# REGULARITY OF BOLTZMANN EQUATION WITH CERCIGNANI-LAMPIS BOUNDARY IN CONVEX DOMAIN. 

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#### Abstract

The Boltzmann equation is a fundamental kinetic equation that describes the dynamics of dilute gas. In this paper we study the regularity of both dynamical and steady Boltzmann equation in strictly convex domain with the Cercignani-Lampis (C-L) boundary condition. The C-L boundary condition describes the intermediate reflection law between diffuse reflection and specular reflection via two accommodation coefficients. We construct local weighted $C^{1}$ dynamical solution using repeated interaction through the characteristic. When we assume small fluctuation to the wall temperature and accommodation coefficients, we construct weighted $C^{1}$ steady solution.


## 1. Introduction

In this paper we consider the classical Boltzmann equation, which describes the dynamics of dilute particles. Denoting $F(t, x, v)$ the phase-space-distribution function of particles at time $t$, location $x \in \Omega$ moving with velocity $v \in \mathbb{R}^{3}$, the equation writes:

$$
\begin{equation*}
\partial_{t} F+v \cdot \nabla_{x} F=Q(F, F) \tag{1.1}
\end{equation*}
$$

The collision operator $Q$ describes the binary collisions between particles:

$$
\begin{align*}
& Q\left(F_{1}, F_{2}\right)(v)=Q_{\text {gain }}-Q_{\text {loss }}=Q_{\text {gain }}\left(F_{1}, F_{2}\right)-\nu\left(F_{1}\right) F_{2} \\
& :=\iint_{\mathbb{R}^{3} \times \mathbb{S}^{2}} B(v-u, \omega) F_{1}\left(u^{\prime}\right) F_{2}\left(v^{\prime}\right) \mathrm{d} \omega \mathrm{~d} u-F_{2}(v)\left(\iint_{\mathbb{R}^{3} \times \mathbb{S}^{2}} B(v-u, \omega) F_{1}(u) \mathrm{d} \omega \mathrm{~d} u\right) . \tag{1.2}
\end{align*}
$$

In the collision process, we assume the energy and momentum are conserved. We denote the post-velocities:

$$
\begin{equation*}
u^{\prime}=u-[(u-v) \cdot \omega] \omega, \quad v^{\prime}=v+[(u-v) \cdot \omega] \omega \tag{1.3}
\end{equation*}
$$

then they satisfy:

$$
\begin{equation*}
u^{\prime}+v^{\prime}=u+v, \quad\left|u^{\prime}\right|^{2}+\left|v^{\prime}\right|^{2}=|u|^{2}+|v|^{2} \tag{1.4}
\end{equation*}
$$

In equation (1.2), $B$ is called the collision kernel. In this paper we only consider the hard sphere case, which is given by

$$
B(v-u, \omega)=|v-u|^{\mathcal{K}} q_{0}\left(\frac{v-u}{|v-u|} \cdot \omega\right), \quad \text { with } \mathcal{K}=1, \quad 0 \leq q_{0}\left(\frac{v-u}{|v-u|} \cdot \omega\right) \leq C\left|\frac{v-u}{|v-u|} \cdot \omega\right|
$$

To describe the boundary condition for $F$, we denote the collection of coordinates on phase space at the boundary:

$$
\gamma:=\left\{(x, v) \in \partial \Omega \times \mathbb{R}^{3}\right\}
$$

And we denote $n=n(x)$ as the outward normal vector at $x \in \Omega$. We split the boundary coordinates $\gamma$ into the incoming $\left(\gamma_{-}\right)$and the outgoing $\left(\gamma_{+}\right)$set:

$$
\gamma_{\mp}:=\left\{(x, v) \in \partial \Omega \times \mathbb{R}^{3}: n(x) \cdot v \lessgtr 0\right\} .
$$

The boundary condition determines the distribution on $\gamma_{-}$, and shows how particles back-scattered into the domain. In our model, we use the scattering kernel $R(u \rightarrow v ; x, t)$ :

$$
\begin{equation*}
F(t, x, v)|n(x) \cdot v|=\int_{n(x) \cdot u>0} R(u \rightarrow v ; x, t) F(t, x, u)\{n(x) \cdot u\} \mathrm{d} u, \quad \text { on } \gamma_{-} . \tag{1.5}
\end{equation*}
$$

$R(u \rightarrow v ; x, t)$ represents the probability of a molecule striking in the boundary at $x \in \partial \Omega$ with velocity $u$, and to be sent back to the domain with velocity $v$ at the same location $x$ and time $t$. In this paper we consider a
scattering kernel proposed by Cercignani and Lampis in [7, 8]:

$$
\begin{align*}
& R(u \rightarrow v ; x, t) \\
:= & \frac{1}{r_{\perp} r_{\|}\left(2-r_{\|}\right) \pi / 2} \frac{|n(x) \cdot v|}{\left(2 T_{w}(x)\right)^{2}} \exp \left(-\frac{1}{2 T_{w}(x)}\left[\frac{\left|v_{\perp}\right|^{2}+\left(1-r_{\perp}\right)\left|u_{\perp}\right|^{2}}{r_{\perp}}+\frac{\left|v_{\|}-\left(1-r_{\|}\right) u_{\|}\right|^{2}}{r_{\|}\left(2-r_{\|}\right)}\right]\right)  \tag{1.6}\\
& \times I_{0}\left(\frac{1}{2 T_{w}(x)} \frac{2\left(1-r_{\perp}\right)^{1 / 2} v_{\perp} u_{\perp}}{r_{\perp}}\right),
\end{align*}
$$

where $T_{w}(x)$ is the wall temperature for $x \in \partial \Omega$ and

$$
\begin{equation*}
I_{0}(y):=\pi^{-1} \int_{0}^{\pi} e^{y \cos \phi} \mathrm{~d} \phi \tag{1.7}
\end{equation*}
$$

In the formula, $v_{\perp}$ and $v_{\|}$denote the normal and tangential components of the velocity respectively:

$$
\begin{equation*}
v_{\perp}=v \cdot n(x), \quad v_{\|}=v-v_{\perp} n(x) . \tag{1.8}
\end{equation*}
$$

Similarly $u_{\perp}=u \cdot n(x)$ and $u_{\|}=u-u_{\perp} n(x)$. There are other derivations of C-L model besides the original one, and we refer interested readers to [13, 7, 6].

The Cercignani-Lampis(C-L) model satisfies the following properties:

- the reciprocity property:

$$
\begin{equation*}
R(u \rightarrow v ; x, t)=R(-v \rightarrow-u ; x, t) \frac{e^{-|v|^{2} /\left(2 T_{w}(x)\right)}}{e^{-|u|^{2} /\left(2 T_{w}(x)\right)}} \frac{|n(x) \cdot v|}{|n(x) \cdot u|}, \tag{1.9}
\end{equation*}
$$

- the normalization property(see Lemma 1)

$$
\begin{equation*}
\int_{n(x) \cdot v<0} R(u \rightarrow v ; x, t) \mathrm{d} v=1 \tag{1.10}
\end{equation*}
$$

The normalization (1.10) property immediately leads to null-flux condition for $F$ :

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} F(t, x, v)\{n(x) \cdot v\} \mathrm{d} v=0, \quad \text { for } x \in \partial \Omega \tag{1.11}
\end{equation*}
$$

This condition guarantees the conservation of total mass:

$$
\begin{equation*}
\int_{\Omega \times \mathbb{R}^{3}} F(t, x, v) \mathrm{d} v \mathrm{~d} x=\int_{\Omega \times \mathbb{R}^{3}} F(0, x, v) \mathrm{d} v \mathrm{~d} x \text { for all } t \geq 0 \tag{1.12}
\end{equation*}
$$

Remark 1. The $C$-L model encompasses pure diffusion and pure reflection.
The pure diffuse boundary condition is given by

$$
\begin{gather*}
F(t, x, v)=\frac{2}{\pi\left(2 T_{w}(x)\right)^{2}} e^{-\frac{|v|^{2}}{2 T_{w}(x)}} \int_{n(x) \cdot u>0} F(t, x, u)\{n(x) \cdot u\} \mathrm{d} u \text { on }(x, v) \in \gamma_{-},  \tag{1.13}\\
R(u \rightarrow v ; x, t)=\frac{2}{\pi\left(2 T_{w}(x)\right)^{2}} e^{-\frac{|v|^{2}}{2 T_{w}(x)}}|n(x) \cdot v|
\end{gather*}
$$

It corresponds to the scattering kernel in (1.6) with $r_{\perp}=1, r_{\|}=1$.
Other basic boundary conditions can be considered as a special case with singular $R$ : specular reflection boundary condition:

$$
\begin{gathered}
F(t, x, v)=F\left(t, x, \Re_{x} v\right) \text { on }(x, v) \in \gamma_{-}, \quad \Re_{x} v=v-2 n(x)(n(x) \cdot v), \\
R(u \rightarrow v ; x, t)=\delta\left(u-\Re_{x} v\right)
\end{gathered}
$$

where $r_{\perp}=0, r_{\|}=0$.
Bounce-back reflection boundary condition:

$$
\begin{gathered}
F(t, x, v)=F(t, x,-v) \text { on }(x, v) \in \gamma_{-}, \\
R(u \rightarrow v ; x, t)=\delta(u+v)
\end{gathered}
$$

where $r_{\perp}=0, r_{\|}=2$.

Due to the generality of the C-L model, it has been vastly used in many field, on the rarefied gas flow in 30, 31, 35, 36, 37; extension to the gas surface interaction model in fluid dynamics 33, 32, 40; on the linearized Boltzmann equation in [19, 39, 34, 18; on S-model kinetic equation in 38 etc.

In this paper we will study the regularity of both the dynamical and steady Boltzmann equation with C-L boundary. The Boltzmann equation with scattering type boundary condition (1.5) has been studied in many aspects. [21, 20, 28, 27, 29] studied the dynamical solution with diffuse, specular and bounce back boundary condition. With such boundary condition, [25, 24, [17, 3] studied the fluid limit of the Boltzmann equation. Moreover, a unique stationary solution has been constructed in [15, 16, 14. Inspired by these studies, in 9 ] the author constructed a unique local dynamical solution and a unique steady solution with C-L boundary in bounded domain.

In non-convex domain the Boltzmann equation possess a boundary singularity [26], and BV is the best estimate we can expect [22]. In convex domain [23] proposed a kinetic weight to construct a unique $C^{1}$ and $W^{1, p}$ dynamical solution. With convex domain the kinetic weight can be further applied to study the Vlasov-Poisson-Boltzmann system [5, 11, 4, 1, 2, In terms of the steady solution, [12] studied the regularity of the stationary linearized Boltzmann equation. Recently a unique weighted $C^{1}$ steady solution in convex domain has been constructed in 10. Our work in this paper originate from these studies and focus on both the dynamical and steady solution.

Throughout this paper we assume the domain is $C^{2}$ and defined as $\Omega=\left\{x \in \mathbb{R}^{3}: \xi(x)<0\right\}$ via a $C^{2}$ function $\xi: \mathbb{R}^{3} \rightarrow \mathbb{R}$. We further assume that the domain is strictly convex in the following sense:

$$
\begin{equation*}
\sum_{i, j=1}^{3} \zeta_{i} \zeta_{j} \partial_{i} \partial_{j} \xi(x) \gtrsim|\zeta|^{2} \text { for all } x \in \bar{\Omega} \text { and } \zeta \in \mathbb{R}^{3} \tag{1.14}
\end{equation*}
$$

Without loss of generality we may assume that $\nabla \xi \neq 0$ near $\partial \Omega$.
Denote the maximum and minimum wall temperature as:

$$
\begin{equation*}
T_{M}:=\max \left\{T_{w}(x)\right\}<\infty \quad T_{m}:=\min \left\{T_{w}(x)\right\}>0 \tag{1.15}
\end{equation*}
$$

It is well known that singularity propagates for the derivative in the boundary value problem [26]. In order to control the generic singularity at the boundary we adopt the following weight of [23]:

Definition 1. For sufficiently small $0<\varepsilon \ll\|\xi\|_{C^{2}}$, we define a kinetic distance:

$$
\begin{equation*}
\alpha(x, v):=\chi_{\varepsilon}(\tilde{\alpha}(x, v)), \quad \tilde{\alpha}(x, v):=\sqrt{\left|v \cdot \nabla_{x} \xi(x)\right|^{2}-2 \xi(x)\left(v \cdot \nabla_{x}^{2} \xi(x) \cdot v\right)}, \quad(x, v) \in \bar{\Omega} \times \mathbb{R}^{3}, \tag{1.16}
\end{equation*}
$$

where $\chi_{a}:[0, \infty) \rightarrow[0, \infty)$ stands for a non-decreasing smooth function such that

$$
\begin{equation*}
\chi_{\alpha}(s)=s \text { for } s \in[0, a], \chi_{a}(s)=2 \text { a for } s \in[4 a, \infty], \text { and }\left|\chi_{a}^{\prime}(s)\right| \leq 1 \text { for } \tau \in[0, \infty) \tag{1.17}
\end{equation*}
$$

The definition of $\xi$ in (1.14) implies that $\xi(x)=0, x \in \partial \Omega$,

$$
\begin{equation*}
\text { when } x \in \partial \Omega \text { and }|n(x) \cdot v| \ll 1, \alpha(x, v) \sim n(x) \cdot v \text {. } \tag{1.18}
\end{equation*}
$$

We will use this kinetic weight to cancel the singularity on the boundary. Lemma 6 indicates that such weight is almost invariant along the trajectory.

Denote

$$
\begin{gather*}
w_{\theta}:=e^{\theta|v|^{2}}  \tag{1.19}\\
\langle v\rangle:=\sqrt{|v|^{2}+1} . \tag{1.20}
\end{gather*}
$$

1.1. Result of dynamical Boltzmann equation. Define the global Maxwellian using the maximum wall temperature:

$$
\begin{equation*}
\mu:=e^{-\frac{|v|^{2}}{2 T_{M}}}, \tag{1.21}
\end{equation*}
$$

and weight $F$ in (1.1) with it: $F=\sqrt{\mu} f$. Then $f$ satisfies

$$
\begin{equation*}
\partial_{t} f+v \cdot \nabla_{x} f=\Gamma(f, f) \tag{1.22}
\end{equation*}
$$

where the collision operator becomes:

$$
\begin{equation*}
\Gamma\left(f_{1}, f_{2}\right)=\Gamma_{\text {gain }}\left(f_{1}, f_{2}\right)-\nu\left(F_{1}\right) F_{2} / \mu=\frac{1}{\sqrt{\mu}} Q_{\operatorname{gain}}\left(\sqrt{\mu} f_{1}, \sqrt{\mu} f_{2}\right)-\nu\left(F_{1}\right) f_{2} \tag{1.23}
\end{equation*}
$$

The weighted $C^{1}$ estimate is given in the following theorem.

Theorem 1. Assume $\Omega \subset \mathbb{R}^{3}$ is bounded, convex and $C^{2}$. Let $0<\theta<\frac{1}{4 T_{M}}$. Assume

$$
\begin{gather*}
0<r_{\perp} \leq 1, \quad 0<r_{\|}<2  \tag{1.24}\\
\frac{T_{m}}{T_{M}}>\max \left(\frac{1-r_{\|}}{2-r_{\|}}, \frac{\sqrt{1-r_{\perp}}-\left(1-r_{\perp}\right)}{r_{\perp}}\right), \tag{1.25}
\end{gather*}
$$

where $T_{M}, T_{m}$ are defined in (1.15).
Also assume the initial condition has bound

$$
\begin{equation*}
\left\|\alpha \nabla_{x, v} f_{0}\right\|_{\infty}<\infty \tag{1.26}
\end{equation*}
$$

Then for some

$$
t_{\infty}=t_{\infty}\left(\left\|w_{\theta} f\right\|_{\infty}, r_{\perp}, r_{\|}, \theta, T_{M}, T_{m}, \Omega\right) \ll 1
$$

we can construct a unique solution $F=\sqrt{\mu} f$ satisfies

$$
\begin{equation*}
\sup _{0 \leq t \leq t_{\infty}}\left\|e^{-\lambda\langle v\rangle t} \alpha \nabla_{x, v} f\right\|_{\infty} \lesssim\left\|\alpha \nabla_{x, v} f_{0}\right\|_{\infty} \tag{1.27}
\end{equation*}
$$

Here $\alpha$ is the kinetic weight defined in (1.16), $\lambda \geq 1$ is a constant specified in (3.8), $\left\|w_{\theta} f\right\|_{\infty}$ is the $L^{\infty}$ estimate given in Theorem 3 with $w_{\theta}(v)=e^{\theta|v|^{2}}$.
Remark 2. The well-posedness of the solution $F=\sqrt{\mu} f$ and $L^{\infty}$ estimate $\left\|w_{\theta} f\right\|_{\infty}$ are proved in [9], in this paper we will focus on the weighted $C^{1}$ estimate (1.27). We record the well-posedness and $L^{\infty}$ estimate in Theorem 3 in section 2.

Remark 3. In Theorem 1 the accommodation coefficient can be any number except $r_{\perp}=0, r_{\|}=0,2$, which corresponds to pure reflection or bounce back reflection. For wall temperature we have a relaxed condition (1.25) rather than the small fluctuation. In particular, for the pure diffuse reflection, i.e, $r_{\|}=r_{\perp}=1$, there is no constraint to the temperature (except $T_{M}<\infty, T_{m}>0$ ).
1.2. Result of steady Boltzmann equation. We also establish the weighted $C^{1}$-estimate for the steady problem. The steady Boltzmann equation is given as

$$
\begin{equation*}
v \cdot \nabla_{x} F_{s}=Q\left(F_{s}, F_{s}\right), \quad(x, v) \in \Omega \times \mathbb{R}^{3} \tag{1.28}
\end{equation*}
$$

with $F_{s}$ satisfying the C-L boundary condition. Here we note that we use $F_{s}$ to represent the steady solution.
We use the short notation $\mu_{0}$ to denote the global Maxwellian with temperature $T_{0}$,

$$
\mu_{0}:=\frac{1}{2 \pi\left(T_{0}\right)^{2}} \exp \left(-\frac{|v|^{2}}{2 T_{0}}\right) .
$$

Here we mark that $\mu_{0}$ is the global Maxwellian for the steady problem while the $\mu$ defined in (1.21) is the global Maxwellian for the dynamical problem.

Let $F_{s}=\mu_{0}+\sqrt{\mu_{0}} f_{s}$. The equation of $f_{s}$ reads

$$
\begin{equation*}
v \cdot \nabla_{x} f_{s}+L f_{s}=\Gamma\left(f_{s}, f_{s}\right) . \tag{1.29}
\end{equation*}
$$

Here $L$ is the standard linearized Boltzmann operator

$$
\begin{equation*}
L f_{s}:=-\frac{1}{\sqrt{\mu_{0}}}\left[Q\left(\mu_{0}, \sqrt{\mu_{0}} f_{s}\right)+Q\left(\sqrt{\mu_{0}} f_{s}, \mu_{0}\right)\right]=\nu(v) f_{s}-K f_{s} \tag{1.30}
\end{equation*}
$$

with the collision frequency $\nu(v) \equiv \iint_{\mathbb{R}^{3} \times \mathbb{S}^{2}} B\left(v-v_{*}, w\right) \mu_{0}\left(v_{*}\right) \mathrm{d} w \mathrm{~d} v_{*} \sim\{1+|v|\}$. When we assume small fluctuation of the wall temperature and the accommodation coefficient, the steady problem is well-posed 9].

In this paper we also derive the weighted- $C^{1}$ regularity of the steady solution in the following theorem.
Theorem 2. For given $T_{0}>0$, there exists $\delta_{0}>0$ such that if

$$
\begin{equation*}
\sup _{x \in \partial \Omega}\left|T_{w}(x)-T_{0}\right|<\delta_{0}, \quad \max \left\{\left|1-r_{\perp}\right|,\left|1-r_{\|}\right|\right\}<\delta_{0} \tag{1.31}
\end{equation*}
$$

then we can construct a unique steady solution $F_{s}=\mu_{0}+\sqrt{\mu_{0}} f_{s}$ satisfies:

$$
\begin{equation*}
\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty} \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} \lesssim 1 \tag{1.32}
\end{equation*}
$$

Here $w_{\vartheta}=e^{\vartheta|v|^{2}}$ for some $\vartheta>0$.
Remark 4. The well-posedness of $f_{s}$ and the $L^{\infty}$ bound $\left\|w_{\vartheta} f_{s}\right\|_{\infty}$ of the steady solution $F_{s}=\mu_{0}+\sqrt{\mu_{0}} f_{s}$ is proved in [9, we record the result in Corollary 4 in section 2. In this paper we focus on proving the regularity estimate (1.32).

Remark 5. In Theorem [2, different to Theorem 1, we need to restrict these two coefficients to be close to 1 in (1.31). To be more specific, we require the C-L boundary to be close to the diffuse boundary condition.
1.3. Difficulty and proof strategy. Dynamical solution. First we illustrate the difficulty and strategy for the dynamical solution in Theorem 1. A common approach for the boundary value problem is to iterate along the backward characteristic until hitting the boundary or the initial datum. In order to clearly state and address the difficulty, we briefly recall the strategy for the well-posedness of the dynamical solution as stated in [9. We define the stochastic cycle $v_{k}, v_{k-1}, \cdots, v_{1}$ in Definition 2. The backward characteristic may hit the boundary for $k$-times before reaching the initial datum. The boundary condition (3.1) will generate a $k$-fold integration. Due to the probability measure $\mathrm{d} \sigma\left(v_{k}, v_{k-1}\right)$ ( see (3.2)), the integral of $v_{k}$ is roughly

$$
\begin{equation*}
\int_{n(x) \cdot v_{k}>0} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]\left|v_{k}\right|^{2}} \mathrm{~d} \sigma\left(v_{k}, v_{k-1}\right) . \tag{1.33}
\end{equation*}
$$

Indeed the integrand is of the form of exponential, we can explicitly compute the above integration as a function of $v_{k-1}$ and adapt the result to the integration over $v_{k-1}$. In such way we can derive an induction formula to compute the $k$-fold integration.

For the rest stochastic cycle, i.e, $v_{k+1}, v_{k+2}, \cdots$, for large $k$, physically it means the characteristic does not reach the initial datum after a large number of interaction with boundary. We follow the idea in [23] to introduce the grazing set

$$
\gamma_{+}^{\delta}=\left\{u \in \gamma_{+}:|n \cdot u|>\delta,|u| \leq \delta^{-1}\right\}
$$

In such subspace characteristic need to take certain time to reach the boundary. One can derive the lower bound of the time as $O\left(\frac{1}{\delta^{3}}\right)$. For bounded $t$ there can be at most $N=O\left(\frac{1}{\delta^{3}}\right)$ many $v_{i}$ belong to such subspace. For the rest $v_{i} \in \gamma_{+} \backslash \gamma_{+}^{\delta}$, the integration over such subspace results in a small magnitude number $O(\delta)$. Thus for large $k$, we get a large power of $O(\delta)$ and thus derive that the measure of the rest cycle $v_{k+1}, v_{k+2}, \cdots$ is small. Hence we will choose a proper $k$ depend on $N=O\left(\frac{1}{\delta^{3}}\right)$.

When it comes to the regularity, it is well-known that singularity occurs at the backward exit position $x_{\mathbf{b}}(x, v):=x-t_{\mathbf{b}}(x, v) v$ which is defined through a backward exit time $t_{\mathbf{b}}$ :

$$
\begin{equation*}
t_{\mathbf{b}}(x, v):=\sup \{s>0: x-s v \in \Omega\} \tag{1.34}
\end{equation*}
$$

Thus while estimating the regularity, the singularity occurs at the boundary. In Theorem 1 we include the kinetic weight (1.16) since such weight can cancel the singularity as stated in Lemma 6. Besides the singularity the boundary condition for $\partial f$ actually has a nice form as stated in Lemma 13 , which looks similar to the boundary condition of $f$ in (3.1).

Even though the boundary condition in our case is similar to the case of $f$, the extra term $\langle v\rangle^{2}$ in Lemma 13 , brings difficulty to our analysis. Since the computation involves various integration with exponential, it is natural to bound polynomial term $\langle v\rangle$ by exponential and adapt it into the computation. For a single integration such upper bound does not have big effect. However, as stated above, we trace back along the characteristic for large $k$ times. Thus in order to follow the induction formula for the $k$-fold integration, we need to bound

$$
\begin{equation*}
\langle v\rangle^{2} \lesssim \frac{1}{\varepsilon} e^{\varepsilon|v|^{2}} \tag{1.35}
\end{equation*}
$$

with small enough coefficient $\varepsilon$. Such extra exponential term $e^{\varepsilon|v|^{2}}$ will slightly increase the coefficient of the exponential after an integration. With a $k$-fold integration we need to impose the $k$-dependence on $\varepsilon$. Since $k$ depends on $N=O\left(1 / \delta^{3}\right)$, the term $\frac{1}{\varepsilon(k)}$ in (1.35) depends on $\delta$ as well. It will be combined with the small magnitude number $O(\delta)$ for the nongrazing set $\gamma_{+} \backslash \gamma_{+}^{\delta}$. Then in order to derive the smallness, we need to ensure $\delta \ll \varepsilon(\delta)$. Unfortunately, with such properties, the $k$-fold integration does not remain bounded.

To overcome such difficulty a key observation is: since we consider local-in-time $[0, t]$, we can obtain a better bound for $N$ as $O\left(\frac{t}{\delta^{3}}\right)$. Thus we can write $\delta=t^{1 / 3} \delta^{\prime}$ for some $\delta^{\prime} \ll 1$. Since we are considering local-in-time regularity, $t$ can be finally designed to be small and depend on all the other variables $k, \delta, \cdots$. In such setting $k=k(N)=k\left(\delta^{\prime}\right)$, which does not depend on $t$. With the extra $t^{1 / 3}$ we can choose proper $\varepsilon$ to satisfy the condition $\delta \ll \varepsilon$ as follow: instead of imposing the $k$ dependence on $\varepsilon$, we directly impose the $t$ dependence as in Lemma 5. Then we assume $\varepsilon=t^{c}$ for some $\frac{1}{3}>c>0$ and incorporate $\frac{1}{t^{c}}$ with $\delta=t^{1 / 3} \delta^{\prime}$ in the computation. Finally, we choose $t$ to be small to ensure the $k$-fold integration is bounded. In order to obtain the smallness for the rest cycle $v_{k+1}, v_{k+2}, \cdots$ in (3.137), we specify $c=1 / 15$.

Steady solution. Then we come to the steady solution in Theorem 2. We express the steady solution as perturbation around a global Maxwellian $F=\mu_{0}+\sqrt{\mu_{0}} f, \mu_{0}=e^{-\frac{|v|^{2}}{2 T_{0}}}$ and trace back along the characteristic
as (4.14)-(4.18). The weighted $C^{1}$ regularity with pure diffuse boundary condition is established in [10], we use the same method to deal with the collision term(not related to the boundary). Then the new difficulty comes from the boundary term. The boundary condition for $f$ can be computed as in Lemma 21] Thus the most singular term from the boundary reads

$$
\begin{equation*}
\nabla_{x} x_{\mathbf{b}} e^{\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{\mathbf{b}}\right)}\right]|v|^{2}} \int_{n\left(x_{\mathbf{b}}\right) \cdot v_{1}>0} \nabla_{x_{\mathbf{b}}} f\left(x_{\mathbf{b}}, v_{1}\right) e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{\mathbf{b}}\right)}\right]\left|v_{1}\right|^{2}} \mathrm{~d} \sigma\left(v_{1}, v\right) . \tag{1.36}
\end{equation*}
$$

Using the characteristic once again for $f\left(x_{\mathbf{b}}, v_{1}\right)$, the contribution of the collision operator (ignoring the singularity of $Q$ for simplicity) can be viewed as

$$
\begin{equation*}
\nabla_{x} x_{\mathbf{b}} e^{\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{\mathbf{b}}\right)}\right]|v|^{2}} \int_{n\left(x_{\mathbf{b}}\right) \cdot v_{1}>0} \int_{0}^{t_{\mathbf{b}}\left(x_{\mathbf{b}}, v_{1}\right)} \nabla_{x} f\left(x_{\mathbf{b}}-s v_{1}, v_{1}\right) e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{\mathbf{b}}\right)}\right]\left|v_{1}\right|^{2}} \mathrm{~d} u \mathrm{~d} s \mathrm{~d} \sigma\left(v_{1}, v\right) \tag{1.37}
\end{equation*}
$$

We can exchange the $x$-derivative into $v_{1}$-derivative as

$$
\begin{equation*}
\nabla_{x} f\left(x_{\mathbf{b}}-\left(t_{1}-s\right) v_{1}, u\right)=\frac{\nabla_{v_{1}}\left[F\left(x_{\mathbf{b}}-\left(t_{1}-s\right) v_{1}, u\right)\right]}{-\left(t_{1}-s\right)} \tag{1.38}
\end{equation*}
$$

Since the accommodation coefficient and wall temperature are assumed to have a small fluctuation as in (1.31), such integration is "close" to the integration of the pure diffuse boundary condition. Then we can apply the change of variable to remove the $v^{1}$-derivative completely from $f$. Different to the pure diffuse boundary condition, the C-L boundary will generate more polynomial factors due to the normal and tangential components in (1.6). Thanks to exponential decay term in the integrand, the polynomial factors will not affect the integrability. In Lemma 22 we compute these integration with extra polynomial terms, extra derivative in detail. Thus the integration can be bounded by $\nabla_{x} x_{\mathbf{b}}\|f\|_{\infty}$.

Another singular term is the boundary contribution of (1.36) along the characteristic:

$$
\begin{equation*}
\nabla_{x} x_{\mathbf{b}} e^{\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{\mathbf{b}}\right)}\right]|v|^{2}} \int_{n\left(x_{\mathbf{b}}\right) \cdot v_{1}>0} \nabla_{x_{\mathbf{b}}} f\left(x_{\mathbf{b}}\left(x_{1}, v_{1}\right), v_{1}\right) e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{\mathbf{b}}\right)}\right]\left|v_{1}\right|^{2}} \mathrm{~d} \sigma\left(v_{1}, v\right) \tag{1.39}
\end{equation*}
$$

The key idea is to convert $v_{1}$-integration to the integration in $\left(x_{2}, t_{\mathbf{b}}\left(x_{1}, v_{1}\right)\right)=\left(x_{\mathbf{b}}\left(x_{1}, v_{1}\right), t_{\mathbf{b}}\left(x_{1}, v_{1}\right)\right)$, with Jacobian given in Lemma 11. Then we are able to remove $\nabla_{x_{\mathrm{b}}}$-derivative from $f$ via the integration by parts. Similar to the collision term (1.37), the integration by parts will generate more polynomial factors. These factors won't affect the integrability. Thus we can again remove the derivative and bound such contribution by $\nabla_{x} x_{\mathbf{b}}\|f\|_{\infty}$.
1.4. Outline. In section 2 we list several lemmas as preparation. In section 3 we derive the weighted $C^{1}$ bound for the dynamical solution and conclude Theorem 1. In section 4 we derive the weighted $C^{1}$ bound for the steady solution and conclude Theorem 2,

## 2. Preliminary

2.1. Basic setting. Throughout this paper we will use the following notation:

$$
\begin{gather*}
f \lesssim g \Leftrightarrow \text { there exists } 0<C<\infty \text { such that } f \leq C g  \tag{2.1}\\
f=O(g) \Leftrightarrow \text { there exists } 0<C<\infty \text { such that } f=C g .  \tag{2.2}\\
f=o(g) \Leftrightarrow \text { there exists } c \ll 1 \text { such that } f=c g . \tag{2.3}
\end{gather*}
$$

First we record the local well-posedness of the dynamical Boltzmann equation with the C-L boundary.
Theorem 3. Assume $\Omega \subset \mathbb{R}^{3}$ is bounded and $C^{2}$. Let $0<\theta<\frac{1}{4 T_{M}}$. Assume wall temperature satisfies (1.24) and (1.25). If $F_{0}=\sqrt{\mu} f_{0} \geq 0$ and $f_{0}$ satisfies

$$
\begin{equation*}
\left\|w_{\theta} f_{0}\right\|_{\infty}<\infty \tag{2.4}
\end{equation*}
$$

then there exists a unique solution $F(t, x, v)=\sqrt{\mu} f(t, x, v) \geq 0$ to (1.1) and (1.5) in $\left[0, t_{d y m}\right] \times \Omega \times \mathbb{R}^{3}$ for some $t_{\text {dym }} \ll 1$. Moreover, the solution $F=\sqrt{\mu} f$ satisfies

$$
\begin{equation*}
\sup _{0 \leq t \leq t_{d y m}}\left\|w_{\theta} e^{-|v|^{2} t} f(t)\right\|_{\infty} \lesssim\left\|w_{\theta} f_{0}\right\|_{\infty} \tag{2.5}
\end{equation*}
$$

Then we record the well-posedness of the steady Boltzmann equation with the C-L boundary.

Corollary 4. For given $T_{0}>0$, if the wall temperature and accommodation coefficient satisfies (1.31), then there exists a unique non-negative solution $F_{s}=\mu_{0}+\sqrt{\mu_{0}} f_{s} \geq 0$ with $\iint_{\Omega \times \mathbb{R}^{3}} f_{s} \sqrt{\mu_{0}} \mathrm{~d} x \mathrm{~d} v=0$ to the steady problem (1.28). And for some $\vartheta>0$,

$$
\left\|e^{\vartheta|v|^{2}} f_{s}\right\|_{\infty} \lesssim \delta_{0} \ll 1
$$

Definition 2. Let $\left(X^{1}(s ; t, x, v), v\right)$ be the location and velocity along the backward trajectory before hitting the boundary,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} s}\binom{X^{1}(s ; t, x, v)}{v}=\binom{v}{0} . \tag{2.6}
\end{equation*}
$$

Therefore, from (2.6), we have

$$
X^{1}(s ; t, x, v)=x-(t-s) v
$$

Define the back-time cycle as

$$
\begin{gathered}
t_{1}(t, x, v)=\sup \left\{s<t: X^{1}(s ; t, x, v) \in \partial \Omega\right\} \\
x_{1}(t, x, v)=X^{1}\left(t_{1}(t, x, v) ; t, x, v\right) \\
v_{1} \in\left\{v_{1} \in \mathbb{R}^{3}: n\left(x_{1}\right) \cdot v_{1}>0\right\}
\end{gathered}
$$

Also define

$$
\mathcal{V}_{1}=\left\{v_{1}: n\left(x_{1}\right) \cdot v_{1}>0\right\}, \quad x_{1} \in \partial \Omega
$$

Inductively, before hitting the boundary for the $k$-th time, define

$$
\begin{gathered}
t_{k}\left(t, x, v, v_{1}, \cdots, v_{k-1}\right)=\sup \left\{s<t_{k-1}: X^{k}\left(s ; t_{k-1}, x_{k-1}, v_{k-1}\right) \in \partial \Omega\right\}, \\
x_{k}\left(t, x, v, v_{1}, \cdots, v_{k-1}\right)=X^{k}\left(t_{k}\left(t, x, v, v_{k-1}\right) ; t_{k-1}(t, x, v), x_{k-1}(t, x, v), v_{k-1}\right), \\
v_{k} \in\left\{v_{k} \in \mathbb{R}^{3}: n\left(x_{k}\right) \cdot v_{k}>0\right\} \\
\mathcal{V}_{k}=\left\{v_{k}: n\left(x_{k}\right) \cdot v_{k}>0\right\} \\
X^{k}\left(s ; t_{k-1}, x_{k-1}, v_{k-1}\right)=x_{k-1}-\left(t_{k-1}-s\right) v_{k-1} .
\end{gathered}
$$

Here we set

$$
\left(t_{0}, x_{0}, v_{0}\right)=(t, x, v)
$$

For simplicity, we denote

$$
X^{k}(s):=X^{k}\left(s ; t_{k-1}, x_{k-1}, v_{k-1}\right)
$$

for the rest lemmas and propositions.
2.2. Properties of the C-L scattering kernel. In this subsection we list some basic properties of the scattering kernel (1.6).

Lemma 1. (Lemma 10 in [9])
For $R(u \rightarrow v ; x, t)$ given by (1.6) and any $u$ such that $n(x) \cdot u>0$, we have

$$
\begin{equation*}
\int_{n(x) \cdot v<0} R(u \rightarrow v ; x, t) \mathrm{d} v=1 \tag{2.7}
\end{equation*}
$$

Lemma 2. (Lemma 11 in [9])
For any $a>0, b>0, \varepsilon>0$ such that $a+\varepsilon<b$, we have

$$
\begin{equation*}
\frac{b}{\pi} \int_{\mathbb{R}^{2}} e^{\varepsilon|v|^{2}} e^{a|v|^{2}} e^{-b|v-w|^{2}} \mathrm{~d} v \leq \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon}|w|^{2}} \tag{2.8}
\end{equation*}
$$

And when $\delta \ll 1$,

$$
\begin{align*}
\frac{b}{\pi} \int_{\left|v-\frac{b}{b-a-\varepsilon} w\right|>\delta^{-1}} e^{\varepsilon|v|^{2}} e^{a|v|^{2}} e^{-b|v-w|^{2}} \mathrm{~d} v & \leq e^{-(b-a-\varepsilon) \delta^{-2}} \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon}|w|^{2}} \\
& \leq \delta \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon}|w|^{2}} \tag{2.9}
\end{align*}
$$

Lemma 3. (Lemma 12 in 9])
For any $a>0, b>0, \varepsilon>0$ with $a+\varepsilon<b$,

$$
\begin{equation*}
2 b \int_{\mathbb{R}^{+}} e^{\varepsilon v^{2}} e^{a v^{2}} e^{-b v^{2}} e^{-b w^{2}} I_{0}(2 b v w) \mathrm{d} v \leq \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon} w^{2}} \tag{2.10}
\end{equation*}
$$

And when $\delta \ll 1$,

$$
\begin{equation*}
2 b \int_{0<v<\delta} e^{\varepsilon v^{2}} e^{a v^{2}} e^{-b v^{2}} e^{-b w^{2}} I_{0}(2 b v w) \mathrm{d} v \leq \delta \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon} w^{2}} \tag{2.11}
\end{equation*}
$$

Lemma 4. (Lemma 13 in 9])
For any $m, n>0$, when $\delta \ll 1$, we have

$$
\begin{equation*}
2 m^{2} \int_{\frac{n}{m} u_{\perp}+\delta^{-1}}^{\infty} v_{\perp} e^{-m^{2} v_{\perp}^{2}} I_{0}\left(2 m n v_{\perp} u_{\perp}\right) e^{-n^{2} u_{\perp}^{2}} \mathrm{~d} v_{\perp} \lesssim e^{-\frac{m^{2}}{4 \delta^{2}}} \tag{2.12}
\end{equation*}
$$

In consequence, for any $a>0, b>0, \varepsilon>0$ with $a+\varepsilon<b$,

$$
\begin{align*}
2 b \int_{\frac{b}{b-a-\varepsilon} w+\delta^{-1}}^{\infty} v e^{\varepsilon v^{2}} e^{a v^{2}} e^{-b v^{2}} e^{-b w^{2}} I_{0}(2 b v w) \mathrm{d} v & \leq e^{\frac{-(b-a-\varepsilon)}{4 \delta^{2}}} \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon} w^{2}}  \tag{2.13}\\
& \leq \delta \frac{b}{b-a-\varepsilon} e^{\frac{(a+\varepsilon) b}{b-a-\varepsilon} w^{2}} \tag{2.14}
\end{align*}
$$

To tackle the difficulty mentioned in (1.35), we bound the polynomial by exponential in the following lemma.
Lemma 5. For $0<c<1$ and $\lambda>1$ we have the following the upper bound:

$$
\begin{equation*}
\langle v\rangle^{4} e^{\lambda\langle v\rangle t} \leq 2 t^{-c / 2} e^{t^{c}|v|^{2}} \leq t^{-c} e^{t^{c}|v|^{2}} \tag{2.15}
\end{equation*}
$$

Proof. For $t<t^{c} \ll 1$ and $c<1$, we bound

$$
\begin{aligned}
e^{\lambda\langle v\rangle t} \leq e^{\lambda t} e^{\lambda|v| t} & \leq 2 e^{t^{c / 2}\left(|v|^{2}+\lambda^{2}\right)} \leq 4 e^{t^{c / 2}|v|^{2}} \\
\langle v\rangle^{4} & \leq t^{-c / 2} e^{t^{c / 2}|v|^{2}}
\end{aligned}
$$

In the first inequality we have used $t \ll 1$ to have

$$
e^{\lambda t}<e^{t^{c / 2} \lambda^{2}}<2, \quad \lambda|v|<\lambda^{2}+|v|^{2}
$$

In the second inequality we have used

$$
\langle v\rangle=1+|v| \leq 2+|v|^{2} \leq t^{-c / 8}+|v|^{2} \leq t^{-c / 8} e^{t^{c / 8}|v|^{2}}
$$

where we have used the Taylor expansion for $e^{t^{\frac{c}{8}}|v|^{2}}$ in the last step.
Thus with $t \ll 1$ we conclude the lemma.
2.3. Properties of the collision kernel and kinetic weight. The next lemma indicates the invariant property of $\alpha$ under the operator $v \cdot \nabla_{x}$.
Lemma 6. (Lemma 2 in [23])
When the transport operator acts on $\alpha$, we have an upper bound

$$
\begin{equation*}
v \cdot \nabla_{x} \alpha \lesssim_{\xi}|v| \alpha(x, v) \tag{2.16}
\end{equation*}
$$

Moreover, there exists $C=C(\xi)$ such that for all $0 \leq s_{1}, s_{2} \leq t$,

$$
\begin{equation*}
e^{-C|v|\left|s_{1}-s_{2}\right|} \alpha\left(s_{1} ; t, x, v\right) \leq \alpha\left(s_{2} ; t, x, v\right) \leq e^{C|v|\left|s_{1}-s_{2}\right|} \alpha\left(s_{1} ; t, x, v\right) \tag{2.17}
\end{equation*}
$$

We summarize the properties of the collision operator in the following lemma.
Lemma 7. (Lemma 12 in [10])
The linearized Boltzmann operator in (1.30) has the following form:

$$
\begin{gather*}
\nu(v) \equiv \iint_{\mathbb{R}^{3} \times \mathbb{S}^{2}} B(v-u, w) \mu_{0}(u) \mathrm{d} w \mathrm{~d} u \sim 1+|v|  \tag{2.18}\\
\left|\nabla_{v} \nu(v)\right| \lesssim 1  \tag{2.19}\\
K f_{s}=\int_{\mathbb{R}^{3}} \mathbf{k}(v, u) f_{s}(u) \mathrm{d} u \tag{2.20}
\end{gather*}
$$

where $\mathbf{k}(v, u) \lesssim \mathbf{k}_{\varrho}(v, u)$ with

$$
\begin{equation*}
\mathbf{k}_{\varrho}(v, u)=\frac{e^{-\varrho|v-u|^{2}}}{|v-u|} \text { for some } \varrho>0 \tag{2.21}
\end{equation*}
$$

The $\mathbf{k}_{\varrho}(v, u)$ satisfies the following condition:

$$
\begin{equation*}
\mathbf{k}_{\varrho}(v, u) \in L_{u}^{1}, \quad \frac{\mathbf{k}(v, u)}{|v-u|} \in L_{u}^{1} \tag{2.22}
\end{equation*}
$$

In consequence,

$$
\begin{equation*}
\left\|K f_{s}\right\|_{\infty} \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} \tag{2.23}
\end{equation*}
$$

The derivative of $\mathbf{k}(v, u)$ satisfies the following condition:

$$
\begin{equation*}
\left|\nabla_{u} \mathbf{k}(v, u)\right| \lesssim \frac{\mathbf{k}_{\varrho}(v, u)}{|v-u|} \tag{2.24}
\end{equation*}
$$

For $(i, j)=(1,2)$ or $(i, j)=(2,1)$, the nonlinear Boltzmann operator can be bounded as

$$
\begin{equation*}
\Gamma_{\text {gain }}\left(f_{1}, f_{2}\right) \lesssim\left\|w_{\theta} f_{i}\right\|_{\infty} \int_{\mathbb{R}^{3}} \mathbf{k}_{\varrho}(v, u)\left|f_{j}(x, u)\right| \mathrm{d} u \tag{2.25}
\end{equation*}
$$

In consequence, we have

$$
\begin{gather*}
\left\|\Gamma\left(f_{s}, f_{s}\right)\right\|_{\infty} \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}  \tag{2.26}\\
\left|\partial_{x, v} \Gamma_{g a i n}(f, f)\right| \lesssim\left\|w_{\theta} f\right\|_{\infty} \int_{\mathbb{R}^{3}} \mathbf{k}_{\varrho}(v, u)\left|\partial_{x, v} f(x, u)\right| \mathrm{d} u+\left\|w_{\theta} f_{s}\right\|_{\infty}^{2}  \tag{2.27}\\
\left|\nabla_{x} \Gamma\left(f_{s}, f_{s}\right)(x, v)\right| \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} \frac{\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)}+\left\|w_{\vartheta} f_{s}\right\|_{\infty} \int_{\mathbb{R}^{3}} \mathbf{k}_{\varrho}(v, u)\left|\nabla_{x} f_{s}(x, u)\right| \mathrm{d} u \tag{2.28}
\end{gather*}
$$

Lemma 8. (Lemma 13 in 10)
If $0<\frac{\tilde{\theta}}{4}<\varrho$, if $0<\tilde{\varrho}<\varrho-\frac{\tilde{\theta}}{4}$,

$$
\begin{equation*}
\mathbf{k}_{\varrho}(v, u) \frac{e^{\tilde{\theta}|v|^{2}}}{e^{\tilde{\theta}|u|^{2}}} \lesssim \mathbf{k}_{\tilde{\varrho}}(v, u) \tag{2.29}
\end{equation*}
$$

where $\mathbf{k}_{\varrho}$ is defined in (2.21).
When we integrate the collision operator $\partial \Gamma_{\text {gain }}(f, f)$ given in (2.25), to construct $\alpha$-weighted $C^{1}$ bound, the extra weight $\alpha$ appears in the denominator. The following lemma is desired to bound the integration of $\frac{1}{\alpha}$.
Lemma 9. (Lemma 14 in [10])

$$
\begin{align*}
& \int_{0}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^{3}} \frac{\mathbf{k}_{\varrho}(v, u)}{\alpha(x-(t-s) v, u)} \mathrm{d} u \lesssim \frac{t}{\alpha(x, v)}  \tag{2.30}\\
& \int_{t-\varepsilon}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^{3}} \frac{\mathbf{k}_{\varrho}(v, u)}{\alpha(x-(t-s) v, u)} \mathrm{d} u \lesssim \frac{O(\varepsilon)}{\alpha(x, v)} \tag{2.31}
\end{align*}
$$

2.4. Reparametrization of boundary and stochastic cycle. In this subsection we reparametrize the boundary and stochastic cycle in Definition 2 We will mainly use the reparametrization in section 4 to prove Theorem 2.

We assume that for all $q \in \partial \Omega$, there exists $0<\delta_{1} \ll 1$

$$
\begin{equation*}
\eta_{q}: B_{+}\left(0 ; \delta_{1}\right) \ni \mathbf{x}_{q}:=\left(\mathbf{x}_{q, 1}, \mathbf{x}_{q, 2}, \mathbf{x}_{q, 3}\right) \rightarrow \mathcal{O}_{q}:=\eta_{q}\left(B_{+}\left(0 ; \delta_{1}\right)\right) \text { is one-to-one and onto for all } q \in \partial \Omega \tag{2.32}
\end{equation*}
$$ and $\eta_{q}\left(\mathbf{x}_{q}\right) \in \partial \Omega$ if and only if $\mathbf{x}_{q, 3}=0$ within the range of $\eta_{q}$.

Since the boundary is compact and $C^{2}$, for fixed $0<\delta_{1} \ll 1$ we may choose a finite number of $p \in \mathcal{P} \subset \partial \Omega$ and $0<\delta_{2} \ll 1$ such that $\mathcal{O}_{p}=\eta_{p}\left(B_{+}\left(0 ; \delta_{1}\right)\right) \subset B\left(p ; \delta_{2}\right) \cap \bar{\Omega}$ and $\left\{\mathcal{O}_{p}\right\}$ forms a finite covering of $\partial \Omega$. We define a partition of unity

$$
\sum_{p \in \mathcal{P}} \iota_{p}(x)=\left\{\begin{array}{ll}
1, & \text { for } x \in \partial \Omega  \tag{2.33}\\
0 & \text { for } x \notin \mathcal{O}_{p}
\end{array} \text { such that } 0 \leq \iota_{p}(x) \leq 1\right.
$$

Without loss of generality (see [28]) we can always reparametrize $\eta_{p}$ such that $\partial_{\mathbf{x}_{p, i}} \eta_{p} \neq 0$ for $i=1,2,3$ at $\mathbf{x}_{p, 3}=0$, and an orthogonality holds as

$$
\begin{equation*}
\partial_{\mathbf{x}_{p, i}} \eta_{p} \cdot \partial_{\mathbf{x}_{p, j}} \eta_{p}=0 \text { at } \mathbf{x}_{p, 3}=0 \text { for } i \neq j \text { and } i, j \in\{1,2,3\} \tag{2.34}
\end{equation*}
$$

For simplicity, we denote

$$
\begin{equation*}
\partial_{i} \eta_{p}\left(\mathbf{x}_{p}\right):=\partial_{\mathbf{x}_{p, i}} \eta_{p} \tag{2.35}
\end{equation*}
$$

Definition 3. For $x \in \bar{\Omega}$, we choose $p \in \mathcal{P}$ as in (2.32). We define

$$
\begin{align*}
g_{p, i i}\left(\mathbf{x}_{p}\right) & =\left\langle\partial_{i} \eta_{p}\left(\mathbf{x}_{p}\right), \partial_{i} \eta_{p}\left(\mathbf{x}_{p}\right)\right\rangle \text { for } \quad i \in\{1,2,3\}  \tag{2.36}\\
T_{\mathbf{x}_{p}} & =\left(\begin{array}{lll}
\frac{\partial_{1} \eta_{p}\left(\mathbf{x}_{p}\right)}{\sqrt{g_{p, 11}\left(\mathbf{x}_{p}\right)}} & \frac{\partial_{2} \eta_{p}\left(\mathbf{x}_{p}\right)}{\sqrt{g_{p, 22}\left(\mathbf{x}_{p}\right)}} & \frac{\partial_{3} \eta_{p}\left(\mathbf{x}_{p}\right)}{\sqrt{g_{p, 33}\left(\mathbf{x}_{p}\right)}}
\end{array}\right)^{t} \tag{2.37}
\end{align*}
$$

Here $A^{t}$ stands the transpose of a matrix $A$. Note that when $\mathbf{x}_{p, 3}=0, T_{\mathbf{x}_{p}} \frac{\partial_{i} \eta_{p}\left(\mathbf{x}_{p}\right)}{\sqrt{g_{p, i i}\left(\mathbf{x}_{p}\right)}}=e_{i}$ for $i=1,2,3$ where $\left\{e_{i}\right\}$ is a standard basis of $\mathbb{R}^{3}$.

We define

$$
\begin{equation*}
\mathbf{v}_{j}\left(\mathbf{x}_{p}\right)=\frac{\partial_{j} \eta_{p}\left(\mathbf{x}_{p}\right)}{\sqrt{g_{p, j j}\left(\mathbf{x}_{p}\right)}} \cdot v \tag{2.38}
\end{equation*}
$$

We note that from (2.34), the map $T_{\mathbf{x}_{p}}$ is an orthonormal matrix when $\mathbf{x}_{p, 3}=0$. Therefore both maps $v \rightarrow \mathbf{v}\left(\mathbf{x}_{p}\right)$ and $\mathbf{v}\left(\mathbf{x}_{p}\right) \rightarrow v$ have a unit Jacobian. Now we reparametrize the stochastic cycle using the local chart defined in Definition 2

Definition 4. Recall the stochastic cycles in Definition 2. For each cycle $x^{k}$ let us choose $p^{k} \in \mathcal{P}$ in (2.32). Then we denote

$$
\begin{align*}
\mathbf{x}_{p^{k}}^{k} & :=\left(\mathbf{x}_{p^{k}, 1}^{k}, \mathbf{x}_{p^{k}, 2}^{k}, 0\right) \text { such that } \eta_{p^{k}}\left(\mathbf{x}_{p^{k}}^{k}\right)=x_{k}, \quad \text { for } k=1,2, \\
\mathbf{v}_{p^{k}, i}^{k} & :=\frac{\partial_{i} \eta_{p^{k}}\left(\mathbf{x}_{p^{k}}^{k}\right)}{\sqrt{g_{p^{k}, i i}\left(\mathbf{x}_{p^{k}}^{k}\right)}} \cdot v_{k} \quad \text { for } k=1,2 \tag{2.39}
\end{align*}
$$

From chain rule we define

$$
\begin{equation*}
\partial_{\mathbf{x}_{p^{k}, i}^{k}}\left[a\left(\eta_{p^{k}}\left(\mathbf{x}_{p^{k}}^{k}\right), v_{k}\right)\right]:=\frac{\partial \eta_{p^{k}}\left(\mathbf{x}_{p^{k}, i}^{k}\right)}{\partial \mathbf{x}_{p^{k}, i}^{k}} \cdot \nabla_{x} a\left(\eta_{p^{k}}\left(\mathbf{x}_{p^{k}}^{k}\right), v_{k}\right), \quad i=1,2 \tag{2.40}
\end{equation*}
$$

When we study the regularity we will need to take derivative to the stochastic cycle. We summarize the derivative in the following lemma.

Lemma 10. (Lemma 1 in [10)
For the $t_{\mathbf{b}}$ and $x_{\mathbf{b}}$ defined in (1.34), the derivative reads

$$
\begin{array}{r}
\nabla_{x} t_{\mathbf{b}}(x, v)=\frac{n\left(x_{\mathbf{b}}\right)}{n\left(x_{\mathbf{b}}\right) \cdot v}, \quad \nabla_{v} t_{\mathbf{b}}(x, v)=-\frac{t_{\mathbf{b}} n\left(x_{\mathbf{b}}\right)}{n\left(x_{\mathbf{b}}\right) \cdot v} \\
\nabla_{x} x_{\mathbf{b}}(x, v)=I d_{3 \times 3}-\frac{n\left(x_{\mathbf{b}}\right) \otimes v}{n\left(x_{\mathbf{b}}\right) \cdot v}, \quad \nabla_{v} x_{\mathbf{b}}(x, v)=-t_{\mathbf{b}} I d+\frac{t_{\mathbf{b}} n\left(x_{\mathbf{b}}\right) \otimes v}{n\left(x_{\mathbf{b}}\right) \cdot v} \tag{2.41}
\end{array}
$$

For $i=1,2, j=1,2$,

$$
\begin{gather*}
\frac{\partial \mathbf{x}_{p^{2}, i}^{2}}{\partial \mathbf{x}_{p^{1}, j}^{1}}=\frac{1}{\sqrt{g_{p^{2}, i i}\left(\mathbf{x}_{p^{2}}^{2}\right)}}\left[\frac{\partial_{i} \eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)}{\sqrt{g_{p^{2}, i i}\left(\mathbf{x}_{p^{2}}^{2}\right)}}-\frac{\mathbf{v}_{p^{2}, i}^{2}}{\mathbf{v}_{p^{2}, 3}^{2}} \frac{\partial_{3} \eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)}{\sqrt{g_{p^{2}, 33}\left(\mathbf{x}_{p^{2}}^{2}\right)}}\right] \cdot \partial_{j} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) .  \tag{2.42}\\
\frac{\partial \mathbf{x}_{p^{1}, i}^{1}}{\partial[x]_{j}}=\frac{1}{\sqrt{g_{p^{1}, i i}\left(\mathbf{x}_{p^{1}}^{1}\right)}}\left[\frac{\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{\sqrt{g_{p^{1}, i i}\left(\mathbf{x}_{p^{1}}^{1}\right)}}-\frac{\mathbf{v}_{p^{1}, i}}{\mathbf{v}_{p^{1}, 3}} \frac{\partial_{3} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{\sqrt{g_{p^{1}, 33}\left(\mathbf{x}_{p^{1}}^{1}\right)}}\right] \cdot e_{j} . \tag{2.43}
\end{gather*}
$$

Here $[x]_{j}$ is defined as the $j$-th coordinate of $x$ as specified in (4.13).
Lemma 11. (Lemma 3 in [10])
The following map is one-to-one

$$
\begin{equation*}
v_{1} \in\left\{n\left(x_{1}\right) \cdot v_{1}>0: x_{\mathbf{b}}\left(x_{1}, v_{1}\right) \in B\left(p^{2}, \delta_{2}\right)\right\} \mapsto\left(\mathbf{x}_{p^{2}, 1}^{2}, \mathbf{x}_{p^{2}, 2}^{2}, t_{\mathbf{b}}^{1}\right) \tag{2.44}
\end{equation*}
$$

with

$$
\begin{equation*}
\operatorname{det}\left(\frac{\partial\left(\mathbf{x}_{p^{2}, 1}^{2}, \mathbf{x}_{p^{2}, 2}^{2}, t_{\mathbf{b}}^{1}\right)}{\partial v_{1}}\right)=\frac{1}{\sqrt{g_{p^{2}, 11}\left(\mathbf{x}_{p^{2}}^{2}\right) g_{p^{2}, 22}\left(\mathbf{x}_{p^{2}}^{2}\right)}} \frac{\left|t_{\mathbf{b}}^{1}\right|^{3}}{\left|n\left(x_{2}\right) \cdot v_{1}\right|} \tag{2.45}
\end{equation*}
$$

Here $t_{\mathbf{b}}^{1}$ is the as the backward exit time starting from $\left(x_{1}, v_{1}\right)$ :

$$
\begin{equation*}
t_{\mathbf{b}}^{1}=t_{\mathbf{b}}\left(x_{1}, v_{1}\right) \tag{2.46}
\end{equation*}
$$

Lemma 12. (Lemma 4 in [10])
Given a $C^{2}$ convex domain defined in (1.14),

$$
\begin{equation*}
\left|n_{p^{j}}\left(\mathbf{x}_{p^{j}}^{j}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)\right| \sim\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{2}, \quad j=1,2 \tag{2.47}
\end{equation*}
$$

For $j^{\prime}=1,2$,

$$
\begin{equation*}
\left|\frac{\partial\left[n_{p^{j}}\left(\mathbf{x}_{p^{j}}^{j}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)\right]}{\partial \mathbf{x}_{p^{2}, j^{\prime}}^{2}}\right| \lesssim\|\eta\|_{C^{2}}\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|, \quad j=1,2 \tag{2.48}
\end{equation*}
$$

## 3. Weighted $C^{1}$-estimate of the dynamical solution.

In this section we prove Theorem [1. We will mainly prove the weighted $C^{1}$ estimate of the iteration equation (3.4) in Proposition 5 .

First we derive the boundary condition for $F=\sqrt{\mu} f$. By the boundary condition of $F(1.5)$ and the reciprocity property (1.9), the boundary condition for $f$ becomes, for $(x, v) \in \gamma_{-}$,

$$
\begin{aligned}
f(t, x, v)|n(x) \cdot v| & =\frac{1}{\sqrt{\mu}} \int_{n(x) \cdot u>0} R(u \rightarrow v ; x, t) f(t, x, u) \sqrt{\mu(u)}\{n(x) \cdot u\} \mathrm{d} u \\
= & \frac{1}{\sqrt{\mu}} \int_{n(x) \cdot u>0} R(-v \rightarrow-u ; x, t) \frac{e^{-|v|^{2} /\left(2 T_{w}(x)\right)}}{e^{-|u|^{2} /\left(2 T_{w}(x)\right)}} f(t, x, u) \sqrt{\mu(u)} \frac{|n(x) \cdot v|}{|n(x) \cdot u|}\{n(x) \cdot u\} \mathrm{d} u .
\end{aligned}
$$

Thus

$$
\begin{equation*}
\left.f(t, x, v)\right|_{\gamma_{-}}=e^{\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]|v|^{2}} \int_{n(x) \cdot u>0} f(t, x, u) e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]|u|^{2}} \mathrm{~d} \sigma(u, v) \tag{3.1}
\end{equation*}
$$

Here we denote

$$
\begin{equation*}
\mathrm{d} \sigma(u, v):=R(-v \rightarrow-u ; x, t) \mathrm{d} u \tag{3.2}
\end{equation*}
$$

which is a probability measure in the space $\{(x, u), n(x) \cdot u>0\}$ (well-defined due to (1.10)).
We consider the following iteration equation:

$$
\begin{equation*}
\partial_{t} F^{m+1}+v \cdot \nabla_{x} F^{m+1}=Q_{\text {gain }}\left(F^{m}, F^{m}\right)-\nu\left(F^{m}\right) F^{m+1},\left.\quad F^{m+1}\right|_{t=0}=F_{0} \tag{3.3}
\end{equation*}
$$

with boundary condition

$$
F^{m+1}(t, x, v)|n(x) \cdot v|=\int_{n(x) \cdot u>0} R(u \rightarrow v ; x, t) F^{m}(t, x, u)\{n(x) \cdot u\} \mathrm{d} u
$$

For $m \leq 0$ we set

$$
F^{m}(t, x, v)=F_{0}(x, v)
$$

We pose $F^{m+1}=\sqrt{\mu} f^{m+1}$, the equation for $f^{m+1}$ reads

$$
\begin{equation*}
\partial_{t} f^{m+1}+v \cdot \nabla_{x} f^{m+1}+\nu\left(F^{m}\right) f^{m+1}=\Gamma_{\text {gain }}\left(f^{m}, f^{m}\right) . \tag{3.4}
\end{equation*}
$$

Taking the derivative $\partial=\left[\nabla_{x}, \nabla_{v}\right]$ with the weight $e^{-\lambda\langle v\rangle t} \alpha$ we obtain

$$
\begin{align*}
& {\left[\partial_{t}+v \cdot \nabla_{x}+\nu^{m}\right] e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}(t, x, v)=\mathcal{G}^{m}}  \tag{3.5}\\
& h(0, x, v)=\alpha(x, v) \partial f_{0}(x, v)
\end{align*}
$$

In (3.5) $\nu^{m}$ and $\mathcal{G}^{m}$ are defined as

$$
\begin{align*}
& \nu^{m}=\nu\left(F^{m}\right)+\lambda\langle v\rangle-\alpha^{-1}\left[v \cdot \nabla_{x} \alpha\right]  \tag{3.6}\\
& \mathcal{G}^{m}(t, x)=e^{-\lambda\langle v\rangle t} \alpha(x, v)\left[-[\partial v] \cdot \nabla_{x} f^{m+1}-\partial\left[\nu \sqrt{\mu} f^{m}\right] f^{m+1}+\partial\left[\Gamma_{\text {gain }}\left(f^{m}, f^{m}\right)\right]\right] \tag{3.7}
\end{align*}
$$

We choose $\lambda=\lambda(\xi) \gg 1$ and apply (2.16) to have

$$
\begin{equation*}
\text { (3.6) }=\nu^{m} \geq \lambda\langle v\rangle-O_{\xi}(1)|v| \geq|v| \text {. } \tag{3.8}
\end{equation*}
$$

The boundary condition is given by the following lemma.

Lemma 13. (Lemma 12 in [11)
For $(x, v) \in \gamma_{-}$, we have the following bound for $e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}$ on the boundary:

$$
\begin{equation*}
\left|e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}(t, x, v)\right| \lesssim P\left(\left\|w_{\theta} f^{m}\right\|_{\infty}\right)+\langle v\rangle^{2} e^{\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]|v|^{2}} \times \tag{3.9}
\end{equation*}
$$

with

$$
\begin{equation*}
\int_{n(x) \cdot u>0}\langle u\rangle^{2}\left|\partial f^{m}(t, x, u)\right| e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]|u|^{2}} \mathrm{~d} \sigma(u, v) \tag{3.10}
\end{equation*}
$$

Then we establish the weighted $L^{\infty}$ bound of the sequence $\partial f^{m}$ in the following proposition.
Proposition 5. Assume $\partial f^{m+1}$ satisfies (3.5) with the boundary condition (3.9) and the wall temperature satisfies (1.24), (1.25). Also assume the initial condition has bound

$$
\left\|\alpha \partial f_{0}\right\|_{L^{\infty}}<\infty
$$

There exists $t_{\infty} \ll 1$ such that if

$$
\begin{equation*}
\sup _{t \leq t_{\infty}} \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}(t, x, v)\right\|_{L^{\infty}} \leq C_{\infty}\left[\left\|\alpha \partial f_{0}\right\|_{L^{\infty}}+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right] \tag{3.11}
\end{equation*}
$$

then

$$
\begin{equation*}
\sup _{0 \leq t \leq t_{\infty}}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}(t, x, v)\right\|_{L^{\infty}} \leq C_{\infty}\left[\left\|\alpha \partial f_{0}\right\|_{L^{\infty}}+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right] \tag{3.12}
\end{equation*}
$$

Here $C_{\infty}$ is a constant defined in (3.139) and

$$
\begin{equation*}
t \leq t_{\infty}=t_{\infty}\left(T_{M}, T_{m}, r_{\perp}, r_{\|}, \Omega, \sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \ll 1 \tag{3.13}
\end{equation*}
$$

Remark 6. The parameters in (3.13) guarantee that the small time only depends on the temperature, accommodation and $L^{\infty}$ bound $\left\|w_{\theta} f^{m}\right\|$. The uniform-in-m $L^{\infty}$ bound is concluded in [9]:

$$
\begin{equation*}
\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}<\infty \tag{3.14}
\end{equation*}
$$

The Proposition 5 implies the uniform-in-m $L^{\infty}$ estimate for $e^{-\lambda\langle v\rangle t} \alpha \partial f^{m}(t, x, v)$,

$$
\begin{equation*}
\sup _{m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{m}(t, x, v)\right\|_{\infty}<\infty \tag{3.15}
\end{equation*}
$$

The strategy to prove Proposition 5 is to express $e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}(t, x, v)$ along the characteristic using the $C-L$ boundary condition. We present the characteristic formula in Lemma 14. We will use Lemma 15 and Lemma 16 to bound the formula.

We represent $e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}(t, x, v)$ with the stochastic cycles defined as follows.
Lemma 14. Assume $e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}(t, x, v)$ satisfies (3.5) with the Cercignani-Lampis boundary condition (3.9). If $t_{1} \leq 0$, then

$$
\begin{equation*}
\left|e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}(t, x, v)\right| \leq\left|\alpha(x, v) \partial f_{0}\left(X^{1}(0), v\right)\right|+\int_{0}^{t}\left|\mathcal{G}^{m}\left(s, X^{1}(s), v\right)\right| \mathrm{d} s \tag{3.16}
\end{equation*}
$$

If $t_{1}>0$, for $k \geq 2$, then

$$
\begin{align*}
\left|e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}(t, x, v)\right| & \leq \int_{t_{1}}^{t} \mathcal{G}^{m}\left(s, X^{1}(s), v\right) \mathrm{d} s+P\left(\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \\
& +\langle v\rangle^{2} e^{\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]|v|^{2}} \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} H \tag{3.17}
\end{align*}
$$

where $H$ is bounded by

$$
\begin{align*}
& \sum_{l=1}^{k-1} \mathbf{1}_{\left\{t_{l}>0, t_{l+1} \leq 0\right\}}\left|\alpha \partial f_{0}\left(X^{l+1}(0), v_{l}\right)\right| \mathrm{d} \Sigma_{l}^{k} \\
& +\sum_{l=1}^{k-1} \int_{\max \left\{0, t_{l+1}\right\}}^{t_{l}} \mid \mathcal{G}^{m-l}\left(s, X^{l+1}(s) \mid \mathrm{d} \Sigma_{l}^{k} \mathrm{~d} s\right.  \tag{3.18}\\
& +\sum_{l=2}^{k-1} \mathbf{1}_{\left\{t_{l}>0\right\}} P\left(\left\|w_{\theta} f^{m-l+1}\right\|_{\infty}\right) \mathrm{d} \Sigma_{l-1}^{k} \\
& +\mathbf{1}_{\left\{t_{k}>0\right\}}\left|e^{-\lambda\left\langle v_{k}\right\rangle t_{k}} \alpha \partial f^{m-k+2}\left(t_{k}, x_{k}, v_{k-1}\right)\right| \mathrm{d} \Sigma_{k-1}^{k}
\end{align*}
$$

with

$$
\begin{align*}
\mathrm{d} \Sigma_{l, m}^{k}= & \left\{\prod_{j=l+1}^{k-1} \mathrm{~d} \sigma\left(v_{j}, v_{j-1}\right)\right\} \\
& \times\left\{e^{\lambda\left\langle v_{l}\right\rangle t_{l}}\left\langle v_{l-1}\right\rangle^{2} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}\right]\left|v_{l}\right|^{2}} \frac{1}{n\left(x_{l}\right) \cdot v_{l}} \mathrm{~d} \sigma\left(v_{l}, v_{l-1}\right)\right\}  \tag{3.19}\\
& \times\left\{\prod_{j=1}^{l-1} e^{\lambda\left\langle v_{j}\right\rangle t_{j}}\left\langle v_{j}\right\rangle^{4} e^{\left[\frac{1}{2 T_{w}\left(x_{j}\right)}-\frac{1}{2 T_{w}\left(x_{j}+1\right)}\right]\left|v_{j}\right|^{2}} \frac{1}{n\left(x_{j}\right) \cdot v_{j}} \mathrm{~d} \sigma\left(v_{j}, v_{j-1}\right)\right\} .
\end{align*}
$$

Proof. From (3.4), for $0 \leq s \leq t$, we apply the fundamental theorem of calculus to get

$$
\frac{\mathrm{d}}{\mathrm{~d} s} \int_{s}^{t}-\nu^{m} \mathrm{~d} \tau=\frac{\mathrm{d}}{\mathrm{~d} s} \int_{t}^{s} \nu^{m} d \tau=\nu^{m}
$$

Thus based on (3.5),

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} s}\left[e^{-\int_{s}^{t} \nu^{m} \mathrm{~d} \tau} e^{-\lambda\langle v\rangle s} \alpha\left(X^{1}(s), v\right) \partial f^{m+1}\left(s, X^{1}(s), v\right)\right]=e^{-\int_{s}^{t} \nu^{m} d \tau} \mathcal{G}^{m}\left(s, X^{1}(s), v\right) \tag{3.20}
\end{equation*}
$$

By (3.8),

$$
\begin{equation*}
e^{-\int_{s}^{t} \nu^{m} d \tau} \leq e^{-|v|(t-s)} \leq 0 \tag{3.21}
\end{equation*}
$$

Combining (3.20) and (3.21), we derive that if $t_{1} \leq 0$, then we have (3.16).
If $t_{1}(t, x, v)>0$, then

$$
\begin{align*}
\left|e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}(t, x, v) 1_{\left\{t_{1}>0\right\}}\right| & \leq\left|e^{-\lambda\langle v\rangle t_{1}} \alpha(x, v) \partial f^{m+1}\left(t_{1}, x_{1}, v\right)\right| e^{-|v|\left(t-t_{1}\right)} \\
& +\int_{t_{1}}^{t} e^{-|v|(t-s)} \mathcal{G}^{m}\left(s, X^{1}(s), v\right) \mid \mathrm{d} s \tag{3.22}
\end{align*}
$$

We use an induction of $k$ to prove (3.17). The first term of the RHS of (3.22) can be bounded by the boundary condition (3.9) as

$$
\begin{align*}
& P\left(\left\|w_{\theta} f\right\|_{\infty}\right)+e^{-\lambda\langle v\rangle\left(t-t_{1}\right)}\langle v\rangle^{2} e^{\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]|v|^{2}} \int_{\mathcal{V}_{1}} e^{-\lambda\left\langle v_{1}\right\rangle t_{1}} \alpha\left(x_{1}, v_{1}\right)\left|\partial f^{m}\left(t_{1}, x_{1}, v_{1}\right)\right| \\
& \times e^{\lambda\left\langle v_{1}\right\rangle t_{1}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left|v_{1}\right|^{2}} \frac{\left\langle v_{1}\right\rangle^{2}}{n\left(x_{1}\right) \cdot v_{1}} \mathrm{~d} \sigma\left(v_{1}, v\right), \tag{3.23}
\end{align*}
$$

where we have used $n\left(x_{1}\right) \cdot v_{1} \lesssim \alpha\left(x_{1}, v_{1}\right)$.
Then we apply (3.16) and (3.22) to derive

$$
\begin{align*}
\text { (3.23) } & \leq\langle v\rangle^{2} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}(x)}\right]|v|^{2}} \\
& \times\left[\int_{\mathcal{V}_{1}} \mathbf{1}_{\left\{t_{2} \leq 0<t_{1}\right\}} \alpha\left(X^{2}\left(t^{1}\right), v_{1}\right)\left|\partial f^{m}\left(0, X^{2}\left(t_{1}\right), v_{1}\right)\right| \frac{\left\langle v_{1}\right\rangle^{2} e^{\lambda\left\langle v_{1}\right\rangle t_{1}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left|v_{1}\right|^{2}}}{n\left(x_{1}\right) \cdot v_{1}} \mathrm{~d} \sigma\left(v_{1}, v\right)\right. \\
& +\int_{\mathcal{V}_{1}} \int_{0}^{t_{1}} \mathbf{1}_{\left\{t_{2} \leq 0<t_{1}\right\}} e^{-\left|v_{1}\right|\left(t_{1}-s\right)}\left|\mathcal{G}^{m-1}\left(s, X^{2}(s), v_{1}\right)\right| \frac{\left\langle v_{1}\right\rangle^{2} e^{\lambda\left\langle v_{1}\right\rangle t_{1}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left|v_{1}\right|^{2}}}{n\left(x_{1}\right) \cdot v_{1}} \mathrm{~d} \sigma\left(v_{1}, v\right) \mathrm{d} s \\
& +\int_{\mathcal{V}_{1}} \mathbf{1}_{\left\{t_{2}>0\right\}} e^{-\left|v_{1}\right|\left(t_{1}-t_{2}\right)} e^{-\lambda\left\langle v_{1}\right\rangle t_{2}} \alpha\left(x_{2}, v_{1}\right)\left|\partial f^{m}\left(t_{2}, x_{2}, v_{1}\right)\right| \frac{\left.\left.\left\langle v_{1}\right\rangle^{2} e^{\lambda\left\langle v_{1}\right\rangle t_{1}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right.}\right)\right]\left.v_{1}\right|^{2}}{n\left(x_{1}\right) \cdot v_{1}} \mathrm{~d} \sigma\left(v_{1}, v\right) \\
& \left.+\int_{\mathcal{V}_{1}} \int_{t_{2}}^{t_{1}} \mathbf{1}_{\left\{t_{2}>0\right\}} e^{-\left|v_{1}\right|\left(t_{1}-s\right)}\left|\mathcal{G}^{m-1}\left(s, X^{2}(s), v_{1}\right)\right| \frac{\left\langle v_{1}\right\rangle^{2} e^{\lambda\left\langle v_{1}\right\rangle t_{1}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left|v_{1}\right|^{2}}}{n\left(x_{1}\right) \cdot v_{1}} \mathrm{~d} \sigma\left(v_{1}, v\right) \mathrm{d} s\right] . \tag{3.24}
\end{align*}
$$

Therefore, the formula (3.17) is valid for $k=2$.
Assume (3.17) is valid for $k \geq 2$ (induction hypothesis). Now we prove that (3.17) holds for $k+1$. We express the last term in (3.18) using the boundary condition. Applying (3.9) (3.10), the contribution of constant term is

$$
\int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{t_{k}>0}\left\|w_{\theta} f^{m-k+1}\right\|_{\infty} \mathrm{d} \Sigma_{k-1}^{k}
$$

Then the summation in the third line of (3.19) extends to $k$ :

$$
\sum_{l=2}^{k} \mathbf{1}_{t_{l}>0} P\left(\left\|w_{\theta} f^{m-l+1}\right\|_{\infty}\right) d \Sigma_{l-1}^{k}
$$

Since $\int_{\mathcal{V}_{k}} \mathrm{~d} \sigma\left(v_{k}, v_{k-1}\right)=1$ from (3.2), we add $v_{k}$ integration to derive that for $l \leq k$

$$
\begin{equation*}
\mathrm{d} \sigma\left(v_{k}, v_{k-1}\right) \Sigma_{l-1}^{k}=\mathrm{d} \Sigma_{l-1}^{k+1} \tag{3.25}
\end{equation*}
$$

Thus the third line of (3.19) is valid for $k+1$.
For the other term in (3.9) (3.10), the front term $\left\langle v_{k-1}\right\rangle^{2} e^{\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k-1}\right|^{2}}$ depends on $v_{k-1}$, we move this term to the integration over $\mathcal{V}_{k-1}$ in (3.17). Using the second line of (3.19), the integration over $\mathcal{V}_{k-1}$ is

$$
\begin{align*}
& \int_{\mathcal{V}_{k-1}} e^{\lambda\left\langle v_{k-1}\right\rangle t_{k-1}} \frac{\left\langle v_{k-1}\right\rangle^{4}}{n\left(x_{k-1}\right) \cdot v_{k-1}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k-1}\right)}\right]\left|v_{k-1}\right|^{2}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k-1}\right|} \mathrm{d} \sigma\left(v_{k-1}, v_{k-2}\right) \\
& =\int_{\mathcal{V}_{k-1}} e^{\lambda\left\langle v_{k-1}\right\rangle t_{k-1}} \frac{\left\langle v_{k-1}\right\rangle^{4}}{n\left(x_{k-1}\right) \cdot v_{k-1}} e^{\left[\frac{1}{2 T_{w}\left(x_{k-1}\right)}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k-1}\right|^{2}} \mathrm{~d} \sigma\left(v_{k-1}, v_{k-2}\right), \tag{3.26}
\end{align*}
$$

which is consistent with third line in (3.19) with $l=k-1$.
For the remaining integration over $\mathcal{V}_{k}$ in (3.9), we split it into two terms as

$$
\begin{equation*}
\int_{\mathcal{V}_{k}}\left\langle v_{k}\right\rangle^{2}\left|\partial f^{m-k}\left(t^{k}, x^{k}, v^{k}\right)\right| e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k}\right|^{2}} \mathrm{~d} \sigma\left(v_{k}, v_{k-1}\right)=\underbrace{\int_{\left\{t_{k+1} \leq 0<t_{k}\right\}}}_{\mathcal{V}_{k}}+\underbrace{\int_{\left\{t_{k+1}>0\right\}}}_{\mathcal{V}_{k}} . \tag{3.27}
\end{equation*}
$$

For the first term of the RHS of (3.27), we use a similar bound as (3.24) and derive that

$$
\begin{align*}
& \text { (3.27) }
\end{align*} \leq \int_{\mathcal{V}_{k}} \mathbf{1}_{\left\{t_{k+1} \leq 0<t_{k}\right\}} \alpha\left(X^{k+1}\left(t^{k}\right), v_{k}\right) \partial f^{m-k+1}\left(0, X^{k+1}\left(t_{k}\right), v_{k}\right) \frac{\left\langle v_{k}\right\rangle^{2} e^{\lambda\left\langle v_{k}\right\rangle t_{k}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k}\right|^{2}}}{n\left(x_{k}\right) \cdot v_{k}} \mathrm{~d} \sigma\left(v_{k}, v_{k-1}\right) .
$$

In (3.28),

$$
\frac{\left\langle v_{k}\right\rangle^{2} e^{\lambda\left\langle v_{k}\right\rangle t_{k}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k}\right|^{2}}}{n\left(x_{k}\right) \cdot v_{k}}
$$

is consistent with the second line of (3.19) with $l=k$.
From the induction hypothesis( (3.17) is valid for $k$ ), we derive the integration over $\mathcal{V}_{j}$ for $j \leq k-1$ is consistent with the third line of (3.19). After taking integration $\int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}}$ we change $\mathrm{d} \Sigma_{k-1}^{k}$ in (3.19) to d $\Sigma_{k}^{k+1}$. Thus the contribution of (3.28) is

$$
\begin{align*}
& \int_{\prod_{j=1}^{k} \mathcal{V}_{j}} 1_{\left\{t_{k+1} \leq 0<t_{k}\right\}}\left|\alpha \partial f_{0}\left(X^{k+1}(0), v_{k}\right)\right| \mathrm{d} \Sigma_{k}^{k+1} \\
& +\int_{\prod_{j=1}^{k} \mathcal{V}_{j}} \int_{0}^{t_{k}} \mathcal{G}^{m-k}(s) \mathrm{d} \Sigma_{k}^{k+1} \mathrm{~d} s \tag{3.29}
\end{align*}
$$

For the second term of the RHS of (3.27), similar to (3.24) we derive
$(3.27)_{2}$

$$
\begin{aligned}
& \leq \int_{\mathcal{V}_{k}} \mathbf{1}_{\left\{t_{k+1}>0\right\}} e^{-\lambda\left\langle v_{k}\right\rangle t_{k}} e^{-\lambda\left\langle v_{k}\right\rangle t_{k}} \alpha\left(x_{k+1}, v_{k}\right) \partial f^{m-k+1}\left(t_{k+1}, x_{k+1}, v_{k}\right) \frac{\left\langle v_{k}\right\rangle^{2} e^{\lambda\left\langle v_{k}\right\rangle t_{k}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k}\right|^{2}}}{n\left(x_{k}\right) \cdot v_{k}} \mathrm{~d} \sigma\left(v_{k}, v_{k-1}\right) \\
& +\int_{\mathcal{V}_{k}} \int_{t_{k+1}}^{t_{k}} \mathbf{1}_{\left\{t_{k+1}>0\right\}} e^{-\left|v_{k}\right|\left(t_{k}-s\right)}\left|\mathcal{G}^{m-k}\left(s, X^{k+1}(s), v_{k}\right)\right| \frac{\left\langle v_{k}\right\rangle^{2} e^{\lambda\left\langle v_{k}\right\rangle t_{k}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{k}\right)}\right]\left|v_{k}\right|^{2}}}{n\left(x_{k}\right) \cdot v_{k}} \mathrm{~d} \sigma\left(v_{k}, v_{k-1}\right) \mathrm{d} s .
\end{aligned}
$$

Similar to (3.29), after taking integration over $\int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}}$ the contribution of (3.30) is

$$
\begin{align*}
& \int_{\prod_{j=1}^{k} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k+1}>0\right\}} e^{-\lambda\left\langle v_{k}\right\rangle t_{k}} \alpha\left(x_{k+1}, v_{k}\right) \partial f^{m-k+1}\left(t_{k+1}, x_{k+1}, v_{k}\right) \mathrm{d} \Sigma_{k}^{k+1} \\
& +\int_{\prod_{j=1}^{k}} \mathcal{V}_{j} \int_{t_{k+1}}^{t_{k}} \mathcal{G}^{m-k}(s) \mathrm{d} \Sigma_{k}^{k+1} \mathrm{~d} s \tag{3.30}
\end{align*}
$$

From (3.30) (3.29), the summation in the first and second lines of (3.18) extends to $k$. And the index of the fourth line of (3.18) changes from $k$ to $k+1$. For the rest terms, the index $l \leq k-1$. We add the $v_{k}$ integration as (3.25) so that the integration change to $\prod_{l=1}^{k} \mathcal{V}_{j}$.

Therefore, the formula (3.18) is valid for $k+1$ and we derive the lemma.
The next lemma is the key to prove the $L^{\infty}$ bound for $h^{m+1}$. Below we define several notation: let

$$
\begin{equation*}
r_{\max }:=\max \left(r_{\|}\left(2-r_{\|}\right), r_{\perp}\right), r_{\min }:=\min \left(r_{\|}\left(2-r_{\|}\right), r_{\perp}\right) \tag{3.31}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
1 \geq r_{\max } \geq r_{\min }>0 \tag{3.32}
\end{equation*}
$$

We inductively define:

$$
\begin{equation*}
T_{l, l}=2 T_{M}, \quad T_{l, l-1}=r_{\min } T_{M}+\left(1-r_{\min }\right) T_{l, l}, \cdots, \quad T_{l, 1}=r_{\min } T_{M}+\left(1-r_{\min }\right) T_{l, 2} \tag{3.33}
\end{equation*}
$$

By a direct computation, for $1 \leq i \leq l$, we have

$$
\begin{equation*}
T_{l, i}=2 T_{M}+\left(T_{M}-2 T_{M}\right)\left[1-\left(1-r_{\min }\right)^{l-i}\right] \tag{3.34}
\end{equation*}
$$

Moreover, we denote

$$
\begin{align*}
\mathrm{d} \Phi_{p, m}^{k, l}(s):= & \left\{\prod_{j=l+1}^{k-1} \mathrm{~d} \sigma\left(v_{j}, v_{j-1}\right)\right\} \\
& \times\left\{\frac{e^{\lambda\left\langle v_{l}\right\rangle t_{l}}\left\langle v_{l-1}\right\rangle^{2} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}\right]\left|v_{l}\right|^{2}}}{n\left(x_{l}\right) \cdot v_{l}} \mathrm{~d} \sigma\left(v_{l}, v_{l-1}\right)\right\}  \tag{3.35}\\
& \times\left\{\prod_{j=p}^{l-1} e^{\left[\frac{1}{2 T_{w}\left(x_{j}\right)}-\frac{1}{2 T_{w}\left(x_{j+1}\right)}\right]\left|v_{j}\right|^{2}} \frac{e^{\lambda\left\langle v_{j}\right\rangle t_{j}}\left\langle v_{j}\right\rangle^{4}}{n\left(x_{j}\right) \cdot v_{j}} \mathrm{~d} \sigma\left(v_{j}, v_{j-1}\right)\right\} .
\end{align*}
$$

Note that if $p=1, \mathrm{~d} \Phi_{1, m}^{k, l}=\mathrm{d} \Sigma_{l}^{k}$ where $\mathrm{d} \Sigma_{l}^{k}$ is defined in (3.19). And we denote

$$
\begin{equation*}
\mathrm{d} \Upsilon_{p}^{p^{\prime}}:=\prod_{j=p}^{p^{\prime}} e^{\left[\frac{1}{2 T_{w}\left(x_{j}\right)}-\frac{1}{2 T_{w}\left(x_{j+1}\right)}\right]\left|v_{j}\right|^{2}} \frac{e^{\lambda\left\langle v_{j}\right\rangle t_{j}}\left\langle v_{j}\right\rangle^{4}}{n\left(x_{j}\right) \cdot v_{j}} \mathrm{~d} \sigma\left(v_{j}, v_{j-1}\right) \tag{3.36}
\end{equation*}
$$

Then by the definition of (3.35) and (3.19), we have

$$
\begin{align*}
\mathrm{d} \Phi_{p, m}^{k, l} & =\mathrm{d} \Phi_{p^{\prime}, m}^{k, l} \mathrm{~d} \Upsilon_{p}^{p^{\prime}-1}  \tag{3.37}\\
\mathrm{~d} \Sigma_{l}^{k} & =\mathrm{d} \Phi_{p, m}^{k, l} \mathrm{~d} \Upsilon_{1}^{p-1} \tag{3.38}
\end{align*}
$$

Remark 7. In Lemma 14 the integration has multiple fold and each fold contains the variable $T_{w}(x), T_{M}, r_{\perp}, r_{\|}$. We define these inductive notations to find a pattern to bound such integration.

Now we state the lemma.
Lemma 15. Given the formula for $e^{-\lambda\langle v\rangle t} \alpha f^{m+1}(t)$ in (3.16) and (3.17) in lemma 14, there exists

$$
\begin{equation*}
t_{*}=t_{*}\left(T_{M}, T_{m}, k, r_{\|}, r_{\perp}, c\right) \tag{3.39}
\end{equation*}
$$

( $t_{*}$ need to satisfy more conditions specified in Lemma 16 and (3.149) ) such that: when $t \leq t_{*}$, we have

$$
\begin{equation*}
\int_{\prod_{j=p}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathrm{d} \Phi_{p, m}^{k, l} \leq t_{*}^{-(l-p+1) c}\left(C_{T_{M}, T_{m}}\right)^{l-p+1} \mathcal{A}_{l, p} \tag{3.40}
\end{equation*}
$$

Here we define:

$$
\begin{equation*}
\mathcal{A}_{l, p}:=\exp \left(\left[\frac{\left[T_{l, p}-T_{w}\left(x_{p}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{p}\right)\left[T_{l, p}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{p}\right)\right]}+\mathcal{C}_{l-p+1} t_{*}^{c}\right]\left|v_{p-1}\right|^{2}\right) . \tag{3.41}
\end{equation*}
$$

$C_{T_{M}, T_{m}}$ is a constant defined in (3.57) and

$$
\begin{equation*}
\mathcal{C}_{n}:=\sum_{i=1}^{n} \mathcal{C}^{i}=\mathcal{C} \frac{\mathcal{C}^{n}-1}{\mathcal{C}-1} \tag{3.42}
\end{equation*}
$$

where $\mathcal{C}$ is constant defined in (3.52). And $c<1$ is a constant. We will specify $c=\frac{1}{15}$ later in (3.137).
Moreover, for any $p<p^{\prime} \leq l$, we have

$$
\begin{align*}
\int_{\prod_{j=p}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathrm{d} \Phi_{p, m}^{k, l} & \leq t_{*}^{-\left(l-p^{\prime}+1\right) q}\left(C_{T_{M}, T_{m}}\right)^{2\left(l-p^{\prime}+1\right)} \int_{\prod_{j=p}^{p^{\prime}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathcal{A}_{l, p^{\prime}} \mathrm{d} \Upsilon_{p}^{p^{\prime}-1}  \tag{3.43}\\
& \leq t_{*}^{-(l-p+1) q}\left(C_{T_{M}, T_{m}}\right)^{2(l-p+1)} \mathcal{A}_{l, p}
\end{align*}
$$

Remark 8. To prove Lemma 15 we do not need the condition (3.149). Such condition will be used in the proof of Proposition 5.
Proof. From (1.10) and (3.2), for the first bracket of the first line in (3.19) with $l+1 \leq j \leq k-1$, we have

$$
\int_{\prod_{j=l+1}^{k-1} \mathcal{V}_{j}} \prod_{j=l+1}^{k-1} \mathrm{~d} \sigma\left(v_{j}, v_{j-1}\right)=1
$$

Without loss of generality we can assume $k=l+1$. Thus $\mathrm{d} \Phi_{p, m}^{k, l}=\mathrm{d} \Phi_{p, m}^{l+1, l}$. We use an induction of $p$ with $1 \leq p \leq l$ to prove (3.40).

When $p=l$, by the second line of (3.35), the integration over $\mathcal{V}_{l}$ is bounded by

$$
\begin{equation*}
\int_{\mathcal{V}_{l}} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}\right]\left|v_{l}\right|^{2}} \frac{e^{\lambda\left\langle v_{l}\right\rangle t_{l}}\left\langle v_{l}\right\rangle^{4}}{n\left(x_{l}\right) \cdot v_{l}} \mathrm{~d} \sigma\left(v_{l}, v_{l-1}\right) \tag{3.44}
\end{equation*}
$$

Clearly $e^{\lambda\left\langle v_{l}\right\rangle t_{l}} \leq e^{\lambda\left\langle v_{l}\right\rangle t_{*}}$. We expand $\mathrm{d} \sigma\left(v_{l}, v_{l-1}\right)$ with (1.6) and (3.2), then we apply (2.15) in Lemma 5 to bound (3.44) by

$$
\begin{align*}
& t_{*}^{-c} \int_{\mathcal{V}_{l, \perp}} \frac{2}{r_{\perp}} \frac{1}{\left\langle v_{l, \perp}\right\rangle^{2}} 2 T_{w}\left(x_{l}\right) e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}-t_{*}^{c}\right]\left|v_{l, \perp}\right|^{2}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{l, \perp} v_{l-1, \perp}}{T_{w}\left(x_{l}\right) r_{\perp}}\right) e^{-\frac{\left|v_{l, \perp}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{l-1, \perp}\right|^{2}}{2 T_{w}\left(x_{l}\right) r_{\perp}}} \mathrm{d} v_{l, \perp} \\
& \times \int_{\mathcal{V}_{l, \|}} \frac{1}{\pi r_{\|}\left(2-r_{\|}\right)\left(2 T_{w}\left(x_{l}\right)\right)} e^{-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}-t_{*}^{c}\right]\left|v_{l, \|}\right|^{2}} e^{-\frac{1}{2 T_{w}\left(x_{l}\right)} \frac{\left|v_{l, \|}-\left(1-r_{\|}\right) v_{l-1, \|}\right|^{2}}{r_{\|}\left(2-r_{\|}\right)}} \mathrm{d} v_{l, \|} \tag{3.45}
\end{align*}
$$

where $v_{l, \|}, v_{l, \perp}, \mathcal{V}_{l, \perp}$ and $\mathcal{V}_{l, \|}$ are defined as

$$
\begin{equation*}
v_{l, \perp}=v_{l} \cdot n\left(x_{l}\right), v_{l, \|}=v_{l}-v_{l, \perp} n\left(x_{l}\right), \mathcal{V}_{l, \perp}=\left\{v_{l, \perp}: v_{l} \in \mathcal{V}_{l}\right\}, \mathcal{V}_{l, \|}=\left\{v_{l, \|}: v_{l} \in \mathcal{V}_{l}\right\} \tag{3.46}
\end{equation*}
$$

$v_{l-1, \|}$ and $v_{l-1, \perp}$ are defined similarly.
First we compute the integration over $\mathcal{V}_{l, \|}$, the second line of (3.45). To apply (2.8) in Lemma 2, we set

$$
\begin{gather*}
\varepsilon=t_{*}^{c}, \quad w=\left(1-r_{\|}\right) v_{l-1, \|}, \quad v=v_{l, \|} \\
a=-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}\right], \quad b=\frac{1}{2 T_{w}\left(x_{l}\right) r_{\|}\left(2-r_{\|}\right)} . \tag{3.47}
\end{gather*}
$$

We take $t_{*}=t_{*}\left(T_{M}, c\right) \ll 1$ such that when $t<t_{*}$, we have

$$
\begin{equation*}
b-a-t_{*}^{c}=\frac{1}{2 T_{w}\left(x_{l}\right) r_{\|}\left(2-r_{\|}\right)}-\frac{1}{2 T_{w}\left(x_{l}\right)}+\frac{1}{4 T_{M}}-t_{*}^{c} \geq \frac{1}{4 T_{M}}-t_{*}^{c} \geq \frac{1}{8 T_{M}} \tag{3.48}
\end{equation*}
$$

Then we further require $t \leq t_{*}\left(T_{M}, c\right) \ll 1$ such that $1+8 T_{M} t_{*}^{c}<2$, then we have

$$
\begin{align*}
\frac{b}{b-a-t_{*}^{c}} & =\frac{b}{b-a}\left[1+\frac{t_{*}^{c}}{b-a-t_{*}^{c}}\right] \leq \frac{2 T_{M}}{2 T_{M}+\left[T_{w}\left(x_{l}\right)-2 T_{M}\right] r_{\|}\left(2-r_{\|}\right)}\left[1+8 T_{M} t_{*}^{c}\right] \\
& \leq \frac{4 T_{M}}{2 T_{M}+\left[T_{m}-2 T_{M}\right] r_{\max }}:=C_{T_{M}} \tag{3.49}
\end{align*}
$$

where we used (3.31).
In regard to (2.8), we have

$$
\begin{equation*}
\frac{\left(a+t_{*}^{c}\right) b}{b-a-t_{*}^{c}}=\frac{a b}{b-a}\left[1+\frac{t_{*}^{c}}{b-a-t_{*}^{c}}\right]+\frac{b}{b-a-t_{*}^{c}} t_{*}^{c} \tag{3.50}
\end{equation*}
$$

By (3.49) we obtain

$$
\frac{b}{b-a-t_{*}^{c}} t_{*}^{c} \leq \frac{4 T_{M}}{2 T_{M}+\left[T_{m}-2 T_{M}\right] r_{\max }} t_{*}^{c}
$$

By (3.47), we have

$$
\frac{a b}{b-a}=\frac{2 T_{M}-T_{w}\left(x_{l}\right)}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}+\left[T_{w}\left(x_{l}\right)-2 T_{M}\right] r_{\|}\left(2-r_{\|}\right)\right]}
$$

Therefore, by (3.48) and (3.50) we obtain

$$
\begin{equation*}
\frac{\left(a+t_{*}^{c}\right) b}{b-a-t_{*}^{c}} \leq \frac{2 T_{M}-T_{w}\left(x_{l}\right)}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}+\left[T_{w}\left(x_{l}\right)-2 T_{M}\right] r_{\|}\left(2-r_{\|}\right)\right]}+\mathcal{C} t_{*}^{c} \tag{3.51}
\end{equation*}
$$

where we defined

$$
\begin{equation*}
\mathcal{C}:=\frac{4 T_{M}\left(2 T_{M}-T_{m}\right)}{2 T_{m}\left[2 T_{M}+\left[T_{m}-2 T_{M}\right] r_{\max }\right]}+\frac{4 T_{M}}{2 T_{M}+\left[T_{m}-2 T_{M}\right] r_{\max }} \tag{3.52}
\end{equation*}
$$

By (3.49), (3.51) and Lemma 2 using $w=\left(1-r_{\|}\right) v_{l-1, \|}$ we bound the second line of (3.45) by

$$
\begin{align*}
& C_{T_{M}} \exp \left(\left[\frac{\left[2 T_{M}-T_{w}\left(x_{l}\right)\right]}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}\left(1-r_{\|}\right)^{2}+r_{\|}\left(2-r_{\|}\right) T_{w}\left(x_{l}\right)\right]}+\mathcal{C} t_{*}^{c}\right]\left|\left(1-r_{\|}\right) v_{l-1, \|}\right|^{2}\right)  \tag{3.53}\\
& \quad \leq C_{T_{M}} \exp \left(\left[\frac{\left[2 T_{M}-T_{w}\left(x_{l}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{l}\right)\right]}+\mathcal{C} t_{*}^{c}\right]\left|v_{l-1, \|}\right|^{2}\right) \tag{3.54}
\end{align*}
$$

where we used (3.31) and (3.32).
Next we compute the first line of (3.45). To apply (2.10) in Lemma 3 we set

$$
\begin{aligned}
& \varepsilon=t_{*}^{c}, \quad w=\sqrt{1-r_{\|}} v_{l-1, \perp}, \quad v=v_{l, \perp} \\
& a=-\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{l}\right)}\right], b=\frac{1}{2 T_{w}\left(x_{l}\right) r_{\perp}} .
\end{aligned}
$$

$\frac{(a+\varepsilon) b}{b-a-\varepsilon}$ can be computed using the same way as (3.51) with replacing $r_{\|}\left(2-r_{\|}\right)$by $r_{\perp}$. Here the difference is the constant term becomes

$$
\begin{align*}
\frac{2 b}{\sqrt{b-a-t_{*}^{c}}} & =2 \sqrt{b} \sqrt{\frac{b}{b-a-t_{*}^{c}}} \leq 2 \sqrt{b} C_{T_{M}} \\
& \leq \frac{2}{\sqrt{T_{m}}} \times \sqrt{C_{T_{M}}} \tag{3.55}
\end{align*}
$$

Hence replacing $r_{\|}\left(2-r_{\|}\right)$by $r_{\perp}$ and replacing $v_{l-1, \|}$ by $v_{l-1, \perp}$ in (3.53), we bound the first line of (3.45) by

$$
\begin{align*}
& \frac{2}{\sqrt{T_{m}}} \sqrt{C_{T_{M}}} \exp \left(\left[\frac{\left[2 T_{M}-T_{w}\left(x_{l}\right)\right]}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}\left(1-r_{\perp}\right)+r_{\perp} T_{w}\left(x_{l}\right)\right]}+\mathcal{C} t_{*}^{c}\right]\left|\sqrt{1-r_{\perp}} v_{l-1, \perp}\right|^{2}\right) \\
& \quad \leq \frac{2}{\sqrt{T_{m}}} C_{T_{M}} \exp \left(\left[\frac{\left[2 T_{M}-T_{w}\left(x_{l}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{l}\right)\right]}+\mathcal{C} t_{*}^{c}\right]\left|v_{l-1, \perp}\right|^{2}\right) \tag{3.56}
\end{align*}
$$

where we used (3.31) and (3.32).
Then we define the constant term in (3.41) as

$$
\begin{equation*}
C_{T_{M}, T_{m}}=\frac{2}{\sqrt{T_{m}}} C_{T_{M}}^{3 / 2} \tag{3.57}
\end{equation*}
$$

Collecting (3.54) (3.56), we derive

$$
\left(\sqrt{3.45)} \leq t_{*}^{-c} C_{T_{M}, T_{m}} \exp \left(\left[\frac{\left[2 T_{M}-T_{w}\left(x_{l}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{l}\right)\left[2 T_{M}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{l}\right)\right]}+\mathcal{C} t_{*}^{c}\right]\left|v_{l-1}\right|^{2}\right)=C_{T_{M}, T_{m}} \mathcal{A}_{l, l}\right.
$$

where $\mathcal{A}_{l, l}$ is defined in (3.41) and $T_{l, l}=2 T_{M}$.
Therefore, (3.40) is valid for $p=l$ by $\mathcal{C}_{1}=\mathcal{C}$.
Suppose (3.40) is valid for $p=q+1$ (induction hypothesis) with $q+1 \leq l$, then

$$
\int_{\prod_{j=q+1}^{l} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathrm{d} \Phi_{q+1, m}^{l+1, l} \leq t_{*}^{-(l-q) c} C_{T_{M}, T_{m}}^{l-q} \mathcal{A}_{l, q+1}
$$

We want to show (3.40) holds for $p=q$. By the hypothesis and the third line of (3.35),

$$
\begin{equation*}
\int_{\prod_{j=q}^{l} \mathcal{V}_{j}} 1_{\left\{t_{l}>0\right\}} \mathrm{d} \Phi_{q, m}^{l+1, l} \leq t_{*}^{-(l-q+1) c} C_{T_{M}, T_{m}}^{l-q} \int_{\mathcal{V}_{q}} \mathcal{A}_{l, q+1} e^{\left[\frac{1}{2 T_{w}\left(x_{q}\right)}-\frac{1}{2 T_{w}\left(x_{q+1}\right)}+t_{*}^{c}\right]\left|v_{q}\right|^{2}} \frac{1}{n\left(x_{q}\right) \cdot v_{q}} \mathrm{~d} \sigma\left(v_{q}, v_{q-1}\right), \tag{3.58}
\end{equation*}
$$

where we have applied Lemma 5 ,
Using the definition of $\mathcal{A}_{l, q+1}$ in (3.41), we obtain

$$
\begin{align*}
& \text { (3.58) } \leq t_{*}^{-(l-q+1) c} C_{T_{M}, T_{m}}^{l-q} \int_{\mathcal{V}_{q}} \exp \left(\frac{\left(T_{l, q+1}-T_{w}\left(x_{q+1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{q+1}\right)\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}\left|v_{q}\right|^{2}+\mathcal{C}_{l-q} t_{*}^{c}\left|v_{q}\right|^{2}\right)  \tag{3.59}\\
& e^{\left[\frac{1}{2 T_{w}\left(x_{q}\right)}-\frac{1}{2 T_{w}\left(x_{q+1}\right)}+t_{*}^{c}\right]\left|v_{q}\right|^{2}} \mathrm{~d} \sigma\left(v_{q}, v_{q-1}\right) .
\end{align*}
$$

We focus on the coefficient of $\left|v_{q}\right|^{2}$ in (3.59), we derive

$$
\begin{aligned}
& \frac{\left(T_{l, q+1}-T_{w}\left(x_{q+1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{q+1}\right)\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}\left|v_{q}\right|^{2}+\left[\frac{1}{2 T_{w}\left(x_{q}\right)}-\frac{1}{2 T_{w}\left(x_{q+1}\right)}\right]\left|v_{q}\right|^{2} \\
& =\frac{\left(T_{l, q+1}-T_{w}\left(x_{q+1}\right)\right)\left(1-r_{\min }\right)-\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}{2 T_{w}\left(x_{q+1}\right)\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}+\frac{\left|v_{q}\right|^{2}}{2 T_{w}\left(x_{q}\right)} \\
& =\frac{-T_{w}\left(x_{q+1}\right)\left(1-r_{\min }\right)-r_{\min } T_{w}\left(x_{q+1}\right)}{2 T_{w}\left(x_{q+1}\right)\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}\left|v_{q}\right|^{2}+\frac{\left|v_{q}\right|^{2}}{2 T_{w}\left(x_{q}\right)} \\
& =\frac{-\left|v_{q}\right|^{2}}{2\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}+\frac{\left|v_{q}\right|^{2}}{2 T_{w}\left(x_{q}\right)} .
\end{aligned}
$$

By the Definition $2 x_{q+1}=x_{q+1}\left(t, x, v, v_{1}, \cdots, v_{q}\right)$, thus $T_{w}\left(x_{q+1}\right)$ depends on $v_{q}$. In order to explicitly compute the integration over $\mathcal{V}_{q}$, we need to get rid of the dependence of the $T_{w}\left(x_{q+1}\right)$ on $v_{q}$. Then we bound

$$
\begin{equation*}
\exp \left(\frac{-\left|v_{q}\right|^{2}}{2\left[T_{l, q+1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q+1}\right)\right]}\right) \leq \exp \left(\frac{-\left|v_{q}\right|^{2}}{2\left[T_{l, q+1}\left(1-r_{m i n}\right)+r_{\min } T_{M}\right]}\right)=\exp \left(\frac{-\left|v_{q}\right|^{2}}{2 T_{l, q}}\right) \tag{3.60}
\end{equation*}
$$

where we used (3.33).
Hence by (3.2) (1.6) and (3.60), we derive

$$
\begin{align*}
& \left(\sqrt{3.59)} \leq t_{*}^{-(l-q+1) c} C_{T_{M}, T_{m}}^{l-q}\right. \\
& \times \int_{\mathcal{V}_{q, \perp}} \frac{1}{r_{\perp} T_{w}\left(x_{q}\right)} e^{-\left[\frac{1}{2 T_{l, q}}-\frac{1}{2 T_{w}\left(x_{q}\right)}-\mathcal{C}_{l-q} t_{*}^{c}-t_{*}^{c}\right]\left|v_{q, \perp}\right|^{2}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{q, \perp} v_{q-1, \perp}}{T_{w}\left(x_{q}\right) r_{\perp}}\right) e^{-\frac{\left|v_{q, \perp}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{q-1, \perp}\right|^{2}}{2 T_{w}\left(x_{q}\right) r_{\perp}}} \mathrm{d} v_{q, \perp} \\
& \times \int_{\mathcal{V}_{q, \|}} \frac{1}{\pi r_{\|}\left(2-r_{\|}\right)\left(2 T_{w}\left(x_{q}\right)\right)} e^{-\left[\frac{1}{2 T_{l, q}}-\frac{1}{2 T_{w}\left(x_{q}\right)}-\mathcal{C}_{l-q} t_{*}^{c}-\left.t_{*}^{c}| | v_{q, \|}\right|^{2}\right.} e^{-\frac{1}{2 T_{w}\left(x_{q}\right)} \frac{\left|v_{q, \|}-\left(1-r_{\|}\right) v_{q-1, \|}\right|^{2}}{r_{\| \|}\left(2-r_{\|}\right)}} \mathrm{d} v_{q, \|} \tag{3.61}
\end{align*}
$$

In the third line of (3.61), to apply (2.8) in Lemma 2 we set

$$
a=-\left[\frac{1}{2 T_{l, q}}-\frac{1}{2 T_{w}\left(x_{q}\right)}\right], \quad b=\frac{1}{2 T_{w}\left(x_{q}\right) r_{\|}\left(2-r_{\|}\right)}, \quad \varepsilon=\mathcal{C}_{l-q} t_{*}^{c}+t_{*}^{c}, w=\left(1-r_{\|}\right) v_{q-1, \|} .
$$

Taking (3.47) for comparison, we can replace $2 T_{M}$ by $T_{l, q}$ and replace $t_{*}^{c}$ by $\mathcal{C}_{l-q} t_{*}^{c}+t_{*}^{c}$. Then we apply the replacement to (3.48) and obtain

$$
b-a-\varepsilon \geq \frac{1}{2 T_{l, q}}-\mathcal{C}_{l-q} t_{*}^{c}-t_{*}^{c} \geq \frac{1}{4 T_{M}}-\mathcal{C}_{k} t_{*}^{c}-t_{*}^{c}=\frac{1}{4 T_{M}}-\mathcal{C} \frac{\mathcal{C}^{k}-1}{\mathcal{C}-1} t_{*}^{c}-t_{*}^{c}>\frac{1}{5 T_{M}}
$$

where we applied (3.42) and we take $t_{*}=t_{*}\left(T_{M}, \mathcal{C}, k, c\right)$ to be small enough with $t \leq t^{*}$. Also we require the $t<t_{*}\left(T_{M}, \mathcal{C}, k, c\right)$ satisfy

$$
\frac{\varepsilon}{b-a-\varepsilon} \leq 5 T_{M}\left(1+\mathcal{C}_{k}\right) t^{c} \leq 2
$$

By the definition of $\mathcal{C}$ in (3.52) we conclude the $t_{*}$ only depends on the parameter in (3.39). Thus by the same computation as (3.49) we obtain

$$
\frac{b}{b-a-\varepsilon} \leq \frac{2 T_{l, q}}{T_{l, q}+\left[\min \left\{T_{w}(x)\right\}-T_{l, q}\right] r_{\|}\left(2-r_{\|}\right)} \leq C_{T_{M}}
$$

where we used $T_{l, q} \leq 2 T_{M}$ from (3.33) and (3.31). $C_{T_{M}}$ is defined in (3.49).

By the same computation as (3.51), we obtain

$$
\begin{aligned}
& \frac{(a+\varepsilon) b}{b-a-\varepsilon}=\frac{a b}{b-a}+\frac{a b}{b-a} \frac{\varepsilon}{b-a-\varepsilon}+\frac{b}{b-a-\varepsilon} \varepsilon \\
\leq & \frac{T_{l, q}-T_{w}\left(x_{q}\right)}{2 T_{w}\left(x_{q}\right)\left[T_{l, q}+\left[T_{w}\left(x_{q}\right)-T_{l, q}\right] r_{\|}\left(2-r_{\|}\right)\right]}+\mathcal{C}_{l-q+1} t_{*}^{c} .
\end{aligned}
$$

Here we used $T_{l, q} \leq 2 T_{M}$ and (3.31) to obtain

$$
\begin{aligned}
& \frac{a b}{b-a} \frac{\varepsilon}{b-a-\varepsilon}+\frac{b \varepsilon}{b-a-\varepsilon} \\
& \leq \frac{4 T_{M}\left(T_{l, q}-\min \left\{T_{w}(x)\right\}\right)}{2 \min \left\{T_{w}(x)\right\}\left[T_{l, q}+\left[\min \left\{T_{w}(x)\right\}-T_{l, q}\right] r_{\|}\left(2-r_{\|}\right)\right]}\left[1+\mathcal{C}_{l-q}\right] t_{*}^{c} \\
& +\frac{2 T_{l, q}}{2+\left[\min \left\{T_{w}(x)\right\}-T_{l, q}\right] r_{\|}\left(2-r_{\|}\right)}\left[1+\mathcal{C}_{l-q}\right] t_{*}^{c} \leq\left[\mathcal{C}+\mathcal{C C}_{l-q}\right] t_{*}^{c}=\mathcal{C}_{l-q+1} t_{*}^{c}
\end{aligned}
$$

with $\mathcal{C}$ defined in (3.52) and $\mathcal{C}_{l-q}$ defined in (3.42).
Thus by Lemma 2 with $w=\left(1-r_{\|}\right) v_{q-1, \|}$, the third line of (3.61) is bounded by

$$
\begin{align*}
& C_{T_{M}} \exp \left(\left[\frac{\left[T_{l, q}-T_{w}\left(x_{q}\right)\right]}{2 T_{w}\left(x_{q}\right)\left[T_{l, q}\left(1-r_{\|}\right)^{2}+r\left(2-r_{\|}\right) T_{w}\left(x_{q}\right)\right]}+\mathcal{C}_{l-q+1} t_{*}^{c}\right]\left|\left(1-r_{\|}\right) v_{q-1, \|}\right|^{2}\right) \\
& \quad \leq C_{T_{M}} \exp \left(\left[\frac{\left[T_{l, q}-T_{w}\left(x_{q}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{q}\right)\left[T_{l, q}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q}\right)\right]}+\mathcal{C}_{l-q+1} t_{*}^{c}\right]\left|v_{q-1, \|}\right|^{2}\right) \tag{3.62}
\end{align*}
$$

By the same computation as (3.56) the second line of (3.61) is bounded by

$$
\begin{equation*}
\frac{2}{\sqrt{T_{m}}} \sqrt{C_{T_{M}}} \exp \left(\left[\frac{\left[T_{l, q}-T_{w}\left(x_{q}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{q}\right)\left[T_{l, q}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q}\right)\right]}+\mathcal{C}_{l-q+1} t_{*}^{c}\right]\left|v_{q-1, \perp}\right|^{2}\right) \tag{3.63}
\end{equation*}
$$

By (3.62) and (3.63) and the notation (3.57), we derive that

$$
\begin{aligned}
(13.61) & \leq t_{*}^{-(l-q+1) c}\left(C_{T_{M}, T_{m}}\right)^{l-q+1} \exp \left(\left[\frac{\left[T_{l, q}-T_{w}\left(x_{q}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{q}\right)\left[T_{l, q}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q}\right)\right]}+\mathcal{C}_{l-q+1} t_{*}^{c}\right]\left|v_{q-1}\right|^{2}\right) \\
& =t_{*}^{-(l-q+1) c} C_{T_{M}, T_{M}}^{l-q+1} \mathcal{A}_{l, q}
\end{aligned}
$$

which is consistent with (3.40) with $p=q$. The induction is valid we derive (3.40).
Now we focus on (3.43). The first inequality in (3.43) follows directly from (3.40) and (3.37). For the second inequality, by (3.36) and Lemma 5 we have

$$
\begin{align*}
& t_{*}^{-\left(l-p^{\prime}+1\right) c} C_{T_{M}, T_{m}}^{l-p^{\prime}+1} \int_{\prod_{j=p}^{p^{\prime}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathcal{A}_{l, p^{\prime}} \mathrm{d} \Upsilon_{p}^{p^{\prime}-1} \\
& \leq t_{*}^{-\left(l-p^{\prime}+1\right) c} C_{T_{M}, T_{m}}^{l-p^{\prime}+1} \int_{\prod_{j=p}^{p^{\prime}-2} \mathcal{V}_{j}} \int_{\mathcal{V}_{p^{\prime}-1}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathcal{A}_{l, p^{\prime}} \frac{e^{\left[\frac{1}{2 T_{w}\left(x_{p^{\prime}}-1\right)}-\frac{1}{2 T_{w}\left(x_{p^{\prime}}\right)}+t_{*}^{c}\right]\left|v_{p^{\prime}-1}\right|^{2}}}{n\left(v_{p^{\prime}-1}\right) \cdot v_{p^{\prime}-1}} \mathrm{~d} \sigma\left(v_{p^{\prime}-1}, v_{p^{\prime}-2}\right) \mathrm{d} \Upsilon_{p}^{p^{\prime}-2} . \tag{3.64}
\end{align*}
$$

In the proof of (3.40) we have

$$
(3.58) \leq(3.59) \leq(3.61) \leq t_{*}^{-(l-q+1) c} C_{T_{M}, T_{m}}^{l-q+1} \mathcal{A}_{l, q} .
$$

Then by replacing $q$ by $p^{\prime}-1$ in the estimate (3.58) $\leq t_{*}^{-(l-q+1) c} C_{T_{M}, T_{m}}^{l-q+1} \mathcal{A}_{l, q}$ we have

$$
\left(\begin{array}{l}
(3.64)
\end{array} t_{*}^{-\left(l-p^{\prime}+2\right) c} C_{T_{M}, T_{m}}^{l-p^{\prime}+2} \int_{\prod_{j=p}^{p^{\prime}-2} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l}>0\right\}} \mathcal{A}_{l, p^{\prime}-1} \mathrm{~d} \Upsilon_{p}^{p^{\prime}-2}\right.
$$

Keep doing this computation until integrating over $\mathcal{V}_{p}$ we obtain the second inequality in (3.43).

The next lemma conclude the smallness of the last term of (3.18).

Lemma 16. Assume

$$
\begin{equation*}
\frac{T_{m}}{T_{M}}>\max \left(\frac{1-r_{\|}}{2-r_{\|}}, \frac{\sqrt{1-r_{\perp}}-\left(1-r_{\perp}\right)}{r_{\perp}}\right) \tag{3.65}
\end{equation*}
$$

For the last term of (3.18), we require $t_{*}$ in (3.39) further satisfies the condition (3.113) and (3.122) (these conditions are consistent with the dependent variables in (3.39). Then there exists

$$
\begin{equation*}
k_{0}=k_{0}\left(\Omega, C_{T_{M}, T_{m}}, \mathcal{C}, T_{M}, r_{\perp}, r_{\|}\right) \gg 1 \tag{3.66}
\end{equation*}
$$

such that for all $t<t_{*}$, we have

$$
\begin{equation*}
\int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} 1_{\left\{t_{k_{0}}>0\right\}} \mathrm{d} \Sigma_{k_{0}-1}^{k_{0}} \leq\left(\frac{1}{2}\right)^{k_{0}} \mathcal{A}_{k_{0}-1,1} \tag{3.67}
\end{equation*}
$$

where $\mathcal{A}_{k_{0}-1,1}$ is defined in (3.41).
Remark 9. The key difference between Lemma 16 and Lemma 15 is that we have the small term $\left(\frac{1}{2}\right)^{k_{0}}$. With this extra term Lemma 16 implies the measure of the last term of (3.18) is small provided $k=k_{0}$ is large enough. Such property is essential in our analysis since we then only need to consider a finite-fold integration and bound the rest fold by small magnitude number.

The $k_{0}$ is specified in (3.66). Combining with (3.39) with $c=\frac{1}{15}$ specified in (3.137) we conclude

$$
\begin{equation*}
t_{*}=t_{*}\left(\Omega, T_{M}, T_{m}, r_{\|}, r_{\perp}\right) \tag{3.68}
\end{equation*}
$$

We need several preparations to prove Lemma 16 .
Lemma 17. For $1 \leq i \leq k-1$, if

$$
\begin{equation*}
\left|v_{i} \cdot n\left(x_{i}\right)\right|<\delta, \tag{3.69}
\end{equation*}
$$

then

$$
\begin{equation*}
\int_{\prod_{j=i}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{v_{i} \in \mathcal{V}_{i}:\left|v_{i} \cdot n\left(x_{i}\right)\right|<\delta\right\}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Phi_{i, m}^{k, k-1} \leq \delta t_{*}^{-(k-i) c} C_{T_{M}, T_{m}}^{k-i} \mathcal{A}_{k-1, i} \tag{3.70}
\end{equation*}
$$

If

$$
\begin{equation*}
\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right|>\delta^{-1} \tag{3.71}
\end{equation*}
$$

then

$$
\begin{equation*}
\int_{\prod_{j=i}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right|>\delta^{-1}\right\}} \mathrm{d} \Phi_{i, m}^{k, k-1} \leq \delta t_{*}^{-(k-i) c} C_{T_{M}, T_{m}}^{k-i} \mathcal{A}_{k-1, i} \tag{3.72}
\end{equation*}
$$

Here $\eta_{i, \|}$ is a constant defined in (3.80).
If

$$
\begin{equation*}
\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right|>\delta^{-1} \tag{3.73}
\end{equation*}
$$

then

$$
\begin{equation*}
\int_{\prod_{j=i}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right|>\delta^{-1}\right\}} \mathrm{d} \Phi_{i, m}^{k, k-1} \leq \delta t_{*}^{-(k-i) c} C_{T_{M}, T_{m}}^{k-i} \mathcal{A}_{k-1, i} \tag{3.74}
\end{equation*}
$$

Here $\eta_{i, \perp}$ is a constant defined in (3.83).
Proof. First we focus on (3.70). By (3.61) in Lemma 15, we can replace $l$ by $k-1$ and replace $q$ by $i$ to obtain

$$
\begin{align*}
& \int_{\prod_{j=i}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Phi_{i, m}^{k, k-1} \leq t_{*}^{-(k-i) c} C_{T_{M}, T_{m}}^{k-i} \\
& \times \int_{\mathcal{V}_{i, \perp}} \frac{1}{r_{\perp} T_{w}\left(x_{i}\right)} e^{-\left[\frac{1}{2 T_{k-1, i}}-\frac{1}{2 T_{w}\left(x_{i}\right)}-\mathcal{C}_{k-i} t_{*}^{c}-t_{*}^{c}\right]\left|v_{i, \perp}\right|^{2}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{i, \perp} v_{i-1, \perp}}{T_{w}\left(x_{i}\right) r_{\perp}}\right) e^{-\frac{\left|v_{i, \perp}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{i-1, \perp}\right|^{2}}{2 T_{w}(x) r_{\perp}}} \mathrm{d} v_{i, \perp} \\
& \times \int_{\mathcal{V}_{i, \|}} \frac{1}{\pi r_{\|}\left(2-r_{\|}\right)\left(2 T_{w}\left(x_{i}\right)\right)} e^{-\left[\frac{1}{2 T_{k-1, i}}-\frac{1}{2 T_{w}\left(x_{i}\right)}-\mathcal{C}_{k-i} t_{*}^{c}-t_{*}^{c}\right]\left|v_{i, \|}\right|^{2}} e^{-\frac{1}{2 T_{w}\left(x_{i}\right)} \frac{\left|v_{i, \|}-\left(1-r_{\|}\right) v_{i-1, \|}\right|^{2}}{r_{\| \mid}^{\left(2-r_{\|}\right)}}} \mathrm{d} v_{i, \|} . \tag{3.75}
\end{align*}
$$

Under the condition (3.69), we consider the second line of (3.75) with integrating over $\left\{v_{i, \perp} \in \mathcal{V}_{i, \perp}:\left|v_{i} \cdot n\left(x_{i}\right)\right|<\right.$ $\left.\frac{1-\eta}{2(1+\eta)} \delta\right\}$. To apply (2.11) in Lemma 3, we set

$$
a=-\left[\frac{1}{2 T_{k-1, i}}-\frac{1}{2 T_{w}\left(x_{i}\right)}\right], \quad b=\frac{1}{2 T_{w}\left(x_{i}\right) r_{\perp}}, \quad \varepsilon=\mathcal{C}_{k-i} t_{*}^{c}+t_{*}^{c}, w=\sqrt{1-r_{\perp}} v_{i-1, \perp}
$$

Under the condition $\left|v_{i} \cdot n\left(x_{i}\right)\right|<\frac{1-\eta}{2(1+\eta)} \delta$, applying (2.11) in Lemma 3 and using (3.63) with $q=i, l=k-1$, we bound the second line of (3.75) by

$$
\begin{equation*}
\delta \frac{2}{\sqrt{T_{m}}} \sqrt{C_{T_{M}}} \exp \left(\left[\frac{\left[T_{k-1, i}-T_{w}\left(x_{i}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{i}\right)\left[T_{k-1, i}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{i}\right)\right]}+\mathcal{C}_{k-i+1} t_{*}^{c}\right]\left|v_{i-1, \perp}\right|^{2}\right) \tag{3.76}
\end{equation*}
$$

Taking (3.63) for comparison, we conclude the second line of (3.75) provides one more constant term $\delta$. The third line of (3.75) is bounded by (3.62) with $q=i, l=k-1$. Therefore, we derive (3.70).

Then we focus on (3.72). We consider the third line of (3.75). To apply (2.9) in Lemma 2, we set

$$
\begin{equation*}
a=-\frac{1}{2 T_{k-1, i}}+\frac{1}{2 T_{w}\left(x_{i}\right)}, \quad b=\frac{1}{2 T_{w}\left(x_{i}\right) r_{\|}\left(2-r_{\|}\right)}, \quad \varepsilon=\mathcal{C}_{k-i} t_{*}^{c}+t_{*}^{c}, w=\left(1-r_{\|}\right) v_{i-1, \|} \tag{3.77}
\end{equation*}
$$

We define

$$
\begin{equation*}
B_{i, \|}:=b-a-\varepsilon . \tag{3.78}
\end{equation*}
$$

In regard to (2.9),

$$
\frac{b}{b-a-\varepsilon} w=\frac{b}{b-a}\left[1+\frac{\varepsilon}{b-a-\varepsilon}\right] w
$$

By (3.77),

$$
\frac{b}{b-a}=\frac{T_{k-1, i}}{T_{k-1, i}\left(1-r_{\|}\right)^{2}+T_{w}\left(x_{i}\right) r_{\|}\left(2-r_{\|}\right)}, \quad \frac{\varepsilon}{b-a-\varepsilon}=\frac{\mathcal{C}_{k-i} t_{*}^{c}+t_{*}^{c}}{B_{i, \|}}
$$

Thus we obtain

$$
\begin{equation*}
\frac{b}{b-a-\varepsilon} w=\eta_{i, \|} v_{i-1, \|}, \tag{3.79}
\end{equation*}
$$

where we defined

$$
\begin{equation*}
\eta_{i, \|}:=\frac{T_{k-1, i}\left[1+\left(\mathcal{C}_{k-i}+1\right) t_{*}^{c} / B_{i, \|}\right]}{T_{k-1, i}\left(1-r_{\|}\right)^{2}+T_{w}\left(x_{i}\right) r_{\|}\left(2-r_{\|}\right)}\left(1-r_{\|}\right) . \tag{3.80}
\end{equation*}
$$

Thus under the condition (3.71), applying (2.9) in Lemma 2 with $\frac{b}{b-a-\varepsilon} w=\eta_{i, \|} v_{i-1, \|}$ and using (3.62) with $q=i, l=k-1$, we bound the third line of (3.75) by

$$
\delta C_{T_{M}} \exp \left(\left[\frac{\left[T_{k-1, i}-T_{w}\left(x_{i}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{i}\right)\left[T_{k-1, i}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{i}\right)\right]}+\mathcal{C}_{k-i+1} t_{*}^{c}\right]\left|v_{i-1, \|}\right|^{2}\right)
$$

Thus we derive (3.72) due to the extra constant $\delta$.
Last we focus on (3.74). We consider the second line of (3.75) with integrating over $\left\{v_{i, \perp}: v_{i, \perp} \in \mathcal{V}_{i, \perp},\left|v_{i, \perp}\right|>\right.$ $\left.\frac{1+\eta}{1-\eta} \delta^{-1}\right\}$. To apply (2.11) in Lemma 4, we set

$$
\begin{equation*}
a=-\frac{1}{2 T_{k-1, i}}+\frac{1}{2 T_{w}\left(x_{i}\right)}, \quad b=\frac{1}{2 T_{w}\left(x_{i}\right) r_{\perp}}, \quad \varepsilon=\mathcal{C}_{k-i} t_{*}^{c}+t_{*}^{c}, w=\sqrt{1-r_{\perp}} v_{i-1, \perp} \tag{3.81}
\end{equation*}
$$

Define

$$
\begin{equation*}
B_{i, \perp}:=b-a-\varepsilon \tag{3.82}
\end{equation*}
$$

By the same computation as (3.79),

$$
\frac{b}{b-a-\varepsilon} w=\eta_{i, \perp} v_{i-1, \perp}
$$

where we defined

$$
\begin{equation*}
\eta_{i, \perp}:=\frac{T_{k-1, i}\left[1+\frac{\left(\mathcal{C}_{k-i}+1\right) t_{*}^{c}}{B_{i, \perp}}\right]}{T_{k-1, i}\left(1-r_{\perp}\right)+T_{w}\left(x_{i}\right) r_{\perp}} \sqrt{1-r_{\perp}} \tag{3.83}
\end{equation*}
$$

Thus under the condition (3.73), applying (2.14) in Lemma 4 with $\frac{b}{b-a-\varepsilon} w=\eta_{i, \perp} v_{i-1, \perp}$ and using (3.63) with $q=i, l=k-1$, we bound the second line of (3.75) by

$$
\delta \frac{2}{\sqrt{T_{m}}} \sqrt{C_{T_{M}}} \exp \left(\left[\frac{\left[T_{k-1, i}-T_{w}\left(x_{i}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{i}\right)\left[T_{k-1, i}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{i}\right)\right]}+\mathcal{C}_{k-i+1} t_{*}^{c}\right]\left|v_{i-1, \perp}\right|^{2}\right)
$$

Then we derive (3.72) due to the extra constant $\delta$.

Lemma 18. For $\eta_{i, \|}$ and $\eta_{i, \perp}$ defined in Lemma 17 , we suppose there exists $\eta<1$ such that

$$
\begin{equation*}
\max \left\{\eta_{i, \|}, \eta_{i, \perp}\right\}<\eta<1 \tag{3.84}
\end{equation*}
$$

Then if

$$
\begin{equation*}
\left|v_{i, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1} \text { and }\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right|<\delta^{-1} \tag{3.85}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|v_{i-1, \|}\right|>\left|v_{i, \|}\right|+\delta^{-1} \tag{3.86}
\end{equation*}
$$

Also if

$$
\begin{equation*}
\left|v_{i, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1} \text { and }\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right|<\delta^{-1} \tag{3.87}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\left|v_{i-1, \perp}\right|>\left|v_{i, \perp}\right|+\delta^{-1} \tag{3.88}
\end{equation*}
$$

Remark 10. Lemma 17 includes the "good" cases since those extra small factor $\delta$ contributes to the decaying constant in Lemma 16. Lemma 18 discusses those "bad" cases since such cases do not directly provide any small factor. Thus those cases are the main difficulty in our estimate. In Lemma 20 we will specify the way to handle them using the properties in this lemma.
Proof. Under the condition (3.85) we have

$$
\eta_{i, \|}\left|v_{i-1, \|}\right|>\left|v_{i, \|}\right|-\delta^{-1}
$$

Thus we derive

$$
\begin{aligned}
\left|v_{i-1, \|}\right| & >\left|v_{i, \|}\right|+\frac{1-\eta_{i, \|}}{\eta_{i, \|}}\left|v_{i, \|}\right|-\frac{1}{\eta_{i, \|}} \delta^{-1} \\
& >\left|v_{i, \|}\right|+\frac{1-\eta_{i, \|}}{\eta_{i, \|}} \frac{1+\eta}{1-\eta} \delta^{-1}-\frac{1}{\eta_{i, \|}} \delta^{-1} \\
& >\left|v_{i, \|}\right|+\frac{1-\eta_{i, \|}}{\eta_{i, \|}} \frac{1+\eta_{i, \|}}{1-\eta_{i, \|}} \delta^{-1}-\frac{1}{\eta_{i, \|}} \delta^{-1} \\
& >\left|v_{i, \|}\right|+\frac{1+\eta_{i, \|}}{\eta_{i, \|}} \delta^{-1}-\frac{1}{\eta_{i, \|}} \delta^{-1}>\left|v_{i, \|}\right|+\delta^{-1}
\end{aligned}
$$

where we used $\left|v_{i, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}$ in the second line and $1>\eta \geq \eta_{i, \|}$ in the third line. Then we obtain (3.86).
Under the condition (3.87), we apply the same computation above to obtain (3.88).

Lemma 19. Suppose there are $n$ number of $v_{j}$ such that

$$
\begin{equation*}
\left|v_{j, \|}-\eta_{j, \|} v_{j-1, \|}\right| \geq \delta^{-1} \tag{3.89}
\end{equation*}
$$

and also suppose the index $j$ in these $v_{j}$ are $i_{1}<i_{2}<\cdots<i_{n}$, then

$$
\begin{equation*}
\int_{\prod_{j=i_{1}}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\sqrt{3.89} \text { holds for } j=i_{1}, i_{2}, \cdots, i_{n}\right\}} \mathrm{d} \Phi_{i_{1}, m}^{k, k-1} \leq(\delta)^{n} t_{*}^{-\left(k-i_{1}\right) c} C_{T_{M}, T_{m}}^{k-i_{1}} \mathcal{A}_{k-1, i_{1}} \tag{3.90}
\end{equation*}
$$

Proof. By (3.43) in Lemma 2 with $l=k-1, p=i_{1}, p^{\prime}=i_{n}$ and using (3.72) with $i=i_{n}$, we have

$$
\begin{aligned}
& \int_{\prod_{j=i_{1}}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\begin{array}{l}
\text { (3.89) holds for } \left.j=i_{1}, \cdots, i_{n}\right\}
\end{array}\right.} \mathrm{d} \Phi_{i_{1}, m}^{k, k-1}
\end{aligned}
$$

$$
\begin{align*}
& =\delta t_{*}^{-\left(k-i_{n}\right) c} C_{T_{M}, T_{m}}^{k-i_{n}} \int_{\prod_{j=i_{1}}^{i_{n-1}^{1-1}} \mathcal{V}_{j}} \int_{\prod_{j=i_{n-1}}^{\left(i_{n}\right)-1} \mathcal{V}_{j}} \mathcal{A}_{k-1, i_{n}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\begin{array}{l}
(3.89) \\
\text { holds for } \left.j=i_{1}, \cdots, i_{n-1}\right\}
\end{array} \mathrm{d}_{i_{n-1}}^{\left(i_{n}\right)-1} \mathrm{~d} \Upsilon_{i_{1}}^{i_{n-1}-1} .\right.} . \tag{3.91}
\end{align*}
$$

Again by (3.43) and (3.72) with $i=i_{n-1}$ we have

$$
\text { (3.91) } \leq \delta^{2} t_{*}^{-\left(k-i_{n-1}\right) c} C_{T_{M}, T_{m}}^{k-i_{n-1}} \int_{\prod_{j=i_{1}}^{i_{n-1}-1} \mathcal{V}_{j}} \mathcal{A}_{k-1, i_{n-1}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\text {(3.89) holds for } j=i_{1}, \cdots, i_{n-2}\right\}} \mathrm{d} \Upsilon_{i_{1}}^{i_{n-1}-1}
$$

Keep doing this computation until integrating over $\mathcal{V}_{i_{1}}$ we derive (3.90).

Lemma 20. For $0<\delta \ll 1$, we define

$$
\begin{equation*}
\mathcal{V}_{j}^{\delta}:=\left\{v_{j} \in \mathcal{V}_{j}:\left|v_{j} \cdot n\left(x_{j}\right)\right|>\delta,\left|v_{j}\right| \leq \delta^{-1}\right\} \tag{3.92}
\end{equation*}
$$

For the sequence $\left\{v_{1}, v_{2}, \cdots, v_{k-1}\right\}$, consider a subsequence $\left\{v_{l+1}, v_{l+2}, \cdots, v_{l+L}\right\}$ with $l+1<l+L \leq k-1$ as follows:

$$
\begin{equation*}
\underbrace{v_{l},}_{\in \mathcal{V}_{l}^{\frac{1-\eta}{2(1+\eta)} \delta}} \underbrace{v_{l+1}, v_{l+2} \cdots v_{l+L}}_{a l l \in \mathcal{V}_{l+j} \backslash \mathcal{V}_{l+j}^{\frac{1-\eta}{2(1+\eta)} \delta}}, \quad \underbrace{v_{l+L+1}}_{\substack{\frac{1-\eta}{\frac{1}{2(1+\eta)}} \delta \\ \in \mathcal{V}_{l+L+1}}} . \tag{3.93}
\end{equation*}
$$

In (3.93), if $L \geq 100 \frac{1+\eta}{1-\eta}$, then we have

$$
\begin{equation*}
\int_{\prod_{j=l}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{v_{l+j} \in \mathcal{V}_{l+j} \backslash \mathcal{V}_{l+j}^{\frac{1-\eta}{2(1+\eta)} \delta}{ }_{\text {for }} 1 \leq j \leq L\right\}} \mathrm{d} \Phi_{l, m}^{k, k-1} \leq(3 \delta)^{L / 2} t_{*}^{-(k-l) c} C_{T_{M}, T_{m}}^{k-l} \mathcal{A}_{k-1, l} \tag{3.94}
\end{equation*}
$$

Here the $\eta$ satisfies the condition (3.84).
Remark 11. In this lemma we combine the estimates in Lemma 17 and Lemma 18 and derive the desired decaying term $(3 \delta)^{L / 2}$. In the proof we will address the difficulty stated in Lemma 18 .

Proof. By the definition (3.92) we have

$$
\mathcal{V}_{i} \backslash \mathcal{V}_{i}^{\frac{1-\eta}{2(1+\eta)} \delta}=\left\{v_{i} \in \mathcal{V}_{i}:\left|v_{i} \cdot n\left(x_{i}\right)\right|<\frac{1-\eta}{2(1+\eta)} \delta \text { or }\left|v_{i}\right| \geq \frac{2(1+\eta)}{1-\eta} \delta^{-1}\right\}
$$

Here we summarize the result of Lemma 17 and Lemma 18. With $\frac{1-\eta}{1+\eta} \delta<\delta$, when $v_{i} \in \mathcal{V}_{i} \backslash \mathcal{V}_{i}^{\frac{1-\eta}{2(1+\eta)} \delta}$
(1) When $\left|v_{i} \cdot n\left(x_{i}\right)\right|<\frac{1-\eta}{2(1+\eta)} \delta$, then we have (3.70).
(2) When $\left|v_{i}\right|>\frac{2(1+\eta)}{1-\eta} \delta^{-1}$,
(a) when $\left|v_{i, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}$, if $\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right|<\delta^{-1}$, then $\left|v_{i-1, \|}\right|>\left|v_{i, \|}\right|+\delta^{-1}$.
(b) when $\left|v_{i, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}$, if $\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right| \geq \delta^{-1}$, then we have (3.72).
(c) when $\left|v_{i, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}$, if $\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right|<\delta^{-1}$, then $\left|v_{i-1, \perp}\right|>\left|v_{i, \perp}\right|+\delta^{-1}$.
(d) when $\left|v_{i, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}$, if $\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right| \geq \delta^{-1}$, then we have (3.74).

We define $\mathcal{W}_{i, \delta}$ as the space that provides the smallness:

$$
\begin{gathered}
\mathcal{W}_{i, \delta}:=\left\{v_{i} \in \mathcal{V}_{i}:\left|v_{i, \perp}\right|<\frac{1-\eta}{2(1+\eta)} \delta\right\} \bigcup\left\{v_{i} \in \mathcal{V}_{i}:\left|v_{i, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1} \text { and }\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right|>\delta^{-1}\right\} \\
\bigcup\left\{v_{i} \in \mathcal{V}_{i}:\left|v_{i, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1} \text { and }\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right|>\delta^{-1}\right\}
\end{gathered}
$$

Then we have

$$
\begin{align*}
& \mathcal{V}_{i} \backslash \mathcal{V}_{i}^{\frac{1-\eta}{2(1+\eta)} \delta} \subset \mathcal{W}_{i, \delta} \bigcup\left\{v_{i, \perp} \in \mathcal{V}_{i, \perp}\left|v_{i, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1} \text { and }\left|v_{i, \perp}-\eta_{i, \perp} v_{i-1, \perp}\right|<\delta^{-1}\right\} \\
& \bigcup\left\{v_{i, \|} \in \mathcal{V}_{i, \|}\left|v_{i, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1} \text { and }\left|v_{i, \|}-\eta_{i, \|} v_{i-1, \|}\right|<\delta^{-1}\right\} \tag{3.95}
\end{align*}
$$

By (3.70), (3.72) and (3.74) with $\frac{1-\eta}{1+\eta} \delta<\delta$, we obtain

$$
\begin{equation*}
\int_{\prod_{j=i}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{v_{i} \in \mathcal{W}_{i, \delta}\right\}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Phi_{i, m}^{k, k-1} \leq 3 \delta t_{*}^{-(k-i) c} C_{T_{M}, T_{m}}^{k-i} \mathcal{A}_{k-1, i} \tag{3.96}
\end{equation*}
$$

For the subsequence $\left\{v_{l+1}, \cdots, v_{l+L}\right\}$ in (3.93), when the number of $v_{j} \in \mathcal{W}_{j, \delta}$ is larger than $L / 2$, by (3.90) in Lemma 19 with $n=L / 2$ and replacing the condition (3.89) by $v_{j} \in \mathcal{W}_{j, \delta}$, we obtain

$$
\begin{align*}
& \int_{\prod_{j=l}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{\text {Number of } v_{j} \in \mathcal{W}_{j, \delta} \text { is larger than } L / 2\right\}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Phi_{l, m}^{k, k-1}  \tag{3.97}\\
& \leq(3 \delta)^{L / 2} t_{*}^{-\left(k-l_{i}\right) c} C_{T_{M}, T_{m}}^{k-l_{i}} \mathcal{A}_{k-1, l} . \tag{3.98}
\end{align*}
$$

We finish the discussion with the case $(1),(2 b),(2 d)$. Then we focus on the case (2a), (2c).
When the number of $v_{j} \notin \mathcal{W}_{j, \delta}$ is larger than $L / 2$, by (3.95) we further consider two cases. The first case is that the number of $v_{j} \in\left\{v_{j}:\left|v_{j, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}\right.$ and $\left.\left|v_{j, \|}-\eta_{j, \|} v_{j-1, \|}\right|<\delta^{-1}\right\}$ is larger than $L / 4$. According to the relation of $v_{j, \|}$ and $v_{j-1, \|}$, we categorize them into

Set1: $\left\{v_{j} \notin \mathcal{W}_{j, \delta}:\left|v_{j, \|}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}\right.$ and $\left.\left|v_{j, \|}-\eta_{j, \|} v_{j-1, \|}\right|<\delta^{-1}\right\}$.
Denote $M=|\operatorname{Set} 1|$ and the corresponding index in Set1 as $j=p_{1}, p_{2}, \cdots, p_{M}$. Then we have

$$
\begin{equation*}
L / 4 \leq M \leq L \tag{3.99}
\end{equation*}
$$

By (3.86) in Lemma 18, for those $v_{p_{j}}$, we have

$$
\begin{equation*}
\left|v_{p_{j}, \|}\right|-\left|v_{p_{j}-1, \|}\right|<-\delta^{-1} \tag{3.100}
\end{equation*}
$$

Set2: $\left\{v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta) \delta}}:\left|v_{j, \|}\right| \geq\left|v_{j-1, \|}\right|\right\}$.
Denote $\mathcal{M}=|\operatorname{Set} 2|$ and the corresponding index in Set2 as $j=q_{1}, q_{2}, \cdots, q_{\mathcal{M}}$. By (3.99) we have

$$
\begin{equation*}
1 \leq \mathcal{M} \leq L-M \leq \frac{3}{4} L \tag{3.101}
\end{equation*}
$$

Then for those $v_{q_{j}}$ we define

$$
\begin{equation*}
a_{j}:=\left|v_{q_{j}, \|}\right|-\left|v_{q_{j}-1, \|}\right|>0 \tag{3.102}
\end{equation*}
$$

Set3: $\left\{v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta) \delta}}:\left|v_{j, \|}\right| \leq\left|v_{j-1, \|}\right| \leq\left|v_{j, \|}\right|+\delta^{-1}\right\}$.
Denote $N=|\operatorname{Set} 3|$ and the corresponding index in $\operatorname{Set} 3$ as $j=o_{1}, o_{2}, \cdots, o_{N}$. Then for those $o_{j}$, we have

$$
\begin{equation*}
\left|v_{o_{j}, \|}\right| \leq\left|v_{o_{j}-1, \|}\right| \leq\left|v_{o_{j}, \|}\right|+\delta^{-1} \tag{3.103}
\end{equation*}
$$

From (3.93), we have $v_{l} \in \mathcal{V}_{l}^{\frac{1-\eta}{2(1+\eta)} \delta}$ and $v_{l+L+1} \in \mathcal{V}_{l+L+1}^{\frac{1-\eta}{2(1+\eta)} \delta}$, thus we can obtain

$$
\begin{equation*}
-\frac{2(1+\eta)}{1-\eta} \delta^{-1}<\left|v_{l+L+1, \|}\right|-\left|v_{l, \|}\right|=\sum_{j=1}^{L+1}\left|v_{l+j, \|}\right|-\left|v_{l+j-1, \|}\right| \tag{3.104}
\end{equation*}
$$

By (3.100), (3.102) and (3.103), we derive that

$$
\begin{aligned}
\frac{-2(1+\eta)}{1-\eta} \delta^{-1} & <\sum_{j=1}^{M}\left(\left|v_{p_{j}, \|}\right|-\left|v_{p_{j}-1, \| \mid}\right|\right)+\sum_{j=1}^{\mathcal{M}}\left(\left|v_{q_{j}, \|}\right|-\left|v_{q_{j}-1, \|}\right|\right)+\sum_{j=1}^{N}\left(\left|v_{o_{j}, \|}\right|-\left|v_{o_{j}-1, \|}\right|\right) \\
& \leq-M \delta^{-1}+\sum_{j=1}^{\mathcal{M}} a_{j}
\end{aligned}
$$

Therefore, by $L \geq 100 \frac{1+\eta}{1-\eta}$ and (3.99), we obtain

$$
\frac{2(1+\eta)}{1-\eta} \delta^{-1} \leq \frac{L}{10} \delta^{-1} \leq \frac{M}{2} \delta^{-1}
$$

and thus

$$
\begin{equation*}
\sum_{j=1}^{\mathcal{M}} a_{j} \geq M \delta^{-1}-\frac{2(1+\eta)}{1-\eta} \delta^{-1}>\frac{M \delta^{-1}}{2} \tag{3.105}
\end{equation*}
$$

We focus on the integration over $\mathcal{V}_{q_{i}}$, such indexes satisfy (3.102). Let $1 \leq i \leq \mathcal{M}$, we consider the third line of (3.75) with $i=q_{i}$ and with integrating over $\left\{v_{q_{i}, \|} \in \mathcal{V}_{q_{i}, \|}:\left|v_{q_{i}, \|}\right|-\left|v_{q_{i}-1, \|}\right|=a_{i}\right\}$. To apply (2.9) in Lemma [2, we set

$$
a=-\frac{1}{2 T_{k-1, q_{i}}}+\frac{1}{2 T_{w}\left(x_{q_{i}}\right)}, \quad b=\frac{1}{\begin{array}{c}
2 T_{w}\left(x_{q_{i}}\right) r_{\|}\left(2-r_{\|}\right)
\end{array}, \quad \varepsilon=\mathcal{C}_{k-q_{i}} t_{*}^{c}+t_{*}^{c} . . . . ~ . ~ . ~}
$$

By the same computation as (3.113), we have

$$
\begin{equation*}
a+\varepsilon-b=-\frac{1}{2 T_{k-1, q_{i}}}+\frac{1}{2 T_{w}\left(x_{q_{i}}\right)}-\frac{1}{2 T_{w}\left(x_{q_{i}}\right) r_{\|}\left(2-r_{\|}\right)}+\mathcal{C}_{k-q_{i}} t_{*}^{c}+t_{*}^{c}<-\frac{1}{5 T_{M}} . \tag{3.106}
\end{equation*}
$$

Then we use $\eta_{q_{i}, \|}<1$ to obtain

$$
\begin{equation*}
\left.\mathbf{1}_{\left\{\left|v_{q_{i}, \|}\right|-\left|v_{q_{i}-1, \|}\right|=a_{i}\right\}} \leq \mathbf{1}_{\left\{\mid v_{q_{i}}, \|\right.}\left|-\eta_{q_{i}, \|}\right| v_{q_{i}-1, \|} \mid>a_{i}\right\} \leq \mathbf{1}_{\left\{\left|v_{q_{i}}, \|-\eta_{q_{i}, \|} v_{q_{i}-1, \|}\right|>a_{i}\right\}} . \tag{3.107}
\end{equation*}
$$

By (2.9) in Lemma 2 and (3.107), we apply (3.62) with $q=q_{i}$ to bound the third line of (3.75) ( the integration over $\left.\mathcal{V}_{q_{i}, \|}\right)$ by

$$
\begin{equation*}
e^{-\frac{a_{i}^{2}}{4 T_{M}}} C_{T_{M}} \exp \left(\left[\frac{\left[T_{k-1, q_{i}}-T_{w}\left(x_{q_{i}}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{q_{i}}\right)\left[T_{k-1, q_{i}}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q_{i}}\right)\right]}+\mathcal{C}_{k-q_{i}+1} t_{*}^{c}\right]\left|v_{q_{i}-1, \|}\right|^{2}\right) . \tag{3.108}
\end{equation*}
$$

Hence by the constant in (3.108) we draw a similar conclusion as (3.96):

$$
\begin{equation*}
\int_{\prod_{j=q_{i}}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\left|v_{q_{i}, \|}\right|-\left|v_{q_{i}-1, \|}\right|=a_{i}\right\}} \mathrm{d} \Phi_{q_{i}, m}^{k, k-1} \leq e^{-\frac{a_{i}^{2}}{4 T_{M}}} t_{*}^{-\left(k-q_{i}+1\right) c} C_{T_{M}, T_{m}}^{k-q_{i}+1} \mathcal{A}_{k-1, q_{i}} \tag{3.109}
\end{equation*}
$$

Therefore, by Lemma 19, after integrating over $\mathcal{V}_{q_{1}, \|}, \mathcal{V}_{q_{2}, \|}, \cdots, \mathcal{V}_{q_{\mathcal{M}}, \|}$ we obtain an extra constant

$$
\begin{gathered}
e^{-\left[a_{i}^{2}+a_{2}^{2}+\cdots+a_{\mathcal{M}}^{2}\right] / 4 T_{M}} \leq e^{-\left[a_{i}+a_{2}+\cdots+a_{\mathcal{M}}\right]^{2} /\left(4 T_{M} \mathcal{M}\right)} \leq e^{-\left[M \delta^{-1} / 2\right]^{2} /\left(4 T_{M} \mathcal{M}\right)} \\
\leq e^{-\left[\frac{L}{8} \delta^{-1}\right]^{2} /\left(4 T_{M} \frac{3}{4} L\right)} \leq e^{-\frac{1}{96 T_{M}} L\left(\delta^{-1}\right)^{2}} \leq e^{-L \delta^{-1}}
\end{gathered}
$$

Here we used (3.105) in the last step of first line and use (3.99), (3.101) in the first step of second line and take $\delta \ll 1$ in the last step of second line. Then $e^{-L \delta^{-1}}$ is smaller than $(3 \delta)^{L / 2}$ in (3.98) and we conclude

$$
\begin{equation*}
\int_{\prod_{j=l}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\{M=|\operatorname{Set} 1| \geq L / 4\}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Phi_{l, m}^{k, k-1} \leq(3 \delta)^{L / 2} t_{*}^{-\left(k-l_{i}\right) c} C_{T_{M}, T_{m}}^{k-l_{i}} \mathcal{A}_{k-1, l} \tag{3.110}
\end{equation*}
$$

The second case is that the number of $v_{j} \in\left\{v_{j} \notin \mathcal{W}_{j, \delta}:\left|v_{j, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}\right\}$ is larger than $L / 4$. We categorize $v_{j, \perp}$ into

Set4: $\left\{v_{j} \notin \mathcal{W}_{j, \delta}:\left|v_{j, \perp}\right|>\frac{1+\eta}{1-\eta} \delta^{-1}\right.$ and $\left.\left|v_{j, \perp}-\eta_{j, \perp} v_{j-1, \perp}\right|<\delta^{-1}\right\}$.
Set5: $\left\{v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta) \delta}}:\left|v_{j, \perp}\right|>\left|v_{j-1, \perp}\right|\right\}$.
Set6: $\left\{v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta) \delta}}:\left|v_{j, \perp}\right| \leq\left|v_{j-1, \perp}\right| \leq\left|v_{j, \perp}\right|+\delta^{-1}\right\}$.
Denote $|\operatorname{Set} 4|=M_{1}$ and the corresponding index as $p_{1}^{\prime}, p_{2}^{\prime}, \cdots, p_{M_{1}}^{\prime},|\operatorname{Set} 5|=\mathcal{M}_{1}$ and the corresponding index as $q_{1}^{\prime}, q_{2}^{\prime}, \cdots, q_{\mathcal{M}_{1}}^{\prime},|\operatorname{Set} 6|=N_{1}$ and the corresponding index as $o_{1}^{\prime}, o_{2}^{\prime}, \cdots, o_{N_{1}}^{\prime}$. Also define $b_{j}:=\left|v_{q_{j}^{\prime}, \perp}\right|-\left|v_{q_{j}^{\prime}-1, \perp}\right|$. By the same computation as (3.105), we have

$$
\sum_{j=1}^{\mathcal{M}_{1}} b_{j} \geq M_{1} \delta^{-1}-\frac{2(1+\eta)}{1-\eta} \delta^{-1}>\frac{M_{1} \delta^{-1}}{2}
$$

We focus on the integration over $v_{q_{j}^{\prime}}$. Let $1 \leq i \leq \mathcal{M}_{1}$, we consider the second line of (3.75) with $i=q_{i}^{\prime}$ and with integrating over $\left\{v_{q_{i}^{\prime}, \perp} \in \mathcal{V}_{q_{i}^{\prime}, \perp}:\left|v_{q_{i}^{\prime}, \perp}\right|-\left|v_{q_{i}^{\prime}-1, \perp}\right|=b_{i}\right\}$. To apply (2.13) in Lemma 2, we set

$$
a=-\frac{1}{2 T_{k-1, q_{i}^{\prime}}}+\frac{1}{2 T_{w}\left(x_{q_{i}^{\prime}}\right)}, \quad b=\frac{1}{2 T_{w}\left(x_{q_{i}^{\prime}}\right) r_{\perp}}, \quad \varepsilon=\mathcal{C}^{k-q_{i}^{\prime}} t_{*}^{c}+t_{*}^{c} .
$$

By the same computation as (3.113), we have

$$
\begin{equation*}
a+\varepsilon-b=-\frac{1}{2 T_{k-1, q_{i}^{\prime}}}+\frac{1}{2 T_{w}\left(x_{q_{i}^{\prime}}\right)}-\frac{1}{2 T_{w}\left(x_{q_{i}^{\prime}}\right) r_{\perp}}+\mathcal{C}_{k-q_{i}^{\prime}} t_{*}^{c}+t_{*}^{c}<-\frac{1}{5 T_{M}} . \tag{3.111}
\end{equation*}
$$

Similar to (3.107), we have

$$
\mathbf{1}_{\left\{\left|v_{q_{i}^{\prime}, \perp}\right|-\left|v_{q_{i}^{\prime}-1, \perp}\right|=b_{i}\right\}} \leq \mathbf{1}_{\left\{\left|v_{q_{i}^{\prime}, \perp}-\eta_{q_{i}^{\prime}, \perp} v_{q_{i}^{\prime}-1, \perp}\right|>b_{i}\right\}} .
$$

Hence by (2.13) in Lemma 4 and applying (3.63), we bound the integration over $\mathcal{V}_{q_{i}^{\prime}, \perp}$ by

$$
e^{-\frac{b_{i}^{2}}{16 T_{M}}} \frac{2}{\sqrt{T_{m}} \sqrt{C_{T_{M}}} \exp \left(\left[\frac{\left[T_{k-1, q_{i}^{\prime}}-T_{w}\left(x_{q_{i}^{\prime}}\right)\right]\left[1-r_{\min }\right]}{2 T_{w}\left(x_{q_{i}^{\prime}}\right)\left[T_{k-1, q_{i}^{\prime}}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{q_{i}^{\prime}}\right)\right]}+\mathcal{C}_{k-q_{i}^{\prime}+1} t_{*}^{c}\right]\left|v_{q_{i}^{\prime}-1, \perp}\right|^{2}\right) . . . . ~ . ~}
$$

Therefore,

$$
\int_{\prod_{j=q_{i}^{\prime}}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{\left|v_{q_{i}^{\prime}, \perp}\right|-\left|v_{q_{i}^{\prime}-1, \perp}\right|=b_{i}\right\}} \mathrm{d} \Phi_{q_{i}^{\prime}, m}^{k, k-1} \leq e^{-\frac{b_{i}^{2}}{16 T_{M}}} t_{*}^{-\left(k-q_{i}^{\prime}\right) c} C_{T_{M}, T_{m}}^{k-q_{i}^{\prime}} \mathcal{A}_{k-1, q_{i}^{\prime}}
$$

The integration over $\mathcal{V}_{q_{1}^{\prime}, \perp}, \mathcal{V}_{q_{2}^{\prime}, \perp}, \cdots, \mathcal{V}_{q_{\mathcal{M}_{1}}^{\prime}, \perp}$ provides an extra constant

$$
e^{-\left[b_{1}^{2}+b_{2}^{2}+\cdots+b_{\mathcal{M}_{1}}^{2}\right] / 16 T_{M}} \leq e^{-\frac{1}{400 T_{M}} L\left(\delta^{-1}\right)^{2}} \leq e^{-L \delta^{-1}}
$$

where we set $\delta \ll 1$ in the last step. Then $e^{-L \delta^{-1}}$ is smaller than $(3 \delta)^{L / 2}$ in (3.98) and we conclude

$$
\begin{equation*}
\int_{\prod_{j=l}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{M_{1}=|\operatorname{Set} 4| \geq L / 4\right\}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Phi_{l, m}^{k, k-1} \leq(3 \delta)^{L / 2} t_{*}^{-(k-l) c} C_{T_{M}, T_{m}}^{k-l} \mathcal{A}_{k-1, l} \tag{3.112}
\end{equation*}
$$

Finally collecting (3.98), (3.110) and (3.112) we derive the lemma.

Now we are ready to prove the Lemma 16 ,

## Proof of Lemma 16. Step 1

To prove (3.67) holds for the C-L boundary condition, we mainly use the decomposition (3.92) done by [5] and [23] for the diffuse boundary condition. In order to apply Lemma 20, here we consider the space $\mathcal{V}_{i}^{\frac{1-\eta}{2(1+\eta)} \delta}$ and ensure $\eta$ satisfy the condition (3.84). In this step we mainly focus on constructing the $\eta$, which will be defined in (3.124).

First we consider $\eta_{i, \|}$, which is defined in (3.80). In regard to (3.77) and (3.78), we require $t_{*}=t_{*}\left(k, T_{M}, c, \mathcal{C}\right)$ ( consistent with (3.39) ) to be small enough such that

$$
\begin{equation*}
B_{i, \|} \geq \frac{1}{2 T_{k-1, i}}-\mathcal{C}_{k-i} t_{*}^{c}-t_{*}^{c} \geq \frac{1}{4 T_{M}}-\mathcal{C}_{k} t_{*}^{c}-t_{*}^{c} \geq \frac{1}{5 T_{M}} \tag{3.113}
\end{equation*}
$$

By (3.34), $T_{k-1, i} \rightarrow T_{M}$ as $k-i \rightarrow \infty$. For any $\varepsilon_{1}>0$, there exists $k_{1}$ s.t when

$$
\begin{equation*}
k \geq k_{1}, \quad i \leq k / 2, \text { we have } T_{k-1, i} \leq\left(1+\varepsilon_{1}\right) T_{M} \tag{3.114}
\end{equation*}
$$

Moreover, by (3.65), there exists $\varepsilon_{2}$ s.t

$$
\begin{equation*}
\frac{T_{m}}{T_{M}}>\frac{1-r_{\|}}{2-r_{\|}}\left(1+\varepsilon_{2}\right) \tag{3.115}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\varepsilon_{2}=\varepsilon_{2}\left(T_{m}, T_{M}, r_{\|}, r_{\perp}\right) \tag{3.116}
\end{equation*}
$$

Thus we can bound $T_{w}\left(x_{i}\right)$ in the $\eta_{i, \|}$ ( defined in (3.80)) below as

$$
\begin{equation*}
T_{w}\left(x_{i}\right)=T_{k-1, i} \frac{T_{w}\left(x_{i}\right)}{T_{k-1, i}} \geq T_{k-1, i} \frac{T_{w}\left(x_{i}\right)}{T_{M}} \frac{1}{1+\varepsilon_{1}}>\frac{1-r_{\|}}{2-r_{\|}} T_{k-1, i} \frac{1+\varepsilon_{2}}{1+\varepsilon_{1}} \tag{3.117}
\end{equation*}
$$

Thus we obtain

$$
\begin{equation*}
\eta_{i, \|}<\frac{1+\frac{\left(\mathcal{C}_{k-i}+1\right) t_{*}^{c}}{B_{i, \|}}}{\left(1-r_{\|}\right)^{2}+\frac{1-r_{\|}}{2-r_{\|}} \frac{1+\varepsilon_{2}}{1+\varepsilon_{1}} r_{\|}\left(2-r_{\|}\right)}\left(1-r_{\|}\right)=\frac{1+\frac{\left(\mathcal{C}_{k-i}+1\right) t_{*}^{c}}{B_{i, \|}}}{1-r_{\|}+r_{\|} \frac{1+\varepsilon_{2}}{1+\varepsilon_{1}}} . \tag{3.118}
\end{equation*}
$$

By (3.114), we take

$$
\begin{equation*}
k=k_{1}=k_{1}\left(\varepsilon_{2}, T_{M}, r_{\min }\right) \tag{3.119}
\end{equation*}
$$

to be large enough such that $\varepsilon_{1}<\varepsilon_{2} / 4$. By (3.113) and (3.118), we derive that when $k=k_{1}$,

$$
\begin{equation*}
\sup _{i \leq k / 2} \eta_{i, \|} \leq \frac{1+5 T_{M}\left(\mathcal{C}_{k}+1\right) t_{*}^{c}}{1-r_{\|}+r_{\|} \frac{1+\varepsilon_{2}}{1+\varepsilon_{2} / 4}}<\eta_{\|}<1 \tag{3.120}
\end{equation*}
$$

Here we define

$$
\begin{equation*}
\eta_{\|}:=\frac{1}{1-r_{\|}+r_{\|} \frac{1+\varepsilon_{2}}{1+\varepsilon_{2} / 2}}<1 \tag{3.121}
\end{equation*}
$$

and we require $t_{*}=t^{\prime}\left(k, T_{M}, \varepsilon_{2}, \mathcal{C}, r_{\|}\right)$to be small enough and such that

$$
\begin{equation*}
5 T_{M} \mathcal{C}_{k} t_{*}^{c} \ll 1 \tag{3.122}
\end{equation*}
$$

to ensure the second inequality in (3.120). Combining (3.116) and (3.119), we conclude the condition for $t_{*}$ (3.122) is consistent with (3.39).

Then we consider $\eta_{i, \perp}$, which is defined in (3.83). In regard to (3.81) and (3.82), by (3.113) we have $B_{i, \perp} \geq \frac{1}{5 T_{M}}$. By $\frac{T_{m}}{T_{M}}>\frac{\sqrt{1-r_{\perp}}-\left(1-r_{\perp}\right)}{r_{\perp}}$ in (3.65) we can use the same computation as (3.117) to obtain

$$
T_{w}\left(x_{i}\right)>\frac{\sqrt{1-r_{\perp}}-\left(1-r_{\perp}\right)}{r_{\perp}} T_{k-1, i} \frac{1+\varepsilon_{2}}{1+\varepsilon_{1}}
$$

with $\varepsilon_{1}<\varepsilon_{2} / 4$. Thus we obtain

$$
\eta_{i, \perp}<\eta_{\perp}<1
$$

where we defined

$$
\begin{equation*}
\eta_{\perp}:=\frac{1}{\sqrt{1-r_{\perp}}+\left(1-\sqrt{1-r_{\perp}}\right) \frac{1+\varepsilon_{2}}{1+\varepsilon_{2} / 2}}<1 \tag{3.123}
\end{equation*}
$$

with small enough $t_{*}=t_{*}\left(k, T_{M}, \varepsilon_{2}, \mathcal{C}, r_{\|}\right)($consistent with (3.39) ).
Finally we define

$$
\begin{equation*}
\eta:=\max \left\{\eta_{\perp}, \eta_{\|}\right\}<1 \tag{3.124}
\end{equation*}
$$

Step 2
Claim: We have

$$
\begin{equation*}
\left|t_{j}-t_{j+1}\right| \gtrsim \Omega\left(\frac{1-\eta}{2(1+\eta)} \delta\right)^{3}, \text { for } v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}, 0 \leq t_{j} \tag{3.125}
\end{equation*}
$$

Proof. For $t_{j} \leq 1$,

$$
\begin{aligned}
& \left|\int_{t_{j}}^{t_{j+1}} v_{j} \mathrm{~d} s\right|^{2}=\left|x_{j+1}-x_{j}\right|^{2} \gtrsim\left|\left(x_{j+1}-x_{j}\right) \cdot n\left(x_{j}\right)\right| \\
& \quad=\left|\int_{t_{j}}^{t_{j+1}} v_{j} \cdot n\left(x_{j}\right) \mathrm{d} s\right|=\left|v_{j} \cdot n\left(x_{j}\right)\right|\left|t_{j}-t_{j+1}\right|
\end{aligned}
$$

Here we used the fact that if $x, y \in \partial \Omega$ and $\partial \Omega$ is $C^{2}$ and $\Omega$ is bounded then $|x-y|^{2} \gtrsim \Omega|(x-y) \cdot n(x)|$ ( see the proof in [15] ). Thus

$$
\begin{equation*}
\left|v_{j} \cdot n\left(x_{j}\right)\right| \lesssim \frac{1}{\left|t_{j}-t_{j+1}\right|}\left|\int_{t_{j}}^{t_{j+1}} v_{j} d s\right|^{2} \lesssim\left|t_{j}-t_{j+1}\right|\left|v_{j}\right|^{2} \tag{3.126}
\end{equation*}
$$

Since $v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}, t_{j} \leq 0$, let $0 \leq t \leq t^{\prime}$, we have

$$
\begin{equation*}
\left|v_{j} \cdot n\left(x_{j}\right)\right| \lesssim\left|t_{j}-t_{j+1}\right|\left(\frac{1-\eta}{2(1+\eta)} \delta\right)^{-2} \tag{3.127}
\end{equation*}
$$

Then we prove (3.125).
In consequence, when $t_{k}>0$ and $t<t_{*}$, by (3.125), there can be at most $t_{*}\left\{\left[C_{\Omega}\left(\frac{2(1+\eta)}{(1-\eta) \delta}\right)^{3}\right]+1\right\}$ numbers of $v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$. Equivalently there are at least $k-2-t_{*}\left(\left[C_{\Omega}\left(\frac{2(1+\eta)}{(1-\eta) \delta}\right)^{3}\right]+1\right)$ numbers of $v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$.

Step 3
In this step we combine Step 1 and Step 2 and focus on the integration over $\prod_{j=1}^{k-1} \mathcal{V}_{j}$.
By (3.125) in Step 2, we define

$$
\begin{equation*}
N:=t_{*}\left[C_{\Omega}\left(\frac{2(1+\eta)}{\delta(1-\eta)}\right)^{3}\right]+t_{*} \tag{3.128}
\end{equation*}
$$

For the sequence $\left\{v_{1}, v_{2}, \cdots, v_{k-1}\right\}$, suppose there are $p$ number of $v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$ with $p \leq N$, we conclude there are at most $\binom{k-1}{p}$ number of these sequences. Below we only consider a single sequence of them.

In order to get (3.121),$(3.123)<1$, we need to ensure the condition (3.114). Thus we take $k=k_{1}\left(T_{M}, r_{\perp}, r_{\|}\right)$ and only use the decomposition $\mathcal{V}_{j}=\left(\mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}\right) \cup \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$ for $\prod_{j=1}^{k / 2} \mathcal{V}_{j}$. Then we only consider the half sequence $\left\{v_{1}, v_{2}, \cdots, v_{k / 2}\right\}$. We derive that when $t_{k}>0$, there are at most $N$ number of $v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$ and at least $k / 2-1-N$ number of $v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$ in $\prod_{j=1}^{k / 2} \mathcal{V}_{j}$.

In this single half sequence $\left\{v_{1}, \cdots, v_{k / 2}\right\}$, in order to apply Lemma 20, we only want to consider the subsequence (3.93) with $l+1<l+L \leq k / 2$ and $L \geq 100 \frac{1+\eta}{1-\eta}$. Thus we need to ignore those subsequence with
$L<100 \frac{1+\eta}{1-\eta}$. By (3.93), we conclude that at the end of this subsequence, it is adjacent to a $v_{l} \in \mathcal{V}_{l}^{\frac{1-\eta}{2(1+\eta)} \delta}$. By (3.128), we conclude

$$
\begin{equation*}
\text { There are at most } N \text { number of subsequences (3.93) with } L \leq 100 \frac{1+\eta}{1-\eta} \text {. } \tag{3.129}
\end{equation*}
$$

We ignore these subsequences. Then we define the parameters for the remaining subsequence( with $L \geq 100 \frac{1+\eta}{1-\eta}$ ) as:

$$
\begin{gathered}
M_{1}:=\text { the number of } v_{j} \in \mathcal{V}_{j} \backslash \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta} \text { in the first subsequence starting from } v_{1}, \\
n:=\text { the number of these subsequences. }
\end{gathered}
$$

Similarly we can define $M_{2}, M_{3}, \cdots, M_{n}$ as the number in the second, third, $\cdots, n$-th subsequence. Recall that we only consider $\prod_{j=1}^{k / 2} \mathcal{V}_{j}$, thus we have

$$
\begin{equation*}
100 \frac{1+\eta}{1-\eta} \leq M_{i} \leq k / 2, \text { for } 1 \leq i \leq n \tag{3.130}
\end{equation*}
$$

By (3.129), we obtain

$$
\begin{equation*}
k / 2 \geq M_{1}+\cdots M_{n} \geq k / 2-1-100 \frac{1+\eta}{1-\eta} N>\frac{k}{2}-101 \frac{1+\eta}{1-\eta} N \tag{3.131}
\end{equation*}
$$

Take $M_{i}$ with $1 \leq i \leq n$ as an example. Suppose this subsequence starts from $v_{l_{i}+1}$ to $v_{l_{i}+M_{i}}$, by (3.94) in Lemma 20 with replacing $l$ by $l_{i}$ and $L$ by $M_{i}$, we obtain

$$
\begin{equation*}
\int_{\prod_{j=l_{i}}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathbf{1}_{\left\{v_{l_{i}+j} \in \mathcal{V}_{l_{i}+j} \backslash \mathcal{V}_{l_{i}+j}^{\frac{1-\eta}{2(1+\eta)} \delta}{ }_{\text {for } \left.1 \leq j \leq M_{i}\right\}}\right.} \mathrm{d} \Phi_{l_{i}, m}^{k, k-1} \leq(3 \delta)^{M_{i} / 2} t_{*}^{-\left(k-l_{i}\right) c} C_{T_{M}, T_{m}}^{k-l_{i}} \mathcal{A}_{k-l, 1_{i}} \tag{3.132}
\end{equation*}
$$

Since (3.132) holds for all $1 \leq i \leq n$, by Lemma 19 we can draw the conclusion for the Step 3 as follows. For a single sequence $\left\{v_{1}, v_{2}, \cdots, v_{k-1}\right\}$, when there are $p$ number $v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}$, we have

$$
\begin{align*}
& \left.\int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}}{ }^{\mathbf{1}}{ }_{\left\{p \text { number } v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}\right.} \text { for a single sequence }\right\} \\
& \leq(3 \delta)^{\left(M_{1}+\cdots+M_{n}\right) / 2} t_{*}^{-k c} C_{\left.T_{M}>0\right\}}^{k} \mathrm{~d} \sum_{k-1}^{k} \mathcal{A}_{k-1,1} \tag{3.133}
\end{align*}
$$

## Step 4

Now we are ready to prove the lemma. By (3.128), we have

$$
\begin{align*}
& \int_{\prod_{j=1}^{k-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Sigma_{k-1}^{k} \\
& \leq \sum_{p=1}^{N} \int_{\left\{\text {Exactly } p \text { number of } v_{j} \in \mathcal{V}_{j}^{\frac{1-\eta}{2(1+\eta)} \delta}\right\}} \mathbf{1}_{\left\{t_{k}>0\right\}} \mathrm{d} \Sigma_{k-1}^{k} \tag{3.134}
\end{align*}
$$

Since (3.133) holds for a single sequence, we derive

$$
\begin{align*}
(\sqrt[3.134)]{ } & \leq t_{*}^{-k c} C_{T_{M}, T_{m}}^{k} \sum_{p=1}^{N}\binom{k-1}{p}(3 \delta)^{\left(M_{1}+M_{2}+\cdots M_{n}\right) / 2} \mathcal{A}_{k-1,1} \\
& \leq t_{*}^{-k c} C_{T_{M}, T_{m}}^{k} N(k-1)^{N}(3 \delta)^{k / 4-101 \frac{1+\eta}{1-\eta} N} \mathcal{A}_{k-1,1} \tag{3.135}
\end{align*}
$$

where we used (3.131) in the second line.
Now we let

$$
\delta=t_{*}^{1 / 3} \delta^{\prime} \text { with } \delta^{\prime} \ll 1
$$

such that

$$
N=\frac{t_{*}}{t_{*}}\left[C_{\Omega}\left(\frac{2(1+\eta)}{\delta^{\prime}(1-\eta)}\right)^{3}+1\right] .
$$

Using (3.128) we derive

$$
3 \delta^{\prime}=C(\Omega, \eta) N^{-1 / 3}
$$

Take $k=N^{3}$, the coefficient in (3.135) is bounded by

$$
\begin{equation*}
t_{*}^{-N^{3} c} C_{T_{M}, T_{m}}^{N^{3}} N^{3 N+1}(3 \delta)^{N^{3} / 4-101 \frac{1+\eta}{1-\eta} N} \leq t_{*}^{-N^{3} c} C_{T_{M}, T_{m}}^{N^{3}} t_{*}^{N^{3} / 15} N^{4 N}\left(3 \delta^{\prime}\right)^{N^{3} / 5}, \tag{3.136}
\end{equation*}
$$

where we choosed $N=N(\eta)$ large such that $N^{3} / 4-101 \frac{1+\eta}{1-\eta} N \geq N^{3} / 5$.
Finally we choose

$$
\begin{equation*}
c:=\frac{1}{15} \tag{3.137}
\end{equation*}
$$

We bound (3.136) by

$$
\begin{aligned}
& t_{*}^{-N^{3}\left(c-\frac{1}{15)}\right)} C_{T_{M}, T_{m}}^{N^{3}} N^{4 N}\left(C(\Omega, \eta) N^{-1 / 3}\right)^{N^{3} / 5} \leq e^{N^{3} \log \left(C_{\left.T_{M}, T_{m}\right)}\right)} e^{4 N \log N} e^{\left(N^{3} / 5\right) \log \left(C(\Omega, \eta) N^{-1 / 3}\right)} \\
& =e^{4 N \log N} e^{\left(N^{3} / 5\right)\left(\log (C(\Omega, \eta))-\frac{1}{3} \log N\right)} e^{N^{3} \log \left(C_{T_{M}, T_{m}}\right)}=e^{4 N \log N-\frac{N^{3}}{15}\left(\log N-3 \log C_{\Omega, \eta}-15 \log C_{T_{M}, T_{m}}\right)} \\
& \leq e^{4 N \log N-\frac{N^{3}}{30} \log N} \leq e^{-\frac{N^{3}}{15} \log N}=e^{-\frac{k}{150} \log k} \leq\left(\frac{1}{2}\right)^{k}
\end{aligned}
$$

where we choosed $\delta^{\prime}$ to be small enough in the second line such that $N=N\left(\Omega, \eta, C_{T_{M}, T_{m}}\right)$ is large enough to satisfy

$$
\begin{gathered}
\log N-3 \log C(\Omega, \eta)-15 \log C_{T_{M}, T_{m}} \geq \frac{\log N}{2} \\
4 N \log N-\frac{N^{3}}{30} \log N \leq-\frac{N^{3}}{50} \log N
\end{gathered}
$$

And thus we choose $k=N^{3}=k_{2}=k_{2}\left(\Omega, \eta, C_{T_{M}, T_{m}}\right)$ and we also require $\log k>150$ in the last step. Then we get (3.67).

Therefore, by the condition (3.114), we choose $k=k_{0}=\max \left\{k_{1}, k_{2}\right\}$. By the definition of $\eta$ (3.124) with (3.121) and (3.123), we obtain $\eta=\eta\left(T_{M}, \mathcal{C}, r_{\perp}, r_{\|}, \varepsilon_{2}\right)$. Thus by (3.116) and (3.119), we conclude the $k_{0}$ we choose here does not depend on $t$ and only depends on the parameter in (3.66). We conclude the lemma.

Proof of Proposition 5. First we take

$$
\begin{equation*}
t_{\infty} \leq t_{*} \tag{3.138}
\end{equation*}
$$

with $t_{*}$ defined in (3.68). Then we let $k=k_{0}$ with $k_{0}$ defined in (3.66) so that we can apply Lemma 16 and Lemma 15. Define the constant in (3.11) as

$$
\begin{equation*}
C_{\infty}=8 t_{*}^{-k_{0} / 15}\left(C_{T_{M}, T_{m}}\right)^{k_{0}} \tag{3.139}
\end{equation*}
$$

We mainly use the formula given in Lemma 14. By (3.7) we have

$$
\begin{equation*}
\left|\mathcal{G}^{m}(s)\right| \leq\left\|w_{\theta} f^{m}\right\|_{\infty}^{2}+\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m}\right\|_{\infty}\left[\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}+1\right]+\left\|w_{\theta} f^{m}\right\|_{\infty} e^{-\lambda\langle v\rangle s} \alpha(x, v) \int_{\mathbb{R}^{3}} \mathbf{k}_{\varrho}(v, u)\left|\partial f^{m}\left(X^{1}(s), u\right)\right| \mathrm{d} u \tag{3.140}
\end{equation*}
$$

where we used (2.27).
We consider two cases.
Case1: $t_{1} \leq 0$,
By (3.16) and (3.140), for some polynomial $P$ we have

$$
\begin{align*}
& \left|e^{-\lambda\langle v\rangle t} \alpha(x, v) \partial f^{m+1}(t, x, v)\right| \\
& \leq\left|\alpha \partial f_{0}\left(X^{1}(0), v\right)\right|+t\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}\right\|_{\infty}+t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)  \tag{3.141}\\
& +P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \alpha(x, v) \int_{0}^{t} \int_{\mathbb{R}^{3}} e^{-|v|(t-s)} \mathbf{k}_{\varrho}(v, u) e^{-\lambda[\langle v\rangle-\langle u\rangle] s} \frac{\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m}(s)\right\|_{\infty}}{\alpha(x-(t-s) v, u)} \mathrm{d} u \mathrm{~d} s \tag{3.142}
\end{align*}
$$

Since $s \leq t \ll 1, e^{-\lambda[\langle v\rangle-\langle u\rangle] s} \lesssim 1+e^{\varrho|v-u|^{2} / 2}$. And thus

$$
\mathbf{k}_{\varrho} e^{-\lambda[\langle v\rangle-\langle u\rangle] s} \lesssim \mathbf{k}_{\varrho}+\mathbf{k}_{\varrho / 2}
$$

Then applying Lemma 9 we have

$$
(3.142) \leq t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \sup _{s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m}(s)\right\|_{\infty}
$$

Collecting (3.141) and (3.142) we obtain

$$
\begin{align*}
\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}(t) \mathbf{1}_{\left\{t_{1} \leq 0\right\}}\right\|_{\infty} & \leq t \sup _{0 \leq s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s)\right\|_{\infty}  \tag{3.143}\\
& +t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)+t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \sup _{0 \leq s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m}(s)\right\|_{\infty}
\end{align*}
$$

Since (3.143) holds for all $t<t_{\infty}$, we derive

$$
\sup _{s}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s) \mathbf{1}_{\left\{t_{1} \leq 0\right\}}\right\|_{\infty} \leq \text { R.H.S of (3.143). }
$$

And thus with $t \ll 1$,

$$
\begin{equation*}
\sup _{s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s) 1_{\left\{t_{1} \leq 0\right\}}\right\|_{\infty} \leq 2 t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)+t\left[1+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right] \sup _{s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m}(s)\right\|_{\infty} \tag{3.144}
\end{equation*}
$$

Case2: $t_{1} \geq 0$,
We consider (3.17) in Lemma 14. For the first line, by (3.140) and the same computation as (3.142) we obtain

$$
\begin{equation*}
\int_{t_{1}}^{t} e^{-(t-s)|v|} \mathcal{G}^{m}(s) \mathrm{d} s \leq t \sup _{0 \leq s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s)\right\|_{\infty}+t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)+t \sup _{s}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m}(s)\right\|_{\infty} \tag{3.145}
\end{equation*}
$$

For the second line of (3.17), we bound it by

$$
\begin{equation*}
\exp \left(\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]|v|^{2}\right) \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} H \tag{3.146}
\end{equation*}
$$

Now we focus on $\int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} H$. We compute $H$ term by term using (3.18).
First we compute the first line of (3.18). By Lemma 15 with $p=1$, for every $1 \leq l \leq k_{0}-1$, we have

$$
\begin{align*}
& \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l+1} \leq 0<t_{l}\right\}}\left|\alpha \partial f_{0}\left(X^{m-l}(0), V^{m-l}(0)\right)\right| \mathrm{d} \Sigma_{l}^{k_{0}} \leq\left\|\alpha \partial f_{0}\right\|_{\infty} \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l+1} \leq 0<t_{l}\right\}} \mathrm{d} \Sigma_{l}^{k_{0}} \\
& \leq t_{*}^{-l / 15} C_{T_{M}, T_{m}}^{l}\left\|\alpha \partial f_{0}\right\|_{\infty} \exp \left(\frac{\left(T_{l, 1}-T_{w}\left(x_{1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{1}\right)\left[T_{l, 1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{1}\right)\right]}|v|^{2}+\mathcal{C}_{l} t_{*}^{1 / 15}|v|^{2}\right) \tag{3.147}
\end{align*}
$$

In regard to (3.146) we have

$$
\begin{aligned}
& \exp \left(\left[\frac{1}{4 T_{M}-2 T_{w}\left(x_{1}\right)}\right]|v|^{2}\right) \times \boxed{(3.147)} \\
& =t_{*}^{-l / 15} C_{T_{M}, T_{m}}^{l}\left\|\alpha \partial f_{0}\right\|_{\infty} \exp \left(\left[\frac{-1}{2\left(T_{w}\left(x_{1}\right) r_{\min }+T_{l, 1}\left(1-r_{\min }\right)\right)}+\frac{1}{4 T_{M}}\right]|v|^{2}+\mathcal{C}_{l} t_{*}^{1 / 15}|v|^{2}\right)
\end{aligned}
$$

Using the definition (3.33) we have $T_{w}\left(x_{1}\right)<2 T_{M}$ and $T_{l, 1}<2 T_{M}$. Then we require

$$
\begin{equation*}
t_{*}=t_{*}\left(T_{M}, k_{0}, \mathcal{C}\right) \tag{3.148}
\end{equation*}
$$

to be small enough such that the coefficient for $|v|^{2}$ is

$$
\begin{align*}
& \frac{-1}{2\left(T_{w}\left(x_{1}\right) r_{\min }+T_{l, 1}\left(1-r_{\min }\right)\right)}+\frac{1}{4 T_{M}}+\mathcal{C}_{l} t_{*}^{1 / 15} \\
\leq & \frac{-1}{2\left(T_{M} r_{\min }+T_{l, 1}\left(1-r_{\min }\right)\right)}+\frac{1}{4 T_{M}}+\mathcal{C}_{k_{0}} t_{*}^{1 / 15} \leq 0 . \tag{3.149}
\end{align*}
$$

Note that the condition (3.148) is consistent with (3.68).
Since (3.147) holds for all $1 \leq l \leq k_{0}-1$, by (3.149) the contribution of the first line of (3.18) in (3.146) is bounded by

$$
\begin{equation*}
t_{*}^{-k_{0} / 15} C_{\substack{T_{M}, T_{m} \\ 30}}^{k_{0}}\left\|\alpha \partial f_{0}\right\|_{\infty} \tag{3.150}
\end{equation*}
$$

Then we compute the second line of (3.18):

$$
\begin{align*}
& \int_{\max \left\{0, t_{l+1}\right\}}^{t_{l}} \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} e^{-\left(t_{l}-s\right)\left|v_{l}\right|}\left|\mathcal{G}^{m-l}(s)\right| \mathrm{d} \Sigma_{l}^{k_{0}} \mathrm{~d} s \\
& \leq t P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty} \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} \mathrm{~d} \Sigma_{l}^{k_{0}} \\
& \leq t t_{*}^{-k_{0} / 15} C_{T_{M}, T_{m}}^{k_{0}} P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty} \\
& \times \exp \left(\frac{\left(T_{l, 1}-T_{w}\left(x_{1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{1}\right)\left[T_{l, 1}\left(1-r_{m i n}\right)+r_{\min } T_{w}\left(x_{1}\right)\right]}|v|^{2}+\mathcal{C}_{l} t_{*}^{1 / 15}|v|^{2}\right) \\
& \leq \frac{\sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty}}{5 k_{0}} \exp \left(\frac{\left(T_{l, 1}-T_{w}\left(x_{1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{1}\right)\left[T_{l, 1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{1}\right)\right]}|v|^{2}+\mathcal{C}_{l} t_{*}^{1 / 15}|v|^{2}\right) \tag{3.151}
\end{align*}
$$

In the second line we applied the same computation as (3.142) to the $s$-integration. In the third line we applied (3.40) In the last line we applied Lemma 15 and take $t_{\infty}=t_{\infty}\left(t_{*}, k_{0}, C_{T_{M}, T_{m}}, P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right)$ to be small enough such that for $t<t_{\infty}$,

$$
\begin{equation*}
t t_{*}^{-k_{0} / 15} C_{T_{M}, T_{m}}^{k_{0}} P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \leq \frac{1}{5 k_{0}} \tag{3.152}
\end{equation*}
$$

In regard to (3.146), by (3.149) we obtain

$$
\exp \left(\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]|v|^{2}\right) \times(\sqrt{3.151}) \leq \frac{1}{5 k_{0}} \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty}
$$

Since (3.151) holds for all $1 \leq l \leq k_{0}-1$, the contribution of the second line of (3.18) in (3.146) is bounded by

$$
\begin{equation*}
\frac{k_{0}-1}{5 k_{0}} \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty} \leq \frac{1}{5} \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty} \tag{3.153}
\end{equation*}
$$

Then we compute the third line of (3.18). Directly applying Lemma 15 we obtain

$$
\begin{align*}
& \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{t_{l+1}<0\right\}} P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \mathrm{d} \Sigma_{l}^{k_{0}} \\
& \leq t_{*}^{-l / 15} C_{T_{M}, T_{m}}^{l} P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \exp \left(\frac{\left(T_{l, 1}-T_{w}\left(x_{1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{1}\right)\left[T_{l, 1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{1}\right)\right]}|v|^{2}+\mathcal{C}_{l} t_{*}^{1 / 15}|v|^{2}\right) \tag{3.154}
\end{align*}
$$

In regard to (3.146), by (3.149) we obtain

$$
\begin{equation*}
\exp \left(\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]|v|^{2}\right) \times(\sqrt{3.154}) \leq t_{*}^{-l / 15} C_{T_{M}, T_{m}}^{l} P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) \tag{3.155}
\end{equation*}
$$

Last we compute the fourth term of (3.18). By Lemma 16 and the assumption (3.11) we obtain

$$
\begin{align*}
& \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{0<t_{k_{0}}\right\}}\left\|e^{-\lambda\langle v\rangle t_{k_{0}}} \alpha \partial f^{m-k_{0}}\left(t_{k_{0}}\right)\right\|_{\infty} \mathrm{d} \Sigma_{k_{0}-1}^{k_{0}} \\
& \leq\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f\right\|_{\infty} \int_{\prod_{j=1}^{k_{0}-1} \mathcal{V}_{j}} \mathbf{1}_{\left\{0<t_{k_{0}}\right\}} \mathrm{d} \Sigma_{k_{0}-1}^{k_{0}} \\
& \leq\left(\frac{1}{2}\right)^{k_{0}} \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty} \exp \left(\frac{\left(T_{l, 1}-T_{w}\left(x_{1}\right)\right)\left(1-r_{\min }\right)}{2 T_{w}\left(x_{1}\right)\left[T_{l, 1}\left(1-r_{\min }\right)+r_{\min } T_{w}\left(x_{1}\right)\right]}|v|^{2}+\mathcal{C}_{l} t_{*}^{1 / 15}|v|^{2}\right) \tag{3.156}
\end{align*}
$$

In regard to (3.146), by (3.149) we have

$$
\exp \left(\left[\frac{1}{4 T_{M}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]|v|^{2}\right) \times(\sqrt{3.156}) \leq\left(\frac{1}{2}\right)^{k_{0}}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f\right\|_{\infty}
$$

Thus the contribution of the third line of (3.18) in (3.146) is bounded by

$$
\begin{equation*}
\left(\frac{1}{2}\right)^{k_{0}} \sup _{i \leq m}\left\|e_{31}^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty} \tag{3.157}
\end{equation*}
$$

Collecting (3.150) (3.153) (3.155) and (3.157) we conclude that the second line of (3.17) is bounded by

$$
\begin{equation*}
\text { (3.146) } \leq\left[\left(\frac{1}{2}\right)^{k_{0}}+\frac{1}{5}\right] \sup _{i \leq m}\left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{i}\right\|_{\infty}+t_{*}^{-k_{0} / 15} C_{T_{M}, T_{m}}^{k_{0}}\left[\left\|\alpha \partial f_{0}\right\|_{\infty}+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right] \tag{3.158}
\end{equation*}
$$

Adding (3.158) to (3.145) we use (3.17) and $t \ll 1$ to derive

$$
\begin{align*}
& \left\|e^{-\lambda\langle v\rangle t} \alpha \partial f^{m+1}(t, x, v) 1_{\left\{t_{1} \geq 0\right\}}\right\|_{\infty} \\
& \leq t \sup _{0 \leq s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s)\right\|_{\infty} \\
& +\left[\frac{1}{4} C_{\infty}+2 t_{*}^{-k_{0} / 15} C_{T_{M}, T_{m}}^{k_{0}}\right]\left[\left\|\alpha \partial f_{0}\right\|_{\infty}+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right]  \tag{3.159}\\
& \leq t \sup _{0 \leq s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s)\right\|_{\infty}+4 t_{*}^{-k_{0} / 15} C_{T_{M}, T_{m}}^{k_{0}}\left[\left\|\alpha \partial f_{0}\right\|_{\infty}+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right]
\end{align*}
$$

where we have used the definition of $C_{\infty}$ (3.139) in the last line.
Since (3.159) holds for all $t<t_{\infty}$, we derive that

$$
\sup _{s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s, x, v) \mathbf{1}_{\left\{t_{1} \geq 0\right\}}\right\|_{\infty} \leq \text { Last line of (3.159). }
$$

Therefore, with $t \ll 1$ we conclude

$$
\begin{equation*}
\sup _{s \leq t}\left\|e^{-\lambda\langle v\rangle s} \alpha \partial f^{m+1}(s, x, v) 1_{\left\{t_{1} \geq 0\right\}}\right\|_{\infty} \leq 8 t_{*}^{-k_{0} / 15} C_{T_{M}, T_{m}}^{k_{0}}\left[\left\|\alpha \partial f_{0}\right\|_{\infty}+P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right] . \tag{3.160}
\end{equation*}
$$

Combining (3.144) and (3.160) we derive (3.12).
Last we focus the parameters for $t_{\infty}$ in (3.13). In the proof the constraint for $t_{\infty}$ comes from (3.152). Thus from the definition of $k_{0}$ in (3.66), definition of $C_{T_{M}, T_{m}}$ in (3.49) and definition of $t_{*}$ in (3.68)

$$
t_{\infty}=t_{\infty}\left(t_{*}, k_{0}, C_{T_{M}, T_{m}}, P\left(\sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right)\right)=t_{\infty}\left(T_{M}, T_{m}, \Omega, r_{\perp}, r_{\|}, \sup _{m}\left\|w_{\theta} f^{m}\right\|_{\infty}\right) .
$$

Thus we derive (3.13).

Proof of Theorem 1. The uniform-in-m bound (3.15) follows from Proposition 5. Then we follow the same argument to $e^{-\lambda\langle v\rangle t} \alpha\left[\partial f^{m+1}-\partial f^{m}\right]$ and conclude that $e^{-\lambda\langle v\rangle t} \alpha \partial f^{m}$ is a Cauchy sequence in $L^{\infty}$. Then we pass the limit and conclude Theorem 1 .

## 4. Weighted $C^{1}$-estimate of the stationary Boltzmann equation

In this section we prove the weighted $C^{1}$-estimate of the stationary Boltzmann equation (1.28). In particular, we will prove Theorem 2 ,

First we give the boundary condition for $f_{s}$ in the following lemma.
Lemma 21. (Lemma 9 in [9)
The boundary condition for $f_{s}$ defined in (1.29) is given by

$$
\left.f_{s}(x, v)\right|_{\gamma_{-}}=r_{s}+e^{\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}(x)}\right]|v|^{2}} \int_{n(x) \cdot u>0} f_{s}(x, u) e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}(x)}\right]|u|^{2}} \mathrm{~d} \sigma(u, v) .
$$

Here the remainder term $r_{s}$ is given by

$$
\begin{equation*}
r_{s}=\frac{\mu_{x, r_{\|}, r_{\perp}}-\mu_{0}}{\sqrt{\mu_{0}}} \tag{4.1}
\end{equation*}
$$

with

$$
\begin{align*}
& \mu_{x, r_{\|}, r_{\perp}}=\frac{1}{2 \pi\left[T_{0}\left(1-r_{\|}\right)^{2}+T_{w}(x) r_{\|}\left(2-r_{\|}\right)\right]} e^{-\frac{\left|v_{\|}\right|^{2}}{2\left[T_{0}\left(1-r_{\|}\right)^{2}+T_{w}(x) r_{\|}\left(2-r_{\|}\right)\right]}}  \tag{4.2}\\
& \times \frac{1}{T_{0}\left(1-r_{\perp}\right)+T_{w}(x) r_{\perp}} e^{-\frac{\left|v_{\perp}\right|^{2}}{2\left[T_{0}\left(1-r_{\perp}\right)+T_{w}(x) r_{\perp}\right]}}
\end{align*}
$$

As mentioned in the introduction, when we perform the integration by parts, polynomial terms appear in the integration. In the next lemma we will bound all the possible integration related to the C-L boundary.

Lemma 22. Denote

$$
\begin{equation*}
\mathcal{A}:=\frac{2}{r_{\perp} r_{\|}\left(2-r_{\|}\right) \pi} \frac{1}{\left(2 T_{w}\left(x_{1}\right)\right)^{2}} e^{\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]|v|^{2}} \tag{4.3}
\end{equation*}
$$

For (4.4) given by

$$
\begin{align*}
& e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left|v_{1}\right|^{2}}\left(n\left(x_{1}\right) \cdot v_{1}\right) I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}\right) \\
& \times \exp \left(-\frac{1}{2 T_{w}\left(x_{1}\right)}\left[\frac{\left|v_{1, \perp}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{\perp}\right|^{2}}{r_{\perp}}+\frac{\left|v_{1, \|}-\left(1-r_{\|}\right) v_{\|}\right|^{2}}{r_{\|}\left(2-r_{\|}\right)}\right]\right) \tag{4.4}
\end{align*}
$$

under the condition (1.31), we have

$$
\begin{gather*}
\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left[1+\left|v_{1}\right|^{2}+|v|^{2}\right](4.4) \lesssim 1  \tag{4.5}\\
\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \frac{1}{\alpha\left(x_{1}, v_{1}\right)}(\sqrt[4.4]{ }) \lesssim 1  \tag{4.6}\\
\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left[1+\left|v_{1}\right|\right] \nabla_{v_{1}}[(4.4)] \lesssim 1 \tag{4.7}
\end{gather*}
$$

For $x_{1}=\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)$ and $i=1,2$

$$
\begin{equation*}
\partial_{\mathbf{x}_{p^{1}, i}^{1}} \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}(4.4) \lesssim 1 \tag{4.8}
\end{equation*}
$$

Remark 12. The condition (1.31) is not necessary in this lemma. Since we will only use this lemma for the stationary problem we impose such condition to simplify the proof.

Proof. From condition (1.31), we have $\left|T_{w}\left(x_{1}\right)-T_{0}\right|,\left|1-r_{\perp}\right|,\left|1-r_{\|}\right| \ll 1$. Then for some $\varepsilon \ll 1$,

$$
\begin{gathered}
\left|\frac{1}{r_{\perp}}-1\right|,\left|1-\frac{1}{r_{\|}\left(2-r_{\|}\right)}\right|=\left|1-\frac{1}{1-\left(1-r_{\|}\right)^{2}}\right| \lesssim O(\varepsilon) \\
\left|\frac{\left(1-r_{\perp}\right)}{r_{\perp}}\right|,\left|\frac{\left(1-r_{\perp}\right)^{1 / 2}}{r_{\perp}}\right|,\left|\frac{2\left(1-r_{\|}\right)}{r_{\|}\left(2-r_{\|}\right)}\right|,\left|\frac{\left(1-r_{\|}\right)^{2}}{r_{\|}\left(2-r_{\|}\right)}\right| \lesssim O(\varepsilon) \\
\left|\frac{1}{T_{w}\left(x_{1}\right)}-\frac{1}{T_{0}}\right| \lesssim O(\varepsilon)
\end{gathered}
$$

Hence

$$
\begin{align*}
\text { (4.4) } & \lesssim\left|n\left(x_{1}\right) \cdot v_{1}\right| e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{0}}\right]\left|v_{1}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{1}\right|^{2}} e^{-\frac{1}{2 T_{0}}\left|v_{1, \perp}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{1, \perp}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{\perp}\right|^{2}} \times e^{\frac{O(\varepsilon)}{T_{0}} v_{1, \perp} v_{\perp}} \\
& \times e^{-\frac{1}{2 T_{0}}\left|v_{1, \|}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{1, \|} v_{\|}\right|} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{\|}\right|^{2}}  \tag{4.9}\\
& \lesssim\left|v_{1}\right| e^{-\frac{1}{4 T_{0}}\left|v_{1}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}|v|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{1}\right|^{2}}
\end{align*}
$$

in the last line we have used $|a b| \lesssim|a|^{2}+|b|^{2},\left|v_{\|}\right|^{2}+\left|v_{\perp}\right|^{2}=|v|^{2}$.
Thus using $\varepsilon \ll 1$ and (4.3) we have

$$
\begin{align*}
& \mathcal{A} \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left[1+|v|^{2}+\left|v_{1}\right|^{2}\right](\boxed{4.4})  \tag{4.10}\\
& \lesssim \frac{1}{T_{0}^{2}} e^{\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{0}}\right]|v|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}|v|^{2}}\left[1+|v|^{2}\right] \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left|v_{1}\right|\left[1+\left|v_{1}\right|^{2}\right] e^{-\frac{1}{4 T_{0}}\left|v_{1}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}|v|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{1}\right|^{2}} \\
& \lesssim \frac{1}{T_{0}^{2}} e^{-\frac{1-O(\varepsilon)}{4 T_{0}}|v|^{2}}\left[1+|v|^{2}\right] \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left|v_{1}\right|\left[1+\left|v_{1}\right|^{2}\right] e^{-\frac{1-O(\varepsilon)}{4 T_{0}}\left|v_{1}\right|^{2}} \lesssim 1 \tag{4.11}
\end{align*}
$$

where we used $T_{0} \gtrsim 1$. Then we conclude (4.5).
(4.6) follows from (4.9), where $\frac{1}{\alpha\left(x_{1}, v_{1}\right)}$ is cancelled by $\left|n\left(x_{1}\right) \cdot v_{1}\right|$, and the rest computation is the same.

Then we prove (4.7). From (4.4), taking the $v_{1}$ derivative we will have extra term

$$
\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left|v_{1}\right|, \quad \frac{\left|v_{1, \perp}\right|}{\substack{T_{w}\left(x_{1}\right) r_{\perp} \\ 33}}, \quad \frac{\left|v_{1, \|}-\left(1-r_{\|}\right) v_{\|}\right|}{T_{w}\left(x_{1}\right) r_{\|}\left(2-r_{\|}\right)}
$$

and from (1.7),

$$
\begin{aligned}
& \nabla_{v_{1}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}\right)=\pi^{-1} \int_{0}^{\pi} e^{\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}} \cos \phi \\
& v_{v_{1}}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}\right) \cos \phi \mathrm{d} \phi \\
& \lesssim \nabla_{v_{1}}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}\right) I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}\right)
\end{aligned}
$$

the extra term is

$$
\frac{\left(1-r_{\perp}\right)^{1 / 2}\left|v_{\perp}\right|}{T_{w}\left(r_{1}\right) r_{\perp}}
$$

Thus all the extra term can be bounded as

$$
\frac{|v|+\left|v_{1}\right|}{T_{0}} \lesssim|v|+\left|v_{1}\right| \lesssim\left[1+|v|^{2}+\left|v_{1}\right|^{2}\right] .
$$

This upper bound is already included in (4.5). Thus we conclude (4.7).
Last we prove (4.8). From (4.3) taking $\partial_{\mathbf{x}_{p^{1}, i}^{1}}^{1}$ derivative we have extra term

$$
\frac{\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{T_{w}\left(x_{1}\right)^{3}}, \quad \frac{\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{T_{w}\left(x_{1}\right)^{2}}|v|^{2} .
$$

From (4.4), taking $\partial_{\mathbf{x}_{p^{1}, i}^{1}}$ derivative we have extra term

$$
\frac{\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{T_{w}^{2}\left(x_{1}\right)}\left|v_{1}\right|^{2}, \quad \frac{\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{T_{w}^{2}\left(x_{1}\right)}
$$

The extra term are bounded by

$$
\|\eta\|_{C^{1}}\left[\frac{1}{T_{0}^{3}}+\frac{1}{T_{0}^{2}}\right]\left[1+|v|^{2}\right]
$$

which is included in (4.5). Thus we conclude (4.8).

Then we start to prove Theorem 2 The main idea is to express the characteristic of (1.29) by using the Duhamel's principle:

$$
\begin{align*}
f_{s}(x, v) & =\mathbf{1}_{t \geq t_{\mathbf{b}}} e^{-\nu(v) t_{\mathbf{b}}} f_{s}\left(x_{\mathbf{b}}, v\right) \\
& +\mathbf{1}_{t<t_{\mathbf{b}}} e^{-\nu(v) t} f_{s}(x-t v, v)  \tag{4.12}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} h(x-(t-s) v, v) \mathrm{d} s
\end{align*}
$$

where $h=K\left(f_{s}\right)+\Gamma\left(f_{s}, f_{s}\right)$.
Here in order to distinguish between Euclidean coordinate and the backward cycles, we denote

$$
\begin{equation*}
x=\left([x]_{1},[x]_{2},[x]_{3}\right) . \tag{4.13}
\end{equation*}
$$

Thus

$$
\nabla_{x}=\left(\partial_{[x]_{1}}, \partial_{[x]_{2}}, \partial_{[x]_{3}}\right)
$$

We take the spatial derivative to (4.12) to have

$$
\begin{align*}
\partial_{[x]_{j}} f_{s}(x, v) & =\mathbf{1}_{t \geq t_{\mathbf{b}}} e^{-\nu(v) t_{\mathbf{b}}} \partial_{[x]_{j}}\left[f_{s}\left(x_{\mathbf{b}}, v\right)\right]  \tag{4.14}\\
& -\mathbf{1}_{t \geq t_{\mathbf{b}}} \nu(v) \partial_{[x]_{j}} t_{\mathbf{b}}(x, v) e^{-\nu(v) t_{\mathbf{b}}} f_{s}\left(x_{\mathbf{b}}, v\right)  \tag{4.15}\\
& +\mathbf{1}_{t<t_{\mathbf{b}}} e^{-\nu(v) t} \partial_{[x]_{j}}\left[f_{s}(x-t v, v)\right]  \tag{4.16}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \partial_{[x]_{j}}[h(x-(t-s) v, v)] \mathrm{d} s  \tag{4.17}\\
& -\mathbf{1}_{t \geq t_{\mathbf{b}}} \partial_{[x]_{j} t_{\mathbf{b}}} e^{-\nu(v) t_{\mathbf{b}}} h\left(x-t_{\mathbf{b}} v, v\right) . \tag{4.18}
\end{align*}
$$

First we give an estimate for (4.14)-(4.18).

Lemma 23. For $h=K f_{s}+\Gamma\left(f_{s}, f_{s}\right)$, we can express $\partial_{[x]_{j}} f_{s}(x, v)$ as

$$
\begin{align*}
\partial_{[x]_{j}} f_{s}(x, v) & =\frac{O(1)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)}  \tag{4.19}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \partial_{[x]_{j}} f(x-(t-s) v, u) \mathrm{d} u \mathrm{~d} s  \tag{4.20}\\
& +\sum_{i=1,2} \frac{\partial \mathbf{x}_{p^{1}, i}^{1}}{\partial[x]_{j}} \partial_{\mathbf{x}_{p^{1}, i}^{1}}\left[f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v\right)\right] . \tag{4.21}
\end{align*}
$$

Proof. We estimate every term in (4.14)-(4.18).
From the chain rule and the definition of $\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)=x_{1}$ in (2.39), the contribution of (4.14) is (4.21).
For (4.15), since $\nu(v) \lesssim w_{\vartheta}(v)$, we apply (2.41) to get

$$
(4.15)=\frac{O(1)\left\|\nu(v) f_{s}\left(x_{\mathbf{b}}, v\right)\right\|}{\alpha(x, v)} \lesssim \frac{O(1)\left\|w_{\vartheta} f_{s}\right\|}{\alpha(x, v)} .
$$

Such contribution is included in (4.19).
For (4.16), using $t \gg 1$ we get

$$
\text { (4.16) }=o(1) \partial_{[x]_{j}} f_{s}(x-t v, v)=o(1) \frac{\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)}
$$

For (4.17), we first consider the contribution of $h=K\left(f_{s}\right)$, which reads

$$
\text { (4.17) } \begin{aligned}
h=K & =\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \partial_{[x]_{j}}\left[\int_{\mathbb{R}^{3}} \mathbf{k}(v, u) f_{s}(x-(t-s) v, u)\right] \mathrm{d} s \\
& =\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \partial_{[x]_{j}} f_{s}(x-(t-s) v, u) \mathrm{d} s
\end{aligned}
$$

Such contribution is included in (4.20).
Then we consider the contribution of $h=\Gamma\left(f_{s}, f_{s}\right)$. By (2.28) we have

$$
\begin{aligned}
\text { (4.17) }_{h=\Gamma} & \lesssim \frac{\left\|w_{\vartheta} f_{s}\right\|_{\infty}\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)}+\left\|w_{\vartheta} f_{s}\right\|_{\infty} \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}_{\rho}(v, u)\left|\partial_{[x]_{j}} f_{s}(x-(t-s) v, u)\right| \mathrm{d} s \\
& =\frac{\left\|w_{\vartheta} f_{s}\right\|_{\infty}\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)}+\left\|w_{\vartheta} f_{s}\right\|_{\infty} \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}_{\rho}(v, u) \frac{\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x-(t-s) v, u)} \mathrm{d} s \\
& \lesssim \frac{\left\|w_{\vartheta} f_{s}\right\|_{\infty}\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)}
\end{aligned}
$$

where we have applied Lemma 9 in the last line. Since $\left\|w_{\vartheta} f_{s}\right\| \ll 1$ from Corollary 4 the contribution of $h=\Gamma\left(f_{s}, f_{s}\right)$ of (4.17) in included in (4.19).

For the last term (4.18), we apply (2.41) and (2.23) (2.26) to get

$$
\text { (4.18) }=\frac{O(1)\|h\|_{\infty}}{\alpha(x, v)}=\frac{O(1)\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}}{\alpha(x, v)}
$$

Such contribution is included in (4.19).
Then we conclude the lemma.

Then we start the proof of Theorem 2
Proof of Theorem (2. By Lemma 23, we only need to estimate (4.20) and (4.21).
First we estimate (4.21). By (2.43) in Lemma 10 we have

$$
\begin{equation*}
(4.21)=\frac{O(1)}{\alpha(x, v)} \sum_{i=1,2} \underbrace{\partial_{\mathbf{x}_{p^{1}, i}}\left[f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v\right)\right]}_{(4.22)_{*}} \tag{4.22}
\end{equation*}
$$

Using the notation (2.38) and Lemma 21, the boundary condition at $f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v\right)$ can be written as

$$
\begin{aligned}
& f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v\right) \\
& =r_{s}+\mathcal{A} \int_{n\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right) \cdot v_{1}>0} f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v_{1}\right) e^{-\left.\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(\eta_{p^{1}\left(\mathbf{x}_{p^{1}}^{1}\right)}\right)}\right] v_{1}\right|^{2}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right) \\
& \times\left|n\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right) \cdot v_{1}\right| \exp \left(-\frac{1}{2 T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\left[\frac{\left|v_{1, \perp}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{\perp}\right|^{2}}{r_{\perp}}+\frac{\left|v_{1, \|}-\left(1-r_{\|}\right) v_{\|}\right|^{2}}{r_{\|}\left(2-r_{\|}\right)}\right]\right) \\
& =r_{s}+\mathcal{A} \times \int_{\mathbf{v}_{p^{1}, 3}^{1}>0} f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}\right) \times(4.23) \mathrm{d} \mathbf{v}_{p^{1}}^{1},
\end{aligned}
$$

with

$$
\begin{align*}
& \mathbf{v}_{p^{1}, 3}^{1} e^{-\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right]\left|\mathbf{v}_{p^{1}}^{1}\right|^{2}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 3}^{1} v_{\perp}}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right) \\
& \times \exp \left(-\frac{1}{2 T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\left[\frac{\left|\mathbf{v}_{p^{1}, 3}^{1}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{\perp}\right|^{2}}{r_{\perp}}+\frac{\left|\left(T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}-\mathbf{v}_{p^{1}, 3}^{1} n\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)\right)-\left(1-r_{\|}\right) v_{\|}\right|^{2}}{r_{\|}\left(2-r_{\|}\right)}\right]\right) \tag{4.23}
\end{align*}
$$

Taking $\partial_{\mathbf{x}_{p^{1}, i}^{1}}$ to $f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v\right)$ we get

$$
\begin{align*}
\partial_{\mathbf{x}_{p^{1}, i}^{1}} f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), v\right) & \lesssim \partial_{\mathbf{x}_{p^{1}, i}^{1}} r_{s}  \tag{4.24}\\
& +\partial_{\mathbf{x}_{p^{1}, i}^{1}}[\mathcal{A}] \int_{\mathbf{v}_{p^{1}, 3}^{1}>0}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \times(4.23)  \tag{4.25}\\
& +\mathcal{A} \int_{\mathbf{v}_{p^{1}, 3}^{1}>0} \partial_{\mathbf{x}_{p^{1}, i}^{1}}\left[f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}\right)\right] \times(4.23)  \tag{4.26}\\
& \left.+\mathcal{A} \int_{\mathbf{v}_{p^{1}, 3}^{1}>0} f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}\right) \partial_{\mathbf{x}_{p^{1}, i}^{1}}[44.23)\right] . \tag{4.27}
\end{align*}
$$

Since

$$
\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)=\nabla T_{w} \cdot \partial_{3} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \lesssim\left\|T_{w}\right\|_{C^{1}}\|\eta\|_{C^{1}}
$$

applying (2.35) we have

$$
\begin{equation*}
(4.24) \lesssim\left\|T_{w}\right\|_{C^{1}}\|\eta\|_{C^{1}} \tag{4.28}
\end{equation*}
$$

For (4.25) we change the $\mathbf{v}_{p^{1}}^{1}$ integration back to $v_{1}$ integration, thus (4.23) is replaced by (4.4), with integral domain changing back to $n\left(x_{1}\right) \cdot v_{1}>0$. Applying (4.8) we conclude

$$
\begin{equation*}
\text { (4.25) } \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} \partial_{\mathbf{x}_{p^{1}, i}^{1}}[\mathcal{A}] \int_{n\left(x_{1}\right) \cdot v_{1}>0} \text { (4.4) } \mathrm{d} v_{1} \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} . \tag{4.29}
\end{equation*}
$$

For (4.27), taking $\partial_{\mathbf{x}_{p^{1}, i}^{1}}$ to (4.23) we have extra term

$$
\begin{gathered}
\frac{\nabla_{x} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right) \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\left|\mathbf{v}_{p^{1}}^{1}\right|^{2}}{T_{w}^{2}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}, \\
\frac{\nabla_{x} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right) \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)}{T_{w}^{2}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\left[\frac{\left|\mathbf{v}_{p^{1}, 3}^{1}\right|^{2}+\left(1-r_{\perp}\right)\left|v_{\perp}\right|^{2}}{r_{\perp}}+\frac{\left|\left(T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}-\mathbf{v}_{p^{1}, 3} n\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)\right)-\left(1-r_{\|}\right) v_{\|}\right|^{2}}{r_{\|}\left(2-r_{\|}\right)}\right], \\
\frac{\left[\left(T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}^{1}-\mathbf{v}_{p^{1}, 3}^{1} n\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)\right)-\left(1-r_{\|}\right) v_{\|}\right]\left[\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{\mathbf{x}_{p^{1}}}^{t} \mathbf{v}_{p^{1}}^{1}-\mathbf{v}_{p^{1}, 3}^{1} \partial_{\mathbf{x}_{p^{1}, i}^{1}} n\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}\right)\right)\right]}{T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right) r_{\|}\left(2-r_{\|}\right)},
\end{gathered}
$$

and from (1.7)

$$
\begin{aligned}
& \partial_{\mathbf{x}_{p^{1}, i}^{1}} I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 3}^{1} v_{\perp}}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right) \\
& =\pi^{-1} \int_{0}^{\pi} e^{\frac{\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 1,}{ }^{v} \perp}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)} \cos \phi} \partial_{\mathbf{x}_{p^{1}, i}^{1}}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 3}^{1} v_{\perp}}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right) \cos \phi \mathrm{d} \phi \\
& \lesssim \partial_{\mathbf{x}_{p^{1}, i}^{1}}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 3}^{1} v_{\perp}}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right) I_{0}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 3}^{1} v_{\perp}}{r_{\perp} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}\right)
\end{aligned}
$$

the extra term is

$$
\frac{\nabla_{x} T_{w}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right) \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\left(1-r_{\perp}\right)^{1 / 2} \mathbf{v}_{p^{1}, 3}^{1} v_{\perp}}{r_{\perp} T_{w}^{2}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right)}
$$

All the extra terms are bounded by

$$
\left\|T_{w}\right\|_{C^{1}}\|\eta\|_{C^{1}}\left[\frac{1}{T_{0}}+\frac{1}{T_{0}^{2}}\right]\left[1+|v|^{2}+\left|\mathbf{v}_{p^{1}}^{1}\right|^{2}\right] \lesssim 1+|v|^{2}+\left|\mathbf{v}_{p^{1}}^{1}\right|^{2}
$$

Thus

$$
\begin{align*}
& \text { (4.27) } \lesssim \mathcal{A} \int_{\mathbf{v}_{p^{1}, 3}>0}\left\|w_{\vartheta} f_{s}\right\|_{\infty}\left[1+|v|^{2}+\left|\mathbf{v}_{p^{1}}^{1}\right|^{2}\right] \times \text { (4.23) } \\
&=\mathcal{A} \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left\|w_{\vartheta} f_{s}\right\|_{\infty}\left[1+|v|^{2}+\left|v_{1}\right|^{2}\right] \times(4.4)  \tag{4.30}\\
& \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} .
\end{align*}
$$

In the second line we changed $\mathbf{v}_{p^{1}}$ integration back to $v_{1}$ integration and used $\left|v_{1}\right|^{2}=\left|\mathbf{v}_{p^{1}}^{1}\right|^{2}$ from (2.37). In the last step we applied (4.5).

Then we focus on (4.26), which reads

$$
\begin{align*}
(4.26) & =\mathcal{A} \times \int_{\mathbf{v}_{p^{1}, 3}^{1}>0}(\underline{4.23)} \\
& \times[\underbrace{\left(\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}\right) \nabla_{v} f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}\right)}_{4_{1.31}}+\underbrace{\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \nabla_{x} f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right), T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}^{1}\right)}_{\underbrace{}_{2}}] . \tag{4.31}
\end{align*}
$$

First we estimate the contribution of (4.31) $1_{1}$. We change the $\mathbf{v}_{p^{1}}^{1}$-integration back to $v_{1}$ integration. The extra term $\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{\mathbf{x}_{p^{1}}^{1}}^{t} \mathbf{v}_{p^{1}}$ becomes

$$
\left(\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{\mathbf{x}_{p^{1}}^{1}}\right) T_{\mathbf{x}_{p^{1}}^{1}} v_{1} .
$$

Thus such contribution is bounded as

$$
\begin{align*}
& \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left(\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{\mathbf{x}_{p^{1}}^{1}}\right) T_{\mathbf{x}_{p^{1}}^{1}} v_{1} \nabla_{v} f_{s}\left(x_{1}, v_{1}\right) \times \boxed{(4.4)} \mathrm{d} v_{1} \\
& \lesssim\|\eta\|_{C^{2}}\|\eta\|_{C^{1}} \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left|\nabla_{v_{1}}\left[v_{1} \times(\boxed{4.4})\right]\right| f_{s}\left(x_{1}, v_{1}\right) \mathrm{d} v_{1} \\
& \lesssim\|\eta\|_{C^{2}}\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left|\left[\boxed{(4.4)}+\left|v_{1}\right| \nabla_{v_{1}}[(\boxed{4.4})]\right]\right| \mathrm{d} v_{1} \\
& \lesssim\|\eta\|_{C^{2}}\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} \tag{4.32}
\end{align*}
$$

In the second line we used (2.37) to get $\left(\partial_{\mathbf{x}_{p^{1}, i}^{1}} T_{\mathbf{x}_{p^{1}}^{1}}\right) T_{\mathbf{x}_{p^{1}}^{1}} \lesssim\|\eta\|_{C^{2}}\|\eta\|_{C^{1}}$. In the third line we used $\nabla_{v} f_{s}\left(x_{1}, v_{1}\right)=$ $\nabla_{v_{1}}\left[f_{s}\left(x_{1}, v_{1}\right)\right]$ and performed the integration by parts with respect to $\mathrm{d} v_{1}$, and used (4.4) $=0$ for $n\left(x_{1}\right) \cdot v_{1}=0$. In the last line we used (4.5) and (4.7).

Then we estimate the contribution of (4.31) 2 . We change the integration $\int_{\mathbf{v}_{p^{1}, 3}^{1}>0}$ back to $\int_{n\left(x_{1}\right) \cdot v_{1}>0}$, such contribution in (4.26) reads

$$
\begin{align*}
& \left.\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \nabla_{x_{1}} f_{s}\left(x_{1}, v_{1}\right) \times[4.4)\right] \mathrm{d} v_{1} \\
& \left.=\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\left|\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \frac{O(1)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha\left(x_{1}, v_{1}\right)}\right| \times[44.4]\right] \mathrm{d} v_{1}  \tag{4.33}\\
& \left.+\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \int_{\max \left\{0, t_{b}^{1}\right\}}^{t_{1}} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \nabla_{x_{1}} f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right) \mathrm{d} u \mathrm{~d} s \times[4.4 .4)\right] \mathrm{d} v_{1}  \tag{4.34}\\
& \left.+\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \nabla_{x_{1}} f_{s}\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right), v_{1}\right) \times[\boxed{4.4}]\right] \mathrm{d} v_{1} . \tag{4.35}
\end{align*}
$$

Here we applied Lemma 23 to $\nabla_{x} f_{s}\left(x_{1}, v_{1}\right)=\nabla_{x_{1}}\left[f_{s}\left(x_{1}, v_{1}\right)\right]$. Then we estimate (4.33)-(4.35).
First we estimate (4.33). We use (4.6) to get

$$
\begin{equation*}
\text { (4.33) } \lesssim\|\eta\|_{C^{1}} O(1)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty} \tag{4.36}
\end{equation*}
$$

Then we estimate (4.34). We split $\mathrm{d} s$ integration into

$$
\begin{equation*}
\int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}}=\underbrace{\int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}-\varepsilon}}_{4_{4.37}^{1}}+\underbrace{\int_{t_{1}-\varepsilon}^{t_{1}}}_{4.37)_{2}} \tag{4.37}
\end{equation*}
$$

For (4.37) $)_{2}$, we apply Lemma 9 to get such contribution in (4.34) is bounded by

$$
\begin{align*}
& \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{t_{1}-\varepsilon}^{t_{1}} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \int_{\mathbb{R}^{3}} \frac{\mathbf{k}\left(v_{1}, u\right)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)} \mathrm{d} u \mathrm{~d} s \times[\boxed{4.4}] \mathrm{d} v_{1} \\
& \lesssim O(\varepsilon) \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \frac{\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha\left(x_{1}, v_{1}\right)} \times[\boxed{4.4}] \mathrm{d} v_{1} \\
& \lesssim O(\varepsilon)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty} . \tag{4.38}
\end{align*}
$$

In the third line we used (4.6).
For (4.37) 1 we exchange the $x_{1}$ derivative to $v_{1}$ derivative:

$$
\begin{equation*}
\nabla_{x_{1}} f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)=-\frac{\nabla_{v_{1}}\left[f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)\right]}{t_{1}-s} \tag{4.39}
\end{equation*}
$$

In this case $t_{1}-s \geq \varepsilon$. The contribution of (4.37) $)_{2}$ in (4.34) is

$$
\begin{align*}
& \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \partial_{i} \eta \int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}-\varepsilon} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{\nabla_{v_{1}}\left[f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)\right]}{t_{1}-s} \mathrm{~d} u \mathrm{~d} s \times[(4.4)] \mathrm{d} v_{1} \\
& \lesssim \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}-\varepsilon} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)}{t_{1}-s}\left|\nabla_{v_{1}} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)}\right| \mathrm{d} u \mathrm{~d} s \times[(4.4)] \mathrm{d} v_{1}  \tag{4.40}\\
& +\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}-\varepsilon} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)}{t_{1}-s} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \mathrm{d} u \mathrm{~d} s \times\left|\nabla_{v_{1}}[(4.4)]\right| \mathrm{d} v_{1}  \tag{4.41}\\
& +\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{f_{s}\left(x_{1}-\left(t_{1}-t_{\mathbf{b}}^{1}\right) v_{1}, u\right)}{t_{1}-s} e^{-\nu\left(v_{1}\right)\left(t_{1}-t_{\mathbf{b}}^{1}\right)}\left|\nabla_{v_{1}} t_{\mathbf{b}}^{1}\right| \mathrm{d} u \times[(4.4)] \mathrm{d} v_{1}  \tag{4.42}\\
& +\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}-\varepsilon} \int_{\mathbb{R}^{3}}\left|\nabla_{v_{1}} \mathbf{k}\left(v_{1}, u\right)\right| \frac{f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)}{t_{1}-s} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \mathrm{d} u \mathrm{~d} s \times(4.4) \mathrm{d} v_{1} . \tag{4.43}
\end{align*}
$$

Here we applied the integration by parts with respect to $\mathrm{d} v_{1}$. And we used (4.4) $=0$ when $n\left(x_{1}\right) \cdot v_{1}=0$. We apply (2.22) and (2.19) to bound

$$
\begin{align*}
\text { (4.40) } & \lesssim \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\max \left\{0, t_{b}^{1}\right\}}^{t_{1}-\varepsilon} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{t_{1}-s}\left|\nabla_{v_{1}} \nu\left(v_{1}\right)\right|\left(t_{1}-s\right) e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \mathrm{d} u \mathrm{~d} s \times[(4.4)] \mathrm{d} v_{1} \\
& \lesssim\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \text { (4.4) } \mathrm{d} v_{1} \\
& \lesssim O(1)\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \tag{4.44}
\end{align*}
$$

where we have used (4.6) in the third line.
For (4.41) we apply (2.31) in Lemma 9 to bound

$$
\begin{align*}
\text { (4.41) } & \lesssim \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\max \left\{0, t_{\mathbf{b}}^{1}\right\}}^{t_{1}-\varepsilon} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{f_{s}\left(x_{1}-\left(t_{1}-s\right) v_{1}, u\right)}{\varepsilon} e^{-\nu\left(v_{1}\right)\left(t_{1}-s\right)} \mathrm{d} u \mathrm{~d} s \times\left|\nabla_{v_{1}}[(4.4)]\right| \mathrm{d} v_{1} \\
& \lesssim O\left(\varepsilon^{-1}\right)\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \nabla_{v_{1}}[(4.4)] \mathrm{d} v_{1} \\
& \lesssim O\left(\varepsilon^{-1}\right)\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty}, \tag{4.45}
\end{align*}
$$

where we have used (4.7) in the third line.
For (4.42) we apply (2.22) to get

$$
\begin{align*}
(4.42) & \left.\lesssim \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0}\|\eta\|_{C^{1}} \int_{\mathbb{R}^{3}} \mathbf{k}\left(v_{1}, u\right) \frac{\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{\varepsilon} e^{-\nu\left(v_{1}\right)\left(t_{1}-t_{\mathbf{b}}^{1}\right)} \frac{1}{\alpha\left(x_{1}, v_{1}\right)} \mathrm{d} u \times[\boxed{4.4})\right] \mathrm{d} v_{1} \\
& \lesssim O\left(\varepsilon^{-1}\right)\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \frac{1}{\alpha\left(x_{1}, v_{1}\right)} \times[(4.4)] \mathrm{d} v_{1} \\
& \lesssim O\left(\varepsilon^{-1}\right)\|\eta\|_{C^{1}}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \tag{4.46}
\end{align*}
$$

where we have used (4.5) in the third line.
Collecting (4.44), (4.45), (4.46) and 4.38) we conclude that

$$
\begin{equation*}
\text { (4.34) } \lesssim O(\varepsilon)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}+O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty} \tag{4.47}
\end{equation*}
$$

Last we estimate 4.35). Applying chain rule we have

$$
\begin{aligned}
\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \nabla_{x_{1}} f_{s}\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right), v_{1}\right) & =\partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right) \sum_{j=1,2} \nabla_{x_{1}} \mathbf{x}_{p^{2}, j}^{2} \partial_{\mathbf{x}_{p^{2}, j}^{2}} f\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right), v_{1}\right) \\
& =\sum_{j=1,2} \partial_{\mathbf{x}_{p^{1}, i}^{1}} \mathbf{x}_{p^{2}, j}^{2} \partial_{\mathbf{x}_{p^{2}, j}^{2}} f_{s}\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right), v_{1}\right) .
\end{aligned}
$$

Note that

$$
\begin{equation*}
v_{1}=\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right) / t_{\mathbf{b}}^{1} . \tag{4.48}
\end{equation*}
$$

Applying Lemma 11 we have

$$
\begin{aligned}
(4.35) & =\mathcal{A} \times \int_{n\left(x_{1}\right) \cdot v_{1}>0} \sum_{j=1,2} \partial_{\mathbf{x}_{p^{1}, i}^{1}} \mathbf{x}_{p^{2}, j}^{2} \partial_{\mathbf{x}_{p^{2}, j}^{2}} f\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right), v\right) \sum_{p^{2} \in \mathcal{P}} \iota_{p^{2}}\left(x_{2}\right)[(4.4)] \\
& =\mathcal{A} \times \sum_{p^{2} \in \mathcal{P}} \iint_{\left|\mathbf{x}_{p^{2}}^{2}\right|<\delta_{1}} \int_{0}^{t_{1}} e^{-\nu\left(v_{1}\right) t_{\mathbf{b}}^{1}} \iota_{p^{2}}\left(x_{2}\right) \times \sum_{j=1,2} \partial_{\mathbf{x}_{p^{1}, i}^{1}} \mathbf{x}_{p^{2}, j}^{2} \partial_{\mathbf{x}_{p^{2}, j}^{2}} f_{s}\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right), v\right) \\
& \times \sqrt{g_{p^{2}, 11}\left(\mathbf{x}_{p^{2}}^{2}\right) g_{p^{2}, 22}\left(\mathbf{x}_{p^{2}}^{2}\right)} \frac{n\left(x_{1}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)}{t_{\mathbf{b}}^{1}} \frac{\left|n\left(x_{2}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)\right|}{\left|t_{\mathbf{b}}^{1}\right|^{4}} \times \frac{\text { (4.4) }}{n\left(x_{1}\right) \cdot v_{1}} .
\end{aligned}
$$

In the last step we used $n\left(x_{1}\right) \cdot v_{1}=\frac{n\left(x_{1}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)}{t_{\mathrm{b}}^{1}}$.

We apply the integration by parts with respect to $\partial_{\mathbf{x}_{p^{2}, j}^{2}}$ for $j=1,2$. For $\iota_{p}^{2}\left(\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)=0$ when $\left|\mathbf{x}_{p^{2}}^{2}\right|=\delta_{1}$ from (2.32), the contribution of $\left|\mathbf{x}_{p^{2}}^{2}\right|=\delta_{1}$ vanishes. Thus we derive

$$
\begin{aligned}
(4.35) & \lesssim \mathcal{A}\left\|w_{\vartheta} f_{s}\right\|_{\infty} \times \sum_{p^{2} \in \mathcal{P}} \sum_{j=1,2} \\
& \times\left[\iiint_{0}^{t_{1}} \partial_{\mathbf{x}_{p^{2}, j}^{2}}\left[\frac{n\left(x_{1}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)}{t_{\mathbf{b}}^{1}} \frac{\mid n_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right) \mid\right.}{\left|t_{\mathbf{b}}^{1}\right|^{4}}\right] \cdots\right. \\
& +\iiint_{0}^{t_{1}} \partial_{\mathbf{x}_{p^{2}, j}^{2}}\left[\sum_{j=1,2} \frac{\partial \mathbf{x}_{p^{2}, j}^{2}}{\partial \mathbf{x}_{p^{1}, i}^{1}} \sqrt{g_{p^{2}, 11 g_{p^{2}, 22}}}\right] \cdots \\
& \left.+\iiint_{0}^{t_{1}} \partial_{\mathbf{x}_{p^{2}, j}^{2}}\left[\frac{(4.4)}{n\left(x_{1}\right) \cdot v_{1}}\right] \cdots\right] .
\end{aligned}
$$

Applying Lemma 12 we have

$$
\begin{gather*}
\frac{n\left(x_{1}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)}{t_{\mathbf{b}}^{1}} \lesssim \frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{2}}{t_{\mathbf{b}}^{1}}, \quad \frac{\left|n_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)\right|}{\left|t_{\mathbf{b}}^{1}\right|^{4}} \lesssim \frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{2}}{\left|t_{\mathbf{b}}^{1}\right|^{4}} \\
\partial_{\mathbf{x}_{p^{2}, j}^{2}}\left[\frac{n\left(x_{1}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)}{t_{\mathbf{b}}^{1}} \frac{\left|n_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right) \cdot\left(x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right)\right|}{\left|t_{\mathbf{b}}^{1}\right|^{4}}\right] \lesssim \frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{3}}{\left|t_{\mathbf{b}}^{1}\right|^{5}} \tag{4.49}
\end{gather*}
$$

From (2.42), (2.47) and (2.48), we derive that

$$
\begin{align*}
& \left|\frac{\partial}{\partial \mathbf{x}_{p^{2}, j}^{2}}\left(\sum_{j=1,2} \frac{\partial \mathbf{x}_{p^{2}, j}^{2}}{\partial \mathbf{x}_{p^{1}, i}^{1}} \sqrt{g_{p^{2}, 11} g_{p^{2}, 22}}\right)\right| \\
\lesssim & \|\eta\|_{C^{2}}\left\{1+\frac{\left|\mathbf{v}_{p^{2}, \|}^{2}\right|}{\left|\mathbf{v}_{p^{2}, 3}^{2}\right|^{2}}\left|\partial_{3} \eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right) \cdot \partial_{i} \eta_{p^{1}}\left(\mathbf{x}_{p^{1}}^{1}\right)\right|\right\}  \tag{4.50}\\
\leq & O\left(\|\eta\|_{C^{2}}\right)\left\{1+\frac{\left|\mathbf{v}_{p^{2}}^{2}\right|}{\left|\mathbf{v}_{p^{2}, 3}^{2}\right|^{2}}\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|\right\} \\
\leq & O\left(\|\eta\|_{C^{2}}\right) \frac{1}{\left|\mathbf{v}_{p^{2}, 3}^{2}\right|}=O\left(\|\eta\|_{C^{2}}\right) \frac{1}{\left|n\left(x_{2}\right) \cdot v_{1}\right|}
\end{align*}
$$

Such term will be cancelled by $n\left(x_{1}\right) \cdot v_{1}$ as:

$$
\frac{\left|n\left(x_{1}\right) \cdot v_{1}\right|}{\left|n\left(x_{2}\right) \cdot v_{1}\right|} \lesssim \frac{\alpha\left(x_{1}, v_{1}\right)}{\alpha\left(x_{2}, v_{1}\right)} \lesssim 1 .
$$



$$
v_{1, \perp}=v_{1} \cdot n\left(x_{1}\right), \quad v_{1, \|}=v_{1}-\left(n\left(x_{1}\right) \cdot v_{1}\right) n\left(x_{1}\right)
$$

Using (4.48) we get

$$
\left|\partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1}\right| \lesssim \frac{1}{t_{\mathbf{b}}^{1}}, \quad\left|\partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1, \perp}\right| \lesssim \frac{1}{t_{\mathbf{b}}^{1}}, \quad\left|\partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1, \|}\right| \lesssim \frac{1}{t_{\mathbf{b}}^{1}} .
$$

Then taking the derivative to $\frac{(4.4)}{n\left(x_{1}\right) \cdot v_{1}}$ we have extra term

$$
\left[\frac{1}{4 T_{0}}-\frac{1}{2 T_{w}\left(x_{1}\right)}\right]\left(\partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1}\right) v_{1}, \quad \frac{\left(1-r_{\|}\right)^{2} \partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1, \|}\left(v_{1, \|}-\left(1-r_{\|}\right) v_{\|}\right)}{T_{w}\left(x_{1}\right) r_{\|}\left(2-r_{\|}\right)}, \quad \frac{\partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1, \perp} v_{1, \perp}}{T_{w}\left(x_{1}\right) r_{\perp}}
$$

The extra term comes from $I_{0}$ is

$$
\partial_{\mathbf{x}_{p^{2}, j}^{2}}\left(\frac{\left(1-r_{\perp}\right)^{1 / 2} v_{1, \perp} v_{\perp}}{r_{\perp} T_{w}\left(x_{1}\right)}\right) \lesssim \frac{\left(1-r_{\perp}\right)^{1 / 2} v_{\perp} \partial_{\mathbf{x}_{p^{2}, j}^{2}} v_{1, \|}}{r_{\perp}} .
$$

Thus all of them are bounded by

$$
\frac{\left[1+|v|^{2}+\left|v_{1}\right|^{2}\right]}{t_{\mathbf{b}}^{1}}
$$

Then for $\varepsilon \ll 1$ applying (4.9) we get

$$
\begin{align*}
\partial_{x_{p^{2}, j}^{2}}\left[\frac{(4.4)}{n\left(x_{1}\right) \cdot v_{1}}\right] & \lesssim\left[1+|v|^{2}+\left|v_{1}\right|^{2}\right]\left[\frac{(4.4)}{n\left(x_{1}\right) \cdot v_{1}}\right] \\
& \lesssim\left[1+|v|^{2}+\left|v_{1}\right|^{2}\right] e^{-\frac{1}{4 T_{0}}\left|v_{1}\right|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}|v|^{2}} e^{\frac{O(\varepsilon)}{T_{0}}\left|v_{1}\right|^{2}} . \tag{4.51}
\end{align*}
$$

Collecting (4.49), (4.50) and (4.51) we obtain

$$
\text { (4.35) } \begin{align*}
& \lesssim \mathcal{A} \times\left\|w_{\vartheta} f_{s}\right\|_{\infty} e^{O(\varepsilon)|v|^{2}} \\
& \times \iiint_{0}^{t_{1}} e^{-\nu t_{\mathrm{b}}^{1}}\left[\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{3}}{t_{\mathbf{b}}^{5}}+\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{2}}{t_{\mathrm{b}}^{4}}+\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{4}}{t_{\mathbf{b}}^{6}}\right] e^{-\frac{1}{8 T_{0}} \frac{\left|x_{1}-\eta_{p^{2}}\left(x_{p^{2}}^{2}\right)\right|^{2}}{\mid t_{\mathrm{b}}^{2}}} \\
& \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} \int_{0}^{\infty} \frac{e^{-\nu\left(v_{1}\right) t_{\mathrm{b}}^{1}}}{\left|t_{\mathrm{b}}^{1}\right|^{1 / 2}} \iint \frac{1}{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{3 / 2}} \lesssim\left\|w_{\vartheta} f_{s}\right\|_{\infty} . \tag{4.52}
\end{align*}
$$

In the last line we have used the definition of $\mathcal{A}$ in (4.3) to have

$$
\mathcal{A} \times e^{O(\varepsilon)|v|^{2}} \lesssim 1 .
$$

And we used

$$
\begin{aligned}
& {\left[\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{3}}{\left|t_{\mathbf{b}}^{1}\right|^{5}}+\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{2}}{\left|t_{\mathbf{b}}^{1}\right|^{4}}+\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{4}}{t_{\mathbf{b}}^{6}}\right] e^{-\frac{\left|x_{1}-\eta_{p} p^{2}\left(x_{p^{2}}\right)\right|^{2}}{8 T_{0}\left|t_{\mathbf{b}}\right|^{2}}}} \\
& \leq \frac{1}{\left|t_{\mathbf{b}}^{1}\right|^{1 / 2}} \frac{1}{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{3 / 2}}\left[\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{9 / 2}}{\left|t_{\mathbf{b}}^{1}\right|^{9 / 2}}+\frac{\mid x_{1}-\eta_{p^{2}}\left(\left.\mathbf{x}_{p^{2}}^{2}\right|^{7 / 2}\right.}{\mid t_{\mathbf{b}}^{1} 7^{/ / 2}}+\frac{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{11 / 2}}{\left|t_{\mathbf{b}}^{1}\right|^{11 / 2}}\right] \\
& \times e^{-\frac{\left|x_{1}-\eta_{p^{2}}\left(x_{p}^{2} p^{2}\right)\right|^{2}}{\left.8 T_{0} t_{\mathbf{b}}\right|^{2}}} \\
& \lesssim \frac{1}{\left|t_{\mathbf{b}}^{1}\right|^{1 / 2}} \frac{1}{\left|x_{1}-\eta_{p^{2}}\left(\mathbf{x}_{p^{2}}^{2}\right)\right|^{3 / 2}} .
\end{aligned}
$$

Then we combine (4.36), (4.47), (4.52) and (4.32) to get

$$
\begin{equation*}
\text { (4.26) } \lesssim o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty} O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty} \tag{4.53}
\end{equation*}
$$

Finally combining (4.28), (4.29), (4.53) and (4.30) we conclude that

$$
\left|\partial_{\mathbf{x}_{p^{1}, i}}\left[f_{s}\left(\eta_{p^{1}}\left(\mathbf{x}_{p^{1}}, v\right)\right)\right]\right| \lesssim O\left(\varepsilon^{-1}\right)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty} .
$$

This, with (4.22), conclude that the boundary term is bounded by

$$
\begin{equation*}
(4.21) ~ \lesssim \frac{O\left(\varepsilon^{-1}\right)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)} . \tag{4.54}
\end{equation*}
$$

Estimate of (4.20). For the collision term we apply Lemma [23] to $\partial_{[x]_{j}} f_{s}(x-(t-s) v, u)$ to get

$$
\begin{align*}
(4.20) & =\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \frac{O(1)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x-(t-s) v, u)}  \tag{4.55}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)} \\
& \times \int_{\mathbb{R}^{3}} \mathbf{k}\left(u, u^{\prime}\right) \partial_{[x]_{j}} f\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right) \mathrm{d} u^{\prime} \mathrm{d} s^{\prime}  \tag{4.56}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \frac{\partial f_{s}\left(x_{\mathbf{b}}(x-(t-s) v, u), u\right)}{\partial[x]_{j}} . \tag{4.57}
\end{align*}
$$

Directly applying (2.30) in Lemma 9 we bound

$$
\begin{equation*}
(4.55) ~ \lesssim \frac{O(1)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)} . \tag{4.58}
\end{equation*}
$$

For (4.57), let $y=x-(t-s) v$, then

$$
\frac{\partial f_{s}\left(x_{\mathbf{b}}(x-(t-s) v, u)\right)}{\partial[x]_{j}}=\frac{\partial f_{s}\left(x_{\mathbf{b}}(y, u)\right)}{\partial[y]_{j}},
$$

which is exactly the same as (4.21) with replacing $x$ by $y, v$ by $u$. Note that we already derive the upper bound for (4.21) in (4.54), such estimate works for any $x \in \Omega, v \in \mathbb{R}^{3}$. Thus we can also bound $\frac{\partial f_{s}\left(x_{\mathrm{b}}(x-(t-s) v, u)\right)}{\partial[x]_{j}}$ by (4.54). Therefore,

$$
\begin{align*}
(4.57) & \lesssim \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \frac{O\left(\varepsilon^{-1}\right)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}\right]+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(y, u)} \mathrm{d} u \mathrm{~d} s  \tag{4.59}\\
& \lesssim \frac{O\left(\varepsilon^{-1}\right)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}+o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}\right]}{\alpha(x, v)}
\end{align*}
$$

where we have used Lemma 9 in the second line.
Last we estimate (4.56). We split the $s^{\prime}$-integration into two cases:

$$
\begin{equation*}
\int_{\max \{\ldots\}}^{s}=\underbrace{\int_{s-\varepsilon}^{s}}_{(4.60)_{1}}+\underbrace{\int_{\max \{\ldots\}}^{s-\varepsilon}}_{(4.60]_{2}} \tag{4.60}
\end{equation*}
$$

For (4.60) ${ }_{1}$, we bound

$$
\left|\partial_{[x]_{j}} f_{s}\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right)\right| \lesssim \frac{\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right)}
$$

Then we apply (2.31) in Lemma 9 to derive

$$
\text { (4.56) } \begin{align*}
1_{s-\varepsilon \leq s^{\prime} \leq s} & \lesssim \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \frac{O(\varepsilon)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x-(t-s) v, u)} \\
& \lesssim \frac{O(\varepsilon)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}}{\alpha(x, v)} \tag{4.61}
\end{align*}
$$

where we have used (2.30) in the second line.
For (4.60) $)_{2}$ we exchange the $x$ derivative into $u$ derivative:

$$
\partial_{[x]_{j}} f_{s}\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right)=\frac{\partial_{u_{j}}\left[f_{s}\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right)\right]}{s^{\prime}-s} .
$$

In this case we have $s-s^{\prime} \geq \varepsilon$. Applying the integration by parts with respect to $\mathrm{d} u$ we get

$$
\begin{align*}
(4.56) 1_{s^{\prime} \leq s-\varepsilon} & \lesssim O\left(\varepsilon^{-1}\right) \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)} \\
& \times \int_{\mathbb{R}^{3}}\left|\nabla_{u}\left[\mathbf{k}(v, u) \mathbf{k}\left(u, u^{\prime}\right)\right]\right| f_{s}\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right) \mathrm{d} u^{\prime} \mathrm{d} s^{\prime}  \tag{4.62}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)}\left(s-s^{\prime}\right)\left|\nabla_{u} \nu(u)\right| \\
& \times \int_{\mathbb{R}^{3}} \mathbf{k}\left(u, u^{\prime}\right) \frac{f_{s}\left(x-(t-s) v-\left(s-s^{\prime}\right) u, u^{\prime}\right)}{s^{\prime}-s} \mathrm{~d} u^{\prime}  \tag{4.63}\\
& +\int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) e^{-\nu(u) t_{\mathbf{b}}(x-(t-s) v, u)}\left|\nabla_{u} t_{\mathbf{b}}(x-(t-s) v, u)\right| \\
& \times \int_{\mathbb{R}^{3}} \mathbf{k}\left(u, u^{\prime}\right) \frac{f_{s}\left(x_{\mathbf{b}}(x-(t-s) v, u) u, u^{\prime}\right)}{s^{\prime}-s} \mathrm{~d} u^{\prime} \tag{4.64}
\end{align*}
$$

First we estimate (4.62). For some $c \ll \vartheta$ we have

$$
\begin{align*}
\text { (4.62) } & \lesssim O\left(\varepsilon^{-1}\right) \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)} \\
& \times \int_{\mathbb{R}^{3}}\left|\nabla_{u}\left[\mathbf{k}(v, u) \mathbf{k}\left(u, u^{\prime}\right)\right]\right| \frac{1}{e^{c|v|^{2}}} \frac{e^{c|v|^{2}}}{e^{c|u|^{2}}} \frac{e^{c|u|^{2}}}{e^{c\left|u^{\prime}\right|^{2}}}\left\|e^{c|v|^{2}} f_{s}\right\|_{\infty} \mathrm{d} u^{\prime} \mathrm{d} s^{\prime} \\
& \lesssim \frac{O\left(\varepsilon^{-1}\right)}{e^{c|v|^{2}}} \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)} \mathbf{k}_{\tilde{\varrho}}(v, u)\left[1+\frac{1}{|v-u|}\right] \\
& \times \int_{\mathbb{R}^{3}} \mathbf{k}_{\tilde{\varrho}}\left(u, u^{\prime}\right)\left[1+\frac{1}{\left|u-u^{\prime}\right|}\right]\left\|w_{\vartheta} f_{s}\right\|_{\infty} \mathrm{d} u^{\prime} \mathrm{d} s^{\prime} \\
& \lesssim \frac{O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{e^{c|v|^{2}}} \times \frac{\alpha(x, v)}{\alpha(x, v)} \lesssim \frac{O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{\alpha(x, v)} . \tag{4.65}
\end{align*}
$$

In the third line we applied (2.29). In the last line we applied Lemma 9 and used $\frac{\alpha(x, v)}{e^{c|v|^{2}}} \lesssim 1$.
Then we estimate (4.63). Similar to computation of (4.65), we have

$$
\begin{align*}
(4.63) & \lesssim \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}(v, u) \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)} \\
& \times \int_{\mathbb{R}^{3}} \mathbf{k}\left(u, u^{\prime}\right) \frac{1}{e^{c|v|^{2}}} \frac{e^{\left.c|v|\right|^{2}}}{e^{c|u|^{2}}} \frac{e^{c|u|^{2}}}{e^{c\left|u^{\prime}\right|^{2}}}\left\|e^{c|v|^{2}} f_{s}\right\|_{\infty} \mathrm{d} u^{\prime} \mathrm{d} s^{\prime} \\
& \lesssim \frac{1}{e^{c|v|^{2}}} \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \int_{\max \left\{0, s-t_{\mathbf{b}}(x-(t-s) v, u)\right\}}^{s} e^{-\nu(u)\left(s-s^{\prime}\right)} \mathbf{k}_{\tilde{\varrho}}(v, u) \\
& \times \int_{\mathbb{R}^{3}} \mathbf{k}_{\tilde{e}}\left(u, u^{\prime}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty} \mathrm{d} u^{\prime} \mathrm{d} s^{\prime} \\
& \lesssim \frac{O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{e^{c|v|^{2}}} \lesssim \frac{O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{\alpha(x, v)} . \tag{4.66}
\end{align*}
$$

Last we estimate (4.64). Applying (2.41) we have

$$
\begin{align*}
\text { (4.64) } & \lesssim O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty} \int_{\max \left\{0, t-t_{\mathbf{b}}\right\}}^{t} e^{-\nu(v)(t-s)} \int_{\mathbb{R}^{3}} \mathbf{k}_{\varrho}(v, u) \frac{1}{\alpha(x-(t-s) v, u)} \\
& \lesssim \frac{O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{\alpha(x, v)} . \tag{4.67}
\end{align*}
$$

In the second line we used Lemma 9 ,
Combining (4.65), (4.66) and (4.67) we conclude

$$
\begin{equation*}
(4.56) 1_{s-s^{\prime} \geq \varepsilon} \lesssim \frac{O\left(\varepsilon^{-1}\right)\left\|w_{\vartheta} f_{s}\right\|_{\infty}}{\alpha(x, v)} . \tag{4.68}
\end{equation*}
$$

Then we combine (4.58), (4.59), (4.61), (4.68) and conclude

$$
\begin{equation*}
(4.20) \lesssim \frac{o(1)\left\|\alpha \nabla_{x} f_{s}\right\|_{\infty}+O\left(\varepsilon^{-1}\right)\left[\left\|w_{\vartheta} f_{s}\right\|_{\infty}+\left\|w_{\vartheta} f_{s}\right\|_{\infty}^{2}\right]}{\alpha(x, v)} \tag{4.69}
\end{equation*}
$$

Finally from Lemma 23, (4.54) and (4.69) we conclude Theorem 2 .

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