# PRECONDITIONING FOR A PRESSURE-ROBUST HDG DISCRETIZATION OF THE STOKES EQUATIONS * 

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#### Abstract

We introduce a new preconditioner for a recently developed pressure-robust hybridized discontinuous Galerkin (HDG) finite element discretization of the Stokes equations. A feature of HDG methods is the straightforward elimination of degrees-of-freedom defined on the interior of an element. In our previous work (J. Sci. Comput., $77(3): 1936-1952,2018$ ) we introduced a preconditioner for the case in which only the degrees-of-freedom associated with the element velocity were eliminated via static condensation. In this work we introduce a preconditioner for the statically condensed system in which the element pressure degrees-of-freedom are also eliminated. In doing so the number of globally coupled degrees-of-freedom are reduced, but at the expense of a more difficult problem to analyse. We will show, however, that the Schur complement of the statically condensed system is spectrally equivalent to a simple trace pressure mass matrix. This result is used to formulate a new, provably optimal preconditioner. Through numerical examples in two- and three-dimensions we show that the new preconditioned iterative method converges in fewer iterations, has superior conservation properties for inexact solves, and is faster in CPU time when compared to our previous preconditioner.


Key words. Stokes equations, preconditioning, hybridized, discontinuous Galerkin, finite element methods.

AMS subject classifications. 65F08, 65N30, 76D07.

1. Introduction. Hybridizable discontinuous Galerkin (HDG) methods were introduced for elliptic problems to reduce cost of discontinuous Galerkin methods while retaining favourable properties [7]. Static condensation can be applied to eliminate degrees-of-freedom on cells, leading to global degrees-of-freedom that are associated with functions that are defined only on the facets of the mesh. The resulting methods have fewer global degrees of freedom than a discontinuous Galerkin method on the same mesh, especially in three dimensions.

For large systems, preconditioned iterative methods are preferred over direct solvers. In the case of HDG methods, new preconditioners need to be designed that are effective for the reduced linear systems following static condensation. This has been a topic of recent research with the design of scalable multigrid and domain decomposition methods for HDG discretizations of elliptic PDEs, see for example [8, 13, 18, 27]. See also [25, 26] for HDG preconditioners for hyperbolic problems and [9] and [33] for preconditioning of statically condensed and hybridized finite element discretizations by, respectively, algebraic multigrid and geometric multigrid.

In this paper we develop a new preconditioner for the pressure-robust HDG discretization of the Stokes equations introduced in [29, 30]. Pressure-robust discretizations have the advantage over other finite element discretizations that the a priori error estimates for the velocity do not depend on the pressure. We refer to, respectively, [17] and [20, 21, 22, 23] for an overview of pressure-robust discretizations and alternative pressure-robust HDG discretizations. In [31] we developed a precondi-

[^0]tioner for a pressure-robust HDG discretization of the Stokes problem, and showed that the pressure Schur complement of the linear system obtained after eliminating only the degrees-of-freedom associated to the element velocity is identical to the pressure Schur complement of the original linear system. Together with a proof showing spectral equivalence of the element/trace pressure Schur complement with an element/trace pressure mass matrix, we were able to formulate a very simple scalable preconditioner when the velocity only was statically condensed. However, a disadvantage of eliminating the cell-wise velocity degrees-of-freedom and not the cell-wise pressure degrees-of-freedom is that the pointwise divergence-free (within cells) property of the method is guaranteed only in the limit of the iterative solver convergence. We address this issue in this work by formulating a preconditioner for a statically condensed system after elimination of both the degrees-of-freedom associated to the element velocity and the degrees-of-freedom associated to the element pressure. The complication that arises in eliminating the degrees-of-freedom associated to the element pressure is that the lifting of the trace velocity unknowns to the element is divergence-free. This requires a new proof to show spectral equivalence of the trace pressure Schur complement with the trace pressure mass matrix. Given this result we can then apply the general theory of Pestana and Wathen [28] for preconditioners of saddle point problems to formulate a new scalable preconditioner for the statically condensed problem. Compared to our previous work [31], there are less globally coupled degrees-of-freedom and our new preconditioner results in a more efficient solver.

The outline of this paper is as follows. In section 2 we present the HDG method for the Stokes equations and discuss some useful properties of the discretization. A preconditioner for the statically condensed form of the HDG method is formulated in section 3. By two- and three-dimensional numerical examples in section 4 we verify our analysis. We draw conclusions in section 5 .
2. HDG for the Stokes problem. Let $\Omega \subset \mathbb{R}^{d}$ be a polygonal $(d=2)$ or polyhedral $(d=3)$ domain with boundary $\partial \Omega$. We consider the Stokes problem: given a body force $f: \Omega \rightarrow \mathbb{R}^{d}$, find the velocity $u: \Omega \rightarrow \mathbb{R}^{d}$ and (kinematic) pressure $p: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
-\nabla^{2} u+\nabla p=f & \text { in } \Omega  \tag{2.1a}\\
\nabla \cdot u=0 & \text { in } \Omega  \tag{2.1b}\\
u=0 & \text { on } \partial \Omega \tag{2.1c}
\end{align*}
$$

To obtain a unique solution to the pressure, we additionally impose that the mean of the pressure over the domain $\Omega$ is zero.
2.1. Preliminaries. Consider a triangulation $\mathcal{T}:=\{K\}$ of the domain $\Omega$ into non-overlapping elements $K$. We introduce the following function spaces on $\Omega$ :

$$
\begin{align*}
V_{h} & :=\left\{v_{h} \in\left[L^{2}(\Omega)\right]^{d}: v_{h} \in\left[P_{k}(K)\right]^{d}, \forall K \in \mathcal{T}\right\},  \tag{2.2}\\
Q_{h} & :=\left\{q_{h} \in L^{2}(\Omega): q_{h} \in P_{k-1}(K), \forall K \in \mathcal{T}\right\}
\end{align*}
$$

where $P_{k}(D)$ denotes the set of polynomials of degree at most $k$ on a domain $D$. Two adjacent elements $K^{+}$and $K^{-}$share an interior facet $F$. A boundary facet is defined as a facet of the boundary of an element, $\partial K$, that lies on $\partial \Omega$. Denoting the set of all facets by $\mathcal{F}=\{F\}$, and the union of all facets by $\Gamma^{0}$, we introduce the following
function spaces on $\Gamma^{0}$ :

$$
\begin{align*}
& \bar{V}_{h}:=\left\{\bar{v}_{h} \in\left[L^{2}\left(\Gamma^{0}\right)\right]^{d}: \bar{v}_{h} \in\left[P_{k}(F)\right]^{d} \forall F \in \mathcal{F}, \bar{v}_{h}=0 \text { on } \partial \Omega\right\},  \tag{2.3}\\
& \bar{Q}_{h}:=\left\{\bar{q}_{h} \in L^{2}\left(\Gamma^{0}\right): \bar{q}_{h} \in P_{k}(F) \forall F \in \mathcal{F}\right\}
\end{align*}
$$

For convenience, we introduce the spaces $\boldsymbol{V}_{h}:=V_{h} \times \bar{V}_{h}, \boldsymbol{Q}_{h}:=Q_{h} \times \bar{Q}_{h}$, and $\boldsymbol{X}_{h}:=\boldsymbol{V}_{h} \times \boldsymbol{Q}_{h}$. Function pairs in $\boldsymbol{V}_{h}$ and $\boldsymbol{Q}_{h}$ will be denoted by boldface, e.g., $\boldsymbol{v}_{h}:=\left(v_{h}, \bar{v}_{h}\right) \in \boldsymbol{V}_{h}$ and $\boldsymbol{q}_{h}:=\left(q_{h}, \bar{q}_{h}\right) \in \boldsymbol{Q}_{h}$.

For scalar-valued functions $p$ and $q$, we write

$$
\begin{equation*}
(p, q)_{\mathcal{T}}:=\sum_{K \in \mathcal{T}}(p, q)_{K}, \quad\langle p, q\rangle_{\partial \mathcal{T}}:=\sum_{K}\langle p, q\rangle_{\partial K}, \tag{2.4}
\end{equation*}
$$

where $(p, q)_{K}:=\int_{K} p q \mathrm{~d} x$ and $\langle p, q\rangle_{\partial K}:=\int_{\partial K} p q \mathrm{~d} s$. Similar inner-products hold for vector-valued functions.

We also introduce the following mesh-dependent norms:

$$
\begin{array}{rlrl}
\left\|\bar{v}_{h}\right\|_{h, 0}^{2} & :=\sum_{K \in \mathcal{T}_{h}} h_{K}\left\|\bar{v}_{h}\right\|_{\partial K}^{2} & \forall \bar{v}_{h} \in \bar{V}_{h} \\
\left\|\bar{v}_{h}\right\|_{h}^{2} & :=\sum_{K \in \mathcal{T}_{h}} h_{K}^{-1}\left\|\bar{v}_{h}-m_{K}\left(\bar{v}_{h}\right)\right\|_{\partial K}^{2} & \forall \bar{v}_{h} \in \bar{V}_{h} \\
\left\|\boldsymbol{v}_{h}\right\|_{v}^{2}:=\sum_{K \in \mathcal{T}}\left\|\nabla v_{h}\right\|_{K}^{2}+\sum_{K \in \mathcal{T}} \alpha h_{K}^{-1}\left\|\bar{v}_{h}-v_{h}\right\|_{\partial K}^{2} & \forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h} \\
\left\|\bar{q}_{h}\right\|_{p}^{2}:=\sum_{K \in \mathcal{T}} h_{K}\left\|\bar{q}_{h}\right\|_{\partial K}^{2} & \forall \bar{q}_{h} \in \bar{Q}_{h} \\
\left\|\boldsymbol{q}_{h}\right\|_{p}^{2}:=\left\|q_{h}\right\|_{\Omega}^{2}+\left\|\bar{q}_{h}\right\|_{p}^{2} & \forall \boldsymbol{q}_{h} \in \boldsymbol{Q}_{h} \tag{2.5e}
\end{array}
$$

where $h_{K}$ is the length measure of an element $K, \alpha>0$ is a constant, and $m_{K}\left(\bar{v}_{h}\right):=$ $\frac{1}{|\partial K|} \int_{\partial K} \bar{v}_{h} \mathrm{~d} s$. We furthermore have the following Poincaré-type inequality (see [8, Lemma 3.7] and [14, Proof of Theorem 2.3]) for the norms on $\bar{V}_{h}$ :

$$
\begin{equation*}
\left\|\bar{v}_{h}\right\|_{h, 0} \leq c\left\|\bar{v}_{h}\right\|_{h} \quad \forall \bar{v}_{h} \in \bar{V}_{h} \tag{2.6}
\end{equation*}
$$

We will use the following reduced version of [16, Theorem 3.1].
THEOREM 2.1. Let $U, P_{1}$, and $P_{2}$ be reflexive Banach spaces, and let $b_{1}: P_{1} \times$ $U \rightarrow \mathbb{R}$ and $b_{2}: P_{2} \times U \rightarrow \mathbb{R}$ be bilinear and continuous. Let

$$
\begin{equation*}
Z_{b_{i}}=\left\{v \in U: b_{i}\left(p_{i}, v\right)=0 \forall p_{i} \in P_{i}\right\} \subset U, \quad i=1,2 \tag{2.7}
\end{equation*}
$$

then the following are equivalent:
i. There exists $c>0$ such that

$$
\sup _{v \in U} \frac{b_{1}\left(p_{1}, v\right)+b_{2}\left(p_{2}, v\right)}{\|v\|_{U}} \geq c\left(\left\|p_{1}\right\|_{P_{1}}+\left\|p_{2}\right\|_{P_{2}}\right) \quad\left(p_{1}, p_{2}\right) \in P_{1} \times P_{2}
$$

ii. There exists $c>0$ such that

$$
\sup _{v \in U} \frac{b_{1}\left(p_{1}, v\right)}{\|v\|_{U}} \geq c\left\|p_{1}\right\|_{P_{1}}, p_{1} \in P_{1} \text { and } \sup _{v \in Z_{b_{1}}} \frac{b_{2}\left(p_{2}, v\right)}{\|v\|_{U}} \geq c\left\|p_{2}\right\|_{P_{2}}, p_{2} \in P_{2} .
$$

2.2. The HDG formulation of the Stokes problem. We consider the HDG method of $[19,29,30]$ for the Stokes problem (2.1), which reads: find $\left(\boldsymbol{u}_{h}, \boldsymbol{p}_{h}\right) \in \boldsymbol{X}_{h}$ such that

$$
\begin{equation*}
a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)+b_{h}\left(\boldsymbol{p}_{h}, v_{h}\right)+b_{h}\left(\boldsymbol{q}_{h}, u_{h}\right)=\left(v_{h}, f\right)_{\mathcal{T}} \quad \forall\left(\boldsymbol{v}_{h}, \boldsymbol{q}_{h}\right) \in \boldsymbol{X}_{h}, \tag{2.8}
\end{equation*}
$$

where

$$
\begin{align*}
a_{h}\left(\boldsymbol{w}_{h}, \boldsymbol{v}_{h}\right):= & \left(\nabla w_{h}, \nabla v_{h}\right)_{\mathcal{T}}+\left\langle\alpha h^{-1}\left(w_{h}-\bar{w}_{h}\right), v_{h}-\bar{v}_{h}\right\rangle_{\partial \mathcal{T}}  \tag{2.9a}\\
& -\left\langle w_{h}-\bar{w}_{h}, \partial_{n} v_{h}\right\rangle_{\partial \mathcal{T}}-\left\langle\partial_{n} w_{h}, v_{h}-\bar{v}_{h}\right\rangle_{\partial \mathcal{T}} \\
b_{h}\left(\boldsymbol{q}_{h}, v_{h}\right):= & -\left(q_{h}, \nabla \cdot v_{h}\right)_{\mathcal{T}}+\left\langle v_{h} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}} \tag{2.9b}
\end{align*}
$$

and where $n$ is the outward unit normal vector on $\partial K$.
The following properties of the bilinear forms will be useful when constructing a preconditioner for the statically condensed version of (2.8). For sufficiently large $\alpha$, $a_{h}(\cdot, \cdot)$ is coercive on $\boldsymbol{V}_{h}$ and bounded, i.e., there exist constants $c_{a}^{s}>0$ and $c_{a}^{b}>0$, independent of $h$, such that for all $\boldsymbol{u}_{h}, \boldsymbol{v}_{h} \in \boldsymbol{V}_{h}$,

$$
\begin{equation*}
a_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right) \geq c_{a}^{s}\left\|\boldsymbol{v}_{h}\right\|_{v}^{2} \quad \text { and } \quad\left|a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)\right| \leq c_{a}^{b}\left\|\boldsymbol{u}_{h} \mid\right\|_{v}\left\|\boldsymbol{v}_{h}\right\|_{v} \tag{2.10}
\end{equation*}
$$

see [30, Lemmas 4.2 and 4.3]. Furthermore, there exist constants $c_{b}^{b}>0$ and $\beta_{p}>0$, independent of $h$, such that for all $\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}$ and for all $\boldsymbol{q}_{h} \in \boldsymbol{Q}_{h}$,

$$
\begin{equation*}
\left|b_{h}\left(\boldsymbol{q}_{h}, v_{h}\right)\right| \leq c_{b}^{b}\| \| \boldsymbol{v}_{h}\| \|_{v}\left\|\boldsymbol{q}_{h}\right\|_{p} \quad \text { and } \quad \beta_{p}\| \| \boldsymbol{q}_{h} \|_{p} \leq \sup _{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}} \frac{b_{h}\left(\boldsymbol{q}_{h}, v_{h}\right)}{\| \| \boldsymbol{v}_{h} \|_{v}} \tag{2.11}
\end{equation*}
$$

see [30, Lemma 4.8 and Eq. 102] and [31, Lemma 1], respectively. Next we note that the velocity-pressure coupling term (2.9b) can be written as

$$
\begin{equation*}
b_{h}\left(\boldsymbol{q}_{h}, v_{h}\right):=b_{1}\left(q_{h}, v_{h}\right)+b_{2}\left(\bar{q}_{h}, v_{h}\right) \tag{2.12}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{1}\left(q_{h}, v_{h}\right):=-\left(q_{h}, \nabla \cdot v_{h}\right)_{\mathcal{T}} \quad \text { and } \quad b_{2}\left(\bar{q}_{h}, v_{h}\right):=\left\langle v_{h} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}} \tag{2.13}
\end{equation*}
$$

It follows immediately from (2.11) that

$$
\begin{equation*}
\left|b_{2}\left(\bar{q}_{h}, v_{h}\right)\right| \leq c_{b}^{b}\| \| \boldsymbol{v}_{h}\left\|_{v}\right\| \bar{q}_{h} \|_{p} . \tag{2.14}
\end{equation*}
$$

Furthermore, in [31, Lemma 3] we proved that there exists a constant $\bar{\beta}>0$, independent of $h$, such that for all $\bar{q}_{h} \in \bar{Q}_{h}$

$$
\begin{equation*}
\bar{\beta}\left\|\bar{q}_{h}\right\|_{p} \leq \sup _{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}} \frac{b_{2}\left(\bar{q}_{h}, v_{h}\right)}{\| \| \boldsymbol{v}_{h}\| \|_{v}} \tag{2.15}
\end{equation*}
$$

Stability of $b_{2}$ holds also for velocities that are divergence-free on each element $K \in \mathcal{T}$, i.e.

$$
\begin{equation*}
\bar{\beta}\left\|\bar{q}_{h}\right\|_{p} \leq \sup _{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}^{0}} \frac{b_{2}\left(\bar{q}_{h}, v_{h}\right)}{\| \| \boldsymbol{v}_{h}\| \|_{v}} \tag{2.16}
\end{equation*}
$$

where $\boldsymbol{V}_{h}^{0}:=V_{h}^{0} \times \bar{V}_{h}$ and

$$
\begin{align*}
V_{h}^{0}: & =\left\{v_{h} \in V_{h}: b_{1}\left(v_{h}, q_{h}\right)=0 \forall q_{h} \in Q_{h}\right\}  \tag{2.17}\\
& =\left\{v_{h} \in V_{h}:\left.\left(\nabla \cdot v_{h}\right)\right|_{K}=0 \forall K \in \mathcal{T}\right\} .
\end{align*}
$$

Equation (2.16) is a direct consequence of the inf-sup condition in (2.11) and Theorem 2.1. This result will play an important role in constructing a preconditioner in section 3.
2.3. The matrix formulation. To express (2.8) as a linear algebra problem, we let $u \in \mathbb{R}^{n_{u}}$ and $\bar{u} \in \mathbb{R}^{\bar{n}_{u}}$ be vectors of the discrete element and trace velocities (degrees-of-freedom) with respect to the basis for $V_{h}$ and $\bar{V}_{h}$, respectively. Similarly, we let $p \in \mathbb{R}^{n_{q}}$ and $\bar{p} \in \mathbb{R}^{\bar{n}_{q}}$ be the degrees-of-freedom associated with the basis for $Q_{h}$ and $\bar{Q}_{h}$, respectively. We define also $\mathbb{V}:=\left\{\mathbf{v}=\left[v^{T} \bar{v}^{T}\right]^{T}: v \in \mathbb{R}^{n_{u}}, \bar{v} \in \mathbb{R}^{\bar{n}_{u}}\right\}$ and $\mathbb{Q}:=\left\{\mathbf{q}=\left[q^{T} \bar{q}^{T}\right]^{T}: q \in \mathbb{R}^{n_{q}}, \bar{q} \in \mathbb{R}^{\bar{n}_{q}}\right\}$ and denote by $\mathbf{u}=\left[\begin{array}{l}u^{T} \\ \bar{u}^{T}\end{array}\right]^{T} \in \mathbb{V}$ and $\mathbf{p}=\left[p^{T} \bar{p}^{T}\right]^{T} \in \mathbb{Q}$.

Next, let $A \in \mathbb{R}^{\left(n_{u}+\bar{n}_{u}\right) \times\left(n_{u}+\bar{n}_{u}\right)}$ be the symmetric matrix defined by

$$
a_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right)=\mathbf{v}^{T} A \mathbf{v} \quad \text { where } \quad A:=\left[\begin{array}{cc}
A_{u u} & A_{\bar{u} u}^{T}  \tag{2.18}\\
A_{\bar{u} u} & A_{\bar{u} \bar{u}}
\end{array}\right]
$$

for any $\mathbf{v} \in \mathbb{V}$. Here $A_{u u}, A_{\bar{u} u}$ and $A_{\bar{u} \bar{u}}$ are the matrices obtained from the discretization of $a_{h}((\cdot, 0),(\cdot, 0)), a_{h}((\cdot, 0),(0, \cdot))$ and $a_{h}((0, \cdot),(0, \cdot))$, respectively. Similarly, let $B_{1} \in \mathbb{R}^{n_{q} \times\left(n_{u}+\bar{n}_{u}\right)}$ and $B_{2} \in \mathbb{R}^{\bar{n}_{q} \times\left(n_{u}+\bar{n}_{u}\right)}$ be the matrices defined by

$$
\begin{array}{lll}
b_{1}\left(q_{h}, v_{h}\right)=q^{T} B_{1} \mathbf{v} & \text { where } & B_{1}:=\left[\begin{array}{ll}
B_{p u} & 0
\end{array}\right] \\
b_{2}\left(\bar{q}_{h}, v_{h}\right)=\bar{q}^{T} B_{2} \mathbf{v} & \text { where } & B_{2}:=\left[\begin{array}{ll}
B_{\bar{p} u} & 0
\end{array}\right] \tag{2.20}
\end{array}
$$

for any $q \in \mathbb{R}^{n_{q}}$ and $\bar{q} \in \mathbb{R}^{\bar{n}_{q}}$. Here $B_{p u}$ and $B_{\bar{p} u}$ are the matrices obtained from the discretization of $b_{h}((\cdot, 0), \cdot)$ and $b_{h}((0, \cdot), \cdot)$, respectively. Finally we define $L_{u}$ such that

$$
\left(v_{h}, f\right)_{\mathcal{T}}=\mathbf{v}^{T} L \quad \text { where } \quad L:=\left[\begin{array}{c}
L_{u}  \tag{2.21}\\
0
\end{array}\right]
$$

We can now express (2.8) in block matrix form as

$$
\left[\begin{array}{cccc}
A_{u u} & A_{\bar{u} u}^{T} & B_{p u}^{T} & B_{\bar{p} u}^{T}  \tag{2.22}\\
A_{\bar{u} u} & A_{\bar{u} \bar{u}} & 0 & 0 \\
B_{p u} & 0 & 0 & 0 \\
B_{\bar{p} u} & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
u \\
\bar{u} \\
p \\
\bar{p}
\end{array}\right]=\left[\begin{array}{c}
L_{u} \\
0 \\
0 \\
0
\end{array}\right] .
$$

The matrices $A_{u u}$ and $B_{p u}$ in (2.22) are block diagonal (one block per cell), therefore $u$ and $p$ and can be eliminated locally via static condensation. This leads to the two-field reduced system:

$$
\underbrace{\left[\begin{array}{cc}
\bar{A}^{d} & -A_{\bar{u} u} A_{u u}^{-1} \mathcal{P}^{T} B_{\overline{\bar{p}} u}^{T}  \tag{2.23}\\
-B_{\bar{p} u} \mathcal{P} A_{u u}^{-1} A_{\bar{u} u}^{T} & -B_{\bar{p} u} \mathcal{P} A_{u u}^{-1} \mathcal{P}^{T} B_{\bar{p} u}^{T}
\end{array}\right]}_{\overline{\mathbb{A}}}\left[\begin{array}{l}
\bar{u} \\
\bar{p}
\end{array}\right]=\left[\begin{array}{l}
-A_{\bar{u} u} \mathcal{P} A_{u u}^{-1} L_{u} \\
-B_{\bar{p} u} \mathcal{P} A_{u u}^{-1} L_{u}
\end{array}\right]
$$

with

$$
\begin{equation*}
\bar{A}^{d}:=-A_{\bar{u} u} \mathcal{P} A_{u u}^{-1} A_{\bar{u} u}^{T}+A_{\bar{u} \bar{u}} \tag{2.24}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{P}:=I-\Pi, \quad \Pi:=A_{u u}^{-1} B_{p u}^{T}\left(B_{p u} A_{u u}^{-1} B_{p u}^{T}\right)^{-1} B_{p u} \tag{2.25}
\end{equation*}
$$

Note that $\bar{A}^{d}$ is symmetric, and we remark that $\mathcal{P}$ is an oblique projection matrix into the null-space of $B_{p u}$. It is important to note that $\mathcal{P}$ can be assembled elementwise and is therefore a local operator. Then, given the trace velocity $\bar{u}$ and the trace
pressure $\bar{p}$, the element velocity $u$ and pressure $p$ can be computed element-wise in a post-processing step by solving:

$$
\left[\begin{array}{cc}
A_{u u} & B_{p u}^{T}  \tag{2.26}\\
B_{p u} & 0
\end{array}\right]\left[\begin{array}{l}
u \\
p
\end{array}\right]=\left[\begin{array}{c}
L_{u}-A_{\bar{u} u}^{T} \bar{u}-B_{\bar{p} u}^{T} \bar{p} \\
0
\end{array}\right]
$$

resulting in

$$
\left[\begin{array}{l}
u  \tag{2.27}\\
p
\end{array}\right]=\left[\begin{array}{c}
\mathcal{P} A_{u u}^{-1}\left(L_{u}-A_{\bar{u} u}^{T} \bar{u}-B_{\overline{\bar{p}} u}^{T} \bar{p}\right) \\
\left(B_{p u} A_{u u}^{-1} B_{p u}^{T}\right)^{-1} B_{p u} A_{u u}^{-1} L_{u}
\end{array}\right]
$$

It is clear that once $\bar{u}$ and $\bar{p}$ are known, $u=\mathcal{P} A_{u u}^{-1}\left(L_{u}-A_{\bar{u} u}^{T} \bar{u}-B_{\bar{p} u}^{T} \bar{p}\right) \in \operatorname{Ker} B_{p u}$ exactly (up to machine precision) due to the application of $\mathcal{P}$ to the vector $A_{u u}^{-1}\left(L_{u}-\right.$ $\left.A_{\bar{u} u}^{T} \bar{u}-B_{\bar{p} u}^{T} \bar{p}\right)$. In other words, the discrete velocity $u_{h}$ is exactly divergence free on each element.
2.4. Matrix properties. The following properties of the different matrices will be useful when constructing a preconditioner for (2.23) in section 3.

Lemma 2.2. The matrix of the two-field reduced system (2.23) is invertible.
Proof. Write (2.22) as

$$
\left[\begin{array}{cccc}
A_{u u} & B_{p u}^{T} & A_{\bar{u} u}^{T} & B_{\bar{p} u}^{T}  \tag{2.28}\\
B_{p u} & 0 & 0 & 0 \\
A_{\bar{u} u} & 0 & A_{\bar{u} \bar{u}} & 0 \\
B_{\bar{p} u} & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
u \\
p \\
\bar{u} \\
\bar{p}
\end{array}\right]=\left[\begin{array}{c}
L_{u} \\
0 \\
0 \\
0
\end{array}\right]
$$

Define

$$
\mathcal{A}=\left[\begin{array}{cc}
A_{u u} & B_{p u}^{T}  \tag{2.29}\\
B_{p u} & 0
\end{array}\right], \quad \mathcal{B}=\left[\begin{array}{ll}
A_{\bar{u} u} & 0 \\
B_{\bar{p} u} & 0
\end{array}\right], \quad \mathcal{C}=\left[\begin{array}{cc}
A_{\bar{u} \bar{u}} & 0 \\
0 & 0
\end{array}\right] .
$$

We can then write the matrix in (2.28) as the following block triangular factorization:

$$
\mathbb{A}=\left[\begin{array}{cc}
\mathcal{A} & \mathcal{B}^{T}  \tag{2.30}\\
\mathcal{B} & \mathcal{C}
\end{array}\right]=\left[\begin{array}{cc}
I & 0 \\
\mathcal{B} \mathcal{A}^{-1} & I
\end{array}\right]\left[\begin{array}{cc}
\mathcal{A} & 0 \\
0 & \mathcal{S}
\end{array}\right]\left[\begin{array}{cc}
I & \mathcal{A}^{-1} \mathcal{B}^{T} \\
0 & I
\end{array}\right]
$$

where the Schur complement $\mathcal{S}=\mathcal{C}-\mathcal{B} \mathcal{A}^{-1} \mathcal{B}^{T}$ is exactly the matrix of the two-field reduced system (2.23). Since $\mathcal{A}$ is invertible ( $A_{u u}$ is positive definite and $B_{p u}$ is full rank since it satisfies the inf-sup condition), we have that $\mathbb{A}$ is nonsingular if and only if $\mathcal{S}$ is nonsingular. Since $\mathbb{A}$ is nonsingular (due to well-posedness of (2.8) [30]), the result follows.

Next, let us recall the following result from [6, Chapter A.5.5].
Proposition 2.3. Consider the following symmetric matrix:

$$
M=\left[\begin{array}{cc}
A & B^{T} \\
B & C
\end{array}\right]
$$

$M$ is positive-definite $\Leftrightarrow A$ and $C-B A^{-1} B^{T}$ are positive-definite.

This proposition is used now to prove the following lemma.
Lemma 2.4. The matrix $\bar{A}=A_{\bar{u} \bar{u}}-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}$ is symmetric-positive definite.
Proof. It is clear that $\bar{A}$ is symmetric. By Proposition 2.3 we know that $A$ in (2.18) is positive definite if and only if $A_{u u}$ and $\bar{A}$ are positive-definite. We know that $A$ is symmetric positive-definite by (2.10) so that $A_{u u}$ and $\bar{A}$ are symmetric positive-definite.

Lemma 2.5. The matrix $\bar{A}^{d}$ in (2.24) is symmetric positive-definite.
Proof. The symmetry of $\bar{A}^{d}$ is clear. Using the definition of $\mathcal{P}$ in (2.25) we have:

$$
\begin{equation*}
\bar{A}^{d}=\bar{A}+A_{\bar{u} u} \Pi A_{u u}^{-1} A_{\bar{u} u}^{T} \tag{2.31}
\end{equation*}
$$

where $\bar{A}=-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}+A_{\bar{u} \bar{u}}$. By Lemma 2.4 we know that $\bar{A}=-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}+$ $A_{\bar{u} \bar{u}}$ is symmetric positive-definite. Consider now $A_{\bar{u} u} \Pi A_{u u}^{-1} A_{\bar{u} u}^{T}$. Let $S_{p p}=B_{p u} A_{u u}^{-1} B_{p u}^{T}$ and note that $S_{p p}$ is symmetric and positive-definite (since $A_{u u}$ is symmetric positivedefinite and $B_{p u}$ is full rank). Then

$$
\begin{align*}
\left\langle A_{\bar{u} u} \Pi A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}, \bar{x}\right\rangle & =\left\langle A_{\bar{u} u} A_{u u}^{-1} B_{p u}^{T} S_{p p}^{-1} B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}, \bar{x}\right\rangle \\
& =\left\langle S_{p p}^{-1} B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}, B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}\right\rangle  \tag{2.32}\\
& =\left\langle B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}, B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}\right\rangle_{S_{p p}^{-1}} \\
& \geq 0,
\end{align*}
$$

hence $A_{\bar{u} u} \Pi A_{u u}^{-1} A_{\bar{u} u}^{T}$ is symmetric positive semidefinite and $\bar{A}^{d}$ must be positivedefinite.

Lemma 2.6. The Schur complement of the matrix of the two-field reduced system (2.23) is invertible.

Proof. The proof is the same as that of Lemma 2.2 but with

$$
\begin{equation*}
\mathcal{A}=\bar{A}^{d}, \quad \mathcal{B}=-B_{\bar{p} u} \mathcal{P} A_{u u}^{-1} A_{\bar{u} u}^{T}, \quad \mathcal{C}=-B_{\bar{p} u} \mathcal{P} A_{u u}^{-1} B_{\bar{p} u}^{T} \tag{2.33}
\end{equation*}
$$

By Lemma 2.2 we know that the matrix of the two-field reduced system (2.23) is invertible and by Lemma 2.5 we know that $\mathcal{A}$ is symmetric positive-definite. The result follows.
3. Preconditioning. We present now a provably optimal preconditioner for the condensed problem in (2.23).
3.1. Block preconditioner. We first introduce some definitions. The (negative) trace pressure Schur complement of the matrix in (2.23) is given by

$$
\begin{equation*}
\bar{S}:=B_{\bar{p} u} \mathcal{P}\left(A_{u u}^{-1}+A_{u u}^{-1} A_{\bar{u} u}^{T}\left(\bar{A}^{d}\right)^{-1} A_{\bar{u} u} A_{u u}^{-1}\right) \mathcal{P}^{T} B_{\bar{p} u}^{T} \tag{3.1}
\end{equation*}
$$

The element $M$ and trace $\bar{M}$ pressure mass matrices are defined by, respectively,

$$
\begin{equation*}
\left\|q_{h}\right\|_{\Omega}^{2}=q^{T} M q, \quad\left\|\bar{q}_{h}\right\|_{p}^{2}=\bar{q}^{T} \bar{M} \bar{q} \tag{3.2}
\end{equation*}
$$

We will also require the following matrix

$$
A_{\mathcal{P}}=\left[\begin{array}{cc}
A_{u u} & \mathcal{P}^{T} A_{\bar{u} u}^{T}  \tag{3.3}\\
A_{\bar{u} u} \mathcal{P} & A_{\bar{u} \bar{u}}
\end{array}\right] .
$$

Next, let $\mathbb{V}^{0}:=\left\{\mathbf{v} \in \mathbb{V}: v \in \operatorname{Ker} B_{p u}\right\}$. It is then easy to show, using that $\mathcal{P} v=v$ for $\mathbf{v} \in \mathbb{V}^{0}$, that

$$
\begin{equation*}
\langle A \mathbf{v}, \mathbf{v}\rangle=\left\langle A_{\mathcal{P}} \mathbf{v}, \mathbf{v}\right\rangle \quad \forall \mathbf{v} \in \mathbb{V}^{0} \tag{3.4}
\end{equation*}
$$

Lemma 3.1. The matrix $A_{\mathcal{P}}$ in (3.3) is symmetric positive-definite.
Proof. We first note that $A_{u u}$ is symmetric and positive-definite. Second, we note that the Schur complement of $A_{\mathcal{P}}$ is $\bar{A}^{d}$ which, by Lemma 2.5 is symmetric positive-definite. The result follows by Proposition 2.3.

In the following two lemmas we show that $\bar{S}$ is spectrally equivalent to $\bar{M}$ and $B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}$.

Lemma 3.2. Let $\bar{S}$ be the matrix defined in (3.1) and let $\bar{M}$ be the trace pressure mass matrix defined in (3.2). Let $c_{a}^{s}$ and $c_{a}^{b}$ be the constants given in (2.10) and let $c_{b}^{b}$ and $\bar{\beta}$ be the constants given in (2.14) and (2.15), respectively. The following holds:

$$
\begin{equation*}
\frac{\bar{\beta}^{2}}{c_{a}^{b}} \leq \frac{\bar{q}^{T} \bar{S} \bar{q}}{\bar{q}^{T} \bar{M} \bar{q}} \leq \frac{\left(c_{b}^{b}\right)^{2}}{c_{a}^{s}} \tag{3.5}
\end{equation*}
$$

Proof. Stability of $b_{2}\left(\right.$ see (2.16)) and equivalence of $a_{h}$ with $\| \| \cdot \mid \|_{v}$ in $\boldsymbol{V}_{h}$ (see (2.10)) imply

$$
\begin{equation*}
\frac{\bar{\beta}}{\sqrt{c_{a}^{b}}} \leq \min _{\substack{\bar{q}_{h} \in \mathbb{R}^{n_{q}} \\ \bar{q}_{h} \neq 0}} \max _{\boldsymbol{v}_{h} \in \boldsymbol{V}_{h}^{0}} \frac{b_{2}\left(\bar{q}_{h}, v_{h}\right)}{a_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right)^{1 / 2}\left\|\bar{q}_{h}\right\|_{p}} \tag{3.6}
\end{equation*}
$$

We may write (3.6) in matrix notation as:

$$
\begin{equation*}
\frac{\bar{\beta}}{\sqrt{c_{a}^{b}}} \leq \min _{\substack{\bar{q}_{h} \in \mathbb{R}^{\bar{n}} \\ \bar{q}_{h} \neq 0}} \max _{\mathbf{v} \in \mathbb{V}^{0}} \frac{\left\langle\bar{q}, B_{2} \mathbf{v}\right\rangle}{\langle A \mathbf{v}, \mathbf{v}\rangle^{1 / 2}\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \tag{3.7}
\end{equation*}
$$

Let $\mathcal{R}$ be a $2 \times 2$ block diagonal matrix with in the top left block $\mathcal{P}$ and the bottom right block the identity matrix $\bar{I}_{u} \in \mathbb{R}^{\bar{n}_{u} \times \bar{n}_{u}}$. Using (3.4) and the property $\left\langle\bar{q}, B_{2} \mathbf{v}\right\rangle=$ $\left\langle\mathcal{R}^{T} B_{2}^{T} \bar{q}, \mathbf{v}\right\rangle$ for $\mathbf{v} \in \mathbb{V}^{0}$, following similar steps as in [11, Chapter 3],

$$
\begin{align*}
\frac{\bar{\beta}}{\sqrt{c_{a}^{b}}} & \leq \min _{\substack{\bar{q}_{h} \in \mathbb{R}^{n_{q}} \\
\bar{q}_{h} \neq 0}} \frac{1}{\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \max _{\mathbf{v} \in \mathbb{V}^{0}} \frac{\left\langle\mathcal{R}^{T} B_{2}^{T} \bar{q}, \mathbf{v}\right\rangle}{\left\langle A_{\mathcal{P}} \mathbf{v}, \mathbf{v}\right\rangle^{1 / 2}} \\
& \leq \min _{\substack{\bar{q}_{h} \in \mathbb{R}^{n_{q}} \\
\bar{q}_{h} \neq 0}} \frac{1}{\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \max _{\mathbf{v} \in \mathbb{V}} \frac{\left\langle\mathcal{R}^{T} B_{2}^{T} \bar{q}, \mathbf{v}\right\rangle}{\left\langle A_{\mathcal{P}} \mathbf{v}, \mathbf{v}\right\rangle^{1 / 2}} \\
& =\min _{\substack{\bar{q}_{h} \in \mathbb{R}^{\bar{n}_{q}} \\
\bar{q}_{h} \neq 0}} \frac{1}{\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \max _{\mathbf{v} \in \mathbb{V}} \frac{\left\langle\mathcal{R}^{T} B_{2}^{T} \bar{q}, A_{\mathcal{P}}^{-1 / 2} A_{\mathcal{P}}^{1 / 2} \mathbf{v}\right\rangle}{\left\langle A_{\mathcal{P}}^{1 / 2} \mathbf{v}, A_{\mathcal{P}}^{1 / 2} \mathbf{v}\right\rangle^{1 / 2}}  \tag{3.8}\\
& =\min _{\bar{q}_{h} \in \mathbb{R}^{\bar{q}_{q}}{ }_{\substack{\bar{q}_{h}} 0} \frac{1}{\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \max _{\mathbf{w}=A_{\mathcal{P}}^{1 / 2} \mathbf{v}, \mathbf{v} \in \mathbb{V}} \frac{\left\langle\mathcal{R}^{T} B_{2}^{T} \bar{q}, A_{\mathcal{P}}^{-1 / 2} \mathbf{w}\right\rangle}{\langle\mathbf{w}, \mathbf{w}\rangle^{1 / 2}}} \\
& =\min _{\substack{\bar{q}_{h} \in \mathbb{R}^{n_{q}} \\
\bar{q}_{h} \neq 0}} \frac{1}{\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \max _{\mathbf{w} \neq 0} \frac{\left\langle A_{\mathcal{P}}^{-1 / 2} \mathcal{R}^{T} B_{2}^{T} \bar{q}, \mathbf{w}\right\rangle}{\langle\mathbf{w}, \mathbf{w}\rangle^{1 / 2}} .
\end{align*}
$$

For a given $\bar{q}$, the maximum is reached for $\mathbf{w}=A_{\mathcal{P}}^{-1 / 2} \mathcal{R}^{T} B_{2}^{T} \bar{q}$, hence

$$
\begin{equation*}
\frac{\bar{\beta}}{\sqrt{c_{a}^{b}}} \leq \min _{\substack{\bar{q}_{h} \in \mathbb{R}^{\bar{n}} q \\ \bar{q}_{h} \neq 0}} \frac{\left\langle B_{2} \mathcal{R} A_{\mathcal{P}}^{-1} \mathcal{R}^{T} B_{2}^{T} \bar{q}, \bar{q}\right\rangle^{1 / 2}}{\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}} \tag{3.9}
\end{equation*}
$$

By direct computation, and using that $\mathcal{P}^{2}=\mathcal{P}$ and $\mathcal{P} A_{u u}^{-1}=A_{u u}^{-1} \mathcal{P}^{T}$, we note that $B_{2} \mathcal{R} A_{\mathcal{P}}^{-1} \mathcal{R}^{T} B_{2}^{T}=B_{\bar{p} u} \mathcal{P}\left(A_{\mathcal{P}}^{-1}\right)_{11} \mathcal{P}^{T} B_{\bar{p} u}^{T}=\bar{S}$, proving the lower bound in (3.5).

For the upper bound, from stability and (2.10) and boundedness (2.14) of $a_{h}$ on $\boldsymbol{V}_{h}$,

$$
\begin{equation*}
b_{2}\left(\bar{q}_{h}, v_{h}\right) \leq c_{b}^{b}\left\|\boldsymbol{v}_{h}\right\|\left\|_{v}\right\| \bar{q}_{h}\left\|_{p} \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}} a_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right)^{1 / 2}\right\| \bar{q}_{h} \|_{p} \tag{3.10}
\end{equation*}
$$

In matrix form this reads as

$$
\begin{equation*}
\left\langle\bar{q}, B_{2} \mathbf{v}\right\rangle \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}}\langle A \mathbf{v}, \mathbf{v}\rangle^{1 / 2}\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}, \quad \forall \mathbf{v} \in \mathbb{V} \tag{3.11}
\end{equation*}
$$

Set $\mathbf{v}=\mathcal{R} \mathbf{w}$ for $\mathbf{w} \in \mathbb{V}$. Then, using (3.4) and since $\mathcal{R} \mathbf{w} \in \mathbb{V}^{0}$, we find

$$
\begin{equation*}
\left\langle\bar{q}, B_{2} \mathcal{R} \mathbf{w}\right\rangle \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}}\langle A \mathcal{R} \mathbf{w}, \mathcal{R} \mathbf{w}\rangle^{1 / 2}\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}=\frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}}\left\langle A_{\mathcal{P}} \mathcal{R} \mathbf{w}, \mathcal{R} \mathbf{w}\right\rangle^{1 / 2}\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}, \tag{3.12}
\end{equation*}
$$

which holds for all $\mathbf{w} \in \mathbb{V}$. We next show that $\left\langle A_{\mathcal{P}} \mathcal{R} \mathbf{w}, \mathcal{R} \mathbf{w}\right\rangle \leq\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle$ for all $\mathbf{w} \in \mathbb{V}$. By definition and using that $\mathcal{P}^{T}=A_{u u} \mathcal{P} A_{u u}^{-1}$,

$$
\begin{align*}
\left\langle A_{\mathcal{P}} \mathcal{R} \mathbf{w}, \mathcal{R} \mathbf{w}\right\rangle & =\left[\begin{array}{ll}
w^{T} \mathcal{P}^{T} & \bar{w}^{T}
\end{array}\right]\left[\begin{array}{cc}
A_{u u} & \mathcal{P}^{T} A_{\bar{u} u}^{T} \\
A_{\bar{u} u} \mathcal{P} & A_{\bar{u} \bar{u}}
\end{array}\right]\left[\begin{array}{c}
\mathcal{P} w \\
\bar{w}
\end{array}\right] \\
& =w^{T} \mathcal{P}^{T} A_{u u} \mathcal{P} w+\bar{w}^{T} A_{\bar{u} u} \mathcal{P} w+(\mathcal{P} w)^{T} A_{\bar{u} u} \bar{w}+\bar{w}^{T} A_{\bar{u} \bar{u}} \bar{w} \\
& =w^{T} A_{u u} \mathcal{P} w+\bar{w}^{T} A_{\bar{u} u} \mathcal{P} w+(\mathcal{P} w)^{T} A_{\bar{u} u} \bar{w}+\bar{w}^{T} A_{\bar{u} \bar{u}} \bar{w} \\
& =w^{T} A_{u u}(I-\Pi) w+\bar{w}^{T} A_{\bar{u} u} \mathcal{P} w+(\mathcal{P} w)^{T} A_{\bar{u} u} \bar{w}+\bar{w}^{T} A_{\bar{u} \bar{u}} \bar{w}  \tag{3.13}\\
& =\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle-\left\langle A_{u u} \Pi w, w\right\rangle \\
& =\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle-\left\langle B_{p u}^{T} S_{p p}^{-1} B_{p u} w, w\right\rangle \\
& =\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle-\left\langle B_{p u} w, B_{p u} w\right\rangle_{S_{p p}^{-1}} \\
& \leq\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle
\end{align*}
$$

where we used that $S_{p p}=B_{p u} A_{u u}^{-1} B_{p u}^{T}$ is symmetric and positive-definite. Combining (3.12) and (3.13) we obtain

$$
\begin{equation*}
\left\langle\bar{q}, B_{2} \mathcal{R} \mathbf{w}\right\rangle \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}}\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle^{1 / 2}\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2} \quad \forall \mathbf{w} \in \mathbb{V} \tag{3.14}
\end{equation*}
$$

The result follows after dividing both sides by $\left\langle A_{\mathcal{P}} \mathbf{w}, \mathbf{w}\right\rangle^{1 / 2}\langle\bar{M} \bar{q}, \bar{q}\rangle^{1 / 2}$ and following similar steps as used to obtain the lower bound.

Lemma 3.3. Let $\bar{S}$ be the matrix defined in (3.1) and let $A_{u u}$ and $B_{\bar{p} u}$ be matrices defined in, respectively, (2.18) and (2.20). Then $\bar{S}$ is spectrally equivalent to $B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}$.

Proof. We first remark that a result of [31, Theorem 3] is that $B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}$ is spectrally equivalent to the trace pressure mass matrix $\bar{M}$ defined in (3.2). Since $\bar{M}$ is spectrally equivalent to $\bar{S}$ by Lemma 3.2 , the result follows.

We now formulate a preconditioner for the statically condensed linear system in (2.23) using Lemmas 3.2 and 3.3.

THEOREM 3.4 (A preconditioner for the statically condensed discrete Stokes problem). Let $\overline{\mathbb{A}}$ be the matrix given in (2.23), let $\bar{M}$ be the trace pressure mass matrix defined in (3.2), let $A_{u u}$ and $B_{\bar{p} u}$ be matrices defined in, respectively, (2.18) and (2.20), and let $\bar{A}^{d}$ be the matrix defined in (2.24). Let $\bar{R}^{d}$ be an operator that is spectrally equivalent to $\bar{A}^{d}$ and define

$$
\mathbb{P}_{\bar{M}}^{-1}=\left[\begin{array}{cc}
\left(\bar{R}^{d}\right)^{-1} & 0  \tag{3.15}\\
0 & \bar{M}^{-1}
\end{array}\right], \quad \mathbb{P}_{B A B}^{-1}=\left[\begin{array}{cc}
\left(\bar{R}^{d}\right)^{-1} & 0 \\
0 & \left(B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}\right)^{-1}
\end{array}\right]
$$

There exist positive constants $C_{1}, C_{2}, C_{3}, C_{4}$, independent of $h$, such that eigenvalues of both $\mathbb{P}_{\bar{M}}^{-1} \overline{\mathbb{A}}$ and $\mathbb{P}_{B A B}^{-1}{ }^{\overline{\mathbb{A}}}$ satisfy $\lambda \in\left[-C_{1},-C_{2}\right] \cup\left[C_{3}, C_{4}\right]$.

Proof. By Lemmas 3.2 and 3.3 we know that $\bar{M}$ and $B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}$ are spectrally equivalent to the pressure Schur complement $\bar{S}$ of $\overline{\mathbb{A}}$. Furthermore, $\bar{S}$ is invertible (see Lemma 2.6) and $\bar{A}^{d}$ is symmetric positive-definite (see Lemma 2.5). The result then follows by direct application of [28, Theorem 5.2].

A consequence of Theorem 3.4 is that the number of MINRES iterations needed to solve (2.23) when preconditioned by $\mathbb{P}_{\bar{M}}^{-1}$ or by $\mathbb{P}_{B A B}^{-1}$ to a given tolerance will be independent of the size of the discrete problem. In other words, $\mathbb{P}_{\bar{M}}^{-1}$ and $\mathbb{P}_{B A B}^{-1}$ are optimal preconditioners for (2.23).
3.2. Choosing $\left(\bar{R}^{d}\right)^{-1}$. The preconditioners in (3.15) require an operator $\bar{R}^{d}$ that is spectrally equivalent to $\bar{A}^{d}$. For notational purposes, let us denote by $(\cdot)^{M G}$ the application of multigrid to approximate the inverse of $(\cdot)$. In this section we then discuss the following three choices: $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{-1},\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{M G}$, and $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$ where $\bar{A}_{\gamma}$ will be defined to approximate $\bar{A}^{d}$.

The optimal choice for $\left(\bar{R}^{d}\right)^{-1}$ is $\left(\bar{A}^{d}\right)^{-1}$. However, computing the inverse (action) of $\bar{A}^{d}$ is too costly in practice. The usual choice for Stokes solvers is to replace the inverse of the discrete vector Laplacian by a V-cycle multigrid method. However, as we show in the following two lemmas, standard multigrid methods designed for $H^{1}$-like operators are not guaranteed to perform well on $\bar{A}^{d}$.

In what follows, we require the following definition of an orthogonal subspace $Y^{\perp}$ of a linear subspace $Y$ of $\mathbb{R}^{n}: Y^{\perp}:=\left\{x \in \mathbb{R}^{n}: x^{T} y=0 \forall y \in Y\right\}$.

Lemma 3.5. Let $\mathcal{P}$ be given by (2.25). Then

$$
\begin{align*}
& \operatorname{Ker} \mathcal{P}^{T}=\operatorname{Im} B_{p u}^{T},  \tag{3.16a}\\
& \operatorname{Ker} \Pi^{T}=\operatorname{Ker}\left(B_{p u} A_{u u}^{-1}\right) . \tag{3.16b}
\end{align*}
$$

Proof. We first show (3.16a). We know that $\mathcal{P}$ is an oblique projector onto Ker $B_{p u}=\left(\operatorname{Im} B_{p u}^{T}\right)^{\perp}$ (the equality is by [5, Theorem 3.1.1]), i.e., $\operatorname{Im} \mathcal{P}=\left(\operatorname{Im} B_{p u}^{T}\right)^{\perp}$. Then, again using [5, Theorem 3.1.1],

$$
\begin{equation*}
\text { Ker } \mathcal{P}^{T}=(\operatorname{Im} \mathcal{P})^{\perp}=\operatorname{Im} B_{p u}^{T} \tag{3.17}
\end{equation*}
$$

The proof for (3.16b) is similar and is given here for completeness. First, note that $\operatorname{Im} \Pi=\operatorname{Im}\left(A_{u u}^{-1} B_{p u}^{T}\right)=\left(\operatorname{Ker}\left(B_{p u} A_{u u}^{-1}\right)\right)^{\perp}$ (see, e.g., [3, Page 19]). Then, using
[5, Theorem 3.1.1],

$$
\begin{equation*}
\operatorname{Ker} \Pi^{T}=(\operatorname{Im} \Pi)^{\perp}=\operatorname{Ker}\left(B_{p u} A_{u u}^{-1}\right) . \tag{3.18}
\end{equation*}
$$

Lemma 3.6. For $\bar{x} \in \mathbb{R}^{\bar{n}_{u}}$ such that $A_{\bar{u} u}^{T} \bar{x} \in \operatorname{Ker}\left(B_{p u} A_{u u}^{-1}\right)$ the following holds:

$$
\begin{equation*}
c_{1}\left\|\mid \bar{x}_{h}\right\|_{h}^{2} \leq\left\langle\bar{A}^{d} \bar{x}, \bar{x}\right\rangle=\langle\bar{A} \bar{x}, \bar{x}\rangle \leq c_{2}\left\|\bar{x}_{h}\right\|_{h}^{2} \tag{3.19}
\end{equation*}
$$

If $\bar{x} \in \mathbb{R}^{\bar{n}_{u}}$ is such that $A_{\bar{u} u}^{T} \bar{x} \in \operatorname{Im} B_{p u}^{T}=\left(\operatorname{Ker} B_{p u}\right)^{\perp}$, then

$$
\begin{equation*}
c_{1}\left\|\bar{x}_{h}\right\|_{h}^{2} \leq\left\langle\bar{A}^{d} \bar{x}, \bar{x}\right\rangle \leq c_{2} h^{-2}\| \| \bar{x}_{h} \|_{h}^{2} \tag{3.20}
\end{equation*}
$$

Proof. We first note that the lower bound is always true because (see the proof of Lemma 2.5)
(3.21) $c_{1}\left\|\bar{x}_{h}\right\| \|_{h}^{2} \leq\langle\bar{A} \bar{x}, \bar{x}\rangle \leq\langle\bar{A} \bar{x}, \bar{x}\rangle+\left\langle A_{\bar{u} u} \Pi A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}, \bar{x}\right\rangle=\langle\bar{A} d \bar{x}, \bar{x}\rangle \quad \forall \bar{x} \in \mathbb{R}^{\bar{n}_{u}}$,
where the first inequality is by [31, Lemma 5].
To prove the upper bound in (3.19) we note that if $\bar{x}$ is such that $A_{\bar{u} u}^{T} \bar{x} \in$ $\operatorname{Ker}\left(B_{p u} A_{u u}^{-1}\right)$ then by Lemma 3.5

$$
\begin{equation*}
\left\langle A_{\bar{u} u} \Pi A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}, \bar{x}\right\rangle=\left\langle A_{\bar{u} u} A_{u u}^{-1} \Pi^{T} A_{\bar{u} u}^{T} \bar{x}, \bar{x}\right\rangle=0 \tag{3.22}
\end{equation*}
$$

where we used that $\Pi A_{u u}^{-1}=A_{u u}^{-1} \Pi^{T}$. It follows that

$$
\begin{align*}
\left\langle\bar{A}^{d} \bar{x}, \bar{x}\right\rangle & =\left\langle\left(A_{\bar{u} \bar{u}}-A_{\bar{u} u} \mathcal{P} A_{u u}^{-1} A_{\bar{u} u}^{T}\right) \bar{x}, \bar{x}\right\rangle=\left\langle\left(A_{\bar{u} \bar{u}}-A_{\bar{u} u}(I-\Pi) A_{u u}^{-1} A_{\bar{u} u}^{T}\right) \bar{x}, \bar{x}\right\rangle  \tag{3.23}\\
& =\left\langle\left(A_{\bar{u} \bar{u}}-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}\right) \bar{x}, \bar{x}\right\rangle=\langle\bar{A} \bar{x}, \bar{x}\rangle .
\end{align*}
$$

The result (3.19) then follows from [31, Lemma 5].
Next we prove the upper bound in (3.20). If $\bar{x}$ is such that $A_{\bar{u} u}^{T} \bar{x} \in \operatorname{Im} B_{p u}^{T}$, then $\mathcal{P} A_{u u}^{-1} A_{\bar{u} u}^{T} \bar{x}=\mathcal{P} A_{u u}^{-1} \mathcal{P}^{T} A_{\bar{u} u}^{T} \bar{x}=0$ by Lemma 3.5. Therefore,

$$
\begin{equation*}
\left\langle\bar{A}^{d} \bar{x}, \bar{x}\right\rangle=\left\langle A_{\bar{u} \bar{u}} \bar{x}, \bar{x}\right\rangle . \tag{3.24}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\left\langle A_{\bar{u} \bar{u}} \bar{x}, \bar{x}\right\rangle & =a_{h}\left(\left(0, \bar{x}_{h}\right),\left(0, \bar{x}_{h}\right)\right) \\
& =\alpha \sum_{K \in \partial T} h_{K}^{-1} \int_{\partial K} \bar{x}_{h} \cdot \bar{x}_{h} \mathrm{~d} s \\
& =\alpha \sum_{K \in \partial T} h_{K}^{-2}\left(h_{K} \int_{\partial K} \bar{x}_{h} \cdot \bar{x}_{h} \mathrm{~d} s\right)  \tag{3.25}\\
& \leq c h^{-2}\left\|\bar{x}_{h}\right\|_{h, 0}^{2}
\end{align*}
$$

The result follows after applying (2.6).
Lemma 3.6 implies that if $\bar{x} \in \mathbb{R}^{\bar{n}_{u}}$ is such that $A_{\bar{u} u}^{T} \bar{x} \in \operatorname{Ker}\left(B_{p u} A_{u u}^{-1}\right)$, then standard multigrid designed for $H^{1}$-like operators will be a good preconditioner for $\bar{A}^{d}$. However, if $\bar{x} \in \mathbb{R}^{\bar{n}_{u}}$ is such that $A_{\bar{u} u}^{T} \bar{x} \in \operatorname{Im} B_{p u}^{T}$, we cannot conclude anything about multigrid convergence. In fact, in numerical examples (see section 4) we see deterioration in performance of the solver when the mesh is refined if we choose $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{M G}$. The upper bound in (3.20) explains this deterioration. The
main conclusion we can draw from Lemma 3.6 then is that we should not choose $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{M G}$ with multigrid methods designed for $H^{1}$-like operators. In what follows we introduce a new approximation to $\bar{A}^{d}$.

We propose to set $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$ where

$$
\begin{equation*}
\bar{A}_{\gamma}=-A_{\bar{u} u} \widehat{A}_{\gamma}^{-1} A_{\bar{u} u}^{T}+A_{\bar{u} \bar{u}} \tag{3.26}
\end{equation*}
$$

and where,

$$
\begin{align*}
\widehat{A}_{\gamma}^{-1} & =\left(A_{u u}+\gamma B_{p u}^{T} M^{-1} B_{p u}\right)^{-1} \\
& =A_{u u}^{-1}-A_{u u}^{-1} B_{p u}^{T}\left(\gamma^{-1} M+B_{p u} A_{u u}^{-1} B_{p u}^{T}\right)^{-1} B_{p u} A_{u u}^{-1}, \tag{3.27}
\end{align*}
$$

where the second equality is by the Woodbury matrix identity which holds for $0<\gamma<$ $\infty$. The $\gamma B_{p u}^{T} M^{-1} B_{p u}$ term is often added to $A_{u u}$ in the discretization of the linearized Navier-Stokes equations to construct augmented Lagrangian preconditioners (see, for example, [4]). The $\gamma B_{p u}^{T} M^{-1} B_{p u}$ term is also similar to grad-div stabilization and can be added to discretizations of incompressible flows to improve mass conservation in the case that $u \in \operatorname{Ker} B_{p u}$ does not exactly imply $\nabla \cdot u_{h}=0$. For example, it was shown in [24] that the solution to the grad-div stabilized Taylor-Hood discretization of the Navier-Stokes equations converges to the Scott-Vogelius solution as $\gamma \rightarrow \infty$. Here we note that $\bar{A}_{\gamma} \rightarrow \bar{A}^{d}$ when $\gamma \rightarrow \infty$. We now show, for finite $\gamma$, that $\bar{A}_{\gamma}$ is equivalent to an $H^{1}$-operator.

Lemma 3.7. Let $\bar{A}_{\gamma}$ be as defined in (3.26) and let $0<\gamma<\infty$. Then $\bar{A}_{\gamma}$ is equivalent to an $H^{1}$-operator.

Proof. Consider the problem: find $\boldsymbol{u}_{h} \in \boldsymbol{V}_{h}$ such that

$$
\begin{equation*}
a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)+\gamma\left(\nabla \cdot u_{h}, \nabla \cdot v_{h}\right)_{\mathcal{T}}=\left(v_{h}, f\right)_{\mathcal{T}} \quad \forall \boldsymbol{v}_{h} \in \boldsymbol{V}_{h} \tag{3.28}
\end{equation*}
$$

Since $\gamma B_{p u}^{T} M^{-1} B_{p u}$ is the matrix representation of $\gamma\left(\nabla \cdot u_{h}, \nabla \cdot v_{h}\right)_{\mathcal{T}}$ (see [12]), the matrix form of (3.28) is given by

$$
\left[\begin{array}{cc}
A_{u u}+\gamma B_{p u}^{T} M^{-1} B_{p u} & A_{\bar{u} u}^{T}  \tag{3.29}\\
A_{\bar{u} u} & A_{\bar{u} \bar{u}}
\end{array}\right]\left[\begin{array}{l}
u \\
\bar{u}
\end{array}\right]=\left[\begin{array}{c}
L_{u} \\
0
\end{array}\right] .
$$

Eliminating $u$ from (3.29) we find the following problem for $\bar{u}$ :

$$
\begin{equation*}
\bar{A}_{\gamma} \bar{u}=\bar{F}_{\gamma}, \tag{3.30}
\end{equation*}
$$

where $\bar{F}_{\gamma}=-A_{\bar{u} u}\left(A_{u u}+\gamma B_{p u}^{T} M^{-1} B_{p u}\right)^{-1} F$.
Let us now eliminate $u_{h}$ from (3.28). For this, let $s \in\left[L^{2}(\Omega)\right]^{d}$ and $\bar{m}_{h} \in \bar{V}_{h}$ be given and define on an element $K$

$$
\begin{align*}
& a_{K}\left(v_{h}, w_{h}\right):=\left(\nabla v_{h}, \nabla w_{h}\right)_{K}+\gamma\left(\nabla \cdot v_{h}, \nabla \cdot w_{h}\right)_{K}  \tag{3.31}\\
&-\left\langle\partial_{n} v_{h}, w_{h}\right\rangle_{\partial K}-\left\langle v_{h}, \partial_{n} w_{h}\right\rangle_{\partial K}+\left\langle\alpha h^{-1} w_{h}, v_{h}\right\rangle_{\partial K}
\end{align*}
$$

and

$$
\begin{equation*}
L_{K}\left(w_{h}\right):=\left(s, w_{h}\right)_{K}-\left\langle\partial_{n} w_{h}, \bar{m}_{h}\right\rangle_{\partial K}+\left\langle\alpha h^{-1} w_{h}, \bar{m}_{h}\right\rangle_{\partial K} \tag{3.32}
\end{equation*}
$$

Let $v_{h}^{L}\left(\bar{m}_{h}, s\right) \in V_{h}$ be the function such that its restriction to element $K$ satisfies the following local problem: given $s \in\left[L^{2}(\Omega)\right]^{d}$ and $\bar{m}_{h} \in \bar{V}_{h}$,

$$
\begin{equation*}
a_{K}\left(v_{h}^{L}, w_{h}\right)=L_{K}\left(w_{h}\right) \quad \forall w_{h} \in\left[P_{k}(K)\right]^{d} \tag{3.33}
\end{equation*}
$$

Suppose now that $\boldsymbol{u}_{h} \in \boldsymbol{V}_{h}$ satisfies (3.28) and that $f \in\left[L^{2}(\Omega)\right]^{d}$. Define $l\left(\bar{u}_{h}\right):=$ $v_{h}^{L}\left(\bar{u}_{h}, 0\right)$ and $u_{h}^{f}:=v_{h}^{L}(0, f)$. Then $u_{h}=u_{h}^{f}+l\left(\bar{u}_{h}\right)$ where $\bar{u}_{h} \in \bar{V}_{h}$ satisfies

$$
\begin{equation*}
a_{h}\left(\left(l\left(\bar{u}_{h}\right), \bar{u}_{h}\right),\left(l\left(\bar{w}_{h}\right), \bar{w}_{h}\right)\right)+\gamma\left(\nabla \cdot l\left(\bar{u}_{h}\right), \nabla \cdot l\left(\bar{w}_{h}\right)\right)=\left(f, l\left(\bar{w}_{h}\right)\right)_{\mathcal{T}} \quad \forall \bar{w}_{h} \in \bar{V}_{h} . \tag{3.34}
\end{equation*}
$$

The steps to show that (3.34) holds are identical to the steps in the proof of [31, Lemma 4], so are omitted here. We next remark that (3.30) is the matrix formulation of (3.34). Let us define

$$
\begin{equation*}
\bar{a}_{h}\left(\bar{u}_{h}, \bar{w}_{h}\right):=a_{h}\left(\left(l\left(\bar{u}_{h}\right), \bar{u}_{h}\right),\left(l\left(\bar{w}_{h}\right), \bar{w}_{h}\right)\right)+\gamma\left(\nabla \cdot l\left(\bar{u}_{h}\right), \nabla \cdot l\left(\bar{w}_{h}\right)\right) \tag{3.35}
\end{equation*}
$$

then $\bar{A}_{\gamma}$ is the matrix obtained from the discretization of $\bar{a}_{h}\left(\bar{u}_{h}, \bar{w}_{h}\right)$. To conclude this proof we need to show that $\bar{a}_{h}(\cdot, \cdot)$ is equivalent to $\|\mid \cdot\|_{h}^{2}$.

Since $\sum_{K \in \mathcal{T}}\left\|\nabla \cdot v_{h}\right\|_{K} \leq\left\|\boldsymbol{v}_{h}\right\|_{v}$ it is clear from (2.10) that

$$
\begin{equation*}
c_{a}^{s}\left\|\boldsymbol{v}_{h}\right\|\left\|_{v}^{2} \leq a_{h}\left(\boldsymbol{v}_{h}, \boldsymbol{v}_{h}\right)+\gamma \sum_{K \in \mathcal{T}}\right\| \nabla \cdot v_{h}\left\|_{K} \leq\left(c_{a}^{b}+\gamma\right)\right\|\left\|\boldsymbol{v}_{h}\right\|_{v} . \tag{3.36}
\end{equation*}
$$

Following now identical steps as in the proof of [31, Lemma 5] (we omit these steps), there exist positive constants $C_{1}$ and $C_{2}$ independent of $h_{K}$ such that

$$
\begin{equation*}
C_{1}\left\|\bar{w}_{h}\right\|_{h}^{2} \leq \bar{a}_{h}\left(\bar{w}_{h}, \bar{w}_{h}\right) \leq C_{2}(1+\gamma)\left\|\bar{w}_{h}\right\|_{h}^{2} \tag{3.37}
\end{equation*}
$$

so that the result follows.
Lemma 3.7 shows that $\bar{A}_{\gamma}$ is equivalent to an $H^{1}$-operator for finite values of $\gamma$, hence, standard algebraic multigrid is expected to be effective on $\bar{A}_{\gamma}$. Lemma 3.7 also shows that the larger we choose $\gamma$, i.e., the better $\bar{A}_{\gamma}$ approximates $\bar{A}^{d}$, the weaker the equivalence between $\bar{A}_{\gamma}$ and the $H^{1}$-operator. We therefore expect multigrid to be less effective for large values of $\gamma$. In our numerical simulations we therefore choose small values for this parameter.
4. Numerical examples. We now examine the performance of MINRES combined with the preconditioners $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ introduced in Theorem 3.4. All examples in this section have been implemented in MFEM [10] and we use the PETSc [1, 2] implementation of MINRES.

In the implementation of the application of $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$, unless specified differently, we use a direct solver to compute $\bar{M}^{-1}$ and $\left(B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}\right)^{-1}$. Furthermore, we consider different choices for $\left(\bar{R}^{d}\right)^{-1}$ as will be discussed below. When we apply multigrid, we use classical algebraic multigrid (four multigrid V-cycles) with one application of a symmetric Gauss-Seidel smoother (pre and post) via the BoomerAMG library [15]. In all examples the MINRES iterations are terminated once the relative true residual reaches a tolerance of $10^{-8}$. We consider unstructured simplicial meshes in two and three dimensions.

Let $k$ be the polynomial degree in our function space $\boldsymbol{X}_{h}$. In two dimensions we set the penalty parameter $\alpha$ to $\alpha=4 k^{2}$ while it is set to $\alpha=6 k^{2}$ in three dimensions. We compare the performance of $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ also to the performance of $\mathbb{P}_{3 \times 3}$, a preconditioner we presented previously in [31]. Where $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ are preconditioners for the two-field reduced system (2.23) in which both $u$ and $p$ have been eliminated from $(2.22), \mathbb{P}_{3 \times 3}$ is a preconditioner for the three-field reduced system (A.1) in which only $u$ has been eliminated from (2.22). For convenience, we summarize the $3 \times 3$ preconditioner in Appendix A.
4.1. Optimality assessment. We examine the number of required MINRES iterations with mesh refinement for the two-dimensional lid-driven cavity problem for polynomial degrees $k=2, k=3$ and $k=4$. In particular, we consider the square domain $\Omega:=[-1,1]^{2}$ and impose the Dirichlet boundary condition $u=\left(1-x_{1}^{4}, 0\right)$ on the boundary with $x_{2}=1$, and $u=0$ on the remaining boundaries.

We first compare the performance of $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ when making the following choices for $\left(\bar{R}^{d}\right)^{-1}$ :

$$
\begin{equation*}
\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{-1} \quad \text { and } \quad\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{-1} \tag{4.1}
\end{equation*}
$$

and with $\gamma=0$ and $\gamma=0.1$. The number of iterations required for MINRES to reach convergence is listed in Table 4.1. We draw the following conclusions from this table:

- Choosing $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{-1}$ we are guaranteed that $\left(\bar{R}^{d}\right)^{-1}$ is spectrally equivalent to $\left(\bar{A}^{d}\right)^{-1}$. With this choice we observe that the iteration count for MINRES to converge to a given tolerance is independent of $h$. This verifies Theorem 3.4 that $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ are optimal preconditioners.
- We observe that choosing $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{-1}$, with $\gamma=0$ and $\gamma=0.1$, result in optimal preconditioners $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$. This verifies that $\bar{A}_{\gamma}$ for small $\gamma$ is a good approximation to $\bar{A}^{d}$ (as discussed in subsection 3.2).
- We observe that as the polynomial degree $k$ increases, there is little variation in iteration count when choosing $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{-1}$, however, there is a slight increase in iteration count when choosing $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{-1}$. Furthermore, when using $\bar{A}_{\gamma}$ to approximate $\bar{A}^{d}$, the iteration count using $\mathbb{P}_{B A B}$ as preconditioner is less than when using $\mathbb{P}_{\bar{M}}$ as preconditioner when $k=3$ and $k=4$.
A direct solver for inverting the blocks in (4.1) is prohibitively expensive in large simulations. We therefore next compare the performance of $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ when replacing the direct solves by multigrid, i.e., making the following choices for $\left(\bar{R}^{d}\right)^{-1}$ :

$$
\begin{equation*}
\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{M G} \quad \text { and } \quad\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G} \tag{4.2}
\end{equation*}
$$

Again we consider $\gamma=0$ and $\gamma=0.1$. The number of iterations required for MINRES to reach convergence are listed in Table 4.2. We draw the following conclusions from this table:

- From Table 4.1 we observe that $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}^{d}\right)^{-1}$ is the best choice, resulting in a solver that converges in three times fewer iterations than the next best choice. However, computing $\left(\bar{A}^{d}\right)^{-1}$ is costly. If we replace $\left(\bar{A}^{d}\right)^{-1}$ by $\left(\bar{A}^{d}\right)^{M G}$ we observe in Table 4.2 that iteration count to convergence grows as $h$ decreases, both for $\mathbb{P}_{\bar{M}}$ and for $\mathbb{P}_{B A B}$. This is as expected from Lemma 3.6, in particular (3.20), which shows that $\bar{A}^{d}$ may not be an $H^{1}$-like operator. We therefore cannot expect standard multigrid to perform well on $\bar{A}^{d}$.
- Since $\bar{A}_{\gamma} \rightarrow \bar{A}$ as $\gamma \rightarrow 0$, we expect multigrid to be effective on $\bar{A}_{\gamma}$ when $\gamma$ is small. This is because $\bar{A}$ is an $H^{1}$-like operator on which multigrid is expected to perform well (see subsection 3.2). This is verified here by the observation that the choice $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$ and $\gamma=0.1$, results in an optimal preconditioner for both $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$.
- As we saw also in Table 4.1, when using $\bar{A}{ }_{\gamma}$ to approximate $\bar{A}^{d}$, there is a slight increase in iteration count as $k$ increases. Finally, we observe that when $k=3$ and $k=4$, the iteration count using $\mathbb{P}_{B A B}$ as preconditioner is less than when using $\mathbb{P}_{\bar{M}}$ as preconditioner.

Table 4.1: Iteration counts for preconditioned MINRES for the relative true residual to reach a tolerance of $10^{-8}$ for the lid-driven cavity problem for different polynomial degrees. We compare different choices for $\left(\bar{R}^{d}\right)^{-1}$, see (4.1), in $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$. The test case is described in subsection 4.1.

| $k=2$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\mathbb{P}_{\bar{M}}$ |  |  | $\mathbb{P}_{\text {BAB }}$ |  |  |
| Elements | DOFs | $\left(\bar{A}^{d}\right)^{-1}$ | $\left(\bar{A}_{0}\right)^{-1}$ | $\left(\bar{A}_{0.1}\right)^{-1}$ | $\left(\bar{A}^{d}\right)^{-1}$ | $\left(\bar{A}_{0}\right)^{-1}$ | $\left(\bar{A}_{0.1}\right)^{-1}$ |
| 176 | 2574 | 31 | 90 | 84 | 54 | 87 | 82 |
| 704 | 9900 | 31 | 94 | 87 | 55 | 92 | 86 |
| 2816 | 38808 | 31 | 96 | 89 | 54 | 95 | 88 |
| 11264 | 153648 | 29 | 93 | 86 | 54 | 95 | 87 |
| 45056 | 611424 | 29 | 93 | 86 | 54 | 95 | 87 |
| $k=3$ |  |  |  |  |  |  |  |
|  |  | $\mathbb{P}_{\bar{M}}$ |  |  | $\mathbb{P}_{\text {BAB }}$ |  |  |
| Elements | DOFs | $\left(\bar{A}^{d}\right)^{-1}$ | $\left(\bar{A}_{0}\right)^{-1}$ | $\left(\bar{A}_{0.1}\right)^{-1}$ | $\left(\bar{A}^{d}\right)^{-1}$ | $\left(\bar{A}_{0}\right)^{-1}$ | $\left(\bar{A}_{0.1}\right)^{-1}$ |
| 176 | 3432 | 26 | 119 | 114 | 50 | 106 | 102 |
| 704 | 13200 | 26 | 114 | 109 | 49 | 107 | 103 |
| 2816 | 51744 | 24 | 114 | 109 | 49 | 107 | 103 |
| 11264 | 204864 | 24 | 114 | 109 | 47 | 107 | 103 |
| 45056 | 815232 | 24 | 106 | 101 | 47 | 101 | 97 |
| $k=4$ |  |  |  |  |  |  |  |
|  |  | $\mathbb{P}_{\bar{M}}$ |  |  | $\mathbb{P}_{\text {BAB }}$ |  |  |
| Elements | DOFs | $\left(\bar{A}^{d}\right)^{-1}$ | $\left(\bar{A}_{0}\right)^{-1}$ | $\left(\bar{A}_{0.1}\right)^{-1}$ | $\left(\bar{A}^{d}\right)^{-1}$ | $\left(\bar{A}_{0}\right)^{-1}$ | $\left(\bar{A}_{0.1}\right)^{-1}$ |
| 176 | 4290 | 24 | 144 | 137 | 49 | 129 | 124 |
| 704 | 16500 | 24 | 146 | 139 | 49 | 131 | 125 |
| 2816 | 64680 | 22 | 137 | 140 | 49 | 131 | 126 |
| 11264 | 256080 | 22 | 137 | 130 | 49 | 128 | 123 |
| 45056 | 1019040 | 20 | 137 | 130 | 47 | 125 | 120 |

4.2. Comparison HDG, EDG, and EDG-HDG. The analysis of the preconditioner in Theorem 3.4 holds also for the embedded discontinuous Galerkin (EDG) and embedded-hybridized discontinuous Galerkin (EDG-HDG) discretizations of the Stokes problem [32]. The EDG-HDG discretization is given by replacing $\bar{V}_{h}$ by the continuous trace velocity space $\bar{V}_{h} \cap C^{0}\left(\Gamma_{0}\right)$ in (2.8). In the EDG method both the trace velocity and trace pressure functions are continuous; in (2.8) $\bar{V}_{h}$ and $\bar{Q}_{h}$ are replaced by, respectively, $\bar{V}_{h} \cap C^{0}\left(\Gamma_{0}\right)$ and $\bar{Q}_{h} \cap C^{0}\left(\Gamma_{0}\right)$. Both the EDG and the EDGHDG discretizations result in velocity approximations that are exactly divergence-free on each cell.

We will compare results of the preconditioners $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$, in which we choose $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$ and $\gamma=0.1$, to results obtained when using the preconditioner $\mathbb{P}_{3 \times 3}$ from [31]. We will apply these two preconditioners to HDG, EDG, and EDG-HDG discretizations of the Stokes problem. We consider the twoand three-dimensional lid-driven cavity problems. The two-dimensional problem is described subsection 4.1. In three dimensions we consider the cube $\Omega:=[0,1]^{3}$ and we impose $u=\left(1-\tau_{1}^{4},\left(1-\tau_{2}^{4}\right) / 10,0\right)$ with $\tau_{i}=2 x_{i}-1$ on the boundary with $x_{3}=1$ and $u=0$ on the remaining boundaries. We consider only the case $k=2$.

The number of iterations for MINRES to reach convergence are presented for the two dimensional problem in Table 4.3 and for the three dimensional problem in Table 4.4. From both tables it is clear that both $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ are optimal

Table 4.2: Iteration counts for preconditioned MINRES for the relative true residual to reach a tolerance of $10^{-8}$ for the lid-driven cavity problem for different polynomial degrees. We compare different choices for $\left(\bar{R}^{d}\right)^{-1}$, see $(4.2)$, in $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$. The test case is described in subsection 4.1.

| $k=2$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $\mathbb{P}_{\bar{M}}$ |  |  | $\mathbb{P}_{B A B}$ |  |  |
| Elements | DOFs | $\left(\bar{A}^{d}\right)^{M G}$ | $\left(\bar{A}_{0}\right)^{M G}$ | $\left(\bar{A}_{0.1}\right)^{M G}$ | $\left(\bar{A}^{d}\right)^{M G}$ | $\left(\bar{A}_{0}\right)^{M G}$ | $\left(\bar{A}_{0.1}\right)^{M G}$ |
| 176 | 2574 | 37 | 90 | 84 | 60 | 87 | 82 |
| 704 | 9900 | 52 | 94 | 87 | 91 | 92 | 86 |
| 2816 | 38808 | 91 | 96 | 85 | 163 | 94 | 88 |
| 11264 | 153648 | 174 | 93 | 87 | 334 | 95 | 87 |
| 45056 | 611424 | 358 | 95 | 88 | $>500$ | 95 | 90 |
| $k=3$ |  |  |  |  |  |  |  |
|  |  | $\mathbb{P}_{\bar{M}}$ |  |  | $\mathbb{P}_{B A B}$ |  |  |
| Elements | DOFs | $\left(\bar{A}^{d}\right)^{M G}$ | $\left(\bar{A}_{0}\right)^{M G}$ | $\left(\bar{A}_{0.1}\right)^{M G}$ | $\left(\bar{A}^{d}\right)^{M G}$ | $\left(\bar{A}_{0}\right)^{M G}$ | $\left(\bar{A}_{0.1}\right)^{M G}$ |
| 176 | 3432 | 38 | 119 | 114 | 65 | 106 | 102 |
| 704 | 13200 | 59 | 114 | 109 | 101 | 107 | 103 |
| 2816 | 51744 | 101 | 115 | 110 | 193 | 107 | 103 |
| 11264 | 204864 | 190 | 116 | 110 | 388 | 107 | 103 |
| 45056 | 815232 | 374 | 110 | 105 | >500 | 105 | 101 |
| $k=4$ |  |  |  |  |  |  |  |
|  |  | $\mathbb{P}_{\bar{M}}$ |  |  | $\mathbb{P}_{B A B}$ |  |  |
| Elements | DOFs | $\left(\bar{A}^{d}\right)^{M G}$ | $\left(\bar{A}_{0}\right)^{M G}$ | $\left(\bar{A}_{0.1}\right)^{M G}$ | $\left(\bar{A}^{d}\right)^{M G}$ | $\left(\bar{A}_{0}\right)^{M G}$ | $\left(\bar{A}_{0.1}\right)^{M G}$ |
| 176 | 4290 | 45 | 145 | 138 | 76 | 129 | 124 |
| 704 | 16500 | 68 | 147 | 140 | 127 | 131 | 125 |
| 2816 | 64680 | 121 | 138 | 132 | 258 | 131 | 126 |
| 11264 | 256080 | 230 | 139 | 133 | $>500$ | 129 | 124 |
| 45056 | 1019040 | 477 | 142 | 135 | $>500$ | 130 | 124 |

preconditioners for HDG, EDG and EDG-HDG discretizations, i.e. the iteration count does not grow systematically with increasing problem size. We also observe that $20-60 \%$ fewer iterations are required to solve the linear system using the two-field preconditioners $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ compared to using the three-field preconditioner $\mathbb{P}_{3 \times 3}$ from [31]. Finally, we note that for all calculations, fewer iterations are required when using $\gamma=0.1$ than when using $\gamma=0$.
4.3. The effect of viscosity. Consider the following modification of the Stokes problem (2.1):

$$
\begin{align*}
-\nu \nabla^{2} u+\nabla p & =f & & \text { in } \Omega  \tag{4.3a}\\
\nabla \cdot u & =0 & & \text { in } \Omega  \tag{4.3b}\\
u & =u_{d} & & \text { on } \partial \Omega \tag{4.3c}
\end{align*}
$$

where $\nu>0$ denotes a constant viscosity and $u_{d}$ a given Dirichlet boundary condition. The discrete formulation for this problem is given by:

$$
\begin{equation*}
\nu a_{h}\left(\boldsymbol{u}_{h}, \boldsymbol{v}_{h}\right)+b_{h}\left(\boldsymbol{p}_{h}, v_{h}\right)+b_{h}\left(\boldsymbol{q}_{h}, u_{h}\right)=\left(v_{h}, f\right)_{\mathcal{T}} \quad \forall\left(\boldsymbol{v}_{h}, \boldsymbol{q}_{h}\right) \in \boldsymbol{X}_{h} \tag{4.4}
\end{equation*}
$$

where $a_{h}(\cdot, \cdot)$ and $b_{h}(\cdot, \cdot)$ are defined in (2.9). Redefining the matrix $A$ in (2.18) as $A \leftarrow \nu A$, the block matrix form of (4.4) is given by (2.22) and its statically condensed form is given by (2.23). Redefining furthermore the element and trace pressure mass

Table 4.3: Iteration counts for preconditioned MINRES for the relative true residual to reach a tolerance of $10^{-8}$ for the lid-driven cavity problem in two dimensions. A comparison between using $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ with $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$ and $\gamma=0.1$, and $\mathbb{P}_{3 \times 3}$ for HDG, EDG, and EDG-HDG discretizations of the Stokes problem. The test case is described in subsection 4.2. The iteration count when using $\gamma=0.1$ is given in brackets.

| $\mathbb{P}_{\bar{M}}$ |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | HDG |  | EDG |  | EDG-HDG |  |  |  |  |  |  |
| Elements | DOFs | Its | DOFs | Its | DOFs | Its |  |  |  |  |  |
| 176 | 2574 | $90(84)$ | 1191 | $86(81)$ | 1652 | $95(86)$ |  |  |  |  |  |
| 704 | 9900 | $94(87)$ | 4491 | $83(79)$ | 6294 | $92(88)$ |  |  |  |  |  |
| 2816 | 38808 | $96(85)$ | 17427 | $85(80)$ | 24553 | $94(85)$ |  |  |  |  |  |
| 11264 | 153648 | $93(87)$ | 68643 | $82(77)$ | 96978 | $92(86)$ |  |  |  |  |  |
| 45056 | 611424 | $95(88)$ | 272451 | $83(78)$ | 385442 | $88(83)$ |  |  |  |  |  |
| $\mathbb{P}_{\text {BAB }}$ |  |  |  |  |  |  |  |  |  |  |  |
| HDG |  |  |  |  |  |  |  | EDG |  | EDG-HDG |  |
| Elements | DOFs | Its | DOFs | Its | DOFs | Its |  |  |  |  |  |
| 176 | 2574 | $87(82)$ | 1191 | $61(56)$ | 1652 | $61(58)$ |  |  |  |  |  |
| 704 | 9900 | $92(86)$ | 4491 | $62(59)$ | 6294 | $64(59)$ |  |  |  |  |  |
| 2816 | 38808 | $94(88)$ | 17427 | $63(60)$ | 24553 | $63(60)$ |  |  |  |  |  |
| 11264 | 153648 | $95(87)$ | 68643 | $63(58)$ | 96978 | $63(60)$ |  |  |  |  |  |
| 45056 | 611424 | $95(90)$ | 272451 | $61(58)$ | 385442 | $63(60)$ |  |  |  |  |  |
| $\mathbb{P}_{3 \times 3}$ |  |  |  |  |  |  |  |  |  |  |  |
| HDG |  |  |  |  |  |  |  | EDG |  |  | EDG-HDG |
| Elements | DOFs | Its | DOFs | Its | DOFs | Its |  |  |  |  |  |
| 176 | 3102 | 130 | 1719 | 124 | 2180 | 131 |  |  |  |  |  |
| 704 | 12012 | 134 | 6603 | 127 | 8406 | 136 |  |  |  |  |  |
| 2816 | 47256 | 136 | 25875 | 124 | 33002 | 131 |  |  |  |  |  |
| 11264 | 187440 | 138 | 102435 | 125 | 130770 | 135 |  |  |  |  |  |
| 45056 | 746592 | 132 | 407619 | 118 | 520610 | 127 |  |  |  |  |  |

matrices in (3.2) as, respectively, $M \leftarrow \nu^{-1} M$ and $\bar{M} \leftarrow \nu^{-1} \bar{M}$, the preconditioners again take on the forms presented in section 3 .

In this section we consider the effect of the viscosity parameter on solving the statically condensed problem. As test case we choose the source term $f$ and the Dirichlet boundary condition in (4.3) such that the exact solution on the domain $\Omega:=[-1,1]^{2}$ is given by

$$
u=\left[\begin{array}{c}
\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right)  \tag{4.5}\\
\cos \left(\pi x_{1}\right) \cos \left(\pi x_{2}\right)
\end{array}\right], p=\sin \left(\pi x_{1}\right) \cos \left(\pi x_{2}\right)
$$

In our simulations we choose polynomial degree $k=2$.
In Table 4.5 we present the number of iterations for MINRES to reach convergence for $\nu=1$ and $\nu=10^{-6}$ for the HDG, EDG, and EDG-HDG discretizations. We also present the $L^{2}$-norm of the velocity error, pressure error, the cell-wise divergence error as well as $\max _{q \in \mathcal{Q}}|\llbracket u \cdot n \rrbracket|$, where $\mathcal{Q}$ is the set of all quadrature points on the facets.

The HDG and EDG-HDG discretizations result in a velocity field $u_{h}$ that is pointwise divergence-free on the cells and $H$ (div)-conforming. A consequence of these properties is that the magnitude of the viscosity does not affect the $L^{2}$-norm of the velocity error [32, Theorem 3]. This 'pressure-robustness' is also observed in Table 4.5;

Table 4.4: Iteration counts for preconditioned MINRES for the relative true residual to reach a tolerance of $10^{-8}$ for the lid-driven cavity problem in three dimensions. A comparison between using $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ with $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$ and $\gamma=0.1$, and $\mathbb{P}_{3 \times 3}$ for HDG, EDG, and EDG-HDG discretizations of the Stokes problem. The test case is described in subsection 4.2. The iteration count when using $\gamma=0.1$ is given in brackets.

| $\mathbb{P}_{\bar{M}}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Elements | HDG |  | EDG |  | EDG-HDG |  |
|  | DOFs | Its | DOFs | Its | DOFs | Its |
| 524 | 28032 | 172 (160) | 3884 | 99 (94) | 9921 | 150 (143) |
| 4192 | 212736 | 161 (149) | 26452 | 91 (86) | 73023 | 133 (121) |
| 33536 | 1655808 | 151 (140) | 194724 | 80 (75) | 559995 | 117 (94) |
| $\mathbb{P}_{\text {BAB }}$ |  |  |  |  |  |  |
|  | HDG |  | EDG |  | EDG-HDG |  |
| Elements | DOFs | Its | DOFs | Its | DOFs | Its |
| 524 | 28032 | 139 (129) | 3884 | 62 (60) | 9921 | 70 (67) |
| 4192 | 212736 | 142 (132) | 26452 | 61 (58) | 73023 | 66 (63) |
| 33536 | 1655808 | 139 (129) | 194724 | 57 (55) | 559995 | 60 (58) |
| $\mathbb{P}_{3 \times 3}$ |  |  |  |  |  |  |
|  | HDG |  | EDG |  | EDG-HDG |  |
| Elements | DOFs | Its | DOFs | Its | DOFs | Its |
| 524 | 30128 | 224 | 5980 | 151 | 12017 | 223 |
| 4192 | 229504 | 218 | 43220 | 146 | 89791 | 206 |
| 33536 | 1789952 | 209 | 328868 | 134 | 694139 | 176 |

the $L^{2}$-norm of the error of the velocity for $\nu=1$ and $\nu=10^{-6}$ are more or less identical. (Note that when viscosity decreases, the conditioning of the matrix worsens, explaining the increase in the error in the divergence of the velocity and the error in the jump of the normal component of the velocity across facets with decreasing viscosity. This, however, does not affect the pressure-robustness of the discretization.)

The EDG method, on the other hand, results in a velocity field $u_{h}$ that is pointwise divergence-free on the cells, but not $H$ (div)-conforming. As a consequence, the upper bound for the $L^{2}$-norm of the error of the velocity is inversely proportional to the viscosity (see also [32, Remark 1]). We indeed observe this in Table $4.5 ;\left\|\nabla \cdot u_{h}\right\|_{\Omega}$ is close to machine precision while the error in $\max _{q \in \mathcal{Q}}|\llbracket u \cdot n \rrbracket|$ is magnitudes larger. Furthermore, we observe an increase in the $L^{2}$-norm of the velocity error when viscosity is decreased.

Finally, we observe for all discretizations that the viscosity has no effect on the number of iterations for MINRES to reach convergence when using the preconditioner $\mathbb{P}_{B A B}$. When using the preconditioner $\mathbb{P}_{\bar{M}}$ there is a slight increase of $9-14 \%$ in iteration count as the viscosity decreases from $\nu=1$ to $\nu=10^{-6}$.
4.4. Performance comparison. In this final section we compare the performance of $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$, in which we choose $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$ and $\gamma=0.1$, to the performance of the $\mathbb{P}_{3 \times 3}$ preconditioner. We do this for the HDG, EDG, and EDG-HDG discretiations. As test case we consider the Stokes problem on the three-dimensional domain $\Omega=[-1,1]^{3}$, with Dirichlet boundary conditions and

Table 4.5: $L^{2}$-norm of the velocity error, pressure error, the cell-wise divergence error, the error in the jump of the normal component of the velocity across facets, and iteration counts for preconditioned MINRES for the relative preconditioned residual norm to reach a tolerance of $10^{-12}$ for the two dimensional test case described in subsection 4.3. By $\mathbb{P}_{\bar{M}}^{\gamma=0}$ we mean that we solve (2.23) using MINRES with preconditioner $\mathbb{P}_{\bar{M}}$ in which we choose $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$. The other methods are described similarly. The numbers listed with an asterisk are the number of iterations required for the relative preconditioned residual norm to reach a tolerance of $10^{-11}$. The relative preconditioned residual stagnates shortly after unable to reach $10^{-12}$. This happens only for $\nu=1$ for the EDG and EDG-HDG discretizations on the coarsest mesh.

| Elements | DOFs | $\left\\|u-u_{h}\right\\|_{\Omega}$ | $\left\\|p-p_{h}\right\\|_{\Omega}$ | $\left\\|\nabla \cdot u_{h}\right\\|_{\Omega}$ | $\left\|\llbracket u_{h} \cdot n \rrbracket\right\|$ | ${ }^{\mathbb{P}} \bar{M}=0$ | $\begin{aligned} & \mathbb{P}_{B A B}^{\gamma=0} \end{aligned}$ | ${ }^{\mathrm{P}} \stackrel{\underset{M}{\gamma}=0.1}{ }$ | $\begin{aligned} & \mathbb{P}_{B A B}^{\gamma=0.1} \\ & \hline \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| HDG, $\nu=1$ |  |  |  |  |  |  |  |  |  |
| 11264 | 153648 | 6.1e-6 | $2.4 \mathrm{e}-3$ | $1.1 \mathrm{e}-13$ | $9.0 \mathrm{e}-14$ | 157 | 148 | 146 | 140 |
| 45056 | 611424 | $7.6 \mathrm{e}-7$ | $5.9 \mathrm{e}-4$ | $2.2 \mathrm{e}-13$ | $9.4 \mathrm{e}-14$ | 159 | 152 | 148 | 142 |
| 180224 | 2439360 | $9.5 \mathrm{e}-8$ | $1.5 \mathrm{e}-4$ | $4.5 \mathrm{e}-13$ | $5.5 \mathrm{e}-14$ | 158 | 155 | 146 | 146 |
| HDG, $\nu=10^{-6}$ |  |  |  |  |  |  |  |  |  |
| 11264 | 153648 | 6.1e-6 | $4.4 \mathrm{e}-4$ | $1.0 \mathrm{e}-10$ | $2.3 \mathrm{e}-9$ | 171 | 150 | 159 | 140 |
| 45056 | 611424 | $7.8 \mathrm{e}-7$ | $1.1 \mathrm{e}-4$ | $1.0 \mathrm{e}-10$ | $4.3 \mathrm{e}-10$ | 173 | 154 | 161 | 145 |
| 180224 | 2439360 | $1.4 \mathrm{e}-7$ | $2.7 \mathrm{e}-5$ | $1.1 \mathrm{e}-10$ | $1.4 \mathrm{e}-9$ | 172 | 157 | 162 | 148 |
| $\mathrm{EDG}, \nu=1$ |  |  |  |  |  |  |  |  |  |
| 11264 | 68643 | $1.1 \mathrm{e}-5$ | $5.2 \mathrm{e}-3$ | $1.1 \mathrm{e}-13$ | $3.0 \mathrm{e}-7$ | 136* | 91* | 129* | 87* |
| 45056 | 272451 | $1.4 \mathrm{e}-6$ | $1.3 \mathrm{e}-3$ | $2.2 \mathrm{e}-13$ | $1.9 \mathrm{e}-8$ | 155 | 101 | 146 | 96 |
| 180224 | 1085571 | $1.7 \mathrm{e}-7$ | $3.3 \mathrm{e}-4$ | $4.5 \mathrm{e}-13$ | $1.2 \mathrm{e}-9$ | 156 | 102 | 147 | 97 |
| $\operatorname{EDG}, \nu=10^{-6}$ |  |  |  |  |  |  |  |  |  |
| 11264 | 68643 | $7.6 \mathrm{e}-3$ | $4.4 \mathrm{e}-4$ | $1.0 \mathrm{e}-10$ | $3.8 \mathrm{e}-4$ | 170 | 101 | 160 | 97 |
| 45056 | 272451 | $4.8 \mathrm{e}-4$ | $1.1 \mathrm{e}-4$ | $1.0 \mathrm{e}-10$ | $1.2 \mathrm{e}-5$ | 176 | 102 | 165 | 100 |
| 180224 | 1085571 | $3.0 \mathrm{e}-5$ | $2.7 \mathrm{e}-5$ | $1.1 \mathrm{e}-10$ | $3.7 \mathrm{e}-7$ | 178 | 105 | 168 | 101 |
| EDG-HDG, $\nu=1$ |  |  |  |  |  |  |  |  |  |
| 11264 | 96978 | $1.1 \mathrm{e}-5$ | $5.3 \mathrm{e}-3$ | $1.1 \mathrm{e}-13$ | $1.5 \mathrm{e}-12$ | 145* | 93* | 137* | 89* |
| 45056 | 385442 | $1.4 \mathrm{e}-6$ | $1.3 \mathrm{e}-3$ | $2.2 \mathrm{e}-13$ | $2.3 \mathrm{e}-13$ | 164 | 103 | 155 | 98 |
| 180224 | 1536834 | $1.7 \mathrm{e}-7$ | $3.4 \mathrm{e}-4$ | $4.4 \mathrm{e}-13$ | $1.3 \mathrm{e}-13$ | 166 | 104 | 157 | 99 |
| $\text { EDG-HDG, } \nu=10^{-6}$ |  |  |  |  |  |  |  |  |  |
| 11264 | 96978 | $1.1 \mathrm{e}-5$ | $4.4 \mathrm{e}-4$ | $1.0 \mathrm{e}-10$ | $2.8 \mathrm{e}-9$ | 184 | 102 | 174 | 97 |
| 45056 | 385442 | $1.4 \mathrm{e}-6$ | $1.1 \mathrm{e}-4$ | $1.0 \mathrm{e}-10$ | $9.3 \mathrm{e}-10$ | 187 | 104 | 177 | 100 |
| 180224 | 1536834 | $1.9 \mathrm{e}-7$ | $2.7 \mathrm{e}-5$ | $1.1 \mathrm{e}-10$ | $3.8 \mathrm{e}-10$ | 189 | 105 | 179 | 101 |

a source term such that the exact solution to the Stokes problem is given by
$u=\pi\left[\begin{array}{l}\sin \left(\pi x_{1}\right) \cos \left(\pi x_{2}\right)-\sin \left(\pi x_{1}\right) \cos \left(\pi x_{3}\right) \\ \sin \left(\pi x_{2}\right) \cos \left(\pi x_{3}\right)-\sin \left(\pi x_{2}\right) \cos \left(\pi x_{1}\right) \\ \sin \left(\pi x_{3}\right) \cos \left(\pi x_{1}\right)-\sin \left(\pi x_{3}\right) \cos \left(\pi x_{2}\right)\end{array}\right], p=\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right) \sin \left(\pi x_{3}\right)-\frac{8}{\pi^{3}}$.
In the implementation of the application of $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ we now approximate $\bar{M}^{-1}$ by $(\bar{M})^{M G}$ and $\left(B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}\right)^{-1}$ by $\left(B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}\right)^{M G}$ as this is slightly more efficient than using a direct solver for these terms.

In Table 4.6 we list the $L^{2}$-norm of the velocity error, pressure error, the cell-wise divergence of the velocity, and maximum value of the jump of the normal component of the velocity across facets. These errors are computed on a mesh consisting of 33536 tetrahedra. We also include the CPU time to convergence.

Considering first the error in the $L^{2}$-norm, for all discretizations we observe that
the $L^{2}$-norm of the error in element velocity $u_{h}$ and element pressure $p_{h}$ are identical when using the two-field (2.23) and three-field (A.1) reduced systems. The error in the divergence of the element velocity, however, is different. In the three-field reduced formulation the divergence of the element velocity is of the order of accuracy at which the MINRES method was terminated. This implies that would we want $\nabla \cdot u_{h}$ on elements to be of the order of machine precision, the stopping criteria of the MINRES method would need to be of the order of machine precision. On the other hand, the error in the divergence of the element velocity is of the order of machine precision when using the two-field reduced formulation. This is due to the elementwise projection $\mathcal{P}$ in (2.25). Indeed, as we saw previously in subsection $2.3, u=$ $\mathcal{P} A_{u u}^{-1}\left(L_{u}-A_{\bar{u} u}^{T} \bar{u}-B_{\bar{p} u}^{T} \bar{p}\right) \in \operatorname{Ker} B_{p u}$. Furthermore, for all discretizations considered here, we have that $u \in \operatorname{Ker} B_{p u}$ implies $\nabla \cdot u_{h}=0$ pointwise on each element. The error in the divergence of the velocity therefore does not depend on the stopping criteria used for MINRES. Finally, we note that the error in the jump of the normal component of the velocity using the EDG discretization is magnitudes larger than when using HDG or EDG-HDG (which are of the order of the stopping criteria of the MINRES method). This is independent of whether the two-field or three-field reduced systems are solved and independent of which preconditioner is used. The higher error in the EDG method is expected as it is the only discretization that is not pressure-robust [32].

Considering now the CPU time and number of iterations required for convergence, we observe for all discretizations that using $\gamma=0.1$ in $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$ results in fewer iterations than when using $\gamma=0$. However, the CPU time to convergence using $\gamma=0.1$ is greater than when using $\gamma=0$. This is in part due to the construction of $\widehat{A}_{\gamma}^{-1}$ in (3.27), slowing down the construction of the preconditioner.

Consider now the case when $\gamma=0$ in $\mathbb{P}_{\bar{M}}$ and $\mathbb{P}_{B A B}$. We then observe for the HDG discretization that both two-field preconditioners outperform the three-field preconditioner $\mathbb{P}_{3 \times 3}$ both in number of iterations (up to $27 \%$ fewer iterations) and CPU time (up to $32 \%$ faster). For the EDG and EDG-HDG discretizations the performance is even better; note, for example, that using $\mathbb{P}_{\bar{M}}$ for the EDG discretization is up to $43 \%$ faster and $\mathbb{P}_{B A B}$ is up to $51 \%$ faster than using the $\mathbb{P}_{3 \times 3}$ preconditioner.
5. Conclusions. The linear system of an HDG discretization of the Stokes equations can efficiently be statically condensed in two ways: (i) eliminating the degrees-of-freedom associated to the element approximation of the velocity (the three-field reduced formulation); or (ii) eliminating the degrees-of-freedom associated to the element approximation of both the velocity and the pressure (the two-field reduced formulation). In our previous work we proposed an optimal preconditioner for the three-field reduced formulation. In this paper we proposed and analyzed a preconditioner for the two-field reduced formulation. In the two-field reduced formulation the lifting of the trace velocity to the elements is algebraically imposed to be divergencefree. Although this complicates the analysis, it has been shown that the trace pressure Schur complement is spectrally equivalent to a simple trace pressure mass matrix and we used this to introduce a new preconditioner. Numerical examples in two and three dimensions show that the new preconditioner is more efficient for solving a hybridized discretization of the Stokes problem than our previous preconditioner.

## References.

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Table 4.6: Errors in the $L^{2}$-norm, CPU times and iteration counts for preconditioned MINRES for the relative preconditioned residual norm to reach a tolerance of $10^{-8}$ for the three dimensional test case described in subsection 4.4. Here HDG $\mathbb{P}_{3 \times 3}$ means that we solve (A.1) using MINRES with preconditioner $\mathbb{P}_{3 \times 3}$. By HDG $\mathbb{P}_{\bar{M}}^{\gamma=0}$ we mean that we solve (2.23) using MINRES with preconditioner $\mathbb{P}_{\bar{M}}$ in which we choose $\left(\bar{R}^{d}\right)^{-1}=\left(\bar{A}_{\gamma}\right)^{M G}$, with $\gamma=0$. The other methods are described similarly. The results, except $\left\|\nabla \cdot u_{h}\right\|_{\Omega}$ and $\left|\llbracket u_{h} \cdot n \rrbracket\right|$, have been normalized with respect to the HDG method using $\mathbb{P}_{3 \times 3}$ (in brackets).

| Method | $\left\\|u-u_{h}\right\\|_{\Omega}$ | $\left\\|p-p_{h}\right\\|_{\Omega}$ | $\left\\|\nabla \cdot u_{h}\right\\|_{\Omega}$ | $\left\|\llbracket u_{h} \cdot n \rrbracket\right\|$ | CPU | Its | DOFs |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| HDG $\mathbb{P}_{3 \times 3}$ | $1(2.5 \mathrm{e}-4)$ | $1(3.3 \mathrm{e}-2)$ | $2.1 \mathrm{e}-6$ | $2.4 \mathrm{e}-8$ | 1 (1171s) | 1 (176) | 1 (1789 952) |
| $\operatorname{HDG} \mathbb{P}_{\bar{M}}^{\gamma=0}$ | 1 | 1 | $1.7 \mathrm{e}-13$ | $2.5 \mathrm{e}-8$ | 0.68 | 0.73 | 0.93 |
| HDG $\mathbb{P}_{\bar{M}}^{\gamma=0.1}$ | 1 | 1 | $1.7 \mathrm{e}-13$ | $2.6 \mathrm{e}-8$ | 1.60 | 0.68 | 0.93 |
| HDG $\mathbb{P}_{B A B}^{\gamma=0}$ | 1 | 1 | $1.7 \mathrm{e}-13$ | $1.4 \mathrm{e}-8$ | 0.82 | 0.76 | 0.93 |
| $\text { HDG } \mathbb{P}_{B A B}^{\gamma=0.1}$ | 1 | 1 | $1.7 \mathrm{e}-13$ | $1.3 \mathrm{e}-8$ | 1.88 | 0.71 | 0.93 |
| EDG $\mathbb{P}_{3 \times 3}$ | 1.84 | 3.00 | $4.0 \mathrm{e}-6$ | $1.3 \mathrm{e}-5$ | 0.063 | 0.68 | 0.18 |
| EDG $\mathbb{P}_{\bar{M}}^{\gamma=0}$ | 1.84 | 3.00 | $1.7 \mathrm{e}-13$ | $1.3 \mathrm{e}-5$ | 0.036 | 0.43 | 0.11 |
| EDG $\mathbb{P}_{\bar{M}}^{\gamma=0.1}$ | 1.84 | 3.00 | $1.7 \mathrm{e}-13$ | $1.3 \mathrm{e}-5$ | 0.078 | 0.40 | 0.11 |
| EDG $\mathbb{P}_{B A B}^{\gamma=0}$ | 1.84 | 3.00 | $1.7 \mathrm{e}-13$ | $1.3 \mathrm{e}-5$ | 0.031 | 0.35 | 0.11 |
| $\text { EDG } \mathbb{P}_{B A B}^{\gamma=0.1}$ | 1.84 | 3.00 | $1.7 \mathrm{e}-13$ | $1.3 \mathrm{e}-5$ | 0.069 | 0.34 | 0.11 |
| EDG-HDG $\mathbb{P}_{3 \times 3}$ | 1.96 | 4.24 | $3.6 \mathrm{e}-6$ | $6.0 \mathrm{e}-8$ | 0.094 | 0.85 | 0.39 |
| EDG-HDG $\mathbb{P}_{\bar{M}}^{\gamma=0}$ | 1.96 | 4.24 | $1.7 \mathrm{e}-13$ | $4.9 \mathrm{e}-8$ | 0.054 | 0.56 | 0.31 |
| EDG-HDG $\underset{\bar{M}}{\gamma=0.1}$ | 1.96 | 4.24 | $1.8 \mathrm{e}-13$ | $4.9 \mathrm{e}-8$ | 0.11 | 0.53 | 0.31 |
| $\text { EDG-HDG } \mathbb{P}_{B A B}^{\gamma=0}$ | 1.96 | 4.24 | $1.8 \mathrm{e}-13$ | $2.1 \mathrm{e}-8$ | 0.068 | 0.38 | 0.31 |
| $\text { EDG-HDG } \mathbb{P}_{B A B}^{\gamma=0.1}$ | 1.96 | 4.24 | $1.8 \mathrm{e}-13$ | $2.0 \mathrm{e}-8$ | 0.11 | 0.37 | 0.31 |

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Appendix A. Eliminating the velocity element degrees-of-freedom.
Instead of eliminating both the velocity and pressure element degrees-of-freedom from (2.22) it is possible also to eliminate only the velocity element degrees-of-freedom $u$. This results in the three-field reduced system:
(A.1)

$$
\left[\begin{array}{ccc}
\bar{A} & -A_{\bar{u} u} A_{u u}^{-1} B_{p u}^{T} & -A_{\bar{u} u} A_{u u}^{-1} B_{\bar{p} u} \\
-B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T} & -B_{p u} A_{u u}^{-1} B_{p u}^{T} & -B_{p u} A_{u u}^{-1} B_{\bar{p} u}^{T} \\
-B_{\bar{p} u} A_{u u}^{-1} A_{\bar{u} u}^{T} & -B_{\bar{p} u} A_{u u}^{-1} B_{p u}^{T} & -B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}
\end{array}\right]\left[\begin{array}{c}
\bar{u} \\
p \\
\bar{p}
\end{array}\right]=\left[\begin{array}{c}
L_{\bar{u}}-A_{\bar{u} u} A_{u u}^{-1} L_{u} \\
-B_{p u} A_{u u}^{-1} L_{u} \\
-B_{\bar{p} u} A_{u u}^{-1} L_{u}
\end{array}\right],
$$

where $\bar{A}=-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}+A_{\bar{u} \bar{u}}$. Given the trace velocity $\bar{u}$, the element pressure $p$, and the trace pressure $\bar{p}$, the element velocity $u$ can be computed element-wise in a post-processing step.

In [31] we developed a preconditioner for (A.1). We proved [31, Theorem 2] that

$$
\mathbb{P}_{3 \times 3}=\left[\begin{array}{ccc}
\bar{R} & 0 & 0  \tag{A.2}\\
0 & M & 0 \\
0 & 0 & \bar{M}
\end{array}\right]
$$

is an optimal preconditioner for (A.1) provided $\bar{R}$ is an operator spectrally equivalent to $\bar{A}$. As discussed in subsection $3.2, \bar{A}$ is an $H^{1}$-like operator motivating the use of multigrid for $\bar{R}^{-1}$, i.e., we set $\bar{R}^{-1}=(\bar{A})^{M G}$ (see also [31, Section 3.3]).


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