \leq N
\end{array}\right.
$$

A crucial estimate used for the emergence of the phase-locked state is (4.12) stated in Lemma 4.2 where the aggregation functional $\mathcal{F}$ tends to zero. Here, $\mathcal{F}$ measures the inter-distance between any two solution configurations $\left\{\left(U_{i}, V_{i}\right)\right\}$ and $\left\{\left(\tilde{U}_{i}, \tilde{V}_{i}\right)\right\}$ :

$$
\mathcal{F}(t)=d(U, \tilde{U})(t)+d(V, \tilde{V})(t)+\mathcal{S}(U, \tilde{U})(t)+\mathcal{S}(V, \tilde{V})(t),
$$

where the diameters are defined in (4.8) and (4.10). We also denote the diameters for natural frequencies:

$$
\mathcal{D}(\Omega):=\max _{1 \leq i, j \leq N}\left\|\Omega_{i}-\Omega_{j}\right\|_{\infty}, \quad \mathcal{D}(\Psi):=\max _{1 \leq i, j \leq N}\left\|\Psi_{i}-\Psi_{j}\right\|_{\infty} .
$$

Without loss of generality, we may assume

$$
\mathcal{D}(\Omega) \geq \mathcal{D}(\Psi)
$$

In the following lemma, we provide a temporal evolution of $\mathcal{F}$.
Lemma C.1. Let $\left\{\left(U_{i}, V_{i}\right)\right\}$ and $\left\{\left(\tilde{U}_{i}, \tilde{V}_{i}\right)\right\}$ be any two solutions to (C.10). Then, the aggregation functionals $\mathcal{L}$ and $\mathcal{F}$ satisfy

$$
\begin{align*}
\frac{\mathrm{d} \mathcal{L}}{d t} & \leq 2(1+3 \sqrt{n}) \mathcal{D}(\Omega)-2 \kappa m \mathcal{L}+\kappa(4 n+9) \mathcal{L}^{2}+\kappa\left(2 n+\frac{8}{3}\right) \mathcal{L}^{3}, \quad t>0, \\
\frac{\mathrm{~d} \mathcal{F}}{d t} & \leq-\kappa\left(2 m-\frac{\max \{\mathcal{D}(\Omega), \mathcal{D}(\Psi)\}}{\kappa}\right) \mathcal{F}+\kappa(4 n+22) \mathcal{L} \mathcal{F}+20 \kappa \mathcal{L}^{2} \mathcal{F} . \tag{C.11}
\end{align*}
$$

Proof. For the inequality for $\mathcal{L}$, it directly follows from (C.8) in Theorem C.1. Similarly, if we closely follow a proof of Lemma 4.2 presented in Appendix B then we find the desired inequality for $\mathcal{F}$.

By applying (C.11) ${ }_{2}$, we establish the desired practical aggregation estimate for (C.10).
Proposition C.1. Suppose that the system parameters and initial data satisfy

$$
\begin{equation*}
\mathcal{D}(\Omega) \geq \mathcal{D}(\Psi), \quad \kappa>\kappa_{\mathrm{c}}, \quad \mathcal{L}^{0}<\nu_{2}, \tag{C.12}
\end{equation*}
$$

where $\kappa_{\mathrm{c}}$ and $\nu_{2}$ are specified in (C.14) and (C.15), respectively, and let $\left\{\left(U_{i}, V_{i}\right)\right\}$ be a solution to (C.10). Then, one has

$$
\lim _{\kappa \rightarrow \infty} \limsup _{t \rightarrow \infty} \mathcal{L}(t)=0
$$

Proof. We introduce an auxiliary polynomial:

$$
\begin{equation*}
g(s):=2 m s-(4 n+9) s^{2}-\left(2 n+\frac{8}{3}\right) s^{3} . \tag{C.13}
\end{equation*}
$$

Then, we notice that $g$ has three roots, say, $\alpha_{0}<0<\alpha_{1}$. Since (C.11) ${ }_{2}$ is rewritten as

$$
\dot{\mathcal{L}} \leq \kappa\left(\frac{2(1+3 \sqrt{n}) \mathcal{D}(\mathcal{H})}{\kappa}-g(\mathcal{L})\right)=: \kappa p(s)
$$

for a sufficiently large $\kappa>0, p$ admits one negative root, say, $\nu_{0}<0$ and two positive roots, say, $0<\nu_{1}<\nu_{2}$ with continuous dependence on $\kappa$ : for $\nu_{1}=\nu_{1}(\kappa)$ and $\nu_{2}=\nu_{2}(\kappa)$,

$$
\lim _{\kappa \rightarrow \infty} \nu_{1}(\kappa)=0, \quad \lim _{\kappa \rightarrow \infty} \nu_{2}(\kappa)=\alpha_{1} .
$$

Since we assume (C.12) ${ }_{3}$, it follows from dynamical systems theory that there exists a finite entrance time $T_{*}>0$ such that

$$
\mathcal{L}(t)<\nu_{1}, \quad t>T_{*} .
$$

Hence, this yields the desired result.
Remark C.2. For the explicit value of $\kappa_{\mathrm{c}}$, we see that the cubic polynomial $g$ in (C.13) admits the local maximum at $s=s_{*}$ :

$$
s_{*}=\frac{-(4 n+9)+\sqrt{(4 n+9)^{2}+4 m(3 n+4)}}{6 n+8}
$$

Hence, $\kappa_{\mathrm{c}}$ is chosen to be

$$
\begin{equation*}
\kappa_{\mathrm{c}}:=\frac{2(1+3 \sqrt{n}) \mathcal{D}(\Omega)}{g\left(s_{*}\right)} \text { so that } g\left(s_{*}\right)>\frac{2(1+3 \sqrt{n})}{\kappa_{\mathrm{c}}} . \tag{C.14}
\end{equation*}
$$

For this $\kappa_{\mathrm{c}}, \nu_{2}=\nu_{2}(\kappa)$ is completely determined as the largest positive root of

$$
\begin{equation*}
p(s)=0 \quad \text { for } \kappa>\kappa_{\mathrm{c}} . \tag{C.15}
\end{equation*}
$$

It follows from Proposition 5.1 that we can make $\mathcal{L}$ sufficiently small by increasing the value of $\kappa$ for $t>T_{*}$. Thus, there exists $\kappa_{\mathrm{p}}>\kappa_{\mathrm{c}}$ such that for $\kappa>\kappa_{\mathrm{p}}$,

$$
(4 n+22) \mathcal{L}+20 \mathcal{L}^{2}<\frac{1}{2}\left(2 m-\frac{\max \{\mathcal{D}(\Omega), \mathcal{D}(\Psi)\}}{\kappa}\right)=: \frac{1}{2} \Lambda .
$$

Hence, (C.11) ${ }_{2}$ yields

$$
\frac{d \mathcal{F}}{d t} \leq-\frac{1}{2} \Lambda \mathcal{F}, \quad t>T_{*}
$$

Since we have established the zero convergence of $\mathcal{F}$, we can obtain the same result in Theorem 4.2. Thanks to exactly the same proof, we only state the results.

Theorem C.2. Suppose that the system parameters and initial data satisfy

$$
n \geq m, \quad \mathcal{D}(\Omega) \geq \mathcal{D}(\Psi), \quad \kappa>\kappa_{\mathrm{p}}, \quad \max \left\{\mathcal{L}^{0}, \tilde{\mathcal{L}}^{0}\right\} \leq \nu_{2}
$$

and let $\left\{\left(U_{i}, V_{i}\right)\right\}$ and $\left\{\left(\tilde{U}_{i}, \tilde{V}_{i}\right)\right\}$ be any two solutions to (C.10). Then, the following assertions hold.
(1) The aggregation functional $\mathcal{F}$ converges to zero exponentially.
(2) The normalized velocities $\dot{U}_{i} U_{i}^{\top}$ and $\dot{U}_{i} \tilde{U}_{i}^{\top}$ synchronize:

$$
\left\|\dot{U}_{i} U_{i}^{\top}-\tilde{\dot{U}}_{i} \tilde{U}_{i}^{\top}\right\|_{\mathrm{F}} \leq 2 \kappa(m+n) \mathcal{F}
$$

(3) There exist special orthogonal matrices $X_{\infty} \in \mathbf{S O}(n)$ and $Y_{\infty} \in \mathbf{S O}(m)$ such that

$$
\begin{array}{ll}
\lim _{t \rightarrow \infty} U_{i}^{\top}(t) \tilde{U}_{i}(t)=X_{\infty}, & \lim _{t \rightarrow \infty}\left\|\tilde{U}_{i}(t)-U_{i}(t) X_{\infty}\right\|_{\mathrm{F}}=0, \\
\lim _{t \rightarrow \infty} V_{i}^{\top}(t) \tilde{V}_{i}(t)=Y_{\infty}, & \lim _{t \rightarrow \infty}\left\|\tilde{V}_{i}(t)-V_{i}(t) Y_{\infty}\right\|_{\mathrm{F}}=0 .
\end{array}
$$

(4) System (C.10) exhibits asymptotic phase-locking: for any indices $i$ and $j$,

$$
\exists \lim _{t \rightarrow \infty} U_{i}(t) U_{j}^{\top}(t) \quad \text { and } \quad \exists \lim _{t \rightarrow \infty} V_{i}(t) V_{j}^{\top}(t) .
$$

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(Seung-Yeal Ha)
Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826 and
Korea Institute for Advanced Study, Hoegiro 85, Seoul, 02455, Republic of Korea
Email address: syha@snu.ac.kr
(Dohyun Kim)
School of Mathematics, Statistics and Data Science,
Sungshin Women's University, Seoul 02844, Republic of Korea
Email address: dohyunkim@sungshin.ac.kr
(Hansol Park)
Department of Mathematical Sciences,
Seoul National University, Seoul 08826, Republic of Korea
Email address: hansol960612@snu.ac.kr


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