# Golden ratio primal-dual algorithm with linesearch

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Abstract Golden ratio primal-dual algorithm (GRPDA) is a new variant of the classical Arrow-Hurwicz method for solving structured convex optimization problem, in which the objective function consists of the sum of two closed proper convex functions, one of which involves a composition with a linear transform. The same as the Arrow-Hurwicz method and the popular primal-dual algorithm (PDA) of Chambolle and Pock, GRPDA is full-splitting in the sense that it does not rely on solving any subproblems or linear system of equations iteratively. Compared with PDA, an important feature of GRPDA is that it permits larger primal and dual stepsizes. However, the stepsize condition of standard GRPDA requires that the spectral norm of the linear transform is known, which can be difficult to obtain in some applications. Furthermore, constant stepsizes prescribed by the stepsize condition are usually overconservative in practice.

In this paper, we propose a linesearch strategy for GRPDA, which not only does not require the spectral norm of the linear transform but also allows adaptive and potentially much larger stepsizes. Within each linesearch step, only the dual variable needs to be updated, and it is thus quite cheap and does not require any extra matrix-vector multiplications for many special yet important applications, e.g., regularized least squares problem. Global convergence and O(1/N)ergodic convergence rate results measured by the primal-dual gap function are established, where N denotes the iteration counter. When one of the component functions is strongly convex, faster  $O(1/N^2)$  ergodic convergence rate results are established by adaptively choosing some algorithmic parameters. Moreover, when both component functions are strongly convex, nonergodic linear converge results are established. Numerical experiments on matrix game and LASSO problems illustrate the effectiveness of the proposed linesearch strategy.

**Keywords** Saddle point problem  $\cdot$  golden ratio primal-dual algorithm  $\cdot$  acceleration  $\cdot$  linesearch  $\cdot$  ergodic convergence rate  $\cdot$  spectral norm

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#### **1** Introduction

Let  $\mathbb{R}^p$  and  $\mathbb{R}^q$  be finite-dimensional Euclidean spaces, each endowed with an inner product and the induced norm denoted by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ , respectively. Let  $f : \mathbb{R}^p \to (-\infty, +\infty)$ and  $g : \mathbb{R}^q \to (-\infty, +\infty)$  be extended real-valued closed proper convex functions,  $K \in \mathbb{R}^{p \times q}$ be a linear transform from  $\mathbb{R}^q$  to  $\mathbb{R}^p$ . Denote the Legendre-Fenchel conjugate of f by  $f^*$ , i.e.,

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 $f^*(y) = \sup_{x \in \mathbb{R}^p} \{ \langle y, x \rangle - f(x) \}, y \in \mathbb{R}^p$ . In this paper, we focus on the following saddle point problem with a bilinear coupling term

$$\min_{x \in \mathbb{R}^q} \max_{y \in \mathbb{R}^p} \mathcal{L}(x, y) := g(x) + \langle Kx, y \rangle - f^*(y).$$
(1)

Since the biconjugate of f is itself, i.e.,  $(f^*)^* = f$ , see [23], problem (1) reduces to the following primal minimization problem

$$\min_{x \in \mathbb{R}^q} g(x) + f(Kx). \tag{2}$$

On the other hand, by swapping the "min" and the "max", problem (1) can be transformed to the following dual maximization problem

$$\max_{y \in \mathbb{R}^p} -f^*(y) - g^*(-K^\top y),$$
(3)

where  $K^{\top}$  denotes the matrix transpose or adjoint operator of K. Under regularity conditions, e.g., Assumption 2.1 given below, strong duality holds between (2) and (3).

Problems (1)-(3) naturally arise from abundant interesting applications, including signal and image processing, machine learning, statistics, mechanics and economics, and so on, see, e.g., [3,4,6,14,27] and the references therein. To solve (1)-(3) simultaneously, popular choices include the well known alternating direction method of multipliers (ADMM) [12,13], the primal-dual algorithm (PDA) of Chambolle and Pock [6,16,22], and their accelerated and generalized variants [18,20]. The focus of this paper is primal-dual type full-splitting algorithms<sup>1</sup> for solving (1)-(3). We emphasize that the literature on numerical algorithms for solving (1)-(3) has become fairly vast and a thorough overview is not only impossible but also far beyond the focus of this work. Instead, we review only some primal-dual type algorithms that are most closely related to this work. Before going into details, we define our notation.

#### 1.1 Notations

As already mentioned above, the transpose operation of a matrix or a vector is denoted by superscript " $\top$ ". The spectral norm of K is denoted by L, i.e.,  $L := ||K|| = \sup\{\langle Kx, y \rangle : ||x|| = ||y|| = 1, x \in \mathbb{R}^q, y \in \mathbb{R}^p\}$ . Let h be any extended real-valued closed proper convex function defined on a finite dimensional Euclidean space  $\mathbb{R}^m$ . The effective domain of h is denoted by dom(h) :=  $\{x \in \mathbb{R}^m : h(x) < +\infty\}$ , and the subdifferential of h at  $x \in \mathbb{R}^m$  is denoted by  $\partial h(x) := \{\xi \in \mathbb{R}^m : h(y) \ge h(x) + \langle \xi, y - x \rangle$  for all  $y \in \mathbb{R}^m\}$ . Furthermore, for  $\lambda > 0$ , the proximal operator of  $\lambda h$  is given by

$$\operatorname{Prox}_{\lambda h}(x) := \arg\min_{y \in \mathbb{R}^m} \left\{ h(y) + \frac{1}{2\lambda} \|y - x\|^2 \right\}, \quad x \in \mathbb{R}^m.$$

The relative interior of C is denoted by ri(C). Finally, throughout this paper, we denote the golden ratio by  $\phi$ , i.e.,  $\phi = \frac{\sqrt{5}+1}{2}$ , which is a key parameter in golden ratio type algorithms. Other notations will be specified in the context.

 $<sup>^{1}</sup>$  An algorithm is said to be full-splitting if it does not rely on solving any subproblems or linear system of equations iteratively and the main computations per iteration are matrix-vector multiplications and the evaluations of proximal operators.

#### 1.2 Related works

The theme of this paper is to incorporate linesearch into the golden ratio primal-dual algorithm, which was originally introduced in [8] for solving (1)-(3). A main feature of primal-dual type algorithms is that the three problems (1)-(3) are solved simultaneously by alternatingly updating the primal and the dual variables. Among others, the classical augmented Lagrangian method and its variants such as ADMM [10,12,13,17] are most popular. However, ADMM is not full-splitting since at each iteration it requires to solve a subproblem of the form  $\min_{x \in \mathbb{R}^q} \frac{1}{2} ||Kx - b_n||^2 + g(x)$ , where  $b_n \in \mathbb{R}^p$  varies with the iteration counter n. Note that even if the proximal operator of g is easy to evaluate, this problem needs to be solved iteratively, unless K is the identity operator. On the other hand, for regularized least-squares problem, ADMM requires to solve a linear system of equations at each iteration, which could be prohibitive for large scale applications.

The most classical and simple full-splitting algorithm designed in the literature for solving (1)-(3) goes back to [26], which is nowadays widely known as the Arrow-Hurwicz method. Started at  $x_0 \in \mathbb{R}^q$  and  $y_0 \in \mathbb{R}^p$ , the Arrow-Hurwicz method iterates for  $n \ge 1$  as

$$\begin{cases} x_n = \operatorname{Prox}_{\tau g}(x_{n-1} - \tau K^\top y_{n-1}), \\ y_n = \operatorname{Prox}_{\sigma f^*}(y_{n-1} + \sigma K x_n), \end{cases}$$

where  $\tau, \sigma > 0$  are stepsize parameters. Since f and g, as well as their conjugate functions, are closed proper and convex, their proximal operators are uniquely well defined everywhere. The heuristics of Arrow-Hurwicz method is to solve the minimax problem (1) by alternatingly minimizing with the primal variable x and maximizing with the dual variable y and, meanwhile, incorporating proximal steps by taking into account the latest information of each of the variables. Convergence of the Arrow-Hurwicz method with small stepsizes was studied in [11], and a sublinear convergence rate result, measured by primal-dual gap function, was obtained in [6, 21] when dom( $f^*$ ) is bounded. Though intuitively make sense, the Arrow-Hurwicz method does not converge in general. In fact, a divergent example has been constructed in [15]. Nonetheless, this method has been popular in image processing community and is known as primal-dual hybrid gradient method [6, 11, 28].

To obtain a convergent full-splitting algorithm under more general setting, Chambolle and Pock [6,7] modified the Arrow-Hurwicz method by adopting an extrapolation step. Specifically,  $x_n$ is replaced by the extrapolated point  $z_n := x_n + \delta(x_n - x_{n-1})$  in the computation of  $y_n$ , where  $\delta \in (0, 1]$  is an extrapolation parameter, resulting the following iterative scheme

$$\begin{cases} x_n = \operatorname{Prox}_{\tau g}(x_{n-1} - \tau K^\top y_{n-1}), \\ z_n = x_n + \delta(x_n - x_{n-1}), \\ y_n = \operatorname{Prox}_{\sigma f^*}(y_{n-1} + \sigma K z_n). \end{cases}$$
(4)

The convergence of (4) with  $\delta = 1$  was established in [6] under the condition  $\tau \sigma L^2 < 1$ . Later, it was shown in [16] that the iterative scheme (4), with  $\delta = 1$ , is an application of a weighted proximal point algorithm to solve an equivalent mixed variational inequality of the optimality conditions of (1). Furthermore, the scheme (4), again with  $\delta = 1$ , is also referred to as split inexact Uzawa method in [11], where the connection of PDA with preconditioned or linearized ADMM has been revealed, see [6, 24]. Note that, without taking a correction step as done in [16], the convergence of (4) with  $\delta \in (0, 1)$  is still open. The overrelaxed, inertial and accelerated versions of (4) were investigated in [7], and its stochastic extension was studied in [5].

Recently, based on a seminal convex combination technique originally introduced by Malitsky [19] for variational inequality problem, we proposed a golden ratio primal-dual algorithm (GRPDA) in [8]. Instead of an extrapolation step as taken in (4), a convex combination of essentially all the primal iterates generated till far, i.e.,  $\{x_i : i = 0, 1, ..., n-1\}$ , is used in the *n*-th iteration of GRPDA. Specifically, given  $x_0 \in \mathbb{R}^q$  and let  $z_0 := x_0$ , GRPDA iterates for  $n \ge 1$  as

$$\begin{cases} z_n = \frac{\psi - 1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \\ x_n = \operatorname{Prox}_{\tau g} (z_n - \tau K^\top y_{n-1}), \\ y_n = \operatorname{Prox}_{\sigma f^*} (y_{n-1} + \sigma K x_n). \end{cases}$$
(5)

Here  $\psi \in (1, \phi]$  determines the convex combination coefficients. Global iterative convergence and ergodic convergence rate results are established under the condition  $\tau \sigma L^2 < \psi$  in [8]. Since  $\psi \in (1, \phi]$  and  $\phi = \frac{\sqrt{5}+1}{2}$ , this stepsize condition is much relaxed than that of the Chambolle and Pock's PDA (4), which is  $\tau \sigma L^2 < 1$ . Therefore, GRPDA permits larger primal and dual stepsizes, which is essential for fast practical convergence. The experimental results given in [8] confirmed the benefits of allowing larger stepsizes.

#### 1.3 Motivations and contributions

The main contribution of this work is to introduce a linesearch strategy into GRPDA (5). Our motivations have two aspects. First, in many applications, especially when the matrix K is large and dense, e.g., CT image reconstruction [1,25], the exact spectral norm of K can be very expensive to compute or estimate. On the other hand, even if the spectral norm of K can be obtained, the stepsizes governed by the condition  $\tau \sigma ||K||^2 < \psi$  is usually too conservative for fast practical convergence. Hence, our goal in this paper is to incorporate linesearch into GRPDA which can significantly accelerate the convergence while still theoretically guarantees convergence with desired convergence rate.

In general, linesearch requires extra evaluations of proximal operators and/or matrix-vector multiplications in every linesearch iteration. Interestingly, for many special yet important applications as pointed out in [20], the proximal operator of  $f^*$  is extremely simple, and as a result the linesearch procedure does not require any additional matrix-vector multiplications. So, motivated by [20], which introduced linesearch into the PDA (4), we propose in this paper to incorporate linesearch into the GRPDA (5). However, as will be seen in later sections, our theoretical analysis on the stepsize behaviors generated by the linesearch is fundamentally different from those presented in [20] or given in other literature. In particular, our algorithm combining with linesearch not only does not assume a priori knowledge about the spectral norm of K, but also generates adaptive and potentially much larger stepsizes. We establish global convergence as well as ergodic  $\mathcal{O}(1/N)$ sublinear convergence rate in the general convex case, where N denotes the iteration counter. When either one of the component functions is strongly convex, GRPDA with linesearch can be shown to converge at the faster  $\mathcal{O}(1/N^2)$  ergodic sublinear rate. Furthermore, if both component functions are strongly convex, iterative and nonergodic linear convergence results are established. Hence, even with the stepsize relaxations by the proposed linesearch, the global convergence as well as theoretical convergence rates are still guaranteed to remain consistent with their counterparts without using linesearch. In addition, our numerical experiments show much practical benefits can be obtained from the proposed linesearch strategies.

#### 1.4 Organization

The rest of this paper is organized as follows. In Section 2, we make our assumptions, provide some useful facts and further define some notations. Section 3 is devoted to the GRPDA with linesearch in the general convex case, while the cases when either one or both of the component functions are strongly convex are discussed in Section 4. Our numerical results on minimax matrix game and LASSO problems are reported in Section 5 to show the benefits obtained by adopting the linesearch strategy. Finally, some concluding remarks are drawn in Section 6.

## 2 Assumptions and preliminaries

# 2.1 Assumptions and further notation

Throughout the paper, we make the following blanket assumptions.

**Assumption 2.1** Assume that the set of solutions of (2) is nonempty and, in addition, there exists  $\tilde{x} \in ri(dom(g))$  such that  $K\tilde{x} \in ri(dom(f))$ .

Under Assumption 2.1, it follows from [23, Corollaries 28.2.2 and 28.3.1] that  $\bar{x} \in \mathbb{R}^q$  is a solution of (2) if and only if there exists  $\bar{y} \in \mathbb{R}^p$  such that  $(\bar{x}, \bar{y})$  is a saddle point of  $\mathcal{L}(x, y)$ , i.e.,  $\mathcal{L}(\bar{x}, y) \leq \mathcal{L}(\bar{x}, \bar{y}) \leq \mathcal{L}(x, \bar{y})$  for all  $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$ . As such,  $(\bar{x}, \bar{y})$  is a solution of the minimax problem (1) and  $\bar{y}$  is a solution of the dual problem (3). We denote the set of solutions of (1) by  $\Omega$ , which is nonempty under Assumption 2.1 and characterized by

$$\Omega := \{ (\bar{x}, \bar{y}) \in \mathbb{R}^q \times \mathbb{R}^p : 0 \in \partial g(\bar{x}) + K^\top \bar{y} \text{ and } 0 \in \partial f^*(\bar{y}) - K\bar{x} \}$$

Hereafter, we let  $(\bar{x}, \bar{y}) \in \Omega$  be a generic saddle point. When g (resp.  $f^*$ ) is strongly convex, the primal optimal solution  $\bar{x}$  (resp. the dual optimal solution  $\bar{y}$ ) is unique. Define

$$P(x) := g(x) - g(\bar{x}) + \langle K^{\top} \bar{y}, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^{q}, D(y) := f^{*}(y) - f^{*}(\bar{y}) - \langle K\bar{x}, y - \bar{y} \rangle, \ \forall y \in \mathbb{R}^{p}.$$

By subgradient inequality, it is clear that  $P(x) \ge 0$  and  $D(y) \ge 0$  for all  $x \in \mathbb{R}^q$  and  $y \in \mathbb{R}^p$ . Apparently, P(x) and D(y) are convex in x and y, respectively. The primal-dual gap function is defined by  $G(x,y) := \mathcal{L}(x,\bar{y}) - \mathcal{L}(\bar{x},y)$  for  $(x,y) \in \mathbb{R}^q \times \mathbb{R}^p$ . It is easy to verify that

$$G(x,y) = P(x) + D(y) \ge 0, \ \forall (x,y) \in \mathbb{R}^q \times \mathbb{R}^p.$$
(6)

Note that the functions  $P(\cdot)$  and  $D(\cdot)$  depend on the saddle point  $(\bar{x}, \bar{y}) \in \Omega$ . Nonetheless, we omit this dependence in the notation  $P(\cdot)$  and  $D(\cdot)$  since it is always clear from the context which saddle point is under consideration, and similarly for the primal-dual gap function  $G(\cdot, \cdot)$ . This measure of primal-dual gap function has been used in, e.g., [6, 7, 20], to establish convergence rate results for primal-dual type methods. We also adopt the measure in this paper.

**Assumption 2.2** Assume that the proximal operators of the component functions f and g either have closed form formulas or can be evaluated efficiently.

In applications such as signal and image processing and machine learning, the component functions f and g usually enforce data fitting and regularization and hence, often preserve simple structures so that their proximal operators can be computed efficiently or just have closed form formulas. Examples of such functions are abundant, see, e.g., [2, Chapter 6]. Therefore, Assumption 2.2 is fulfilled in diverse applications. Note that the proximal operators of  $f^*$  and  $g^*$  are also easily computable under Assumption 2.2 due to the Moreau decomposition theorem [23, Theorem 31.5].

#### 2.2 Facts and identities

The following simple facts and identities will be used repeatedly in the convergence analysis.

**Fact 2.1** Let  $h : \mathbb{R}^m \to (-\infty, +\infty]$  be an extended real-valued closed proper and  $\gamma$ -strongly convex function with modulus  $\gamma \ge 0$ . Then for any  $\tau > 0$  and  $x \in \mathbb{R}^m$ , it holds that  $z = \operatorname{Prox}_{\tau h}(x)$  if and only if  $h(y) \ge h(z) + \frac{1}{\tau} \langle x - z, y - z \rangle + \frac{\gamma}{2} ||y - z||^2$  for all  $y \in \mathbb{R}^m$ .

**Fact 2.2** Let  $\{a_n : n \ge 1\}$  and  $\{b_n : n \ge 1\}$  be real and nonnegative sequences. If  $a_{n+1} \le a_n - b_n$  for all  $n \ge 1$ , then  $\lim_{n\to\infty} a_n$  exists and  $\lim_{n\to\infty} b_n = 0$ .

For any  $x, y, z \in \mathbb{R}^m$  and  $\alpha \in \mathbb{R}$ , there hold

$$2\langle x - y, x - z \rangle = \|x - y\|^2 + \|x - z\|^2 - \|y - z\|^2,$$
(7)

 $\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$ (8)

Verifications of these identities are elementary.

#### 3 GRPDA with linesearch — Convex case

As pointed out in Section 1, in some applications it can be very expensive to estimate the spectral norm of K. Furthermore, the stepsizes governed by the condition  $\tau\sigma \|K\|^2 < \psi \in (1, \phi]$  are usually too conservative in the applications. To address these issues, we introduce a linesearch strategy into GRPDA to choose stepsizes adaptively. Within each linesearch step, only the dual variable needs to be updated. The resulting algorithm, called GRPDA-L, is stated in Algorithm 3.1.

# Algorithm 3.1 (GRPDA-L)

Step 0. Choose  $x_0 = z_0 \in \mathbb{R}^q$ ,  $y_0 \in \mathbb{R}^p$ ,  $\psi \in (1, \phi)$ ,  $\sigma \in (0, 1)$ ,  $\beta > 0$ ,  $\mu \in (0, 1)$  and  $\tau_0 > 0$ . Set  $\varphi = \frac{1+\psi}{\psi^2} \in (1,2) \text{ and } n = 1.$ 

Step 1. Čompute

$$z_n = \frac{\psi - 1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1},\tag{9}$$

$$x_n = \operatorname{Prox}_{\tau_{n-1}g}(z_n - \tau_{n-1}K^{\top}y_{n-1}).$$
(10)

Step 2. Let  $\tau = \varphi \tau_{n-1}$  and compute

$$y_n = \operatorname{Prox}_{\beta\tau_n f^*}(y_{n-1} + \beta\tau_n K x_n), \tag{11}$$

where  $\tau_n = \tau \mu^i$  and *i* is the smallest nonnegative integer such that

$$\sqrt{\beta\tau_n} \| K^\top y_n - K^\top y_{n-1} \| \le \sigma \sqrt{\psi/\tau_{n-1}} \| y_n - y_{n-1} \|.$$
(12)

Step 3. Set  $n \leftarrow n+1$  and return to Step 1.

From Step 2 of Algorithm 3.1, the linesearch procedure may require to compute  $\operatorname{Prox}_{\beta\tau_n f^*}$  and  $K^+y_n$  repeatedly to find a proper  $\tau_n$  at each iteration. However, as pointed out in [19, Remark 2], the linesearch procedure becomes extremely simple when the proximal operator of  $\lambda f^*$ , where  $\lambda > 0$ , is linear or affine. Some examples are listed below.

- (a)  $\operatorname{Prox}_{\lambda f^*}(u) = u \lambda c$  when  $f^*(y) = \langle c, y \rangle$  for some  $c \in \mathbb{R}^p$ , (b)  $\operatorname{Prox}_{\lambda f^*}(u) = \frac{1}{1+\lambda}(u+\lambda b)$  when  $f^*(y) = \frac{1}{2}||y-b||^2$  for some  $b \in \mathbb{R}^p$ ,
- (c)  $\operatorname{Prox}_{\lambda f^*}(u) = u + \frac{b \langle u, a \rangle}{\|a\|^2} a$  when  $f^*(y)$  is the indicator function of the hyperplane  $H = \{u : u \in U\}$  $\langle a, u \rangle = b$  for some  $a \in \mathbb{R}^p$  and  $b \in \mathbb{R}$ .

In all these cases, the evaluation of  $\operatorname{Prox}_{\lambda f^*}$  is very simple, and it is unnecessary to compute  $K^{\top} y_n$ repeatedly since it can be obtained by combining some already computed quantities. Therefore, in these cases the linesearch step is quite cheap and does not require any additional matrix-vector multiplications. Furthermore, if necessary, one can always exchange the roles of the primal and the dual variables in problem (1) to take advantage of the above mentioned structure. Also note that at each iteration the increasing ratio of stepsizes  $\{\tau_n\}$  is upper bounded by  $\varphi \in (1,2)$  as  $\tau_n \leq \varphi \tau_{n-1}$ , and as suggested in [19], one default choice could let  $\psi = 1.5$  so that  $\varphi = (1+\psi)/\psi^2 = 10/9$ . Here, the parameter  $\beta > 0$  in Algorithm 3.1, also appeared in Algorithm 4.2, is introduced to scale the primal and the dual variables so that they will converge in a weighted balance way. Similar settings has also been done in [8] for GRPDA without linesearch.

The following lemma shows that the linesearch step of Algorithm 3.1 is well-defined. In addition, it establishes some important properties on sequences  $\{\tau_n : n \ge 1\}$  and  $\{\delta_n : n \ge 1\}$ , with  $\delta_n := \tau_n / \tau_{n-1}$ , which are essential for establishing the convergence results of Algorithm 3.1. Since the proof of Lemma 3.1 is rather technical and complicated, for fluency of the overall paper, we put the proof in Appendix A.

**Lemma 3.1** Let  $\underline{\tau} := \frac{\sigma\sqrt{\psi}}{L\sqrt{\beta\varphi}} > 0$ . Then, we have the following properties. (i) The linesearch step of Algorithm 3.1, i.e., Step 2, always terminates. (ii) For any  $\rho \in (0, 1)$ , there exists an infinite subsequence  $\{n_k : k \ge 1\} \subseteq \{1, 2, ...\}$  such that  $\tau_{n_k} \ge \underline{\tau}$  and  $\delta_{n_k} \ge \rho$ . (iii) For any integer N > 0, we have  $|\mathcal{K}_N| \ge \hat{c}N$  for some constant  $\hat{c} > 0$ , where  $\mathcal{K}_N = \{1 \le n \le N : \tau_n \ge \underline{\tau}\}$  and  $|\mathcal{K}_N|$  is the cardinality of the set  $\mathcal{K}_N$ , which implies  $\sum_{n=1}^N \tau_n \ge \underline{c}N$  with  $\underline{c} = \hat{c}\underline{\tau}$ .

We emphasize that the linesearch procedure adopted by Algorithm 3.1 is motivated but theoretically very different from that of [20, Algorithm 1]. In fact, the sequence  $\{\tau_n : n \ge 1\}$  generated by [20, Algorithim 1] is uniformly bounded below by some positive constant, see [20, Lemma 3.3 (ii)]. In contrast, as seen in (iii) of Lemma 3.1, only a subsequence  $\{\tau_{n_k}: k \geq 1\}$ , a fraction of  $\{\tau_n : n \geq 1\}$ , is guaranteed to have uniform lower bound  $\underline{\tau} > 0$ . Similar arguments also apply to Algorithms 4.1 and 4.2 in Section 4.1.

# 3.1 Useful lemmas

We next present two useful lemmas, which play critical roles in the convergence analysis. Hereafter, we always fix  $(\bar{x}, \bar{y}) \in \Omega$  arbitrarily without further mentioning.

**Lemma 3.2** Let  $\{(z_n, x_n, y_n) : n \ge 1\}$  be generated by Algorithm 3.1. Then, for any  $(\bar{x}, \bar{y}) \in \Omega$ , there holds

$$\tau_n G(x_n, y_n) \le \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle + \frac{1}{\beta} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \psi \delta_n \langle x_n - z_{n+1}, x_{n+1} - x_n \rangle + \tau_n \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle.$$
(13)

Proof. It follows from (10), (11) and Fact 2.1 that

$$\langle x_{n+1} - z_{n+1} + \tau_n K^{\top} y_n, \ \bar{x} - x_{n+1} \rangle \ge \tau_n (g(x_{n+1}) - g(\bar{x})),$$
 (14)

$$\langle x_n - z_n + \tau_{n-1} K^{\top} y_{n-1}, \ x_{n+1} - x_n \rangle \ge \tau_{n-1} \big( g(x_n) - g(x_{n+1}) \big), \tag{15}$$

$$\left\langle \frac{1}{\beta}(y_n - y_{n-1}) - \tau_n K x_n, \ \bar{y} - y_n \right\rangle \ge \tau_n \left( f^*(y_n) - f^*(\bar{y}) \right).$$
 (16)

Multiplying (15) by  $\delta_n = \tau_n / \tau_{n-1}$  and using the fact  $x_n - z_n = \psi(x_n - z_{n+1})$ , which follows from (9), we obtain

$$\langle \psi \delta_n(x_n - z_{n+1}) + \tau_n K^\top y_{n-1}, x_{n+1} - x_n \rangle \ge \tau_n (g(x_n) - g(x_{n+1})).$$
 (17)

Direct calculations show that a summation of (14), (16) and (17) gives

$$\begin{aligned} \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle &+ \frac{1}{\beta} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \psi \delta_n \langle x_n - z_{n+1}, x_{n+1} - x_n \rangle \\ &+ \tau_n \langle K^\top (y_n - y_{n-1}), x_n - x_{n+1} \rangle - \tau_n \langle K^\top \bar{y}, x_n - \bar{x} \rangle + \tau_n \langle K \bar{x}, y_n - \bar{y} \rangle \\ &\geq \tau_n \big( f^*(y_n) - f^*(\bar{y}) \big) + \tau_n \big( g(x_n) - g(\bar{x}) \big), \end{aligned}$$

which, by the definition of  $G(\cdot, \cdot)$  in (6), implies (13) immediately.

**Lemma 3.3** Let  $\{(z_n, x_n, y_n) : n \ge 1\}$  be generated by Algorithm 3.1. For  $n \ge 1$ , define

$$a_n := \frac{\psi}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2,$$
(18)

$$b_n := \psi \delta_n \|z_{n+1} - x_n\|^2 + (1 - \sigma) \left(\psi \delta_n \|x_{n+1} - x_n\|^2 + \frac{1}{\beta} \|y_n - y_{n-1}\|^2\right).$$
(19)

Then, it holds that  $a_{n+1} \leq a_n - b_n$  for  $n \geq 1$ .

Proof. Fix  $n \ge 1$ . First, it is easy to verify from  $\varphi = (1 + \psi)/\psi^2$  and  $\delta_n = \tau_n/\tau_{n-1} \le \varphi$  that

$$1 + \frac{1}{\psi} - \psi \delta_n \ge 1 + \frac{1}{\psi} - \psi \varphi = 0.$$
<sup>(20)</sup>

It follows from (12) and Cauchy-Schwarz inequality that

$$2\tau_n \|K^{\top} y_n - K^{\top} y_{n-1}\| \|x_{n+1} - x_n\| \le \sigma \left(\psi \delta_n \|x_{n+1} - x_n\|^2 + \frac{1}{\beta} \|y_n - y_{n-1}\|^2\right).$$
(21)

Furthermore, by Lemma 3.2, identity (7) and Cauchy-Schwarz inequality, we have

$$\|x_{n+1} - \bar{x}\|^{2} + \frac{1}{\beta} \|y_{n} - \bar{y}\|^{2} + 2\tau_{n}G(x_{n}, y_{n})$$

$$\leq \|z_{n+1} - \bar{x}\|^{2} + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^{2} + 2\tau_{n} \|K^{\top}y_{n} - K^{\top}y_{n-1}\| \|x_{n+1} - x_{n}\| \qquad (22)$$

$$-\psi \delta_{n} \|z_{n+1} - x_{n}\|^{2} - (1 - \psi \delta_{n}) \|x_{n+1} - z_{n+1}\|^{2} - \psi \delta_{n} \|x_{n+1} - x_{n}\|^{2} - \frac{1}{\beta} \|y_{n} - y_{n-1}\|^{2}.$$

Since  $x_{n+1} = \frac{\psi}{\psi-1} z_{n+2} - \frac{1}{\psi-1} z_{n+1}$ , which follows from (8) and (9), we deduce

$$\|x_{n+1} - \bar{x}\|^2 = \frac{\psi}{\psi - 1} \|z_{n+2} - \bar{x}\|^2 - \frac{1}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{\psi}{(\psi - 1)^2} \|z_{n+2} - z_{n+1}\|^2$$
$$= \frac{\psi}{\psi - 1} \|z_{n+2} - \bar{x}\|^2 - \frac{1}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\psi} \|x_{n+1} - z_{n+1}\|^2,$$
(23)

where the second equality is due to  $z_{n+2} - z_{n+1} = \frac{\psi - 1}{\psi}(x_{n+1} - z_{n+1})$ . Combining (23) with (22), we obtain

$$\frac{\psi}{\psi-1} \|z_{n+2} - \bar{x}\|^2 + \frac{1}{\beta} \|y_n - \bar{y}\|^2 + 2\tau_n G(x_n, y_n) 
\leq \frac{\psi}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2 + 2\tau_n \|K^\top y_n - K^\top y_{n-1}\| \|x_{n+1} - x_n\| 
-\psi \delta_n \|z_{n+1} - x_n\|^2 - (1 + \frac{1}{\psi} - \psi \delta_n) \|x_{n+1} - z_{n+1}\|^2 - \psi \delta_n \|x_{n+1} - x_n\|^2 - \frac{1}{\beta} \|y_n - y_{n-1}\|^2. 
\leq \frac{\psi}{\psi-1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2 - \psi \delta_n \|z_{n+1} - x_n\|^2 
-(1 - \sigma) \psi \delta_n \|x_{n+1} - x_n\|^2 - \frac{1 - \sigma}{\beta} \|y_n - y_{n-1}\|^2,$$
(24)

where the second inequality follows from (20) and (21). Finally, by the definitions of  $a_n$  and  $b_n$  in (18) and (19), respectively, and the fact that  $G(x_n, y_n) \ge 0$ , (24) implies  $a_{n+1} \le a_n - b_n$  immediately.

#### 3.2 Convergence results

Now, we are ready to establish global convergence and ergodic sublinear convergence rate of Algorithm 3.1.

**Theorem 3.1 (Global convergence)** Let  $\{(z_n, x_n, y_n) : n \ge 1\}$  be generated by Algorithm 3.1. Then  $\{(x_n, y_n) : n \ge 1\}$  converges to a solution of (1). Proof. Let  $\rho \in (0,1)$  and  $\underline{\tau} > 0$  be defined in Lemma 3.1. By (ii) of Lemma 3.1, there exists an infinite sequence  $\{n_k : k \ge 1\}$  such that  $\tau_{n_k} \ge \underline{\tau}$  and  $\delta_{n_k} \ge \rho$ . By Lemma 3.3, we have

$$\sum_{k=1}^{\infty} \delta_{n_k} \left( \|z_{n_k+1} - x_{n_k}\|^2 + (1 - \sigma) \|x_{n_k+1} - x_{n_k}\|^2 \right) \le \sum_{n=1}^{\infty} b_n < \infty.$$
(25)

Since  $\lim_{k\to\infty} \sum_{k=1}^{\infty} \delta_{n_k} = \infty$ , which together with (25) implies that there exists a subsequence of  $\{n_k : k \ge 1\}$ , still denoted as  $\{n_k : k \ge 1\}$ , such that

$$\lim_{k \to \infty} \|z_{n_k+1} - x_{n_k}\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|x_{n_k+1} - x_{n_k}\| = 0.$$
(26)

In addition, it follows from Lemma 3.3 and Fact 2.2 that  $\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0$ ,  $\lim_{n\to\infty} ||z_n - \bar{x}||$ and  $\lim_{n\to\infty} ||y_n - \bar{y}||$  exist. Thus,  $\{z_n : n \ge 1\}$  and  $\{y_n : n \ge 1\}$  are bounded and, by (9), so is  $\{x_n : n \ge 1\}$ . Therefore, there exist a subsequence of  $\{n_k : k \ge 1\}$ , still denoted as  $\{n_k : k \ge 1\}$ , and  $(x^*, y^*)$  such that  $\lim_{k\to\infty} x_{n_k} = x^*$  and  $\lim_{k\to\infty} y_{n_k} = y^*$ , which together with (26) also implies that  $\lim_{k\to\infty} x_{n_k+1} = \lim_{k\to\infty} z_{n_k+1} = x^*$ . Now, similar to (14) and (16), for any (x, y), there hold

$$\begin{cases} \langle x_{n_k+1} - z_{n_k+1} + \tau_{n_k} K^\top y_{n_k}, \ x - x_{n_k+1} \rangle \ge \tau_{n_k} \big( g(x_{n_k+1}) - g(x) \big), \\ \big\langle \frac{1}{\beta} (y_{n_k} - y_{n_k-1}) - \tau_{n_k} K x_{n_k}, \ y - y_{n_k} \big\rangle \ge \tau_{n_k} \big( f^*(y_{n_k}) - f^*(y) \big). \end{cases}$$
(27)

Then, dividing  $\tau_{n_k} \geq \underline{\tau} > 0$  from both sides of (27), taking into account that both g and  $f^*$  are closed (and thus lower semicontinuous) and letting  $k \to \infty$ , we obtain

$$\langle K^{\top}y^{*}, x - x^{*} \rangle \ge g(x^{*}) - g(x) \text{ and } - \langle Kx^{*}, y - y^{*} \rangle \ge f^{*}(y^{*}) - f^{*}(y).$$
 (28)

Since (28) holds for any  $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$ , we have  $(x^*, y^*) \in \Omega$ . Note that Lemma 3.3 holds for any  $(\bar{x}, \bar{y}) \in \Omega$ . Therefore,  $(\bar{x}, \bar{y})$  can be replaced by  $(x^*, y^*)$  in the definition of  $\{a_n : n \ge 1\}$  in (18). As such, we have  $\lim_{k\to\infty} a_{n_k} = 0$  since  $\lim_{k\to\infty} z_{n_k+1} = x^*$  and  $\lim_{k\to\infty} y_{n_k-1} = \lim_{k\to\infty} y_{n_k} = y^*$ . Since  $\{a_n : n \ge 1\}$  is monotonically nonincreasing, it follows that  $\lim_{n\to\infty} a_n = 0$ . Therefore,  $\lim_{n\to\infty} (z_n, y_n) = (x^*, y^*)$ . Again by (9), we have  $\lim_{n\to\infty} x_n = x^*$ . This completes the proof.  $\Box$ 

We now establish an  $\mathcal{O}(1/N)$  ergodic sublinear convergence rate of Algorithm 3.1.

**Theorem 3.2 (Sublinear convergence)** Let  $\{(z_n, x_n, y_n, \tau_n) : n \ge 1\}$  be the sequence generated by Algorithm 3.1 and  $\underline{c} > 0$  be the constant given in Lemma 3.1. For any  $N \ge 1$ , define

$$X_N = \frac{1}{S_N} \sum_{n=1}^N \tau_n x_n \quad and \quad Y_N = \frac{1}{S_N} \sum_{n=1}^N \tau_n y_n \quad with \quad S_N = \sum_{n=1}^N \tau_n.$$
(29)

Then, it holds that

$$G(X_N, Y_N) \le \frac{1}{2\underline{c}N} \Big( \frac{\psi}{\psi - 1} \| z_2 - \bar{x} \|^2 + \frac{1}{\beta} \| y_0 - \bar{y} \|^2 \Big).$$
(30)

Proof. First, it follows from (24) and the definition of  $b_n$  in (19) that

$$2\tau_n G(x_n, y_n) \le \frac{\psi}{\psi - 1} \left( \|z_{n+1} - \bar{x}\|^2 - \|z_{n+2} - \bar{x}\|^2 \right) + \frac{1}{\beta} \left( \|y_{n-1} - \bar{y}\|^2 - \|y_n - \bar{y}\|^2 \right).$$

By taking summation over  $n = 1, \ldots, N$ , we obtain

$$2\sum_{n=1}^{N} \tau_n G(x_n, y_n) \le \frac{\psi}{\psi - 1} \|z_2 - \bar{x}\|^2 + \frac{1}{\beta} \|y_0 - \bar{y}\|^2.$$
(31)

Since G(x,y) = P(x) + D(y) and  $P(\cdot)$  and  $D(\cdot)$  are convex, it follows that

$$\sum_{n=1}^{N} \tau_n G(x_n, y_n) = \sum_{n=1}^{N} \tau_n P(x_n) + \sum_{n=1}^{N} \tau_n D(y_n) \ge S_N \big( P(X_N) + D(Y_N) \big), \tag{32}$$

where  $S_N$ ,  $X_N$  and  $Y_N$  are defined in (29). Combining (31) and (32), we obtain

$$G(X_N, Y_N) = P(X_N) + D(Y_N) \le \frac{1}{2S_N} \left(\frac{\psi}{\psi - 1} \|z_2 - \bar{x}\|^2 + \frac{1}{\beta} \|y_0 - \bar{y}\|^2\right).$$

By property (iii) in Lemma 3.1, we have  $S_N = \sum_{n=1}^N \tau_n \ge \underline{c}N$ . Hence, (30) holds.

# 4 GRPDA with linesearch — Strongly convex case

In this section, we introduce linesearch strategy into GRPDA (5) when either g or  $f^*$  or both are strongly convex. When g or  $f^*$  is strongly convex, it was shown in [6] that one can adaptively choose the primal and the dual stepsizes, as well as the inertial parameter, so that the PDA (4) achieves a faster  $\mathcal{O}(1/N^2)$  convergence rate. Similar results have been obtained in [20] for the PDA with linesearch. In this section, we propose an adaptive linesearch strategy for GRPDA (5).

Apparently, the minimax problem (1) is equivalent to

$$\max_{x \in \mathbb{R}^q} \min_{y \in \mathbb{R}^p} f^*(y) + \langle -K^\top y, x \rangle - g(x).$$
(33)

Then, by swapping " $\max_{x \in \mathbb{R}^{q}}$ " with " $\min_{y \in \mathbb{R}^{p}}$ " and (g, K, x, q) with  $(f^{*}, -K^{T}, y, p)$ , (33) is reducible to (1). Therefore, we only present the analysis for the case when g is strongly convex in Section 4.1, while the case when  $f^{*}$  is strongly convex can be treated similarly and is thus omitted. Finally, we establish in Section 4.2 nonergodic linear convergence results when both g and  $f^{*}$  are strongly convex.

# 4.1 The case when g is strongly convex

In this subsection, we assume that g is  $\gamma_q$ -strongly convex, i.e., it holds for some  $\gamma_q > 0$  that

$$g(y) \ge g(x) + \langle u, y - x \rangle + \frac{\gamma_g}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{R}^q, \ \forall u \in \partial g(x).$$

where the strongly convex parameter  $\gamma_g > 0$  is known in our algorithm. Then, the following algorithm that incorporates linesearch into GRPDA (5) and exploits the strong convexity of g achieves faster convergence.

# Algorithm 4.1 (Accelerated GRPDA with linesearch when g is $\gamma_g$ -strongly convex)

Step 0. Let  $\psi_0 = 1.3247...$  be the unique real root of  $\psi^3 - \psi - 1 = 0$ . Choose  $\psi \in (\psi_0, \phi)$ ,  $\beta_0 > 0$ ,  $\tau_0 > 0$  and  $\mu \in (0, 1)$ . Choose  $x_0 = z_0 \in \mathbb{R}^q$  and  $y_0 \in \mathbb{R}^p$ . Set  $\varphi = \frac{1+\psi}{\psi^2}$  and n = 1.

Step 1. Compute

$$z_n = \frac{\psi - 1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \tag{34}$$

$$x_n = \operatorname{Prox}_{\tau_{n-1}g}(z_n - \tau_{n-1}K^{\top}y_{n-1}), \tag{35}$$

$$\omega_n = \frac{\psi - \psi}{\psi + \varphi \gamma_g \tau_{n-1}},\tag{36}$$

$$\beta_n = \beta_{n-1} (1 + \gamma_g \omega_n \tau_{n-1}). \tag{37}$$

Step 2. Let  $\tau = \varphi \tau_{n-1}$  and compute

$$y_n = \operatorname{Prox}_{\beta_n \tau_n f^*} (y_{n-1} + \beta_n \tau_n K x_n), \tag{38}$$

where  $\tau_n = \tau \mu^i$  and *i* is the smallest nonnegative integer such that

$$\sqrt{\beta_n \tau_n} \| K^\top y_n - K^\top y_{n-1} \| \le \sqrt{\psi} / \tau_{n-1} \| y_n - y_{n-1} \|.$$
(39)

Step 3. Set  $n \leftarrow n+1$  and return to Step 1.

**Remark 4.1** As in [8, Algorithm 4.1],  $\omega_n$  is used to update  $\beta_n$ , which plays a key role in establishing the  $\mathcal{O}(1/N^2)$  ergodic convergence rate of Algorithm 4.1. The condition  $\psi > \psi_0$ , where  $\psi_0$  is the unique real root of  $\psi^3 - \psi - 1 = 0$ , ensures that  $\psi > (1 + \psi)/\psi^2 = \varphi$ . Therefore, we have  $\omega_n > 0$  and  $\beta_n > \beta_{n-1} > 0$  for all  $n \ge 1$ .

Next, we state a key lemma with respect to Algorithm 4.1, whose detailed proof is left to Appendix B since it takes similar spirit with that of Lemma 3.1.

**Lemma 4.1** Let  $\{(\tau_n, \beta_n) : n \ge 0\}$  be the sequence generated by Algorithm 4.1. Then, we have the following properties. (i) The linesearch step of Algorithm 4.1, i.e., Step 2, always terminates. (ii) There exists constant c > 0 such that  $\beta_n \ge cn^2$  for all  $n \ge 1$ . (iii) There exists a constant  $\tilde{c} > 0$  such that for any  $N \ge 1$ , we have  $\sum_{n=1}^{N} \tau_n \le \tilde{c} \sum_{n \in S_N} \tau_n$ , where  $S_N := \{1 \le n \le N : \sqrt{\beta_n} \tau_n \ge 1/L\}$ .

We next establish the promised  $\mathcal{O}(1/N^2)$  ergodic convergence rate. Note that in this case  $\bar{x}$  is unique and will be simply denoted as  $x^*$ .

**Theorem 4.1 (Accelerated sublinear convergence)** Let  $\{(z_n, x_n, y_n, \beta_n, \tau_n) : n \ge 1\}$  be the sequence generated by Algorithm 4.1. Then, the following holds: (a) there exist constants  $C_1, C_2 > 0$  such that  $||z_{n+1} - x^*|| \le C_1/n$  and  $||x_{n+1} - x^*|| \le C_2/n$  for all  $n \ge 1$ ;

(b) the sequence  $\{y_n : n \ge 1\}$  is bounded and there exists a subsequence of  $\{y_n : n \ge 1\}$  converging to  $y^*$  such that  $(x^*, y^*)$  is a solution of (1);

(c) there exist a constant  $C_3 > 0$  such that  $G(X_N, Y_N) \leq C_3/N^2$  for any integer  $N \geq 1$ , where

$$S_N = \sum_{n=1}^N \beta_n \tau_n, \quad X_N = \frac{1}{S_N} \sum_{n=1}^N \beta_n \tau_n x_n \quad and \quad Y_N = \frac{1}{S_N} \sum_{n=1}^N \beta_n \tau_n y_n.$$
(40)

Proof. Since g is strongly convex, it follows from (35) and Fact 2.1 that

$$\langle x_n - z_n + \tau_{n-1} K^\top y_{n-1}, x - x_n \rangle \ge \tau_{n-1} \Big( g(x_n) - g(x) + \frac{\gamma_g}{2} \|x_n - x\|^2 \Big), \forall x.$$
 (41)

By passing n + 1 to n and  $\bar{x}$  to x in (41), we obtain

$$\langle x_{n+1} - z_{n+1} + \tau_n K^\top y_n, \ \bar{x} - x_{n+1} \rangle \ge \tau_n \Big( g(x_{n+1}) - g(\bar{x}) + \frac{\gamma_g}{2} \|x_{n+1} - \bar{x}\|^2 \Big).$$
 (42)

Similarly, by passing  $x_{n+1}$  to x in (41) and multiplying both sides by  $\delta_n = \tau_n / \tau_{n-1}$ , we obtain

$$\langle \delta_n(x_n - z_n) + \tau_n K^\top y_{n-1}, \ x_{n+1} - x_n \rangle \ge \tau_n \Big( g(x_n) - g(x_{n+1}) + \frac{\gamma_g}{2} \|x_{n+1} - x_n\|^2 \Big).$$
(43)

Similar to (16), it follows from (38) that

$$\left\langle \frac{1}{\beta_n} (y_n - y_{n-1}) - \tau_n K x_n, \ \bar{y} - y_n \right\rangle \ge \tau_n \left( f^*(y_n) - f^*(\bar{y}) \right).$$
 (44)

From (34), it is easy to derive  $x_n - z_n = \psi(x_n - z_{n+1})$ . Then, by adding (42)-(44) and using similar arguments as in Lemma 3.2, we obtain

$$\tau_n G(x_n, y_n) \le \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle + \frac{1}{\beta_n} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \psi \delta_n \langle x_n - z_{n+1}, x_{n+1} - x_n \rangle + \tau_n \langle K^\top y_n - K^\top y_{n-1}, x_n - x_{n+1} \rangle - \frac{\gamma_g \tau_n}{2} \| x_{n+1} - \bar{x} \|^2 - \frac{\gamma_g \tau_n}{2} \| x_{n+1} - x_n \|^2.$$
(45)

By removing  $-\frac{\gamma_g \tau_n}{2} \|x_{n+1} - x_n\|^2 \le 0$ , using (7) and Cauchy-Schwarz inequality, we obtain from (45) that

$$(1 + \gamma_{g}\tau_{n})\|x_{n+1} - \bar{x}\|^{2} + \frac{1}{\beta_{n}}\|y_{n} - \bar{y}\|^{2} + 2\tau_{n}G(x_{n}, y_{n})$$

$$\leq \|z_{n+1} - \bar{x}\|^{2} + \frac{1}{\beta_{n}}\|y_{n-1} - \bar{y}\|^{2} - \psi\delta_{n}\|z_{n+1} - x_{n}\|^{2} + (\psi\delta_{n} - 1)\|x_{n+1} - z_{n+1}\|^{2}$$

$$-\psi\delta_{n}\|x_{n+1} - x_{n}\|^{2} - \frac{1}{\beta_{n}}\|y_{n} - y_{n-1}\|^{2} + 2\tau_{n}\|K^{\top}(y_{n} - y_{n-1})\|\|x_{n+1} - x_{n}\|.$$
(46)

Plugging in (23) and using Cauchy-Schwarz inequality, we deduce

$$(1 + \gamma_{g}\tau_{n})\frac{\psi}{\psi - 1}\|z_{n+2} - \bar{x}\|^{2} + \frac{1}{\beta_{n}}\|y_{n} - \bar{y}\|^{2} + 2\tau_{n}G(x_{n}, y_{n})$$

$$\leq \frac{\psi + \gamma_{g}\tau_{n}}{\psi - 1}\|z_{n+1} - \bar{x}\|^{2} + \frac{1}{\beta_{n}}\|y_{n-1} - \bar{y}\|^{2} + \left(\psi\delta_{n} - 1 - \frac{1 + \gamma_{g}\tau_{n}}{\psi}\right)\|x_{n+1} - z_{n+1}\|^{2}$$

$$-\psi\delta_{n}\|z_{n+1} - x_{n}\|^{2} - \psi\delta_{n}\|x_{n+1} - x_{n}\|^{2} - \frac{1}{\beta_{n}}\|y_{n} - y_{n-1}\|^{2}$$

$$+2\tau_{n}\|K^{\top}y_{n} - K^{\top}y_{n-1}\|\|x_{n+1} - x_{n}\|.$$
(47)

Recall that  $\delta_n = \tau_n / \tau_{n-1}$ . It follows from (39) and Cauchy-Schwarz inequality that

$$2\tau_n \|K^{\top} y_n - K^{\top} y_{n-1}\| \|x_{n+1} - x_n\| \le \psi \delta_n \|x_{n+1} - x_n\|^2 + \frac{1}{\beta_n} \|y_n - y_{n-1}\|^2.$$
(48)

Furthermore,  $\psi \delta_n - 1 - \frac{1 + \gamma_g \tau_n}{\psi} \le \psi \varphi - 1 - \frac{1}{\psi} - \frac{\gamma_g \tau_n}{\psi} = -\frac{\gamma_g \tau_n}{\psi}$  since  $\delta_n \le \varphi$ . Therefore, (47) implies

$$(1+\gamma_{g}\tau_{n})\frac{\psi}{\psi-1}\|z_{n+2}-\bar{x}\|^{2}+\frac{1}{\beta_{n}}\|y_{n}-\bar{y}\|^{2}+2\tau_{n}G(x_{n},y_{n})$$

$$\leq \frac{\psi+\gamma_{g}\tau_{n}}{\psi-1}\|z_{n+1}-\bar{x}\|^{2}+\frac{1}{\beta_{n}}\|y_{n-1}-\bar{y}\|^{2}-\frac{\gamma_{g}\tau_{n}}{\psi}\|x_{n+1}-z_{n+1}\|^{2}.$$
(49)

Note that  $(1 + \gamma_g \tau_n) \frac{\psi}{\psi - 1} = \frac{\psi(1 + \gamma_g \tau_n)}{\psi + \gamma_g \tau_{n+1}} \frac{\psi + \gamma_g \tau_{n+1}}{\psi - 1}$ . It follows from  $\tau_{n+1} \leq \varphi \tau_n$  that

$$\frac{\psi(1+\gamma_g\tau_n)}{\psi+\gamma_g\tau_{n+1}} \ge \frac{\psi(1+\gamma_g\tau_n)}{\psi+\gamma_g\varphi\tau_n} = 1 + \frac{\psi-\varphi}{\psi+\gamma_g\varphi\tau_n}\gamma_g\tau_n = 1 + \omega_{n+1}\gamma_g\tau_n,$$
(50)

and thus

$$(1 + \gamma_g \tau_n) \frac{\psi}{\psi - 1} \ge (1 + \omega_{n+1} \gamma_g \tau_n) \frac{\psi + \gamma_g \tau_{n+1}}{\psi - 1} = \frac{\beta_{n+1}}{\beta_n} \frac{\psi + \gamma_g \tau_{n+1}}{\psi - 1}.$$
 (51)

Define  $A_n := \frac{\psi + \gamma_g \tau_n}{2(\psi - 1)} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{2\beta_n} \|y_{n-1} - \bar{y}\|^2$ . Combining (49) and (51), we deduce

$$\beta_{n+1}A_{n+1} + \beta_n \tau_n G(x_n, y_n) \le \beta_n A_n - \frac{\beta_n \gamma_g \tau_n}{2\psi} \|x_{n+1} - z_{n+1}\|^2.$$
(52)

By summing (52) over  $n = 1, \ldots, N$ , we obtain

$$\beta_{N+1}A_{N+1} + \sum_{n=1}^{N} \beta_n \tau_n G(x_n, y_n) + \sum_{n=1}^{N} \frac{\beta_n \gamma_g \tau_n}{2\psi} \|x_{n+1} - z_{n+1}\|^2 \le \beta_1 A_1.$$
(53)

Then, the convexity of  $G(\cdot, \cdot)$ ,  $G(x_n, y_n) \ge 0$ , the definition of  $A_n$ , (40) and (53) imply that

$$G(X_N, Y_N) \le \frac{1}{S_N} \sum_{n=1}^N \beta_n \tau_n G(x_n, y_n) \le \frac{\beta_1 A_1}{S_N},$$
(54)

$$\|z_{N+2} - \bar{x}\|^2 \le \frac{2(\psi - 1)}{\psi + \gamma_g \tau_{N+1}} \frac{\beta_1 A_1}{\beta_{N+1}} \le \frac{2\beta_1 A_1}{\beta_{N+1}}.$$
(55)

From Lemma 4.1, there exists c > 0 such that  $\beta_n \ge cn^2$  for all  $n \ge 1$ . Taking  $x^* = \bar{x}$ , then (55) implies  $||z_{N+1} - x^*|| \le C_1/N$  with  $C_1 := \sqrt{2\beta_1 A_1/c} > 0$ . Thus, (34) implies  $||x_{N+1} - x^*|| \le C_2/N$  for some  $C_2 > 0$ . Hence, property (a) holds and the convergence of  $\{z_n : n \ge 1\}$  and  $\{x_n : n \ge 1\}$  to  $x^*$  follows immediately.

Let  $S = \{n \in \mathbb{Z}^+ : \sqrt{\beta_n}\tau_n \ge 1/L\}$ . Properties (ii) and (iii) in Lemma 4.1 imply that  $|S| = \infty$ . We next show by contradiction that there exists a subsequence  $\{n_k : k \ge 1\} \subseteq S$  such that

$$\lim_{k \to \infty} \|x_{n_k+1} - z_{n_k+1}\| / \tau_{n_k} = 0.$$
(56)

Let  $N \ge 1$  be arbitrarily fixed and  $\theta = (\psi - \varphi)\gamma_g/\psi > 0$ . By Lemma 4.1 (ii) and (37), we have

$$c(N+1)^2 \le \beta_{N+1} = \beta_N \left( 1 + \frac{(\psi - \varphi)\gamma_g \tau_N}{\psi + \varphi \gamma_g \tau_N} \right) \le \beta_N \left( 1 + \theta \tau_N \right) \le \beta_1 \prod_{n=1}^N \left( 1 + \theta \tau_n \right).$$

Then, by Lemma 4.1 (iii) we deduce

$$2\ln(N+1) - \ln(\beta_1/c) \le \sum_{n=1}^N \ln(1+\theta\tau_n) \le \theta \sum_{n=1}^N \tau_n \le \theta \tilde{c} \sum_{n \in \mathcal{S}_N} \tau_n.$$
(57)

On the other hand, it follows from (53) that

$$\frac{2\psi\beta_1 A_1}{\gamma_g} \ge \sum_{n \in \mathcal{S}} \beta_n \tau_n^3 \left( \frac{\|x_{n+1} - z_{n+1}\|}{\tau_n} \right)^2 \ge \frac{1}{L^2} \sum_{n \in \mathcal{S}} \tau_n \left( \frac{\|x_{n+1} - z_{n+1}\|}{\tau_n} \right)^2,$$

from which the existence of a subsequence  $\{n_k : k \ge 1\} \subseteq S$  such that (56) holds is guaranteed since (57) indicates that  $\sum_{n \in S} \tau_n = \lim_{N \to \infty} \sum_{n \in S_N} \tau_n = \infty$ . Now, it follows from (53) that  $\{\beta_n A_n : n \ge 1\}$  is bounded, which implies that  $\{y_n : n \ge 1\}$  is

Now, it follows from (53) that  $\{\beta_n A_n : n \ge 1\}$  is bounded, which implies that  $\{y_n : n \ge 1\}$  is bounded. Let  $\{n_k : k \ge 1\} \subseteq S$  be the subsequence satisfying (56), which then has a subsequence, still denoted as  $\{n_k : k \ge 1\} \subseteq S$ , such that  $\lim_{k\to\infty} y_{n_k} = y^*$ . Similar to (27), for any  $(x, y) \in \mathbb{R}^q \times \mathbb{R}^p$ , we have

$$\begin{cases} \langle x_{n_k+1} - z_{n_k+1} + \tau_{n_k} K^\top y_{n_k}, \ x - x_{n_k+1} \rangle \ge \tau_{n_k} \big( g(x_{n_k+1}) - g(x) \big), \\ \big\langle \frac{1}{\beta_{n_k}} (y_{n_k} - y_{n_k-1}) - \tau_{n_k} K x_{n_k}, \ y - y_{n_k} \big\rangle \ge \tau_{n_k} \big( f^*(y_{n_k}) - f^*(y) \big). \end{cases}$$
(58)

Since  $n_k \in S$ , we have  $\sqrt{\beta_{n_k}}\tau_{n_k} \ge 1/L$ , which together with  $\beta_n \ge cn^2$  for some c > 0 implies  $\lim_{k\to\infty} \beta_{n_k}\tau_{n_k} = \infty$ . Hence,  $\lim_{k\to\infty} ||y_{n_k} - y_{n_k-1}||/(\beta_{n_k}\tau_{n_k}) = 0$  since  $\{y_n : n \ge 1\}$  is bounded. Then, dividing  $\tau_{n_k}$  from both sides of the two inequalities in (58), taking  $k \to \infty$ , it follows from  $\lim_{n\to\infty} x_n = x^*$ ,  $\lim_{k\to\infty} y_{n_k} = y^*$ , lower semicontinuous of g and  $f^*$ , and (56) that (28) holds, which implies  $(x^*, y^*)$  is a saddle point of (1). Hence, property (b) holds.

Finally, it follows from (37) and  $\omega_{n+1} < 1$  that

$$\beta_n \tau_n = \frac{\beta_{n+1} - \beta_n}{\omega_{n+1} \gamma_g} \ge \frac{\beta_{n+1} - \beta_n}{\gamma_g}.$$

Hence, we have from  $\beta_n \geq cn^2$  that  $S_N = \sum_{n=1}^N \beta_n \tau_n \geq \frac{\beta_{N+1}-\beta_1}{\gamma_g} \geq c_1 N^2$  for some  $c_1 > 0$ . Consequently,  $G(X_N, Y_N) \leq C_3/N^2$  with  $C_3 = \beta_1 A_1/c_1$  follows from (54). Hence, property (c) holds. We complete the proof.

4.2 Linear convergence when g and  $f^*$  are strongly convex

In this section, we assume that both g and  $f^*$  are strongly convex, with modulus  $\gamma_g > 0$  and  $\gamma_f > 0$ , respectively, and establish nonergodic linear convergence rate of GRPDA with linesearch. Note that when only one of the component function is strongly convex, say g, the strong convexity parameter  $\gamma_g$  is used in the algorithm, see (37). However, when both g and  $f^*$  are strongly convex, the strong convexity modulus  $\gamma_g$  and  $\gamma_f$  do not need to be known in advance since they are only used in the analysis and play no role in the algorithm itself. We now summarize the modified algorithm as follows.

## Algorithm 4.2 (GRPDA with linesearch when g and $f^*$ are strongly convex)

Step 0. Let  $\psi_0 = 1.3247...$  be the unique real root of  $\psi^3 - \psi - 1 = 0$ . Choose  $\psi \in (\psi_0, \phi), \beta > 0$ ,  $\tau_0 > 0$  and  $\mu \in (0, 1)$ . Choose  $x_0 = z_0 \in \mathbb{R}^q$  and  $y_0 \in \mathbb{R}^p$ . Set  $\varphi = \frac{1+\psi}{\psi^2}$  and n = 1. Step 1. Compute

$$z_n = \frac{\psi - 1}{\psi} x_{n-1} + \frac{1}{\psi} z_{n-1}, \tag{59}$$

$$x_n = \operatorname{Prox}_{\tau_{n-1}g}(z_n - \tau_{n-1}K^{\top}y_{n-1}).$$
(60)

Step 2. Let  $\tau = \varphi \tau_{n-1}$  and compute

$$y_n = \operatorname{Prox}_{\beta\tau_n f^*}(y_{n-1} + \beta\tau_n K x_n), \tag{61}$$

where  $\tau_n = \tau \mu^i$  and *i* is the smallest nonnegative integer such that

$$\sqrt{\beta \tau_n} \| K^\top y_n - K^\top y_{n-1} \| \le \sqrt{\psi/\tau_{n-1}} \| y_n - y_{n-1} \|.$$
(62)

Step 3. Set  $n \leftarrow n+1$  and return to Step 1.

With respect to Algorithm 4.2, we have the following key lemma.

**Lemma 4.2** Let  $\{\tau_n : n \ge 0\}$  be the sequence generated by Algorithm 4.2. Then, we have the following properties. (i) The linesearch step of Algorithm 4.1, i.e., Step 2, always terminates. (ii) There exists a constant  $\theta > 1$  such that for any  $N \ge 1$ , we have  $\Gamma_{N+1} := \prod_{n=2}^{N+1} \theta_n \ge \theta^N$  with

$$\theta_n = \min\left\{1 + \omega_n \gamma_g \tau_{n-1}, 1 + \beta \gamma_f \tau_{n-1}\right\}, \quad n \ge 2, \tag{63}$$

where  $\omega_n = \frac{\psi - \varphi}{\psi + \gamma_g \varphi \tau_{n-1}}$  is defined in (36)

Proof. The conclusion (i) follows from the same proof of conclusion (i) in Lemma 3.1 with setting  $\sigma = 1$ . We next prove conclusion (ii). Given any integer  $N \ge 1$ , it follows from the same proof of conclusion (iii) in Lemma 3.1 with setting  $\sigma = 1$  that  $|\mathcal{K}_N| \ge \hat{c}N$  for some constant  $\hat{c} > 0$ , where

$$\mathcal{K}_N = \{ 1 \le n \le N : \tau_n \ge \hat{\tau} \} \quad \text{with} \quad \hat{\tau} = \frac{1}{L} \frac{\sqrt{\psi}}{\sqrt{\beta\varphi}} > 0 \tag{64}$$

and  $|\mathcal{K}_N|$  is the cardinality of the set  $\mathcal{K}_N$ . Then, by the definition of  $\Gamma_N$  and  $\theta_n$ , we have

$$\begin{split} \Gamma_{N+1} &= \prod_{n=2}^{N+1} \theta_n = \prod_{n=2}^{N+1} \min\left\{ 1 + \frac{(\psi - \varphi)\gamma_g \tau_{n-1}}{\psi + \gamma_g \varphi \tau_{n-1}}, 1 + \beta \gamma_f \tau_{n-1} \right\} \\ &\geq \prod_{n \in \mathcal{K}_N} \min\left\{ 1 + \frac{(\psi - \varphi)\gamma_g \tau_n}{\psi + \gamma_g \varphi \tau_n}, 1 + \beta \gamma_f \tau_n \right\} \\ &\geq \tilde{\theta}^{|\mathcal{K}_N|} \geq \tilde{\theta}^{\hat{c}N} = \theta^N, \end{split}$$

where  $\theta = \tilde{\theta}^{\hat{c}} > 1$  and  $\tilde{\theta} = \min\left\{1 + \frac{(\psi - \varphi)\gamma_g \hat{\tau}}{\psi + \gamma_g \varphi \hat{\tau}}, 1 + \beta \gamma_f \hat{\tau}\right\} > 1.$ 

Based on Lemma 4.2, we have the following iterative and nonergodic linear convergence results.

**Theorem 4.2 (Linear convergence)** Let  $\{(z_n, x_n, y_n) : n \ge 1\}$  be the sequence generated by Algorithm 4.2. Then, the following holds:

(a)  $\{(z_n, y_n) : n \ge 1\}$  converge to the unique primal and dual optimal solutions  $(\bar{x}, \bar{y});$ (b) there exit constants  $C_1, C_2 > 0$  and  $\theta > 1$  such that for all  $n \ge 1$  there hold  $||z_{n+2} - \bar{x}|| \le C_1/\theta^n$ and  $||y_n - \bar{y}|| \le C_2/\theta^n;$ 

(c) there exist constants  $C_3 > 0$  and  $\hat{\theta} > 1$  such that  $\min_{1 \le n \le N} G(x_n, y_n) \le C_3/\hat{\theta}^N$ .

Proof. Following the line of proof of Theorem 4.1, similar to (45), one can easily show that

$$\tau_n G(x_n, y_n) \le \langle x_{n+1} - z_{n+1}, \bar{x} - x_{n+1} \rangle + \frac{1}{\beta} \langle y_n - y_{n-1}, \bar{y} - y_n \rangle + \psi \delta_n \langle x_n - z_{n+1}, x_{n+1} - x_n \rangle \\ + \tau_n \langle K^\top y_n - K^\top y_{n-1}, x_n - x_{n+1} \rangle - \frac{\gamma_g \tau_n}{2} \| x_{n+1} - \bar{x} \|^2 - \frac{\gamma_f \tau_n}{2} \| y_n - \bar{y} \|^2,$$

where  $\delta_n = \tau_n / \tau_{n-1}$ . Then, by the same arguments as in (46)-(49), we have

$$(1 + \gamma_g \tau_n) \frac{\psi}{\psi - 1} \|z_{n+2} - \bar{x}\|^2 + (1/\beta + \gamma_f \tau_n) \|y_n - \bar{y}\|^2 + 2\tau_n G(x_n, y_n)$$
  
$$\leq \frac{\psi + \gamma_g \tau_n}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2.$$
(65)

From (50) and the definition of  $\theta_{n+1}$ , it holds that

$$\frac{\psi(1+\gamma_g\tau_n)}{\psi-1} = \frac{\psi(1+\gamma_g\tau_n)}{\psi+\gamma_g\tau_{n+1}}\frac{\psi+\gamma_g\tau_{n+1}}{\psi-1} \ge \left(1+\omega_{n+1}\gamma_g\tau_n\right)\frac{\psi+\gamma_g\tau_{n+1}}{\psi-1} \ge \theta_{n+1}\frac{\psi+\gamma_g\tau_{n+1}}{\psi-1}, \quad (66)$$

where  $\omega_n$  and  $\theta_n$  are defined in (36) and (63), respectively. Let

$$A_n := \frac{\psi + \gamma_g \tau_n}{\psi - 1} \|z_{n+1} - \bar{x}\|^2 + \frac{1}{\beta} \|y_{n-1} - \bar{y}\|^2.$$
(67)

Then, we have from (65), (66) and  $G(x_n, y_n) \ge 0$  that

$$\theta_{n+1}A_{n+1} \le \theta_{n+1}A_{n+1} + 2\tau_n G(x_n, y_n) \le A_n,$$
(68)

which implies  $(\prod_{n=2}^{N+1} \theta_n) A_{N+1} \leq A_1$ . By (ii) of Lemma 4.2, we have  $\Gamma_{N+1} = \prod_{n=2}^{N+1} \theta_n \geq \theta^N$  for some  $\theta > 1$ . Hence, we have  $A_{N+1} \leq A_1/\theta^N$ , which implies conclusions (a) and (b). Finally, let

 $\mathcal{K}_N = \{1 \le n \le N : \tau_n \ge \hat{\tau} > 0\}$  be defined in (64) and  $\hat{N} = \max\{n : n \in \mathcal{K}_N\}$ . Then, we have  $\hat{N} \ge |\mathcal{K}_N| \ge \hat{c}N$  for some  $\hat{c} > 0$ . Thus, it follows from (b) and (68) that

$$\min_{1 \le n \le N} G(x_n, y_n) \le G(x_{\hat{N}}, y_{\hat{N}}) \le \frac{A_{\hat{N}}}{2\hat{\tau}} \le \frac{A_1}{2\hat{\tau}\theta^{\hat{N}-1}} \le \frac{A_1}{2\hat{\tau}\theta^{\hat{c}N-1}} := \frac{C_3}{\hat{\theta}^N},$$

where  $C_3 = \theta A_1/(2\hat{\tau})$  and  $\hat{\theta} = \theta^{\hat{c}} > 1$ . So, (c) holds. We complete the proof.

#### **5** Numerical results

In this section, numerical results are presented to demonstrate the performance of the proposed Algorithm 3.1 (GRPDA-L) and Algorithm 4.1 (AGRPDA-L). We set  $\sigma = 0.99$  for GRPDA-L and  $\psi = 1.5$  and  $\mu = 0.7$  for both algorithms. First, choose  $y_{-1}$  arbitrarily in a small neighborhood of the starting point  $y_0$  such that  $K^{\top}y_{-1} \neq K^{\top}y_0$  and compute

$$m = \frac{\|y_{-1} - y_0\|}{\|K^\top y_{-1} - K^\top y_0\|} \ge \frac{1}{L}.$$
(69)

Then, set  $\tau_0 = \frac{\sqrt{\psi}}{\sqrt{\beta}}m \geq \frac{\sqrt{\psi}}{\sqrt{\beta L}}$  for GRPDA-L and  $\tau_0 = \frac{\sqrt{\psi}}{\sqrt{\beta_0}}m \geq \frac{\sqrt{\psi}}{\sqrt{\beta_0 L}}$  with some  $\beta_0 > 0$  for AGRPDA-L. We compare the proposed algorithms with their corresponding counterparts without linesearch, i.e., GRPDA [8, Algorithm 3.1] with  $\tau = \frac{\sqrt{\psi}}{\sqrt{\beta L}}$  and  $\psi = 1.618$ , as well as the state-of-the-art primal-dual algorithm with linesearch PDA-L, i.e., [20, Algorithm 1], with  $\mu = 0.7$ ,  $\delta = 0.99$  and  $\tau_0 = \|y_{-1} - y_0\|/(\sqrt{\beta}\|K^{\top}y_{-1} - K^{\top}y_0\|)$ . Other parameters will be specified in the following.

All experiments were performed within Python 3.8 on an Intel(R) Core(TM) i5-4590 CPU 3.30GHz PC with 8GB of RAM running on 64-bit Windows operating system. For reproducible purpose, the codes are provided at https://github.com/cxk9369010/GRPDA-Linesearch. We solve the minimax matrix game problem and the LASSO problem for comparison.

**Problem 5.1 (Minimax matrix game)** The minimax matrix game problem is given by

$$\min_{x \in \Delta_q} \max_{y \in \Delta_p} \langle Kx, y \rangle, \tag{70}$$

where  $K \in \mathbb{R}^{p \times q}$ ,  $\Delta_q = \{x \in \mathbb{R}^q : \sum_i x_i = 1, x \ge 0\}$  and  $\Delta_p = \{y \in \mathbb{R}^p : \sum_i y_i = 1, y \ge 0\}$  denote the standard unit simplex in  $\mathbb{R}^q$  and  $\mathbb{R}^p$ , respectively.

Clearly, (70) is a special case of (1) with  $g = \iota_{\Delta_q}$  and  $f^* = \iota_{\Delta_p}$ , where  $\iota_C$  denotes the indicator function of a set C. Since neither g nor  $f^*$  is strongly convex, only the non-accelerated algorithms GRPDA, GRPDA-L and PDA-L are relevant here. For  $(x, y) \in \Delta_q \times \Delta_p$ , the primal-dual gap function is given by  $G(x, y) := \max_i (Kx)_i - \min_j (K^\top y)_j$ . Initial points for all the algorithms are set to be  $x_0 = \frac{1}{q}(1, \ldots, 1)^\top \in \mathbb{R}^q$  and  $y_0 = \frac{1}{p}(1, \ldots, 1)^\top \in \mathbb{R}^p$ . The projection onto the unit simplex is computed by the algorithm from [9]. We set  $\psi = 1.618$  and  $\tau = \sigma = 1/||K||$  for GRPDA and  $\beta = 1$  for PDA-L and GRPDA-L. As in [20], we generated  $K \in \mathbb{R}^{p \times q}$  randomly in the following four different ways with random number generator seed = 50:

- (i) All entries of K were generated independently from the uniform distribution in [-1, 1], and (p, q) = (100, 100);
- (ii) All entries of K were generated independently from the normal distribution  $\mathcal{N}(0, 1)$ , and (p, q) = (100, 100);
- (iii) All entries of K were generated independently from the normal distribution  $\mathcal{N}(0, 10)$ , and (p, q) = (500, 100);
- (iv) The matrix K is sparse with 10% nonzero elements generated independently from the uniform distribution in [0, 1], and (p, q) = (1000, 2000).

For a given  $\epsilon > 0$ , we terminate the algorithms when  $G(x_n, y_n) < \epsilon$  or  $n = n_{\max}$ , where  $n_{\max}$  is the maximum number of iterations allowed. In this section, we set  $n_{\max} = 3 \times 10^5$  and examine how the values of the primal-dual gap function G(x, y) decrease as CPU time proceeds. Table 1 presents the total CPU time (Time, in seconds), the number of iterations (Iter) and the number of extra linesearch trial steps (#LS) of PDA-L and GRPDA-L as compared with their counterparts without linesearch. We emphasize that for each trial of linesearch a projection onto the unit simplex is required for this example. The decreasing behavior of the primal-dual gap function values (abbreviated as PD gap  $G(x_n, y_n)$ ) versus CPU time is shown in Figure 1 for the compared algorithms with  $\epsilon = 10^{-10}$ .

**Table 1** Results of GRPDA, PDA-L and GRPDA-L on problem 5.1. In the table, "—" represents that the algorithm reached the maximum number of iterations without satisfying the stopping condition.

ε	Test	GRPDA			PDA-L		GRPDA-L			
		Iter	Time	Iter	#LS	Time	Iter	#LS	Time	
10 <sup>-7</sup>	(i)	25688	3.7	18612	18376	4.5	11010	3250	2.1	
	(ii)	103788	14.9	40676	40209	10.2	32656	9646	7.0	
	(iii)		79.7	73197	72903	32.4	64628	19088	23.4	
	(iv)		486.5	45705	44633	118.7	30356	8961	65.5	
10 <sup>-10</sup>	(i)	151134	20.2	58282	57561	16.0	45645	13481	9.8	
	(ii)	245612	33.5	89644	88622	24.4	75467	22292	15.9	
	(iii)	_	79.9	155281	154655	69.3	145527	42985	51.2	
	(iv)		486.5	—	292982	699.6	—	88613	617.7	

It can be seen from Table 1 that PDA-L requires approximately one extra linesearch trial step per outer iteration, while GRPDA-L requires roughly one extra linesearch trial step per three outer iterations. As a result, GRPDA-L consumed less CPU time than PDA-L. From the results in Figure 1, for all the four tests GRPDA-L performs the best, followed by PDA-L, both are faster than GRPDA. It can be seen from Figure 1 that test (iv) is a difficult case, which makes all the compared algorithms fail to reduce the primal-dual gap function value to less than  $10^{-10}$  within the prescribed maximum number of iterations.

**Problem 5.2 (LASSO)** Let  $K \in \mathbb{R}^{p \times q}$  be a sensing matrix and  $b \in \mathbb{R}^p$  be an observation vector. One form of the LASSO problem is to recover a sparse signal via solving

$$\min_{x} F(x) := \mu \|x\|_{1} + \frac{1}{2} \|Kx - b\|^{2},$$
(71)

where  $\mu > 0$  is a regularization parameter.

It is easy to verify that the LASSO problem (71) can be represented as the saddle point problem (1) with  $g(x) = \mu ||x||_1$  and  $f^*(y) = \frac{1}{2} ||y||^2 + \langle b, y \rangle$ . Thus, the proximal operator  $\operatorname{Prox}_{\tau f^*}(\cdot)$  is linear. In fact, it holds that

$$y_n = \operatorname{Prox}_{\beta_n \tau_n f^*} (y_{n-1} + \beta_n \tau_n K x_n) = \frac{y_{n-1} + \beta_n \tau_n (K x_n - b)}{1 + \beta_n \tau_n}$$

Therefore, there is no extra matrix-vector multiplications introduced within a linesearch step for GRPDA-L as  $K^{\top}y_n$  can always be obtained via a convex combination of the already computed quantities  $K^{\top}y_{n-1}$  and  $K^{\top}(Kx_n - b)$ . On the other hand, problem (1) is equivalent to

$$\max_{x \in \mathbb{R}^q} \min_{y \in \mathbb{R}^p} \quad f^*(y) + \langle -K^\top y, x \rangle - g(x).$$

Then, by swapping "max<sub> $x \in \mathbb{R}^q$ </sub>" with "min<sub> $y \in \mathbb{R}^p$ </sub>" and (g, K, x, q) with  $(f^*, -K^T, y, p)$ , the strong convexity of  $\frac{1}{2} ||y||^2 + \langle b, y \rangle$  (previously  $f^*$ ) can be transferred to g, which enables the application of



Fig. 1 Comparison results of GRPDA, GRPDA-L and PDA-L on Problem 5.1: PD gap versus CPU time.

the accelerated version, i.e., AGRPDA-L. Therefore, the algorithms to compare in this experiment are GRPDA-L, AGRPDA-L and PDA-L (i.e., [20, Algorithm 1]). GRPDA without linesearch will not be compared since it is the most inefficient.

We set seed = 100 and generate a random vector  $x^* \in \mathbb{R}^q$  for which s random coordinates are drawn from  $\mathcal{N}(0,1)$  and the rest are set to be zero. Then, we generate  $\omega \in \mathbb{R}^p$  with entries drawn from  $\mathcal{N}(0,0.1)$  and set  $b = Kx^* + \omega$ . The matrix  $K \in \mathbb{R}^{p \times q}$  is constructed in the following ways:

- (i) All entries of K are generated independently from  $\mathcal{N}(0,1)$ . The s entries of  $x^*$  are drawn from the uniform distribution in [-10, 10];
- (ii) First, we generate a matrix  $A \in \mathbb{R}^{p \times q}$ , whose entries are independently drawn from  $\mathcal{N}(0, 1)$ . Then, for a scalar  $v \in (0, 1)$  we construct the matrix K column by column as follows:  $K_1 = A_1/\sqrt{1-v^2}$  and  $K_j = vK_{j-1} + A_j$ ,  $j = 2, \ldots, q$ . Here  $K_j$  and  $A_j$  represent the *j*th column of K and A, respectively. As  $v \in (0, 1)$  becomes larger, K becomes more ill-conditioned. In this experiment we take v = 0.5 and v = 0.9, respectively. The sparse vector  $x^*$  is generated in the same way as in case (i).

In both cases, the regularization parameter  $\mu$  was set to be 0.1. Similar to [20], we set  $\beta = 400$  for PDA-L and GRPDA-L. For AGRPDA-L, we set  $\gamma = 0.01$  and  $\beta_0 = 1$  as in [8]. The initial points for all algorithms are  $x_0 = (0, \ldots, 0)^{\top}$  and  $y_0 = Kx_0 - b$ .

In this experiment, we first ran all the algorithms by a sufficiently large number of iterations and then chose the minimum attainable function value as an approximation of the optimal value  $F^*$ of (71). Again, for a given  $\epsilon > 0$ , we terminate the algorithms when  $F(x_n) - F^* < \epsilon$  or  $n = n_{\text{max}}$ . In this experiment, we set  $\epsilon = 10^{-12}$  and  $n_{\text{max}} = 8 \times 10^4$  to examine their convergence behavior.

It can be seen from the results in Table 2 that similar conclusion can be drawn, i.e., PDA-L requires approximately one extra linesearch trial step per outer iteration, while GRPDA-L and

$\epsilon$	Test	PDA-L			GRPDA-L			AGRPDA-L		
		Iter	#LS	Time	Iter	#LS	Time	Iter	#LS	Time
$10^{-8}$	(i)	4757	4688	14.7	4043	1186	12.5	2450	723	10.9
	(ii) $v = 0.5$	6167	6130	21.3	5213	1532	17.1	1759	517	8.1
	(ii) $v = 0.9$	27899	27889	94.2	26080	7697	86.7	7480	2208	34.5
$10^{-12}$	(i)	11234	11082	34.8	9287	2735	28.8	3539	1043	15.8
	(ii) $v = 0.5$	14913	14831	49.1	12330	3634	40.3	3124	922	14.3
	(ii) $v = 0.9$	64003	64007	213.8	55758	16464	183.1	12216	3608	55.6

Table 2 Results of GRPDA-L, PDA-L and AGRPDA-L on the LASSO problem (71).

AGRPDA-L require roughly one extra linesearch trial step per three outer iterations. Since the proximal operator  $\operatorname{Prox}_{\tau f^*}(\cdot)$  is linear and does not incur extra computations, GRPDA-L and PDA-L perform similarly in terms of outer iteration and CPU time. In comparison, AGRPDA-L performs the best, i.e., takes much less number of iterations and CPU time. The evolution of function value residuals  $F(x_n) - F^*$  versus CPU time is given in Figure 2, from which it can be seen that AGRPDA-L, which takes advantage of strong convexity, is much faster than GRPDA-L and PDA-L, which do not. These results lead to the conclusion that strong convexity of the component functions, if properly explored, helps to improve the performance of primal-dual type algorithms.



Fig. 2 Experimental results on the LASSO problem. Decreasing behavior of function value versus CPU time. From left to right: Case (i) with (p, q, s) = (1000, 2000, 100), Case (ii) with (p, q, s, v) = (1000, 2000, 10, 0.5), and Case (ii) with (p, q, s, v) = (1000, 2000, 10, 0.9).

## 6 Conclusions

In this paper, we have incorporated linesearch strategy into the golden ratio primal-dual algorithm (GRPDA) recently proposed in [8]. Global convergence and  $\mathcal{O}(1/N)$  ergodic convergence rate measured by primal-dual function gap are established in the general convex case. When either one of the component functions is strongly convex, accelerated GRPDA with linesearch is proposed, which achieves  $\mathcal{O}(1/N^2)$  ergodic rate of convergence. Furthermore, when both component functions are strongly convex, nonergodic linear convergence results are obtained. The proposed linesearch strategy does not require to evaluate the spectral norm of K and adopts potentially much larger stepsizes. In cases such as regularized least-squares problem, the proposed linesearch strategy only requires minimal extra computational cost and thus is particularly useful. Our numerical experimental on minimax matrix game and LASSO problems demonstrate the benefits gained by taking advantage of strong convexity and incorporating our proposed linesearch. Experimentally, the extra linesearch trial steps used by golden ratio type primal-dual algorithms are about one-third of those proposed by Malitsky [20] and larger stepsizes can be accepted, which could be significant when the evaluations of proximal operators are nontrivial.

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#### A Proof of Lemma 3.1

Proof. (i) Since (12) is fulfilled whenever  $\tau_n$  satisfies  $\sqrt{\beta\tau_n}L \leq \sigma\sqrt{\psi/\tau_{n-1}}$ , we have from  $\tau_n = \varphi\tau_{n-1}\mu^i$  that (12) is fulfilled whenever  $\sqrt{\mu^i} \leq \underline{\tau}/\tau_{n-1}$ . Hence, (12) will be fulfilled by the linesearch procedure in Step 2 since  $\mu \in (0, 1)$ . (ii) We consider two cases.

Case 1: There exists a  $\bar{k}$  such that  $\tau_n \geq \underline{\tau}$  for all  $n \geq \bar{k}$ . If there is no infinite subsequence  $\{n_k : k \geq 1\} \subseteq \{1, 2, \ldots\}$  such that  $\delta_{n_k} \geq \rho$ , we have  $\delta_n < \rho < 1$  for all n sufficiently large. Then, we will have from  $\tau_n = \tau_0 \prod_{i=1}^n \delta_i$  and  $\rho < 1$  that  $\lim_{n \to \infty} \tau_n = 0$ , which contradicts with  $\tau_n \geq \underline{\tau}$  for all  $n \geq \bar{k}$  Hence, in this case, property (ii) holds.

Case 2: There exists an infinite subsequence  $\{n_i : i \ge 1\}$  such that  $\tau_{n_i} < \underline{\tau}$ . By the linesearch procedure in Step 2, for any  $\tau_n \le \underline{\tau}$ , the initial trial  $\tau_{n+1} = \varphi \tau_n$  will satisfy (12) and be accepted by the linesearch. Hence, defining  $\ell(t) = \lfloor \log_{\varphi}(\underline{\tau}/t) \rfloor$  where  $t < \underline{\tau}$ , we have the following property:

If 
$$\tau_n < \underline{\tau}$$
, then with  $k' := n + \ell(\tau_n)$  we have  
 $\tau_k < \underline{\tau}$  and  $\tau_{k+1} = \varphi \tau_k$  for all  $k = n, n+1, \dots, k'$ , and  $\underline{\tau} \le \tau_{k'+1} < \varphi \underline{\tau}$ . (72)

Here,  $\lfloor t \rfloor$  is the largest integer less or equal to t. So, for any  $\tau_{n_i} < \underline{\tau}$ , we have  $\tau_{k'+1} \ge \underline{\tau}$ , where  $k' = n_i + \ell(\tau_{n_i})$ . In addition, we have  $\delta_{k'+1} = \tau_{k'+1}/\tau_{k'} = \varphi > \rho$ . Hence, in this case property (ii) also holds.

(iii) First, if  $\tau_0 < \underline{\tau}$ , by property (72), we have  $\tau_s = \tau_0 \varphi^s \ge \underline{\tau}$ , where  $s = \ell(\tau_0) + 1$ . Hence, without losing of generality, to show property (iii), we can simply assume  $\tau_1 \ge \underline{\tau}$ .

Now, we show the following property:

For any 
$$\tau_{n-1} \ge \underline{\tau}$$
 and  $\tau_n < \underline{\tau}$ , we have (74) and (75) hold. (73)

Since  $\tau_n < \tau_{n-1}$ , by the linesearch procedure in Step 2, we have  $\tau_n = \varphi \tau_{n-1} \mu^j$  with  $j \ge 1$  and  $\sqrt{\beta \varphi \tau_{n-1} \mu^{j-1}} L > \sigma \sqrt{\psi/\tau_{n-1}}$ , which is equivalent to

 $\tau_{n-}$ 

$$_{1} > \underline{\tau}\mu^{-(j-1)/2}.$$
 (74)

On the other hand, by (74) and  $j \ge 1$ , we have

$$1 + \ell(\tau_n) = 1 + \lfloor \log_{\varphi}(\underline{\tau}/\tau_n) \rfloor = 1 + \lfloor \log_{\varphi}(\underline{\tau}/(\varphi\tau_{n-1}\mu^j)) \rfloor$$
$$= \lfloor \log_{\varphi}(\underline{\tau}/(\tau_{n-1}\mu^j)) \rfloor \leq \left\lfloor \frac{j+1}{2} \log_{\varphi}(1/\mu) \right\rfloor$$
$$\leq j \log_{\varphi}(1/\mu).$$
(75)

Let  $\mathcal{Z}^+ := \{1, 2, 3, \ldots\}$  be the set of positive integers. Given two integers  $i_1 \leq i_2$ , let interval  $[i_1, i_2] := \{i \in \mathcal{Z}^+ : i_1 \leq i \leq i_2\}$  and interval  $[i_1, \infty) := \{i \in \mathcal{Z}^+ : i_1 \leq i < \infty\}$ . Then, based on properties (72), (73) and the assumption  $\tau_1 \geq \underline{\tau}$ , there exist a set of positive integers  $\mathcal{K} := \bigcup_{i=1}^{|\mathcal{K}|} \{k_i\} \subseteq \mathcal{Z}^+$  and an associated integer set  $\mathcal{M} := \bigcup_{i=1}^{|\mathcal{K}|-1} \{m_i\} \subseteq \mathcal{Z}^+$ , where  $|\mathcal{K}| \geq 1$  denotes the cardinality of  $\mathcal{K}$  that is either a finite number or infinity, such that they partition  $\mathcal{Z}^+$ , i.e.,  $\mathcal{Z}^+ = \bigcup_{i=1}^{\infty} [k_i, k_{i+1} - 1]$  if  $|\mathcal{K}| = \infty$  or  $\mathcal{Z}^+ = \bigcup_{i=1}^{|\mathcal{K}|-1} [k_i, k_{i+1} - 1] \cup [k_{|\mathcal{K}|}, \infty)$  if  $|\mathcal{K}| < \infty$ , and the following properties hold:

(a)  $k_1 = 1$  and  $k_i < m_i < k_{i+1}$  for all *i*;

(b)  $\tau_k \geq \underline{\tau}$  for all  $k \in [k_i, m_i - 1]$  and  $\tau_k < \underline{\tau}$  for all  $k \in [m_i, k_{i+1} - 1]$ , see the diagram below

$$\overbrace{\ldots, k_i-1}^{t_k \ge \underline{\tau}, (m_i-k_i) \text{ times}} \overbrace{k_i, \ldots, m_i-1}^{t_k \ge \underline{\tau}, (m_i-k_i) \text{ times}} \overbrace{m_i, \ldots, k_{i+1}-1}^{t_k < \underline{\tau}, (k_{i+1}-m_i) \text{ times}} \overbrace{k_{i+1}, \ldots}^{t_k < \underline{\tau}, (k_{i+1}-m_i) \text{ times}}$$

(c) If  $|\mathcal{K}| < \infty$ ,  $\tau_k \ge \underline{\tau}$  for all  $k \ge k_{|\mathcal{K}|}$ ; Otherwise,  $|\mathcal{K}| = \infty$  and  $\mathcal{Z}^+ = \bigcup_{i=1}^{\infty} [k_i, k_{i+1} - 1];$ 

- (d)  $\tau_{k_i} < \varphi \underline{\tau}$  by property (72) for all  $k_i \in \mathcal{K} \setminus \{k_1\}$ ;
- (e)  $\tau_{m_i-1} > \underline{\tau}\mu^{-(j-1)/2}$  by (74) and  $k_{i+1} m_i = \ell(\tau_{m_i}) + 1 \le j \log_{\varphi}(1/\mu)$  by (75) for all  $m_i \in \mathcal{M}$  and some  $j \ge 1$  depending on  $m_i$ .

Now, we consider any interval  $[k_i, k_{i+1} - 1] = [k_i, m_i - 1] \cup [m_i, k_{i+1} - 1]$ . Let  $j \ge 1$  be the integer associated with  $m_i$  such that property (e) holds. Since  $\tau_{k+1} \le \varphi \tau_k$  for all k, we have

$$\tau_{m_i-1} < \tau_{k_i} \varphi^{m_i-1-k_i}.$$

Then, by properties (d) and (e), for  $k_i \neq k_1 = 1$ , we have  $\tau_{k_i} < \varphi \underline{\tau}$  and  $\tau_{m_i-1} \geq \underline{\tau} \mu^{-(j-1)/2}$ , which together with the above inequality gives  $\underline{\tau} \mu^{-(j-1)/2} \leq \tau_{m_i-1} < \tau_{k_i} \varphi^{m_i-1-k_i} < \underline{\tau} \varphi^{m_i-k_i}$ , which is equivalent to

$$(j-1)\log_{\varphi}(1/\sqrt{\mu}) < m_i - k_i$$

For  $k_i = k_1 = 1$ , we have  $\underline{\tau}\mu^{-(j-1)/2} < \tau_1 \varphi^{m_i-2}$ , or  $(j-1)\log_{\varphi}(1/\sqrt{\mu}) \leq \log_{\varphi}(\tau_1/\underline{\tau}) + m_i - 2$ . So, there exists an integer constant  $\overline{j} \geq 1$ , which does not depend on either  $k_i$  or  $m_i$ , such that

$$m_i - k_i \ge \frac{j}{4} \log_{\varphi}(1/\mu)$$
 whenever  $j \ge \overline{j}$ . (76)

Next, we show that there exists a constant  $\bar{c} > 0$ , which does not depend on  $m_i$  or the interval  $[k_i, k_{i+1} - 1]$ , such that

$$m_i - k_i \ge \bar{c} (k_{i+1} - m_i).$$
 (77)

Noticing that by property (e), we have  $k_{i+1} - m_i \leq j \log_{\varphi}(1/\mu)$ . Hence, it follows from (76) that if  $j \geq \overline{j}$  we have

$$m_i - k_i \ge \frac{\log_{\varphi}(1/\mu)}{4\log_{\varphi}(1/\mu)}(k_{i+1} - m_i) = \frac{1}{4}(k_{i+1} - m_i).$$

On the other hand, if  $1 \leq j < \overline{j}$ , we have from  $k_{i+1} - m_i \leq j \log_{\varphi}(1/\mu)$  that

$$n_i - k_i \ge 1 > j/\bar{j} \ge \frac{k_{i+1} - m_i}{\bar{j}\log_{\varphi}(1/\mu)}.$$

Hence, (77) holds with  $\bar{c} = \min\{1/(\bar{j}\log_{\varphi}(1/\mu)), 1/4\} > 0$ . Apparently, if follows from (77) that  $m_i - k_i \geq \hat{c}(k_{i+1} - k_i)$  with  $\hat{c} := \bar{c}/(1 + \bar{c}) > 0$ . If  $|\mathcal{K}| = \infty$ , given any  $N \geq 1$ , it follows from property (c) that  $N \in [k_i, k_{i+1} - 1]$  for certain  $i \geq 1$ . Hence, by the definition of  $\mathcal{K}_N = \{1 \leq n \leq N : \tau_n \geq \underline{\tau}\}$ , we have  $\mathcal{K}_N = \bigcup_{j=1}^i (\mathcal{K}_N \cap [k_j, k_{j+1} - 1])$ . Then, it follows from (77) and property (b) that

$$\begin{aligned} |\mathcal{K}_N| &= \sum_{j=1}^{i} |\mathcal{K}_N \cap [k_j, k_{j+1} - 1]| \ge \sum_{j=1}^{i} |\mathcal{K}_N \cap [k_j, m_j - 1]| \\ &= \min\{N, m_i\} - k_i + \sum_{j=1}^{i-1} |[k_j, m_j - 1]| = \min\{N, m_i\} - k_i + \sum_{j=1}^{i-1} (m_j - k_j) \\ &\ge \hat{c}(N - k_i) + \hat{c} \sum_{j=1}^{i-1} (k_{j+1} - k_j) = \hat{c}N, \end{aligned}$$
(78)

where  $\hat{c} = \bar{c}/(1+\bar{c})$ . If  $|\mathcal{K}| < \infty$ , by property (c), for all  $n \ge k_{|\mathcal{K}|}$  we have  $\tau_n \ge \underline{\tau}$ . This property together with (78) implies  $|\mathcal{K}_N| \ge \hat{c}N$  for any  $N \ge 1$ . 

### B Proof of Lemma 4.1

Proof. (i) This conclusion follows almost from an identical proof of conclusion (i) in Lemma 3.1 except by replacing  $\beta$  by  $\beta_n$  and setting  $\sigma = 1$ .

(ii) Let  $h(\tau) := 1 + \frac{(\psi - \varphi)\gamma_g \tau}{\psi + \varphi \gamma_g \tau}$ . Since  $h(\tau)$  is strictly increasing with respect to  $\tau > 0$ , we have that  $1 < 1 + \gamma_g \omega_n \tau_{n-1} = h(\tau_{n-1}) < \varsigma := \psi/\varphi$ . So, by (37),  $\beta_{n-1} < \beta_n = \beta_{n-1} h(\tau_{n-1}) < \varsigma \beta_{n-1}$ . Therefore, for any  $\sqrt{\beta_{n-1}}\tau_{n-1} \leq 1/L$ , we have  $\sqrt{\beta_n}\tau_{n-1} \leq \sqrt{\zeta\beta_{n-1}}\tau_{n-1} \leq \sqrt{\zeta}/L = \frac{1}{L}\sqrt{\frac{\psi}{\varphi}}$ , which by the linesearch procedure in Step 2 implies that the initial trial  $\tau_n = \varphi \tau_{n-1}$  will satisfy (39) and be accepted by the linesearch. Hence, analogous to property (72), we have the following property:

If 
$$\sqrt{\beta_n}\tau_n < \frac{1}{L}$$
, then there exists an integer  $\ell_n \ge 0$  such that  $k' := n + \ell_n$  satisfies:  
 $\sqrt{\beta_k}\tau_k < \frac{1}{L}$  and  $\tau_{k+1} = \varphi\tau_k$  for all  $k = n, n+1, \dots, k'$ , and  $\sqrt{\beta_{k'+1}}\tau_{k'+1} \ge \frac{1}{L}$ , (79)

which, by  $\beta_n < \beta_{n+1} < \varsigma \beta_n$  and  $\tau_{n+1} \leq \varphi \tau_n$  for all  $n \geq 1$ , also implies

$$\ell_n \leq \vartheta \left( \sqrt{\beta_n} \tau_n \right) \quad \text{and} \quad \sqrt{\beta_{k'+1}} \tau_{k'+1} < \sqrt{\varsigma \beta_{k'}} \varphi \tau_{k'} = \sqrt{\psi \varphi} \sqrt{\beta_{k'}} \tau_{k'} < \theta,$$

$$\tag{80}$$

where  $\vartheta(t) := |\log_{\varphi}(1/(Lt))|$  for t < 1/L and  $\theta := \sqrt{\psi\varphi}/L$ .

Analogous to property (73), we show the following property:

For any 
$$\sqrt{\beta_{n-1}}\tau_{n-1} \ge \frac{1}{L}$$
 and  $\sqrt{\beta_n}\tau_n < \frac{1}{L}$ , we have (82) and (83) hold. (81)

Since  $\beta_n > \beta_{n-1}$  and  $\sqrt{\beta_n}\tau_n < \sqrt{\beta_{n-1}}\tau_{n-1}$ , by the linesearch procedure in Step 2, we have  $\tau_n = \varphi \tau_{n-1} \mu^j$  with  $j \geq 1$  and  $\sqrt{\beta_n \varphi \tau_{n-1} \mu^{j-1}} L > \sqrt{\psi/\tau_{n-1}}$ , which gives

$$\sqrt{\beta_n}\tau_{n-1} \ge \sqrt{\frac{\psi}{\varphi}} \frac{\mu^{-\frac{j-1}{2}}}{L} \quad \text{and} \quad \sqrt{\beta_{n-1}}\tau_{n-1} > \sqrt{\beta_n/\varsigma}\tau_{n-1} \ge \frac{\mu^{-\frac{j-1}{2}}}{L}.$$
(82)

It follows from (80),  $\psi > \varphi > 1$ , j > 1 and (82) that

1

$$+ \ell_n \leq 1 + \vartheta(\sqrt{\beta_n}\tau_n) = 1 + \left\lfloor \log_{\varphi} \left( 1/(L\sqrt{\beta_n}\tau_n) \right) \right\rfloor$$

$$= 1 + \left\lfloor \log_{\varphi} \left( 1/(L\sqrt{\beta_n}\varphi\tau_{n-1}\mu^j) \right) \right\rfloor = \left\lfloor \log_{\varphi} (1/(L\sqrt{\beta_n}\tau_{n-1}\mu^j)) \right\rfloor$$

$$\leq \left\lfloor \frac{j+1}{2} \log_{\varphi} (1/\mu) + \frac{1}{2} \log_{\varphi} (\varphi/\psi) \right\rfloor$$

$$\leq j \log_{\varphi} (1/\mu).$$

$$(83)$$

Let  $\mathcal{Z}^+ := \{1, 2, 3, \ldots\}$  be the set of positive integers. Given two integers  $i_1 \leq i_2$ , let interval  $[i_1, i_2] := \{i \in \mathcal{Z}^+ : i_1 \leq i_2, i_3 \leq i_3 \}$ Let  $\mathcal{Z}^+ := \{1, 2, 3, ...\}$  be the set of positive integers. Given two integers  $i_1 \leq i_2$ , let interval  $[i_1, i_2] := \{i \in \mathcal{Z}^+ : i_1 \leq i < \infty\}$ . To show this lemma, without losing of generality, by property (79), we can simply assume  $\sqrt{\beta_1}\tau_1 \geq 1/L$ . Then, based on properties (79) and (81), there exist a set of positive integers  $\mathcal{K} := \bigcup_{i=1}^{|\mathcal{K}|} \{k_i\} \subseteq \mathcal{Z}^+$  and an associated integer set  $\mathcal{M} := \bigcup_{i=1}^{|\mathcal{K}|-1} \{m_i\} \subseteq \mathcal{Z}^+$ , where  $|\mathcal{K}| \geq 1$  denotes the cardinality of  $\mathcal{K}$  that is either a finite number or infinity, such that they partition  $\mathcal{Z}^+$ , i.e.,  $\mathcal{Z}^+ = \bigcup_{i=1}^{\infty} [k_i, k_{i+1} - 1]$  if  $|\mathcal{K}| = \infty$  or  $\mathcal{Z}^+ = \bigcup_{i=1}^{|\mathcal{K}|-1} [k_i, k_{i+1} - 1] \cup [k_{|\mathcal{K}|}, \infty)$  if  $|\mathcal{K}| < \infty$ , and the following properties hold: hold:

- (a)  $k_1 = 1$  and  $k_i < m_i < k_{i+1}$  for all i; (b)  $\sqrt{\beta_k}\tau_k \ge 1/L$  for all  $k \in [k_i, m_i 1]$  and  $\sqrt{\beta_k}\tau_k < 1/L$  for all  $k \in [m_i, k_{i+1} 1]$ ; (c) If  $|\mathcal{K}| < \infty$ ,  $\sqrt{\beta_k}\tau_k \ge 1/L$  for all  $k \ge k_{|\mathcal{K}|}$ ; Otherwise,  $|\mathcal{K}| = \infty$  and  $\mathcal{Z}^+ = \bigcup_{i=1}^{\infty} [k_i, k_{i+1} 1]$ ;

- (d)  $\sqrt{\beta_{k_i}}\tau_{k_i} < \theta$  by (80) for all  $k_i \in \mathcal{K} \setminus \{k_1\}$ , where  $\theta = \sqrt{\psi\varphi}/L$ ; (e)  $\sqrt{\beta_{m_i-1}}\tau_{m_i-1} > \mu^{-(j-1)/2}/L$  by (82) and  $k_{i+1} m_i = \ell_{m_i} + 1 \le j \log_{\varphi}(1/\mu)$  by (83) for all  $m_i \in \mathcal{M}$  and some  $j \ge 1$  depending on  $m_i$ , where  $\ell_{m_i}$  is defined in property (79) associated with  $\sqrt{\beta_{m_i}} \tau_{m_i}$ .

Now, we consider any interval  $[k_i, k_{i+1} - 1] = [k_i, m_i - 1] \cup [m_i, k_{i+1} - 1]$ . Let  $j \ge 1$  be the integer associated with  $m_i$  such that property (e) holds. Since  $\sqrt{\beta_{k+1}}\tau_{k+1} < \sqrt{\varsigma\beta_k}\varphi\tau_k = \sqrt{\psi\varphi}\sqrt{\beta_k}\tau_k$  for all k, we have

$$\sqrt{\beta_{m_i-1}}\tau_{m_i-1} < \sqrt{\beta_{k_i}}\tau_{k_i}\rho^{(m_i-1-k_i)/2},$$

where  $\rho := \psi \varphi > 1$ . By properties (d) and (e), for  $k_i \neq k_1 = 1$ , we have  $\sqrt{\beta_{k_i} \tau_{k_i}} < \theta$  and  $\sqrt{\beta_{m_i-1} \tau_{m_i-1}} \geq 1$  $\mu^{-(j-1)/2}/L,$  which together with the above inequality gives

$$\mu^{-(j-1)/2}/L < \theta \rho^{(m_i - 1 - k_i)/2} \iff j \log_{\rho} (1/\mu) < \log_{\rho} \left( (\theta L)^2 / (\mu \rho) \right) + m_i - k_i$$
  
=  $\log_{\rho} \left( 1/\mu \right) + m_i - k_i.$ 

For  $k_i = k_1 = 1$ , we have  $j \log_{\rho}(1/\mu) \leq \log_{\rho}\left((\theta_1 L)^2/(\mu\rho)\right) + m_i - 1$ , where  $\theta_1 := \sqrt{\beta_1}\tau_1$ . So, there exists an integer constant  $\overline{j} \geq 1$ , which does not depend on either  $k_i$  or  $m_i$ , such that

$$m_i - k_i \ge \frac{j}{2} \log_{\rho}(1/\mu)$$
 whenever  $j \ge \overline{j}$ . (84)

Next, we show that there exists a constant  $\bar{c} > 0$ , which does not depend on  $m_i$ , such that

$$m_i - k_i \ge \bar{c} (k_{i+1} - m_i + 1).$$
 (85)

Noticing that by property (e), we have

$$k_{i+1} - m_i + 1 \le j \log_{\varphi}(1/\mu) + 1 \le j (\log_{\varphi}(1/\mu) + 1).$$
(86)

Hence, it follows from (84) and (86) that whenever  $j \ge \overline{j}$  we have

$$m_i - k_i \ge \frac{\log_{\rho}(1/\mu)}{2(\log_{\varphi}(1/\mu) + 1)}(k_{i+1} - m_i + 1).$$

On the other hand, if  $1 \le j < \overline{j}$ , then by (86) we have

$$m_i - k_i \ge 1 > j/\bar{j} \ge \frac{k_{i+1} - m_i + 1}{\bar{j} (\log_{\varphi}(1/\mu) + 1)}$$

Hence, (85) holds with  $\bar{c} = \min\{1/(\bar{j}(\log_{\varphi}(1/\mu)) + 1)), \log_{\rho}(1/\mu)/(2(\log_{\varphi}(1/\mu) + 1))\} > 0.$ Ν

Now, by (37) we have 
$$\beta_{n+1} = \beta_n h(\tau_n) = \beta_n (1+t_n)$$
, where  $t_n = \frac{(\psi - \varphi)\gamma_g \tau_n}{\psi + \varphi \gamma_g \tau_n}$ . Then, we have

$$\sqrt{\beta_{n+1}} - \sqrt{\beta_n} = \frac{\beta_{n+1} - \beta_n}{\sqrt{\beta_{n+1}} + \sqrt{\beta_n}} = \frac{t_n \beta_n}{\sqrt{\beta_{n+1}} + \sqrt{\beta_n}}$$
$$\geq \frac{t_n \beta_n}{(\sqrt{\zeta} + 1)\sqrt{\beta_n}} \ge \tilde{c} \min\{\tau_n, 1\} \sqrt{\beta_n}, \tag{87}$$

where  $\tilde{c} > 0$  is some constant. Consider any  $k \in [k_i, k_{i+1} - 1] = [k_i, m_i - 1] \cup [m_i, k_{i+1} - 1]$ . If  $k \in [m_i, k_{i+1} - 1]$ , by (85) we have

$$\frac{m_i - k_i}{k_{i+1} - k_i + 1} = \frac{m_i - k_i}{(m_i - k_i) + (k_{i+1} - m_i + 1)} \ge \frac{1}{1 + 1/\bar{c}} = \frac{\bar{c}}{1 + \bar{c}},$$
(88)

and it thus follows from (87), property (b),  $\beta_n \ge \beta_0 > 0$  and (88) that

$$\sqrt{\beta_k} - \sqrt{\beta_{k_i}} = \sum_{n=k_i}^{k-1} \left( \sqrt{\beta_{n+1}} - \sqrt{\beta_n} \right) \ge \tilde{c} \sum_{n=k_i}^{k-1} \min\{\tau_n, 1\} \sqrt{\beta_n} \\
\ge \tilde{c} \sum_{n=k_i}^{m_i-1} \min\{\tau_n, 1\} \sqrt{\beta_n} \ge \tilde{c} \sum_{n=k_i}^{m_i-1} \min\{1/L, \sqrt{\beta_0}\} \\
= \tilde{c}(m_i - k_i) \min\{1/L, \sqrt{\beta_0}\} \ge \hat{c}(k_{i+1} - k_i + 1) \ge \hat{c} (k - k_i + 1),$$
(89)

where  $\hat{c} := \tilde{c} \bar{c} \min\{1/L, \sqrt{\beta_0}\}/(1+\bar{c}) > 0$ ,  $\bar{c}$  and  $\tilde{c}$  are constants given in (85) and (87), respectively. On the other hand, if  $k \in [k_i, m_i - 1]$ , similar to (89) we can show

$$\sqrt{\beta_k} - \sqrt{\beta_{k_i}} \ge \tilde{c} \min\{1/L, \sqrt{\beta_0}\}(k-k_i) \ge \hat{c} (k-k_i).$$

$$\tag{90}$$

If  $|\mathcal{K}| = \infty$ , given any  $n \ge 1$ , it follows from property (c) that  $n \in [k_i, k_{i+1} - 1]$  for certain  $i \ge 1$ . Hence, by (89), (90) and  $0 < \beta_j < \beta_{j+1}$  for any  $j \ge 1$ , we have

$$\sqrt{\beta_n} = \sqrt{\beta_n} - \sqrt{\beta_{k_i}} + \sum_{j=1}^{i-1} (\sqrt{\beta_{k_{j+1}}} - \sqrt{\beta_{k_j}}) + \sqrt{\beta_{k_1}} \\
= \sqrt{\beta_n} - \sqrt{\beta_{k_i}} + \sum_{j=1}^{i-1} \left[ \left( \sqrt{\beta_{k_{j+1}-1}} - \sqrt{\beta_{k_j}} \right) + \left( \sqrt{\beta_{k_{j+1}}} - \sqrt{\beta_{k_{j+1}-1}} \right) \right] + \sqrt{\beta_{k_1}} \\
\ge \sqrt{\beta_n} - \sqrt{\beta_{k_i}} + \sum_{j=1}^{i-1} \left( \sqrt{\beta_{k_{j+1}-1}} - \sqrt{\beta_{k_j}} \right) \\
\ge \hat{c} \left[ (n-k_i) + \sum_{j=1}^{i-1} (k_{j+1} - k_j) \right] = \hat{c} (n-1).$$
(91)

If  $|\mathcal{K}| < \infty$ , by property (c), for any  $k \ge k_{|\mathcal{K}|}$  we have  $\sqrt{\beta_k} \tau_k \ge 1/L$  and thus

$$\sqrt{\beta_k} - \sqrt{\beta_{k_{|\mathcal{K}|}}} \ge \hat{c}(k - k_{|\mathcal{K}|}).$$

This together with (89) and property (c) also implies (91) holds for any  $n \ge 1$ . Then, conclusion (ii) follows from (91).

(iii). Note that for any  $\sqrt{\beta_p}\tau_p \ge \sqrt{\beta_q}\tau_q$  with  $p \le q$ , we have from  $\beta_p \le \beta_q$  that  $\tau_p \ge \tau_q$ . So, based on property (b) in (ii) and (85), we have the property:

(b') 
$$\tau_p \ge \tau_q$$
 for all  $p \in [k_i, m_i - 1], q \in [m_i, k_{i+1} - 1]$  and therefore  $\sum_{n=k_i}^{m_i - 1} \tau_n \ge \bar{c} \sum_{n=m_i}^{k_{i+1} - 1} \tau_n$ ,

where  $\bar{c} > 0$  is the constant in (85).

Let  $\mathcal{K} \subset \mathcal{Z}^+$  be the set given in the proof of (ii). If  $|\mathcal{K}| = \infty$ , then for any  $N \ge 1$  it follows from property (c) that  $N \in [k_j, k_{j+1} - 1]$  for certain  $j \ge 1 = k_1$ . Hence, we have from property (b') that

$$\sum_{n=1}^{N} \tau_n = \sum_{n=k_j}^{N} \tau_n + \sum_{s=1}^{j-1} \sum_{n=k_s}^{k_{s+1}-1} \tau_n = \sum_{n=k_j}^{N} \tau_n + \sum_{s=1}^{j-1} \left( \sum_{n=k_s}^{m_s-1} \tau_n + \sum_{n=m_s}^{k_{s+1}-1} \tau_n \right)$$
$$\leq (1+1/\bar{c}) \sum_{n=k_j}^{\min\{N,m_j-1\}} \tau_n + \sum_{s=i}^{j-1} \left( \sum_{n=k_s}^{m_s-1} \tau_n + 1/\bar{c} \sum_{n=k_s}^{m_s-1} \tau_n \right)$$
$$= (1+1/\bar{c}) \left( \sum_{n=k_j}^{\min\{N,m_j-1\}} \tau_n + \sum_{s=1}^{j-1} \sum_{n=k_s}^{m_s-1} \tau_n \right) = (1+1/\bar{c}) \sum_{n\in\mathcal{S}_N} \tau_n.$$

When  $|\mathcal{K}| < \infty$ , we can also similarly prove that  $\sum_{n=1}^{N} \tau_n \leq (1+1/\bar{c}) \sum_{n \in S_N} \tau_n$  for any  $N \geq 1$ , because  $\sqrt{\beta_n} \tau_n \geq 1/L$  for all  $n \geq k_{|\mathcal{K}|}$ . The proof is completed by letting  $\tilde{c} = 1 + 1/\bar{c}$ .