Hamiltonian cycles in 4-connected planar and projective planar triangulations with few 4-separators

On-Hei Solomon Lo* Jianguo Qian*

Abstract

Whitney proved in 1931 that every 4-connected planar triangulation is hamiltonian. Later in 1979, Hakimi, Schmeichel and Thomassen conjectured that every such triangulation on n vertices has at least $2(n-2)(n-4)$ hamiltonian cycles. Along this direction, Brinkmann, Souffriau and Van Cleemput established a linear lower bound on the number of hamiltonian cycles in 4-connected planar triangulations. In stark contrast, Alahmadi, Aldred and Thomassen showed that every 5-connected triangulation of the plane or the projective plane has exponentially many hamiltonian cycles. This gives the motivation to study the number of hamiltonian cycles of 4-connected triangulations with few 4-separators. Recently, Liu and Yu showed that every 4-connected planar triangulation with $O(n/\log n)$ 4-separators has a quadratic number of hamiltonian cycles. By adapting the framework of Alahmadi et al. we strengthen the last two aforementioned results. We prove that every 4-connected planar or projective planar triangulation with $O(n)$ 4-separators has exponentially many hamiltonian cycles.

1 Introduction

A classical theorem of Whitney in 1931 proved that every 4-connected planar triangulation has a hamiltonian cycle [19]. In 1956, this result was extended by Tutte [18] to 4-connected planar graphs. One may subsequently ask how many hamiltonian cycles a 4-connected planar triangulation or planar graph may have. In 1979, Hakimi, Schmeichel and Thomassen [9] proposed the following conjecture:

Conjecture 1 ([9])**.** Every 4-connected planar triangulation G on n vertices has at least $2(n-2)(n-4)$ hamiltonian cycles, with equality if and only if G is the double-wheel graph on n vertices, that is, the join of a cycle of length $n-2$ and an empty graph on two vertices.

In the same paper, they also proved that every 4-connected planar triangulation on n vertices has at least $n/\log_2 n$ hamiltonian cycles. This lower bound was recently improved by Brinkmann, Souffriau and Van Cleemput [3] to a linear bound of $12(n-2)/5$, which was then refined to $161(n-2)/60$ for $n \geq 7$ by Cuvelier [6]. For 4-connected planar graphs, Sander [16] showed that there exists a hamiltonian cycle containing any two prescribed edges in any 4-connected planar graph. This implies that every 4-connected planar graph G has at least $\binom{\Delta}{2}$ hamiltonian cycles, where Δ denotes the maximum degree of G. However, it assures only a constant lower bound as there are infinitely many 4-connected planar graphs of maximum degree upper bounded by 4.

*School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China. This work was partially supported by NSFC grant 11971406.

Generalizing the method used in [3], Brinkmann and Van Cleemput [4] gave a linear lower bound on the number of hamiltonian cycles in 4-connected planar graphs.

We will focus on the number of hamiltonian cycles of 4-connected triangulations of the plane or the projective plane with a bounded number of 4-separators. Interestingly, if Conjecture 1 held, we would have that a 4-connected planar triangulation has a minimum number of hamiltonian cycles if and only if it has a maximum number of 4-separators, as the double-wheel graphs are the 4-connected planar triangulations that maximize the number of 4-separators [8]. Trivially, 5-connected planar triangulations have a minimum number of 4-separators among 4-connected planar triangulations as they have no 4-separators. Indeed, Alahmadi, Aldred and Thomassen [1] proved that every 5-connected triangulation embedded on the plane or on the projective plane has $2^{\Omega(n)}$ hamiltonian cycles. Following the approach of Alahmadi et al., it was shown in [12] that every 4-connected planar triangulation has at least $\Omega((n/\log n)^2)$ hamiltonian cycles if it has only $O(\log n)$ 4-separators. Recently, Liu and Yu [11] improved this result by showing that every 4-connected planar triangulation with $O(n/\log n)$ 4-separators has $\Omega(n^2)$ hamiltonian cycles. These results, in a sense, give evidence supporting that triangulations of the plane or the projective plane with fewer 4-separators may have more hamiltonian cycles. We will prove the following result, indicating that a 4-connected planar or projective planar triangulation may have exponentially many hamiltonian cycles as long as it has at most a linear number of 4-separators, thereby extending the results mentioned above.

Theorem 2. *Let G be a 4-connected planar or projective planar triangulation on n vertices and let c be an arbitrary constant less than $1/324$. If G has at most cn 4-separators, then it has $2^{\Omega(n)}$ hamiltonian cycles.*

2 Results

In this section we first prepare some lemmas which will be used to construct a vertex subset with several properties (see also Lemma 8). The proof of Theorem 2 will be given at the end of this section.

For notation and terminology not explicitly defined in this paper, we refer the reader to [2, 13]. A vertex subset or a subgraph of a connected graph is *separating* if its removal disconnects the graph. We call a separating vertex set on k vertices a *k-separator*. A *k-cycle* is a cycle of length k. Let S be an independent set. We say S *saturates* a 4- or 5-cycle C if S contains two vertices of C. A *diamond-6-cycle* is the graph depicted in Figure 1, where the white vertices are called *crucial*. We say S *saturates* a diamond-6-cycle D if S contains three crucial vertices of D. Recall that the *Euler genus* $eg(\Sigma)$ of a surface Σ is defined to be $2-\chi(\Sigma)$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. A graph H is *d-degenerate* if every induced subgraph of H has a vertex of degree at most d. It is well known that every d-degenerate graph H is $(d+1)$-colorable and hence has an independent set of at least $|V(H)|/(d+1)$ vertices.

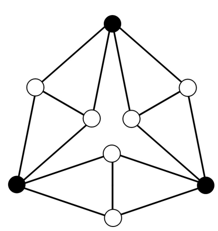

Figure 1: A diamond-6-cycle with six crucial vertices (white).

The following tool is due to Alahmadi et al. [1], which helps finding homotopic curves from a sufficiently large family of curves.

Lemma 3 ([1, Corollary 2])**.** *Let Σ be a surface of Euler genus σ and let $\mathcal{C}$ be a family of simple closed curves on Σ with the property that every $C \in \mathcal{C}$ can have at most one point that is contained in other curves in $\mathcal{C}$. Let r be a positive integer. If $|\mathcal{C}| \geq 5(r-1)\sigma + 1$, then there are r homotopic curves in $\mathcal{C}$.*

The number of the vertices adjacent to at least three vertices of a 4-cycle on a surface can be bounded as follows.

Lemma 4. *Let G be a triangulation of a surface of Euler genus σ, and C be a 4-cycle in G. Denote $V_C := \{v \in V(G) : |N_G(v) \cap V(C)| \geq 3\}$. Then $|V_C| \leq 8(\sigma + 1)$.*

Proof. Suppose to the contrary that $|V_C| > 8(\sigma+1)$. We have that G contains $K_{3,q}$ as a subgraph, where $q = 2(\sigma+1)+1$. This is however impossible since, by a theorem of Ringel [14, 15], the Euler genus of $K_{3,q}$ equals $eg(K_{3,q}) = \lceil \frac{q-2}{2} \rceil = \sigma + 1 > \sigma$. $\square$

The next three lemmas aim at finding a vertex set that saturates no 4-cycle, or 5-cycle, or diamond-6-cycle. The main idea of the proofs comes from [1].

Lemma 5. *Let G be a triangulation of a surface of Euler genus σ and let $S \subseteq V(G)$ be an independent set of vertices of degree at most 6. If S saturates no separating 4-cycle in G, then S has a subset of size at least $|S|/c$ that saturates no 4-cycle, where $c := \binom{6}{2}(10\sigma + 1) + 1$.*

Proof. Let H be the graph on the vertex set S such that two vertices are adjacent if they saturate some 4-cycle in G. It suffices to show that H has maximum degree at most $d := c - 1 = \binom{6}{2}(10\sigma + 1)$ and hence chromatic number at most c. Suppose, to the contrary, there exists $v \in S = V(H)$ with $d_H(v) > d$. As $d_G(v) \leq 6$, there are $u, w \in N_G(v)$ such that the path uvw are contained in at least $\lceil (d+1)/\binom{6}{2} \rceil = (10\sigma + 1) + 1$ 4-cycles in G. We may contract the path uvw and apply Lemma 3 to show that there are at least three homotopic 4-cycles containing the path uvw and saturated by S. This yields a separating 4-cycle saturated by S and hence contradicts our assumption. Thus the lemma follows. $\square$

Lemma 6. *Let G be a triangulation of a surface of Euler genus σ and let $S \subseteq V(G)$ be an independent set of vertices of degree at most 6. If S saturates no 4-cycle in G, then S has a subset of size at least $|S|/c$ that saturates no 5-cycle, where $c := 2\binom{6}{2}(40\sigma + 1) + 1$.*

Proof. Let H be the graph on the vertex set S in which two vertices are adjacent if they saturate some 5-cycle in G. Let $d := c - 1 = 2\binom{6}{2}(40\sigma + 1)$. It suffices to show that H is d-degenerate. Let K be any induced subgraph of H. We will show that K has a vertex of degree at most d.

We first consider the following. Let v be a vertex in K with $d_K(v) \geq d$. Since $d_G(v) \leq 6$, there exist $u, w \in N_G(v)$, distinct vertices $x_1, \ldots, x_{2t} \in N_K(v)$ and $y_1, \ldots, y_{2t} \in V(G) \setminus S$, where $t := 40\sigma + 1$, such that for every $1 \leq i \leq 2t$ either $uvwx_iy_iu$ or $uvwy_ix_iu$ is a 5-cycle saturated by $V(K)$. Denote by C_i the 5-cycle saturated by v and x_i, and denote by P_i the path obtained from C_i by deleting v ($1 \leq i \leq 2t$). We claim that at least t vertices of $y_1, \ldots, y_{2t}$ are distinct. Otherwise there are $1 \leq i < j < k \leq 2t$ such that $y_i = y_j = y_k$. Then u or w is adjacent to two of x_i, x_j, x_k, say $x_i, x_j \in N_G(w)$. However, this implies that $wx_iy_ix_jw$ is a 4-cycle saturated by $V(K)$, which contradicts our assumption. Therefore, we may assume that the paths $P_1, \ldots, P_t$ are pairwise internally disjoint. By contracting the path uvw and applying Lemma 3, we may obtain nine homotopic curves from $P_1, \ldots, P_t$, say $P_1, \ldots, P_9$. Relabelling if necessary, we may further assume that the closed disc D_v bounded by $P_1 \cup P_9$ contains $P_1, \dots, P_9$, and the closed disc bounded by $P_1 \cup P_5$ contains $P_1, \dots, P_5$ but not $P_6, \dots, P_9$.

We now show that K has minimum degree at most d. Suppose to the contrary that $d_K(v) > d$ for every $v \in V(K)$. We choose a vertex $v \in V(K)$ and the associated nine homotopic curves such that the number of vertices of G contained in D_v is minimum.

Let C be a 5-cycle in G containing x_5 and another vertex $v' \in V(K)$ outside of D_v. We claim that $v' = v$. Let P be the minimal path in C containing x_5 with end-vertices in $P_1 \cup P_9$. Notice that C has length five and x_5 lies in the interior but not the boundary of the disc D_v. So by the arrangement of $P_1, \dots, P_9$, the end-vertices of P must be u and w. Since $V(K)$ is an independent set of G and $v, x_5 \in V(K)$, we have $v' \notin N_G(x_5) \cup \{u, w\}$. Therefore P has length less than four. If P has length two, then $uvwx_5u$ would be a 4-cycle saturated by $V(K) \subseteq S$, which contradicts our assumption. If P has length three, then we have $v' = v$ (otherwise $uvwv'u$ would be a 4-cycle saturated by $V(K)$). This thus establishes our claim.

Our previous claim implies that all 5-cycles in G containing x_5 and another vertex of $V(K)$ other than v must lie in D_v. As $d_{K-v}(x_5) = d_K(x_5) - 1 \geq d$, we may apply our discussion above to $K - v$ and x_5 (instead of K and v) to obtain nine homotopic curves associated with x_5 and a closed disc containing them. Then we may have $D_{x_5} \subseteq D_v$. Moreover, it is not hard to see that D_{x_5} does not contain x_1. Therefore the number of vertices in D_{x_5} is strictly less than that of D_v. This contradicts our choice of v and D_v and hence the result follows. $\square$

The proof of following lemma is omitted as it can be readily deduced from the proof of [1, Lemma 10].

Lemma 7 ([1, Lemma 10])**.** *Let G be a triangulation of a surface of Euler genus σ and let $S \subseteq V(G)$ be an independent set of vertices of degree at most 6. If S saturates no 4-cycle in G, then S has a subset of size at least $|S|/c$ that saturates no diamond-6-cycle, where c is a positive constant depending only on σ.*

One of the key ingredients of the proof of exponential lower bound on the number of hamiltonian cycles in 5-connected triangulations given by Alahmadi et al. [1] is to find many edge sets $F \subseteq E(G)$ so that $G - F$ is 4-connected. Their approach was refined by [12, 11] for 4-connected planar triangulations. In order to prove our result for triangulations of the projective plane, we need to generalize a lemma given by Liu and Yu [11] to surfaces of higher genus.

Let G be a triangulation of any surface and $A \subseteq V(G)$ be a 3-separator of G. It is shown in the proof of [1, Lemma 1] and its subsequent discussion that $G[A]$ is a surface separating 3-cycle. Using this fact and a theorem of Thomas and Yu [17] that every 4-connected projective planar graph is hamiltonian, the proof of [11, Lemma 2.1] can be easily modified to show the following result. We omit the proof.

Lemma 8 ([11, Lemma 2.1])**.** *Let G be a 4-connected triangulation of a surface Σ of Euler genus σ. Let $S \subseteq V(G)$ be a vertex subset satisfying the following conditions:*

(i) $d_G(v) \leq 6$ *for any* $v \in S$;

(ii) *S is an independent vertex set;*

(iii) *no vertex in S is contained in any separating 4-cycle in G;*

(iv) *no vertex in S is adjacent to three vertices of any separating 4-cycle in G; and*

(v) *S saturates no 4-, 5- or diamond-6-cycle.*

Let $F \subseteq E(G)$ be any edge subset such that $|F| = |S|$ and for any $v \in S$ there is precisely one edge in F incident with v. Then $G - F$ is 4-connected. Moreover, if Σ is the plane or the projective plane, then G has $2^{\Omega(|S|)}$ hamiltonian cycles.

We are now ready to prove Theorem 2.

Proof of Theorem 2. To prove the theorem, it suffices to construct a vertex set $S \subseteq V(G)$ of size $\Omega(n)$ satisfying conditions (i) to (v) in Lemma 8.

Since G has average degree less than 6 and minimum degree at least 4, the set S_1 of vertices of degree at most 6 has size at least $n/3$. We may subsequently obtain a vertex set $S_2 \subseteq S_1$ of size at least $|S_1|/6 \geq n/18$ satisfying (i) and (ii) of Lemma 8, as planar and projective planar graphs are 6-colorable.

Let C be any separating 4-cycle in G. Since S_2 is an independent set, it follows from Lemma 4 that S_2 has at most $16+2 = 18$ vertices that are contained in C or adjacent to three vertices of C. Deleting these vertices from S_2 for every separating 4-cycle, we may obtain a vertex set $S_3 \subseteq S_2$ satisfying (i) to (iv) of Lemma 8 and $|S_3| \geq n/18 - 18cn = (1/18 - 324c/18)n$. Recall that c is a constant less than $1/324$. This means that $1/18 - 324c/18$ is a positive constant and hence $|S_3| = \Omega(n)$.

As no vertex in S_3 is contained in any separating 4-cycle, no 4-cycle saturated by S_3 is separating. Successively applying Lemmas 5, 6 and 7, we obtain a vertex set $S \subseteq S_3$ of size $\Omega(n)$ satisfying (i) to (v) of Lemma 8, implying that G has $2^{\Omega(n)}$ hamiltonian cycles. This completes our proof. $\square$

We remark that Grünbaum [7] and Nash-Williams [5] independently conjectured that every 4-connected toroidal graph is hamiltonian. The truth of this conjecture (respectively, an analogue for the Klein bottle) would extend Theorem 2 to triangulations of the torus (respectively, the Klein bottle). Note that there are non-hamiltonian 4-connected graphs that are embedded in the double torus or in the surface obtained from the sphere by attaching three crosscaps (see [10]).

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