# ELLIPTIC EQUATIONS WITH DEGENERATE WEIGHTS 

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#### Abstract

We obtain new local Calderón-Zygmund estimates for elliptic equations with matrix-valued weights for linear as well as non-linear equations. We introduce a novel $\log -\mathrm{BMO}$ condition on the weight $\mathbb{M}$. In particular, we assume smallness of the logarithm of the matrix-valued weight in BMO. This allows to include degenerate, discontinuous weights. We provide examples that show the sharpness of the estimates in terms of the log-BMO-norm.


## 1. Introduction and Statement of the Results

We study weak solutions of elliptic equations with degenerate matrix-valued weights. We consider both the linear case as well as the non-linear case. The main concern is the transfer of regularity from the data $G: \Omega \rightarrow \mathbb{R}^{n}$ to the weak solution $u: \Omega \rightarrow \mathbb{R}$ of the equation

$$
\begin{equation*}
-\operatorname{div}(\mathbb{A}(x) \nabla u)=-\operatorname{div}(\mathbb{A}(x) G) \tag{1.1}
\end{equation*}
$$

in the linear case and of the equation

$$
\begin{equation*}
-\operatorname{div}\left(|\mathbb{M}(x) \nabla u|^{p-2} \mathbb{M}^{2}(x) \nabla u\right)=-\operatorname{div}\left(|\mathbb{M}(x) G|^{p-2} \mathbb{M}^{2}(x) G\right) \tag{1.2}
\end{equation*}
$$

for the non-linear case. Here $\Omega \subset \mathbb{R}^{n}$ is a domain, $1<p<\infty, \mathbb{A}, \mathbb{M}: \Omega \rightarrow \mathbb{R}_{\text {sym }}^{n \times n}$ are symmetric, positive definite, matrix-valued weights (almost everywhere) with $\mathbb{M}=\mathbb{A}^{\frac{1}{2}}$, and $\nabla u$ is the column vector $\left(\partial_{1} u, \ldots, \partial_{n} u\right)^{T}$ and $|\cdot|$ denote the usual euclidean distance on $\mathbb{R}^{n}$. We do not need to specify boundary values, since we only look at local solutions. If $p=2$, then the non-linear equation (1.2) reduces to linear one (1.1).

The main objective of our paper is to transfer regularity of the data $G$ in terms of weighted Lebesgue spaces $L_{\omega}^{\rho}$ to the gradient of the solution. Very roughly speaking we want to prove an estimate of the form

$$
\begin{equation*}
\|\nabla u\|_{L_{\omega}^{\rho}(B)} \lesssim\|G\|_{L_{\omega}^{\rho}(2 B)}+\text { lower order terms } \tag{1.3}
\end{equation*}
$$

where $L_{\omega}^{\rho}$ is the natural corresponding weighted Lebesgue space. We treat the weights $\omega$ in the multiplicative sense, i.e. $\|\nabla u\|_{L_{\omega}^{\rho}(B)}=\|\omega \nabla u\|_{L^{\rho}}(B)$ and $L_{\omega}^{\rho}(B)$ corresponds to $L^{\rho}\left(B, \omega^{\rho} d x\right)$.

[^0]Let us start with the linear case in the form of (1.1) with $\mathbb{A}=\mathbb{M}=\operatorname{Id}$ (unweighted case; Id is the identity matrix), then (1.3) just follows from the standard $L^{q}$-estimates for the Laplacian using Calderón-Zygmund theory for all $\rho \in$ $(1, \infty)$. If $\mathbb{A}$ is a uniformly elliptic weight $\mathbb{A}(x)$, i.e. $\lambda_{1}|\xi|^{2} \leq\langle\mathbb{A}(x) \xi, \xi\rangle \leq \Lambda_{1}|\xi|^{2}$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^{n}$, then Meyers in [39] proved (1.3) for $\rho \in[2,2+\varepsilon$ ) for some $\varepsilon>0$. He achieved this by representing the equation with weight as a perturbation of the Laplacian. The same idea earlier was used by Boyarskiĭ in [3] for the problem on the plane. The case of bounded and uniformly continuous (and therefore non-degenerate) positive definite weights has been studied for example by Morrey [40]. This has been extended by Di Fazio [13] to the case of uniformly elliptic weights, i.e. $\lambda_{1}|\xi|^{2} \leq\langle\mathbb{A}(x) \xi, \xi\rangle \leq \Lambda_{1}|\xi|^{2}$ for all $x \in \Omega$ and all $\xi \in \mathbb{R}^{n}$, with $\mathbb{A} \in \mathrm{VMO}$ (vanishing mean oscillation). The case of systems has been obtained by Di Fazio, Fanciullo and Zamboni [14]. The global result for equations has been obtained by Iwaniec and Sbordone [31].

Due to the assumed uniform ellipticity, the results above exclude the possibility of degenerate weights like $|x|^{ \pm \varepsilon}$ Id. Fabes, Kenig and Serapioni [23] studied the case, where

$$
\begin{equation*}
\Lambda^{-1} \mu(x)|\xi|^{2} \leq\langle\mathbb{A}(x) \xi, \xi\rangle \leq \Lambda \mu(x)|\xi|^{2} \tag{1.4}
\end{equation*}
$$

for some non-negative weight $\mu$. This is equivalent to say that $\mathbb{A}(x)$ has a uniformly bounded condition number $\Lambda^{2}$. Fabes, Kenig and Serapioni proved that $u$ is Hölder continuous provided that $\mu$ is in the Muckenhoupt class $\mathcal{A}_{2}$. Cao, Mengesha and Phan [6] have considered gradient estimates in the linear case also under the condition (1.4). Additionally, they assumed that $\mu$ is of Muckenhoupt class $\mathcal{A}_{2}$ and that $\mathbb{A}$ has small $\mathrm{BMO}_{\mu}^{2}$ norm, where

$$
|\mathbb{A}|_{\mathrm{BMO}_{\mu}^{s}}=\sup _{B}\left(\frac{1}{\mu(B)} \int_{B}\left|\frac{\left|\mathbb{A}(x)-\langle\mathbb{A}\rangle_{B}\right|}{\mu(x)}\right|^{s} \mu(x) d x\right)^{\frac{1}{s}}
$$

for $s \geq 1$, where the supremum is taken over all balls $B$ and $\langle\mathbb{A}\rangle_{B}$ denotes the meanvalue over the ball $B$. (If $\mu$ is of Muckenhoupt class $\mathcal{A}_{1}$, then $\mathrm{BMO}_{\mu}^{1}=\mathrm{BMO}_{\mu}^{s}$ with equivalence of norms, see [24].) Under these conditions Cao, Mengesha and Phan proved that $|G|^{q} \mu \in L_{\text {loc }}^{1}$ implies $|\nabla u|^{q} \mu \in L_{\text {loc }}^{1}$. Their condition allowed to include weights like $|x|^{ \pm \varepsilon}$ Id for small $\varepsilon>0$. The case of systems has been covered by the same authors in [7]. Our condition on the weight $\mathbb{A}$ differs somehow from the previous ones. Instead of a BMO or $\mathrm{BMO}_{\mu}^{2}$ smallness condition for $\mathbb{A}$, we use a BMO smallness condition on its $\operatorname{logarithm} \log \mathbb{A}$ or equivalently on $\log \mathbb{M}=$ $\frac{1}{2} \log \mathbb{A}$. This new $\log$-BMO-condition allows us also to include the degenerate weights, for example $\mathbb{M}(x):=|x|^{\varepsilon} \operatorname{Id}$ and $\mathbb{M}(x):=|x|^{-\varepsilon}$ Id for small $\varepsilon>0$. We will show in Section 4 by a counterexample that this log-BMO condition is sharp in terms of the achievable higher integrability exponent $q$.

To our knowledge the log-BMO condition is new even in the context of linear equations.

Let us also mention that our log-BMO condition as well as the $\mathrm{BMO}_{\mu}^{2}$-condition of Cao, Mengesha and Phan are invariant under scaling of the equation in the following sense. If we scale $\mathbb{M}$ by a factor $t>0$ and $u$ and $G$ by $1 / t$ (which will scale $\omega$ by $t$ and $\mathbb{A}$ and $\mu$ by $\sqrt{t}$ ), then the equation remains valid. Moreover, $|\nabla u| \omega$ and $|G| \omega$ are scaling invariant. Thus, the condition on the weight $\mathbb{M}$ for the
higher integrability of $|\nabla u| \omega$ with respect to $|G| \omega$ should be invariant under this scaling. Now, our log-BMO condition is scaling invariant, since $|\log (t \mathbb{M})|_{\mathrm{BMO}}=$ $|\log (\mathbb{M})|_{\text {BMO }}$. Note however, that the condition $\mathbb{A} \in$ VMO by Di Fazio[13] is not scaling invariant and therefore not natural.

Our main result differs also from the one of Cao, Mengesha and Phan [6] since we treat the weight $\omega$ rather as a multiplier than a measure. Cao, Mengesha and Phan show for $p=2$ that $|G|^{\rho} \omega^{2} \in L_{\text {loc }}^{1}$ implies $|\nabla u|^{\rho} \omega^{2} \in L_{\mathrm{loc}}^{1}\left(\right.$ recall $\left.\mu=\Lambda^{-1} \omega^{2}\right)$, so the weight stays the same for all exponents. We, on the other hand, show that $|G| \omega \in L_{\mathrm{loc}}^{\rho}$ implies $|\nabla u| \omega \in L_{\mathrm{loc}}^{\rho}$. So in our situation the original weight $\omega^{2}$ (for $p=2)$ changes to $\omega^{\rho}$.

Let us also mention that Baison, Clop, Giova, Orobitg and Passarelli di Napoli in [1] derived estimates in Besov and Triebel- Lizorkin spaces for slightly nonlinear system (non-linear but linear growth) for uniformly elliptic weights given in a suitable Besov space.

We now turn to the non-linear equations in the form of (1.2). Recall that the linear equation (1.1) is just the special case $p=2$. Let us abbreviate

$$
\begin{align*}
A(\xi) & :=|\xi|^{p-2} \xi \\
\mathcal{A}(x, \xi) & :=|\mathbb{M}(x) \xi|^{p-2} \mathbb{M}^{2} \xi=\mathbb{M} A(\mathbb{M} \xi) \tag{1.5}
\end{align*}
$$

Note that we use the upright letter $A$ for the unweighted version and the calligraphic letter $\mathcal{A}$ for the weighted version. Then we can rewrite (1.2) as

$$
\begin{equation*}
-\operatorname{div} \mathcal{A}(\cdot, \nabla u)=-\operatorname{div} \mathcal{A}(\cdot, G) \tag{1.6}
\end{equation*}
$$

Sometimes in literature, e.g. [33], the system is also given as

$$
\begin{equation*}
-\operatorname{div}\left(\langle\mathbb{A} \nabla u, \nabla u\rangle^{\frac{p-2}{2}} \mathbb{A} \nabla u\right)=-\operatorname{div}\left(|F|^{p-2} F\right) \tag{1.7}
\end{equation*}
$$

with some positive definite matrix-valued weight $\mathbb{A}: \Omega \rightarrow \mathbb{R}_{\text {sym }}^{n \times n}$ (almost everywhere) and $F: \Omega \rightarrow \mathbb{R}^{n}$ is the given data. With the transformation $\mathbb{M}=\mathbb{A}^{\frac{1}{2}}$ and $\mathcal{A}(\cdot, G)=A(F)$ we can pass from (1.7) to (1.2) and vice versa.

Note that $u$ is just the local minimizer of the energy

$$
\begin{aligned}
\mathcal{J}(v) & :=\int_{\Omega} \frac{1}{p}\left(\langle\mathbb{A} \nabla v, \nabla v\rangle^{\frac{p}{2}} d x-\int_{\Omega}|F|^{p-2} F \cdot \nabla v d x\right. \\
& =\int_{\Omega} \frac{1}{p}|\mathbb{M} \nabla v|^{p} d x-\int_{\Omega}|\mathbb{M} G|^{p-2} \mathbb{M} G \cdot(\mathbb{M} \nabla v) d x .
\end{aligned}
$$

We assume that our matrix-valued weight has a uniformly bounded condition number, i.e.

$$
\begin{equation*}
|\mathbb{M}(x)|\left|\mathbb{M}^{-1}(x)\right| \leq \Lambda \quad \text { for all } x \in \Omega \tag{1.8}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
|\mathbb{A}(x)|\left|\mathbb{A}^{-1}(x)\right| \leq \Lambda^{2} \quad \text { for all } x \in \Omega \tag{1.9}
\end{equation*}
$$

Let us define

$$
\begin{equation*}
\omega(x):=|\mathbb{M}(x)|=|\mathbb{A}(x)|^{\frac{1}{2}} \tag{1.10}
\end{equation*}
$$

Then (1.9) is also equivalent to

$$
\begin{equation*}
\Lambda^{-2} \omega^{2}(x)|\xi|^{2} \leq\langle\mathbb{A}(x) \xi, \xi\rangle \leq \omega^{2}(x)|\xi|^{2} \quad \text { for all } \xi \in \mathbb{R}^{n} \text { and almost all } x \in \Omega \tag{1.11}
\end{equation*}
$$

Note that (1.4) with the choice $\mu(x):=\Lambda^{-1} \omega^{2}(x)$ is exactly our condition (1.11). Since $\mathbb{M}(x)$ is positive definite, we can define its logarithm $\log \mathbb{M}(x)$ either by transformation to a diagonal matrix or by Taylor series. Note that $\log \mathbb{M}=\frac{1}{2} \log \mathbb{A}$. As usual let us denote by $f_{B} \cdots d x$ the mean value integral.

Our main result is as follows.
Theorem 1 (Linear Case). Let $u$ be a local weak solution of (1.1), let $\mathbb{A}$ satisfy (1.9), define $\omega$ by (1.10). Then there exists $\kappa=\kappa(p, n, \Lambda)$ such that for all balls $B_{0}$ with $4 B_{0} \subset \Omega$ and all $\rho \in(1, \infty)$ with

$$
\begin{equation*}
|\log \mathbb{A}|_{\mathrm{BMO}\left(4 B_{0}\right)} \leq \kappa \min \left\{\frac{1}{\rho}, 1-\frac{1}{\rho}\right\} \tag{1.12}
\end{equation*}
$$

there holds

$$
\begin{equation*}
\left(f_{B_{0}}(|\nabla u| \omega)^{\rho} d x\right)^{\frac{1}{\rho}} \leq c_{\rho_{0}} f_{2 B_{0}}|\nabla u| \omega d x+c_{q_{0}}\left(f_{2 B_{0}}(|G| \omega)^{\rho} d x\right)^{\frac{1}{\rho}} \tag{1.13}
\end{equation*}
$$

for all balls $B_{0}$ with $4 B_{0} \subset \Omega$.
Theorem 2 (Non-Linear Case). Let $u$ be a local weak solution of (1.2), let $\mathbb{M}$ satisfy (1.8), define $\omega$ by (1.10). Then there exists $\kappa=\kappa(p, n, \Lambda)$ such that for all balls $B_{0}$ with $8 B_{0} \subset \Omega$ and all $\rho \in[p, \infty)$ with

$$
\begin{equation*}
|\log \mathbb{M}|_{\mathrm{BMO}\left(8 B_{0}\right)} \leq \kappa \frac{1}{\rho} \tag{1.14}
\end{equation*}
$$

there holds

$$
\begin{equation*}
\left(f_{\frac{1}{2} B_{0}}(|\nabla u| \omega)^{\rho} d x\right)^{\frac{1}{\rho}} \leq c_{\rho} f_{4 B_{0}}|\nabla u| \omega d x+c_{\rho}\left(f_{4 B_{0}}(|G| \omega)^{\rho} d x\right)^{\frac{1}{\rho}} \tag{1.15}
\end{equation*}
$$

for all balls $B_{0}$ with $8 B_{0} \subset \Omega$, where $c_{\rho}=c_{\rho}(p, n, \Lambda, \rho)$. The constant $c_{\rho}$ is continuous in $\rho$.

Note that Theorem 1 holds for all $\rho \in(1, \infty)$, while the non-linear case of Theorem 2 requires $\rho \geq p$. We write below more on this difference.

The case $1<p<\infty$ with $\mathbb{A}=\mathbb{M}=\operatorname{Id}$ (unweighted case) has been obtained by Iwaniec [29] and Di Benedetto and Manfredi [15]. The limiting case $\rho=\infty$ is slightly different and better expressed in terms of $\mathcal{A}(\cdot, \nabla u)$ and $F$. It has been shown by Di Benedetto and Manfredi [15] for $p>2$ and by Diening, Kaplický and Schwarzacher [18] for all $1<p<\infty$ that $F \in$ BMO implies $\mathcal{A}(\cdot, \nabla u) \in$ BMO. In [18] is has also been shown that BMO can be replaced by $C^{0, \alpha}$ for small $\alpha>0$. All of these results are also consequences of the point-wise estimates obtained in [4] by Breit, Cianchi, Diening, Kuusi and Schwarzacher. The same authors proved estimates up to the boundary in [5]. Calderón-Zygmund estimates in the space $W^{1,2}$ have been studied by Cianchi and Maz'ya [9]. Estimates in Besov and TriebelLizorkin spaces up to differentiability with arbitrary integrability one have been studied in the planar case of equations for $p>2$ by Balci, Diening and Weimar [2]. Gradient estimates for the right hand side in non-divergence form were obtained for
equations by Kuusi and Mingione see [36], [35]. The case of systems was considered by Duzaar and Mingione in [22] and by Kuusi and Mingione in [37], [38].

Let us turn to the weighted case. In [33] Kinnunen and Zhou extended (1.15) to the case $1<p<\infty$, also for uniformly elliptic weights with $\mathbb{A} \in \mathrm{VMO}$ (vanishing mean oscillation). It is also enough to assume that $\mathbb{A}$ has small BMO-norm. They have also obtained global results in [34]. Note that both conditions are not scaling invariant (as mentioned above).

The condition $\rho \geq p$ in our theorem is due to the non-linear situation $p \neq 2$. The case $\rho=p$ corresponds to the context of weak solutions, while $\max \{1, p-1\}<\rho<$ $p$ corresponds the case of very weak solutions. Although it is conjectured by Iwaniec and Sbordone [30] that $\rho>\max \{1, p-1\}$ should be the maximal range for $\rho$, this has not been shown yet. In the same paper they prove (1.15) in the unweighted case for $\rho>p-\varepsilon$ for small $\varepsilon>0$. A qualitative control of $\varepsilon$ has been obtained in [32] by Kinnunen and Zhou, which implies the optimal range $\rho>\max \{1, p-1\}$ but only if $|p-2|$ is small. This results has been extended to uniformly elliptic weights with $\mathbb{A} \in \mathrm{VMO}$ by Greco and Verde [27].

There are only a few results on the non-linear case with degenerate weights. Cruz-Uribe, Moen and Naibo proved Hölder continuity of the solution for $1<$ $p<\infty$ in [11] also using a Muckenhoupt condition. For matrix-valued weights there exists also a weaker notion of matrix-valued Muckenhoupt classes $\mathcal{A}_{p}$ by Roudenko [42]. This weaker notion was for example used by Cruz-Uribe, Moen and Rodney [12] to prove partial regularity for mappings of finite distortion, where (1.11) is replaced by a condition with different lower and upper growth.

The outline of this article is as follows. In Section 2 we introduce and present new facts on scalar and matrix-valued weights and their logarithm. This also includes Poincaré-type estimates and new John-Nirenberg type estimates.

In Section 3 we then derive our Calderón-Zygmund estimates. We begin in Subsection 3.1 with Caccioppoli and reverse Hölder inequalities. The comparison system is constructed in Subsection 3.3. The comparison estimate is proved in Proposition 17, Subsection 3.3 and conclude the decay estimate in Subsection 3.4. Finally, the proof of our main theorems are presented in the Subsection 3.5.

In the final Section 4 we show by means of examples that our results are sharp. In particular, we show that the smallness condition on $\rho|\log \mathbb{A}|_{\text {BMO }}$ is optimal to obtain $L^{\rho}$ integrability of $|\nabla u|^{p} \omega^{p}$.

## 2. On Scalar and Matrix-Valued Weights

In this section we present the necessary tools on scalar and matrix-valued weights. We also introduce a novel smallness condition in terms of the logarithm of the weight. After this we show that this condition implies suitable Poincaré type estimates.
2.1. Matrix-Valued Weights and Logarithms. By $\mathbb{R}_{\text {sym }}^{n \times n}$ we denote the symmetric, real-valued matrices. By $\mathbb{R}_{\geq 0}^{n \times n}$ we denote the cone of symmetric, realvalued, positive semidefinite matrices and by $\mathbb{R}_{>0}^{n \times n}$ the subset of positive definite matrices. For $\mathbb{X}, \mathbb{Y} \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$, we write $\mathbb{X} \geq \mathbb{Y}$ if $\mathbb{X}-\mathbb{Y} \in \mathbb{R}_{\geq 0}^{n \times n}$.

We say that $\mathbb{M}: \Omega \rightarrow \mathbb{R}_{\geq 0}^{n \times n}$ is a (matrix-valued) weight if $\mathbb{M}$ is almost everywhere positive definite. We say that $\omega: \Omega \rightarrow[0, \infty)$ is a (scalar) weight if $\omega$ is positive almost everywhere.

For simplicity we assume in this section that our weights are defined on the whole $\mathbb{R}^{n}$ (instead of the subset $\Omega$ ). If they are defined only on $\Omega$, they have to be extended in a suitable way to $\mathbb{R}^{n}$. This is not difficult due to the locality of our main theorem, Theorem 2.

By $|\cdot|$ we denote the euclidean norm on $\mathbb{R}^{n}$. For $\mathbb{L} \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$, let $|\mathbb{L}|$ denote the spectral norm (which is just the matrix norm induced by the euclidean norm for vectors). We write $B_{R}\left(x_{0}\right) \subset \mathbb{R}^{n}$ for the open ball of radius $R>0$ and center $x_{0} \in \mathbb{R}^{n}$. For a ball $B$ we denote by $r_{B}$ the radius and by $x_{B}$ the center of $B$. For the mean value of a function over a ball $B$ we write $\langle f\rangle_{B}:=f_{B} f(x) d x$. We write $\mathbb{1}_{U}$ for the indicator function of the set $U$.

We will denote by $c$ a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. We also write $f \lesssim g$ if $f \leq c g$. We write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

By $L^{p}\left(\mathbb{R}^{n}\right)$ we denote the usual Lebesgue space with norm $\|\cdot\|_{p}$ and by $L_{\text {loc }}^{p}\left(\mathbb{R}^{n}\right)$ its local version ( $L^{p}$ on compact subsets). By $p^{\prime}$ we denote the conjugate exponent.

For $1<p<\infty$ and a weight $\omega \in L_{\mathrm{loc}}^{p}\left(\mathbb{R}^{n}\right)$ with $\omega^{-1} \in L_{\mathrm{loc}}^{p^{\prime}}\left(\mathbb{R}^{n}\right)$ we define the weighted spaces

$$
L_{\omega}^{p}\left(\mathbb{R}^{n}\right):=\left\{f: \mathbb{R}^{n} \rightarrow \mathbb{R}: \omega f \in L^{p}\left(\mathbb{R}^{n}\right)\right\}
$$

with norm $\|f\|_{p, \omega}:=\|f \omega\|_{p}$. We write $L^{p}\left(\mathbb{R}^{n}, \mu\right)$ for the $L^{p}$-space with measure $\mu$. So $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)=L^{p}\left(\mathbb{R}^{n}, \omega^{p} d x\right)$. Note that we use $\omega$ as a multiplicative weight (not as a measure). The dual space of $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ is $L_{1 / \omega}^{p^{\prime}}\left(\mathbb{R}^{n}\right)$. Both $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ and $L_{1 / \omega}^{p^{\prime}}\left(\mathbb{R}^{n}\right)$ are Banach functions spaces mapping to $L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{n}\right)$. Let $W^{1, p}(\Omega)$ denote the usual Sobolev space. Let $W_{\text {loc }}^{1, p}(\Omega)$ be its local version and $W_{0}^{1, p}(\Omega)$ be the one with zero boundary values. Let $W_{\omega}^{1, p}(\Omega)$ be the weighted Sobolev space, which consists of functions $u \in W^{1,1}(\Omega)$ such that $u,|\nabla u| \in L_{\omega}^{p}(\Omega)$. We equip $W_{\omega}^{1, p}(\Omega)$ with the norm $\|u\|_{L_{\omega}^{p}(\Omega)}+\|\nabla u\|_{L_{\omega}^{p}(\Omega)}$. Let $W_{0, \omega}^{1, p}(\Omega)$ denote the subspace of functions with zero boundary values.

For every $\mathbb{L} \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$ we can consider the matrix exponential $\exp (\mathbb{L}) \in \mathbb{R}_{>0}^{n \times n}$, i.e $\exp : \mathbb{R}_{\mathrm{sym}}^{n \times n} \rightarrow \mathbb{R}_{>0}^{n \times n}$. Moreover, there exists a unique inverse mapping log : $\mathbb{R}_{>0}^{n \times n} \rightarrow \mathbb{R}_{\text {sym }}^{n \times n}$. Thus, since $\mathbb{M}: \mathbb{R}^{n} \rightarrow \mathbb{R}_{\text {sym }}^{n \times n}$ is almost everywhere positive definite, we can define its $\operatorname{logarithm} \log \mathbb{M}: \mathbb{R}^{n} \rightarrow \mathbb{R}_{\text {sym }}^{n \times n}$. Both exp and log can be defined by transformation to a diagonal matrix or by Taylor series.

Of particular interest to us are the logarithm means of $\omega$ and $\mathbb{M}$. We define the logarithmic means

$$
\begin{align*}
\langle\omega\rangle_{B}^{\log } & :=\exp \left(\langle\log \omega\rangle_{B}\right) \\
\langle\mathbb{M}\rangle_{B}^{\log } & :=\exp \left(\langle\log \mathbb{M}\rangle_{B}\right) \tag{2.1}
\end{align*}
$$

Recall, that the dual space of $L_{\omega}^{p}$ is $L_{1 / \omega}^{p^{\prime}}$. It is interesting to observe, that the logarithmic mean is compatible with this operation, since

$$
\begin{equation*}
\left\langle\frac{1}{\omega}\right\rangle_{B}^{\log }=\exp \left(-\langle\log \omega\rangle_{B}\right)=\frac{1}{\langle\omega\rangle_{B}^{\log }} \tag{2.2}
\end{equation*}
$$

The logarithmic mean also commutes with inversion. Indeed, using the identities $\log \left(\mathbb{M}^{-1}\right)=-\log \mathbb{M}$ and $(\exp (\mathbb{L}))^{-1}=\exp (-\mathbb{L})$ we get

$$
\left\langle\mathbb{M}^{-1}\right\rangle_{B}^{\log }=\exp \left(-\langle\log (\mathbb{M})\rangle_{B}\right)=\left(\exp \left(\langle\log (\mathbb{M})\rangle_{B}\right)\right)^{-1}=\left(\langle\mathbb{M}\rangle_{B}^{\log }\right)^{-1}
$$

2.2. Muckenhoupt Weights. We give a brief review on Muckenhoupt weights. Let $1<p<\infty$. A weight $\mu \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ is called an $\mathcal{A}_{p}$-Muckenhoupt weight if and only if

$$
[\mu]_{\mathcal{A}_{p}}:=\sup _{B}\left(f_{B} \mu d x\right)\left(f_{B} \mu^{-\frac{1}{p-1}} d x\right)^{p-1}<\infty
$$

where the supremum is taken over all balls $B$.
If $\mu$ is an $\mathcal{A}_{p}$-Muckenhoupt weight then the maximal operator $M$ is bounded on $L^{p}\left(\mathbb{R}^{n}, \mu\right)$. Let us reformulate it in the language of $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$. The weight $\omega^{p}$ is an $\mathcal{A}_{p}$-Muckenhoupt weight if and only if

$$
\begin{equation*}
\left[\omega^{p}\right]_{\mathcal{A}_{p}}^{\frac{1}{p}}=\sup _{B}\left(f_{B} \omega^{p} d x\right)^{\frac{1}{p}}\left(f_{B} \omega^{-p^{\prime}} d x\right)^{\frac{1}{p^{\prime}}}<\infty \tag{2.3}
\end{equation*}
$$

The property of being a Muckenhoupt weight can also be characterized by its logarithmic means. Indeed, if $\omega^{p}$ is an $\mathcal{A}_{p}$-Muckenhoupt weight, then by the help of Jensen's inequality for all balls $B$

$$
\begin{gather*}
\left(f_{B} \omega^{p} d x\right)^{\frac{1}{p}} \leq c_{1}\langle\omega\rangle_{B}^{\log } \\
\left(f \omega_{B}^{-p^{\prime}} d x\right)^{\frac{1}{p^{\prime}}} \leq c_{2}\left\langle\omega^{-1}\right\rangle_{B}^{\log }=c_{2} \frac{1}{\langle\omega\rangle_{B}^{\log }} \tag{2.4}
\end{gather*}
$$

with $c_{1}, c_{2}=\left[\omega^{p}\right]_{\mathcal{A}_{p}}^{\frac{1}{p}}$. On the other hand, if (2.4) holds, then $\omega^{p}$ is an $\mathcal{A}_{p^{-}}$ Muckenhoupt weight and $\left[\omega^{p}\right]_{\mathcal{A}_{p}}^{\frac{1}{p}} \leq c_{1} c_{2}$ using $\langle\omega\rangle_{B}^{\log }\left\langle\omega^{-1}\right\rangle_{B}^{\log }=1$.
2.3. Weighted Poincaré Estimate. In this section we present a Poincaré type estimate in terms of multiplicative weights. The following Proposition is a scaling invariant version of [21, Theorem 3.3].
Proposition 3. Let $1<p<\infty$ and $\theta \in(0,1]$ such that $\theta p \geq \max \left\{1, \frac{n p}{n+p}\right\}$. Furthermore, let $B$ be a ball and assume that $\omega$ is a weight on $2 B$ with

$$
\begin{equation*}
\sup _{B^{\prime} \subset 2 B}\left(f_{B^{\prime}} \omega^{p} d x\right)^{\frac{1}{p}}\left(f_{B^{\prime}} \omega^{-(\theta p)^{\prime}} d x\right)^{\frac{1}{(\theta p)^{\prime}}} \leq c_{1} \tag{2.5}
\end{equation*}
$$

where the supremum is taken over all balls $B^{\prime}$ contained in $2 B$. Then

$$
\left(f_{B}\left|\frac{u-\langle u\rangle_{B}}{r_{B}}\right|^{p} \omega^{p} d x\right)^{\frac{1}{p}} \leq c_{2}\left(f_{B}(|\nabla u| \omega)^{\theta p} d x\right)^{\frac{1}{\theta p}}
$$

where $c_{2}=c_{2}\left(c_{1}, n, p\right)$.

Proof. The result follows from [21, Theorem 3.3] for a fixed ball with radius 1, which is formulated in a slightly different way. Since the condition (2.5) is scaling invariant w.r.t. $x \leftrightarrow R x$, we obtain the general estimate simply by scaling. Note that in the statement of [21, Theorem 3.3] the case $\alpha=0$ (in notation from [21]), which we need is excluded in the statement but included in the proof. For the sake of completeness let us restate their proof in a scaling invariant formulation.

Recall that the Riesz potential of a measurable function $f \in \mathbb{R}^{n}$ is

$$
\left(I_{1} f\right)(x):=\int_{\mathbb{R}^{n}} \frac{f(y)}{|x-y|^{n-1}} d y
$$

We use the following well-known estimate (see for example [28, Section 15.23])

$$
\begin{equation*}
f_{B}|v(x)-v(y)| d y \leq c \int_{B} \frac{|\nabla v(y)|}{|x-y|^{n-1}} d y=c I_{1}\left(\mathbb{1}_{B}|\nabla v|\right)(x) . \tag{2.6}
\end{equation*}
$$

Let $g \in L_{\frac{1}{\omega}}^{p^{\prime}}(B)$ with

$$
\left(f_{B}\left|\frac{g(x)}{\omega(x)}\right|^{p^{\prime}} d x\right)^{\frac{1}{p^{\prime}}} \leq 1
$$

Applying (2.6), we get

$$
\begin{aligned}
\left|f_{B}\left(v(x)-\langle v\rangle_{B}\right) g(x) d x\right| & \lesssim\left|f_{B} I_{1}\left(\mathbb{1}_{B} \nabla v\right)(x) g(x) d x\right| \\
& =\left|f_{B}(\nabla v)(x) I_{1}\left(\mathbb{1}_{B} g\right)(x) d x\right| \\
& \leq c\left(f_{B}(|\nabla v| \omega)^{\theta p} d x\right)^{\frac{1}{\theta p}}\left(f_{B}\left|\frac{I_{1}\left(\mathbb{1}_{B} g\right)(x)}{\omega(x)}\right|^{(\theta p)^{\prime}} d x\right)^{\frac{1}{(\theta p)^{\prime}}}
\end{aligned}
$$

where we used the selfadjointness of $I_{1}$ and Hölder's inequality . Now, condition (2.5), our assumption $\theta p \geq \max \left\{1, \frac{n p}{n+p}\right\}$ and [41, Theorem 4] give

$$
\left(f_{B}\left|\frac{I_{1}\left(\mathbb{1}_{B} g\right)(x)}{\omega(x)}\right|^{(\theta p)^{\prime}} d x\right)^{\frac{1}{(\theta p)^{\prime}}} \lesssim\left(f_{B}\left|\frac{g(x)}{\omega(x)}\right|^{p^{\prime}} d x\right)^{\frac{1}{p^{\prime}}} \leq 1
$$

This and the previous estimate shows

$$
f_{B}\left|\left(v(x)-\langle v\rangle_{B}\right) g(x)\right| d x \lesssim\left(f_{B}(|\nabla v| \omega)^{\theta p} d x\right)^{\frac{1}{\theta_{p}}}
$$

Taking the supremum over all admissible $g$ proves the claim.
2.4. John-Nirenberg-Type Estimates. We present several estimates of JohnNirenberg type for matrix-valued and scalar weights in terms of its logarithm.

For a ball $B_{R}$ with radius $R$ we define the local $\operatorname{BMO}\left(B_{R}\right)$ space as the set of function $f \in L^{1}\left(B_{R}\right)$ such that the semi-norm

$$
\begin{equation*}
|f|_{\mathrm{BMO}\left(B_{R}\right)}:=\sup _{\substack{0<r \leq R \\ x \in B_{R}}}\left(\frac{1}{\left|B_{r}(x)\right|} \int_{B_{r}(x) \cap B_{R}}\left|f(x)-\langle f\rangle_{B_{r}(x)}\right| d x\right) \tag{2.7}
\end{equation*}
$$

is finite.
First we will show that the BMO-estimates for the matrix-valued weight $\log \mathbb{M}$ transfers to scalar weight $\log \omega$.
Lemma 4. For a matrix-valued weight $\mathbb{M}$ and $\omega=|\mathbb{M}|$ there holds

$$
\begin{equation*}
f_{B}\left|\log \omega(x)-\langle\log \omega\rangle_{B}\right| d x \leq 2 f_{B}\left|\log \mathbb{M}(x)-\langle\log \mathbb{M}\rangle_{B}\right| d x \tag{2.8}
\end{equation*}
$$

Moreover, $|\log \omega|_{\mathrm{BMO}(B)} \leq 2|\log \mathbb{M}|_{\mathrm{BMO}(B)}$.
Proof. Let us abbreviate $\mathbb{H}(x):=\log \mathbb{M}(x)$ and $\mathbb{H}(y):=\log \mathbb{M}(y)$. For $\mathbb{X} \in \mathbb{R}_{\text {sym }}^{n \times n}$ let $\mu(\mathbb{X})$ denote $^{1}$ the maximal eigenvalue of $\mathbb{X} \in \mathbb{R}_{\mathrm{sym}}^{n \times n}$. Then $\mu$ is sub-additive. As a consequence,

$$
\begin{equation*}
|\mu(\mathbb{H}(x))-\mu(\mathbb{H}(y))| \leq|\mathbb{H}(x)-\mathbb{H}(y)| . \tag{2.9}
\end{equation*}
$$

Since $\mu(\mathbb{X})=\log |\exp (\mathbb{X})|$ for $\mathbb{X} \in \mathbb{R}_{\text {sym }}^{n \times n}$, we have

$$
\mu(\mathbb{H}(x))=\log |\exp (\mathbb{H}(x))|=\log \omega(x)
$$

Therefore, we can rewrite (2.9) as

$$
\begin{equation*}
|\log (\omega(x))-\log (\omega(y))| \leq|\mathbb{H}(x)-\mathbb{H}(y)|=|\log \mathbb{M}(x)-\log \mathbb{M}(y)| \tag{2.10}
\end{equation*}
$$

This implies that

$$
\begin{aligned}
f_{B}\left|\log \omega(x)-\langle\log \omega\rangle_{B}\right| d x & \leq f_{B} f_{B}|\log \omega(x)-\log \omega(y)| d y d x \\
& \leq f_{B} f_{B}|\log \mathbb{M}(x)-\log \mathbb{M}(y)| d y d x \\
& \leq 2 f_{B}\left|\log \mathbb{M}(x)-\langle\log \mathbb{M}\rangle_{B}\right| d x
\end{aligned}
$$

using Jensen's inequality in the first step and (2.10) in the second step. This proves estimate (2.8). As a consequence $|\log \omega|_{\mathrm{BMO}\left(\mathbb{R}^{n}\right)} \leq 2|\log \mathbb{M}|_{\mathrm{BMO}\left(\mathbb{R}^{n}\right)}$. The local version $|\log \omega|_{\mathrm{BMO}(B)} \leq 2|\log \mathbb{M}|_{\mathrm{BMO}(B)}$ follows by simple modifications.

Proposition 5. There exist constants $\kappa_{1}=\kappa_{1}(n, \Lambda)>0$ and $c_{3}>0$ such that the following holds. If $q \geq 1$ and $\mathbb{M}$ is a matrix-valued weight with $|\log \mathbb{M}|_{\mathrm{BMO}(B)} \leq \frac{\kappa_{1}}{q}$, then

$$
\left(f_{B}\left(\frac{\left|\mathbb{M}-\langle\mathbb{M}\rangle_{B}^{\log }\right|}{\left|\langle\mathbb{M}\rangle_{B}^{\log }\right|}\right)^{q} d x\right)^{\frac{1}{q}} \leq c_{3} q|\log \mathbb{M}|_{\mathrm{BMO}(B)}
$$

The same holds with $\mathbb{M}$ replaced by a scalar weight $\omega$.
Proof. Let us abbreviate $\mathbb{H}(x):=\log \mathbb{M}(x)$. Then we have $\mathbb{M}(x)=\exp (\mathbb{H}(x))$ and $\langle\mathbb{M}\rangle_{B}^{\log }=\exp \left(\langle\mathbb{H}\rangle_{B}\right)$ and

$$
\mathrm{I}:=\left(f_{B}\left(\frac{\left|\mathbb{M}-\langle\mathbb{M}\rangle_{B}^{\log }\right|}{\left|\langle\mathbb{M}\rangle_{B}^{\log }\right|}\right)^{q} d x\right)^{\frac{1}{q}}=\left(f_{B}\left(\frac{\left|\exp (\mathbb{H})-\exp \left(\langle\mathbb{H}\rangle_{B}\right)\right|}{\exp \left(\left|\langle\mathbb{H}\rangle_{B}\right|\right)}\right)^{q} d x\right)^{\frac{1}{q}}
$$

[^1]Note that for all (hermetian) matrices $\mathbb{X}, \mathbb{Y}$ we have

$$
|\exp (\mathbb{X}+\mathbb{Y})-\exp (\mathbb{X})| \leq|\mathbb{Y}| \exp (|\mathbb{Y}|) \exp (|\mathbb{X}|)
$$

Therefore, with $\mathbb{X}=\langle\mathbb{H}\rangle_{B}$ and $\mathbb{Y}=\mathbb{H}-\langle\mathbb{H}\rangle_{B}$ we estimate

$$
\mathrm{I} \leq\left(f_{B}\left(\left|\mathbb{H}-\langle\mathbb{H}\rangle_{B}\right| \exp \left(\left|\mathbb{H}-\langle\mathbb{H}\rangle_{B}\right|\right)\right)^{q} d x\right)^{\frac{1}{q}}
$$

So by Hölder's inequality

$$
\begin{equation*}
\mathrm{I} \leq\left(f_{B}\left|\mathbb{H}-\langle\mathbb{H}\rangle_{B}\right|^{2 q} d x\right)^{\frac{1}{2 q}} \cdot\left(f_{B} \exp \left(2 q\left|\mathbb{H}-\langle\mathbb{H}\rangle_{B}\right|\right) d x\right)^{\frac{1}{2 q}} \tag{2.11}
\end{equation*}
$$

It follows from the classical John-Nirenberg estimate in the form of [26, Corollary 3.1.8] that

$$
\begin{equation*}
\left(f_{B}\left|\mathbb{H}-\langle\mathbb{H}\rangle_{B}\right|^{2 q} d x\right)^{\frac{1}{2 q}} \leq c(c q!)^{\frac{1}{2 q}}|\mathbb{H}|_{\mathrm{BMO}(B)} \leq c q|\mathbb{H}|_{\mathrm{BMO}(B)} \tag{2.12}
\end{equation*}
$$

where we have used Stirling's formula in the last step. Another consequence of the John-Nirenberg estimate in the form of [26, Corollary 3.1.7] is that there exists $\kappa_{1}>$ 0 such that $q|\mathbb{H}|_{\mathrm{BMO}(B)} \leq \kappa_{1}$ implies

$$
\begin{equation*}
\left(f_{B} \exp \left(2 q\left|\mathbb{H}-\langle\mathbb{H}\rangle_{B}\right|\right) d x\right)^{\frac{1}{2 q}} \leq c^{\frac{1}{2 q}} \leq c \tag{2.13}
\end{equation*}
$$

Note that the results in [26] are stated for $\operatorname{BMO}\left(\mathbb{R}^{n}\right)$, but a simple extension from $\mathrm{BMO}(B)$ to $\mathrm{BMO}\left(\mathbb{R}^{n}\right)$ allows to deduce the local version. Moreover, the estimates for the vector valued BMO follow immediately from the scalar valued ones.

Now, the claim follows from (2.11), (2.12) and (2.13).
We will now apply Proposition 5 to deduce certain properties for scalar weights.
Proposition 6. There exists a constant $\gamma>0$ such that the following holds for all weights $\omega$.
(a) If $|\log \omega|_{\mathrm{BMO}(B)} \leq \frac{\gamma}{s}$ with $s \geq 1$, then

$$
\left(f_{B} \omega^{s} d x\right)^{\frac{1}{s}} \leq 2\langle\omega\rangle_{B}^{\log }
$$

(b) If $|\log \omega|_{\mathrm{BMO}(B)} \leq \frac{\gamma}{s}$ with $s \geq 1$, then

$$
\left(f_{B} \omega^{-s} d x\right)^{\frac{1}{s}} \leq 2 \frac{1}{\langle\omega\rangle_{B}^{\log } .}
$$

(c) If $|\log \omega|_{\mathrm{BMO}} \leq \gamma \min \left\{\frac{1}{p}, \frac{1}{p^{\prime}}\right\}$ with $1<p<\infty$, then $\omega^{p}$ is an $\mathcal{A}_{p}$-Muckenhoupt weight and

$$
\left[\omega^{p}\right]_{\mathcal{A}_{p}}^{\frac{1}{p}}=\sup _{B}\left(f_{B} \omega^{p} d x\right)^{\frac{1}{p}}\left(f_{B} \omega^{-p^{\prime}} d x\right)^{\frac{1}{p^{\prime}}} \leq 4 .
$$

(d) Let $1<p<\infty$ and $\theta \in(0,1)$ such that $\theta p>1$. If $|\log \omega|_{\mathrm{BMO}} \leq$ $\gamma \min \left\{\frac{1}{p}, 1-\frac{1}{\theta p}\right\}$, then

$$
\left(f_{B} \omega^{p} d x\right)^{\frac{1}{p}}\left(f_{B} \omega^{-(\theta p)^{\prime}} d x\right)^{\frac{1}{(\theta p)^{\prime}}} \leq 4
$$

(This is the ensures that (2.5) in Proposition 3 holds.)
Proof. We begin with (a). Let $\kappa_{1}$ and $c_{3}$ be as in Proposition 5. We define $\gamma:=\min \left\{\kappa_{1}, 1 / c_{3}\right\}$. Now, assume that $|\log \omega|_{\mathrm{BMO}(B)} \leq \frac{\gamma}{s}$. Then it follows with Proposition 5 that

$$
\begin{aligned}
\left(f \omega_{B}^{s} d x\right)^{\frac{1}{s}} & \leq\left(f\left|\omega-\langle\omega\rangle_{B}^{\log }\right|^{s} d x\right)^{\frac{1}{s}}+\left|\langle\omega\rangle_{B}^{\log }\right| \\
& \leq\langle\omega\rangle_{B}^{\log }\left(c_{3} s|\log \omega|_{\mathrm{BMO}(B)}+1\right) \\
& \leq 2\langle\omega\rangle_{B}^{\log } .
\end{aligned}
$$

This proves (a).
Now, (b) is just (a) applied to $\frac{1}{\omega}$ using also that $\left\langle\omega^{-1}\right\rangle_{B}^{\log }=\left(\langle\omega\rangle_{B}^{\log }\right)^{-1}$.
Let us now prove (c). If follows from (a) applied to $\omega$ and $p$, resp. $1 / \omega$ and $p^{\prime}$, that

$$
\left(f_{B} \omega^{p} d x\right)^{\frac{1}{p}}\left(f_{B} \omega^{-p^{\prime}} d x\right)^{\frac{1}{p^{\prime}}} \leq 2\langle\omega\rangle_{B}^{\log } \cdot 2\langle 1 / \omega\rangle_{B}^{\log }=4 .
$$

The proof of (d) is analogous to the one of (c).
Remark 7. Since $\log \left(\mathbb{M}^{-1}\right)=-\log (\mathbb{M})$ and $\log \left(\omega^{-1}\right)=-\log (\omega)$ for weights $\mathbb{M}$ and $\omega$, it is possible to apply Proposition 5 and Proposition 6 to $\mathbb{M}^{-1}$ and $\omega^{-1}$.

## 3. Calderón-Zygmund Estimates

In this section we develop the full higher integrability result for the solutions of our weighted $p$-Laplace equation. Let $\Omega \subset \mathbb{R}^{n}$ be a domain with Lipschitz boundary and $1<p<\infty$. Let $\mathbb{M}$ be a matrix-valued weight on $\mathbb{R}^{n}$ with uniformly bounded condition number, i.e. (1.8) holds. Since $\mathbb{M}$ is symmetric and positive definite (1.8) is in fact equivalent to

$$
\begin{equation*}
\Lambda^{-1} \omega(x)|\xi| \leq|\mathbb{M} \xi| \leq \omega(x)|\xi| \quad \text { for all } \xi \in \mathbb{R}^{n} \tag{3.1}
\end{equation*}
$$

and also

$$
\begin{equation*}
\Lambda^{-1} \omega(x) \operatorname{Id} \leq \mathbb{M}(x) \leq \omega(x) \operatorname{Id} \tag{3.2}
\end{equation*}
$$

both for all $x \in \Omega$.
We assume in the following that the logarithmic weight $\log \mathbb{M}$ has small BMOnorm, i.e.

$$
\begin{equation*}
|\log \mathbb{M}|_{\mathrm{BMO}(\Omega)} \leq \kappa \tag{3.3}
\end{equation*}
$$

Hence, by Lemma 4 we have $|\log \omega|_{\mathrm{BMO}(\Omega)} \leq 2 \kappa$.
Note that we do keep track of the dependence of the constants in terms of $\Lambda$ but we keep track of the dependence on $\kappa$.

We assume that $\kappa$ is so small that by Proposition $6 \omega^{p}$ is an $\mathcal{A}_{p}$-Muckenhoupt weight. In particular, smooth functions are dense in $W_{\omega}^{1, p}(\Omega)$.

In the following let $u \in W_{\omega}^{1, p}(\Omega)$ be a weak solution of (1.2) with $G \in L_{\omega}^{p}(\Omega)$, i.e.

$$
\begin{equation*}
\int_{\Omega}|\mathbb{M} \nabla u|^{p-2} \mathbb{M}^{2} \nabla u \cdot \nabla \xi d x=\int_{\Omega}|\mathbb{M} G|^{p-2} \mathbb{M}^{2} G \cdot \nabla \xi d x \tag{3.4}
\end{equation*}
$$

for all $\xi \in C_{0}^{\infty}(\Omega)$ or equivalently all $\xi \in W_{0, \omega}^{1, p}(\Omega)$. Note that the existence of a weak solution is ensured by standard arguments from the calculus of variations, since $\omega^{p}$ is an $\mathcal{A}_{p}$-Muckenhoupt weight.
3.1. Caccioppoli Estimate and Reverse Hölder's Inequality. We begin with the standard Caccioppoli estimates.

Proposition 8 (Caccioppoli). For all balls $B$ with $2 B \subset \Omega$ there holds

$$
f_{B}|\nabla u|^{p} \omega^{p} d x \leq c f_{2 B}\left|\frac{u-\langle u\rangle_{2 B}}{r_{B}}\right|^{p} \omega^{p} d x+c{\underset{2 B}{ }|G|^{p} \omega^{p} d x . . . . . . ~}
$$

Proof. Fix a smooth cut-off function $\eta$ with $\mathbb{1}_{B} \leq \eta \leq \mathbb{1}_{2 B}$ and $|\nabla \eta| \leq \frac{c}{r_{B}}$. Using the test function $\eta^{p}\left(u-\langle u\rangle_{2 B}\right)$ in (3.4) we get

$$
\int|\mathbb{M} \nabla u|^{p-2} \mathbb{M} \nabla u \cdot \mathbb{M} \nabla\left(\eta^{p}\left(u-\langle u\rangle_{2 B}\right) d x=\int|\mathbb{M} G|^{p-2} \mathbb{M} G \cdot \mathbb{M} \nabla\left(\eta^{p}\left(u-\langle u\rangle_{2 B}\right) d x\right.\right.
$$

Using (3.1) we obtain by standard calculations

$$
\begin{aligned}
\int_{2 B} \eta^{p}|\nabla u|^{p} \omega^{p} d x \leq & c \int_{2 B} \eta^{p-1}|\nabla u|^{p-1}\left|\frac{u-\langle u\rangle_{2 B}}{r_{B}}\right| \omega^{p} d x \\
& +c \int_{2 B} \eta^{p}|G|^{p-1}|\nabla u| \omega^{p} d x \\
& +c \int_{2 B} \eta^{p-1}|G|^{p-1}\left|\frac{u-\langle u\rangle_{2 B}}{r_{B}}\right| \omega^{p} d x .
\end{aligned}
$$

We use Young's inequality, absorb the term with $\eta^{p}|\nabla u|^{p} \omega^{p}$ and obtain

$$
\int_{B}|\nabla u|^{p} \omega^{p} d x \leq c \int_{2 B}\left|\frac{u-\langle u\rangle_{2 B}}{r_{B}}\right|^{p} \omega^{p} d x+c \int_{2 B}|G|^{p} \omega^{p} d x .
$$

This proves the claim.
From the Caccioppoli estimate we derive as usual the reverse Hölder estimate.
Proposition 9. There exists $\kappa_{2}>0$ and $\theta \in(0,1)$ such that for all balls $B$ with $2 B \subset \Omega$ there holds: if $|\log \mathbb{M}|_{\mathrm{BMO}(2 B)} \leq \kappa_{2}=\kappa_{2}(p, n, \Lambda)$, then

$$
f_{B}|\nabla u|^{p} \omega^{p} d x \lesssim\left(f_{2 B}|\nabla u|^{\theta p} \omega^{p} d x\right)^{\frac{1}{\theta}}+f_{2 B}|G|^{p} \omega^{p} d x
$$

Proof. We can choose $\kappa_{2}$ so small such that Proposition 6 (d) ensures the applicability of the weighted Poincaré-Estimate of Proposition 3. This and the Caccioppoli estimate of Proposition 8 prove the claim.

An application of Gehring's lemma (e.g. [25, Theorem 6.6]) immediately gives the following consequence.

Corollary 10 (Small Higher Integrability). There exists $\kappa_{2}>0$ and $s>1$ such that for all balls $B$ with $2 B \subset \Omega$ there holds: if $|\log \mathbb{M}|_{\mathrm{BMO}(2 B)} \leq \kappa_{2}=\kappa_{2}(p, n, \Lambda)$, then

$$
\left(f_{B}\left(|\nabla u|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \lesssim f_{2 B}|\nabla u|^{p} \omega^{p} d x+\left(f_{2 B}\left(|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
$$

3.2. Interlude on Orlicz Functions. For the analysis in the following sections it is useful to introduce a few auxiliary functions and some basic properties on Orlicz functions. The N -function

$$
\varphi(t):=\frac{1}{p} t^{p}
$$

is the natural one for our problems.
Then

$$
\begin{equation*}
A(\xi)=|\xi|^{p-2} \xi=\frac{\varphi^{\prime}(|\xi|)}{|\xi|} \xi \tag{3.5}
\end{equation*}
$$

Let us define

$$
V(\xi):=\sqrt{\frac{\varphi^{\prime}(|\xi|)}{|\xi|}} \xi=|\xi|^{\frac{p-2}{2}} \xi
$$

In general a function $\psi: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is called an $N$-function if and only if there is a right-continuous, positive on the positive real line, and non-decreasing function $\psi^{\prime}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ with $\psi^{\prime}(0)=0$ and $\lim _{t \rightarrow \infty} \psi^{\prime}(t)=\infty$ such that $\psi(t)=\int_{0}^{t} \psi^{\prime}(\tau) d \tau$. An N -function is said to satisfy the $\Delta_{2}$-condition if and only if there is a constant $c>1$ such that $\psi(2 t) \leq c \psi(t)$.

The conjugate of an N -function $\psi$ is defined as

$$
\psi^{*}(t):=\sup _{s \geq 0}(t s-\psi(s)), \quad t \geq 0
$$

In our case $\varphi^{*}(t)=\frac{1}{p^{\prime}} t^{p^{\prime}}$.
Moreover, we need the notion of shifted N-functions first introduced in [16]. Here, we use the slight variant of [17, Appendix B] with even nicer properties.

We define the shifted N -functions $\varphi_{a}$ for $a \geq 0$ by

$$
\begin{equation*}
\varphi_{a}(t):=\int_{0}^{t} \frac{\varphi^{\prime}(a \vee s)}{a \vee s} s d s \tag{3.6}
\end{equation*}
$$

where $s_{1} \vee s_{2}:=\max \left\{s_{1}, s_{2}\right\}$ for $s_{1}, s_{2} \in \mathbb{R}$. Then ${ }^{2}$

$$
\begin{align*}
\varphi_{a}(t) & \approx(a \vee t)^{p-2} t^{2} \\
\varphi_{a}^{\prime}(t) & \approx(a \vee t)^{p-2} t \tag{3.7}
\end{align*}
$$

[^2]with constants only depending on $p$. The index $a$ is called the shift. Obviously, $\varphi_{0}=\varphi$. Moreover, if $a \approx b$, then $\varphi_{a}(t) \approx \varphi_{b}(t)$. For the shifted N-functions $\varphi_{a}$ we have
\[

$$
\begin{equation*}
\left(\varphi_{a}\right)^{*}=\left(\varphi^{*}\right)_{\varphi^{\prime}(a)} \tag{3.8}
\end{equation*}
$$

\]

Thus, we get the useful equation

$$
\begin{equation*}
\left(\varphi_{|\xi|}\right)^{*}=\left(\varphi^{*}\right)_{|A(\xi)|} \tag{3.9}
\end{equation*}
$$

Moreover, the family $\varphi_{a}, a \geq 0$, as well as its conjugate functions also satisfy the $\Delta_{2^{-}}$ condition with a $\Delta_{2}$-constant uniformly bounded with respect to $a$. In particular, we can apply Young's inequality to obtain: for every $\delta>0$ there exists $c_{\delta}=$ $c_{\delta}(\delta, p) \geq 1$ such that for all $s, t, a \geq 0$

$$
\begin{equation*}
s t \leq c_{\delta}\left(\varphi_{a}\right)^{*}(s)+\delta \varphi_{a}(t) \tag{3.10}
\end{equation*}
$$

Using $\varphi_{a}(t) \approx \varphi_{a}^{\prime}(t) t$ and $\left(\varphi_{a}\right)^{*} \approx\left(t \varphi_{a}^{\prime}(t)\right)$ we get the following equivalent versions

$$
\begin{align*}
\varphi_{a}^{\prime}(s) t & \leq c_{\delta} \varphi_{a}(s)+\delta \varphi_{a}(t) \\
\varphi_{a}^{\prime}(s) t & \leq \delta \varphi_{a}(s)+c_{\delta} \varphi_{a}(t) \tag{3.11}
\end{align*}
$$

for $s, t, a \geq 0$.
Moreover, the following simple equivalence holds for $a \geq 0$

$$
\varphi_{a}(\lambda a)= \begin{cases}\lambda^{2} \varphi(a), & \text { for } \lambda<1  \tag{3.12}\\ \varphi(\lambda a) & \text { for } \lambda>1\end{cases}
$$

The important relation between $A, V$ and the $\varphi_{a}$ is best summarized in the following lemma.
Lemma 11 ([17, Lemma 41]). For all $P, Q \in \mathbb{R}^{n}$ there holds

$$
\begin{aligned}
(A(P)-A(Q)) \cdot(P-Q) & \approx|V(P)-V(Q)|^{2} \\
& \approx \varphi_{|Q|}(|P-Q|) \\
& \approx\left(\varphi^{*}\right)_{|A(Q)|}(|A(P)-A(Q)|)
\end{aligned}
$$

and

$$
A(Q) \cdot Q=|V(Q)|^{2} \approx \varphi_{|Q|}(|Q|) \approx \varphi(|Q|)
$$

and

$$
|A(P)-A(Q)| \bar{\sim}\left(\varphi_{|Q|}\right)^{\prime}(|P-Q|)
$$

The implicit constants depend only on $p$.
Also of strong use is the possibility to change the shift:
Lemma 12 (Change of shift, [17, Corollary 44]). For $\delta>0$ there exists $c_{\delta}=c_{\delta}(p)$ such that for all $P, Q \in \mathbb{R}^{n}$ there holds

$$
\begin{aligned}
\varphi_{|P|}(t) & \leq c_{\delta} \varphi_{|Q|}(t)+\delta|V(P)-V(Q)|^{2} \\
\left(\varphi_{|P|}\right)^{*}(t) & \leq c_{\delta}\left(\varphi_{|Q|}\right)^{*}(t)+\delta|V(P)-V(Q)|^{2}
\end{aligned}
$$

The implicit constants depend only on $p$.
We are in particular interested in the following special case for $Q=0$.

Lemma 13 (Removal of shift). For all $a \in \mathbb{R}^{n}$, all $t \geq 0$ and all $\delta \in(0,1]$ there holds

$$
\begin{align*}
\varphi_{|a|}^{\prime}(t) & \leq \varphi^{\prime}\left(\frac{t}{\delta}\right) \vee\left(\delta \varphi^{\prime}(|a|)\right)  \tag{3.13}\\
\varphi_{|a|}(t) & \leq \delta \varphi(|a|)+c \delta \varphi\left(\frac{t}{\delta}\right)  \tag{3.14}\\
\left(\varphi_{|a|}\right)^{*}(t) & \leq \delta \varphi(|a|)+c \delta \varphi^{*}\left(\frac{t}{\delta}\right), \tag{3.15}
\end{align*}
$$

where $c$ only depends on the $\Delta_{2}$ constants of $\varphi$ and $\varphi^{*}$.
Proof. We start with (3.13). If $t \leq \delta|a|$, then

$$
\varphi_{|a|}^{\prime}(t)=\frac{\varphi^{\prime}(|a| \vee t)}{|a| \vee t} t=\frac{\varphi^{\prime}(|a|)}{|a|} t \leq \delta \varphi^{\prime}(|a|)
$$

If $t \geq \delta|a|$, then with $0<\delta \leq 1$

$$
\varphi_{|a|}^{\prime}(t)=\frac{\varphi^{\prime}(|a| \vee t)}{|a| \vee t} t \leq \varphi^{\prime}(|a| \vee t) \leq \varphi^{\prime}\left(\frac{t}{\delta}\right)
$$

This proves (3.13). We continue with (3.14). From Lemma 43 of [17] (with $b=0$ ) we have

$$
\varphi_{|a|}^{\prime}(t) \leq c \varphi^{\prime}(|a|)
$$

Thus, for $\delta>0$ we obtain with Young's inequality

$$
\begin{aligned}
\varphi_{|a|}(t) & \leq \varphi_{|a|}^{\prime}(t) t \\
& \leq c \varphi^{\prime}(|a|) t \\
& \leq \delta \varphi^{*}\left(\varphi^{\prime}(|a|)\right)+c \delta \varphi(t / \delta) \\
& \leq \delta c \varphi(|a|)+c \delta \varphi(t / \delta)
\end{aligned}
$$

This proves the claim (if we replace $\delta$ by smaller one). Now, (3.15) follows from (3.14) using $\left(\varphi_{|a|}\right)^{*}=\left(\varphi^{*}\right)_{\varphi^{\prime}(|a|)}$ and the equivalence $\varphi^{*}\left(\varphi^{\prime}(|a|)\right) \approx \varphi(|a|)$.
3.3. Comparison Estimate. To obtain higher integrability beyond Corollary 10 we need to derive comparison estimates. This is where we need the logarithmic mean $\langle\mathbb{M}\rangle_{B}^{\log }$ of our matrix-values weight $\mathbb{M}$, which is a positive definite matrix.

Recall, that $\mathcal{A}(x, \xi):=|\mathbb{M}(x) \xi|^{p-2} \mathbb{M}^{2} \xi=\mathbb{M} A(\mathbb{M} \xi)$ with $A(\xi):=|\xi|^{p-2} \xi$. Now, for a ball $B \subset \Omega$ we abbreviate

$$
\begin{align*}
\mathbb{M}_{B} & :=\langle\mathbb{M}\rangle_{B}^{\log },  \tag{3.16}\\
\omega_{B} & :=\langle\omega\rangle_{B}^{\log }
\end{align*}
$$

and define

$$
\begin{equation*}
\mathcal{A}_{B}(\xi):=\left|\mathbb{M}_{B} \xi\right|^{p-2} \mathbb{M}_{B}^{2} \xi=\mathbb{M}_{B} A\left(\mathbb{M}_{B} \xi\right) \tag{3.17}
\end{equation*}
$$

We will use $\mathcal{A}_{B}$ below to define a suitable comparison problem. It naturally arises if we minimize the energy $\int \frac{1}{p}\left|\mathbb{M}_{B} \nabla h\right|^{p} d x$.

We abbreviate

$$
\begin{aligned}
& \mathcal{A}(x, \xi)=\mathbb{M}(x) A(\mathbb{M}(x) \xi) \\
& \mathcal{V}(x, \xi)=V(\mathbb{M}(x) \xi)
\end{aligned}
$$

$$
\mathcal{V}_{B}(\xi)=V\left(\mathbb{M}_{B} \xi\right) .
$$

Then we have

$$
\begin{aligned}
\mathcal{A}(x, \xi) \cdot \xi & =|\mathcal{V}(x, \xi)|^{2} \\
\mathcal{A}_{B}(\xi) \cdot \xi & =\left|\mathcal{V}_{B}(\xi)\right|^{2}
\end{aligned}
$$

and

$$
\left|\mathcal{A}_{B}(\xi)\right| \lesssim \omega_{B}^{p}|\xi|^{p-1}
$$

We will now compare $u$ locally to its $\mathcal{A}_{B}$-harmonic counterpart. Due to some later localization argument will not compare directly with $u$ but with to a truncated version of it. This technique goes back to Kinnunen and Zhou [33]. Originally, we wanted to compare directly with $u$, since this seemed us more natural to us. However, we encountered problems with the localization of the maximal operators later and decided to proceed as Kinnunen and Zhou in [33].

We fix a ball $B_{0}:=B_{R}\left(x_{0}\right)$ with $2 B_{0} \subset \Omega$ and choose a cut-off function $\zeta \in$ $C_{0}^{\infty}\left(B_{0}\right)$ with $\mathbb{1}_{\frac{1}{2} B_{0}} \leq \zeta \leq \mathbb{1}_{B_{0}}$ and $\|\nabla \zeta\|_{\infty} \leq c R^{-1}$. Let us define the localized function

$$
\begin{equation*}
z:=\left(u-\langle u\rangle_{2 B_{0}}\right) \zeta^{p^{\prime}} \tag{3.18}
\end{equation*}
$$

Moreover, we will use

$$
\begin{equation*}
g:=\zeta^{p^{\prime}} \nabla u-\nabla z=-\left(u-\langle u\rangle_{2 B_{0}}\right) \nabla\left(\zeta^{p^{\prime}}\right)=-\left(u-\langle u\rangle_{2 B_{0}}\right) p^{\prime} \zeta^{p^{\prime}-1} \nabla \zeta \tag{3.19}
\end{equation*}
$$

Now, let $B=B_{r}$ denote an arbitrary ball with $4 B \subset 2 B_{0}$. We want to compare our localized function $z$ on the ball $B$ with the weak solution $h$ of

$$
\begin{align*}
-\operatorname{div}\left(\mathcal{A}_{B}(\nabla h)\right)=0 & \text { in } B \\
h=z & \text { on } \partial B \tag{3.20}
\end{align*}
$$

The natural function space for $h$ is $W_{\omega_{B}}^{1, p}(B)$ and $h$ is the unique minimizer of

$$
\begin{equation*}
v \mapsto \int_{B} \varphi\left(\mathbb{M}_{B}|\nabla v|\right) d x \tag{3.21}
\end{equation*}
$$

subject to the boundary condition $v=z$ on $\partial B$. We will explain the well posedness of the boundary conditions below in Lemma 14.

Recall that

$$
\Lambda^{-1} \omega(x) \operatorname{Id} \leq \mathbb{M}(x) \leq \omega(x) \operatorname{Id}
$$

Considering the eigenvalues it follows that

$$
\log \left(\Lambda^{-1} \omega(x)\right) \operatorname{Id} \leq \log \mathbb{M}(x) \leq(\log \omega) \operatorname{Id}
$$

This also follows from the operator monotonicity of the matrix logarithm, see the survey article of Chansangiam [8, Example 13]. Taking the mean value we obtain

$$
\Lambda^{-1} \exp \left(\langle\log \omega\rangle_{B}\right) \operatorname{Id} \leq \exp \left(\langle\log \mathbb{M}\rangle_{B}\right) \leq \exp \left(\langle\log \omega\rangle_{B}\right) \operatorname{Id}
$$

Comparing again the eigenvalues we obtain by taking the exponential ${ }^{3}$

$$
\left(\log \left(\Lambda^{-1}\right)+\langle\log \omega\rangle_{B}\right) \operatorname{Id} \leq\langle\log \mathbb{M}\rangle_{B} \leq\langle\log \omega\rangle_{B} \mathrm{Id}
$$

[^3]In other words,

$$
\begin{equation*}
\Lambda^{-1} \omega_{B} \operatorname{Id} \leq \mathbb{M}_{B} \leq \omega_{B} \operatorname{Id} \tag{3.22}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\Lambda^{-1} \omega_{B}|\xi| \leq\left|\mathbb{M}_{B} \xi\right| \leq \omega_{B}|\xi| \quad \text { for all } \xi \in \mathbb{R}^{n} \tag{3.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\Lambda^{-1} \omega_{B} \leq\left|\mathbb{M}_{B}\right| \leq \omega_{B} \tag{3.24}
\end{equation*}
$$

For the well posedness of the boundary condition of the equation (3.20) it is necessary that $u$ has enough regularity. The following lemma ensures that $u$ has indeed the required regularity natural to (3.20).
Lemma 14. There exists $\kappa_{3}>0$ and $s>1$ such that for all balls $B$ with $2 B \subset \Omega$ there holds: if $|\log \mathbb{M}|_{\mathrm{BMO}(4 B)} \leq \kappa_{3}=\kappa_{3}(p, n, \Lambda)$, then

$$
f_{B}|\nabla u|^{p} \omega_{B}^{p} d x \lesssim f_{2 B}|\nabla u|^{p} \omega^{p} d x+\left(f_{2 B}\left(|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
$$

Proof. By Hölder's inequality and Proposition 6

$$
\begin{aligned}
\int_{B}|\nabla u|^{p} \omega_{B}^{p} d x & \leq\left(f_{B}\left(|\nabla u|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}\left(f_{B}\left(\omega_{B} \omega^{-1}\right)^{p s^{\prime}} d x\right)^{\frac{1}{s^{\prime}}} \\
& \leq\left(f_{B}\left(|\nabla u|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} 2^{p}
\end{aligned}
$$

Now, the claim follows with Corollary 10.
It follows from this lemma that $u \in W_{\omega_{B}}^{1, p}(B)$ (assuming the smallness condition on $\log \mathbb{M})$. Thus also $z=\left(u-\langle u\rangle_{B}\right) \zeta^{p^{\prime}} \in W_{\omega_{B}}^{1, p}(B)$. In particular, it follows that equation (3.20) is well posed with a unique solution $h \in W_{\omega_{B}}^{1, p}(B)$ and $h=z$ on $\partial B$ in the usual trace sense.

The following proposition summarizes the higher regularity properties of $h$.
Proposition 15. Let $h$ be the solution of (3.20). Then

$$
\sup _{\frac{1}{2} B}|\nabla h|^{p} \omega_{B}^{p} \leq c f_{B}|\nabla h|^{p} \omega_{B}^{p} d x .
$$

Moreover, there exists $\alpha \in(0,1)$ such that for all $\lambda \in(0,1)$

$$
\underset{\lambda B}{ }\left|\mathcal{V}_{B}(\nabla h)-\left\langle\mathcal{V}_{B}(\nabla h)\right\rangle_{\lambda B}\right|^{2} d x \leq c \lambda^{2 \alpha} \int_{B}\left|\mathcal{V}_{B}(\nabla h)-\left\langle\mathcal{V}_{B}(\nabla h)\right\rangle_{B}\right|^{2} d x
$$

The constants $c, \alpha$ only depend on $p, n$ and $\Lambda$.
Proof. If $\omega=1$, then both estimates just follow from Lemma 5.8 and Theorem 6.4 of [20]. In the general case we have by (3.22)

$$
\Lambda^{-1} \omega_{B} \operatorname{Id} \leq \mathbb{M}_{B} \leq \omega_{B} \operatorname{Id}
$$

Since (3.20) and our estimates scale by scalar factors, we can assume without loss of generality $\omega_{B}=1$. We can also assume that $B$ is centered at 0 .

Since $\mathbb{M}_{B}$ is also symmetric, we can find an orthogonal matrix $Q$ and a diagonal matrix $\mathbb{D}_{B}$ such that $\mathbb{M}_{B}=Q \mathbb{D}_{B} Q^{*}$. If we define $w(x):=h(Q x)$, then it follows that $w$ solves (3.20) with $\mathbb{M}_{B}$ replaced by $\mathbb{D}_{B}$. The boundary values are also rotated but this is not important for our estimates. Hence, we can assume without loss of generality that $\mathbb{M}_{B}$ is already diagonal. We have reduced the claim so far to a diagonal matrix $\mathbb{M}_{B}=\mathbb{D}_{B}$ with

$$
\Lambda^{-1} \operatorname{Id} \leq \mathbb{D}_{B} \leq \mathrm{Id}
$$

Since $\Lambda$ is fixed we can use an anisotropic scaling $x \mapsto \mathbb{D}_{B}^{-1} x$. This turns estimates on balls into estimates on ellipses of uniformly bounded eccentricity. Thus, we deduce from Lemma 5.8 and Theorem 6.4 of [20] that our estimates are valid for ellipses instead of balls. Since all balls can be covered by slightly enlarged ellipses and vice versa, it follows that the estimates are also true for balls with slightly enlarged constants (depending on $\Lambda$ ). This proves the claim.

The following lemma allows us to control the difference of the mapping $\mathcal{A}(\cdot, \cdot)$ and its frozen version $\mathcal{A}_{B}(\cdot)$.

Lemma 16. For all $\xi \in \mathbb{R}^{n}$ and all $x \in B$ there holds

$$
\begin{aligned}
\left|\mathcal{A}_{B}(\xi)-\mathcal{A}(x, \xi)\right| & =\left|\mathbb{M}_{B} A\left(\mathbb{M}_{B} \xi\right)-\mathbb{M}(x) A(\mathbb{M}(x) \xi)\right| \\
& \lesssim \frac{\left|\mathbb{M}_{B}-\mathbb{M}(x)\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}(x)|}\left(\left|\mathcal{A}_{B}(\xi)\right|+|\mathcal{A}(x, \xi)|\right)
\end{aligned}
$$

Proof. Note that $\left|\mathbb{M}_{B} \xi\right| \approx\left|\mathbb{M}_{B}\right||\xi|$ and $|\mathbb{M} \xi| \approx|\mathbb{M}||\xi|$ due to (3.1), (3.23) and (3.24). Thus, we can estimate

$$
\begin{aligned}
& \mid \mathcal{A}_{B}(\xi)-\mathcal{A}(x, \xi)\left|=\left|\mathbb{M}_{B} A\left(\mathbb{M}_{B} \xi\right)-\mathbb{M}(x) A(\mathbb{M}(x) \xi)\right|\right. \\
& \quad \leq\left|\mathbb{M}_{B}\right|\left|A\left(\mathbb{M}_{B} \xi\right)-A(\mathbb{M}(x) \xi)\right|+\left|\mathbb{M}_{B}-\mathbb{M}(x)\right||A(\mathbb{M}(x) \xi)| \\
& \quad \lesssim\left|\mathbb{M}_{B}\right| \varphi_{\left|\mathbb{M}_{B} \xi\right| \vee|\mathbb{M}(x) \xi|}\left(\left|\mathbb{M}_{B} \xi-\mathbb{M}(x) \xi\right|\right)+\left|\mathbb{M}_{B}-\mathbb{M}(x)\right||A(\mathbb{M}(x) \xi)| \\
& \quad \lesssim\left|\mathbb{M}_{B}\right| \frac{\left|\mathbb{M}_{B}-\mathbb{M}(x)\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}(x)|}\left(\left(\left|\mathbb{M}_{B}\right|+|\mathbb{M}(x)|\right)|\xi|\right)^{p-1} \\
&+\left|\mathbb{M}_{B}-\mathbb{M}(x)\right|(|\mathbb{M}(x)||\xi|)^{p-1} \\
& \quad \lesssim \frac{\left|\mathbb{M}_{B}-\mathbb{M}(x)\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}(x)|}\left(\left|\mathbb{M}_{B}\right|+|\mathbb{M}(x)|\right)^{p}|\xi|^{p-1} \\
& \quad \frac{\left|\mathbb{M}_{B}-\mathbb{M}(x)\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}(x)|}\left(\left|\mathcal{A}_{B}(\xi)\right|+|\mathcal{A}(x, \xi)|\right)
\end{aligned}
$$

using also (3.7) in the fourth step. This proves the lemma.

We are now prepared for our comparison estimate. From the equation (3.20) for $h$, the homogeneity of $\mathcal{A}$, and the equation (1.6) for $u$ we deduce that

$$
\begin{align*}
-\operatorname{div} & \left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}_{B}(\nabla h)\right) \\
= & -\operatorname{div}\left(\mathcal{A}_{B}(\nabla z)\right) \\
= & -\operatorname{div}\left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}(\cdot, \nabla z)\right)-\operatorname{div}(\mathcal{A}(\cdot, \nabla z)-\mathcal{A}(\cdot, \nabla z+g)) \\
& -\operatorname{div}\left(\mathcal{A}\left(\cdot, \zeta^{p^{\prime}} \nabla u\right)\right) \\
= & -\operatorname{div}\left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}(\cdot, \nabla z)\right)-\operatorname{div}(\mathcal{A}(\cdot, \nabla z)-\mathcal{A}(\cdot, \nabla z+g))  \tag{3.25}\\
& -\operatorname{div}\left(\mathcal{A}(\cdot, \nabla u) \zeta^{p}\right) \\
= & -\operatorname{div}\left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}(\cdot, \nabla z)\right)-\operatorname{div}(\mathcal{A}(\cdot, \nabla z)-\mathcal{A}(\cdot, \nabla z+g)) \\
& -\zeta^{p} \operatorname{div}(\mathcal{A}(\cdot, G))-\nabla\left(\zeta^{p}\right) \mathcal{A}(\cdot, \nabla u)
\end{align*}
$$

Proposition 17 (Comparison). Recall, that $B=B_{r}, B_{0}=B_{R}\left(x_{0}\right), 4 B \subset 2 B_{0}$ and $z, g, h$ are given by (3.18), (3.19) and (3.20), respectively. There exist $s>1$ and $\kappa_{4}=\kappa_{4}(p, n, \Lambda)$, such that if $|\log \mathbb{M}|_{\mathrm{BMO}(2 B)} \leq \kappa_{4}$, then for every $\delta \in(0,1)$ there holds

$$
\begin{aligned}
& \int_{B}\left|\mathcal{V}_{B}(\nabla h)-\mathcal{V}_{B}(\nabla z)\right|^{2} d x \\
& \quad \leq c\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}+\delta\right)\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \quad+c \delta^{1-p}\left(f_{4 B}\left(\frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}+c \delta^{1-p}\left(f_{4 B}\left(\zeta^{p}|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
\end{aligned}
$$

Proof. Let $s>1$ be as in Corollary 10 (so $s$ just depends on $p$ ). We assume that $|\log \mathbb{M}|_{\mathrm{BMO}(B)} \leq \kappa_{3}$ with $\kappa_{3}$ from Lemma 14 so that the comparison equation is well defined. We will add in the proof several other smallness conditions on $|\log \mathbb{M}|_{\mathrm{BMO}(2 B)}$ that will finally determine the value of $\kappa_{4}$.

Using (3.20) and (1.6) with the test function $|B|^{-1}(z-h)$ we obtain

$$
\begin{aligned}
\mathrm{I}_{0}:= & f_{B}\left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}_{B}(\nabla h)\right) \cdot(\nabla z-\nabla h) d x \\
= & \int_{B}\left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}(x, \nabla z)\right) \cdot(\nabla z-\nabla h) d x \\
& +f_{B}(\mathcal{A}(x, \nabla z)-\mathcal{A}(x, \nabla z+g)) \cdot(\nabla z-\nabla h) d x \\
& +\int_{B} \zeta^{p} \mathcal{A}(x, G) \cdot(\nabla z-\nabla h) d x \\
& +\int_{B} \nabla\left(\zeta^{p}\right) \mathcal{A}(x, \nabla u)(z-h) d x \\
= & \mathrm{I}_{1}+\mathrm{I}_{2}+\mathrm{I}_{3}+\mathrm{I}_{4} .
\end{aligned}
$$

Now,

$$
\begin{equation*}
\mathrm{I}_{0} \approx \int_{B}|V(\nabla z)-V(\nabla h)|^{2} \omega_{B}^{p} d x \approx \int_{B} \varphi_{|\nabla z|}(|\nabla z-\nabla h|) \omega_{B}^{p} d x \tag{3.26}
\end{equation*}
$$

Let us estimate $\mathrm{I}_{1}$. Using Lemma 16 we get

$$
\begin{aligned}
\mathrm{I}_{1} & =\int_{B}\left(\mathcal{A}_{B}(\nabla z)-\mathcal{A}(x, \nabla z)\right) \cdot(\nabla z-\nabla h) d x \\
& \lesssim \int_{B}\left|\mathcal{A}_{B}(\nabla z)-\mathcal{A}(x, \nabla z)\right| \cdot|\nabla z-\nabla h| d x \\
& \lesssim \int_{B} \frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\left(\left|\mathcal{A}_{B}(\nabla z)\right|+|\mathcal{A}(x, \nabla z)|\right)|\nabla z-\nabla h| d x \\
& \lesssim \int_{B} \frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\left(\omega_{B}^{p} \varphi^{\prime}(|\nabla z|)+\omega^{p} \varphi^{\prime}(|\nabla z|)\right)|\nabla z-\nabla h| d x
\end{aligned}
$$

Now we use Young's inequality and Lemma 11

$$
\begin{aligned}
\mathrm{I}_{1} \leq & \sigma f_{B} \varphi_{|\nabla z|}(|\nabla z-\nabla h|) \omega_{B}^{p} d x \\
& +c_{\sigma} f_{B}\left(\varphi_{|\nabla z|}\right)^{*}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|} \varphi^{\prime}(|\nabla z|)\right) \omega_{B}^{p} d x \\
& +c_{\sigma} f_{B}\left(\varphi_{|\nabla z|}\right)^{*}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|} \frac{\omega^{p}}{\omega_{B}^{p}} \varphi^{\prime}(|\nabla z|)\right) \omega_{B}^{p} d x:=\mathrm{I}_{1,1}+\mathrm{I}_{1,2}+\mathrm{I}_{1,3}
\end{aligned}
$$

Now, using $\left(\varphi_{a}\right)^{*}(\lambda t) \leq c\left(\lambda^{2}+\lambda^{p^{\prime}}\right) \varphi_{a}^{*}(t)$ for $a, \lambda, t \geq 0$ we obtain

$$
\mathrm{I}_{1,2}+\mathrm{I}_{1,3} \leq c_{\sigma} f_{B}\left(\varphi_{|\nabla z|}\right)^{*}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|} \varphi^{\prime}(|\nabla z|)\right)\left(1+\frac{\omega^{2 p}}{\omega_{B}^{2 p}}+\frac{\omega^{p p^{\prime}}}{\omega_{B}^{p p^{\prime}}}\right) \omega_{B}^{p} d x
$$

Now, (3.12) and $\left(\varphi_{|a|}\right)^{*}\left(\varphi^{\prime}(|a|)\right) \approx \varphi(|a|)$ imply

$$
\begin{aligned}
\mathrm{I}_{1,2}+\mathrm{I}_{1,3} & \leq c_{\sigma} \int_{B}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\right)^{2}\left(\varphi_{|\nabla z|}\right)^{*}\left(\varphi^{\prime}(|\nabla z|)\right)\left(1+\frac{\omega^{2 p}}{\omega_{B}^{2 p}}+\frac{\omega^{p p^{\prime}}}{\omega_{B}^{p p^{\prime}}}\right) \omega_{B}^{p} d x \\
& \approx c_{\sigma} \int_{B}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\right)^{2} \varphi(|\nabla z|)\left(\frac{\omega_{B}^{p}}{\omega^{p}}+\frac{\omega^{p}}{\omega_{B}^{p}}+\frac{\omega^{p^{\prime}}}{\omega_{B}^{p^{\prime}}}\right) \omega^{p} d x
\end{aligned}
$$

Now, we can use Hölder's inequality to conclude with Proposition 6

$$
\begin{aligned}
\mathrm{I}_{1,2}+\mathrm{I}_{1,3} \lesssim c( & \left.f_{B}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\right)^{4 s^{\prime}} d x\right)^{\frac{1}{2 s^{\prime}}}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \cdot\left(f_{B}\left(\frac{\omega_{B}^{p}}{\omega^{p}}+\frac{\omega^{p}}{\omega_{B}^{p}}+\frac{\omega^{p^{\prime}}}{\omega_{B}^{p^{\prime}}}\right)^{2 s^{\prime}} d x\right)^{\frac{1}{2 s^{\prime}}}
\end{aligned}
$$

$$
\lesssim c\left(2^{p}+2^{p^{\prime}}\right)\left(f_{B}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\right)^{4 s^{\prime}} d x\right)^{\frac{1}{2 s^{\prime}}}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

Now, Proposition 5 and the additional smallness assumption $|\log \mathbb{M}|_{\mathrm{BMO}(B)} \leq \kappa_{1}$ implies

$$
\mathrm{I}_{1,2}+\mathrm{I}_{1,3} \lesssim c\left(2 s^{\prime}\right)^{2}|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

Now we estimate $I_{2}$ as

$$
\begin{aligned}
\mathrm{I}_{2} & =\int_{B}(\mathcal{A}(x, \nabla z)-\mathcal{A}(x, \nabla z+g)) \cdot(\nabla z-\nabla h) d x \\
& \leq c \int_{B_{r}} \varphi_{|\nabla z|}^{\prime}(|g|)|\nabla z-\nabla h| \omega^{p} d x
\end{aligned}
$$

By Young's inequality with $\varphi_{|\nabla z|}$ we get for some $\sigma>0$

$$
\begin{aligned}
\mathrm{I}_{2} & \leq \sigma \int_{B_{r}} \varphi_{|\nabla z|}(|\nabla z-\nabla h|) \omega_{B}^{p} d x+c_{\sigma} f_{B_{r}}\left(\varphi_{|\nabla z|}\right)^{*}\left(\varphi_{|\nabla z|}^{\prime}(|g|) \frac{\omega^{p}}{\omega_{B}^{p}}\right) \omega_{B}^{p} d x \\
& =: \mathrm{I}_{2,1}+\mathrm{I}_{2,2}
\end{aligned}
$$

We fix $\sigma>0$ so small such that $\mathrm{I}_{2,1} \leq \frac{1}{8} \mathrm{I}_{0}$. Now, using that $\sigma$ is fixed, we can replace $c_{\sigma}$ in $\mathrm{I}_{2,2}$ by $c$. We calculate

$$
\begin{aligned}
\mathrm{I}_{2,2} & \left.\leq c \int_{B_{r}}\left(\varphi_{|\nabla z|}\right)^{*}\left(\varphi_{|\nabla z|}^{\prime}| | g \mid\right)\right)\left(\frac{\omega^{p p^{\prime}}}{\omega_{B}^{p p^{\prime}}}+\frac{\omega^{2 p}}{\omega_{B}^{2 p}}\right) \omega_{B}^{p} d x \\
& \leq c \int_{B_{r}} \varphi_{|\nabla z|}(|g|)\left(\frac{\omega^{p p^{\prime}}}{\omega_{B}^{p p^{\prime}}}+\frac{\omega^{2 p}}{\omega_{B}^{2 p}}\right) \omega_{B}^{p} d x .
\end{aligned}
$$

With Lemma 13 we can remove the shift from $\varphi_{|\nabla z|}$ and obtain

$$
\mathrm{I}_{2,2} \leq \delta c \int_{B_{r}}|\nabla z|^{p}\left(\frac{\omega^{p p^{\prime}}}{\omega_{B}^{p p^{\prime}}}+\frac{\omega^{2 p}}{\omega_{B}^{2 p}}\right) \frac{\omega_{B}^{p}}{\omega^{p}} \omega^{p} d x+c \delta^{1-p} \int_{B_{r}}|g|^{p}\left(\frac{\omega^{p p^{\prime}}}{\omega_{B}^{p p^{\prime}}}+\frac{\omega^{2 p}}{\omega_{B}^{2 p}}\right) \frac{\omega_{B}^{p}}{\omega^{p}} \omega^{p} d x
$$

As before we can use Hölder's inequality and Proposition 6 to get rid of the extra weight factors at the expense of a slightly larger power.

$$
\mathrm{I}_{2,2} \leq \delta c\left(f_{B_{r}}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}+c \delta^{1-p}\left(f_{B_{r}}\left(|g|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

Using $g=\left(u-\langle u\rangle_{2 B_{0}}\right) p^{\prime} \zeta^{p^{\prime}-1} \nabla \zeta$ we get

$$
\mathrm{I}_{2,2} \leq \delta c\left(f_{B_{r}}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}+c \delta^{1-p}\left(f_{B_{r}}\left(\frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
$$

Let us estimate $\mathrm{I}_{3}$. We estimate with Young's inequality and $0 \leq \zeta \leq 1$

$$
\mathrm{I}_{3} \lesssim \int_{B} \zeta^{p-1} \omega^{p}|G|^{p-1}\left(\zeta|\nabla z-\nabla h|+\frac{|z-h|}{r}\right) d x
$$

$$
\begin{aligned}
& \leq \delta^{1-p} c \int_{B} \frac{\omega^{p^{\prime}}}{\omega_{B}^{p^{\prime}}} \zeta^{p}|G|^{p} \omega^{p} d x+\delta c f_{B}|\nabla z-\nabla h|^{p} \omega_{B}^{p} d x+\delta c f_{B}\left|\frac{z-h}{r}\right|^{p} \omega_{B}^{p} d x \\
& =: \mathrm{I}_{3,1}+\mathrm{I}_{3,2}+\mathrm{I}_{3,3} .
\end{aligned}
$$

For the calculations that follow consider the term $z-h$ to be extended outside of $B$ by zero. By the weighted Poincaré estimate of Proposition 3 we estimate $\mathrm{I}_{3,3} \lesssim \mathrm{I}_{3.2}$. By triangle inequality and the minimizing property of $h$, see (3.21), we obtain

$$
\mathrm{I}_{3,2} \leq \delta c\left(f_{B}|\nabla z|^{p} \omega_{B}^{p} d x+\int_{B}|\nabla h|^{p} \omega_{B}^{p} d x\right) \leq \delta c f_{B}|\nabla z|^{p} \omega_{B}^{p} d x
$$

As before we can use Hölder's inequality and Proposition 6 to correct the weight slightly at the expense of a slightly larger power. We get

$$
\mathrm{I}_{3,2} \leq \delta c\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

By the same trick

$$
\mathrm{I}_{3,1} \leq \delta^{1-p} c\left(f_{B}\left(\zeta^{p}|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

It remains to estimate $\mathrm{I}_{4}$. For the calculations that follow consider the term $z-h$ to be extended outside of $B$ by zero. From here we need to distinguish cases $p>2$ and $1<p<2$.

We start from the case $p>2$. Using $\nabla z+g=\zeta^{p^{\prime}} \nabla u$ we get

$$
\begin{aligned}
\mathrm{I}_{4} & \leq \int_{B}\left|\nabla\left(\zeta^{p}\right)\right||\mathcal{A}(x, \nabla u)| \cdot|(z-h)| d x \\
& \lesssim \int_{B} \zeta^{\frac{1}{p-1}}|\nabla \zeta|\left|\zeta^{p^{\prime}} \nabla u\right|^{p-2} \omega^{p-1}|\nabla u| \cdot|z-h| \omega d x \\
& \lesssim \int_{B} \frac{r}{R}|\nabla z+g|^{(p-2)}|\nabla u|\left|\frac{z-h}{r}\right| \omega^{p} d x \\
& \lesssim \delta\left(f_{B}|\nabla z+g|^{p} \omega^{p} d x\right)+c_{\delta}\left(\int_{B} \frac{r^{p}}{R^{p}}|\nabla u|^{p} \omega^{p} d x\right)+\delta\left(f_{B}\left|\frac{z-h}{r}\right|^{p} \omega^{p} d x\right) \\
& :=\mathrm{I}_{4,1}+\mathrm{I}_{4,2}+\mathrm{I}_{4,3},
\end{aligned}
$$

where we have applied Young's inequality with exponents $\frac{p}{p-2}, p, p$ at the last step. We estimate now using triangle inequality and $g=-\left(u-\langle u\rangle_{2 B_{0}}\right) p^{\prime} \zeta^{p^{\prime}-1} \nabla \zeta$, see (3.19),

$$
\begin{aligned}
\mathrm{I}_{4,1} & \lesssim \delta\left(f_{B}|g|^{p} \omega^{p} d x\right)+\delta\left(f_{B}|\nabla z|^{p} \omega^{p} d x\right) \\
& \lesssim \delta\left(f_{B}\left|\frac{u-\langle u\rangle_{2 B_{0}}}{R}\right|^{p} \omega^{p} d x\right)+\delta\left(f_{B}|\nabla z|^{p} \omega^{p} d x\right) .
\end{aligned}
$$

Using Caccioppoli inequality we get

$$
\begin{aligned}
\mathrm{I}_{4,2} & \lesssim c_{\delta} \frac{r^{p}}{R^{p}}\left(f_{B}|\nabla u|^{p} \omega^{p} d x\right) \\
& \lesssim c_{\delta} \frac{r^{p}}{R^{p}}\left(f_{2 B}\left(\frac{\left|u-\langle u\rangle_{2 B}\right|}{r}\right)^{p} \omega^{p} d x+\int_{2 B}|G|^{p} \omega^{p} d x\right) \\
& \lesssim c_{\delta} f_{2 B}\left|\frac{u-\langle u\rangle_{2 B_{0}}}{R}\right|^{p} \omega^{p} d x+c_{\delta} f_{2 B}|G|^{p} \omega^{p} d x,
\end{aligned}
$$

where we used that $r<R$ and changed the mean value $\langle u\rangle_{2 B}$ to the worse approximation $\langle u\rangle_{2 B_{0}}$. It remains to estimate $\mathrm{I}_{4,3}$. Using Poincaré's inequality of Proposition 3 we get triangle inequality, and the minimizing property of $h$ we get for some $\theta \in(0,1)$

$$
\mathrm{I}_{4,3} \lesssim \delta f_{B}\left|\frac{z-h}{r}\right|^{p} \omega^{p} d x \lesssim \delta\left(f_{B}(|\nabla z-\nabla h| \omega)^{\theta p} d x\right)^{\frac{1}{\theta}}
$$

As before we can correct the weight $\omega^{p}$ to $\omega_{B}^{p}$ by Hölder's inequality and Proposition 6 and then use the triangle inequality and the minimizing property of $h$.

$$
\mathrm{I}_{4,3} \lesssim \delta f_{B}|\nabla z-\nabla h|^{p} \omega_{B}^{p} d x \lesssim \delta \int_{B}|\nabla z|^{p} \omega_{B}^{p} d x
$$

Now, we can change the weight $\omega_{B}^{p}$ back to $\omega^{p}$ by Hölder's inequality and Proposition 6 at the expense of a slightly increased exponent $s>1$.

$$
\mathrm{I}_{4,3} \lesssim \delta f_{B}|\nabla z-\nabla h|^{p} \omega_{B}^{p} d x \lesssim \delta\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
$$

This completes the case $p>2$.
For case $1<p<2$ we estimate with Young's inequality

$$
\begin{aligned}
\mathrm{I}_{4} & \lesssim f|\nabla \zeta| \zeta^{p-1}|\nabla u|^{p-1}|z-h| \omega^{p} d x \\
& \leq c \delta^{1-p} \int_{B}|\nabla u|^{p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}} \frac{\omega^{p p^{\prime}}}{\omega_{B}^{p^{\prime}}} d x+\delta \int_{B}\left|\frac{z-h}{r}\right|^{p} \omega_{B}^{p} d x \\
& \leq c \delta^{1-p} \int_{B}|\nabla u|^{p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}} \frac{\omega^{p p^{\prime}}}{\omega_{B}^{p^{\prime}}} d x+\delta f_{B}|\nabla z-\nabla h|^{p} \omega_{B}^{p} d x \\
& =: \mathrm{I}_{4,1}+\mathrm{I}_{4,2} .
\end{aligned}
$$

By triangle inequality and the minimizing property of $h$, see (3.21), we obtain

$$
\mathrm{I}_{4,2} \leq \delta c\left(f_{B}|\nabla z|^{p} \omega_{B}^{p} d x+\int_{B}|\nabla h|^{p} \omega_{B}^{p} d x\right) \leq \delta c f_{B}|\nabla z|^{p} \omega_{B}^{p} d x
$$

As before, we can correct the weight $\omega_{B}^{p}$ to $\omega^{p}$ by Hölder's inequality and Proposition 6 at the expense of a slightly increased exponent $s>1$. We get

$$
\mathrm{I}_{4,2} \leq \delta c\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

Also at the term $\mathrm{I}_{4,1}$ we can correct the weight $\omega^{p p^{\prime}} \omega_{B}^{-p^{\prime}}$ to $\omega^{p}$ and obtain

$$
\mathrm{I}_{4,1} \leq c \delta^{1-p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}}\left(f_{B}\left(|\nabla u|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
$$

Now, by Corollary 10 and the Caccioppoli inequality from Proposition 8 and $r \leq R$ we get

$$
\begin{aligned}
\mathrm{I}_{4,1} & \leq c \delta^{1-p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}} f_{2 B}|\nabla u|^{p} \omega^{p} d x+c \delta^{1-p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}}\left(f_{2 B}\left(|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \leq c \delta^{1-p} \frac{r^{p}}{R^{p}} \int_{2 B}|\nabla u|^{p} \omega^{p} d x+c \delta^{1-p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}}\left(f_{2 B}\left(|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \leq c \delta^{1-p} f_{4 B} \frac{\left|u-\langle u\rangle_{4 B}\right|^{p}}{R^{p}} \omega^{p} d x+c \delta^{1-p} \frac{r^{p^{\prime}}}{R^{p^{\prime}}}\left(f_{4 B}\left(|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \leq c \delta^{1-p} f_{4 B} \frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p} d x+c \delta^{1-p}\left(f_{4 B}\left(|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}},
\end{aligned}
$$

where, in the second line, we used that, since $1<p<2$, we have $p^{\prime}>p$.
Collecting all estimates proves the proposition.
3.4. Decay Estimates. We will now use the comparison estimate to derive certain decay estimates of $\mathcal{V}(\cdot, \nabla u)$.
Proposition 18 (Decay estimate). Recall, that $B=B_{r}, B_{0}=B_{R}\left(x_{0}\right)$ and let $4 B \subset 2 B_{0}$. There exist $\lambda \in\left(0, \frac{1}{2}\right), s>1$ and $\kappa_{5}=\kappa_{5}(p, n, \Lambda, s)$ such that the following holds: If $|\log \mathbb{M}|_{\mathrm{BMO}(2 B)} \leq \kappa_{5}$, then for every $\delta \in(0,1)$ there holds

$$
\begin{aligned}
& \int_{\lambda B}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{\lambda B}\right|^{2} d x \\
& \quad \leq \frac{1}{4} f_{B}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{B}\right|^{2} d x \\
& \quad+c\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}+\delta\right) f_{B}|\mathcal{V}(x, \nabla z)|^{2} d x \\
& \quad+c \lambda^{-n} \delta^{1-p}\left(f_{4 B}\left(\frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}+c \delta^{1-p}\left(f_{4 B}\left(\zeta^{p}|\mathcal{V}(x, G)|^{2}\right)^{s} d x\right)^{\frac{1}{s}}
\end{aligned}
$$

Proof. Let $|\log \mathbb{M}|_{\mathrm{BMO}(2 B)} \leq \kappa_{4}$ with $\kappa_{4}$ as in Proposition 17. Also let $s>1$ be as in Proposition 17. Let $\lambda \in\left(0, \frac{1}{2}\right)$, whose precise value will be chosen later. Then

$$
\begin{aligned}
\mathrm{I}_{1} & :=\int_{\lambda B}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{\lambda B}\right|^{2} d x \\
& \leq c{\underset{\lambda B}{ }\left|\mathcal{V}(x, \nabla h)-\langle\mathcal{V}(x, \nabla h)\rangle_{\lambda B}\right|^{2} d x+c{\underset{\lambda B}{ }|\mathcal{V}(x, \nabla z)-\mathcal{V}(x, \nabla h)|^{2} d x}=: \mathrm{I}_{2}+\mathrm{I}_{3} .} .
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathrm{I}_{3} \leq & c \lambda^{-n} f_{\frac{1}{2} B}|\mathcal{V}(x, \nabla z)-\mathcal{V}(x, \nabla h)|^{2} d x \\
\leq & c \lambda^{-n} f_{\frac{1}{2} B}\left|\mathcal{V}_{B}(\nabla z)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x \\
& +c \lambda^{-n} f_{\frac{1}{2} B}\left|\mathcal{V}(x, \nabla z)-\mathcal{V}_{B}(\nabla z)\right|^{2} d x+c \lambda^{-n} f_{\frac{1}{2} B}\left|\mathcal{V}(x, \nabla h)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x \\
= & \mathrm{I}_{3,1}+\mathrm{I}_{3,2}+\mathrm{I}_{3,3} .
\end{aligned}
$$

With the comparison of Proposition 17 we get

$$
\begin{aligned}
\mathrm{I}_{3,1} \leq & c \lambda^{-n} f_{B}\left|\mathcal{V}_{B}(\nabla z)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x \\
\leq & c \lambda^{-n}\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}+\delta\right)\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& +c \lambda^{-n} \delta^{1-p}\left(f_{4 B}\left(\frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}+c \lambda^{-n} \delta^{1-p}\left(f_{4 B}\left(\zeta^{p}|G|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
\end{aligned}
$$

For the terms $\mathrm{I}_{3,2}$ and $\mathrm{I}_{3,3}$ we will proceed similar to the proof of Lemma 16. Note that $\left|\mathbb{M}_{B} \xi\right| \approx\left|\mathbb{M}_{B}\right||\xi|$ and $|\mathbb{M} \xi| \approx|\mathbb{M}||\xi|$ due to (3.1), (3.23) and (3.24). So Lemma 11 and (3.12) imply

$$
\begin{aligned}
\mathrm{I}_{3,2} & \leq c \lambda^{-n} f_{B}\left|V(\mathbb{M} \nabla z)-V\left(\mathbb{M}_{B} \nabla z\right)\right|^{2} d x \\
& \leq c \lambda^{-n} f_{B} \varphi|\mathbb{M} \nabla z| \vee\left|\mathbb{M}_{B} \nabla z\right|\left(\left|\mathbb{M}(x) \nabla z-\mathbb{M}_{B} \nabla z\right|\right) d x \\
& \leq c \lambda^{-n} f_{B}\left(\frac{\left|\mathbb{M} \nabla z-\mathbb{M}_{B} \nabla z\right|}{|\mathbb{M} \nabla z| \vee\left|\mathbb{M}_{B} \nabla z\right|}\right)^{2}\left(\varphi(|\mathbb{M}(x) \nabla z|)+\varphi\left(\left|\mathbb{M}_{B} \nabla z\right|\right)\right) d x \\
& \leq c \lambda^{-n} f_{B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(\varphi(|\mathbb{M}(x) \nabla z|)+\varphi\left(\left|\mathbb{M}_{B} \nabla z\right|\right)\right) d x
\end{aligned}
$$

$$
\begin{aligned}
& \leq c \lambda^{-n} f_{B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(|\nabla z|^{p} \omega^{p}+|\nabla z|^{p} \omega_{B}^{p}\right) d x \\
& \leq c \lambda^{-n} f_{B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(1+\frac{\omega_{B}^{p}}{\omega^{p}}\right)|\nabla z|^{p} \omega^{p} d x
\end{aligned}
$$

Let $s>1$ be as in Corollary 10. Then Hölder's inequality with exponents ( $2 s^{\prime}, 2 s^{\prime}, s$ ) implies

$$
\begin{aligned}
\mathrm{I}_{3,2} \leq c & \lambda^{-n}\left(f_{B}\left(\frac{\left|\mathbb{M}_{B}-\mathbb{M}\right|}{\left|\mathbb{M}_{B}\right|+|\mathbb{M}|}\right)^{4 s^{\prime}} d x\right)^{\frac{1}{2 s^{\prime}}}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
\cdot & \left(1+\left(f_{B}\left(\frac{\omega_{B}^{p}}{\omega^{p}}\right)^{2 s^{\prime}} d x\right)^{\frac{1}{2 s^{\prime}}}\right)
\end{aligned}
$$

With Propositions 5 and 6 and Corollary 10 we obtain

$$
\begin{aligned}
\mathrm{I}_{3,2} & \leq c|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2} \lambda^{-n}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \left.\leq c \lambda^{-n}|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}\left(\int_{2 B}|\mathcal{V}(x, \nabla z)|^{2} d x+\left.\left(f_{2 B} \mid \mathcal{V}(x, G)\right)\right|^{2 s}\right)^{\frac{1}{s}}\right)
\end{aligned}
$$

Analogously, as with $\mathrm{I}_{3,2}$ we estimate

$$
\begin{aligned}
\mathrm{I}_{3,3} & \leq c \lambda^{-n} f_{\frac{1}{2} B}\left|V(\mathbb{M} \nabla h)-V\left(\mathbb{M}_{B} \nabla h\right)\right|^{2} d x \\
& \leq c \lambda^{-n} f_{\frac{1}{2} B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(|\nabla h|^{p} \omega^{p}+|\nabla h|^{p} \omega_{B}^{p}\right) d x \\
& \leq c \lambda^{-n}\left(\max _{\frac{1}{2} B} \omega_{B}^{p}|\nabla h|^{p}\right) f_{\frac{1}{2} B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(\frac{\omega^{p}}{\omega_{B}^{p}}+1\right) d x
\end{aligned}
$$

The interior regularity of $h$, see Proposition 15, and the minimizing property of $h$ implies

$$
\begin{aligned}
\mathrm{I}_{3,3} & \leq c \lambda^{-n} f_{B}|\nabla h|^{p} \omega_{B}^{p} d x f_{\frac{1}{2} B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(\frac{\omega^{p}}{\omega_{B}^{p}}+1\right) d x \\
& \leq c \lambda^{-n} f_{B}|\nabla z|^{p} \omega_{B}^{p} d x f_{\frac{1}{2} B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{2}\left(\frac{\omega^{p}}{\omega_{B}^{p}}+1\right) d x
\end{aligned}
$$

With Hölder's inequality, Proposition 5 and Proposition 6 we obtain

$$
\mathrm{I}_{3,3} \leq c \lambda^{-n} f_{B}|\nabla z|^{p} \omega_{B}^{p} d x\left(f_{B}\left(\frac{\left|\mathbb{M}-\mathbb{M}_{B}\right|}{|\mathbb{M}| \vee\left|\mathbb{M}_{B}\right|}\right)^{4} d x\right)^{\frac{1}{2}}\left(\left(f_{B}\left(\frac{\omega^{p}}{\omega_{B}^{p}}\right)^{2} d x\right)^{\frac{1}{2}}+1\right)
$$

$$
\leq c|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2} \lambda^{-n} f_{B}|\nabla z|^{p} \omega_{B}^{p} d x
$$

With Hölder's inequality, Proposition 6 and Corollary 10 we obtain

$$
\begin{aligned}
\mathrm{I}_{3,3} & \leq c|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2} \lambda^{-n}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}\left(f\left(\frac{\omega_{B}^{p}}{\omega^{p}}\right)^{s^{\prime}} d x\right)^{\frac{1}{s^{\prime}}} \\
& \leq c|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2} \lambda^{-n}\left(f_{B}\left(|\nabla z|^{p} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} \\
& \leq c|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2} \lambda^{-n}\left(f_{B}|\mathcal{V}(x, \nabla z)|^{2} d x+\left(f_{B}|\mathcal{V}(x, G)|^{2 s} d x\right)^{\frac{1}{s}}\right) .
\end{aligned}
$$

The final estimate for the term $\mathrm{I}_{3}$ takes form

$$
\begin{aligned}
\mathrm{I}_{3} \leq & c\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}+\delta\right) \lambda^{-n} f_{B}|\mathcal{V}(x, \nabla z)|^{2} d x \\
& +c\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}+\delta^{1-p}\right) \lambda^{-n}\left(f|\mathcal{V}(x, G)|^{2 s} d x\right)^{\frac{1}{s}} \\
& +c \lambda^{-n} \delta^{1-p}\left(f_{4 B}\left(\frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}}
\end{aligned}
$$

We estimate now the term $\mathrm{I}_{2}$ :

$$
\begin{aligned}
\mathrm{I}_{2} & =c f_{\lambda B}\left|\mathcal{V}(x, \nabla h)-\langle\mathcal{V}(x, \nabla h)\rangle_{\lambda B}\right|^{2} d x \\
& \leq c \int_{\lambda B}\left|\mathcal{V}_{B}(\nabla h)-\left\langle\mathcal{V}_{B}(\nabla h)\right\rangle_{\lambda B}\right|^{2} d x+c{\underset{\lambda B}{ }\left|\mathcal{V}(x, \nabla h)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x}:=\mathrm{I}_{2,1}+\mathrm{I}_{2,2} .
\end{aligned}
$$

We use the decay estimate from Proposition 15 to get

$$
\mathrm{I}_{2,1} \leq c \lambda^{2 \alpha} \int_{\frac{1}{2} B}\left|\mathcal{V}_{B}(\nabla h)-\left\langle\mathcal{V}_{B}(\nabla h)\right\rangle_{B}\right|^{2} d x
$$

By several triangle inequalities we obtain

$$
\begin{aligned}
\mathrm{I}_{2,1} \leq & c \lambda^{2 \alpha} f_{B}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{B}\right|^{2} d x+c \lambda^{2 \alpha} f_{\frac{1}{2} B}\left|\mathcal{V}_{B}(\nabla z)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x \\
& +c \lambda^{2 \alpha} f_{\frac{1}{2} B}\left|\mathcal{V}(x, \nabla z)-\mathcal{V}_{B}(\nabla z)\right|^{2} d x+c \lambda^{2 \alpha} f_{\frac{1}{2} B}\left|\mathcal{V}(x, \nabla h)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x \\
= & \mathrm{I}_{2,1,0}+\mathrm{I}_{2,1,1}+\mathrm{I}_{2,1,2}+\mathrm{I}_{2,1,3} .
\end{aligned}
$$

We can estimate $\mathrm{I}_{2,1,1}, \mathrm{I}_{2,1,2}, \mathrm{I}_{2,1,3}$ and as $\mathrm{I}_{3,1}, \mathrm{I}_{3,2}$ and $\mathrm{I}_{3,3}$, respectively, except that the factor $\lambda^{-n}$ is replaced by $\lambda^{2 \alpha}$. Moreover, we have

$$
\mathrm{I}_{2,2} \leq c \lambda^{-n} \int_{\frac{1}{2} B}\left|\mathcal{V}(x, \nabla h)-\mathcal{V}_{B}(\nabla h)\right|^{2} d x
$$

which can be estimated as $\mathrm{I}_{3,3}$. Overall, we can estimate $\mathrm{I}_{2}$ exactly as $\mathrm{I}_{3}$ (with some better factors as some places) but get the additional term $\mathrm{I}_{2,1,0}$. We arrive at the final estimate

$$
\begin{aligned}
\mathrm{I}_{1} \leq & \mathrm{I}_{2}+\mathrm{I}_{3} \\
\leq & c \lambda^{2 \alpha} \int_{B}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{B}\right|^{2} d x \\
& +c \delta \lambda^{-n} f_{B}|\mathcal{V}(x, \nabla z)|^{2} d x \\
& +c\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}^{2}+\delta^{1-p}\right) \lambda^{-n}\left(f_{4 B}|\mathcal{V}(x, G)|^{2 s} d x\right)^{\frac{1}{s}} \\
& +c \lambda^{-n} \delta^{1-p}\left(f_{4 B}\left(\frac{\left|u-\langle u\rangle_{2 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{s} d x\right)^{\frac{1}{s}} .
\end{aligned}
$$

Now, we fix $\lambda \in\left(0, \frac{1}{2}\right)$ such that the factor $c \lambda^{2 \alpha}$ is smaller than $\frac{1}{4}$. This proves the claim.

For locally integrable function $f$ we define the Hardy-Littlewood maximal function and the sharp maximal function for $\rho \in[1, \infty)$ by

$$
\mathcal{M}_{\rho} f(x):=\sup _{B(x)}\left(f_{B(x)}|f|^{\rho} d y\right)^{\frac{1}{\rho}}, \quad \mathcal{M}_{\rho}^{\sharp} f(x):=\sup _{B(x)}\left(f_{B(x)}\left|f-\langle f\rangle_{B(x)}\right|^{\rho} d y\right)^{\frac{1}{\rho}} .
$$

We can use these operators to express the decay estimates of Proposition 18 in another form.

Proposition 19. There exists $s>1$ and $\kappa_{5}=\kappa_{5}(p, n, \Lambda, s)$ such that the following holds: If $|\log \mathbb{M}|_{\mathrm{BMO}\left(4 B_{0}\right)} \leq \kappa_{5}$, then for almost all $x \in \mathbb{R}^{n}$

$$
\begin{aligned}
M_{2}^{\sharp}(\mathcal{V}(\cdot, \nabla z))(x) \leq & c\left(\mid \log \mathbb{M}_{\left.\right|_{\mathrm{BMO}\left(2 B_{0}\right)}}+\delta\right) M_{2}(\mathcal{V}(\cdot, \nabla z))(x) \\
& \left.+c \delta^{1-p} R^{-p}\left(M_{2 s} \mathbb{1}_{4 B_{0}}\left|u-\langle u\rangle_{2 B_{0}}\right|^{p} \omega^{p}\right)(x)\right)^{\frac{1}{2}} \\
& +c \delta^{1-p} \mathcal{M}_{2 s}\left(\mathbb{1}_{B_{0}} \mathcal{V}(\cdot, G)\right)(x) \\
& +c \frac{R^{n}}{(R+|x|)^{n}}\left(f_{B_{0}}\left|\mathcal{V}(\cdot, \nabla z)-\langle\mathcal{V}(\cdot, \nabla z)\rangle_{B_{0}}\right|^{2} d x\right)^{\frac{1}{2}} .
\end{aligned}
$$

Proof. We choose $\kappa_{5}, s$ and $\lambda \in\left(0, \frac{1}{2}\right)$ as in Proposition 18. Since $\mathcal{V}(\cdot, \nabla v) \in$ $L^{2}\left(\mathbb{R}^{n}\right), \mathcal{V}(\cdot, G) \in L^{2}\left(4 B_{0}\right)$ and by Proposition $3\left|u-\langle u\rangle_{2 B_{0}}\right|^{p} \omega^{p} \in L^{s}\left(2 B_{0}\right)$ all
terms in the following calculations are finite at least for almost every $x$. Fix $x \in \mathbb{R}^{n}$. Then

$$
\mathrm{I}:=M_{2}^{\sharp}(\mathcal{V}(\cdot, \nabla z))(x)=\sup _{r>0}\left(\int_{B_{r}(x)}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{B_{r}(x)}\right|^{2} d y\right)^{\frac{1}{2}} .
$$

We split the choice of $r \in(0, \infty)$ into three parts
(a) $J_{1}:=\left\{r>0: B_{r}\left(x_{0}\right) \cap B_{0}=\emptyset\right\}$.
(b) $J_{2}:=\left\{r>0: \frac{2}{\lambda} B_{r}(x) \subset 4 B_{0}\right\}$.
(c) $J_{3}:=\left\{r>0: B_{r}\left(x_{0}\right) \cap B_{0}=\emptyset\right.$ and $\left.\frac{2}{\lambda} B_{r}(x) \not \subset 4 B_{0}\right\}$.

For $k=1,2,3$ abbreviate

$$
\mathrm{I}_{k}:=\sup _{r \in J_{k}} f_{B_{r}(x)}\left|\mathcal{V}(x, \nabla z)-\langle\mathcal{V}(x, \nabla z)\rangle_{B_{r}(x)}\right| d y
$$

Since $z=0$ outside of $B_{0}$, we obviously have $\mathrm{I}_{1}=0$. If $r \in J_{2}$, then by the decay estimate of Proposition 18 applied to $B=\lambda^{-1} B_{r}(x)$ (with $\delta$ replaced by $\delta^{2}$ ) we get

$$
\begin{aligned}
\mathrm{I}_{2} \leq & \frac{1}{4} \mathrm{I}+c\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}+\delta\right) M_{2}(\mathcal{V}(\cdot, \nabla z))(x) \\
& +c \delta^{1-p} R^{-p}\left(\mathcal{M}_{2 s}\left(\mathbb{1}_{4 B_{0}}\left|u-\langle u\rangle_{2 B_{0}}\right|^{p} \omega^{p}\right)(x)\right)^{\frac{1}{2}}+c \delta^{1-p} M_{2 s}\left(\mathbb{1}_{B_{0}} \mathcal{V}(\cdot, G)\right)(x)
\end{aligned}
$$

If $r \in J_{3}$, then $r \geq c R$. It follows with $\operatorname{supp} z \subset \overline{B_{0}}$ that

$$
\mathrm{I}_{3} \leq c \frac{R^{n}}{(R+|x|)^{n}}\left(f_{B_{0}}\left|\mathcal{V}(\cdot, \nabla z)-\langle\mathcal{V}(\cdot, \nabla z)\rangle_{B_{0}}\right|^{2} d x\right)^{\frac{1}{2}}
$$

Combining the estimate and absorbing $\frac{1}{4} \mathrm{I}$ (which is finite for almost all $x$ ) we prove the claim.
3.5. Main Result Non-Linear. In this section we prove our main theorem 2. We will use Proposition 19 to prove higher integrability of $\mathcal{V}(\cdot, \nabla z)$ and then as a consequence of $|\nabla u|^{p} \omega^{p}$. For this we need the famous Fefferman-Stein inequality that allows to estimate the $L^{q}$-norm of the maximal operator the $L^{q}$-norm of the sharp maximal operator, i.e. for $q \in(2, \infty)$ there holds

$$
\begin{equation*}
\left\|\mathcal{M}_{2} f\right\|_{q} \leq c(q)\left\|\mathcal{M}_{2}^{\sharp} f\right\|_{q} \tag{3.27}
\end{equation*}
$$

for all $f \in L^{q}$. This allows to absorb the term with $\left(|\log \mathbb{M}|_{\mathrm{BMO}(B)}+\delta\right) M_{2}(\mathcal{V}(\cdot, \nabla z))$ later on the left-hand side. This trick was already used in [33] and more recently in [1] in a slightly different form. Kinnunen and Zhou used a local version of the Fefferman-Stein inequality. Unfortunately, the constant $c(q)$ in the version of [33, Lemma 2.4] depends heavily on $q$ and is not adequate to obtain sharp estimates ${ }^{4}$. We therefore present a version of Fefferman-Stein inequality with linear dependency on $q$ (for $q$ large).

Theorem 20. Let $q>1$. Then

$$
\begin{equation*}
\|f\|_{q} \leq c q\left\|\mathcal{M}_{1}^{\sharp} f\right\|_{q} \tag{3.28}
\end{equation*}
$$

for all $f \in L^{q}\left(\mathbb{R}^{n}\right)$.

[^4]Proof. The proof ${ }^{5}$ is based on the duality of the Hardy space $\mathcal{H}^{1}$ and BMO, see Chapter IV, Section 2 of [43]. By the same truncation arguments as in [43] it suffices to consider $f \in L^{q}\left(\mathbb{R}^{n}\right) \cap L^{\infty}\left(\mathbb{R}^{n}\right)$. Let $f \in \mathcal{H}^{1}\left(\mathbb{R}^{n}\right) \cap L^{q^{\prime}}\left(\mathbb{R}^{n}\right)$, where $\mathcal{H}^{1}$ is the Hardy space. Then by (16) of [43]

$$
\langle f, g\rangle \leq c\left\langle\mathcal{M}^{\sharp} f, \mathcal{M} g\right\rangle \leq c\left\|\mathcal{M}^{\sharp} f\right\|_{q}\|\mathcal{M} g\|_{q^{\prime}} .
$$

It is well known that

$$
\begin{equation*}
\|\mathcal{M} g\|_{q^{\prime}} \leq c q\|g\|_{q^{\prime}} \tag{3.29}
\end{equation*}
$$

see for example Chapter I, Section 3, Theorem 1and Remark [43]. Thus,

$$
\langle f, g\rangle \leq c q\left\|\mathcal{M}^{\sharp} f\right\|_{q}\|g\|_{q^{\prime}} .
$$

The claim follows by taking the supremum over all $g \in L^{q^{\prime}}\left(\mathbb{R}^{n}\right)$ with $\|g\|_{p^{\prime}} \leq 1$.
To proceed, we need the following lemma for improving reverse Hölder estimates from [18]. The lemma is a minor modification of the [25, Remark 6.12] and [22, Lemma 3.2].

Lemma 21. Let $B \subset \mathbb{R}^{n}$ be a ball, let $g, h: \Omega \rightarrow \mathbb{R}$ be a integrable functions and $\theta \in(0,1)$ such that

$$
f_{B}|g| d x \leq c_{0}\left(f_{2 B}|g|^{\theta} d x\right)^{\frac{1}{\theta}}+f_{2 B}|h| d x
$$

for all balls $B$ with $2 B \subset \Omega$. Then for every $\gamma \in(0,1)$ there exists $c_{1}=c_{1}\left(c_{0}, \gamma\right)$ such that

$$
f_{B}|g| d x \leq c_{1}\left(f_{2 B}|g|^{\gamma} d x\right)^{\frac{1}{\gamma}}+c_{1} f_{2 B}|h| d x .
$$

We are now prepared to prove the estimate of our main result under the assumption that the function $u$ is already regular enough. We get rid of this extra assumption later.

Proposition 22. Let $u$ be a local weak solution of (1.2), let $\mathbb{M}$ satisfy (1.8), define $\omega$ by (1.10). Then there exists $\kappa_{6}=\kappa_{6}(p, n, \Lambda)$ such that for all balls $B_{0}$ with $8 B_{0} \subset \Omega$ and all $\rho \in[p, \infty)$ with

$$
\begin{equation*}
|\log \mathbb{M}|_{\mathrm{BMO}\left(8 B_{0}\right)} \leq \kappa_{6} \frac{1}{\rho} \tag{3.30}
\end{equation*}
$$

and $|\nabla u| \omega \in L^{\rho}\left(B_{0}\right)$ there holds

$$
\left(f_{\frac{1}{2} B_{0}}(|\nabla u| \omega)^{\rho} d x\right)^{\frac{1}{\rho}} \leq c_{\rho} f_{4 B_{0}}|\nabla u| \omega d x+c_{\rho}\left(f_{4 B_{0}}(|G| \omega)^{\rho} d x\right)^{\frac{1}{\rho}}
$$

for all balls $B_{0}$ with $8 B_{0} \subset \Omega$, where $c_{\rho}=c_{\rho}(p, n, \Lambda, \rho)$. The constant $c_{\rho}$ is continuous in $\rho$.

[^5]Proof. Define $z$ as in the previous section and let $\kappa_{5}$ as in Proposition 19. We will choose $\kappa_{6} \leq \kappa_{5} / p$. Let $q:=\rho / p \geq 1$. If $1 \leq q \leq s$, then the claim already follows from Corollary 10. Thus, it suffices to consider the case $q \geq s$. By replacing $s$ by a smaller one in the steps above, we can even assume that $1<s<s^{2}<q$. The only reason for this assumption is to avoid exploding constants for $q$ close to 1 .

It follows from Proposition 19

$$
\begin{aligned}
\mathrm{I}:=\left\|\mathcal{M}_{2}^{\sharp} \mathcal{V}(\cdot, \nabla z)\right\|_{2 q} \leq & c\left(|\log \mathbb{M}|_{\mathrm{BMO}\left(2 B_{0}\right)}+\delta\right) \| M_{2}\left(\mathcal{V}(\cdot, \nabla z) \|_{2 q}\right. \\
& +c \delta^{1-p} R^{-p}\left\|\mathcal{M}_{2 s}\left(\mathbb{1}_{4 B_{0}}\left|u-\langle u\rangle_{4 B_{0}}\right|^{p} \omega^{p}\right)\right\|_{q}^{\frac{1}{2}} \\
& +c \delta^{1-p}\left\|\mathcal{M}_{2 s}\left(\mathbb{1}_{4 B_{0}} \mathcal{V}(\cdot, G)\right)\right\|_{2} \\
& +c\left\|\frac{R^{n}}{(R+|x|)^{n}}\right\|_{2 q}\left(f\left|\mathcal{V}(\cdot, \nabla z)-\langle\mathcal{V}(\cdot, \nabla z)\rangle_{B_{0}}\right|^{2} d x\right)^{\frac{1}{2}} \\
= & \mathrm{I}_{1}+\mathrm{I}_{2}+\mathrm{I}_{3}+\mathrm{I}_{4} .
\end{aligned}
$$

Since $|\nabla u|^{p} \omega^{p} \in L^{q}\left(B_{0}\right)$, we have $\mathcal{V}(\cdot, \nabla z) \in L^{2 q}\left(\mathbb{R}^{n}\right)$. As a consequence I $<\infty$.
Now, by (3.29) (using $\left.\mathcal{M}_{2}(g)=\left(\mathcal{M}\left(|g|^{2}\right)\right)^{\frac{1}{2}}\right)$ and $\left.s^{2}<q\right)$ and Theorem 20 we obtain

$$
\| \mathcal{M}_{2}\left(\mathcal{V}(\cdot, \nabla z)\left\|_{2 q} \leq c_{s} q\right\| \mathcal{V}(\cdot, \nabla z)\left\|_{2 q} \leq c_{s}\right\| \mathcal{M}_{1}^{\sharp}(\mathcal{V}(\cdot, \nabla z))\left\|_{2 q} \leq c_{s}\right\| \mathcal{M}_{2}^{\sharp}(\mathcal{V}(\cdot, \nabla z)) \|_{2 q} .\right.
$$

We obtain

$$
\mathrm{I}_{1} \leq c q\left(|\log \mathbb{M}|_{\mathrm{BMO}\left(2 B_{0}\right)}+\delta\right) \mathrm{I}
$$

Since $|\nabla u|^{p} \omega^{p} \in L^{q}\left(B_{0}\right)$, we have $|\mathcal{V}(\cdot, \nabla z)|^{2} \in L^{q}\left(\mathbb{R}^{n}\right)$.
Now, we can fix $\kappa_{6}$ and $\delta$ (choose $\delta \in O(1 / q)$ ) so small such that

$$
\mathrm{I}_{1} \leq \frac{1}{2} \mathrm{I}
$$

Thus, we can absorb $I_{1}$ into I. On the other hand we get

$$
\mathrm{I}_{2} \leq c q^{p-1}\left(\int_{4 B_{0}}\left(\frac{\left|u-\langle u\rangle_{4 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{q} d x\right)^{\frac{1}{2 q}}
$$

With $|\mathcal{V}(\cdot, G)|^{2} \leq c|G|^{p} \omega^{p}$ we get and

$$
\mathrm{I}_{3} \leq c q^{p-1}\left(\int_{4 B_{0}}\left(|\mathcal{V}(\cdot, G)|^{2} \omega^{p}\right)^{q} d x\right)^{\frac{1}{2 q}}
$$

Finally,

$$
\mathrm{I}_{4} \leq c\left|B_{0}\right|^{\frac{1}{2 q}}\left(f_{B_{0}}|\mathcal{V}(\cdot, \nabla z)|^{2} d x\right)^{\frac{1}{2}}
$$

We also use

$$
\mathrm{I}=\left\|\mathcal{M}_{2}^{\sharp} \mathcal{V}(\cdot, \nabla z)\right\|_{2 q} \geq c\left\|\mathcal{M}_{1}^{\sharp} \mathcal{V}(\cdot, \nabla z)\right\|_{2 q} \geq \frac{c}{q}\|\mathcal{V}(\cdot, \nabla z)\|_{2 q} .
$$

Overall, we obtain

$$
\|\mathcal{V}(\cdot, \nabla z)\|_{2 q} \leq c q^{p}\left(\int_{4 B_{0}}\left(\frac{\left|u-\langle u\rangle_{4 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{q} d x\right)^{\frac{1}{2 q}}
$$

$$
\begin{aligned}
& +c q^{p}\left(\int_{4 B_{0}}\left(|\mathcal{V}(\cdot, G)|^{2} \omega^{p}\right)^{q} d x\right)^{\frac{1}{2 q}} \\
& +c q\left|B_{0}\right|^{\frac{1}{2 q}}\left(f_{B_{0}}|\mathcal{V}(\cdot, \nabla z)|^{2} d x\right)^{\frac{1}{2}}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\left(f_{B_{0}}\left(|\nabla z|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}} \leq & c q^{p}\left(f_{4 B_{0}}\left(\frac{\left|u-\langle u\rangle_{4 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}} \\
& +c q^{p}\left(\int_{4 B_{0}}\left(|G|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}}+c q{\underset{B}{B_{0}}}|\mathcal{V}(\cdot, \nabla z)|^{2} d x .
\end{aligned}
$$

This proves the claim. Using the definition of $z$ from (3.18) and (3.19) we can translate this back to an estimate in terms of $\nabla u$.

$$
\begin{align*}
\left(f_{\frac{1}{2} B_{0}}\left(|\nabla u|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}} \leq & c q^{p}\left(f_{4 B_{0}}\left(\frac{\left|u-\langle u\rangle_{4 B_{0}}\right|^{p}}{R^{p}} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}}  \tag{3.31}\\
& +c q^{p}\left(f_{4 B_{0}}\left(|G|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}}+c q{\underset{B}{B_{0}}}|\nabla u|^{p} \omega^{p} d x
\end{align*}
$$

Then the smallness assumption (1.12) on $\log \mathbb{M}$ together with Proposition 6 allows to apply the Poincaré type lemma of Proposition 3. We obtain for some $\theta \in(0,1)$

$$
\left(f_{\frac{1}{2} B_{0}}\left(|\nabla u|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}} \leq c_{q}\left(f_{4 B_{0}}\left(|\nabla u|^{p} \omega^{p}\right)^{\theta q} d x\right)^{\frac{1}{\theta_{q}}}+c_{q}\left(f_{4 B_{0}}\left(|G|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}}
$$

We obtained a reverse Hölder's estimate for $\left(|\nabla u|^{p} \omega^{p}\right)^{q}$. Now, Lemma 21 allows to reduce the exponent $\theta q$ to $\frac{1}{p}$. In particular, we get

$$
\begin{equation*}
\left(f_{\frac{1}{2} B_{0}}\left(|\nabla u|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}} \leq c_{q}\left(f_{4 B_{0}}|\nabla u| \omega d x\right)^{p}+c_{q}\left(f_{4 B_{0}}\left(|G|^{p} \omega^{p}\right)^{q} d x\right)^{\frac{1}{q}} \tag{3.32}
\end{equation*}
$$

This proves the claim.
We are now prepared to prove our main theorem,
Proof of main Theorem 2. Propositon 22 agrees in most parts with our main theorem. First, the Proposition 22 is stated with higher integrablity on $\frac{1}{2} B_{0}$, right-hand side on $4 B_{0}$ and smallness on $8 B_{0}$. A simple covering argument shows that we can replace this by higher integrbility on $B_{0}$, right-hand side on $2 B_{0}$ and smallness on $4 B_{0}$.

Second, we require in Proposition 22 the a priori knowledge, that $|\nabla u|^{p} \omega^{p}$ is already locally in $L^{q}$. This artificial assumption can be overcome for example by an approximation argument. This way was for example used in [33] and [1], see
also [10]. Due to our precise sharp estimates we are able to circumvent this argument and argue directly. Indeed, it follows by an iteration argument that $|\nabla u|^{p} \omega^{p} \in$ $L^{q}\left(B_{0}\right)$. For this, let $q_{1} \in\left[1, q_{0}\right]$ be such that $|\nabla u|^{p} \omega^{p} \in L^{q_{1}}\left(B_{0}\right)$. Then Proposition 22 ensures that we have a reverse Hölder's estimate for $\left(|\nabla u|^{p} \omega^{p}\right)^{q_{1}}$. The constants of this estimate only depend on $q_{0}$ and are independent of $q_{1}$. Therefore, we can apply Gehring's lemma (e.g. [25, Theorem 6.6]) to deduce $|\nabla u|^{p} \omega^{p} \in L^{s_{1} q_{1}}$ with $s_{1}>1$ only depending on $q$. Repeating this argument we see that $|\nabla u|^{p} \omega^{p} \in L^{q_{0}}$ and Proposition 22 can be applied. Our main Theorem 2 follows.
3.6. Main Result Linear. In this subsection we give the proof of the main Theorem 1 for the linear setting.

Proof of main Theorem 1. The case $\rho \geq 2$ just follows from Theorem 2 with $p=2$, so it remains to prove the case $1<\rho<2$. We will deduce this from the case $\rho>2$ by means of a local duality argument.

Recall that

$$
-\operatorname{div}(\mathbb{A}(x) \nabla u)=-\operatorname{div}(\mathbb{A}(x) G)
$$

and that $B_{0}$ be a ball with radius $R$ and $4 B_{0} \subseteq \Omega$. Let $H \in L_{\omega}^{\rho^{\prime}}\left(2 B_{0}\right)$ with

$$
\begin{equation*}
\left(f_{2 B_{0}}(|H| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} \leq 1 \tag{3.33}
\end{equation*}
$$

Now, let $z$ solve the dual equation

$$
\begin{align*}
-\operatorname{div}(\mathbb{A}(x) \nabla z) & =-\operatorname{div}\left(\mathbb{A}(x) \mathbb{1}_{2 B_{0}} H\right) & & \text { on } 4 B_{0} \\
z & =0 & & \text { on } \partial\left(4 B_{0}\right) \tag{3.34}
\end{align*}
$$

We want to control $|\nabla z| \omega$ in terms of $H$. For this we can use the super-quadratic case that we have already proven. In particular, by Theorem 1 applied to the exponent $\rho^{\prime} \geq 2$ we have

$$
\begin{aligned}
\left(f_{2 B_{0}}(|\nabla z| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} & \leq c f_{4 B_{0}}|\nabla z| \omega d x+c\left(f_{2 B_{0}}(|H| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} \\
& \leq c\left(f_{4 B_{0}}(|\nabla z| \omega)^{2} d x\right)^{\frac{1}{2}}+c\left(f_{2 B_{0}}(|H| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
\end{aligned}
$$

Using the test function $z$ in (3.34) we immediately see that

$$
\left(f_{4 B_{0}}(|\nabla z| \omega)^{2} d x\right)^{\frac{1}{2}} \leq c\left(f_{2 B_{0}}(|H| \omega)^{2} d x\right)^{\frac{1}{2}} \leq c\left(f_{2 B_{0}}(|H| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
$$

This and the previous estimate imply

$$
\begin{equation*}
\left(\int_{2 B_{0}}(|\nabla z| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} \leq c\left(f_{2 B_{0}}(|H| \omega)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} \leq c \tag{3.35}
\end{equation*}
$$

We choose a cut-off function $\eta \in C_{0}^{\infty}\left(2 B_{0}\right)$ with $\mathbb{1}_{B_{0}} \leq \eta \leq \mathbb{1}_{2 B_{0}}$ and $\|\nabla \eta\|_{\infty} \leq$ $c R^{-1}$. Using the equation for $z$ we calculate

$$
\begin{aligned}
\mathrm{I}:= & f_{2 B_{0}} \mathbb{A}(x) \nabla\left(\eta^{2}\left(u-u_{0}\right)\right) \cdot H d x \\
= & \int_{2 B_{0}} \mathbb{A}(x) \nabla\left(\eta^{2}\left(u-u_{0}\right)\right) \cdot \nabla z d x \\
= & \int_{2 B_{0}} \mathbb{A}(x) \nabla u \cdot \nabla\left(\eta^{2}\left(z-z_{0}\right)\right) d x+\int_{2 B_{0}} \mathbb{A}(x) \nabla\left(\eta^{2}\right)\left(u-u_{0}\right) \cdot \nabla z d x \\
& -\int_{2 B_{0}} \mathbb{A}(x) \nabla u \cdot \nabla\left(\eta^{2}\right)\left(z-z_{0}\right) d x \\
= & : \mathrm{I}_{1}+\mathrm{I}_{2}+\mathrm{I}_{3} .
\end{aligned}
$$

Using the equation for $u$ we get

$$
\begin{aligned}
\left|I_{1}\right| & =\left|\int_{2 B_{0}} \mathbb{A}(x) G \cdot \nabla\left(\eta^{2}\left(z-z_{0}\right)\right) d x\right| \\
& \leq f_{2 B_{0}} \omega^{2}|G|\left|\nabla\left(\eta^{2}\left(z-z_{0}\right)\right)\right| d x \\
& \leq\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}}\left(f_{2 B_{0}}\left(\omega\left|\nabla\left(\eta^{2}\left(z-z_{0}\right)\right)\right|\right)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
\end{aligned}
$$

Using triangle inequality and the weighted Poincaré's inequality of Proposition 3

$$
\left|\mathrm{I}_{1}\right| \leq\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}}\left(f_{2 B_{0}}(\omega|\nabla z|)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
$$

Next

$$
\left|\mathrm{I}_{2}\right| \leq\left(f_{2 B_{0}}\left(\omega \frac{\left|u-u_{0}\right|}{R}\right)^{\rho} d x\right)^{\frac{1}{\rho}}\left(f_{2 B_{0}}(\omega|\nabla z|)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
$$

Thus, by Poincaré's inequality of Proposition 3

$$
\left|\mathrm{I}_{2}\right| \leq\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta \rho} d x\right)^{\frac{1}{\theta \rho}}\left(f_{2 B_{0}}(\omega|\nabla z|)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
$$

for some $\theta \in\left(\frac{1}{\rho}, 1\right)$. Moreover, for some $\theta_{2} \in(0,1)$ close to one we have

$$
\left|\mathrm{I}_{3}\right| \lesssim\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta_{2} \rho} d x\right)^{\frac{1}{\theta_{2} \rho}}\left(f_{2 B_{0}}\left(\omega \frac{\left|z-z_{0}\right|}{R}\right)^{\left(\theta_{2} \rho\right)^{\prime}} d x\right)^{\frac{1}{\left(\theta_{2} \rho\right)^{\prime}}}
$$

Again, by Poincaré's inequality of Proposition 3 with $\theta_{2}$ close to one we get

$$
\left|\mathrm{I}_{3}\right| \lesssim\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta_{2} \rho} d x\right)^{\frac{1}{\theta_{2} \rho}}\left(f_{2 B_{0}}(\omega|\nabla z|)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}}
$$

It is possible to choose $\theta=\theta_{2}$ in the above steps. We finally obtained

$$
|\mathrm{I}| \lesssim\left[\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta \rho} d x\right)^{\frac{1}{\theta \rho}}+\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}}\right]\left(f_{2 B_{0}}(\omega|\nabla z|)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} .
$$

With (3.35) we get

$$
\begin{aligned}
|I| & \lesssim\left[\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta \rho} d x\right)^{\frac{1}{\theta \rho}}+\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}}\right]\left(f_{2 B_{0}}(\omega|\nabla z|)^{\rho^{\prime}} d x\right)^{\frac{1}{\rho^{\prime}}} \\
& \lesssim\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta \rho} d x\right)^{\frac{1}{\theta_{\rho}}}+\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}} .
\end{aligned}
$$

Since $H$ was arbitrary satisfying (3.33) and $\left(L_{\omega}^{\rho^{\prime}}\right)^{*}=L_{\omega^{-1}}^{\rho}$, it follows that

$$
\left.\left(f_{2 B_{0}}\left|\mathbb{A} \nabla\left(\eta^{2}\left(u-u_{0}\right)\right)\right| \omega^{-1}\right)^{\rho} d x\right)^{\frac{1}{\rho}} \lesssim\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta \rho} d x\right)^{\frac{1}{\theta \rho}}+\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}}
$$

Using $\mathbb{A} \nabla\left(\eta^{2}\left(u-u_{0}\right)\right)=\mathbb{A} \nabla u$ on $B_{0}$ and $|\mathbb{A} \nabla u| \approx \omega^{2}|\nabla u|$, we obtain

$$
\left(f_{B_{0}}(\omega|\nabla u|)^{\rho} d x\right)^{\frac{1}{\rho}} \lesssim\left(f_{2 B_{0}}(\omega|\nabla u|)^{\theta \rho} d x\right)^{\frac{1}{\theta_{\rho}}}+\left(f_{2 B_{0}}(\omega|G|)^{\rho} d x\right)^{\frac{1}{\rho}}
$$

Now, Lemma 21 allows to reduce the exponent $\theta \rho$ to 1 . This proves Theorem 1 also in the sub-quadratic case $p<2$.

## 4. Sharpness of the log-BMO Condition

In this section we show by means of examples that our log-BMO condition is sharp. In particular, we show in Example 23 that that the condition on the exponent $\rho$ of higher integrability $|\log \mathbb{M}|_{\mathrm{BMO}\left(8 B_{0}\right)} \leq \frac{\kappa}{\rho}$ in Theorem 1 and Theorem 2 is optimal. We also present an example with a degenerate a weight which does not belong to BMO, but satisfies our smallness condition $|\log \mathbb{M}|_{\mathrm{BMO}}<\varepsilon$ (see Example 24).

Our examples are formulated for the nice linear situation, i.e. $p=2$, which shows that Theorem 2 is even optimal in the linear case, which corresponds to Theorem 1 for $\rho>2$.

Before we start with our examples let us make a short remarks on the logarithm of certain matrices. If $a \in(-1,1)$ and $x \in \mathbb{R}^{n}$, then by Taylor expansion

$$
\begin{aligned}
\log (\operatorname{Id}+a \hat{x} \otimes \hat{x}) & =\sum_{k \geq 1} \frac{(-1)^{k+1}}{k}(a \hat{x} \otimes \hat{x})^{k} \\
& =\left(\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} a^{k}\right) \hat{x} \otimes \hat{x}=\log (1+a) \hat{x} \otimes \hat{x}
\end{aligned}
$$

It is possible to conclude from this that for all $a>-1$

$$
\begin{equation*}
\log (\operatorname{Id}+a \hat{x} \otimes \hat{x})=\left(\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} a^{k}\right) \hat{x} \otimes \hat{x}=\log (1+a) \hat{x} \otimes \hat{x} \tag{4.1}
\end{equation*}
$$

We will also need that the matrix $\operatorname{Id}+a \hat{x} \otimes \hat{x}$ has eigenvalues $1+a$ with eigenvector $\hat{x}:=x /|x|$ and eigenvalue 1 with eigenspace $(\operatorname{span}\{\hat{x}\})^{\perp}$. This implies that for example that Id $-\hat{x} \otimes \hat{x}$ has eigenvalues zero and one, so for the spectral norm we have $|\operatorname{Id}-\hat{x} \otimes \hat{x}|=1$.

Example 23. This example is a modification of the one of Meyers [39, Section 5], who considered the case $n=2$. Let $B_{1}(0)$ denote the unit ball in $\mathbb{R}^{n}$. Let us define $u: B_{1}(0) \rightarrow \mathbb{R}$ for $n \geq 2$ by

$$
\begin{equation*}
u(x):=|x|^{1-\varepsilon} \hat{x}_{1} \tag{4.2}
\end{equation*}
$$

with $\hat{x}_{1}:=x_{1} /|x|$ and $\varepsilon \in\left(0, \frac{1}{2}\right]$. Then

$$
\nabla u(x)=|x|^{-\varepsilon}\left(e_{1}-\varepsilon \hat{x} \hat{x}_{1}\right)
$$

where $e_{1}=(1,0, \ldots, 0)$. Since $\varepsilon \in\left(0, \frac{1}{2}\right]$ we have $u \in W^{1,2}\left(B_{1}(0)\right)$. More precisely, we have $\nabla u \in L^{\frac{n}{\varepsilon}, \infty}\left(B_{1}(0)\right)$ (Marcinkiewicz space), so $u \in W^{1, \rho}\left(B_{1}(0)\right)$ for all $\rho<\frac{n}{\varepsilon}$. Moreover, $u \notin W^{1, \rho}\left(B_{1}(0)\right)$ for $\rho \geq \frac{n}{\varepsilon}$.

Let us define the symmetric matrices $\mathbb{M}, \mathbb{A}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n \times n}$ by

$$
\begin{align*}
\mathbb{M}(x) & :=\theta \operatorname{Id}+(1-\theta) \hat{x} \otimes \hat{x} \\
\mathbb{A}(x) & :=\mathbb{M}^{2}(x)=\theta^{2} \operatorname{Id}+\left(1-\theta^{2}\right) \hat{x} \otimes \hat{x} \tag{4.3}
\end{align*}
$$

with $\theta \in(0,1)$ to be chosen later. The eigenvalues of $\mathbb{M}$ are 1 with eigenvector $\hat{x}$ and $\theta$ with multiplicity $n-1$ and eigenspace $(\operatorname{span}\{\hat{x}\})^{\perp}$. For $\mathbb{A}$ the eigenvalue $\theta$ changes to $\theta^{2}$. Thus,

$$
\begin{align*}
& \theta \mathrm{Id} \leq \mathbb{M}(x) \\
& \leq \mathrm{Id}  \tag{4.4}\\
& \theta^{2} \mathrm{Id} \leq \mathbb{A}(x) \leq \mathrm{Id}
\end{align*}
$$

We calculate

$$
\mathbb{M}(x)^{2} \nabla u(x)=\mathbb{A}(x) \nabla u(x)=|x|^{-\varepsilon}\left(\theta^{2} e_{1}+\left(1-\varepsilon-\theta^{2}\right) \hat{x} \hat{x}_{1}\right)
$$

and

$$
-\operatorname{div}\left(\mathbb{M}^{2} \nabla u\right)=-\operatorname{div}(\mathbb{A} \nabla u)=-|x|^{-\varepsilon-1}\left(\left(-\varepsilon(1-\varepsilon)+\left(1-\varepsilon-\theta^{2}\right)(n-1)\right) \hat{x}_{1}\right.
$$

To get $-\operatorname{div}(\mathbb{A} \nabla u)=0$ we need

$$
\begin{equation*}
-\varepsilon(1-\varepsilon)+\left(1-\varepsilon-\theta^{2}\right)(n-1)=0 \tag{4.5}
\end{equation*}
$$

For $n=2$ we can set $\theta:=1-\varepsilon$. In general, we can define

$$
\begin{equation*}
\theta:=\sqrt{1-\varepsilon-\frac{\varepsilon(1-\varepsilon)}{n-1}} \tag{4.6}
\end{equation*}
$$

and obtain

$$
-\operatorname{div}\left(\mathbb{M}^{2} \nabla u\right)=-\operatorname{div}(\mathbb{A} \nabla u)=0
$$

Since $\varepsilon \in\left(0, \frac{1}{2}\right]$ and $n \geq 2$, we have $\theta \in\left[\frac{1}{2}, 1\right)$. This implies with (4.4) that

$$
|\mathbb{M}(x)|\left|\mathbb{M}^{-1}(x)\right| \leq \frac{1}{\theta} \leq 2=: \Lambda
$$

In particular, the condition number of $\mathbb{M}(x)$ is bounded independently of the specific choice of $\varepsilon \in\left(\frac{1}{2}, 1\right]$.

By (4.1) we calculate

$$
\begin{aligned}
\log \mathbb{M} & =\log (\theta \operatorname{Id}+(1-\theta) \hat{x} \otimes \hat{x})=\log (\theta) \operatorname{Id}+\log \left(\operatorname{Id}+\frac{1-\theta}{\theta} \hat{x} \otimes \hat{x}\right) \\
& =\log (\theta) \operatorname{Id}+\log \left(1+\frac{1-\theta}{\theta}\right) \hat{x} \otimes \hat{x} \\
& =\log (\theta)(\operatorname{Id}-\hat{x} \otimes \hat{x})
\end{aligned}
$$

Thus,

$$
\|\log \mathbb{M}\|_{\mathrm{BMO}} \leq\|\log \mathbb{M}\|_{\infty}=|\log (\theta)||\mathrm{Id}-\hat{x} \otimes \hat{x}|=|\log (\theta)|
$$

Thus, by (4.6)

$$
|\log \theta|=\left|\log \sqrt{1-\varepsilon-\frac{\varepsilon(1-\varepsilon)}{n-1}}\right| \leq \frac{1}{2}|\log (1-2 \varepsilon)| \leq \varepsilon
$$

Overall, we have a function $u: B \rightarrow \mathbb{R}$ and a positive matrix valued weights $\mathbb{A}=\mathbb{M}^{2}$ with the following properties
(a) The function $u \in W^{1,2}(B)$ solves

$$
-\operatorname{div}\left(\mathbb{M}^{2} \nabla u\right)=-\operatorname{div}(\mathbb{A} \nabla u)=0
$$

(b) The condition number of the weight satisfies $|\mathbb{M}(x)|\left|\mathbb{M}^{-1}(x)\right| \leq 2$.
(c) The weight satisfies the smallness condition $\|\log \mathbb{M}\|_{\infty} \leq \varepsilon$.
(d) We have limited higher integrability of the gradients. More precisely, we have $u \in W^{1, \rho}(B)$ for all $\rho \varepsilon<n$ and $u \notin W^{1, \rho}(B)$ for all $\rho \varepsilon \geq n$.
This shows that the smallness assumption $|\log \mathbb{M}|_{\mathrm{BMO}} \leq \kappa_{0} \frac{1}{\rho}$ in our Theorems 1 and 2 are optimal.

We will now present an example with a degenerate matrix-valued weight $\mathbb{A}$, which is not from BMO but satisfies our logarithmic smallness assumption. Note, that it was already mentioned in $[6$, Remark 2.12$]$ that the condition $\mathbb{A} \in \mathrm{BMO}$ is not necessary.

Example 24. We proceed similar to Example 23. Let $B_{1}(0)$ denote the unit ball in $\mathbb{R}^{n}$ with $n \geq 2$. For $\varepsilon \in\left(0, \frac{1}{2}\right]$ define $u: B_{1}(0) \rightarrow \mathbb{R}$ by

$$
u(x):=|x|^{1-\varepsilon / 2} \hat{x}_{1} .
$$

with $\hat{x}_{1}:=x_{1} /|x|$. Moreover, let us define the matrix-valued weights $\mathbb{M}, \mathbb{A}: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}^{n \times n}$ by

$$
\begin{aligned}
\mathbb{M}(x) & :=|x|^{-\varepsilon / 2}(\theta \operatorname{Id}+(1-\theta) \hat{x} \otimes \hat{x}), \\
\mathbb{A}(x) & :=\mathbb{M}^{2}(x)=|x|^{-\varepsilon / 2}\left(\theta^{2} \operatorname{Id}+\left(1-\theta^{2}\right) \hat{x} \otimes \hat{x}\right) .
\end{aligned}
$$

So compared to Example 23 our function $u$ has an additional factor $|x|^{\varepsilon / 2}$ and the weight $\mathbb{M}$ has an additional factor $|x|^{-\varepsilon / 2}$. Similar calculations lead to

$$
\begin{equation*}
\nabla u(x)=|x|^{-\varepsilon / 2}\left(e_{1}-\frac{\varepsilon}{2} \hat{x} \hat{x}_{1}\right) \tag{4.7}
\end{equation*}
$$

with $e_{1}=(1,0, \ldots, 0)$ and

$$
-\operatorname{div}\left(\mathbb{M}^{2} \nabla u\right)=-\operatorname{div}(\mathbb{A} \nabla u)=-|x|^{-\varepsilon-1}\left(-\frac{\varepsilon}{2}(1-\varepsilon)+\left(1-\frac{\varepsilon}{2}-\theta^{2}\right)(n-1)\right) \hat{x}_{1} .
$$

Thus, for

$$
\begin{equation*}
\theta:=\sqrt{1-\frac{\varepsilon}{2}-\frac{\varepsilon(1-\varepsilon)}{2(n-1)}} \tag{4.8}
\end{equation*}
$$

we obtain

$$
-\operatorname{div}\left(\mathbb{M}^{2} \nabla u\right)=-\operatorname{div}(\mathbb{A} \nabla u)=0
$$

Since $\delta>0$ and $\varepsilon \in\left(0, \frac{1}{2}\right]$ we have $\theta \in\left[\frac{1}{2}, 1\right)$
Our weight satisfies

$$
|x|^{-\varepsilon / 2} \theta \operatorname{Id} \leq \mathbb{M}(x) \leq|x|^{-\varepsilon / 2} \mathrm{Id} .
$$

So, although the weight $\mathbb{A}$ is singular, it has finite condition number

$$
|\mathbb{M}(x)|\left|\mathbb{M}^{-1}(x)\right| \leq \frac{1}{\theta} \leq 2=: \Lambda
$$

Similar to Example 23 we conclude

$$
\log \mathbb{M}=-\frac{\varepsilon}{2}(\log |x|) \operatorname{Id}+\log (\theta)(\operatorname{Id}-\hat{x} \otimes \hat{x})
$$

Thus,

$$
|\log \mathbb{M}|_{\mathrm{BMO}} \leq \frac{\varepsilon}{2}|\log | x|\mathrm{Id}|_{\mathrm{BMO}}+|\log (\theta)(\operatorname{Id}-\hat{x} \otimes \hat{x})|_{\mathrm{BMO}} \leq \varepsilon+|\log (\theta)|
$$

We calculate

$$
|\log (\theta)|=\frac{1}{2}\left|\log \left(1-\frac{\varepsilon}{2}-\frac{\varepsilon(1-\varepsilon)}{2(n-1)}\right)\right| \leq \frac{1}{2}|\log (1-\varepsilon)| \leq \frac{\varepsilon}{2}
$$

Overall,

$$
|\log \mathbb{M}|_{\mathrm{BMO}} \leq \frac{3}{2} \varepsilon
$$

This shows that the weight $\mathbb{M}$ satisfies our log-smallness condition and our Theorem 2 can be applied. In particular, we get $\nabla u \in L^{\rho}(B)$ for all $q>p$ with $\rho \leq \frac{\kappa_{0}}{\varepsilon}$.

Due to (4.7) we have $\nabla u \in L^{\frac{2 n}{\varepsilon}, \infty}(B)$ (Marcinkiewicz space). More precisely, $\nabla u \in L^{2}$ if and only if $\rho<\frac{2 n}{\varepsilon}$. So we have limited higher integrability in agreement with our Theorems 1 and 2.

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[^1]:    ${ }^{1} \mu(\mathbb{X})$ is just the logarithmic norm induced by the euclidean norm $|\cdot|$.

[^2]:    ${ }^{2}$ The version from [16] used + instead of $\vee$. This implies that the equality in (3.8) has to be replaced by $\bar{\sim}$. This would still be sufficient for the purpose of this paper.

[^3]:    ${ }^{3}$ Note that the matrix exponential is not operator monotone on $\mathbb{R}_{\mathrm{sym}}^{n \times n}$. However, we compare here only with a multiple of the identity matrix.

[^4]:    ${ }^{4}$ There also exists other local version of the Fefferman-Stein estimate in [29, Lemma 4] or [19, Theorem 5.25]. However, as far as we can see these versions depend exponentially on $q$.

[^5]:    ${ }^{5}$ It is also possible to proof the theorem by redistributional estimates as in Chapter IV, Section 3.6, Corollary 1 of [43]. However, the dependency on $q$ is again exponential.

