

An irrational Lagrangian density of a single hypergraph

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Abstract

The *Turán number* of an r -uniform graph F , denoted by $ex(n, F)$, is the maximum number of edges in an F -free r -uniform graph on n vertices. The *Turán density* of F is defined as $\pi(F) = \lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{r}}$. Denote $\Pi_\infty^{(r)} = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-uniform graphs}\}$, $\Pi_{fin}^{(r)} = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a finite family of } r\text{-uniform graphs}\}$ and $\Pi_t^{(r)} = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-uniform graphs and } |\mathcal{F}| \leq t\}$. For graphs, Erdős-Stone-Simonovits ([7], [8]) showed that $\Pi_\infty^{(2)} = \Pi_{fin}^{(2)} = \Pi_1^{(2)} = \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{l-1}{l}, \dots\}$. We know quite few about the Turán density of an r -uniform graph for $r \geq 3$. Baber and Talbot [2], and Pikhurko [27] showed that there is an irrational number in $\Pi_3^{(3)}$ and $\Pi_{fin}^{(3)}$ respectively, disproving a conjecture of Chung and Graham [5]. Baber and Talbot [2] asked whether $\Pi_1^{(r)}$ contains an irrational number. The Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. The *Lagrangian density* of an r -uniform graph F is $\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ is } F\text{-free}\}$, where $\lambda(G)$ is the Lagrangian of an r -uniform graph G . Sidorenko [31] showed that the Lagrangian density of an r -uniform hypergraph F is the same as the Turán density of the extension of F . In this paper, we show that the Lagrangian density of $F = \{123, 124, 134, 234, 567\}$ (the disjoint union of K_4^3 and an edge) is $\frac{\sqrt{3}}{3}$, consequently, the Turán density of the extension of F is an irrational number, answering the question of Baber and Talbot.

Keywords: Hypergraph Lagrangian, Lagrangian density, Turán density

1 Introduction

For a set V and a positive integer r , let $V^{(r)}$ denote the family of all r -subsets of V . An r -uniform graph or r -graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. Let $e(G)$ denote the number of edges of G . An edge $e = \{a_1, a_2, \dots, a_r\}$ will be simply denoted by $a_1 a_2 \dots a_r$. An r -graph H is a *subgraph* of an r -graph G , denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph of G *induced* by $V' \subseteq V$, denoted as $G[V']$, is the r -graph with vertex set V' and edge set $E' = \{e \in E(G) : e \subseteq V'\}$. For $S \subseteq V(G)$, let $G - S$ denote the subgraph of G induced by $V(G) \setminus S$. Let G^c denote the *complement* r -graph of an r -graph G with $V(G^c) = V(G)$ and $E(G^c) = \{e : e \in V(G)^{(r)} \setminus E(G)\}$. Let K_t^r denote the complete r -graph on t vertices. Let K_t^{r-} be obtained by removing one edge from K_t^r . For a positive integer n , let $[n]$ denote $\{1, 2, 3, \dots, n\}$.

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For a family \mathcal{F} of r -graphs, an r -graph G is called \mathcal{F} -free if it does not contain an isomorphic copy of any r -graph of \mathcal{F} . For a fixed positive integer n and a family of r -graphs \mathcal{F} , the *Turán number* of \mathcal{F} , denoted by $ex(n, \mathcal{F})$, is the maximum number of edges in an \mathcal{F} -free r -graph on n vertices. An averaging argument of Katona, Nemetz and Simonovits [17] shows that the sequence $\frac{ex(n, \mathcal{F})}{\binom{n}{r}}$ is non-increasing. Hence $\lim_{n \rightarrow \infty} \frac{ex(n, \mathcal{F})}{\binom{n}{r}}$ exists. The *Turán density* of \mathcal{F} is defined as

$$\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{ex(n, \mathcal{F})}{\binom{n}{r}}.$$

If \mathcal{F} consists of an single r -graph F , we simply write $ex(n, \{F\})$ and $\pi(\{F\})$ as $ex(n, F)$ and $\pi(F)$. Denote

$$\begin{aligned}\Pi_{\infty}^{(r)} &= \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-uniform graphs}\}, \\ \Pi_{fin}^{(r)} &= \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a finite family of } r\text{-uniform graphs}\}\end{aligned}$$

and

$$\Pi_t^{(r)} = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-uniform graphs and } |\mathcal{F}| \leq t\}.$$

Clearly,

$$\Pi_1^{(r)} \subseteq \Pi_2^{(r)} \subseteq \dots \subseteq \Pi_{fin}^{(r)} \subseteq \Pi_{\infty}^{(r)}.$$

For 2-graphs, Erdős-Stone-Simonovits ([7], [8]) determined the Turán numbers of all non-bipartite graphs asymptotically. Their result implies that

$$\Pi_{\infty}^{(2)} = \Pi_{fin}^{(2)} = \Pi_1^{(2)} = \{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{l-1}{l}, \dots\}.$$

Very few results are known for $r \geq 3$. In [5] Chung and Graham proposed the conjecture that every element in $\Pi_{fin}^{(r)}$ is a rational number. Baber and Talbot [2], and Pikhurko [27] disproved this conjecture by showing that there is an irrational number in $\Pi_3^{(3)}$ and $\Pi_{fin}^{(3)}$, respectively. Baber and Talbot [2] asked whether $\Pi_1^{(r)}$ contains an irrational number. In this paper, we answer this question by showing that the Lagrangian density of the disjoint union of K_4^3 and an edge is an irrational number.

The hypergraph Lagrangian method has been helpful in hypergraph extremal problems.

Definition 1.1 Let G be an r -graph on $[n]$ and let $\vec{x} = (x_1, \dots, x_n) \in [0, \infty)^n$. Define the Lagrangian function

$$\lambda(G, \vec{x}) = \sum_{e \in E(G)} \prod_{i \in e} x_i.$$

The *Lagrangian* of G , denoted by $\lambda(G)$, is defined as

$$\lambda(G) = \max\{\lambda(G, \vec{x}) : \vec{x} \in \Delta\},$$

where

$$\Delta = \{\vec{x} = (x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for every } i \in [n]\}.$$

The value x_i is called the *weight* of the vertex i and a vector $\vec{x} \in \Delta$ is called a *feasible weight vector*

on G . A feasible weight vector $\vec{y} \in \Delta$ is called an *optimum weight vector* for G if $\lambda(G, \vec{y}) = \lambda(G)$.

In [22], Motzkin and Straus established a connection between the Lagrangian of a 2-graph and its maximum complete subgraphs.

Theorem 1.2 ([22]) *If G is a 2-graph in which a maximum complete subgraph has t vertices, then $\lambda(G) = \lambda(K_t^2) = \frac{1}{2}(1 - \frac{1}{t})$.*

They also applied this connection to give another proof of the theorem of Turán on the Turán density of complete graphs. Since then the Lagrangian method has been a useful tool in hypergraph extremal problems. Earlier applications include that Frankl and Rödl [11] applied it in disproving the long standing jumping constant conjecture of Erdős. Sidorenko [31] applied Lagrangians of hypergraphs to first find infinitely many Turán densities of hypergraphs. More recent developments of this method were obtained in [26, 2, 13, 24, 3, 25, 15, 16, 14, 37]. Determining the Lagrangian of a hypergraph is much more difficult than graphs and there is no conclusion similar to Theorem 1.2 for hypergraphs. It is of great interests to estimate Lagrangians of hypergraphs that have some certain properties. In 1980's, Frankl and Füredi [9] asked the question that for a given integer m , what is the maximum Lagrangian among all r -graphs with m edges? They conjectured that the r -graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r -graphs with m edges. For distinct $A, B \in \mathbb{N}^r$ we say that A is less than B in the *colex ordering* if $\max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. By Theorem 1.2, this conjecture is true when $r = 2$. For hypergraphs, Talbot [32] first proved the conjecture for $r = 3$ and $\binom{\ell}{3} \leq m \leq \binom{\ell}{3} + \binom{\ell-1}{2} - \ell$, where $\ell > 0$ is an integer. Subsequent progress in this conjecture were made in the papers of Tang, Peng, Zhang and Zhao [33, 34], Tyomkyn [36], Lei, Lu and Peng [20], Nikiforov [23], Lei and Lu [19], and Lu [21]. Recently, Gruslys, Letzter and Morrison [12] confirmed this conjecture for r and $\binom{\ell}{r} \leq m \leq \binom{\ell}{r} + \binom{\ell-1}{r-1}$ if ℓ is sufficiently large. They also found infinitely many counterexamples for all $r \geq 4$. As remarked in [12], it would be interesting to find the maximisers for other values of m though it might be a very hard problem. In this paper, we will apply the connection of the Lagrangian density and the Turán density of an r -graph to answer the question of Baber and Talbot. Our proof relies heavily on the estimation of Lagrangians of 3-graphs.

The *Lagrangian density* $\pi_\lambda(F)$ of an r -graph F is defined to be

$$\pi_\lambda(F) = \sup\{r!\lambda(G) : G \text{ is } F\text{-free}\}.$$

A pair of vertices $\{i, j\}$ is *covered* in a hypergraph F if there exists an edge e in F such that $\{i, j\} \subseteq e$. We say that F covers pairs if every pair of vertices in F is covered. Let $r \geq 3$ and F be an r -graph. The *extension* of F , denoted by H^F is obtained as follows: For each pair of vertices v_i and v_j not covered in F , we add a set B_{ij} of $r - 2$ new vertices and the edge $\{v_i, v_j\} \cup B_{ij}$, where the B_{ij} 's are pairwise disjoint over all such pairs $\{i, j\}$.

The Lagrangian density is closely related to the Turán density. The following proposition is implied by Theorem 2.6 in [30] (see Proposition 5.6 in [3] and Corollary 1.8 in [31] for the explicit statement).

Proposition 1.3 ([30, 3, 31]) *Let F be an r -graph. Then*

- (i) $\pi(F) \leq \pi_\lambda(F)$;
- (ii) $\pi(H^F) = \pi_\lambda(F)$. In particular, if F covers pairs, then $\pi(F) = \pi_\lambda(F)$.

To answer the question of Baber and Talbot, we show that the Lagrangian density of $\{123, 124, 134, 234, 567\}$, the disjoint union of K_4^3 and an edge, denoted as $K_4^3 \cup e$, is $\frac{\sqrt{3}}{3}$. The following is our main Theorem.

Theorem 1.4 $\pi_\lambda(K_4^3 \cup e) = \frac{\sqrt{3}}{3}$.

Applying Theorem 1.4 and Proposition 1.3, we see that the Turán density of the extension of $K_4^3 \cup e$ is $\frac{\sqrt{3}}{3}$.

For an r -graph H on t vertices, it is clear that $\pi_\lambda(H) \geq r!\lambda(K_{t-1}^r)$. An r -graph H on t vertices is λ -perfect if $\pi_\lambda(H) = r!\lambda(K_{t-1}^r)$. Theorem 1.2 implies that all 2-graphs are λ -perfect. Theorem 1.4 indicates that $K_4^3 \cup e$ is not λ -perfect. We can show however that $K_4^3 \cup k \cdot e$, the disjoint union of K_4^3 and k disjoint edges is λ -perfect for $k \geq 2$.

Theorem 1.5 $K_4^3 \cup k \cdot e$ is λ -perfect for $k \geq 2$.

In Section 2, we give a sketch of the proof of Theorem 1.4. In Section 3, we will give the proof of Theorem 1.5. In Section 4, we give some preliminaries on KKT conditions for continuous optimization problems and properties of hypergraph Lagrangians. In Section 5, we prove the main Lemmas needed in the proof of Theorem 1.4.

2 Sketch of the proof of Theorem 1.4

The following three 3-graphs are to be used throughout the paper.

B(2, n-2): the 3-graph with vertex set $[n]$ and edge set $E(B(2, n-2)) = \{e \in \binom{[n]}{3} : e \cap \{1, 2\} \neq \emptyset\}$, i.e., every edge in $B(2, n-2)$ contains vertex 1 or 2 or both. Note that $B(2, n-2)$ is $K_4^3 \cup e$ -free, we will show that it is an extremal 3-graph for $K_4^3 \cup e$ (in terms of Lagrangian density).

X_i: the 3-graph with vertex set $[2i+2]$ such that $\{1, 2, 2j+1, 2j+2\}$ form K_4^3 for all j , $1 \leq j \leq i$, i.e., it consists of i copies of K_4^3 all sharing vertices $\{1, 2\}$.

Y_i: the 3-graph with vertex set $[i+3]$ such that $\{1, 2, 3, j+3\}$ form K_4^3 for all j , $1 \leq j \leq i$, i.e., it consists of i copies of K_4^3 all sharing vertices $\{1, 2, 3\}$.

An r -graph G is *dense* if $\lambda(G') < \lambda(G)$ for every proper subgraph G' of G .

Sketch of the proof of Theorem 1.4: For the lower bound, note that $B(2, n-2)$ is $K_4^3 \cup e$ -free, we shall prove $\lim_{n \rightarrow \infty} \lambda(B(2, n)) = \frac{\sqrt{3}}{18}$ in Lemma 4.10. So $\pi_\lambda(K_4^3 \cup e) \geq 3! \lim_{n \rightarrow \infty} \lambda(B(2, n-2)) = \frac{\sqrt{3}}{3}$.

For the upper bound, let G be a $K_4^3 \cup e$ -free 3-graph, our goal is to show that $\lambda(G) \leq \frac{\sqrt{3}}{18}$. If G is not dense, then there exists a proper subgraph G' of G such that $\lambda(G') = \lambda(G)$ and $|V(G')| < |V(G)|$. If G' is dense, then we stop. Otherwise, we continue this process until we find a dense subgraph G'' such that $\lambda(G'') = \lambda(G)$. This process terminates since the number of vertices is reduced by at least one in each step. To show $\lambda(G) \leq \frac{\sqrt{3}}{18}$, it's sufficient to show that $\lambda(G'') \leq \frac{\sqrt{3}}{18}$. So we may assume that G is a dense $K_4^3 \cup e$ -free 3-graph. Suppose that $\lambda(G) > \frac{\sqrt{3}}{18}$, we will prove the following lemmas in Section 5.

Lemma 2.1 Let G be a dense $K_4^3 \cup e$ -free 3-graph with $\lambda(G) > \frac{\sqrt{3}}{18}$. Then G is X_2 -free.

Lemma 2.2 Let G be a dense $K_4^3 \cup e$ -free 3-graph with $\lambda(G) > \frac{\sqrt{3}}{18}$. Then G contains at least two copies of K_4^3 .

Lemma 2.3 Let G be a dense $K_4^3 \cup e$ -free 3-graph with $\lambda(G) > \frac{\sqrt{3}}{18}$. Then G is Y_2 -free.

By Lemma 2.2, G contains two copies of K_4^3 . Since G is $K_4^3 \cup e$ -free, these two copies of K_4^3 must have 2 or 3 vertices in common. So G contains a copy of X_2 or Y_2 , a contradiction to Lemmas 2.1 and 2.3.

To complete the proof of Theorem 1.4, what remains is to show those three main lemmas. They will be given in Section 5.

3 The proof of Theorem 1.5

In order to prove Theorem 1.5, we need some lemmas from [38]. Let $S_{2,t}$ denote the 3-graph with vertex set $\{v_1, v_2, u_1, u_2, \dots, u_t\}$ and edge set $\{v_1 v_2 u_1, v_1 v_2 u_2, \dots, v_1 v_2 u_t\}$. A result of Sidorenko in [31] implies that $S_{2,t}$ is λ -perfect.

Theorem 3.1 ([38]) *If H is λ -perfect, then $H \cup S_{2,t}$ is λ -perfect for any $t \geq 1$.*

Claim 3.2 ([38]) *Let G be a 3-graph with $\lambda(G) > \lambda(K_{k+1}^3)$ and let \vec{x} be an optimal weight vector. Then for any $v \in V(G)$, its weight x_v satisfies that $x_v < 1 - \frac{\sqrt{k(k-1)}}{k+1}$.*

Claim 3.3 ([38]) *Let v be a vertex in a 3-graph G and x_v be the weight of v in an optimal weight vector \vec{x} of G . If $G - \{v\}$ is H -free, then $\lambda(G) \leq \frac{\pi_\lambda(H)(1-x_v)^3}{6(1-3x_v)}$.*

Remark 3.4 ([38]) *$f(x) = \frac{(1-x)^3}{1-3x}$ is increasing in $(0, \frac{1}{3})$.*

Definition 3.5 *For $v \in V(G)$, the link graph of v in G , denote by G_v , is the graph on vertex set $V(G)$ and the edge set $\{e \setminus \{v\} : v \in e \in E(G)\}$. Let $\omega(G_v)$ be the number of vertices in a maximum complete subgraph in G_v .*

Claim 3.6 ([38]) *Let a 3-graph G be $H \cup S_{2,t}$ -free, where H is a 3-graph with s vertices. Let $v \in V(H)$. If $H \subseteq G - \{v\}$, then $\omega(G_v) \leq s + t$.*

Claim 3.7 ([38]) *Let a 3-graph G be $H \cup S_{2,t}$ -free, where H is a 3-graph with s vertices. Let $S_{2,s+t} = \{v_1 v_2 b_1, v_1 v_2 b_2, \dots, v_1 v_2 b_{s+t}\} \subseteq G$. Then $G - \{v_1, v_2\}$ is H -free.*

Claim 3.8 ([38]) *Let a 3-graph G be $H \cup S_{2,t}$ -free, where H is a 3-graph with s vertices. If $H \subseteq G - \{v_1\}$ and $H \not\subseteq G - \{v_1, v_2\}$, then $\omega((G - \{v_2\})_{v_1}) \leq s + t - 1$.*

Proof of Theorem 1.5. By Theorem 3.1, it's sufficient to show that $K_4^3 \cup 2 \cdot e$ is λ -perfect. Note that $K_4^3 \cup 2 \cdot e$ has 10 vertices. It's sufficient to show that if G is $K_4^3 \cup 2 \cdot e$ -free dense 3-graph then $\lambda(G) \leq \lambda(K_9^3)$. Suppose on the contrary that $\lambda(G) > \lambda(K_9^3) = \frac{28}{243}$. Let \vec{x} be an optimal weight vector of G .

Case 1. There exists $v \in V(G)$ with weight x_v such that $G - \{v\}$ is $K_4^3 \cup e$ -free.

By Claim 3.2, $x_v < 1 - \frac{2\sqrt{14}}{9}$. By Theorem 1.4 and Claim 3.3,

$$\lambda(G) \leq \frac{\frac{\sqrt{3}}{18}(1-x_v)^3}{1-3x_v} = f(x_v). \quad (3).$$

Since $f(x_v)$ is increasing in $[0, 1 - \frac{2\sqrt{14}}{9}]$, then

$$\begin{aligned}\lambda(G) &\leq f(1 - \frac{2\sqrt{14}}{9}) \\ &= \frac{28\sqrt{42}}{729(3\sqrt{14} - 9)} \\ &\leq \frac{28}{243},\end{aligned}$$

a contradiction.

Case 2. For any $v \in V(G)$, $K_4^3 \cup e \subseteq G - \{v\}$.

Since $\lambda(G) > \lambda(K_9^3)$ and $S_{2,8}$ is λ -perfect, then $S_{2,8} = \{v_1v_2b_1, v_1v_2b_2, \dots, v_1v_2b_8\} \subseteq G$. By Claim 3.7, $G - \{v_1, v_2\}$ is $K_4^3 \cup e$ -free. Applying Claim 3.6 ($s = 7, t = 1$), we have

$$\omega(G_{v_1}) \leq 8 \quad \text{and} \quad \omega(G_{v_2}) \leq 8.$$

Applying Claim 3.8 ($s = 7, t = 1$), we have $\omega((G - \{v_2\})_{v_1}) \leq 7$ and $\omega((G - \{v_1\})_{v_2}) \leq 7$.

Assume the weight of v_1 and v_2 are a_1 and a_2 respectively, and $a_1 + a_2 = 2a$. Since $G - \{v_1, v_2\}$ is $K_4^3 \cup e$ -free and by Theorem 1.4, the contribution of edges containing neither v_1 nor v_2 to $\lambda(G, \vec{x})$ is at most $\frac{\sqrt{3}}{18}(1 - 2a)^3$. Since $\omega((G - \{v_2\})_{v_1}) \leq 7$ and $\omega((G - \{v_1\})_{v_2}) \leq 7$, by Theorem 1.2, the contribution of edges containing either v_1 or v_2 to $\lambda(G, \vec{x})$ is at most $2 \times \frac{1}{2}a(1 - \frac{1}{7})(1 - 2a)^2$. The contribution of edges containing both v_1 and v_2 to $\lambda(G, \vec{x})$ is at most $a^2(1 - 2a)$. Therefore

$$\begin{aligned}\lambda(G) &\leq \frac{\sqrt{3}}{18}(1 - 2a)^3 + a^2(1 - 2a) + \frac{6}{7}a(1 - 2a)^2 \\ &\leq \frac{1}{10}(1 - 2a)^3 + a^2(1 - 2a) + \frac{6}{7}a(1 - 2a)^2 = f(a) \\ f'(a) &= \frac{66a^2 - 86a + 9}{35}.\end{aligned}$$

Since $f(a)$ is increasing in $[0, \frac{43 - \sqrt{1255}}{66}]$ and is decreasing in $[\frac{43 - \sqrt{1255}}{66}, 1]$, then $\lambda(G) \leq f(\frac{43 - \sqrt{1255}}{66}) < \frac{28}{243}$. \square

4 Preliminaries

4.1 Karush-Kuhn-Tucker Conditions

Let us consider the optimisation problem:

$$\begin{aligned}&\text{maximise } f(x) \\ &\text{subject to } g_i(x) \leq 0, i = 1, \dots, m,\end{aligned}\tag{3.1}$$

where $x \in \mathbb{R}^n$ and f and g_i are differentiable functions from \mathbb{R}^n to \mathbb{R} for all i . Let $\nabla f(x)$ be the gradient of f at x i.e. the vector in \mathbb{R}^n whose i th coordinate is $\frac{\partial}{\partial x_i} f(x)$. We say that KKT conditions hold at $x^* \in \mathbb{R}^n$ if there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$(i) \quad \nabla f(x^*) = \sum_{i=1}^m \lambda_i \nabla g_i(x^*),$$

(ii) $\lambda_i \geq 0, i = 1, \dots, m,$

(iii) $\lambda_i g_i(x^*) = 0, i = 1, \dots, m.$

We call the constraints linear if g_1, \dots, g_m are all affine functions.

Theorem 4.1 ([4],[15]) *If the constraints of (3.1) are linear, then any optimum point of (3.1) must satisfy the KKT conditions.*

4.2 Properties of the Lagrangian function

The following fact follows immediately from the definition of the Lagrangian.

Fact 4.2 *Let G_1, G_2 be r -graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

Fact 4.3 ([11]) *Let G be an r -graph on $[n]$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimum weight vector on G . Then*

$$\frac{\partial \lambda(G, \vec{x})}{\partial x_i} = r \lambda(G)$$

for every $i \in [n]$ satisfying $x_i > 0$.

Given an r -graph G , and $i, j \in V(G)$, define

$$L_G(j \setminus i) = \{e : i \notin e, e \cup \{j\} \in E(G) \text{ and } e \cup \{i\} \notin E(G)\}.$$

Fact 4.4 *Let G be an r -graph on $[n]$. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be a feasible weight vector on G . Let $i, j \in [n], i \neq j$ satisfying $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$. Let $\vec{y} = (y_1, y_2, \dots, y_n)$ be defined by letting $y_\ell = x_\ell$ for every $\ell \in [n] \setminus \{i, j\}$ and $y_i = y_j = \frac{1}{2}(x_i + x_j)$. Then $\lambda(G, \vec{y}) \geq \lambda(G, \vec{x})$. Furthermore, if the pair $\{i, j\}$ is contained in an edge of G , $x_i > 0$ for each $1 \leq i \leq n$, and $\lambda(G, \vec{y}) = \lambda(G, \vec{x})$, then $x_i = x_j$.*

Proof of Fact 4.4. Since $L_G(i \setminus j) = L_G(j \setminus i) = \emptyset$, then

$$\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = \sum_{\{i, j\} \subseteq e \in G} \left(\frac{(x_i + x_j)^2}{4} - x_i x_j \right) \prod_{k \in e \setminus \{i, j\}} x_k \geq 0.$$

If the pair $\{i, j\}$ is contained in an edge of G and $x_i > 0$ for each $1 \leq i \leq n$, then the equality holds only if $x_i = x_j$. \square

Fact 4.5 *Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimum vector for an r -graph G on $[n]$. If $L_G(j \setminus i) = \emptyset$, then we may assume that $x_i \geq x_j$.*

Proof of Fact 4.5. If $x_i < x_j$, then let $\epsilon = \frac{x_j - x_i}{2}$ and $\vec{x}' = (x_1, x_2, \dots, x_i + \epsilon, \dots, x_j - \epsilon, \dots, x_n)$. Since $L_G(j \setminus i) = \emptyset$, then

$$\lambda(G, \vec{x}') - \lambda(G, \vec{x}) \geq \sum_{\{i, j\} \subseteq e \in G} ((x_i + \epsilon)(x_j - \epsilon) - x_i x_j) \prod_{k \in e \setminus \{i, j\}} x_k \geq \frac{(x_j - x_i)^2}{4} \sum_{\{i, j\} \subseteq e \in G} \prod_{k \in e \setminus \{i, j\}} x_k \geq 0.$$

\square

Fact 4.6 ([11]) *Let $G = (V, E)$ be a dense r -graph. Then G covers pairs.*

Let $C_{r,m}$ denote the r -graph with m edges formed by taking the first m sets in the colex ordering of \mathbb{N}^r . The following two results were given in [35] and [32], respectively.

Lemma 4.7 ([35]) *Let m and t be positive integers satisfying the condition that $\binom{t}{3} - 6 \leq m \leq \binom{t}{3} - 3$. Let G be a 3-graph with m edges. Then $\lambda(G) \leq \lambda(C_{3,m})$.*

Lemma 4.8 ([32]) *For any integers m , t and r satisfying the condition that $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$, then we have $\lambda(C_{r,m}) = \lambda(K_{t-1}^r)$.*

Fact 4.9 *Let $f(x) = \frac{x^2(1-x)}{4} + \frac{x(1-x)^2}{2}$, where $0 \leq x \leq 1$. Then $f(x) \leq \frac{\sqrt{3}}{18}$ and equality holds only if $x = \frac{3-\sqrt{3}}{3}$.*

Proof of Fact 4.9. Since $f'(x) = \frac{3x^2-6x+2}{4}$, then $f(x)$ is increasing when $x \in [0, \frac{3-\sqrt{3}}{3}]$ and decreasing when $x \in [\frac{3-\sqrt{3}}{3}, 1]$. Therefore $f(x) \leq f(\frac{3-\sqrt{3}}{3}) = \frac{\sqrt{3}}{18}$. \square

Lemma 4.10 $\lambda(B(2, n-2)) \leq \frac{\sqrt{3}}{18}$ and $\lim_{n \rightarrow +\infty} \lambda(B(2, n-2)) = \frac{\sqrt{3}}{18}$.

Proof of Lemma 4.10. Let $\vec{x} = \{x_1, x_2, \dots, x_n\}$ be an optimum vector of $\lambda(B(2, n-2))$. Let $x_1 + x_2 = a$ and $b = 1 - a$. Then

$$\begin{aligned} \lambda(B(2, n-2)) &\leq \frac{a^2(1-a)}{4} + a \left(\frac{1-a}{n-2} \right)^2 \binom{n-2}{2} \\ &\leq \frac{a^2(1-a)}{4} + \frac{a(1-a)^2}{2}. \end{aligned}$$

By Fact 4.9, $\lambda(B(2, n-2)) \leq \frac{\sqrt{3}}{18}$. Note that $' = '$ holds only if $a = \frac{3-\sqrt{3}}{3}$ and $n \rightarrow \infty$. \square

4.3 Preliminaries for the main Lemmas

In this section we introduce two hypergraphs and show their Lagrangian are less than $\frac{\sqrt{3}}{18}$. When giving some proofs in Section 5, we will change our hypergraph by replacing some edges such that the new hypergraph has non-decreasing Lagrangian and is isomorphic with the subgraph of the following two hypergraphs.

H₁ : the 3-graph with vertex set $[n]$ and edge set $E(H_1) = E(B(2, n-2)) \setminus \{2ij : i, j \in [n] \setminus \{1, 2, 3, 4, 5, 6\}\} \cup \{345, 346\}$.

H₂ : a 3-graph with vertex set $[n]$ and edge set $E(H_2) = E(B(2, n-2)) \setminus \{2ij : i, j \in D\} \cup \{34i : i \in D\}$, where D is a subset of $[n] \setminus \{1, 2, 3, 4\}$ with $|D| \geq 2$.

Lemma 4.11 $\lambda(H_1) < \frac{\sqrt{3}}{18}$.

Proof of Lemma 4.11. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimum vector of $\lambda(H_1)$. Assume that $x_1 = a$, $x_2 = b$, $x_3 + x_4 = c$, $x_5 + x_6 = d$, $e = 1 - a - b - c - d$. By Fact 4.4, we may assume that $x_5 = x_6 = \frac{d}{2}$ and

$x_3 = x_4 = \frac{c}{2}$. By Fact 4.5, we may assume that $a \geq b$. If $c = 0$ or $d = 0$, then $\lambda(H_1) < \lambda(B(2, n-2)) \leq \frac{\sqrt{3}}{18}$. If $b = 0$, then

$$\begin{aligned}\lambda(H_1) &\leq a\left(\frac{c^2}{4} + \frac{d^2}{4} + \frac{e^2}{2} + cd + ce + de\right) + \frac{c^2d}{4} \\ &\leq \frac{a(c+d+e)^2}{2} + \frac{(\frac{c}{2} + \frac{c}{2} + d+e)^3}{27} = \frac{a(1-a)^2}{2} + \frac{(1-a)^3}{27} = f(a). \\ f'(a) &= \frac{25a^2}{18} - \frac{16a}{9} + \frac{7}{18}.\end{aligned}$$

Note that $f(a)$ is increasing in $[0, \frac{7}{25}]$ and decreasing in $[\frac{7}{25}, 1]$, then $\lambda \leq f(\frac{7}{25}) < \frac{\sqrt{3}}{18}$. So we may assume that $a, b, c, d > 0$. By Fact 4.3,

$$\frac{\partial \lambda}{\partial x_1} + \frac{\partial \lambda}{\partial x_2} = \frac{\partial \lambda}{\partial x_5} + \frac{\partial \lambda}{\partial x_6}.$$

Note that

$$\frac{\partial \lambda}{\partial x_1} + \frac{\partial \lambda}{\partial x_2} \geq bc + bd + be + cd + ce + de + \frac{c^2}{4} + \frac{d^2}{4} + ac + ad + ae + cd + ce + de + \frac{c^2}{4} + \frac{d^2}{4},$$

and

$$\frac{\partial \lambda}{\partial x_5} + \frac{\partial \lambda}{\partial x_6} = ab + ac + ae + bc + be + (a+b)\frac{d}{2} + \frac{c^2}{4} + ab + ac + ae + bc + be + (a+b)\frac{d}{2} + \frac{c^2}{4}.$$

Therefore

$$2cd + 2ce + 2de + \frac{d^2}{2} \leq 2ab + ac + ae + bc + be.$$

If $a + b \leq d$, then $2ab \leq \frac{d^2}{2}$, $ac + bc \leq cd$ and $ae + be \leq de$. So we have $a = b$ and $e = c = 0$, a contradiction to $c > 0$. So we may assume that $a + b > d$, then

$$\begin{aligned}\lambda(H_1) &\leq ab(c+d+e) + a\left(\frac{c^2}{4} + \frac{d^2}{4} + \frac{e^2}{2} + cd + ce + de\right) + b\left(\frac{c^2}{4} + \frac{d^2}{4} + cd + ce + de\right) + \frac{c^2d}{4} \\ &\leq ab(1-a-b) + \frac{(a+b)(c+d+e)^2}{2} + \frac{c^2d}{4} - \frac{c^2a}{4} - \frac{c^2b}{4} \\ &< ab(1-a-b) + \frac{(a+b)(1-a-b)^2}{2} \\ &\leq \frac{(a+b)^2(1-a-b)}{4} + \frac{(a+b)(1-a-b)^2}{2}.\end{aligned}$$

By Fact 4.9, then $\lambda(H_1) < \frac{\sqrt{3}}{18}$. □

Lemma 4.12 $\lambda(H_2) \leq \frac{\sqrt{3}}{18}$.

Proof of Lemma 4.12. Let $\vec{x} = (x_1, x_2, \dots, x_n)$ be an optimum vector of $\lambda(H_2)$. Let $x_1 = a$, $x_2 = b$, $x_3 + x_4 = c$, $\sum_{v \in D} x_v = d$, $\sum_{v \in E} x_v = e$. We have

$$\begin{aligned}\lambda(H_2) &\leq ab(c+d+e) + a\left(\frac{c^2}{2} + \frac{d^2}{2} + \frac{e^2}{2} + cd + ce + de\right) + b\left(\frac{c^2}{4} + \frac{e^2}{2} + cd + ce + de\right) + \frac{c^2d}{4} \\ &= \lambda(a, b, c, d, e) = \lambda\end{aligned}$$

under the constraint

$$\begin{cases} a + b + c + d + e = 1, \\ a \geq 0, b \geq 0, c \geq 0, d \geq 0 \text{ and } e \geq 0. \end{cases} \quad (1)$$

Note that if $c = 0$ or $d = 0$, then

$$\lambda < \frac{(a+b)^2(1-a-b)}{4} + \frac{(a+b)(1-a-b)^2}{2},$$

by Fact 4.9, then $\lambda \leq \frac{\sqrt{3}}{18}$. So we may assume that $c, d > 0$. By Theorem 4.1, then $\frac{\partial \lambda}{\partial c} = \frac{\partial \lambda}{\partial d}$. By direct calculation,

$$\begin{aligned} \frac{\partial \lambda}{\partial c} &= ab + a(c + d + e) + b\left(\frac{c}{2} + d + e\right) + \frac{cd}{2} \\ \frac{\partial \lambda}{\partial d} &= ab + a(c + d + e) + b(c + e) + \frac{c^2}{4}, \end{aligned}$$

then $\frac{c}{2}(\frac{c}{2} + b) = d(\frac{c}{2} + b)$, so $c = 2d$.

By Fact 4.5, we may assume that $a \geq b$. We claim that $b > 0$. If $b = 0$, then

$$\begin{aligned} \lambda &= a\left(\frac{c^2}{2} + \frac{d^2}{2} + \frac{e^2}{2} + cd + ce + de\right) + \frac{c^2d}{4} \\ &\leq \frac{a(c + d + e)^2}{2} + \left(\frac{\frac{c}{2} + \frac{c}{2} + d + e}{3}\right)^3 \\ &= \frac{a(1-a)^2}{2} + \frac{(1-a)^3}{27} = f(a), \\ f'(a) &= \frac{25a^2}{18} - \frac{16a}{9} + \frac{7}{18}. \end{aligned}$$

Note that $f(a)$ is increasing in $[0, \frac{7}{25}]$ and decreasing in $[\frac{7}{25}, 1]$, then $\lambda \leq f(\frac{7}{25}) \leq \frac{\sqrt{3}}{18}$. So we may assume that $a, b > 0$. By Theorem 4.1, then $\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial b}$. Therefore $(a-b)(c+d+e) = \frac{c^2}{4} + \frac{d^2}{2}$.

We claim that $e > 0$. If $e = 0$, recall that we have shown that $c = 2d$ and $(a-b)(c+d) = \frac{c^2}{4} + \frac{d^2}{2}$, then $a = b + \frac{d}{2}$. Since $a + b + c + d = 1$, then $a = \frac{1}{2} - \frac{5d}{4}$ and $b = \frac{1}{2} - \frac{7d}{4}$. So

$$\begin{aligned} \lambda &= \frac{-53d^3 - 12d^2 + 12d}{16} \\ \lambda' &= \frac{-159d^2}{16} - \frac{3d}{2} + \frac{3}{4}. \end{aligned}$$

Note that λ is increasing in $[0, \frac{2\sqrt{57}-4}{53}]$ and decreasing in $[\frac{2\sqrt{57}-4}{53}, 1]$, so $\lambda \leq 0.094$.

Therefore we may assume that $a, b, c, d, e > 0$. Recall that $c = 2d$. By Theorem 4.1, then $\frac{\partial \lambda}{\partial d} = \frac{\partial \lambda}{\partial e}$. Since

$$\begin{aligned} \frac{\partial \lambda}{\partial d} &= ab + a(c + d + e) + b(c + e) + \frac{c^2}{4} \\ \frac{\partial \lambda}{\partial e} &= ab + a(c + d + e) + b(c + d + e), \end{aligned}$$

then $d = b$. So

$$\begin{aligned}\lambda &< \frac{(a+b)^2(1-a-b)}{4} + \frac{(a+b)(1-a-b)^2}{2} + \frac{c^2d}{4} - \frac{bc^2}{4} - \frac{bd^2}{2} \\ &< \frac{(a+b)^2(1-a-b)}{4} + \frac{(a+b)(1-a-b)^2}{2}.\end{aligned}$$

By Fact 4.9, then $\lambda \leq \frac{\sqrt{3}}{18}$. \square

5 Proofs of the main Lemmas

To complete the proof of Theorem 1.4, what remains is to show Lemma 2.1 to 2.3. In this section, we prove these lemmas. Throughout this section, let G be a dense $K_4^3 \cup e$ -free 3-graph on vertex set $[n]$ with Lagrangian $\lambda(G) > \frac{\sqrt{3}}{18}$. Let $\vec{x} = \{x_1, x_2, \dots, x_n\}$ be an optimum vector for $\lambda(G)$. Since $\lambda(K_6^3) = \frac{5}{54} \leq \frac{\sqrt{3}}{18}$, then $n \geq 7$.

Lemma 5.1 ($[1], [28]$) $\pi(K_4^3) \leq 0.5615$.

By Fact 4.6, Proposition 1.3 and Lemma 5.1, $\pi_\lambda(K_4^3) = \pi(K_4^3) \leq 0.5615 < 3! \frac{\sqrt{3}}{18} < 3!\lambda(G)$. So $K_4^3 \subseteq G$. Without loss of generality, assume that $\{1, 2, 3, 4\}$ forms a K_4^3 in G .

For $x, y \in V(G)$, let $N^*(x, y) = \{v : vxy \in E(G)\}$.

Claim 5.2 $K_5^3 \not\subseteq G$.

Proof of Claim 5.2. Assume that $K_5^3 \subseteq G$. Since $n \geq 7$, then there are $x, y \in [n] \setminus V(K_5^3)$, since G is dense, then there exists $z \in N^*(x, y)$. The edge xyz together with the K_5^3 contains a $K_4^3 \cup e$, a contradiction. \square

Claim 5.3 For $v \in [n]$, $\omega(G_v) \geq 3$ and $x_v < 1 - \sqrt{\frac{\sqrt{3}}{3} \frac{\omega(G_v)}{\omega(G_v) - 1}}$. Furthermore, if $v \in [n] \setminus \{1, 2, 3, 4\}$, then $3 \leq \omega(G_v) \leq 4$. Therefore $x_v < 0.0694$ if $\omega(G_v) = 3$, and $x_v < 0.12262$ if $\omega(G_v) = 4$.

Proof of Claim 5.3. For any $v \in [n]$, applying Fact 4.3 and Theorem 1.2, we have

$$\frac{\sqrt{3}}{6} < 3\lambda = \frac{\partial \lambda}{\partial x_v} \leq \left(\frac{1 - x_v}{\omega(G_v)} \right)^2 \binom{\omega(G_v)}{2}.$$

Then

$$x_v < 1 - \sqrt{\frac{\sqrt{3}}{3} \frac{\omega(G_v)}{\omega(G_v) - 1}}.$$

If $v \in \{1, 2, 3, 4\}$, then it's clear that $\omega(G_v) \geq 3$. Let $v \notin [n] \setminus \{1, 2, 3, 4\}$. If $\omega(G_v) \leq 2$, then $x_v < 0$, a contradiction. Therefore $\omega(G_v) \geq 3$. If $\omega(G_v) \geq 5$, then $\omega(G_v)$ contains at least 1 vertex in $[n] \setminus \{1, 2, 3, 4\}$, therefore G contains either a K_5^3 (if $\omega(G_v)$ contains only 1 vertex in $[n] \setminus \{1, 2, 3, 4\}$) or a $K_4^3 \cup e$ (if $\omega(G_v)$ contains at least 2 vertices in $[n] \setminus \{1, 2, 3, 4\}$), a contradiction. Therefore $3 \leq \omega(G_v) \leq 4$ for all $v \in [n] \setminus \{1, 2, 3, 4\}$. \square

Claim 5.4 For 2 vertices a and b in $[n]$, $x_a + x_b \leq \frac{3 - \sqrt{3}}{3}$.

Proof of Claim 5.4. Applying Fact 4.3, we have

$$\begin{aligned}
\frac{\sqrt{3}}{3} &\leq 6\lambda(G) = \frac{\partial\lambda(G)}{\partial x_a} + \frac{\partial\lambda(G)}{\partial x_b} \\
&\leq x_b(1 - x_a - x_b) + \left(\frac{1 - x_a - x_b}{n - 2}\right)^2 \binom{n-2}{2} + x_a(1 - x_a - x_b) + \left(\frac{1 - x_a - x_b}{n - 2}\right)^2 \binom{n-2}{2} \\
&\leq (x_a + x_b)(1 - x_a - x_b) + (1 - x_a - x_b)^2 \\
&= 1 - (x_a + x_b).
\end{aligned}$$

Therefore $x_a + x_b \leq \frac{3-\sqrt{3}}{3}$. □

Claim 5.5 *If $G - \{v\}$ is K_4^3 -free for some $v \in [n]$, then $x_v > 0.0848$.*

Proof of Claim 5.5. Applying Lemma 5.1 and Proposition 1.3, we have

$$\lambda(G - \{v\}, \vec{x}) \leq (1 - x_v)^3 \frac{0.5615}{6}.$$

Therefore

$$\begin{aligned}
\lambda(G) &\leq (1 - x_v)^3 \frac{0.5615}{6} + x_v \frac{\partial\lambda(G)}{\partial x_v} \\
&= (1 - x_v)^3 \frac{0.5615}{6} + 3x_v \lambda(G),
\end{aligned}$$

the last equality follows from Fact 4.3. So

$$\lambda(G) \leq \frac{0.5615}{6} \frac{(1 - x_v)^3}{1 - 3x_v}.$$

Note that $\frac{(1-x_v)^3}{1-3x_v}$ is increasing in $[0, \frac{1}{3})$. If $x_v \leq 0.0848$, then $\lambda(G) \leq 0.09622 \leq \frac{\sqrt{3}}{18}$, a contradiction. □

Let M_t^r be an r -graph on tr vertices with t disjoint edges.

Claim 5.6 $n \geq 8$.

Proof of Claim 5.6. Assume that and $V(G) = [7]$. Recall that $\{1, 2, 3, 4\}$ forms K_4^3 . By Claim 5.3, then $\omega(G_5), \omega(G_6), \omega(G_7) \leq 4$ and $x_5 + x_6 + x_7 \leq 3 \times 0.12262$. Therefore $x_1 + x_2 + x_3 + x_4 \geq 0.63214 > 4 \times 0.158$, then, without loss of generality, let $x_1 \geq 0.158$. By Claim 5.3, $x_v < 0.151$ if $\omega(G_v) \leq 5$. Therefore $\omega(G_1) = 6$. Since G is $K_4^3 \cup e$ -free, then $G - \{1\}$ is M_2^3 -free. Hefetz and Keevash ([13]) proved that $\pi_\lambda(M_2^3) \leq \frac{12}{25}$. So $\lambda(G - \{1\}) \leq \frac{2}{25}$. Therefore

$$\begin{aligned}
\lambda(G) &\leq x_1 \left(\frac{1 - x_1}{6}\right)^2 \binom{6}{2} + \frac{2}{25}(1 - x_1)^3 \\
&= \frac{5}{12}x_1(1 - x_1)^2 + \frac{2}{25}(1 - x_1)^3 = f(x_1). \\
f'(x_1) &= \frac{5}{12}(1 - x_1)^2 - \frac{5}{6}x_1(1 - x_1) - \frac{6}{25}(1 - x_1)^2 \\
&= \frac{(1 - x_1)(53 - 303x_1)}{300}.
\end{aligned}$$

So $f(x_1)$ is increasing in $[0, \frac{53}{303}]$ and decreasing in $[\frac{53}{303}, 1]$, then $f(x_1) \leq f(\frac{53}{303}) < 0.095 < \frac{\sqrt{3}}{18}$. \square

5.1 G is K_5^{3-} -free

Lemma 5.7 G is K_5^{3-} -free.

Proof of Lemma 5.7. Assume that $K_5^{3-} \subseteq G$ with vertex set $\{1, 2, 3, 4, 5\}$ and $345 \notin E(G)$. Since G is $K_4^3 \cup e$ -free, then for any $x, y \in [n] \setminus \{1, 2, 3, 4, 5\}$, we have $N^*(x, y) \subseteq \{1, 2\}$. By Claim 5.6, $n \geq 8$. Recall that $\vec{x} = (x_1, x_2, \dots, x_n)$ is an optimal vector for G .

Case 1. $x_1, x_2 \geq x_3, x_4, x_5$.

We say that a vertex $v \in [n] \setminus \{1, 2, 3, 4, 5\}$ is a *good vertex* if for the set of edges in

$$B_v = \{v_{ij} \in E(G) : \{i, j\} \in \{3, 4, 5\}^{(2)}\},$$

there exist the same number of triples in

$$A_v = \{v_{ij} \in E(G^c) : \{i, j\} \cap \{1, 2\} \neq \emptyset, \{i, j\} \subset \{1, 2, 3, 4, 5\}\}$$

such that $\sum_{v_{ij} \in B_v} x_v x_i x_j \leq \sum_{v_{ij} \in A_v} x_v x_i x_j$. In this case, we say that B_v can be replaced by A_v . Otherwise, we call v a *bad vertex*.

We call $v34, v35, v45$ bad edges for $v \in [n] \setminus \{1, 2, 3, 4, 5\}$. Note that for a good vertex $v \in [n] \setminus \{1, 2, 3, 4, 5\}$, replacing B_v by A_v in G does not reduce the Lagrangian. Let B be the set of all *bad vertices*. If $B = \emptyset$, then we can replace B_v by A_v in G for each $v \in [n] \setminus \{1, 2, 3, 4, 5\}$, obtain G^0 , and all edges in $E(G^0)$ are incident to 1 or 2. So $G^0 \subseteq B(2, n-2)$. Hence $\lambda(G) \leq \lambda(G^0) \leq \lambda(B(2, n-2)) = \frac{\sqrt{3}}{18}$, a contradiction.

Case 1.1. There exists $v \in B$ such that exactly 1 of $v34, v35, v45$ is in $E(G)$.

Without loss of generality, let $v34 \in E(G)$. Since G is K_5^3 -free, then at least 1 of $v12, v13, v23, v14, v24 \notin E(G)$. Since $x_1, x_2 \geq x_3, x_4$, then $v34$ can be replaced by that missing edge, so $v \notin B$, a contradiction.

Case 1.2. There exists $v \in B$ such that exactly 2 of $v34, v35, v45$ in $E(G)$.

Without loss of generality, let $v34, v35 \in E(G)$. Since $\{v, 1, 2, 3, 4\}$ can't form a K_5^3 , then at least 1 of $v12, v13, v23, v14, v24$ is not in $E(G)$.

If $v12 \notin E(G)$, then $v13, v14, v23, v24 \in E(G)$. Otherwise we can replace $v34, v35$ by that missing edge and $v12$ with $\lambda(G)$ non-decreasing. Contradict to $v \in B$. Therefore $\{v, 1, 3, 4\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction to that G is dense.

If $v14 \notin E(G)$ (or $v24 \notin E(G)$), then $\{v, 1, 3, 5\}, \{v, 2, 3, 5\}$ form K_4^3 . Otherwise we can replace $v34, v35$ by $v14$ and one of the missing edges, a contradiction to $v \in B$. Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction to that G is dense.

If $v23 \notin E(G)$ (or $v13 \notin E(G)$), then $\{v, 1, 3, 4\}, \{v, 1, 3, 5\}, \{v, 1, 2, 4\}, \{v, 1, 2, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, we have $N^*(x, y) = \{1\}$ and $x34, x35 \notin E(G)$. So for any $x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$ only, possibly, $x45 \in E(G)$ is a bad edge incident to x . By the proof of *Case 1.1*, $x \notin B$, so $B = \{v\}$. Replacing $v35$ by $v23$, adding 345 and replacing B_x by A_x for all $x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, we obtain G^0 . Note that G^0 is contained in an isomorphic copy of H_1 (view v in G^0 as 6 in H_1), then $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction to $\lambda(G) > \frac{\sqrt{3}}{18}$.

Case 1.3. There exists $v \in B$ such that $v34, v35, v45 \in E(G)$.

Since $v \in B$, then at most two of $\{v12, v13, v23, v14, v24, v15, v25\}$ are not in $E(G)$, otherwise we can replace B_v by the three missing edges. We claim that there are 2 of those edges not in $E(G)$. Suppose that there is only 1 of those edges missing in $E(G)$. If only $v12 \notin E(G)$, then $\{v, 1, 3, 4\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free and $|G| \geq 8$, then there are $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$ such that $N^*(x, y) = \emptyset$, a contradiction to that G is dense. If only $vi j \notin E(G)$ for some $i \in \{1, 2\}$ and some $j \in \{3, 4, 5\}$, then $\{v, 1, 2, k, l\}$ form a K_5^3 , where $\{k, l\} = \{3, 4, 5\} \setminus \{j\}$, a contradiction to Claim 5.2. So we may assume that there are two of those edges not in $E(G)$.

If $v12 \notin E(G)$ and $vi j \notin E(G)$, where $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, then $\{v, 1, k, l\}$ and $\{v, 2, k, l\}$ ($\{k, l\} = \{3, 4, 5\} \setminus \{j\}$) form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense.

If $v1i, v2j \notin E(G)$, where $i, j \in \{3, 4, 5\}$, then $\{v, 1, p, q\}$ and $\{v, 2, s, t\}$ form K_4^3 , where $\{p, q\} = \{3, 4, 5\} \setminus \{i\}$ and $\{s, t\} = \{3, 4, 5\} \setminus \{j\}$. Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense.

If $vi j, vik \notin E(G)$, where $i \in \{1, 2\}$ and $\{j, k\} \in \{3, 4, 5\}^{(2)}$, without loss of generality, assume that $v13, v14 \notin E(G)$, then $\{v, 2, 3, 4\}, \{v, 2, 3, 5\}, \{v, 2, 4, 5\}, \{v, 1, 2, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, we have $N^*(x, y) = \{2\}$ and $x34 \notin E(G)$. By the proof of *Case 1.2*, $x \notin B$, so $B = \{v\}$. Replacing B_x by A_x for all $x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, replacing $v35, v45$ by $v13, v14$ and adding 345, we obtain G^0 which is contained in an isomorphic copy of H_1 (view v in G^0 as 6 in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction to $\lambda(G) > \frac{\sqrt{3}}{18}$.

Case 2. $x_1, x_3 \geq x_2, x_4, x_5$.

A vertex $v \in [n] \setminus \{1, 2, 3, 4, 5\}$ is a *good vertex* if for the edges in

$$B_v = \{vi j \in E(G) : i j \in \{2, 4, 5\}^{(2)}\},$$

there exist the same number of triples in

$$A_v = \{vi'j' \in E(G^c) : \{i', j'\} \cap \{1, 3\} \neq \emptyset, \{i', j'\} \subset \{1, 2, 3, 4, 5\}\}$$

such that the substitute $vi'j'$ for $vi j \in B_v$ satisfies $|\{i', j'\} \cap \{i, j\}| = 1$ or $\{i', j'\} = \{1, 3\}$. Note that $\sum_{vi j \in B_v} x_v x_i x_j \leq \sum_{vi j \in A_v} x_v x_i x_j$. In this case, we say that B_v can be replaced by A_v . Otherwise, we call v a *bad vertex*. We call $v24, v25, v45$ *bad edges* for $v \in [n] \setminus \{1, 2, 3, 4, 5\}$.

Let B be the vertex set containing all *bad vertices*.

Case 2.1. There exists $v \in B$ and exactly one of $\{v24, v25, v45\}$ is in $E(G)$.

Case 2.1.1. $v45 \in E(G)$.

Since $v \in B$, then $v13, v14, v34, v15, v35 \in E(G)$. Therefore $\{v, 1, 3, 4\}, \{v, 1, 3, 5\}, \{v, 1, 4, 5\}$ form K_4^3 , then for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$ we have $N^*(x, y) = \{1\}$ and $x23, x24, x25 \notin E(G)$. Otherwise $K_4^3 \cup e \subseteq G$. For any $u \in B$, since $u24, u25 \notin E(G)$, then $u45 \in E(G)$ and $\{u, 1, 3, 4\}, \{u, 1, 3, 5\}, \{u, 1, 4, 5\}$ form K_4^3 .

If $B = \{v\}$, then let $E = [n] \setminus \{1, 2, 3, 4, 5, v\}$. Note that $N^*(x, y) = 1$ for $x, y \in E$. Replacing B_x by A_x for $x \in E$, we obtain G^0 . Then G^0 is contained in an isomorphic copy of H_1 (view 3 in G^0 as 2 in H_1 , view 245, $v45$ in G^0 as 345, 346 in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

If $B = \{v, u\}$, then let $E = [n] \setminus (B \cup \{1, 3, 4, 5\})$. Note that $2 \in E$. Since x is a good vertex for $x \in E \setminus \{2\}$, then $x45$ can be replaced by 1 of $\{x13, x14, x15, x34, x35\}$. Note that $N^*(x, y) = 1$ for $x, y \in E \cup \{2\}$. Replace $uv2$ by $uv3$, then the obtained G^0 is contained in an isomorphic copy of H_1 (view 3 in G^0 as 2 in H_1 , view $v45, u45$ in G^0 as $345, 346$ in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

If $|B| \geq 3$, then $N^*(x, y) = \{1\}$ for $x, y \in B$. Otherwise if $xy2 \in E(G)$, then $\{z, 1, 4, 5\} \cup \{x, y, 2\}$ forms $K_4^3 \cup e$ in G for $x, y, z \in B$. Let $E = [n] \setminus (B \cup \{1, 3, 4, 5\})$. Replacing B_x by A_x for each $x \in E$, we obtain G^0 which is contained in an isomorphic copy of H_2 (view B in G^0 as D in H_2 , view 3 in G^0 as 2 in H_2 , view $i45 (i \in B)$ in G^0 as $i34 (i \in D)$ in H_2). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_2) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.12, a contradiction.

Case 2.1.2. $v2i \in E(G)$, where i is 4 or 5.

Since $v \in B$, then $v12, v23, v1i, v3i, v13 \in E(G)$, so $\{v, 1, 2, 3, i\}$ forms a K_5^3 , a contradiction to Claim 5.2.

Case 2.2. There exists $v \in B$ and exactly two of $\{v24, v25, v45\}$ are in $E(G)$.

Case 2.2.1. $v24, v25 \in E(G)$.

Since $\{v, 1, 2, 3, 4\}$ can't form a K_5^3 , then at least one of $\{v12, v23, v14, v34, v13\}$ is not in $E(G)$.

If $v12 \notin E(G)$, then $v13, v14, v23, v34 \in E(G)$. Otherwise we can replace $v24, v25$ by $v12$ and that missing edge, a contradiction to $v \notin B$. Therefore $\{v, 1, 3, 4\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction to that G is dense.

If $v23 \notin E(G)$, then $v12, v13, v14, v15, v34, v35 \in E(G)$. Therefore $\{v, 1, 2, 4\}, \{v, 1, 2, 5\}, \{v, 1, 3, 4\}, \{v, 1, 3, 5\}$ form K_4^3 , then for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$ we have $N^*(x, y) = \{1\}$ and $x24, x25 \notin E(G)$, by the proof of *Case 2.1*, $x \notin B$, so $B = \{v\}$. Let $E = [n] \setminus \{1, 2, 3, 4, 5, v\}$. Deleting $v25$, adding $v23$, and replacing B_x by A_x for $x \in E$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view 3, 2 in G^0 as 2, 3 in H_1 respectively, view $v24$ in G^0 as 346 in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

If $v34 \notin E(G)$ (or $v14 \notin E(G)$ or both), then $\{v12, v13, v15, v23, v35\} \subset E(G)$, so $\{v, 1, 3, 5\}, \{v, 2, 3, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction to that G is dense.

If $v13 \notin E(G)$, then $\{v12, v14, v15, v23, v34, v35\} \subseteq E(G)$, otherwise we can replace $v24, v25$ by that missing edge and $v13$, so $\{v, 1, 2, 4\}, \{v, 1, 2, 5\}, \{v, 2, 3, 4\}, \{v, 2, 3, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \{2\}$ and $x34, x35, x14 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. Note that $x \notin B$ for $x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, otherwise we can replace the all bad edges incident to x by $x34, x35, x14$. Deleting $v25$, adding $v13$, replacing B_x by A_x and replacing $xv2, xy2$ by $xv3, xy1$ for $x, y \in [n] \setminus \{1, 2, 3, 4, 5, v\}$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view 3, 2 in G^0 as 2, 3 in H_1 respectively, view $v24$ in G^0 as 346 in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

Case 2.2.2. $v24, v45 \in E(G)$ (the proof for $v25, v45 \in E(G)$ is identical).

Since $\{v, 1, 2, 3, 4\}$ can't form a K_5^3 , then at least 1 of $\{v23, v12, v13, v14, v34\}$ is not in $E(G)$.

If $v23 \notin E(G)$, since $v \in B$, then $v13, v14, v15, v34, v35 \in E(G)$. So $\{v, 1, 3, 4\}, \{v, 1, 3, 5\}, \{v, 1, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \{1\}$ and $x23, x24, x25 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. By the proof of case 2.1, then $x \notin B$, so $B = \{v\}$. Let $E = [n] \setminus \{1, 2, 3, 4, 5, v\}$. Deleting $v24$, adding $v23$, deleting $xv2$, adding $xv3$ for $x \in E$, and replacing B_x by A_x for $x \in E$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view 3 in G^0 as 2 in H_1 , view $\{245, v45\}$ in G^0 as

$\{345, 346\}$ in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction. So we may assume that $v23 \in E(G)$.

If $v12 \notin E(G)$, since $v \in B$, then $v13, v14, v23, v34 \in E(G)$, so $\{v, 1, 3, 4\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. So we may assume that $v12 \in E(G)$.

If $v13 \notin E(G)$, since $v \in B$, then $v14, v15, v23, v34 \in E(G)$, so $\{v, 1, 4, 5\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. So we may assume that $v13 \in E(G)$.

If $v14 \notin E(G)$, since $v \in B$, then $v13, v15, v23, v34, v35 \in E(G)$, so $\{v, 1, 3, 5\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. So we may assume that $v14 \in E(G)$.

If $v34 \notin E(G)$, since $v \in B$, then $v12, v13, v14, v15, v23, v35 \in E(G)$, so $\{v, 1, 2, 3\}, \{v, 1, 2, 4\}, \{v, 1, 3, 5\}, \{v, 1, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \{1\}$ and $x24, x45 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, which means, by the proof of *Case 2.1.*, $x \notin B$ and $B = \{v\}$. Let $E = [n] \setminus \{1, 2, 3, 4, 5, v\}$. Deleting $v24$, adding $v34$, replacing B_x by A_x , deleting $xv2$ and adding $xv3$ for $x \in E$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view 3 in G^0 as 2 in H_1 , view $\{245, v45\}$ in G^0 as $\{345, 346\}$ in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

Case 2.3. There exists $v \in B$ and $v24, v25, v45 \in E(G)$.

Since $v \in B$, then at most 2 of $\{v12, v13, v14, v15, v23, v34, v35\}$ are not in $E(G)$, otherwise we can replace $B_v = \{v24, v25, v45\}$ by those 3 missing edges in A_v . We claim that exactly 2 of $\{v12, v13, v14, v15, v23, v34, v35\}$ are not in $E(G)$. Otherwise there is only 1 of those edges not in $E(G)$. If only $v12 \notin E(G)$ (or only $v23 \notin E(G)$), then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction to G being dense. If only $v13$ (or one of $\{v14, v15, v34, v35\}$) is not in $E(G)$, then $\{v, 1, 2, 4, 5\}$ forms an K_5^3 , a contradiction. So we can assume that there are exactly two of $\{v12, v13, v14, v15, v23, v34, v35\}$ not in $E(G)$.

If $v12 \notin E(G)$, and $v23 \notin E(G)$ (the discussion for $v23$ replaced by 1 of $\{v13, v14, v15, v34, v35\}$ is similar), then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. So we may assume that $v12 \in E(G)$.

If $v13 \notin E(G)$, and $v23 \notin E(G)$ (or 1 of $\{v34, v35\} \notin E(G)$), then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . Therefore $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction. If $v13 \notin E(G)$ and $v14 \notin E(G)$ (or $v15 \notin E(G)$, since 4 and 5 are symmetric, we only discuss $v14$ here), then $\{v, 2, 3, 4\}, \{v, 2, 3, 5\}, \{v, 2, 4, 5\}, \{v, 1, 2, 5\}$ form K_4^3 . Therefore $N^*(x, y) = \{2\}$ and $x13, x14, x15, xv3 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. Since we can replace B_x by $x13, x14, x15$ for $x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, then $x \notin B$, so $B = \{v\}$. Let $E = [n] \setminus \{1, 2, 3, 4, 5, v\}$. Deleting $v24$ and $v25$, adding $v13$ and $v14$, deleting $xv2$, adding $xv3$, deleting $xy2$, adding $xy1$ for $x, y \in E$, and replacing B_x by A_x for $x \in E$, we obtain G^0 . Note that G^0 is contained in an isomorphic copy of H_1 (view 3 in G^0 as 2 in H_1 , view $\{245, v45\}$ in G^0 as $\{345, 346\}$ in H_1). So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_1) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction. So we may assume that $v13 \in E(G)$.

If $v14 \notin E(G)$, and $v23 \notin E(G)$ (or $v34 \notin E(G)$), then $\{v, 1, 3, 5\}$ and $\{v, 2, 4, 5\}$ form K_4^3 . Therefore $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction. If $v14, v15 \notin E(G)$ ($v14, v35 \notin E(G)$ is similar), then $\{v, 1, 2, 3\}, \{v, 2, 3, 4\}, \{v, 2, 3, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . Therefore $N^*(x, y) = \{2\}$ and $x13, x14, x15, x45 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. So we can replace B_x by $x13, x14, x15$ for

$x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, then $x \notin B$, and $B = \{v\}$. Let $E = [n] \setminus \{1, 2, 3, 4, 5, v\}$. Deleting $v24$ and $v25$, adding $v14$ and $v15$, deleting $xv2$, adding $xv3$, deleting $xy2$, adding $xy1$ for $x, y \in E$, and replacing B_x by A_x for $x \in E$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view 3 in G^0 as 2 in H_1 , view $\{245, v45\}$ in G^0 as $\{345, 346\}$ in H_1). By Lemma 4.11, then $\lambda(G) \leq \lambda(G^0) \leq \frac{\sqrt{3}}{18}$, a contradiction. So we may assume that $v14 \in E(G)$.

Similar to $v14$, we may assume that $v15 \in E(G)$, but then $\{v, 1, 2, 4, 5\}$ forms a K_5^3 , a contradiction to Claim 5.2.

Case 3. $x_3, x_4 \geq x_1, x_2, x_5$.

A vertex $v \in [n] \setminus \{1, 2, 3, 4, 5\}$ is a *good vertex* if for the edges in

$$B_v = \{vij \in E(G) : ij \in \{1, 2, 5\}^{(2)}\},$$

there exist the same number of triples in

$$A_v = \{vi'j' \in E(G^c) : \{i', j'\} \cap \{3, 4\} \neq \emptyset, \{i', j'\} \subset \{1, 2, 3, 4, 5\}\}$$

such that the substitute $vi'j'$ for $vij \in B_v$ satisfies $|\{i', j'\} \cap \{i, j\}| = 1$ or $\{i', j'\} = \{3, 4\}$. Note that $\sum_{vij \in B_v} x_v x_i x_j \leq \sum_{vi'j' \in A_v} x_v x_{i'} x_{j'}$. In this case, we say that B_v can be replaced by A_v . Otherwise we call v a *bad vertex*. We call $v12, v15, v25$ *bad edges* for $v \in [n] \setminus \{1, 2, 3, 4, 5\}$.

Let B be the set of all *bad vertices*. Let $E = [n] \setminus (B \cup \{1, 2, 3, 4, 5\})$.

Case 3.1. There exists $v \in B$ and there is exactly 1 of $v12, v15, v25 \in E(G)$.

Case 3.1.1. $v12 \in E(G)$.

Since $v \in B$, then $\{v13, v14, v23, v24, v34\} \subseteq E(G)$, so $\{v, 1, 2, 3, 4\}$ forms a K_5^3 , a contradiction to Claim 5.2.

Case 3.1.2. $v15 \in E(G)$ (the case $v25 \in E(G)$ is similar).

Since $v \in B$, then $v13, v14, v34, v35, v45 \in E(G)$. So $\{v, 1, 3, 4\}, \{v, 1, 3, 5\}, \{v, 1, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \{1\}$ and $v12, v25, x23, x24, x25 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. Therefore for all $u \in B$, only possibly, $u12, u15 \in E(G)$. Since we can replace $u12$ by $u23$ and $u \in B$, then $u15$ can't be replaced. So $\{u, 1, 3, 4\}, \{u, 1, 3, 5\}, \{u, 1, 4, 5\}$ form K_4^3 .

If $|B| = 1$, i.e. $B = \{v\}$, then replacing B_x by A_x , deleting $xv1, xv2, xy1$, adding $xv3, xv4, xy3$ for all $x, y \in E$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view 3, 4 in G^0 as 1, 2 in H_1 respectively, view $\{v15, 125\}$ in G^0 as $\{345, 346\}$ in H_1). Then $\lambda(G) \leq \lambda(G^0) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

If $|B| = 2$, i.e. $B = \{v, v'\}$, then $v'15 \in E(G)$ (or $v'12, v'15 \in E(G)$ but we can replace $v'12$ by $v'23$). Deleting $vv'1, vv'2$, adding $vv'3, vv'4$, and replacing $xy1, xv1, xv'1, x12, 125$ by $xy3, xv3, xv'3, x23, 345$ respectively for all $x, y \in E$, we obtain G^0 . View 3, 4, 1, 5, v, v' in G^0 as 1, 2, 3, 4, 5, 6 in H_1 , respectively. Note that $N^*(x, 2) = \{3\}$ in G^0 for $x \in E$, so G^0 is contained in an isomorphic copy of H_1 . Hence $\lambda(G) \leq \lambda(G^0) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.11, a contradiction.

If $|B| \geq 3$, then $vv'2, xv2 \notin E(G)$ for any $v, v' \in B$ and $x \in E$, since otherwise $\{v'', 1, 3, 4\} \cup \{v, v', 2\}$ forms $K_4^3 \cup e$ for $v'' \in B$ and $x \in E$. Replacing $vv'1$ by $vv'3$, deleting $xy1, xv1$, adding $xy3, xv3$ for all $x, y \in E$ and $v, v' \in B$, and replacing B_x by A_x for $x \in E$, we obtain G^0 . So $N^*(v, v') = \{3\}$ for $v, v' \in B$ in G^0 . View $\{3, 4\}$ in G^0 as $\{1, 2\}$ in H_2 , view $v15$ in G^0 ($v \in B$) as $i34$ ($i \in D$) in H_2 ,

then G^0 is contained in an isomorphic copy of H_2 . So $\lambda(G) \leq \lambda(G^0) \leq \lambda(H_2) \leq \frac{\sqrt{3}}{18}$ by Lemma 4.12, a contradiction.

Case 3.2. $v \in B$ and exactly 2 of $v12, v15, v25 \in E(G)$.

Case 3.2.1. $v12, v15 \in E(G)$. (The case that $v12, v25 \in E(G)$ is similar.)

Since $\{v, 1, 2, 3, 4\}$ can't form a K_5^3 , then there is at least 1 of $\{v13, v14, v23, v24, v34\}$ not in $E(G)$. Since $v \in B$, then we may assume that $v35, v45 \in E(G)$. Otherwise we may replace $v12, v15$ by 1 of $\{v13, v14, v23, v24, v34\}$ and 1 of $v35, v45 \in E(G)$, a contradiction. If $v13 \notin E(G)$ (or $v14 \notin E(G)$), since $v \in B$, then $\{v, 2, 3, 4\}$ and $\{v, 1, 4, 5\}$ (or $\{v, 1, 3, 5\}$) form K_4^3 , then $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. So we may assume that $v13, v14 \in E(G)$.

If $v23 \notin E(G)$ (or $v24 \notin E(G)$ or both), since $v \in B$, then $v34 \in E(G)$. So $\{v, 1, 3, 4\}, \{v, 1, 3, 5\}, \{v, 1, 4, 5\}$ form K_4^3 . So $N^*(x, y) = \{1\}$ and $x23, x24, x25 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. If $v' \in B \setminus \{v\}$, note that $v'25 \notin E(G)$, only possibly, $v'12, v'15 \in E(G)$. And $v'12$ can be replaced by $v'23$, so $v'15$ can't be replaced, therefore $\{v', 1, 3, 4\}, \{v', 1, 3, 5\}, \{v', 1, 4, 5\}$ form K_4^3 .

If $B = \{v\}$, then replacing B_x by A_x for $x \in E$, deleting $xy1, xv1, xv2$, and adding $xy3, xv3, xv4$ for $x, y \in E$, we obtain G^0 which is contained in an isomorphic copy of H_1 (view $3, 4, v$ in G^0 as $1, 2, 6$ in H_1 respectively, view $\{v15, 125\}$ in G^0 as $\{345, 346\}$ in H_1), a contradiction.

If $B = \{v, v'\}$, then replace $v12$ by $v23$. Delete $125, 1vv', 2vv', xy1$ and add $345, 3vv', 4vv', xy3$ for $x, y \in E$, respectively. Replace $x12$ by $x23$. Since x is a good vertex for $x \in E$, then $x15$ can be replaced by 1 of the missing edges in $\{x13, x14, x34, x35, x45\}$. Let G^0 be the resulting 3-graph, note that $N^*(x, y) = \{3\}$ for $x, y \in E \cup \{2\}$ in G^0 (view $3, 4, 1, 5, v, v'$ in G^0 as $1, 2, 3, 4, 5, 6$ in H_1 respectively). Then G^0 is contained in an isomorphic copy of H_1 , a contradiction.

If $|B| \geq 3$, then replace $v12$ by $v23$ for $v \in B$. Delete 125 and add 345 for $x \in E$. Replace B_x by A_x for $x \in E$. Since G is $K_4^3 \cup e$ -free, then $N^*(v, v') = \{1\}$ for any $v, v' \in B$. Replace $vv'1$ by $vv'3$ for $v, v' \in B$. Let G^0 be the resulting 3-graph. Note that $N^*(x, y) = \{3\}$ for $x, y \in B$ in G^0 . View $\{3, 4\}$ in G^0 as $\{1, 2\}$ in H_2 , view $v15$ in G^0 ($v \in B$) as $i34$ ($i \in D$) in H_2 . Then G^0 is contained in an isomorphic copy of H_2 , a contradiction.

So we may assume that $v23, v24 \in E(G)$, then $v34 \notin E(G)$. Therefore $\{v, 1, 2, 3\}, \{v, 1, 2, 4\}, \{v, 1, 3, 5\}, \{v, 1, 4, 5\}$ form K_4^3 . So $N^*(x, y) = \{1\}$ and $x23, x24, x35, x45 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, so $B = \{v\}$. Deleting $v12, 1xv, 2xv, xy1$, adding $v34, 3xv, 4xv, xy3$ for $x, y \in E$, replacing B_x by A_x for $x \in E$, we obtain G^0 . View $\{3, 4\}$ in G^0 as $\{1, 2\}$ in H_1 , view $\{v15, 125\}$ in G^0 as $\{345, 346\}$ in H_1 . Then G^0 is contained in an isomorphic copy of H_1 , a contradiction.

Case 3.2.2. $v15, v25 \in E(G)$.

If $v34 \notin E(G)$, since $v \in B$, then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. Therefore $v34 \in E(G)$.

If $v35 \notin E(G)$ (or $v45 \notin E(G)$), then $\{v, 1, 3, 4\}, \{v, 2, 3, 4\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. Therefore $v35, v45 \in E(G)$.

If $v13 \notin E(G)$ (or $v14 \notin E(G)$ or neither $v13$ nor $v14$ is in $E(G)$ or $v23 \notin E(G)$ or $v24 \notin E(G)$ or neither $v23$ nor $v24$ is in $E(G)$), then $\{v, 2, 3, 4\}, \{v, 2, 3, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . So $N^*(x, y) = \{2\}$ and $x13, x14, x15 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. By *Case 3.2.1*, $x \notin B$ for $x \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, so $B = \{v\}$. Replacing B_x by A_x for $x \in E$, deleting $xy2, v15, 1xv, 2xv$, and adding $xy3, v13, 3xv, 4xv$, respectively, we obtain G^0 . Note that $N^*(x, y) = \{3\}$ for $x, y \in E$ in G^0 . View $\{3, 4\}$ in G^0 as $\{1, 2\}$ in H_1 , view $\{v25, 125\}$ in G^0 as $\{345, 346\}$ in H_1 . Then G^0 is contained in an isomorphic copy of H_1 , a

contradiction.

Case 3.3. There exists $v \in B$ and $v12, v15, v25 \in E(G)$.

Since $v \in B$, then there are at most 2 of $\{v13, v14, v23, v24, v34, v35, v45\}$ not in $E(G)$. We claim that there are exactly 2 of those edges not in $E(G)$. Otherwise if there is at most 1 of $\{v13, v14, v23, v24, v34, v35, v45\}$ not in $E(G)$, without loss of generality, say at most $v13 \notin E(G)$, then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . So $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction. So we may assume that there are exactly 2 of $v13, v14, v23, v24, v34, v35, v45$ not in $E(G)$.

Assume that $v34 \notin E(G)$. If $v13 \notin E(G)$ (the case that 1 of $\{v14, v23, v24\}$ is not in $E(G)$ is similar), then $\{v, 1, 4, 5\}, \{v, 2, 3, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. If $v35 \notin E(G)$ (or $v45 \notin E(G)$), then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. So we may assume that $v34 \in E(G)$.

Assume that 1 of $\{v13, v14, v23, v24\}$ is not in $E(G)$. Without loss of generality, let $v13 \notin E(G)$. If $v14 \notin E(G)$, then $\{v, 1, 2, 5\}, \{v, 2, 3, 4\}, \{v, 2, 3, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . So $N^*(x, y) = \{2\}$ and $x13, x14, x34, x35 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. Therefore $B = \{v\}$. Replacing $v12, v15$ by $v13, v14$, replacing B_x by A_x for $x \in E$, deleting $1xv, 2xv, xy2$ and adding $3xv, 4xv, xy3$ for $x, y \in E$, respectively. Let G^0 be the resulting 3-graph. Note that $N^*(x, y) = \{3\}$ for $x, y \in E$ in G^0 . View $\{3, 4\}$ in G^0 as $\{1, 2\}$ in H_1 , view $\{v25, 125\}$ in G^0 as $\{345, 346\}$ in H_1 . Then G^0 is contained in an isomorphic copy of H_1 , a contradiction. If $v23 \notin E(G)$ (or $v35 \notin E(G)$), then $\{v, 1, 4, 5\}, \{v, 2, 4, 5\}$ form K_4^3 . So $N^*(x, y) = \emptyset$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, a contradiction. If $v24 \notin E(G)$, then $\{v, 1, 4, 5\}, \{v, 2, 3, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. If $v45 \notin E(G)$, then $\{v, 1, 2, 4\}, \{v, 1, 2, 5\}, \{v, 2, 3, 4\}, \{v, 2, 3, 5\}$ form K_4^3 . So $N^*(x, y) = \{2\}$ and $x14, x15, x34, x35 \notin E(G)$ for all $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$. Therefore $B = \{v\}$. Replacing $v12, v25$ by $v13, v45$, replacing B_x by A_x for $x \in E$, deleting $1xv, 2xv, xy2$ and adding $3xv, 4xv, xy3$ for $x, y \in E$, respectively. Let G^0 be the resulting 3-graph. Note that $N^*(x, y) = \{3\}$ for $x, y \in E$ in G^0 . View $\{3, 4\}$ in G^0 as $\{1, 2\}$ in H_1 , view $\{v15, 125\}$ in G^0 as $\{345, 346\}$ in H_1 . Then G^0 is contained in an isomorphic copy of H_1 , a contradiction. So it's sufficient to consider $v35, v45 \notin E(G)$. However $\{v, 1, 3, 4\}, \{v, 2, 3, 4\}$ form K_4^3 in this situation. Since G is $K_4^3 \cup e$ -free, then $N^*(x, y) = \emptyset$ for $x, y \in [n] \setminus \{v, 1, 2, 3, 4, 5\}$, contradicting to G being dense. \square

5.2 G does not contain two copies of K_4^3 sharing two vertices

Before giving the proof of Lemma 2.1, we will prove the following Lemmas.

Lemma 5.8 G is X_4 -free.

Proof of Lemma 5.8. Assume that G contains an X_4 with the vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{1, 2, 7, 8\}, \{1, 2, 9, 10\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then there is no edge in $V(G) \setminus \{1, 2\}$. Therefore G is a subgraph of $B(2, n - 2)$. By Lemma 4.10, $\lambda(G) \leq \frac{\sqrt{3}}{18}$. \square

Lemma 5.9 G is X_3 -free.

Proof of Lemma 5.9. Assume that G contains an X_3 with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{1, 2, 7, 8\}$ form K_4^3 . Denote $C = \{3, 4, 5, 6, 7, 8\}$, $D = \{x \in [n] \setminus \{1, 2, 3, 4, 5, 6, 7, 8\} : x12 \in$

$E(G)\}$ and $E = [n] \setminus (\{1, 2\} \cup C \cup D)$. Note that $x12 \notin E(G)$ for $x \in E$. Since G is $K_4^3 \cup e$ -free and X_4 -free, then $N^*(x, y) = \{1\}$ or $\{2\}$ for $x, y \in D$, and there is no edge between C and $D \cup E$. And if there is an edge e_1 in C , then $|e_1 \cap \{3, 4\}| = |e_1 \cap \{5, 6\}| = |e_1 \cap \{7, 8\}| = 1$, i.e. $G[C]$ is a 3-partite 3-graph and $\lambda(G[C]) \leq \frac{1}{27}$. Set

$$x_1 = a, \quad x_2 = b, \quad \sum_{v \in C} x_v = c, \quad \sum_{v \in D} x_v = d \text{ and } \sum_{v \in E} x_v = e.$$

Without loss of generality, assume that $a \geq b$, then replacing $xy2$ by $xy1$ for all $x, y \in D$ does not decrease the Lagrangian. So

$$\begin{aligned} \lambda(G) &\leq ab(c+d) + a\left(\left(\frac{c}{6}\right)^2 \binom{6}{2} + cd + ce + de + \frac{d^2}{2} + \frac{e^2}{2}\right) + b\left(\left(\frac{c}{6}\right)^2 \binom{6}{2} + cd + ce + de + \frac{e^2}{2}\right) + \frac{c^3}{27} \\ &\leq ab(c+d) + a\left(\frac{5c^2}{12} + c(d+e) + \frac{(d+e)^2}{2}\right) + b\left(\frac{5c^2}{12} + c(d+e) + \frac{(d+e)^2}{2}\right) + \frac{c^3}{27}. \end{aligned}$$

Let $\delta = d + e$. Then

$$\lambda(G) \leq \lambda(a, b, c, \delta) = ab(c+\delta) + a\left(\frac{5c^2}{12} + c\delta + \frac{\delta^2}{2}\right) + b\left(\frac{5c^2}{12} + c\delta + \frac{\delta^2}{2}\right) + \frac{c^3}{27} \triangleq \lambda$$

subject to

$$\begin{cases} a + b + c + \delta = 1, \\ a \geq 0, \quad b \geq 0, \quad c \geq 0 \text{ and } \delta \geq 0. \end{cases} \quad (2)$$

For simplicity of the notation, we assume that λ reaches the maximum at (a, b, c, δ) .

If $c = 0$, then $\lambda \leq \frac{(a+b)^2[1-(a+b)]}{4} + \frac{(a+b)[1-(a+b)]^2}{2}$. By Fact 4.9, $\lambda \leq \frac{\sqrt{3}}{18}$.

If $\delta = 0$, then

$$\begin{aligned} \lambda &= abc + \frac{5(a+b)c^2}{12} + \frac{c^3}{27} \\ &\leq \frac{(a+b)^2c}{4} + \frac{5(a+b)c^2}{12} + \frac{c^3}{27} \\ &= \frac{(1-c)^2c}{4} + \frac{5(1-c)c^2}{12} + \frac{c^3}{27} = f(c). \\ f'(c) &= \frac{(1-c)^2 - 2c(1-c)}{4} + \frac{10c(1-c) - 5c^2}{12} + \frac{c^2}{9} \\ &= \frac{-14c^2 - 6c + 9}{36}. \end{aligned}$$

So $f(c)$ is increasing in $[0, \frac{3(\sqrt{15}-1)}{14}]$, then $f_{\max} = f(\frac{3(\sqrt{15}-1)}{14}) < 0.0921$.

If $c, \delta > 0$, then by Theorem 4.1, we have $\frac{\partial \lambda}{\partial c} = \frac{\partial \lambda}{\partial \delta}$, solving it, we obtain that $\frac{ac+bc}{6} = \frac{c^2}{9}$. Therefore

$c = \frac{3(a+b)}{2}$. So

$$\begin{aligned}
\lambda &= ab(1-a-b) + \frac{(a+b)(c+\delta)^2}{2} - \frac{c^2(a+b)}{12} + \frac{c^3}{27} \\
&\leq \frac{(a+b)^2[1-(a+b)]}{4} + \frac{(a+b)[1-(a+b)]^2}{2} + \frac{c^3}{27} - \frac{c^3}{18} \\
&< \frac{(a+b)^2[1-(a+b)]}{4} + \frac{(a+b)[1-(a+b)]^2}{2} \\
&\leq \frac{\sqrt{3}}{18},
\end{aligned}$$

the last inequality follows from Fact 4.9. \square

Proof of Lemma 2.1. Assume that G contains an X_2 with vertex set $\{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$ form K_4^3 . We prove the following claim first.

Claim 5.10 $\lambda(G[\{1, 2, 3, 4, 5, 6\}]) \leq \frac{2}{25}$.

Proof of Claim 5.10. Denote $A = \{1, 2, 3, 4, 5, 6\}$. By Lemma 5.7, G is K_5^{3-} -free, then $e(G[A']) \leq 8$ for all $A' \subseteq A$ and $|A'| = 5$. By double counting, we have

$$\binom{6-3}{2} e(G[A]) \leq \binom{6}{5} \times 8.$$

Therefore $e(G[A]) \leq 16 = \binom{6-1}{3} + \binom{6-2}{2} < \binom{6}{3} - 3 = 17$. By Lemma 4.7 and Lemma 4.8, $\lambda(G[A]) \leq \lambda(K_5^3) = \frac{2}{25}$. \square

Denote $C = \{3, 4, 5, 6\}$, $D = \{x \in V(G) \setminus \{1, 2, 3, 4, 5, 6\} : x12 \in E(G)\}$ and $E = V(G) \setminus (\{1, 2\} \cup C \cup D)$. Note that $x12 \notin E(G)$ for $x \in E$. Since G is $K_4^3 \cup e$ -free, then $x34, x56 \notin E(G)$ and only possibly, $x35, x36, x45, x46 \in E(G)$ for $x \in D \cup E$. Let

$$x_1 = a, x_2 = b, \sum_{v \in C} x_v = c, \sum_{v \in D} x_v = d, \sum_{v \in E} x_v = e.$$

Without loss of generality, assume that $a \geq b$. Note that the contribution of the edges between C and $D \cup E$ to $\lambda(G)$ is at most $(d+e)(x_3+x_4)(x_5+x_6) \leq \frac{(d+e)c^2}{4}$. Since G is X_3 -free, then $N^*(x, y) = \{1\}$ or $\{2\}$ for $x, y \in D$. If $xy2 \in E(G)$, we delete $xy2$ and add $xy1$, this does not reduce the Lagrangian. Hence

$$\begin{aligned}
\lambda(G) &\leq abd + a(cd + ce + de + \frac{d^2}{2} + \frac{e^2}{2}) + b(cd + ce + de + \frac{e^2}{2}) + \frac{2(a+b+c)^3}{25} + \frac{(d+e)c^2}{4} \quad (3) \\
&= \lambda(a, b, c, d, e) = \lambda
\end{aligned}$$

under the constraints $a + b + c + d + e = 1$, $a, b, c, d, e \geq 0$.

To simplify the notation, we assume that λ reaches the maximum at (a, b, c, d, e) , Note that $a \geq b$.

Claim 5.11 $\lambda(a, 0, c, d, e) \leq \frac{\sqrt{3}}{18}$.

The proof of Claim 5.11 is given in Appendix.

Let us continue the proof of Lemma 2.1. We have shown that $a \geq b > 0$ (Claim 5.11). If $d = 0$, substitute it into (3), then

$$\lambda(a, b, c, 0, e) = a\left(ce + \frac{e^2}{2}\right) + b\left(ce + \frac{e^2}{2}\right) + \frac{2(a+b+c)^3}{25} + \frac{ec^2}{4}.$$

So $\lambda(a+b, 0, c, 0, e)$ also gets the maximum value, a contradiction to $a \geq b > 0$ when λ gets the maximum. So $a, b, d > 0$. By Theorem 4.1, $\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial b}$, combining with

$$\begin{aligned} \frac{\partial \lambda}{\partial a} &= bd + cd + ce + de + \frac{d^2}{2} + \frac{e^2}{2} + \frac{6(a+b+c)^2}{25}, \\ \frac{\partial \lambda}{\partial b} &= ad + cd + ce + de + \frac{e^2}{2} + \frac{6(a+b+c)^2}{25}, \end{aligned}$$

we get

$$a = b + \frac{d}{2}. \quad (4)$$

Claim 5.12 $\lambda(a, b, c, d, 0) \leq \frac{\sqrt{3}}{18}$.

The proof of Claim 5.12 is given in Appendix.

Let us continue the proof of Lemma 2.1. We have shown that $a, b, d, e > 0$. By Theorem 4.1, we have $\frac{\partial \lambda}{\partial d} = \frac{\partial \lambda}{\partial e}$. Since

$$\begin{aligned} \frac{\partial \lambda}{\partial d} &= ab + ac + ae + ad + bc + be + \frac{c^2}{4}, \\ \frac{\partial \lambda}{\partial e} &= ac + ad + ae + bc + bd + be + \frac{c^2}{4}, \end{aligned}$$

then $a = d$. Recall (4), so

$$a = d = 2b. \quad (5)$$

Claim 5.13 $\lambda(a, b, 0, d, e) \leq \frac{\sqrt{3}}{18}$.

Proof of Claim 5.13. Assume $c = 0$. By (5), then $e = 1 - a - b - c - d = 1 - 5b$. Substituting it into (3), then

$$\begin{aligned} \lambda &= abd + a\left(de + \frac{d^2}{2} + \frac{e^2}{2}\right) + b\left(de + \frac{e^2}{2}\right) + \frac{2(a+b)^3}{25} \\ &= 4b^3 + b\left(6b(1-5b) + 4b^2 + \frac{3(1-5b)^2}{2}\right) + \frac{2(3b)^3}{25} \\ &= \frac{883b^3}{50} - 9b^2 + \frac{3b}{2} = f(b), \\ f'(b) &= \frac{2649b^2}{50} - 18b + \frac{3}{2}. \end{aligned}$$

So $f(b)$ is increasing in $[0, \frac{450-5\sqrt{153}}{2649}]$ or $[\frac{450+5\sqrt{153}}{2649}, 1]$. Note that $a + b + d + e = 1$, then $5b < 1$, so $\lambda \leq f(\frac{450-5\sqrt{153}}{2649}) \leq 0.083$. \square

Let us continue the proof of Lemma 2.1. We have shown that $a = d = 2b$ and $e = 1 - 5b - c$ and $a, b, c, d, e > 0$. Substituting them into (3), we have

$$\begin{aligned}\lambda &= 4b^3 + b(6bc + 3ce + 6be + 4b^2 + \frac{3e^2}{2}) + \frac{2(3b+c)^3}{25} + \frac{(2b+e)c^2}{4} \\ &= \frac{31b^3}{2} - \frac{3bc^2}{2} - 9b^2 + \frac{3b}{2} + \frac{2(3b+c)^3}{25} + \frac{(1-3b-c)c^2}{4} \\ &\leq 16b^3 - \frac{3bc^2}{2} - 9b^2 + \frac{3b}{2} + \frac{(3b+c)^3}{12} + \frac{(1-3b-c)c^2}{4} = \lambda_0(b, c)\end{aligned}$$

under the constraints $5b + c \leq 1, b, c \geq 0$. Now we estimate the optimum value of λ_0 . For simplicity of the notation, let λ_0 reach the maximum value at (b, c) .

If $b = 0$, then $\lambda_0 = \frac{c^3}{12} + \frac{(1-c)c^2}{4} \leq \frac{3c^2-2c^3}{12} = f(c)$. Since $f'(c) = \frac{c(1-c)}{2}$, then λ is increasing in $[0, 1]$. Therefore $\lambda_0 \leq f(1) = \frac{1}{12} < 0.09$.

If $c = 0$, then $\lambda_0 = 16b^3 - 9b^2 + \frac{3b}{2} + \frac{(3b)^3}{12} = \frac{73b^3}{4} - 9b^2 + \frac{3b}{2} \leq 19b^3 - 9b^2 + \frac{3b}{2} = f(b)$. Since $f'(b) = 57b^2 - 18b + \frac{3}{2} > 0$, so $f(b)$ is increasing in $[0, 1]$. Therefore $\lambda_0 \leq f(\frac{1}{5}) = 0.092$.

If $5b + c = 1$, then

$$\begin{aligned}\lambda_0 &= 16b^3 - \frac{3b(1-5b)^2}{2} - 9b^2 + \frac{3b}{2} + \frac{(1-2b)^3}{12} + \frac{b(1-5b)^2}{2}, \\ \lambda'_0 &= -29b^2 + 4b.\end{aligned}$$

Then $\lambda_0 \leq \lambda_0(4/29) \leq 0.0961 < \frac{\sqrt{3}}{18}$.

Therefore we may assume that λ_0 gets maximum when $b, c > 0$ and $5b + c < 1$. By Theorem 4.1,

$$\begin{aligned}\frac{\partial \lambda_0}{\partial b} &= 48b^2 - \frac{9c^2}{4} - 18b + \frac{3}{2} + \frac{3(3b+c)^2}{4} = 0 \\ \frac{\partial \lambda_0}{\partial c} &= -3bc + \frac{(3b+c)^2}{4} + \frac{2c-6bc-3c^2}{4} = 0.\end{aligned}$$

Equivalently, $\frac{\partial \lambda_0}{\partial b} - 3 \times \frac{\partial \lambda_0}{\partial c} = 0$ and $\frac{\partial \lambda_0}{\partial c} = 0$. Solving these two equations, we have

$$c = \frac{1-12b+32b^2}{1-9b} = \frac{(1-8b)(1-4b)}{1-9b} \quad \text{and} \quad 9b^2 - 12bc + 2c - 2c^2 = 0.$$

Recall that $c > 0$ and $b < \frac{1}{5}$. So $0 < b < \frac{1}{9}$ or $\frac{1}{8} \leq b \leq \frac{1}{5}$. Combining the above equations, we have $b(2137b^3 - 882b^2 + 125b - 6) = 0$. Let $f(b) = 2137b^3 - 882b^2 + 125b - 6$. Since $f'(b) = 6411b^2 - 1764b + 125 > 0$, then $f(b)$ is increasing in $[0, \frac{1}{4}]$. However $f(0), f(\frac{1}{9}) < 0$ and $f(\frac{1}{8}) > 0$, so $f(b) = 0$ has no solution in $0 < b < \frac{1}{9}$ and $\frac{1}{8} \leq b \leq \frac{1}{5}$, a contradiction. This completes the proof of Lemma 2.1. \square

5.3 G contains at least two copies of K_4^3

In this section, we give the proof of Lemma 2.2.

Proof of Lemma 2.2. Recall that $\{1, 2, 3, 4\}$ forms a K_4^3 . Assume that G contains no other K_4^3 , in other words, v does not belong to any K_4^3 for any $v \in [n] \setminus \{1, 2, 3, 4\}$. We claim that $|G_v \cap \{12, 13, 14, 23, 24, 34\}| \leq 4$. Since otherwise $G_v[\{1, 2, 3, 4\}]$ contains a triangle and v is contained in the K_4^3 formed by v and the vertices in this triangle. Since G is $K_4^3 \cup e$ -free, then G_v does not contain

an edge in $[n] \setminus \{1, 2, 3, 4\}$. By Claim 5.3, $\omega(G_v) \geq 3$, so the maximum clique of G_v contains at least 2 vertices in $\{1, 2, 3, 4\}$, therefore $|G_v \cap \{12, 13, 14, 23, 24, 34\}| \geq 1$. Let

$$A = \{1, 2, 3, 4\},$$

$$A_1 = \{v \in [n] \setminus A : |G_v \cap \{12, 13, 14, 23, 24, 34\}| = 1\},$$

$$A_2 = \{v \in [n] \setminus A : |G_v \cap \{12, 13, 14, 23, 24, 34\}| = 2\},$$

$$A_3 = \{v \in [n] \setminus A : |G_v \cap \{12, 13, 14, 23, 24, 34\}| = 3\},$$

$$A_4 = \{v \in [n] \setminus A : |G_v \cap \{12, 13, 14, 23, 24, 34\}| = 4\}.$$

Without loss of generality, let's assume that $x_1 \geq x_2 \geq x_3 \geq x_4$. Then $x_1x_2 \geq x_1x_3 \geq x_2x_3, x_1x_4 \geq x_2x_4 \geq x_3x_4$. We aim to give an upper bound of $\lambda(G, \vec{x})$, therefore we can assume that $v12, v13 \in E(G)$ for $v \in A_2$, $v12 \in E(G)$ for $v \in A_1$. Set

$$x_1 = a, x_2 = b, x_3 = c, x_4 = d, \sum_{v \in A_1} x_v = h, \sum_{v \in A_2} x_v = g, \sum_{v \in A_3} x_v = f, \sum_{v \in A_4} x_v = e.$$

Since G doesn't contain two copies of K_4^3 , then the deletion of any 1 of $\{123, 124, 134, 234\}$ of $E(G)$ makes G K_4^3 -free. So $abc, abd, acd, bcd > 0.00264$ since otherwise $\lambda(G) \leq 0.00264 + \frac{0.5615}{6}$ (in view of Lemma 5.1) $\leq \frac{\sqrt{3}}{18}$. So $(a+b)cd > 2 \times 0.00264$. By Claim 5.4, $a+b \leq \frac{3-\sqrt{3}}{3}$, then $cd \geq \frac{6 \times 0.00264}{3-\sqrt{3}}$. Therefore $c+d \geq 2\sqrt{cd} > 0.22354$. If $d < 0.11177$, then $b+c \geq 2\sqrt{bc} > 0.307$.

To complete the proof, we show the following three claims in Appendix.

Claim 5.14 $\lambda(G[A \cup A_4], \vec{x}) \leq 0.0789(a+b+c+d+e)^3$.

Claim 5.15 $\lambda(G[A \cup A_4 \cup A_3], \vec{x}) \leq 0.092(a+b+c+d+e+f)^3$.

Claim 5.16 $\lambda(G) \leq \frac{\sqrt{3}}{18}$.

5.4 G does not contain two copies of K_4^3 sharing three vertices

Proof of Lemma 2.3. Assume that G contains an Y_2 with the vertex set $\{1, 2, 3, 4, 5\}$, where $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 5\}$ form K_4^3 . Since G is $K_4^3 \cup e$ -free, then any two K_4^3 in G must intersect 2 or 3 vertices. Since G is X_2 -free (Lemma 2.1), then any two K_4^3 in G must intersect 3 vertices. Therefore $G - \{3\}$ cannot contain a K_4^3 since it cannot intersect with $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 5\}$ at three vertices in the same time. Let $x_1 = a, x_2 = b, x_3 = c, x_4 + x_5 = d$ and assume that $a \geq b \geq c$. By Claim 5.5, $c > 0.08$. Let $D = \{4, 5\}$, and

$$E_0 = \{v : v \in [n] \setminus \{1, 2, 3, 4, 5\} \text{ and } |G_v \cap \{12, 13, 23\}| = 0\},$$

$$E_1 = \{v : v \in [n] \setminus \{1, 2, 3, 4, 5\} \text{ and } |G_v \cap \{12, 13, 23\}| = 1\},$$

$$E_2 = \{v : v \in [n] \setminus \{1, 2, 3, 4, 5\} \text{ and } |G_v \cap \{12, 13, 23\}| \geq 2\}.$$

Set $\sum_{v \in E_0} x_v = g, \sum_{v \in E_1} x_v = f$ and $\sum_{v \in E_2} x_v = e$. Since G is $K_4^3 \cup e$ -free, then for $x, y \in [n] \setminus \{1, 2, 3, 4, 5\}$, $N^*(x, y) \subseteq \{1, 2, 3\}$. If $x \in D$ and $y \in E_2$, then $N^*(x, y) \subseteq \{1, 2, 3, 4, 5\}$. We claim that $|N^*(x, y) \cap \{1, 2, 3\}| \leq 2$ for $x, y \in E_2$ or $x \in D$ and $y \in E_2$, otherwise there exists two vertices

$z, w \in N^*(x, y) \cap \{1, 2, 3\}$ such that $zw \in G_x \cap G_y$. So $\{x, y, z, w\}$ forms a K_4^3 , which forms an X_2 with $G[\{1, 2, 3, 4\}]$, a contradiction. Therefore we may let $N^*(x, y) \cap \{1, 2, 3\} \subseteq \{1, 2\}$ for those x, y with Lagrangian non-decreasing. Since G is $K_4^3 \cup e$ -free, then all edges in $G[V - \{1, 2, 3\}]$ must contain $\{4, 5\}$. Since G is K_5^3 -free, then at least one of $\{145, 245, 345\}$ is not in $E(G)$. (Indeed, G is K_5^3 -free, at least two of $\{145, 245, 345\}$ are not in $E(G)$. But it seems to be easier to estimate the Lagrangian below if we relax it to be one.) We may assume that 345 is not in $E(G)$. Therefore

$$\begin{aligned}
\lambda(G) &\leq ab(c + d + e + f) + (ac + bc)(d + e) + (a + b)(de + \frac{e^2}{2}) + (a + b + c)(df + dg + eg + fg + ef \\
&\quad + \frac{f^2}{2} + \frac{g^2}{2}) + \frac{d^2(a + b + e + f + g)}{4} \\
&= ab(c + d + e + f) + (ac + bc)(d + e) + (a + b)(de + \frac{e^2}{2}) + (a + b + c) \left(d(g + f) + e(g + f) \right. \\
&\quad \left. + \frac{(f + g)^2}{2} \right) + \frac{d^2(a + b + e + f + g)}{4} \\
&= \lambda(a, b, c, d, e, f, g) = \lambda.
\end{aligned}$$

Note that $\lambda(a, b, c, d, e, f, g) \leq \lambda(\frac{a+b}{2}, \frac{a+b}{2}, c, d, e, f + g, 0)$, then we may assume that $g = 0$ and $a = b$. Then let $\alpha = a + b$, so

$$\lambda = \frac{\alpha^2(c + d + e + f)}{4} + \alpha c(d + e) + \alpha(de + \frac{e^2}{2}) + (\alpha + c)(df + ef + \frac{f^2}{2}) + \frac{d^2(\alpha + e + f)}{4}$$

subject to

$$\begin{cases} \alpha + c + d + e + f = 1, \\ c \geq 0.08, \alpha, d, e, f \geq 0. \end{cases} \quad (6)$$

Note that

$$\lambda = \frac{\alpha^2(c + d + e + f)}{4} + \alpha c(d + e) + \alpha(\frac{d^2}{2} + de + \frac{e^2}{2}) + (\alpha + c)(df + ef + \frac{f^2}{2}) + \frac{d^2(e + f - \alpha)}{4}.$$

If $e + f - \alpha < \frac{d}{2}$, then $\lambda \leq \lambda(a, c, d - \epsilon, e + \epsilon, f)$ for $\epsilon > 0$ small enough, a contradiction. So we may assume that $e + f \geq \alpha + \frac{d}{2}$ or $d = 0$.

Claim 5.17 $\lambda(0, c, d, e, f) \leq \frac{\sqrt{3}}{18}$.

Proof of Claim 5.17. If $\alpha = 0$, then $c + d + e + f = 1$ and

$$\begin{aligned}
\lambda &= c(df + ef + \frac{f^2}{2}) + \frac{d^2(e + f)}{4} \\
&\leq c \frac{(d + e + f)^2}{2} + \left(\frac{\frac{d}{2} + \frac{d}{2} + e + f}{3} \right)^3 \\
&= \frac{c(1 - c)^2}{2} + \frac{(1 - c)^3}{27} = \frac{25c^3 - 48c^2 + 21c + 2}{54} = f(c), \\
f'(c) &= \frac{25c^2 - 32c + 7}{18} = \frac{(25c - 7)(c - 1)}{18}.
\end{aligned}$$

Therefore $\lambda \leq f(\frac{7}{25}) = 0.0864$. □

Let's continue the proof of Lemma 2.3. We claim that if $c > 0.08$, then $\alpha \geq c$. If $\alpha < c$, then $\alpha \leq \frac{1}{2}$, and for $0 < \epsilon < \frac{a}{3}$,

$$\begin{aligned}\lambda(\alpha + \epsilon, c - \epsilon, d, e, f) - \lambda(\alpha, c, d, e, f) &\geq \frac{(\alpha + \epsilon)^2(c - \epsilon + d + e + f) - (\alpha)^2(c + d + e + f)}{4} \\ &> \epsilon(2\alpha - 3\alpha^2 - 3\epsilon\alpha) \\ &> \epsilon(2\alpha - 4\alpha^2) \geq 0,\end{aligned}$$

a contradiction.

Claim 5.18 $\lambda(\alpha, c, d, e, 0) \leq \frac{\sqrt{3}}{18}$.

The proof of Claim 5.18 is given in Appendix.

Claim 5.19 $\lambda(\alpha, c, d, 0, f) \leq \frac{\sqrt{3}}{18}$.

The proof of Claim 5.19 is given in Appendix.

By the above claims, we may assume that $\alpha, e, f > 0$, so by Theorem 4.1, we have $\frac{\partial \lambda}{\partial e} = \frac{\partial \lambda}{\partial f}$. In view of (6),

$$\begin{aligned}\frac{\partial \lambda}{\partial e} &= \frac{\alpha^2}{4} + \alpha c + \alpha d + \alpha e + (\alpha + c)f + \frac{d^2}{4} \\ \frac{\partial \lambda}{\partial f} &= \frac{\alpha^2}{4} + (\alpha + c)(d + e + f) + \frac{d^2}{4}.\end{aligned}$$

So $\alpha = d + e$.

Claim 5.20 $\lambda(\alpha, c, 0, e, f) \leq \frac{\sqrt{3}}{18}$.

The proof of Claim 5.20 is given in Appendix.

Let's continue the proof of Lemma 2.3. The above claims indicate that we may assume $d, e > 0$, then by Theorem 4.1, we have $\frac{\partial \lambda}{\partial d} = \frac{\partial \lambda}{\partial e}$. Since

$$\begin{aligned}\frac{\partial \lambda}{\partial d} &= \frac{\alpha^2}{4} + \alpha c + \alpha e + (\alpha + c)f + \frac{d(\alpha + e + f)}{2} \\ \frac{\partial \lambda}{\partial e} &= \frac{\alpha^2}{4} + \alpha c + \alpha d + \alpha e + (\alpha + c)f + \frac{d^2}{4},\end{aligned}$$

then $\alpha + \frac{d}{2} = e + f$. Since $\alpha = d + e$, then $f = \frac{3d}{2}$. Note that $\alpha + c + d + e + f = 1$, then $f = 1 - 2\alpha - c$ and $d = \frac{2(1-2\alpha-c)}{3} > 0$ and $e = \frac{7\alpha+2c-2}{3}$. Substituting these into (6), we have

$$\begin{aligned}\lambda &= \frac{\alpha^2(1-\alpha)}{4} + \alpha^2 c + \alpha \left(\left(\frac{2(1-2\alpha-c)}{3} \right) \left(\frac{7\alpha+2c-2}{3} \right) + \frac{(\frac{7\alpha+2c-2}{3})^2}{2} \right) \\ &+ (\alpha + c) \left(\alpha(1-2\alpha-c) + \frac{(1-2\alpha-c)^2}{2} \right) + \frac{(1-2\alpha-c)^2}{9} \left(1 - c - \frac{2(1-2\alpha-c)}{3} \right) \\ &= \frac{-5\alpha^3}{108} + \frac{14\alpha^2 c}{9} - \frac{11\alpha^2}{36} + \frac{23\alpha c^2}{18} - \frac{14\alpha c}{9} + \frac{5\alpha}{18} + \frac{25c^3}{54} - \frac{8c^2}{9} + \frac{7c}{18} + \frac{1}{27} \\ &= \lambda(\alpha, c).\end{aligned}$$

If $c = 0.08$, then

$$\begin{aligned}\lambda &= \frac{-5\alpha^3}{108} - \frac{1.63\alpha^2}{9} + \frac{1.4536\alpha}{9} + \frac{1.6928}{27}, \\ \lambda' &= \frac{-5\alpha^2}{36} - \frac{3.26\alpha}{9} + \frac{1.4536}{9}.\end{aligned}$$

Note that λ is increasing in $[0, \frac{3\sqrt{4971}-163}{125}]$, then $\lambda \leq 0.096$. So we may assume that $c > 0.08$. Recall that $\alpha \geq c$ when $c > 0.08$ and $e = \frac{7\alpha+2c-2}{3}$, $f = 1 - 2\alpha - c > 0$, so $\alpha > \frac{2}{9}$ and $2\alpha + c < 1$. So we can maximize $\lambda(\alpha, c)$ subject to

$$\begin{cases} 2\alpha + c \leq 1, \\ \alpha \geq \frac{2}{9}, \\ c \geq 0.08. \end{cases} \quad (7)$$

Consider

$$\begin{aligned}\lambda & \quad (\alpha + \frac{c-0.08}{2}, 0.08) - \lambda(\alpha, c) \\ &= \frac{(2-25c)(32500\alpha^2 + 26250\alpha c - 25500\alpha + 9375c^2 - 16150c + 4728)}{500000} = \lambda_0, \\ \lambda'_0|_\alpha &= \frac{(2-25c)(260\alpha + 105c - 102)}{2000}.\end{aligned}$$

If $260\alpha + 105c - 102 > 0$, then $\lambda'_0|_\alpha < 0$. So $\lambda_0 \geq \lambda_0(\frac{1-c}{2}, c)$. Therefore

$$\begin{aligned}\lambda_0 &\geq \frac{-7c^3}{32} + \frac{11c^2}{32} - \frac{c}{32} + \frac{103}{250000} = \lambda_1, \\ \lambda'_1 &= \frac{-21c^2}{32} + \frac{11c}{16} - \frac{1}{32}.\end{aligned}$$

Note that λ_1 is increasing in $[0.08, 1]$, so $\lambda_1 \geq \lambda_1(0.08) = 0$. So λ gets maximum when $c = 0.08$, a contradiction.

If $260\alpha + 105c - 102 < 0$, then $\lambda'_0|_\alpha > 0$. So $\lambda_0 \geq \lambda_0(\frac{2}{9}, c)$. Therefore

$$\begin{aligned}\lambda_0 &\geq \frac{(2-25c)(9375c^2 - \frac{30950c}{3} + \frac{53968}{81})}{500000} = \lambda_2 \\ \lambda'_2 &= \frac{83c}{75} - \frac{45c^2}{32} - \frac{6041}{81000}.\end{aligned}$$

Note that λ_2 is increasing in $[0.08, 0.7]$, then $\lambda_2 \geq \lambda_2(0.08) = 0$. So λ gets maximum when $c = 0.08$, a contradiction. The proof of Lemma 2.3 is completed. \square

6 Remark

Let $\Lambda_t^{(r)} = \{\pi_\lambda(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-uniform graphs and } |\mathcal{F}| \leq t\}$. Proposition 1.3 (Proposition 1.3 can be generalized to a family of r -graphs) implies that $\Lambda_t^{(r)} \subseteq \Pi_t^{(r)}$.

Question 1 Is $\Lambda_t^{(r)}$ the same as $\Pi_t^{(r)}$?

Let us propose the following conjecture implying that there exists an r -graph whose Turán density is an irrational number.

Conjecture 6.1 *If $c \cdot \frac{r!}{r^r}$ is in $\Pi_1^{(r)}$ for $r \geq 2$, then $c \cdot \frac{p!}{p^p}$ is in $\Pi_1^{(p)}$ for $p \geq r$.*

7 Appendix

We give theoretical proofs for Claims 5.11, 5.12, 5.14-5.16 and 5.18-5.20 in this section, we have also used Lingo to run the optimization problems. The outcome by Lingo is consistent with the expected optimum values. We can provide the programming upon request.

7.1 Proof of Claim 5.11

Substitute $b = 0$ into (3), then

$$\begin{aligned}\lambda &= a(cd + ce + de + \frac{d^2}{2} + \frac{e^2}{2}) + \frac{2(a+c)^3}{25} + \frac{(d+e)c^2}{4}, \\ &= a[c(d+e) + \frac{(d+e)^2}{2}] + \frac{2(a+c)^3}{25} + \frac{(d+e)c^2}{4}, \\ &= a[c\delta + \frac{\delta^2}{2}] + \frac{2(a+c)^3}{25} + \frac{\delta c^2}{4},\end{aligned}\tag{8}$$

where $\delta = d + e$, then $a + c + \delta = 1$.

If $a = 0$, then

$$\begin{aligned}\lambda &= \frac{2c^3}{25} + \frac{\delta c^2}{4} \leq \frac{c^3}{12} + \frac{(1-c)c^2}{4} = \frac{3c^2 - 2c^3}{12} = f(c) \\ f'(c) &= \frac{c - c^2}{2} = \frac{c(1-c)}{2}.\end{aligned}$$

So $f(c)$ is increasing in $[0, 1]$, then $\lambda \leq f(1) = \frac{1}{12} < \frac{\sqrt{3}}{18}$.

If $\delta = 0$, then

$$\lambda = \frac{2(a+c)^3}{25} \leq \frac{2}{25}.$$

If $c = 0$, then

$$\begin{aligned}\lambda &= \frac{a\delta^2}{2} + \frac{2a^3}{25} \leq \frac{a(1-a)^2}{2} + \frac{a^3}{12} = \frac{7a^3 - 12a^2 + 6a}{12} = f(a) \\ f'(a) &= \frac{7a^2 - 8a + 2}{4}.\end{aligned}$$

So $f(a)$ is increasing in $[0, \frac{4-\sqrt{2}}{7}]$ or $[\frac{4+\sqrt{2}}{7}, 1]$, then $\lambda \leq \max\{f(1), f(\frac{4-\sqrt{2}}{7})\}$. Note that $f(\frac{4-\sqrt{2}}{7}) \leq 0.078$, then $\lambda \leq f(1) = \frac{1}{12} \leq \frac{\sqrt{3}}{18}$.

Therefore we may assume that $a, c, \delta > 0$. Substituting $c = 1 - a - \delta$ into (8), then

$$\lambda = a[(1-a-\delta)\delta + \frac{\delta^2}{2}] + \frac{2(1-\delta)^3}{25} + \frac{\delta(1-a-\delta)^2}{4}$$

gets its maximum inside interior points. By Theorem 4.1,

$$\begin{aligned}\frac{\partial \lambda}{\partial a} &= \frac{\delta}{2} - \frac{3a\delta}{2} = 0, \\ \frac{\partial \lambda}{\partial \delta} &= \frac{(1-a-\delta)(1+3a-3\delta)}{4} - \frac{6(1-\delta)^2}{25} = 0.\end{aligned}$$

Then $a = \frac{1}{3}$ and $\delta = \frac{26-10\sqrt{2}}{51}$, and $\lambda < 0.09$. \square

7.2 Proof of Claim 5.12

Substitute $e = 0$ into (3), then

$$\lambda = abd + a(cd + \frac{d^2}{2}) + bcd + \frac{2(a+b+c)^3}{25} + \frac{dc^2}{4}. \quad (9)$$

If $c = 0$, applying (4), then $b = 3a - 1$ and $d = 2 - 4a$. By Theorem 4.1, $\frac{\partial \lambda}{\partial b} = \frac{\partial \lambda}{\partial d}$. Combining with

$$\begin{aligned}\frac{\partial \lambda}{\partial b} &= ad + \frac{6(a+b)^2}{25}, \\ \frac{\partial \lambda}{\partial d} &= ab + ad,\end{aligned}$$

we get $ab = \frac{6(a+b)^2}{25}$. Substituting $b = 3a - 1$, we have $21a^2 - 23a + 6 = 0$, then $a = \frac{2}{3}$ or $\frac{3}{7}$. Note that if $a = \frac{2}{3}$, then $d < 0$, a contradiction. So $a = \frac{3}{7}$, $b = \frac{2}{7}$ and $d = \frac{2}{7}$. Therefore $\lambda = \frac{4}{49} \leq 0.082$.

If $c \neq 0$, substitute $c = 1 - a - b - d$ into (9),

$$\begin{aligned}\lambda &= abd + a[(1-a-b-d)d + \frac{d^2}{2}] + b(1-a-b-d)d + \frac{2(1-d)^3}{25} + \frac{d(1-a-b-d)^2}{4} \\ &= ad - a^2d - abd - \frac{ad^2}{2} + bd - b^2d - bd^2 + \frac{2(1-d)^3}{25} + \frac{d(1-a-b-d)^2}{4}.\end{aligned}$$

subject to $a + b + d \leq 1$, and $a, b, d \geq 0$. Since $a + b + d < 1$ and we have showned that $a, b, d > 0$, then by Theorem 4.1,

$$\begin{aligned}\frac{\partial \lambda}{\partial a} &= d - 2ad - bd - \frac{d^2}{2} - \frac{d(1-a-b-d)}{2} = 0, \\ \frac{\partial \lambda}{\partial b} &= -ad + d - 2bd - d^2 - \frac{d(1-a-b-d)}{2} = 0, \\ \frac{\partial \lambda}{\partial d} &= a - a^2 - ab - ad + b - b^2 - 2bd - \frac{6(1-d)^2}{25} + \frac{(1-a-b-d)^2}{4} - \frac{d(1-a-b-d)}{2} = 0.\end{aligned}$$

Then we have $b = 1 - 3a$ (by solving $\frac{\partial \lambda}{\partial a} = 0$) and $d = 8a - 2$ (by solving $\frac{\partial \lambda}{\partial b} = 0$), substitute this into $\frac{\partial \lambda}{\partial d} = 0$, we have $1266a^2 - 787a + 121 = 0$. Therefore $a = \frac{787-5\sqrt{265}}{2532}$ and $\lambda \leq 0.0939$. \square

7.3 Proof of Claim 5.14

Since G contains no other K_4^3 , then for any $v \in A_4$, there is no K_3 in $G_v[A]$, so $G_v[A]$ forms a C_4 . Hence

$$\lambda(G_v[A], \vec{x}) \leq \max\{(a+b)(c+d), (a+c)(b+d), (a+d)(b+c)\} \leq \frac{(a+b+c+d)^2}{4}.$$

Since G is $K_4^3 \cup \{e\}$ -free, then $N^*(x, y) \subseteq A$ for $x, y \in [n] \setminus A$. Since G contains only one K_4^3 , we claim that $|N^*(x, y)| \leq 2$ for all $x, y \in A_4$. Recall that G_x and G_y are C_4 's, then $G_x[A] \cap G_y[A]$ must be two vertex disjoint edges or $G_x[A] = G_y[A]$. If $|N^*(x, y)| \geq 3$, then there are $z, w \in N^*(x, y) \subseteq A$ such that $zw \in G_x \cap G_y$. Then $\{x, y, z, w\}$ forms a K_4^3 , a contradiction. Recall that $a \geq b \geq c \geq d$, then we may assume that $N^*(x, y) = \{1, 2\}$ for $x, y \in A_4$ with the Lagrangian non-decreasing. Therefore

$$\lambda(G[A \cup A_4], \vec{x}) \leq abc + abd + acd + bcd + \frac{(a+b)e^2}{2} + e \frac{(a+b+c+d)^2}{4},$$

subject to $a + b + c + d + e = 1$ and $c + d > 0.22354$ and $a + b > c + d$. Let $\alpha = \frac{a+b}{a+b+c+d+e}$, $\gamma = \frac{c+d}{a+b+c+d+e} \geq 0.22354$ and $\eta = \frac{e}{a+b+c+d+e}$. So

$$\frac{\lambda(G[A \cup A_4], \vec{x})}{(a+b+c+d+e)^3} \leq \frac{\alpha^2\gamma}{4} + \frac{\alpha\gamma^2}{4} + \frac{\alpha\eta^2}{2} + \eta \frac{(\alpha+\gamma)^2}{4} = \lambda(\alpha, \gamma, \eta) \triangleq \lambda$$

subject to

$$\begin{cases} \alpha + \gamma + \eta = 1, \\ \gamma \geq 0.22354, \\ \alpha \geq \gamma. \end{cases} \quad (10)$$

If $\eta = 0$, then $\lambda \leq \lambda(\frac{1}{2}, \frac{1}{2}, 0) = \frac{1}{16}$.

If $\alpha = \gamma$, then $\eta = 1 - 2\alpha$. So

$$\lambda = \frac{\alpha^3 + \alpha\eta^2}{2} + \eta\alpha^2 = \frac{\alpha(\alpha + \eta)^2}{2} = \frac{\alpha(1 - \alpha)^2}{2} \leq \frac{(\frac{2\alpha+1-\alpha+1-\alpha}{3})^3}{4} = \frac{2}{27} < 0.075.$$

If $\gamma = 0.22354$, then $\eta = 0.77646 - \alpha$

$$\begin{aligned} \lambda &\leq \frac{\alpha^3}{4} - 0.6382\alpha^2 + 0.38823\alpha + 0.0097 = \lambda_0 \\ \lambda'_0 &= \frac{3\alpha^2}{4} - 1.2764\alpha + 0.38823. \end{aligned}$$

So λ_0 is increasing in $[0, 0.3965684]$. Then $\lambda \leq 0.0789$.

Therefore we may assume that λ gets the maximum in its interior points. By Theorem 4.1, then $\frac{\partial \lambda}{\partial \gamma} = \frac{\partial \lambda}{\partial \eta}$. Combining with

$$\begin{aligned} \frac{\partial \lambda}{\partial \gamma} &= \frac{\alpha^2}{4} + \frac{\alpha\gamma}{2} + \eta \frac{(\alpha + \gamma)}{2} \\ \frac{\partial \lambda}{\partial \eta} &= \alpha\eta + \frac{(\alpha + \gamma)^2}{4}, \end{aligned}$$

we get $\gamma^2 = 2\eta(\gamma - \alpha)$, contradicting to $\alpha > \gamma$. This completes the proof of Claim 5.14. \square

7.4 Proof of Claim 5.15.

Note that for $x, y \in A_3$, if $|N^*(x, y)| = 4$ (i.e. $N^*(x, y) = A$) and there exists $zw \in G_x[A] \cap G_y[A]$, then $\{x, y, z, w\}$ forms a K_4^3 , a contradiction. So for $x, y \in A_3$, if $|N^*(x, y)| = 4$, then $G_x[A] \cap G_y[A] = \emptyset$ and $G_x[A] \cup G_y[A] = \binom{A}{2}$. Such pairs $\{x, y\}$ are partitioned into at most $\binom{6}{3}/2$ groups $\{B_{0i}, B_{1i}\}$ such that $G_x[A]$ are the same for all $x \in B_{0i}$, $G_y[A]$ are the same for all $y \in B_{1i}$, and $G_x[A] \cup G_y[A]$ is a partition of $\binom{A}{2}$ for all $x \in B_{0i}$ and $y \in B_{1i}$ for each i . Assume that there are s such groups. So $|N^*(x, y)| \leq 3$ for $x, y \in A_4 \cup A_3$ except $x \in B_{0i}$ and $y \in B_{1i}$, then we may assume that $N^*(x, y) = \{1, 2, 3\}$ for such x, y with the Lagrangian non-decreasing. Note that $|N^*(x, y)| \leq 2$ for $x, y \in B_{0i}$ or $x, y \in B_{1i}$ for $1 \leq i \leq s$. Therefore we may assume that $N^*(x, y) = \{1, 2\}$ for such x, y with the Lagrangian non-decreasing. And for all $x \in A_3$, if $x23 \notin E(G)$, then we may assume that $x12, x13, x14 \in E(G)$. If $x23 \in E(G)$, since $\{x, 1, 2, 3\}$ doesn't span K_4^3 , then one of $x12, x13$ does not belong to $E(G)$, we can replace $x23$ by that edge and replace other 2 edges $xij, ij \in A^{(2)}$ from $\{x12, x13, x14\}$ with the Lagrangian non-decreasing. Let $f_{0i} = \sum_{v \in B_{0i}} x_v$ and $f_{1i} = \sum_{v \in B_{1i}} x_v$ and $f_i = f_{0i} + f_{1i}$ for $1 \leq i \leq s$. Let $f' = f - \sum_{i=1}^s f_i$. By Claim 5.14 and the above analysis, we have

$$\begin{aligned}
\lambda(G[A \cup A_4 \cup A_3], \vec{x}) &\leq 0.0789(a + b + c + d + e)^3 + (ab + ac + ad)f + (ef + f' \sum_{i=1}^s f_i + \sum_{1 \leq i \neq j \leq s} f_i f_j) \\
&\quad + \frac{f'^2}{2}(a + b + c) + \sum_{i=1}^s \frac{f_{0i}^2 + f_{1i}^2}{2}(a + b) + \sum_{i=1}^s f_{0i} f_{1i}(a + b + c + d) \\
&\leq 0.0789(a + b + c + d + e)^3 + (ab + ac + ad)f + ef(a + b + c) + \frac{(a + b)f^2}{2} \\
&\quad + c(f' \sum_{i=1}^s f_i + \sum_{1 \leq i \neq j \leq s} f_i f_j + \frac{f'^2}{2}) + (c + d) \sum_{i=1}^s f_{0i} f_{1i} \\
&\leq 0.0789(a + b + c + d + e)^3 + (ab + ac + ad)f + ef(a + b + c) + \frac{(a + b)f^2}{2} \\
&\quad + c(f' \sum_{i=1}^s f_i + \sum_{1 \leq i \neq j \leq s} f_i f_j + \frac{f'^2}{2}) + (c + d) \sum_{i=1}^s \frac{f_i^2}{4} \\
&= \lambda(f', f_1, \dots, f_s).
\end{aligned}$$

Note that

$$\begin{aligned}
\lambda(f, 0, \dots, 0) &= \lambda(f', f_1, \dots, f_s) \\
&= \frac{cf^2}{2} - c(f' \sum_{i=1}^s f_i + \sum_{1 \leq i \neq j \leq s} f_i f_j + \frac{f'^2}{2}) - (c + d) \sum_{i=1}^s \frac{f_i^2}{4} \\
&= c \frac{\sum_{i=1}^s f_i^2}{2} - (c + d) \sum_{i=1}^s \frac{f_i^2}{4} \\
&\geq 0
\end{aligned}$$

since $c \geq d$. So we may assume that $f' = f$ and $f_i = 0$ for $1 \leq i \leq s$. So

$$\begin{aligned}
\lambda(G[A \cup A_4 \cup A_3], \vec{x}) &\leq 0.0789(a+b+c+d+e)^3 + (ab+ac+ad)f + (ef + \frac{f^2}{2})(a+b+c) \\
&= 0.0789(a+(b+c)+d+e)^3 + (a(b+c)+ad)f + (ef + \frac{f^2}{2})(a+(b+c)) \\
&= \lambda.
\end{aligned} \tag{11}$$

Let $\alpha = \frac{a}{(a+b+c+d+e+f)}$, $\beta = \frac{b+c}{a+b+c+d+e+f}$, $\delta = \frac{d}{a+b+c+d+e+f}$, $\eta = \frac{e}{a+b+c+d+e+f}$ and $\rho = \frac{f}{a+b+c+d+e+f}$, and let $\tau = \frac{\lambda}{(a+b+c+d+e+f)^3}$. Then

$$\tau = 0.0789(\alpha + \beta + \delta + \eta)^3 + (\alpha\beta + \alpha\delta)\rho + (\eta\rho + \frac{\rho^2}{2})(\alpha + \beta),$$

and it's sufficient to prove that $\tau \leq 0.092$.

Case 1. $d \geq 0.11177$.

In this case, note that $\delta \geq d \geq 0.11177$. Recall that $b+c \geq 2d$, then $\beta \geq b+c \geq 0.22354$. Note that τ is non-decreasing if we change (β, δ) to $(\beta + \epsilon, \delta - \epsilon)$ for $\epsilon > 0$. So, we may assume that $\delta = 0.11177$. By Claim 5.4, $a \leq \frac{3-\sqrt{3}}{3} - d \leq 3 \times 0.11177 \leq b+c+d$, so $\alpha \leq \beta + \delta$. Therefore τ is non-decreasing if we change (α, β) to $(\alpha + \epsilon, \beta - \epsilon)$ for small ϵ , we may assume that $\beta = 0.22354$. Therefore

$$\tau = 0.0789(\alpha + 0.33531 + \eta)^3 + 0.33531\alpha\rho + (\eta\rho + \frac{\rho^2}{2})(\alpha + 0.22354) \tag{12}$$

subject to $\alpha + 0.33531 + \eta + \rho = 1$, $\alpha, \eta, \rho \geq 0$. If $\rho = 0$, then $\tau = 0.0789$. So we may assume that $\rho > 0$.

If $\alpha = 0$, then $\eta = 0.66469 - \rho$. So

$$\begin{aligned}
\tau &\leq -0.0789\rho^3 + 0.125\rho^2 - 0.08\rho + 0.0789 = f(\rho), \\
f'(\rho) &= -0.2367\rho^2 + 0.25\rho - 0.08 < 0.
\end{aligned}$$

Note that $f(\rho)$ is decreasing in $[0, 1]$, then $\tau_0 \leq f(0) \leq 0.0789$. So we may assume that $\alpha > 0$.

If $\eta > 0$, then by Theorem 4.1, we have $\frac{\partial \tau}{\partial \alpha} = \frac{\partial \tau}{\partial \eta}$, so $\alpha = 0.11177 + \eta + \frac{\rho}{2}$. Therefore $\rho = 1.55292 - 4\alpha$ and $\eta = 3\alpha - 0.88823$. So

$$\begin{aligned}
\tau &\leq 1.05\alpha^3 - 3.55\alpha^2 + 1.55\alpha - 0.094 = \tau_0 \\
\tau'_0 &= 3.15\alpha^2 - 7.1\alpha + 1.55.
\end{aligned}$$

Note that τ_0 is increasing in $[0, \frac{71-4\sqrt{193}}{63}]$, then $\tau_0 \leq 0.09$.

If $\eta = 0$, then $\rho = 0.66469 - \alpha > 0$. So

$$\begin{aligned}
\tau &= 0.0789(\alpha + 0.33531)^3 + 0.33531\alpha(0.66469 - \alpha) + \frac{(0.66469 - \alpha)^2}{2}(\alpha + 0.22354) \\
&\leq 0.5789\alpha^3 - 0.8088\alpha^2 + 0.3219\alpha + 0.0524 = \tau_0 \\
\tau'_0 &= 1.7367\alpha^2 - 1.6176\alpha + 0.3219.
\end{aligned}$$

Note that τ_0 is increasing in $[0, \frac{2696-\sqrt{1056819}}{5789}]$ or $[\frac{2696+\sqrt{1056819}}{5789}, 0.66469]$, then $\tau_0 \leq 0.092$.

Case 2. $d < 0.11177$.

In this case, we know that $b + c \geq 0.307$, so $\beta \geq 0.307$. By Claim 5.5, we know that $d \geq 0.0848$, so $\delta \geq 0.0848$. Note that τ is non-decreasing if we change (β, δ) to $(\beta + \epsilon, \delta - \epsilon)$ for $\epsilon > 0$, so we may let $\delta = 0.0848$ in τ . Therefore

$$\tau = 0.0789(\alpha + \beta + 0.0848 + \eta)^3 + (\alpha\beta + 0.0848\alpha)\rho + (\eta\rho + \frac{\rho^2}{2})(\alpha + \beta)$$

subject to

$$\begin{cases} \alpha + \beta + 0.0848 + \eta + \rho = 1, \\ \beta \geq 0.307, \\ \alpha, \rho \geq 0 \end{cases} \quad (13)$$

Note that $\rho > 0$.

If $\alpha = 0$, then we claim that $\beta = 0.307$, this is because that τ is non-decreasing if we change (α, β) to $(\alpha + \epsilon, \beta - \epsilon)$ for small ϵ . So $\eta = 0.6082 - \rho$. Substituting these into (13), we have

$$\begin{aligned} \tau &\leq -0.0789\rho^3 + 0.042\rho^2 + 0.002\rho + 0.0789 = f(\rho), \\ f'(\rho) &= -0.2367\rho^2 + 0.084\rho + 0.002. \end{aligned}$$

Note that $f(\rho)$ is increasing in $[0, \frac{2\sqrt{6215}+140}{789}]$ and decreasing in $[\frac{2\sqrt{6215}+140}{789}, 1]$, then $\tau \leq f(\frac{2\sqrt{6215}+140}{789}) < 0.085$. So we may assume that $\alpha > 0$.

If $\beta > 0.307$, then by Theorem 4.1, $\frac{\partial \tau}{\partial \alpha} = \frac{\partial \tau}{\partial \beta}$, simplifying it, we have $\alpha = \beta + 0.0848 > 0.3918$. If $\eta > 0$, then by Theorem 4.1, we have $\frac{\partial \tau}{\partial \alpha} = \frac{\partial \tau}{\partial \eta}$, simplifying it, we get $\alpha = 0.0848 + \eta + \frac{\rho}{2}$. Hence $1.1754 < 3\alpha = \alpha + \beta + 2 \times 0.0848 + \eta + \frac{\rho}{2} < 1 + 0.0848 = 1.0848$, a contradiction. So $\eta = 0$. Then $\rho = 1 - \alpha - \beta - 0.0848 = 1 - 2\alpha > 0$ and $\beta = \alpha - 0.0848$. So $0.39 < \alpha < 0.5$. And

$$\begin{aligned} \tau &= 0.0789(2\alpha)^3 + \alpha^2(1 - 2\alpha) + \frac{(1 - 2\alpha)^2}{2}(2\alpha - 0.0848) \\ &\leq 2.65\alpha^3 - 3.1695\alpha^2 + 1.1695\alpha - 0.0424 = \tau_0, \\ \tau'_0 &= 7.95\alpha^2 - 6.339\alpha + 1.1695. \end{aligned}$$

Note that τ_0 is decreasing in $[0.39, 0.5]$, then $\tau_0 \leq \tau_0(0.39) \leq 0.089$.

So we may assume that $\beta = 0.307$. If $\eta > 0$, then by Theorem 4.1, $\frac{\partial \tau}{\partial \alpha} = \frac{\partial \tau}{\partial \eta}$, simplifying it, we have $\alpha = 0.0848 + \eta + \frac{\rho}{2}$. Since $\alpha + \eta + \rho = 1 - \beta - 0.0848 = 0.6082$, then $\eta = 3\alpha - 0.7778$ and $\rho = 1.386 - 4\alpha > 0$. Since $\frac{\partial \tau}{\partial \eta} = \frac{\partial \tau}{\partial \rho}$, then $0.2367(\alpha + \beta + 0.0848 + \eta)^2 = \alpha\beta + 0.0848\alpha + \eta(\alpha + \beta)$. By direct calculation, $\frac{492\alpha^2}{625} - \frac{395603\alpha}{312500} + \frac{685129883}{2500000000} = 0$, and $\alpha = \frac{395603 - 20\sqrt{180576895}}{492000}$. But $3\alpha < 0.774$, then $\eta = 3\alpha - 0.7778 < 0$, a contradiction. So $\eta = 0$, then $\rho = 1 - \alpha - \beta - 0.0848 = 0.6082 - \alpha > 0$. So

$$\begin{aligned} \tau &= 0.0789(\alpha + 0.3918)^3 + 0.3918\alpha(0.6082 - \alpha) + \frac{(0.6082 - \alpha)^2}{2}(\alpha + 0.307) \\ &\leq 0.5789\alpha^3 - 0.75375\alpha^2 + 0.27286\alpha + 0.06153 = \tau_0, \\ \tau'_0 &= 1.7367\alpha^2 - 1.5075\alpha + 0.27286. \end{aligned}$$

Note that τ_0 is increasing in $[0, \frac{5025}{11578} - \frac{\sqrt{942631005}}{173670}]$, then $\tau_0 \leq 0.092$. This complete the proof of Claim

5.15. □

7.5 Proof of Claim 5.16

By Claim 5.15, we have

$$\begin{aligned}
\lambda(G) &\leq 0.092(a+b+c+d+e+f)^3 + (ab+ac)g + abh + (eg+eh+fg+fh + \frac{g^2}{2} + gh \\
&\quad + \frac{h^2}{2})(a+b+c+d) \\
&\leq 0.092(a+\beta+d+\zeta)^3 + a\beta\eta + (\zeta\eta + \frac{\eta^2}{2})(a+\beta+d) \\
&= \lambda,
\end{aligned}$$

where $\eta = g + h$, $\beta = b + c$ and $\zeta = e + f$.

Case 1. $d \geq 0.11177$.

Since replacing (a, d) by $(a+\epsilon, d-\epsilon)$ for $\epsilon > 0$ will not decrease λ , so we may assume that $d = 0.11177$. Let $\alpha = a + \beta$. Then

$$\lambda \leq 0.092(\alpha + \delta + \zeta)^3 + \frac{\alpha^2\eta}{4} + (\zeta\eta + \frac{\eta^2}{2})(\alpha + \delta),$$

subject to

$$\begin{cases} \alpha + \delta + \zeta + \eta = 1, \\ \delta = 0.11177, \alpha, \zeta, \eta \geq 0. \end{cases} \tag{14}$$

If $\eta = 0$, then $\lambda = 0.092$, we are done. So assume that $\eta > 0$.

If $\alpha = 0$, then $\zeta = 0.88823 - \eta$.

$$\begin{aligned}
\lambda &\leq -0.092\eta^3 + 0.221\eta^2 - 0.1767\eta + 0.092 = \lambda_0 \\
\lambda'_0 &= -0.276\eta^2 + 0.442\eta - 0.1767.
\end{aligned}$$

Note that λ_0 is decreasing in $[0, 0.7700236]$ or $[0.8314257, 1]$. Therefore $\lambda_0 \leq 0.095$. So assume that $\alpha > 0$.

If $\zeta = 0$, then $\eta = 0.88823 - \alpha > 0$. So

$$\begin{aligned}
\lambda &\leq 0.342\alpha^3 - 0.58\alpha^2 + 0.3\alpha + 0.045 = \lambda_0 \\
\lambda'_0 &= 1.026\alpha^2 - 1.16\alpha + 0.3.
\end{aligned}$$

Note that λ_0 is increasing in $[0, \frac{290-5\sqrt{286}}{513}]$ or $[\frac{290+5\sqrt{286}}{513}, 0.88823]$. Therefore $\lambda_0 \leq 0.095$.

If $\zeta > 0$, then by Theorem 4.1, we have $\frac{\partial\lambda}{\partial\alpha} = \frac{\partial\lambda}{\partial\zeta}$, i.e. $2\zeta + \eta = \alpha + 2\delta$. Since $\alpha + \delta + \zeta + \eta = 1$, then $\zeta = 2\alpha - 0.66469$ and $\eta = 1.55292 - 3\alpha > 0$. So $0.33234 < \alpha < 0.51764$. And

$$\begin{aligned}
\lambda &\leq 0.234\alpha^3 - 0.7115\alpha^2 + 0.4765\alpha + 0.00385 = \lambda_0 \\
\lambda'_0 &= 0.702\alpha^2 - 1.423\alpha + 0.4765.
\end{aligned}$$

Note that λ_0 is increasing in $[0, \frac{1423-11\sqrt{5677}}{1404}]$. Therefore $\lambda_0 \leq 0.096$.

Case 2. $d < 0.11177$.

In this case, we have $\beta \geq 0.307$. By Claim 5.5, then $d \geq 0.0848$. Since replacing (a, d) by $(a + \epsilon, d - \epsilon)$ for $\epsilon > 0$ will not decrease λ , so we may assume $d = 0.0848$. So

$$\lambda = 0.092(a + \beta + \delta + \zeta)^3 + a\beta\eta + (\zeta\eta + \frac{\eta^2}{2})(a + \beta + \delta),$$

subject to

$$\begin{cases} a + \beta + \delta + \zeta + \eta = 1, \\ \beta \geq 0.307, \\ \delta = 0.0848. \end{cases} \quad (15)$$

If $a = 0$, we claim that $\beta = 0.307$ since λ is non-decreasing if we change (a, β) to $(a + \epsilon, \beta - \epsilon)$ for small ϵ . So $\zeta = 0.6082 - \eta$. Then

$$\begin{aligned} \lambda &\leq -0.092\eta^3 + 0.0801\eta^2 - 0.037\eta + 0.092 = f(\eta), \\ f'(\eta) &= -0.276\eta^2 + 0.1602\eta - 0.037 < 0. \end{aligned}$$

Therefore $\lambda_0 \leq f(0) = 0.092$. So we may assume that $a > 0$.

If $\beta > 0.307$, then by Theorem 4.1, $\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial \beta}$, simplifying it, we get $a = \beta > 0.307$. If $\zeta > 0$, then we have $\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial \zeta}$, i.e. $a + \delta = \zeta + \frac{\eta}{2}$. So $1 < 3a + 2\delta = a + \beta + \delta + \zeta + \frac{\eta}{2} < 1$, a contradiction. Then $\zeta = 0$. So $\eta = 1 - a - \beta - \delta = 1 - 2a - \delta = 0.9152 - 2a > 0$, then $a < 0.4576$. Recall that $\delta = 0.0848$. So

$$\lambda = 0.092(2a + 0.0848)^3 + (0.9152 - 2a)a^2 + \frac{(0.9152 - 2a)^2}{2}(2a + 0.0848).$$

By a direct calculation on the derivative of λ_0 , we obtain that λ_0 is increasing in $[0, 0.2138466]$. Therefore $\lambda_0 \leq 0.096$.

So we may assume that $\beta = 0.307$ and $\delta = 0.0848$. If $\zeta = 0$, then $\eta = 1 - a - \beta - \delta = 0.6082 - a$. By Theorem 4.1, then $\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial \eta}$. Combining them, we get $0.276(a + 0.3918)^2 + \frac{(0.6082 - a)^2}{2} = (0.6082 - a)(a + 0.0848) + 0.307a$. Solving the equation for a and substituting the values into λ , we obtain that $\lambda \leq 0.095$. So we may assume that $\zeta > 0$. By Theorem 4.1, then $\frac{\partial \lambda}{\partial a} = \frac{\partial \lambda}{\partial \zeta}$, i.e. $a + \delta = \zeta + \frac{\eta}{2}$. So $\eta = 1.0468 - 4a$ and $\zeta = 3a - 0.4386$. Then $0.1462 < a < 0.2617$ and

$$\begin{aligned} \lambda &= 0.092(4a - 0.0468)^3 + 0.307(1.0468 - 4a)a + ((3a - 0.4386)(1.0468 - 4a) \\ &\quad + \frac{(1.0468 - 4a)^2}{2})(a + 0.3918). \end{aligned}$$

By a direct calculation on the derivative of λ , we obtain that λ is increasing in $[0, \frac{716959}{1770000} - \frac{\sqrt{211982461}}{70800}]$ or $[\frac{716959}{1770000} + \frac{\sqrt{211982461}}{70800}, 1]$. Recall that $0.1462 < a < 0.2617$, then $\lambda < 0.0961$. \square

7.6 Proof of Claim 5.18

If $f = 0$, then

$$\lambda = \frac{\alpha^2(c+d+e)}{4} + \alpha c(d+e) + \alpha(de + \frac{e^2}{2}) + \frac{d^2(\alpha+e)}{4}$$

subject to

$$\begin{cases} \alpha + c + d + e = 1, \\ c \geq 0.08. \end{cases} \quad (16)$$

If $d = 0$, then $\lambda = \frac{\alpha^2(c+e)}{4} + \alpha ce + \alpha \frac{e^2}{2} \leq \lim_{n \rightarrow \infty} B(2, n-2) = \frac{\sqrt{3}}{18}$. So we are done.

If $e = 0$, then $\lambda = \frac{\alpha^2(c+d)}{4} + \alpha cd + \frac{\alpha d^2}{4} \leq \lambda(K_5^3) < \frac{\sqrt{3}}{18}$. So we are done.

So we may assume that $d, e > 0$. By direct calculation,

$$\begin{aligned} \frac{\partial \lambda}{\partial \alpha} &= \frac{\alpha(c+d+e)}{2} + cd + ce + de + \frac{e^2}{2} + \frac{d^2}{4} \\ \frac{\partial \lambda}{\partial c} &= \frac{\alpha^2}{4} + \alpha d + \alpha e \\ \frac{\partial \lambda}{\partial d} &= \frac{\alpha^2}{4} + \alpha c + \alpha e + \frac{d(\alpha+e)}{2} \\ \frac{\partial \lambda}{\partial e} &= \frac{\alpha^2}{4} + \alpha c + \alpha d + \alpha e + \frac{d^2}{4}. \end{aligned}$$

By Theorem 4.1, $\frac{\partial \lambda}{\partial d} = \frac{\partial \lambda}{\partial e}$, combining with the above equations, we get $d = 2e - 2\alpha$. If $c > 0.08$, then by Theorem 4.1, $\frac{\partial \lambda}{\partial c} = \frac{\partial \lambda}{\partial e}$, combining the equations, we get $\alpha c + \frac{d^2}{4} = 0$, a contradiction. So $c = 0.08$, therefore $\alpha + d + e = 0.92$, combining with $d = 2e - 2\alpha$, we get $d = \frac{1.84-4\alpha}{3}$ and $e = \frac{\alpha+0.92}{3}$. Since $\frac{\partial \lambda}{\partial \alpha} = \frac{\partial \lambda}{\partial e}$, then $\frac{5\alpha^2}{36} + \frac{307\alpha}{450} - \frac{3473}{11250} = 0$. Then $\alpha = \frac{3\sqrt{14331}-307}{125}$. By direct calculation, $\lambda \leq 0.096 < \frac{\sqrt{3}}{18}$, a contradiction. \square

7.7 Proof of Claim 5.19

If $e = 0$, in view of (6), then

$$\lambda = \frac{\alpha^2(c+d+f)}{4} + \alpha cd + (\alpha+c)(df + \frac{f^2}{2}) + \frac{d^2(\alpha+f)}{4},$$

subject to

$$\begin{cases} \alpha + c + d + f = 1, \\ c \geq 0.08; \alpha, d, f \geq 0. \end{cases} \quad (17)$$

If $d = 0$, then we have

$$\lambda = \frac{\alpha^2(c+f)}{4} + (\alpha+c)\frac{f^2}{2}.$$

We relax the constraint $c \geq 0.08$ to $c \geq 0$ and $\alpha + c + f = 1$. If $c = 0$, then $\lambda = \frac{\alpha^2(1-\alpha)}{4} + \alpha \frac{(1-\alpha)^2}{2} \leq \frac{\sqrt{3}}{18}$ by Fact 4.9. If $c > 0$, recall that $f > 0$, then by Theorem 4.1, we have $\frac{\partial \lambda}{\partial c} = \frac{\partial \lambda}{\partial f}$, then $f = 2(\alpha + c)$,

combining with $\alpha + c + f = 1$, we have $f = \frac{2}{3}$ and $c = \frac{1}{3} - \alpha$. So $\lambda = \frac{\alpha^2(1-\alpha)}{4} + \frac{2}{27}$, where $\alpha < \frac{1}{3}$. Since $\lambda' = \frac{\alpha(2-3\alpha)}{4}$, then λ is increasing in $[0, \frac{1}{3}]$, i.e. $\lambda \leq \frac{(\frac{1}{3})^2 \frac{2}{3}}{4} + \frac{2}{27} = \frac{5}{54} \leq \frac{\sqrt{3}}{18}$.

So we may assume that $d > 0$. Recall that $e + f > \alpha + \frac{d}{2}$ and $e = 0$, so $f > \alpha + \frac{d}{2}$. By direct calculation,

$$\begin{aligned}\frac{\partial \lambda}{\partial \alpha} &= \frac{\alpha(c+d+f)}{2} + cd + df + \frac{f^2}{2} + \frac{d^2}{4} \\ \frac{\partial \lambda}{\partial c} &= \frac{\alpha^2}{4} + \alpha d + df + \frac{f^2}{2} \\ \frac{\partial \lambda}{\partial d} &= \frac{\alpha^2}{4} + \alpha c + (\alpha + c)f + \frac{d(\alpha + f)}{2} \\ \frac{\partial \lambda}{\partial f} &= \frac{\alpha^2}{4} + (\alpha + c)(d + f) + \frac{d^2}{4}.\end{aligned}$$

By Theorem 4.1, we have $\frac{\partial \lambda}{\partial d} = \frac{\partial \lambda}{\partial f}$, i.e.

$$\frac{d^2}{4} + (\alpha + c)d = \alpha c + \frac{d(\alpha + f)}{2} > \alpha c + \frac{d(2\alpha + \frac{d}{2})}{2},$$

then $d > \alpha$. If $c > 0.08$, then $\frac{\partial \lambda}{\partial \alpha} = \frac{\partial \lambda}{\partial c}$. Therefore $\frac{\alpha^2}{4} + \alpha d = \frac{\alpha(c+d+f)}{2} + cd + \frac{d^2}{4}$. Since $f > \alpha + \frac{d}{2}$, then $\frac{\alpha^2}{4} + \alpha d > \frac{\alpha(c+\alpha+\frac{3d}{2})}{2} + cd + \frac{d^2}{4} > \frac{\alpha^2}{2} + \frac{d^2+3\alpha d}{4}$. So $\alpha d > \alpha^2 + d^2$, a contradiction. So $c = 0.08$. Since $\frac{\partial \lambda}{\partial \alpha} = \frac{\partial \lambda}{\partial f}$ and $f = 0.92 - \alpha - d$, then

$$\frac{\alpha^2}{4} + \alpha d + \alpha f + cf - \frac{\alpha(1-\alpha)}{2} - df - \frac{f^2}{2} = 0.$$

Substituting $c = 0.08$ and $f = 0.92 - \alpha - d$ into it, we have

$$\frac{\alpha^2}{4} + \alpha d + \alpha(0.92 - \alpha - d) + 0.08(0.92 - \alpha - d) - \frac{\alpha(1-\alpha)}{2} - d(0.92 - \alpha - d) - \frac{(0.92 - \alpha - d)^2}{2} = 0.$$

Simplifying it, we have $\frac{d^2}{2} - \frac{2d}{25} + \frac{63\alpha}{50} - \frac{3\alpha^2}{4} - \frac{437}{1250} = 0$, then $d = \sqrt{\frac{3\alpha^2}{2} - \frac{63\alpha}{25} + \frac{441}{625}} + \frac{2}{25}$. Note that $0.92 = \alpha + d + f > \alpha + \alpha + \frac{3\alpha}{2} = \frac{7\alpha}{2}$, then $0 < \alpha < 0.2629$. Therefore $\phi(\alpha) = \frac{3\alpha^2}{2} - \frac{63\alpha}{25} + \frac{441}{625} > \phi(0.2629) > 0.1467$. So $d > \sqrt{0.1467} + \frac{2}{25} > 0.46$ and $f > 0.23 + \alpha$, then $\alpha + d + f > \alpha + \sqrt{\frac{3\alpha^2}{2} - \frac{63\alpha}{25} + \frac{441}{625}} + \frac{2}{25} + 0.23 + \alpha \triangleq \psi(\alpha) > \psi(0) = 1.15 > 1$, a contradiction. \square

7.8 Proof of Claim 5.20

If $d = 0$, then we have $\alpha + c + e + f = 1$ and $\alpha = e$. In view of (6),

$$\lambda = \frac{\alpha^2(c+e+f)}{4} + \alpha ce + \alpha \frac{e^2}{2} + (\alpha + c)(ef + \frac{f^2}{2}).$$

Then

$$\begin{aligned}\frac{\partial \lambda}{\partial \alpha} &= \frac{\alpha(c+e+f)}{2} + ce + \frac{e^2}{2} + ef + \frac{f^2}{2} \\ \frac{\partial \lambda}{\partial c} &= \frac{\alpha^2}{4} + \alpha e + ef + \frac{f^2}{2} \\ \frac{\partial \lambda}{\partial e} &= \frac{\alpha^2}{4} + \alpha c + \alpha e + (\alpha + c)f,\end{aligned}$$

If $c > 0.08$, then by Theorem 4.1, we have $\frac{\partial \lambda}{\partial \alpha} = \frac{\partial \lambda}{\partial c}$, so $\frac{\alpha(1-\alpha)}{2} + \alpha c + \frac{\alpha^2}{2} = \frac{\alpha^2}{4} + \alpha^2$, then $0.08 < c = \frac{5\alpha-2}{4}$ and $0 < f = \frac{6-13\alpha}{4}$. So $0.464 < \alpha < \frac{6}{13} < 0.462$, a contradiction.

If $c = 0.08$, then $f = 0.92 - 2\alpha$. So

$$\begin{aligned}\lambda &= \frac{\alpha^2(1-\alpha)}{4} + 0.08\alpha^2 + \frac{\alpha^3}{2} + (\alpha + 0.08)(\alpha(0.92 - 2\alpha) + \frac{(0.92 - 2\alpha)^2}{2}) \\ &= \frac{\alpha^3}{4} - \frac{59\alpha^2}{100} + \frac{437\alpha}{1250} + \frac{529}{15625}. \\ \lambda' &= \frac{3\alpha^2}{4} - \frac{59\alpha}{50} + \frac{437}{1250}.\end{aligned}$$

Note that λ is increasing in $[0, \frac{59-\sqrt{859}}{75}]$, then $\lambda \leq 0.0955$. □

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