The Cahn–Hilliard equation with forward-backward dynamic boundary condition via vanishing viscosity

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Abstract

An asymptotic analysis for a system with equation and dynamic boundary condition of Cahn–Hilliard type is carried out as the coefficient of the surface diffusion acting on the phase variable tends to 0, thus obtaining a forward-backward dynamic boundary condition at the limit. This is done in a very general setting, with nonlinear terms admitting maximal monotone graphs both in the bulk and on the boundary. The two graphs are related by a growth condition, with the boundary graph that dominates the other one. It turns out that in the limiting procedure the solution of the problem looses some regularity and the limit equation has to be properly interpreted in the sense of a subdifferential inclusion. However, the limit problem is still well-posed since a continuous dependence estimate can be proved. Moreover, in the case when the two graphs exhibit the same growth, it is shown that the solution enjoys more regularity and the boundary condition holds almost everywhere. An error estimate can also be shown, for a suitable order of the diffusion parameter. **Key words:** Cahn–Hilliard system, dynamic boundary conditions, asymptotics, forward-backward equation, well-posedness, error estimates.

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1 Introduction

We consider a pure Cahn–Hilliard equation in the form

$$\partial_t u - \Delta \mu = 0 \qquad \text{in } Q := \Omega \times (0, T),$$
(1.1)

$$\mu = -\Delta u + F'(u) - f \qquad \text{in } Q, \tag{1.2}$$

where T > 0 is some fixed time, $\Omega \subset \mathbb{R}^d$ (d = 2 or 3) is a bounded smooth domain with smooth enough boundary Γ , and the symbols ∂_t and Δ denote the partial time-derivative and the Laplacian with respect to the space variables, respectively.

On the boundary, we deal with a dynamic condition also of Cahn–Hilliard type, depending on a positive parameter δ , in the form

$$u_{\Gamma} = u_{|_{\Gamma}}, \quad \mu_{\Gamma} = \mu_{|_{\Gamma}} \quad \text{on } \Sigma := \Gamma \times (0, T),$$

$$(1.3)$$

$$\partial_t u_{\Gamma} + \partial_{\nu} \mu - \Delta_{\Gamma} \mu_{\Gamma} = 0 \qquad \text{on } \Sigma, \tag{1.4}$$

$$\mu_{\Gamma} = \partial_{\nu} u - \delta \Delta_{\Gamma} u_{\Gamma} + F_{\Gamma}'(u_{\Gamma}) - f_{\Gamma} \qquad \text{on } \Sigma.$$
(1.5)

Here, the notation $v_{|\Gamma}$ is employed for the trace of a function $v : \Omega \to \mathbb{R}$ on the boundary Γ ; besides, ∂_{ν} and Δ_{Γ} denote the the outward normal derivative and the Laplace–Beltrami operator on Γ . At the initial time t = 0, we assume that

$$u(0) = u_0 \qquad \text{in } \Omega, \tag{1.6}$$

$$u_{\Gamma}(0) = u_{0\Gamma} \qquad \text{on } \Gamma. \tag{1.7}$$

The variables $u, \mu : Q \to \mathbb{R}$ and the respective ones $u_{\Gamma}, \mu_{\Gamma} : \Sigma \to \mathbb{R}$ on the boundary represent the phase parameter and the chemical potential. Moreover, $f : Q \to \mathbb{R}$ and $f_{\Gamma} : \Sigma \to \mathbb{R}$ stand for two known source terms and $u_0, u_{0\Gamma}$ are the given initial data, in the bulk and on the boundary. The nonlinearities F' and F'_{Γ} owe their presence in the equations as derivatives of the double-well potentials F and F_{Γ} .

This paper is dedicated to investigate the asymptotic behaviour of the system (1.1)–(1.7) as δ tends to 0. The limit condition which is obtained on the boundary is rather interesting since, as we will explain later, it consists of a forward-backward dynamic boundary condition.

From now, let us spend some words on the Cahn-Hilliard system and recall that it is a phenomenological model describing the spinodal decomposition in the framework of partial differential equations. It originates from the work of J. W. Cahn [7] and his collaboration with J. E. Hilliard [8]. A number of research contributions in recent times has been devoted to Cahn-Hilliard and viscous Cahn-Hilliard [33,34] systems. An impressive amount of related references can be found in the literature, in particular we may refer to the review paper [31] and references therein. The coupling of Cahn-Hilliard and other systems with dynamic boundary conditions turns out to be a research theme that has been developed quite intensively in the last twenties. If one considers a boundary dynamics of heat equation type, the well-posedness issue has been treated in [37] and a study of the convergence to equilibrium is shown in [43]. Since then, the Cahn–Hilliard system with nonlinear equations as dynamic boundary condition (including the Allen–Cahn equation), was addressed from different viewpoints and studied in several papers: among other contributions we quote [10, 13, 14, 16, 17, 19, 20, 22, 25–30, 35, 38]. The articles [12, 15, 18] refer instead to similar approaches but for different equations in the domain. Let us also mention the papers [17, 21, 24] devoted to the analysis of optimal control problems for some Cahn–Hilliard systems coupling equation and dynamic boundary condition. For completeness, let us also mention that vanishing diffusion studies on Cahn–Hilliard and Allen–Cahn equations have been pursued also in the case of stochastic forcing, for which we refer to [39] and [36], respectively.

Therefore, the problem (1.1)-(1.7) yields the Cahn-Hilliard system in the bulk and on the boundary, where the variables on the boundary are the traces of the respective ones and equations dynamic boundary conditions have the same structure as Cahn-Hillird systems. The related initial-boundary value problem has been investigated – in the case $\delta > 0$ – in the paper [10] and also in [20] when convection effects are taken into account. Both analyses allow the presence of singular and non-smooth potentials for F and F_{Γ} . In fact, typical examples for these potentials are the so-called classical regular potential, the logarithmic potential, and the double obstacle potential, which are defined by

$$\begin{split} F_{\rm reg}(r) &:= \frac{1}{4} \left(r^2 - 1 \right)^2, \quad r \in \mathbb{R} \,, \\ F_{\rm log}(r) &:= \begin{cases} (1+r) \ln(1+r) + (1-r) \ln(1-r) - c_1 r^2, & r \in (-1,1) \\ 2 \ln(2) - c_1, & r \in \{-1,1\} \\ +\infty, & r \notin [-1,1] \end{cases} \\ F_{\rm obs}(r) &:= \begin{cases} c_2(1-r^2), & r \in [-1,1] \\ +\infty, & r \notin [-1,1] \end{cases} , \end{split}$$

where $c_1 > 1$ and $c_2 > 0$ are constants, with the role of rendering F_{\log} and F_{obs} nonconvex. Here, as in [10,20] we split the nonlinear contributions F' in (1.2) and F'_{Γ} in (1.5) into two parts, i.e., we let $F' = \beta + \pi$ and $F'_{\Gamma} = \beta_{\Gamma} + \pi_{\Gamma}$, where β , β_{Γ} are the monotone parts, i.e. the derivatives, or in general the subdifferentials, of the convex parts of F and F_{Γ} , and π , π_{Γ} stand for the (smooth) anti-monotone parts. In particular, for the classical regular potential $F'_{reg} = \beta_{reg} + \pi_{reg}$ is exactly the derivative of F_{reg} , that is

$$F'_{\rm reg}(r) = r^3 - r$$
, with $\beta_{\rm reg}(r) := r^3$, $\pi_{\rm reg}(r) := -r$,

while for the non-smooth double obstacle potential F_{obs} we have that β_{obs} is the subdifferential of the indicator function of [-1, 1], so to have

$$F'_{\rm obs}(r) = \partial I_{[-1,1]}(r) - 2c_2 r, \quad \beta_{\rm obs}(r) := \partial I_{[-1,1]}(r), \quad \pi_{\rm obs}(r) := -2c_2 r.$$

As a general rule, we employ subdifferentials for β , β_{Γ} , which reduce to the derivatives whenever these exist. Please note that the subdifferentials may also be multivalued graphs, as it happens for $\partial I_{[-1,1]}(r)$ when r = -1 or r = 1. Thus, we generally interpret equation (1.2) as

$$\mu = -\Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q \tag{1.8}$$

and the boundary condition (1.5) as

$$\mu_{\Gamma} = \partial_{\nu} u - \delta \Delta_{\Gamma} u_{\Gamma} + \xi_{\Gamma} + \pi_{\Gamma}(u_{\Gamma}) - f_{\Gamma}, \quad \xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma}) \quad \text{on } \Sigma.$$
(1.9)

Of course, different potentials can be considered for F and F_{Γ} , leading in particular to different graphs β and β_{Γ} . About possible relations between β and β_{Γ} , following a rather usual approach (cf., e.g., [9, 10, 12, 14–16, 18, 20, 30, 38]) we assume in general that β_{Γ} dominates β in the sense of assumption **A2** in Section 2. For better regularity results (cf. the later Theorems 2.10 and 2.12) we let β and β_{Γ} have the same growth (cf. assumption (2.40)). Thus, in our framework it is always possible to choose similar or even equal graphs β and β_{Γ} .

This paper is dedicated to the asymptotic analysis as the surface diffusion term on the dynamic boundary condition (1.9) tends to 0. By the asymptotic limit as $\delta \searrow 0$, one aims to obtain at the limit the solution of the problem without surface diffusion, i.e., with (1.9) possibly replaced by

$$\mu_{\Gamma} = \partial_{\nu} u + \xi_{\Gamma} + \pi_{\Gamma}(u_{\Gamma}) - f_{\Gamma}, \quad \xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma}) \quad \text{on } \Sigma.$$
(1.10)

It turns out that this program is doable, as shown by our analysis, by accepting that the solution of the limiting problem looses some regularity, due to the absence of the diffusive term in (1.10). Indeed, in general the terms $\partial_{\nu} u$ and ξ_{Γ} in (1.9) are not functions but elements of a dual space, and the inclusion $\xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma})$ has to be suitably reinterpreted in the sense of inclusion for a subdifferential operator acting from a space on the boundary to its dual space. However, the solution of the limiting problem turns out to be uniquely determined at least for what concerns the components (u, u_{Γ}) . Moreover, in the case where the graphs β and β_{Γ} exhibit the same growth, we demonstrate that the boundary condition (1.10) holds almost everywhere on Σ . Besides that, in such a case we are even able to prove an error estimate of order $\delta^{1/2}$ between the solution of the problem with surface diffusion in (1.9) and that of the limiting problem with (1.10).

Two special references related to our investigation are the recent papers [38] and [10], which deal with Cahn–Hilliard systems in the bulk with dynamic boundary condition of Allen–Cahn type, and extensions of them in [38] as well. In fact, our asymptotic results can be compared with the ones contained in these papers, where [10] also examines the case of the graphs β and β_{Γ} having the same growth. However, no error estimate is discussed in [10,38] (as instead we do here).

The limiting equation (1.10) that we obtain on the boundary, when coupled to (1.4), yields a forward-backward type equation since the surface diffusion operator present in (1.4) eventually applies to μ_{γ} in (1.10), and μ_{Γ} is given here in terms of a non-monotone function (i.e., $\beta_{\Gamma} + \pi_{\Gamma}$) of the phase variable u_{Γ} . About forward-backward equations and possible regularizations of them we can quote [2–4, 23, 41, 42] and referenced therein.

The main novelty of this paper is that we can give rigorous sense to a forward-backward dynamic on the boundary in terms of well-posedness of the whole system. Indeed, we underline that, unless special cases, the evolution problems for a single forward-backward equation are ill-posed. Here, nonetheless, we show that the coupling of the badly-behaving boundary condition with the Cahn-Hilliard equation in the interior is somehow strong enough to ensure solvability of the boundary forward-backward dynamics. We also point out that the Cahn-Hilliard equation itself may be actually seen as an elliptic spaceregularisation of a forward-backward equation by means of the local diffusion operator $-\Delta$. Consequently, the choice of the limit forward-backward boundary condition is extremely natural, and corresponds to the intuitive degenerate limit of the Cahn-Hilliard equation with no diffusion regularization on the boundary.

The present paper is structured as follows. In Section 2, after setting up the notation and the basic tools for a precise interpretation of the problem, we present the main theorems. First, we recall the well-posedness result for the case $\delta > 0$; then, we state the convergence-existence result as δ goes to, and becomes 0 at the limit, including the continuous dependence with respect to data and the uniqueness of the solution for the limit problem. There are two more statements, dedicated to an improvement of the convergence-existence theorem and to the error estimate in the case when the graphs β and β_{Γ} show the same growth. Section 3 is devoted to the proofs, in this order: we start with proving the uniform estimates for all $\delta \in (0, 1)$, hence passing to the limit as $\delta \searrow 0$; then, we deal with the continuous dependence estimate, we examine the refined convergence and show the error estimate of order $\delta^{1/2}$. There is also a Section A with auxiliary results for equivalence of norms and approximation of initial data.

2 Setting and main results

In this section, we rigorously introduce the variational setting and the main assumptions of the work, and we state our main results.

Throughout the paper, $\Omega \subset \mathbb{R}^d$ (d = 2, 3) is a smooth bounded domain with smooth boundary Γ , and T > 0 is a fixed finite final time. We use the classical notations

$$Q_t := \Omega \times (0, t), \quad \Sigma_t := \Gamma \times (0, t) \quad \text{for } t \in [0, T], \qquad Q := Q_T, \quad \Sigma := \Sigma_T$$

We are interested in the asymptotic behaviour as $\delta \searrow 0$ of the following initial-boundary value problem:

$$\partial_t u - \Delta \mu = 0 \qquad \text{in } Q, \tag{2.1}$$

$$\mu \in -\Delta u + \beta(u) + \pi(u) - f \quad \text{in } Q, \tag{2.2}$$

$$u_{\Gamma} = u_{|_{\Gamma}}, \quad \mu_{\Gamma} = \mu_{|_{\Gamma}} \quad \text{on } \Sigma,$$
 (2.3)

$$\partial_t u_\Gamma + \partial_\nu \mu - \Delta_\Gamma \mu_\Gamma = 0 \qquad \text{on } \Sigma, \tag{2.4}$$

$$\mu_{\Gamma} \in \partial_{\nu} u - \delta \Delta_{\Gamma} u_{\Gamma} + \beta_{\Gamma} (u_{\Gamma}) + \pi_{\Gamma} (u_{\Gamma}) - f_{\Gamma} \quad \text{on } \Sigma,$$
(2.5)

 $u(0) = u_0 \qquad \text{in } \Omega, \tag{2.6}$

$$u_{\Gamma}(0) = u_{0\Gamma}$$
 on Γ . (2.7)

The following assumptions on the data are in order throughout the work.

A1 $\widehat{\beta}, \widehat{\beta}_{\Gamma} : \mathbb{R} \to [0, +\infty]$ are proper, convex, and lower semicontinuous functions on \mathbb{R} satisfying the condition $\widehat{\beta}(0) = \widehat{\beta}_{\Gamma}(0) = 0$. This implies that their subdifferentials

$$\beta := \partial \widehat{\beta}, \qquad \beta_{\Gamma} := \partial \widehat{\beta}_{\Gamma},$$

are maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, with some effective domains $D(\beta)$ and $D(\beta_{\Gamma})$, respectively, and that $0 \in \beta(0) \cap \beta_{\Gamma}(0)$.

A2 $D(\beta_{\Gamma}) \subseteq D(\beta)$ and there exists a constant M > 0 such that

$$|\beta^{\circ}(r)| \le M \left(1 + |\beta^{\circ}_{\Gamma}(r)|\right) \quad \forall r \in D(\beta_{\Gamma}),$$
(2.8)

where β° and β_{Γ}° denote the minimal sections of the graphs β and β_{Γ} , respectively.

A3 $\pi, \pi_{\Gamma} : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with Lipschitz-constants L and L_{Γ} , respectively, and we set

$$\widehat{\pi}, \widehat{\pi}_{\Gamma} : \mathbb{R} \to \mathbb{R}, \qquad \widehat{\pi}(r) := \int_0^r \pi(s) \, ds, \quad \widehat{\pi}_{\Gamma}(r) := \int_0^r \pi_{\Gamma}(s) \, ds, \quad r \in \mathbb{R}.$$

We define for convenience of notation $\boldsymbol{\pi} := (\pi, \pi_{\Gamma}) : \mathbb{R}^2 \to \mathbb{R}^2$.

2.1 Variational setting

We describe here the variational setting that we consider and the concept of weak solution for the problem (2.1)-(2.7).

We define the functional spaces

$$\begin{aligned} H &:= L^{2}(\Omega), \quad V &:= H^{1}(\Omega), \quad W &:= H^{2}(\Omega), \\ H_{\Gamma} &:= L^{2}(\Gamma), \quad Z_{\Gamma} &:= H^{1/2}(\Gamma), \quad V_{\Gamma} &:= H^{1}(\Gamma), \quad W_{\Gamma} &:= H^{2}(\Gamma), \end{aligned}$$

endowed with their natural norms $\|\cdot\|_{H}$, $\|\cdot\|_{V}$, $\|\cdot\|_{W}$, $\|\cdot\|_{H_{\Gamma}}$, $\|\cdot\|_{Z_{\Gamma}}$, $\|\cdot\|_{V_{\Gamma}}$, $\|\cdot\|_{W_{\Gamma}}$, and their scalar products $(\cdot, \cdot)_{H}$, $(\cdot, \cdot)_{V}$, $(\cdot, \cdot)_{W_{\Gamma}}$, $(\cdot, \cdot)_{Z_{\Gamma}}$, $(\cdot, \cdot)_{V_{\Gamma}}$, $(\cdot, \cdot)_{W_{\Gamma}}$. We will denote by $z_{|\Gamma}$ the trace of the generic element $z \in V$. Moreover, we set

$$\begin{split} \boldsymbol{H} &:= \boldsymbol{H} \times \boldsymbol{H}_{\Gamma}, \\ \boldsymbol{Z} &:= \left\{ (z, z_{\Gamma}) \in \boldsymbol{V} \times \boldsymbol{Z}_{\Gamma} : z_{\Gamma} = z_{|_{\Gamma}} \text{ a.e. on } \Gamma \right\}, \\ \boldsymbol{V} &:= \left\{ (z, z_{\Gamma}) \in \boldsymbol{V} \times \boldsymbol{V}_{\Gamma} : z_{\Gamma} = z_{|_{\Gamma}} \text{ a.e. on } \Gamma \right\}, \\ \boldsymbol{W} &:= \left\{ (z, z_{\Gamma}) \in \boldsymbol{W} \times \boldsymbol{W}_{\Gamma} : z_{\Gamma} = z_{|_{\Gamma}} \text{ a.e. on } \Gamma \right\}. \end{split}$$

Let us make clear now once and for all that we will use the bold notation $\boldsymbol{z} = (z, z_{\Gamma})$ for the generic element in \boldsymbol{H} . Note that if $\boldsymbol{z} \in \boldsymbol{H}$, then z_{Γ} is not necessarily the trace of z on the boundary: this is true only if at least $\boldsymbol{z} \in \boldsymbol{Z}$. Clearly, $\boldsymbol{H}, \boldsymbol{Z}, \boldsymbol{V}$, and \boldsymbol{W} are Hilbert spaces with respect to the scalar products

$$\begin{split} (\boldsymbol{w}, \boldsymbol{z})_{\boldsymbol{H}} &:= (u, z)_{H} + (u_{\Gamma}, z_{\Gamma})_{H_{\Gamma}}, & \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{H}, \\ (\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{Z}} &:= (u, z)_{V} + (u_{\Gamma}, z_{\Gamma})_{Z_{\Gamma}}, & \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{Z}, \\ (\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{V}} &:= (u, z)_{V} + (u_{\Gamma}, z_{\Gamma})_{V_{\Gamma}}, & \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{V}, \\ (\boldsymbol{u}, \boldsymbol{z})_{\boldsymbol{W}} &:= (u, z)_{W} + (u_{\Gamma}, z_{\Gamma})_{W_{\Gamma}}, & \boldsymbol{w}, \boldsymbol{z} \in \boldsymbol{W}, \end{split}$$

and the respective norms $\|\cdot\|_{H}$, $\|\cdot\|_{Z}$, $\|\cdot\|_{V}$, and $\|\cdot\|_{W}$. The Hilbert space H is identified to its dual through the Riesz isomorphism, so that we have the continuous and dense embeddings

$$W \hookrightarrow V \hookrightarrow Z \hookrightarrow H \hookrightarrow V^*,$$

where the inclusions $W \hookrightarrow V \hookrightarrow H$, $Z \hookrightarrow H$, and $H \hookrightarrow V^*$ are also compact.

We introduce the generalized "mean" operator $m: V^* \to \mathbb{R}$ as

$$m(\boldsymbol{z}) := \frac{1}{|\Omega| + |\Gamma|} \langle \boldsymbol{z}, \boldsymbol{1} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} = \frac{1}{|\Omega| + |\Gamma|} \big(\langle \boldsymbol{z}, \boldsymbol{1} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} + \langle \boldsymbol{z}_{\Gamma}, \boldsymbol{1} \rangle_{\boldsymbol{V}^*_{\Gamma}, \boldsymbol{V}_{\Gamma}} \big), \quad \boldsymbol{z} \in \boldsymbol{V}^*,$$

and define the subspace of null-mean elements as

$$\boldsymbol{H}_0 := \boldsymbol{H} \cap \ker(m), \qquad \boldsymbol{V}_0 := \boldsymbol{V} \cap \boldsymbol{H}_0, \qquad \boldsymbol{Z}_0 := \boldsymbol{Z} \cap \boldsymbol{H}_0,$$

endowed with the norms

$$egin{aligned} &\|oldsymbol{z}\|_{oldsymbol{H}_0} \coloneqq \|oldsymbol{z}\|_{oldsymbol{H}} \,, &oldsymbol{z} \in oldsymbol{H}, \ &\|oldsymbol{z}\|_{oldsymbol{V}_0} \coloneqq \left(\|
abla z\|_{oldsymbol{H}}^2 + \|
abla_{\Gamma} z_{\Gamma}\|_{H_{\Gamma}}^2
ight)^{1/2}, &oldsymbol{z} \in oldsymbol{V}_0 \end{aligned}$$

Let us recall the following Poincaré-type inequalities:

$$\exists C_p > 0: \quad \|\boldsymbol{z}\|_{\boldsymbol{V}} \le C_p \, \|\boldsymbol{z}\|_{\boldsymbol{V}_0} \quad \forall \, \boldsymbol{z} = (z, z_{\Gamma}) \in \boldsymbol{V}_0, \tag{2.9}$$

$$\exists C_p > 0: \quad \|z\|_V \le C_p \|\nabla z\|_H \quad \forall \, \boldsymbol{z} = (z, z_\Gamma) \in \boldsymbol{Z}_0.$$

$$(2.10)$$

For the proof of (2.9) the reader can refer to [10, Lem. A], while the proof of (2.10) is given in Lemma A.1 in the Appendix. These imply in particular that an equivalent norm in the space V is given by

$$\boldsymbol{z} \mapsto \left(\|\boldsymbol{z} - \boldsymbol{m}(\boldsymbol{z})\boldsymbol{1}\|_{\boldsymbol{V}_0}^2 + |\boldsymbol{m}(\boldsymbol{z})|^2 \right)^{1/2}, \quad \boldsymbol{z} \in \boldsymbol{V},$$
 (2.11)

while an equivalent norm in V is given by

$$z \mapsto \left(\|\nabla z\|_{H}^{2} + |m(z, z_{|\Gamma})|^{2} \right)^{1/2}, \quad z \in V.$$
 (2.12)

Moreover, we define the linear operator

$$\mathcal{L}: \mathbf{V} \to \mathbf{V}^*, \qquad \langle \mathcal{L} \boldsymbol{v}, \boldsymbol{z} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} := \int_{\Omega} \nabla v \cdot \nabla z + \int_{\Gamma} \nabla_{\Gamma} v_{\Gamma} \cdot \nabla_{\Gamma} z_{\Gamma}, \quad \boldsymbol{v}, \boldsymbol{z} \in \boldsymbol{V},$$

and note that $\mathcal{L}\mathbf{1} = \mathbf{0}$ in \mathbf{V}^* : hence, since $\mathbf{V} = \mathbf{V}_0 \oplus \operatorname{span}{\mathbf{1}}$, we have that the restriction of \mathcal{L} to \mathbf{V}_0 is injective and with range

$$\boldsymbol{V}_{0,*} := \mathcal{L}(\boldsymbol{V}_0) = \{ \boldsymbol{z} \in \boldsymbol{V}^* : m(\boldsymbol{z}) = 0 \}.$$

Consequently, $\mathcal{L} : \mathbf{V}_0 \to \mathbf{V}_{0,*}$ is a linear isomorphism with well-defined inverse $\mathcal{L}^{-1} : \mathbf{V}_{0,*} \to \mathbf{V}_0$. With this notation, we introduce the norm

$$\|\boldsymbol{z}\|_{*} := \left(\|\mathcal{L}^{-1}(\boldsymbol{z} - m(\boldsymbol{z}))\|_{\boldsymbol{V}_{0}}^{2} + |m(\boldsymbol{z})|^{2}\right)^{1/2}, \qquad \boldsymbol{z} \in \boldsymbol{V}^{*},$$

which is equivalent to the usual dual norm on V^* and satisfies

$$\langle \partial_t \boldsymbol{z}, \mathcal{L}^{-1} \boldsymbol{z} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} = \frac{d}{dt} \frac{1}{2} \| \boldsymbol{z} \|_*^2 \qquad \forall \, \boldsymbol{z} \in H^1(0, T; \boldsymbol{V}_{0,*}).$$
 (2.13)

2.2 Concepts of solution

Let us precise here the concepts of variational (weak) solution for the system (2.1)–(2.7), in the cases $\delta \in (0, 1)$ and $\delta = 0$, respectively.

Definition 2.1 ($\delta > 0$). Let $\delta > 0$, and

$$\boldsymbol{u}_0^\delta \in \boldsymbol{V}, \qquad \boldsymbol{f}^\delta \in L^2(0,T;\boldsymbol{H}).$$

A weak solution to the problem (2.1)–(2.7) is a triplet $(\boldsymbol{u}^{\delta}, \boldsymbol{\mu}^{\delta}, \boldsymbol{\xi}^{\delta})$, with

$$\begin{split} & \boldsymbol{u}^{\delta} \in H^{1}(0,T;\boldsymbol{V}^{*}) \cap L^{\infty}(0,T;\boldsymbol{V}) \cap L^{2}(0,T;\boldsymbol{W}), \\ & \boldsymbol{\mu}^{\delta} \in L^{2}(0,T;\boldsymbol{V}), \\ & \boldsymbol{\xi}^{\delta} \in L^{2}(0,T;\boldsymbol{H}), \end{split}$$

such that $\boldsymbol{u}^{\delta}(0) = \boldsymbol{u}_0^{\delta}$,

$$\langle \partial_t \boldsymbol{u}^{\delta}, \boldsymbol{z} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} + \int_{\Omega} \nabla \mu^{\delta} \cdot \nabla \boldsymbol{z} + \int_{\Gamma} \nabla_{\Gamma} \mu_{\Gamma}^{\delta} \cdot \nabla_{\Gamma} \boldsymbol{z}_{\Gamma} = 0 \qquad \forall \, \boldsymbol{z} \in \boldsymbol{V},$$
(2.14)

$$(\boldsymbol{\mu}^{\delta}, \boldsymbol{z})_{\boldsymbol{H}} = \int_{\Omega} \nabla u^{\delta} \cdot \nabla z + \delta \int_{\Gamma} \nabla_{\Gamma} u^{\delta}_{\Gamma} \cdot \nabla_{\Gamma} z_{\Gamma} + (\boldsymbol{\xi}^{\delta} + \boldsymbol{\pi}(\boldsymbol{u}^{\delta}) - \boldsymbol{f}^{\delta}, \boldsymbol{z})_{\boldsymbol{H}} \qquad \forall \, \boldsymbol{z} \in \boldsymbol{V},$$
(2.15)

almost everywhere in (0,T), and

$$\xi^{\delta} \in \beta(u^{\delta}) \qquad a.e. \ in \ Q, \tag{2.16}$$

$$\xi_{\Gamma}^{\delta} \in \beta_{\Gamma}(u_{\Gamma}^{\delta}) \qquad a.e. \ on \ \Sigma.$$
(2.17)

Definition 2.2 $(\delta = 0)$. Let $\delta = 0$, and

$$oldsymbol{u}_0 \in oldsymbol{Z}, \qquad oldsymbol{f} \in L^2(0,T;oldsymbol{H}).$$

A weak solution to the problem (2.1)–(2.7) is a triplet $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$, with

$$\begin{split} & u \in H^{1}(0,T; V^{*}) \cap L^{\infty}(0,T; Z), \qquad \Delta u \in L^{2}(0,T; H) \\ & \mu \in L^{2}(0,T; V), \\ & \xi \in L^{2}(0,T; H \times Z_{\Gamma}^{*}), \end{split}$$

such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$,

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{z} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} + \int_{\Omega} \nabla \mu \cdot \nabla z + \int_{\Gamma} \nabla_{\Gamma} \mu_{\Gamma} \cdot \nabla_{\Gamma} z_{\Gamma} = 0 \qquad \forall \, \boldsymbol{z} \in \boldsymbol{V}, \quad (2.18)$$

$$(\boldsymbol{\mu}, \boldsymbol{z})_{\boldsymbol{H}} = \int_{\Omega} \nabla u \cdot \nabla z + (\xi, z)_{\boldsymbol{H}} + \langle \xi_{\Gamma}, z_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}} + (\boldsymbol{\pi}(\boldsymbol{u}) - \boldsymbol{f}, \boldsymbol{z})_{\boldsymbol{H}} \qquad \forall \, \boldsymbol{z} \in \boldsymbol{Z}, \quad (2.19)$$

almost everywhere in (0, T), and

$$\xi \in \beta(u) \qquad a.e. \ in \ Q, \tag{2.20}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{T} \int_{-\infty}^{T} \int_{-\infty}^{\infty} \int_{-\infty}$$

$$\int_{\Sigma} \widehat{\beta}_{\Gamma}(u_{\Gamma}) + \int_{0} \langle \xi_{\Gamma}, z_{\Gamma} - u_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}} \leq \int_{\Sigma} \widehat{\beta}_{\Gamma}(z_{\Gamma}) \quad \forall z_{\Gamma} \in L^{2}(0, T; Z_{\Gamma}),$$
(2.21)

where the last integral is intended to be $+\infty$ whenever $\widehat{\beta}_{\Gamma}(z_{\Gamma}) \notin L^{1}(\Sigma)$.

Remark 2.3. Let us point out that the variational equalities (2.14) and (2.18) can be formally obtained from the equations (2.1) and (2.4) multiplying by the generic pair $(z, z_{\Gamma}) \in \mathbf{V}$ and integrating by parts. As a matter of fact, the equations (2.14) and (2.18) actually provide a representation of the time derivative $\partial_t \boldsymbol{u}^{\delta}$ and $\partial_t \boldsymbol{u}$ as elements of the dual space $L^2(0, T; \mathbf{V}^*)$.

Remark 2.4. We note that the variational equation (2.15) entails

$$\mu^{\delta} = -\Delta u^{\delta} + \xi^{\delta} + \pi(u^{\delta}) - f^{\delta} \qquad \text{a.e. in } Q, \qquad (2.22)$$

$$\mu_{\Gamma}^{\delta} = \partial_{\nu} u^{\delta} - \delta \Delta_{\Gamma} u_{\Gamma}^{\delta} + \xi_{\Gamma}^{\delta} + \pi_{\Gamma} (u_{\Gamma}^{\delta}) - f_{\Gamma}^{\delta} \qquad \text{a.e on } \Sigma.$$
(2.23)

Indeed, (2.22) follows testing (2.15) by the generic pair (z, 0) with $z \in H_0^1(\Omega)$, integrating by parts, and using the regularity of u^{δ} . Then, due to the regularity of $\partial_{\nu} u^{\delta}$ and u_{Γ}^{δ} , the boundary condition (2.23) can be easily derived from (2.15) using (2.22).

Remark 2.5. In the same spirit, one can argue on (2.19) to deduce

$$\mu = -\Delta u + \xi + \pi(u) - f \qquad \text{a.e. in } Q, \qquad (2.24)$$

$$\mu_{\Gamma} = \partial_{\nu} u + \xi_{\Gamma} + \pi_{\Gamma}(u_{\Gamma}) - f_{\Gamma} \qquad \text{in } Z^*_{\Gamma}, \text{ a.e. in } (0, T).$$
(2.25)

Indeed, (2.24) follows as above testing (2.19) by (z, 0) with $z \in H_0^1(\Omega)$, using integration by parts, and the regularity of Δu . As for (2.25), since for almost every $t \in (0, T)$ it holds that $u(t) \in V$ and $\Delta u(t) \in H$, by [6, Thm. 2.27, p. 1.64] we have that $\partial_{\nu} u(t)$ is well-defined in $Z_{\Gamma}^* \cong H^{-1/2}(\Gamma)$. Hence, (2.25) can be deduced from (2.19) using (2.24).

Remark 2.6. Let us comment on condition (2.21). Whenever $\xi_{\Gamma} \in L^2(0, T; H_{\Gamma})$, it turns out that (2.21) is actually equivalent to the classical inclusion

$$\xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma})$$
 a.e. on Σ ,

or equivalently

$$\xi_{\Gamma} \in \partial I_{\Sigma}(u_{\Gamma}),$$

where

$$I_{\Sigma}: L^{2}(0,T; H_{\Gamma}) \to [0,+\infty], \qquad I_{\Sigma}(z_{\Gamma}) := \begin{cases} \int_{\Sigma} \widehat{\beta}_{\Gamma}(z_{\Gamma}) & \text{if } \widehat{\beta}_{\Gamma}(z_{\Gamma}) \in L^{1}(\Sigma), \\ +\infty & \text{otherwise.} \end{cases}$$

More generally, if we only have $\xi_{\Gamma} \in L^2(0,T; Z_{\Gamma}^*)$, then (2.21) means that

$$\xi_{\Gamma} \in \partial J_{\Sigma}(u_{\Gamma}),$$

where

$$J_{\Sigma}: L^{2}(0,T;Z_{\Gamma}) \to [0,+\infty], \qquad J_{\Sigma}(z_{\Gamma}) := \begin{cases} \int_{\Sigma} \widehat{\beta}_{\Gamma}(z_{\Gamma}) & \text{if } \widehat{\beta}_{\Gamma}(z_{\Gamma}) \in L^{1}(\Sigma), \\ +\infty & \text{otherwise.} \end{cases}$$

Here, the main point is that, since we are identifying H_{Γ} to its dual, the subdifferential ∂I_{Σ} is intended as a multivalued operator

$$\partial I_{\Sigma}: L^2(0,T;H_{\Gamma}) \to 2^{L^2(0,T;H_{\Gamma})}$$

while ∂J_{Σ} is seen as an operator

$$\partial J_{\Sigma}: L^2(0,T;Z_{\Gamma}) \to 2^{L^2(0,T;Z_{\Gamma}^*)}$$

For further details we refer to [1, 5].

2.3 Main results

Let us recall the well-posedness result for the system (2.1)–(2.7) when $\delta > 0$ is fixed: the reader can refer to [10, Thm. 2.1–2.2].

Theorem 2.7. Assume A1–A3, let $\delta > 0$ be fixed, and suppose that

$$\boldsymbol{u}_{0}^{\delta} \in \boldsymbol{V}, \qquad \widehat{\beta}(\boldsymbol{u}_{0}^{\delta}) \in L^{1}(\Omega), \qquad \widehat{\beta}_{\Gamma}(\boldsymbol{u}_{0\Gamma}^{\delta}) \in L^{1}(\Gamma), \qquad \boldsymbol{m}(\boldsymbol{u}_{0}^{\delta}) \in \operatorname{Int} D(\beta_{\Gamma}), \qquad (2.26)$$
$$\boldsymbol{f}^{\delta} := \boldsymbol{g}^{\delta} + \boldsymbol{h}^{\delta}, \qquad \boldsymbol{g}^{\delta} \in W^{1,1}(0,T;\boldsymbol{H}), \qquad \boldsymbol{h}^{\delta} \in L^{2}(0,T;\boldsymbol{V}). \qquad (2.27)$$

Then, there exists a weak solution $(\mathbf{u}^{\delta}, \boldsymbol{\mu}^{\delta}, \boldsymbol{\xi}^{\delta})$ of the system (2.1)–(2.7), in the sense of Definition 2.1. Moreover, there exists a constant $K_{\delta} > 0$ such that, for any data $\{(\mathbf{u}_{0,i}^{\delta}, \mathbf{f}_{i}^{\delta})\}_{i=1,2}$ satisfying (2.26)–(2.27) and $m(\mathbf{u}_{0,1}^{\delta}) = m(\mathbf{u}_{0,1}^{\delta})$, any respective weak solutions $\{(\mathbf{u}_{i}^{\delta}, \boldsymbol{\mu}_{i}^{\delta}, \boldsymbol{\xi}_{i}^{\delta})\}_{i=1,2}$ satisfy

$$\|\boldsymbol{u}_{1}^{\delta}-\boldsymbol{u}_{2}^{\delta}\|_{L^{\infty}(0,T;\boldsymbol{V}^{*})\cap L^{2}(0,T;\boldsymbol{V})} \leq K_{\delta}\left(\|\boldsymbol{u}_{0,1}^{\delta}-\boldsymbol{u}_{0,2}^{\delta}\|_{\boldsymbol{V}^{*}}+\|\boldsymbol{f}_{1}^{\delta}-\boldsymbol{f}_{2}^{\delta}\|_{L^{2}(0,T;\boldsymbol{V}^{*})}\right).$$

In particular, the solution components \mathbf{u}^{δ} and $\boldsymbol{\mu}^{\delta} - \boldsymbol{\xi}^{\delta}$ are unique. If also β is single-valued then the whole triplet $(\mathbf{u}^{\delta}, \boldsymbol{\mu}^{\delta}, \boldsymbol{\xi}^{\delta})$ is unique as well.

We are now ready to state our main results.

Theorem 2.8. Assume A1–A3 and let

$$\boldsymbol{u}_{0} \in \boldsymbol{Z}, \qquad \widehat{\boldsymbol{\beta}}(\boldsymbol{u}_{0}) \in L^{1}(\Omega), \qquad \widehat{\boldsymbol{\beta}}_{\Gamma}(\boldsymbol{u}_{0\Gamma}) \in L^{1}(\Gamma), \qquad \boldsymbol{m}(\boldsymbol{u}_{0}) \in \operatorname{Int} D(\boldsymbol{\beta}_{\Gamma}), \qquad (2.28)$$
$$\boldsymbol{f} := \boldsymbol{g} + \boldsymbol{h}, \qquad \boldsymbol{g} \in W^{1,1}(0,T;\boldsymbol{H}), \qquad \boldsymbol{h} \in L^{2}(0,T;\boldsymbol{V}). \qquad (2.29)$$

Consider a family of data $\{(\boldsymbol{u}_0^{\delta}, \boldsymbol{f}^{\delta})\}_{\delta \in (0,1)}$ which satisfy assumptions (2.26)–(2.27) and denote by $\{(\boldsymbol{u}^{\delta}, \boldsymbol{\mu}^{\delta}, \boldsymbol{\xi}^{\delta})\}_{\delta \in (0,1)}$ the respective weak solutions of the system (2.1)–(2.7) given by Theorem 2.1. Suppose also that there exists a constant $M_0 > 0$ such that

$$\delta \|\nabla_{\Gamma} u_{0\Gamma}^{\delta}\|_{H_{\Gamma}}^{2} + \|\widehat{\beta}(u_{0}^{\delta})\|_{L^{1}(\Omega)} + \|\widehat{\beta}_{\Gamma}(u_{0\Gamma}^{\delta})\|_{L^{1}(\Gamma)} \le M_{0} \qquad \forall \delta \in (0,1),$$
(2.30)

$$\|\boldsymbol{g}^{\delta}\|_{W^{1,1}(0,T;\boldsymbol{H})} + \|\boldsymbol{h}^{\delta}\|_{L^{2}(0,T;\boldsymbol{V})} \le M_{0} \qquad \forall \, \delta \in (0,1),$$
(2.31)

and that, as $\delta \to 0$,

$$\boldsymbol{u}_0^{\delta} \rightharpoonup \boldsymbol{u}_0 \quad in \ \boldsymbol{Z}, \qquad \boldsymbol{f}^{\delta} \rightharpoonup \boldsymbol{f} \quad in \ L^2(0,T;\boldsymbol{H}).$$
 (2.32)

Then, there exists a weak solution $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$ of the system (2.1)–(2.7) with $\delta = 0$ in the sense of Definition 2.2, such that, as $\delta \to 0$,

$$\boldsymbol{u}^{\delta} \to \boldsymbol{u} \quad in \ C^0([0,T]; \boldsymbol{H}),$$

$$(2.33)$$

$$\boldsymbol{u}^{\delta} \stackrel{*}{\rightharpoonup} \boldsymbol{u} \quad in \ H^1(0,T; \boldsymbol{V}^*) \cap L^{\infty}(0,T; \boldsymbol{Z}), \tag{2.34}$$

$$\Delta u^{\delta} \rightharpoonup \Delta u \quad in \ L^2(0,T;H), \tag{2.35}$$

$$\boldsymbol{\mu}^{\delta} \rightharpoonup \boldsymbol{\mu} \quad in \ L^2(0, T; \boldsymbol{V}), \tag{2.36}$$

$$\boldsymbol{\xi}^{\delta} \rightharpoonup \boldsymbol{\xi} \quad in \ L^2(0, T; H \times V_{\Gamma}^*), \tag{2.37}$$

$$-\delta\Delta_{\Gamma}u^{\delta}_{\Gamma} + \xi^{\delta}_{\Gamma} \rightharpoonup \xi_{\Gamma} \quad in \ L^2(0,T;Z^*_{\Gamma}), \tag{2.38}$$

$$\delta \boldsymbol{u}^{\delta} \to \boldsymbol{0} \quad in \ L^{\infty}(0,T; \boldsymbol{V}).$$
 (2.39)

Moreover, there exists a constant K > 0 such that, for any data $\{(\boldsymbol{u}_{0,i}, \boldsymbol{f}_i)\}_{i=1,2}$ satisfying (2.28)-(2.29) and $m(\boldsymbol{u}_{0,1}) = m(\boldsymbol{u}_{0,1})$, any respective weak solutions $\{(\boldsymbol{u}_i, \boldsymbol{\mu}_i, \boldsymbol{\xi}_i)\}_{i=1,2}$ of (2.1)-(2.7) with $\delta = 0$ in the sense of Definition 2.2 satisfy

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^{\infty}(0,T;\boldsymbol{V}^*) \cap L^2(0,T;\boldsymbol{Z})} \leq K \left(\|\boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}\|_{\boldsymbol{V}^*} + \|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{L^2(0,T;\boldsymbol{H})} \right).$$

In particular, the solution components \boldsymbol{u} and $\boldsymbol{\mu} - \boldsymbol{\xi}$ are unique. If also β is single-valued then the whole triplet $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$ is unique as well.

Remark 2.9. The existence of an approximating sequence $\{\boldsymbol{u}_0^{\delta}\}_{\delta \in (0,1)}$ satisfying (2.30) is discussed in the Appendix under the additional assumption (2.40) specified below. In the general case, we point out that the most natural choice for $\{\boldsymbol{u}_0^{\delta}\}_{\delta \in (0,1)}$ is given by the constant sequence \boldsymbol{u}_0 in the case $\boldsymbol{u}_0 \in \boldsymbol{V}$. Similarly, a typical choice for $\{\boldsymbol{f}^{\delta}\}_{\delta \in (0,1)}$ is the constant one \boldsymbol{f} .

Theorem 2.10. In the setting of Theorem 2.8, suppose also that

$$D(\beta) = D(\beta_{\Gamma}), \quad \text{there exists a constant } M \ge 1 \text{ such that}$$
$$\frac{1}{M} |\beta_{\Gamma}^{\circ}(r)| - M \le |\beta^{\circ}(r)| \le M(|\beta_{\Gamma}^{\circ}(r)| + 1) \quad \forall r \in D(\beta). \tag{2.40}$$

Then, the limiting triplet $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$ obtained in Theorem 2.8 also satisfies

$$u \in L^{2}(0,T; H^{3/2}(\Omega)), \qquad u_{\Gamma} \in L^{2}(0,T; V_{\Gamma}), \qquad \partial_{\nu} u \in L^{2}(0,T; H_{\Gamma}), \\ \xi_{\Gamma} \in L^{2}(0,T; H_{\Gamma}), \qquad \xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma}) \quad a.e. \text{ in } \Sigma,$$

and, in addition to (2.33)–(2.39), the following convergences hold:

$$\begin{split} \boldsymbol{\xi}^{\delta} &\rightharpoonup \boldsymbol{\xi} \quad in \ L^{2}(0,T;\boldsymbol{H}), \\ \delta u_{\Gamma}^{\delta} &\rightharpoonup 0 \quad in \ L^{2}(0,T;H^{3/2}(\Gamma)), \\ \partial_{\nu}u^{\delta} - \delta \Delta_{\Gamma} u_{\Gamma}^{\delta} &\rightharpoonup \partial_{\nu} u \quad in \ L^{2}(0,T;H_{\Gamma}), \end{split}$$

In particular, (2.25) entails

$$\mu_{\Gamma} = \partial_{\nu} u + \xi_{\Gamma} + \pi_{\Gamma}(u_{\Gamma}) - f_{\Gamma} \qquad a.e. \ on \ \Sigma.$$
(2.41)

Remark 2.11. We note that the additional regularity in Theorem 2.10 in particular implies that $\partial_{\nu} u, \xi_{\Gamma}$ are well-defined in H_{Γ} , almost everywhere in (0, T). Consequently, equation (2.25) holds not only in Z_{Γ}^* , but also in H_{Γ} , almost everywhere in (0, T), and this directly implies the validity of (2.41).

Theorem 2.12. In the setting of Theorem 2.10, assume that $m(\mathbf{u}_0^{\delta}) = m(\mathbf{u}_0)$ for all $\delta \in (0, 1)$. Then, there exists a constant C > 0, independent of δ , such that

$$\|\boldsymbol{u}^{\delta} - \boldsymbol{u}\|_{L^{\infty}(0,T;\boldsymbol{V}^{*}) \cap L^{2}(0,T;\boldsymbol{Z})} \leq C \left(\delta^{1/2} + \|\boldsymbol{u}_{0}^{\delta} - \boldsymbol{u}_{0}\|_{\boldsymbol{V}^{*}} + \|\boldsymbol{f}^{\delta} - \boldsymbol{f}\|_{L^{2}(0,T;\boldsymbol{H})}\right)$$

for every $\delta \in (0, 1)$ and, as $\delta \searrow 0$,

$$\boldsymbol{u}^{\delta} \rightharpoonup \boldsymbol{u} \qquad in \ L^2(0,T;\boldsymbol{V})$$

In particular, if

$$\|\boldsymbol{u}_0^{\delta} - \boldsymbol{u}_0\|_{\boldsymbol{V}^*} + \|\boldsymbol{f}^{\delta} - \boldsymbol{f}\|_{L^2(0,T;\boldsymbol{H})} = O(\delta^{1/2}) \quad as \ \delta \searrow 0,$$

then

$$\|\boldsymbol{u}^{\delta} - \boldsymbol{u}\|_{L^{\infty}(0,T;\boldsymbol{V}^{*}) \cap L^{2}(0,T;\boldsymbol{Z})} = O(\delta^{1/2}) \qquad as \ \delta \searrow 0.$$

3 Proofs

This section is devoted to proving Theorems 2.8, 2.10, and 2.12.

3.1 Uniform estimates

First estimate. Testing equation (2.14) by $1/(|\Omega| + |\Gamma|)$ and integrating in time we get

$$m(\boldsymbol{u}^{\delta}(t)) = m(\boldsymbol{u}_{0}^{\delta}) \qquad \forall t \in [0, T].$$
(3.1)

By assumption (2.32) on the initial data $\{\boldsymbol{u}_0^{\delta}\}_{\delta \in (0,1)}$, it holds that $m(\boldsymbol{u}_0^{\delta}) \to m(\boldsymbol{u}_0)$ as $\delta \searrow 0$. Hence, from (2.26), (2.28), and assumption **A2** it follows that

$$\exists [a,b] \subset \operatorname{Int} D(\beta_{\Gamma}) \subseteq \operatorname{Int} D(\beta) : \quad m(\boldsymbol{u}_0^{\delta}) \in [a,b] \quad \forall \, \delta \in (0,1).$$
(3.2)

We deduce that there exists a constant C > 0, independent of δ , such that

$$\|m(\boldsymbol{u}^{\delta})\|_{L^{\infty}(0,T)} \le C.$$
(3.3)

In the same spirit, equation (2.14) directly implies by comparison and the Schwarz inequality that

$$\|\partial_t \boldsymbol{u}^{\delta}(t)\|_{\boldsymbol{V}^*} \le \|\nabla \mu^{\delta}(t)\|_H + \|\nabla_{\Gamma} \mu_{\Gamma}(t)\|_{H_{\Gamma}} \quad \text{for a.e. } t \in (0, T).$$
(3.4)

Second estimate. We first note that by (3.1) we have that $\boldsymbol{u}^{\delta} - m(\boldsymbol{u}_0^{\delta}) \mathbf{1} \in \boldsymbol{V}_0$ in [0,T]. Hence, we can test equation (2.14) by $\mathcal{L}^{-1}(\boldsymbol{u}^{\delta} - m(\boldsymbol{u}_0^{\delta})\mathbf{1})$, equation (2.15) by

 $-(\boldsymbol{u}^{\delta} - m(\boldsymbol{u}_0^{\delta})\mathbf{1})$, and sum. By doing this, we note that there is a cancellation of two terms since for $\boldsymbol{z} = \mathcal{L}^{-1}(\boldsymbol{u}^{\delta} - m(\boldsymbol{u}_0^{\delta})\mathbf{1})$ we have

$$\begin{split} &\int_{\Omega} \nabla \mu^{\delta} \cdot \nabla z + \int_{\Gamma} \nabla_{\Gamma} \mu_{\Gamma}^{\delta} \cdot \nabla_{\Gamma} z_{\Gamma} \\ &= \int_{\Omega} \nabla (\mu^{\delta} - m(\boldsymbol{\mu}^{\delta})) \cdot \nabla z + \int_{\Gamma} \nabla_{\Gamma} (\mu_{\Gamma}^{\delta} - m(\boldsymbol{\mu}^{\delta})) \cdot \nabla_{\Gamma} z_{\Gamma} \\ &= \langle \mathcal{L}(\boldsymbol{\mu}^{\delta} - m(\boldsymbol{\mu}^{\delta})\mathbf{1}), \boldsymbol{z} \rangle_{\boldsymbol{V}^{*}, \boldsymbol{V}} = (\boldsymbol{\mu}^{\delta} - m(\boldsymbol{\mu}^{\delta})\mathbf{1}, \boldsymbol{u}^{\delta} - m(\boldsymbol{u}^{\delta})\mathbf{1})_{\boldsymbol{H}} \\ &= (\boldsymbol{\mu}^{\delta}, \boldsymbol{u}^{\delta} - m(\boldsymbol{u}^{\delta})\mathbf{1})_{\boldsymbol{H}}. \end{split}$$

Hence, we obtain

$$\langle \partial_t \boldsymbol{u}^{\delta}, \mathcal{L}^{-1}(\boldsymbol{u}^{\delta} - m(\boldsymbol{u}_0^{\delta})\boldsymbol{1}) \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + \int_{\Omega} |\nabla u^{\delta}|^2 + \delta \int_{\Gamma} |\nabla_{\Gamma} u^{\delta}_{\Gamma}|^2$$
$$+ \int_{\Omega} \xi^{\delta}(u^{\delta} - m(\boldsymbol{u}_0^{\delta})) + \int_{\Omega} \xi^{\delta}_{\Gamma}(u^{\delta}_{\Gamma} - m(\boldsymbol{u}_0^{\delta}))$$
$$= (\boldsymbol{f}^{\delta} - \boldsymbol{\pi}(\boldsymbol{u}^{\delta}), \boldsymbol{u}^{\delta} - m(\boldsymbol{u}_0^{\delta})\boldsymbol{1})_{\boldsymbol{H}}.$$

Thanks to the remark (3.2) we can use the inequalities devised by Miranville and Zelik [32] (a proof can be checked also in [28, § 5]) to infer that there is $C_{MZ} > 0$ such that

$$\int_{\Omega} \xi^{\delta}(u^{\delta} - m(\boldsymbol{u}_{0}^{\delta})) + \int_{\Gamma} \xi^{\delta}_{\Gamma}(u^{\delta}_{\Gamma} - m(\boldsymbol{u}_{0}^{\delta})) \ge C_{MZ} \left(\|\xi^{\delta}\|_{L^{1}(\Omega)} + \|\xi^{\delta}_{\Gamma}\|_{L^{1}(\Gamma)} \right) - C$$

almost everywhere in (0, T), from which we obtain

$$\left\langle \partial_{t} \boldsymbol{u}^{\delta}, \mathcal{L}^{-1}(\boldsymbol{u}^{\delta} - m(\boldsymbol{u}_{0}^{\delta})\boldsymbol{1}) \right\rangle_{\boldsymbol{V}^{*},\boldsymbol{V}} + \int_{\Omega} |\nabla u^{\delta}|^{2} + \delta \int_{\Gamma} |\nabla_{\Gamma} u_{\Gamma}^{\delta}|^{2} + C_{MZ} \left(\|\xi^{\delta}\|_{L^{1}(\Omega)} + \|\xi_{\Gamma}^{\delta}\|_{L^{1}(\Gamma)} \right) \leq C + (\boldsymbol{f}^{\delta} - \boldsymbol{\pi}(\boldsymbol{u}^{\delta}), \boldsymbol{u}^{\delta} - m(\boldsymbol{u}_{0}^{\delta})\boldsymbol{1})_{\boldsymbol{H}}.$$

$$(3.5)$$

We now integrate (3.5) in time using the chain rule (2.13): recalling also (3.1) and (3.3) and adding $|m(\boldsymbol{u}^{\delta}(t))|^2$ to both sides, we easily obtain

$$\begin{split} |m(\boldsymbol{u}^{\delta}(t))|^{2} &+ \frac{1}{2} \|\boldsymbol{u}^{\delta}(t) - m(\boldsymbol{u}^{\delta}(t))\|_{*}^{2} + \int_{Q_{t}} |\nabla u^{\delta}|^{2} + \delta \int_{\Sigma_{t}} |\nabla_{\Gamma} u^{\delta}_{\Gamma}|^{2} \\ &+ C_{MZ} \left(\int_{Q_{t}} |\xi^{\delta}| + \int_{\Sigma_{t}} |\xi^{\delta}_{\Gamma}| \right) \\ &\leq C + \frac{1}{2} \|u^{\delta}_{0} - m(\boldsymbol{u}^{\delta}_{0})\|_{*}^{2} + \int_{0}^{t} (\boldsymbol{f}^{\delta} - \boldsymbol{\pi}(\boldsymbol{u}^{\delta}), \boldsymbol{u}^{\delta} - m(\boldsymbol{u}^{\delta}_{0})\boldsymbol{1})_{\boldsymbol{H}}. \end{split}$$

Thanks to the Poincaré inequality (2.10), the Hölder and Young inequalities together with the Lipschitz continuity of π we infer that

$$\begin{aligned} \|\boldsymbol{u}^{\delta}(t)\|_{*}^{2} &+ \int_{0}^{t} \|\boldsymbol{u}^{\delta}(s)\|_{V}^{2} \, ds + \delta \int_{\Sigma_{t}} |\nabla_{\Gamma} \boldsymbol{u}_{\Gamma}^{\delta}|^{2} \\ &\leq C \left(1 + \|\boldsymbol{f}^{\delta}\|_{L^{2}(0,T;\boldsymbol{H})}^{2} + \int_{0}^{t} \|\boldsymbol{u}^{\delta}(s)\|_{\boldsymbol{H}}^{2} \, ds\right) \qquad \forall t \in [0,T]. \end{aligned}$$

At this point, since $Z \hookrightarrow H$ with compact embedding, the following Ehrling lemma holds:

$$\forall \varepsilon > 0, \quad \exists C_{\varepsilon} > 0: \quad \|\boldsymbol{z}\|_{\boldsymbol{H}}^{2} \leq \varepsilon \|\boldsymbol{z}\|_{\boldsymbol{Z}}^{2} + C_{\varepsilon} \|\boldsymbol{z}\|_{\boldsymbol{V}^{*}}^{2} \quad \forall \boldsymbol{z} \in \boldsymbol{Z}.$$
(3.6)

Noting that by the trace theorems it holds $\|\boldsymbol{z}\|_{\boldsymbol{Z}} \leq C \|\boldsymbol{z}\|_{V}$ for every $\boldsymbol{z} = (z, z_{\Gamma}) \in \boldsymbol{Z}$, applying this inequality on the right-hand side and taking also assumption (2.31) into account we infer that, for every $\varepsilon > 0$,

$$\begin{aligned} \|\boldsymbol{u}^{\delta}(t)\|_{*}^{2} &+ \int_{0}^{t} \|\boldsymbol{u}^{\delta}(s)\|_{V}^{2} \, ds + \delta \int_{\Sigma_{t}} |\nabla_{\Gamma} \boldsymbol{u}_{\Gamma}^{\delta}|^{2} \\ &\leq C + \varepsilon \int_{0}^{t} \|\boldsymbol{u}^{\delta}(s)\|_{V}^{2} \, ds + C_{\varepsilon} \int_{0}^{t} \|\boldsymbol{u}^{\delta}(s)\|_{*}^{2} \, ds \qquad \forall t \in [0,T]. \end{aligned}$$

Choosing for example $\varepsilon = 1/2$ and rearranging the terms, an application of the Gronwall lemma yields

$$\|\boldsymbol{u}^{\delta}\|_{L^{\infty}(0,T;\boldsymbol{V}^{*})\cap L^{2}(0,T;\boldsymbol{Z})} \leq C.$$
(3.7)

Third estimate. We proceed now in a formal but perhaps more explicative way, referring to [10] for a rigorous approach. Testing (2.14) by $\boldsymbol{\mu}^{\delta}$, (2.15) by $-\partial_t \boldsymbol{u}^{\delta}$, summing, and integrating, we obtain the formal energy inequality

$$\begin{split} &\int_{Q_t} |\nabla \mu^{\delta}|^2 + \int_{\Sigma_t} |\nabla_{\Gamma} \mu^{\delta}_{\Gamma}|^2 + \frac{1}{2} \int_{\Omega} |\nabla u^{\delta}(t)|^2 + \frac{\delta}{2} \int_{\Gamma} |\nabla_{\Gamma} u^{\delta}_{\Gamma}(t)|^2 \\ &\quad + \int_{\Omega} (\widehat{\beta} + \widehat{\pi}) (u^{\delta}(t)) + \int_{\Gamma} (\widehat{\beta}_{\Gamma} + \widehat{\pi}_{\Gamma}) (u^{\delta}_{\Gamma}(t)) \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u^{\delta}_{0}|^2 + \frac{\delta}{2} \int_{\Gamma} |\nabla_{\Gamma} u^{\delta}_{0\Gamma}|^2 + \int_{\Omega} (\widehat{\beta} + \widehat{\pi}) (u^{\delta}_{0}) + \int_{\Gamma} (\widehat{\beta}_{\Gamma} + \widehat{\pi}_{\Gamma}) (u^{\delta}_{0\Gamma}) + \int_{0}^{t} (\partial_{t} \boldsymbol{u}^{\delta}, \boldsymbol{f}^{\delta})_{\boldsymbol{H}}. \end{split}$$

Let us stress that this inequality is only formal, since that the last term is not well-defined in general for the regularities of $\partial_t u^{\delta}$ and f^{δ} . Nonetheless, we can give rigorous sense to it by exploiting the representation $f^{\delta} = g^{\delta} + h^{\delta}$ and using integration by parts in time: indeed, we formally have

$$\int_0^t (\partial_t \boldsymbol{u}^{\delta}, \boldsymbol{f}^{\delta})_{\boldsymbol{H}} = (\boldsymbol{u}^{\delta}(t), \boldsymbol{g}^{\delta}(t))_{\boldsymbol{H}} - (\boldsymbol{u}_0^{\delta}, \boldsymbol{g}^{\delta}(0))_{\boldsymbol{H}} - \int_0^t (\boldsymbol{u}^{\delta}, \partial_t \boldsymbol{g}^{\delta})_{\boldsymbol{H}} + \int_0^t \langle \partial_t \boldsymbol{u}^{\delta}, \boldsymbol{h}^{\delta} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}}.$$

It is then clear that all terms above make sense, and a classical argument based on suitable approximations of the problem at $\delta > 0$ fixed (see [10]) yields the rigorous estimate

$$\begin{split} &\int_{Q_t} |\nabla \mu^{\delta}|^2 + \int_{\Sigma_t} |\nabla_{\Gamma} \mu^{\delta}_{\Gamma}|^2 + \frac{1}{2} \int_{\Omega} |\nabla u^{\delta}(t)|^2 + \frac{\delta}{2} \int_{\Gamma} |\nabla_{\Gamma} u^{\delta}_{\Gamma}(t)|^2 \\ &\quad + \int_{\Omega} \widehat{\beta}(u^{\delta}(t)) + \int_{\Gamma} \widehat{\beta}_{\Gamma}(u^{\delta}_{\Gamma}(t)) \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u^{\delta}_{0}|^2 + \frac{\delta}{2} \int_{\Gamma} |\nabla_{\Gamma} u^{\delta}_{0\Gamma}|^2 + \int_{\Omega} (\widehat{\beta} + \widehat{\pi})(u^{\delta}_{0}) + \int_{\Gamma} (\widehat{\beta}_{\Gamma} + \widehat{\pi}_{\Gamma})(u^{\delta}_{0\Gamma}) \\ &\quad - \int_{\Omega} \widehat{\pi}(u^{\delta}(t)) - \int_{\Gamma} \widehat{\pi}_{\Gamma}(u^{\delta}_{\Gamma}(t)) \\ &\quad + (\boldsymbol{u}^{\delta}(t), \boldsymbol{g}^{\delta}(t))_{\boldsymbol{H}} - (\boldsymbol{u}^{\delta}_{0}, \boldsymbol{g}^{\delta}(0))_{\boldsymbol{H}} - \int_{0}^{t} (\boldsymbol{u}^{\delta}, \partial_{t} \boldsymbol{g}^{\delta})_{\boldsymbol{H}} + \int_{0}^{t} \langle \partial_{t} \boldsymbol{u}^{\delta}, \boldsymbol{h}^{\delta} \rangle_{\boldsymbol{V}^{*}, \boldsymbol{V}}. \end{split}$$

Summing now the estimate (3.3) and using the fact that (2.12) yields an equivalent norm in V, we obtain a control on the V-norm of $u^{\delta}(t)$ on the left-hand side. Observe also that $\hat{\beta}, \hat{\beta}_{\Gamma}$ are nonnegative and that $\hat{\pi}, \hat{\pi}_{\Gamma}$ are at most with quadratic growth since π, π_{Γ} are Lipschitz continuous. Consequently, by virtue of the bounds (2.30)–(2.31) on the data and the Young inequality we obtain that

$$\begin{split} &\int_{Q_t} |\nabla \mu^{\delta}|^2 + \int_{\Sigma_t} |\nabla_{\Gamma} \mu_{\Gamma}^{\delta}|^2 + \|u^{\delta}(t)\|_V^2 + \delta \int_{\Gamma} |\nabla_{\Gamma} u_{\Gamma}^{\delta}(t)|^2 \\ &\leq C \left(1 + \|\boldsymbol{u}^{\delta}(t)\|_{\boldsymbol{H}}^2 + \int_0^t \|\partial_t \boldsymbol{g}^{\delta}(s)\|_{\boldsymbol{H}} \|\boldsymbol{u}^{\delta}(s)\|_{\boldsymbol{H}} \, ds \right) + \frac{1}{4} \|\partial_t \boldsymbol{u}^{\delta}\|_{L^2(0,t;\boldsymbol{V}^*)}^2 \end{split}$$

for a certain constant C > 0 independent of δ . Now, recalling again that the classical trace theory implies that $\|\boldsymbol{z}\|_{\boldsymbol{Z}} \leq C \|\boldsymbol{z}\|_{V}$ for every $\boldsymbol{z} = (z, z_{\Gamma}) \in \boldsymbol{Z}$, by the Ehrling inequality (3.6) and the already proved estimate (3.7) we have, for all $\varepsilon > 0$,

$$\|\boldsymbol{u}^{\delta}(t)\|_{\boldsymbol{H}}^{2} \leq \varepsilon \|\boldsymbol{u}^{\delta}(t)\|_{V}^{2} + C_{\varepsilon} \qquad \forall t \in [0, T],$$

where C_{ε} is independent of t and δ . Hence, choosing ε small enough and rearranging the terms, in view also of the inequality (3.4) on the right-hand side, we infer that

$$\begin{split} &\int_{Q_t} |\nabla \mu^{\delta}|^2 + \int_{\Sigma_t} |\nabla_{\Gamma} \mu_{\Gamma}^{\delta}|^2 + \frac{1}{2} \|u^{\delta}(t)\|_V^2 + \delta \int_{\Gamma} |\nabla_{\Gamma} u_{\Gamma}^{\delta}(t)|^2 \\ &\leq C \left(1 + \int_0^t \|\partial_t \boldsymbol{g}^{\delta}(s)\|_{\boldsymbol{H}} \|u^{\delta}(s)\|_V \, ds \right) + \frac{1}{2} \int_{Q_t} |\nabla \mu^{\delta}|^2 + \frac{1}{2} \int_{\Sigma_t} |\nabla_{\Gamma} \mu_{\Gamma}^{\delta}|^2. \end{split}$$

Hence, rearranging the terms yields, thanks to the Gronwall lemma, that

$$\|\boldsymbol{u}^{\delta}\|_{L^{\infty}(0,T;\boldsymbol{Z})} + \delta^{1/2} \|\boldsymbol{u}^{\delta}\|_{L^{\infty}(0,T;\boldsymbol{V})} \le C, \qquad (3.8)$$

$$\|\nabla\mu^{\delta}\|_{L^{2}(0,T;H)} + \|\nabla_{\Gamma}\mu^{\delta}_{\Gamma}\|_{L^{2}(0,T;H_{\Gamma})} \le C.$$
(3.9)

Moreover, the estimate (3.4) implies also that

$$\|\partial_t \boldsymbol{u}^{\delta}\|_{L^2(0,T;\boldsymbol{V}^*)} \le C. \tag{3.10}$$

Fourth estimate. By comparison in (3.5) we have, almost everywhere on (0, T),

$$\|\xi^{\delta}\|_{L^{1}(\Omega)}+\|\xi^{\delta}_{\Gamma}\|_{L^{1}(\Gamma)}\leq C\left(1+\|\partial_{t}\boldsymbol{u}^{\delta}\|_{\boldsymbol{V}^{*}}+\|\boldsymbol{f}^{\delta}\|_{\boldsymbol{H}}\right)\left(1+\|\boldsymbol{u}^{\delta}\|_{L^{\infty}(0,T;\boldsymbol{H})}\right),$$

and the estimates (3.8)–(3.10) yield

$$\|\xi^{\delta}\|_{L^{2}(0,T;L^{1}(\Omega))} + \|\xi^{\delta}_{\Gamma}\|_{L^{2}(0,T;L^{1}(\Gamma))} \leq C.$$
(3.11)

Testing (2.15) by $1/(|\Omega| + |\Gamma|)$ and using the estimates (3.8), (3.11), together with the Lipschitz continuity of π , it follows that

$$\|m(\boldsymbol{\mu}^{\delta})\|_{L^{2}(0,T)} \le C,$$
 (3.12)

so that by (3.9) and the equivalent norm (2.11) in V we get

$$\|\boldsymbol{\mu}^{\delta}\|_{L^{2}(0,T;\boldsymbol{V})} \leq C.$$
(3.13)

Fifth estimate. The idea now is to test equation (2.15) by $(\xi^{\delta}, \xi^{\delta}_{|\Gamma})$: however, this is only formal due to the regularity of ξ^{δ} . To make it rigorous, we recall that from [10] the system (2.1)–(2.7) can be seen as limit as $\lambda \searrow 0$ of a suitable approximated system where β and β_{Γ} are replaced by their Yosida approximations β_{λ} and $\beta_{\Gamma,\lambda}$, with $\lambda \in (0, \lambda_0)$. In this case, ξ^{δ} is exactly the weak limit in $L^2(0, T; H)$ of the respective sequence $\beta_{\lambda}(u^{\delta}_{\lambda})$. At this level, the Lipschitz-continuity of β_{λ} yields the desired regularity $(\beta_{\lambda}(u^{\delta}_{\lambda}), \beta_{\lambda}(u^{\delta}_{\lambda,\Gamma})) \in \mathbf{V}$ almost everywhere in (0, T). Hence, testing the λ -regularised equation (2.15) by $(\beta_{\lambda}(u^{\delta}_{\lambda}), \beta_{\lambda}(u^{\delta}_{\lambda,\Gamma}))$ yields

$$\begin{split} \int_{Q_t} \beta_{\lambda}'(u_{\lambda}^{\delta}) |\nabla u_{\lambda}^{\delta}|^2 + \delta \int_{\Sigma_t} \beta_{\lambda}'(u_{\lambda,\Gamma}^{\delta}) |\nabla_{\Gamma} u_{\lambda,\Gamma}^{\delta}|^2 + \int_{Q_t} |\beta_{\lambda}(u_{\lambda}^{\delta})|^2 + \int_{\Sigma_t} \beta_{\lambda}(u_{\lambda,\Gamma}^{\delta}) \beta_{\Gamma,\lambda}(u_{\lambda,\Gamma}^{\delta}) \\ &= \int_0^t (\mu_{\lambda}^{\delta} - f^{\delta} - \pi(u_{\lambda}^{\delta}), \beta_{\lambda}(u_{\lambda}^{\delta}))_H + \int_0^t (\mu_{\lambda,\Gamma}^{\delta} - f_{\Gamma}^{\delta} - \pi_{\Gamma}(u_{\lambda,\Gamma}^{\delta}), \beta_{\lambda}(u_{\lambda,\Gamma}^{\delta}))_{H_{\Gamma}}. \end{split}$$

Now, we note that (2.8) yields an analogous inequality on the Yosida approximations β_{λ} and $\beta_{\Gamma,\lambda}$ (see for example [9]), from which we have the control from below

$$\int_{\Sigma_t} \beta_\lambda(u_{\lambda,\Gamma}^\delta) \beta_{\Gamma,\lambda}(u_{\lambda,\Gamma}^\delta) \ge \frac{1}{2M} \int_{\Sigma_t} |\beta_\lambda(u_{\lambda,\Gamma}^\delta)|^2 - C.$$

Consequently, by the monotonicity of β_{λ} , the Young inequality, the Lipschitz continuity of $\boldsymbol{\pi}$, and the estimate (3.11) we obtain, after rearranging the terms,

$$\|\beta_{\lambda}(u_{\lambda}^{\delta})\|_{L^{2}(0,T;H)}^{2} + \|\beta_{\lambda}(u_{\lambda,\Gamma}^{\delta})\|_{L^{2}(0,T;H_{\Gamma})}^{2} \leq C\left(1 + \|\boldsymbol{u}_{\lambda}^{\delta}\|_{L^{2}(0,T;\boldsymbol{H})}^{2}\right)$$

where C > 0 is independent of both δ and λ . Consequently, by the estimate (3.8) we obtain

$$\|\beta_{\lambda}(u_{\lambda}^{\delta})\|_{L^{2}(0,T;H)} + \|\beta_{\lambda}(u_{\lambda,\Gamma}^{\delta})\|_{L^{2}(0,T;H_{\Gamma})} \leq C \qquad \forall \lambda \in (0,\lambda_{0}),$$
(3.14)

from which it follows in particular, by weak lower semicontinuity as $\lambda \searrow 0$, that

$$\|\xi^o\|_{L^2(0,T;H)} \le C. \tag{3.15}$$

Now, in view of Remark 2.4, by comparison in equation (2.22) and the estimates just proved we have

$$\|\Delta u^{\delta}\|_{L^{2}(0,T;H)} \le C. \tag{3.16}$$

By the classical trace theorems [6, Thm. 2.27] and elliptic regularity [6, Thm. 3.2], the estimates (3.8) and (3.16) yield

$$\|\partial_{\nu} u^{\delta}\|_{L^{2}(0,T;Z_{\Gamma}^{*})} + \delta^{1/2} \|\partial_{\nu} u^{\delta}\|_{L^{2}(0,T;H_{\Gamma})} \leq C, \qquad (3.17)$$

so that by comparison in (2.23) and estimate (3.13) we infer that

$$\| -\delta\Delta_{\Gamma} u_{\Gamma}^{\delta} + \xi_{\Gamma}^{\delta} \|_{L^{2}(0,T;Z_{\Gamma}^{*})} \leq C.$$
(3.18)

Eventually, this implies together with (3.8) that

$$\delta^{1/2} \|\Delta_{\Gamma} u_{\Gamma}^{\delta}\|_{L^{2}(0,T;V_{\Gamma}^{*})} + \|\xi_{\Gamma}^{\delta}\|_{L^{2}(0,T;V_{\Gamma}^{*})} \leq C.$$
(3.19)

3.2 Passage to the limit

From the estimates (3.8)–(3.19) and weak and weak* compactness, we infer that there exists a triplet (u, μ, ξ) with

$$\begin{split} & \boldsymbol{u} \in H^{1}(0,T; \boldsymbol{V}^{*}) \cap L^{\infty}(0,T; \boldsymbol{Z}), \qquad \Delta u \in L^{2}(0,T; H) \\ & \boldsymbol{\mu} \in L^{2}(0,T; \boldsymbol{V}), \\ & \boldsymbol{\xi} \in L^{2}(0,T; H \times Z_{\Gamma}^{*}), \end{split}$$

such that, as $\delta \searrow 0$, on a possibly relabelled subsequence,

 $\boldsymbol{u}^{\delta} \stackrel{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in } H^1(0,T; \boldsymbol{V}^*) \cap L^{\infty}(0,T; \boldsymbol{Z}),$ (3.20)

$$\Delta u^{\delta} \rightharpoonup \Delta u \quad \text{in } L^2(0,T;H), \tag{3.21}$$

$$\boldsymbol{\mu}^{\delta} \rightharpoonup \boldsymbol{\mu} \quad \text{in } L^2(0,T;\boldsymbol{V}), \tag{3.22}$$

$$\boldsymbol{\xi}^{\delta} \rightharpoonup \boldsymbol{\xi} \quad \text{in } L^2(0, T; H \times V_{\Gamma}^*), \tag{3.23}$$

$$\delta \boldsymbol{u}^{\delta} \to \boldsymbol{0} \quad \text{in } L^{\infty}(0,T;\boldsymbol{V}),$$
(3.24)

$$-\delta\Delta_{\Gamma}u^{\delta}_{\Gamma} + \xi^{\delta}_{\Gamma} \rightharpoonup \xi_{\Gamma} \quad \text{in } L^2(0,T;Z^*_{\Gamma}).$$
(3.25)

In particular, by the Aubin–Lions and Simon compactness results (see e.g. [40, § 8, Cor. 4]), the compact inclusion $Z \hookrightarrow H$ implies that

$$\boldsymbol{u}^{\delta} \to \boldsymbol{u} \quad \text{in } C^0([0,T]; \boldsymbol{H}),$$
 (3.26)

which can be rewritten as

 $u^{\delta} \to u$ in $C^0([0,T];H), \quad u^{\delta}_{\Gamma} \to u_{\Gamma}$ in $C^0([0,T];H_{\Gamma}).$

Hence, the Lipschitz continuity of π and π_{Γ} implies also that

 $\pi(u^{\delta}) \to \pi(u)$ in $C^{0}([0,T];H), \qquad \pi_{\Gamma}(u^{\delta}_{\Gamma}) \to \pi_{\Gamma}(u_{\Gamma})$ in $C^{0}([0,T];H_{\Gamma}),$

and therefore

$$\boldsymbol{\pi}(\boldsymbol{u}^{\delta}) \to \boldsymbol{\pi}(\boldsymbol{u}) \quad \text{in } C^{0}([0,T];\boldsymbol{H}).$$
 (3.27)

Hence, passing to the weak limit in (2.14)-(2.15) yields exactly (2.18)-(2.19).

In order to conclude, we only need to prove conditions (2.20)–(2.21). To this end, the demi-closedness of the maximal monotone operator β yields $\xi \in \beta(u)$ almost everywhere in Q by the classical results in [1,5]. Moreover, testing (2.15) by \boldsymbol{u}^{δ} gives

$$\int_{Q} |\nabla u^{\delta}|^{2} + \delta \int_{\Sigma} |\nabla_{\Gamma} u^{\delta}_{\Gamma}|^{2} + \int_{Q} \xi^{\delta} u^{\delta} + \int_{\Sigma} \xi^{\delta}_{\Gamma} u^{\delta}_{\Gamma}$$
$$= \int_{Q} (\mu^{\delta} + f^{\delta} - \pi(u^{\delta})) u^{\delta} + \int_{\Sigma} (\mu^{\delta}_{\Gamma} + f^{\delta}_{\Gamma} - \pi_{\Gamma}(u^{\delta}_{\Gamma})) u^{\delta}_{\Gamma},$$

while testing (2.19) by \boldsymbol{u} gives

$$\int_{Q} |\nabla u|^{2} + \int_{Q} \xi u + \int_{0}^{T} \langle \xi_{\Gamma}, u_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}}$$
$$= \int_{Q} (\mu + f - \pi(u))u + \int_{\Sigma} (\mu_{\Gamma} + f_{\Gamma} - \pi_{\Gamma}(u_{\Gamma}))u_{\Gamma}.$$

By lower semicontinuity and weak-strong convergence we deduce that

$$\begin{split} \limsup_{\delta\searrow 0} \int_{\Sigma} \xi_{\Gamma}^{\delta} u_{\Gamma}^{\delta} &\leq \limsup_{\delta\searrow 0} \int_{Q} (\mu^{\delta} + f^{\delta} - \pi(u^{\delta})) u^{\delta} + \limsup_{\delta\searrow 0} \int_{\Sigma} (\mu_{\Gamma}^{\delta} + f_{\Gamma}^{\delta} - \pi_{\Gamma}(u_{\Gamma}^{\delta})) u_{\Gamma}^{\delta} \\ &- \liminf_{\delta\searrow 0} \int_{Q} |\nabla u^{\delta}|^{2} - \liminf_{\delta\searrow 0} \int_{Q} \xi^{\delta} u^{\delta} \\ &\leq \int_{Q} (\mu + f - \pi(u)) u + \int_{\Sigma} (\mu_{\Gamma} + f_{\Gamma} - \pi_{\Gamma}(u_{\Gamma})) u_{\Gamma} - \int_{Q} |\nabla u|^{2} - \int_{Q} \xi u, \end{split}$$

from which

$$\limsup_{\delta \searrow 0} \int_{\Sigma} \xi_{\Gamma}^{\delta} u_{\Gamma}^{\delta} \le \int_{0}^{T} \langle \xi_{\Gamma}, u_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}}.$$
(3.28)

Now, condition (2.17) and the subdifferential property $\beta_{\Gamma} = \partial \hat{\beta}_{\Gamma}$ yields

$$\int_{\Sigma} \widehat{\beta}_{\Gamma}(u_{\Gamma}^{\delta}) + \int_{\Sigma} \xi_{\Gamma}^{\delta}(z_{\Gamma} - u_{\Gamma}^{\delta}) \le \int_{\Sigma} \widehat{\beta}_{\Gamma}(z_{\Gamma}) \qquad \forall z_{\Gamma} \in L^{2}(0, T; H_{\Gamma}).$$

Choosing now $z_{\Gamma} \in L^2(0,T;V_{\Gamma})$, using the convergences (3.23) and (3.26), the weak lower semicontinuity of $\hat{\beta}_{\Gamma}$, and (3.28), we have

$$\begin{split} &\int_{\Sigma} \widehat{\beta}_{\Gamma}(u_{\Gamma}) \leq \liminf_{\delta \searrow 0} \int_{\Sigma} \widehat{\beta}_{\Gamma}(u_{\Gamma}^{\delta}), \\ &\int_{0}^{T} \langle \xi_{\Gamma}, z_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}} = \lim_{\delta \searrow 0} \int_{\Sigma} \xi_{\Gamma}^{\delta} z_{\Gamma}, \\ &- \int_{0}^{T} \langle \xi_{\Gamma}, u_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}} \leq -\limsup_{\delta \searrow 0} \int_{\Sigma} \xi_{\Gamma}^{\delta} u_{\Gamma}^{\delta} = \liminf_{\delta \searrow 0} \left(- \int_{\Sigma} \xi_{\Gamma}^{\delta} u_{\Gamma}^{\delta} \right) \end{split}$$

Hence, passing to the lim inf as $\delta \searrow 0$ we obtain

$$\int_{\Sigma} \widehat{\beta}_{\Gamma}(u_{\Gamma}) + \int_{0}^{T} \langle \xi_{\Gamma}, z_{\Gamma} - u_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}} \leq \int_{\Sigma} \widehat{\beta}_{\Gamma}(z_{\Gamma}) \qquad \forall z_{\Gamma} \in L^{2}(0, T; V_{\Gamma}).$$

We infer now that such inequality holds also for all $z_{\Gamma} \in L^2(0,T;Z_{\Gamma})$. Indeed, given an arbitrary $z_{\Gamma} \in L^2(0,T;Z_{\Gamma})$, for $\varepsilon > 0$ we can set $z_{\Gamma}^{\varepsilon} \in L^2(0,T;W_{\Gamma})$ as the unique solution to the elliptic problem

$$z_{\Gamma}^{\varepsilon} - \varepsilon \Delta_{\Gamma} z_{\Gamma}^{\varepsilon} = z_{\Gamma} \quad \text{on } \Sigma$$

Then, it is not difficult to show by standard testing techniques (see, e.g., [11, Lemma A.1]) that

$$z_{\Gamma}^{\varepsilon} \to z_{\Gamma} \quad \text{in } L^2(0,T;Z_{\Gamma}), \qquad \widehat{\beta}_{\Gamma}(z_{\Gamma}^{\varepsilon}) \le \widehat{\beta}_{\Gamma}(z_{\Gamma}) \quad \text{a.e. on } \Sigma,$$

so that letting $\varepsilon \to 0$ in the subdifferential relation we can conclude. This shows that $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$ is a weak solution to the system with $\delta = 0$ in the sense of Definition 2.2.

3.3 Continuous dependence

Let $\{(\boldsymbol{u}_i, \boldsymbol{\mu}_i, \boldsymbol{\xi}_i)\}_{i=1,2}$ be two weak solutions of the system (2.1)–(2.7) in the sense of Definition 2.2, with respect to the data $\{(\boldsymbol{u}_{0,i}, \boldsymbol{f}_i)\}_{i=1,2}$. Then, setting $\bar{\boldsymbol{u}} := \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\bar{\boldsymbol{\mu}} := \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$, $\bar{\boldsymbol{\xi}} := \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$, $\bar{\boldsymbol{u}}_0 := \boldsymbol{u}_{0,1} - \boldsymbol{u}_{0,2}$, and $\bar{\boldsymbol{f}} := \boldsymbol{f}_1 - \boldsymbol{f}_2$, it holds that

$$\langle \partial_t \bar{\boldsymbol{u}}, \boldsymbol{z} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} + \int_{\Omega} \nabla \bar{\mu} \cdot \nabla z + \int_{\Gamma} \nabla_{\Gamma} \bar{\mu}_{\Gamma} \cdot \nabla_{\Gamma} z_{\Gamma} = 0$$
 for every $\boldsymbol{z} \in \boldsymbol{V}$, a.e. in $(0, T)$, (3.29)

$$(\bar{\boldsymbol{\mu}}, \boldsymbol{z})_{\boldsymbol{H}} = \int_{\Omega} \nabla \bar{u} \cdot \nabla z + (\bar{\xi}, z)_{\boldsymbol{H}} + \langle \bar{\xi}_{\Gamma}, z_{\Gamma} \rangle_{Z_{\Gamma}^{*}, Z_{\Gamma}} + (\boldsymbol{\pi}(\boldsymbol{u}_{1}) - \boldsymbol{\pi}(\boldsymbol{u}_{2}) - \bar{\boldsymbol{f}}, \boldsymbol{z})_{\boldsymbol{H}}$$

for every $\boldsymbol{z} \in \boldsymbol{Z}$, a.e. in $(0, T)$. (3.30)

Recalling that $m(\boldsymbol{u}_{0,1}) = m(\boldsymbol{u}_{0,2})$, testing (3.29) by $1/(|\Omega| + |\Gamma|)$ it follows that

$$m(\bar{\boldsymbol{u}}(t)) = 0 \quad \forall t \in [0, T].$$
(3.31)

Consequently, we can test (3.29) by $\mathcal{L}^{-1}\bar{\boldsymbol{u}}$, (3.30) by $-\bar{\boldsymbol{u}}$, integrate in time, and add the respective equations. Noting that there is a cancellation (as pointed out in (3.5)), thanks to the chain rule (2.13) we obtain that

$$\frac{1}{2} \|\bar{\boldsymbol{u}}(t)\|_{*}^{2} + \int_{Q_{t}} |\nabla \bar{\boldsymbol{u}}|^{2} + \int_{Q_{t}} \bar{\boldsymbol{\xi}} \bar{\boldsymbol{u}} + \int_{0}^{t} \langle \bar{\boldsymbol{\xi}}_{\Gamma}, \bar{\boldsymbol{u}}_{\Gamma} \rangle = \frac{1}{2} \|\bar{\boldsymbol{u}}_{0}\|_{*}^{2} + \int_{0}^{t} (\bar{\boldsymbol{f}} + \boldsymbol{\pi}(\boldsymbol{u}_{2}) - \boldsymbol{\pi}(\boldsymbol{u}_{1}), \bar{\boldsymbol{u}})_{\boldsymbol{H}}.$$

Exploiting condition (3.31), the monotonicity of β and β_{Γ} , the Lipschitz continuity of π , and the Young inequality, we infer that

$$\|\bar{\boldsymbol{u}}(t)\|_{\boldsymbol{V}^{*}}^{2} + \int_{0}^{t} \|\bar{\boldsymbol{u}}\|_{\boldsymbol{Z}}^{2} \leq C\left(\|\bar{\boldsymbol{u}}_{0}\|_{\boldsymbol{V}^{*}}^{2} + \int_{0}^{t} \|\bar{\boldsymbol{f}}\|_{\boldsymbol{H}}^{2} + \int_{0}^{t} \|\bar{\boldsymbol{u}}\|_{\boldsymbol{H}}^{2}\right).$$

At this point, applying the Ehrling inequality (3.6) on the right-hand side, choosing $\varepsilon > 0$ sufficiently small, and rearranging the terms, we deduce, possibly renominating the constant C, that

$$\|\bar{\boldsymbol{u}}(t)\|_{\boldsymbol{V}^{*}}^{2} + \frac{1}{2}\int_{0}^{t}\|\bar{\boldsymbol{u}}\|_{\boldsymbol{Z}}^{2} \leq C\left(\|\bar{\boldsymbol{u}}_{0}\|_{\boldsymbol{V}^{*}}^{2} + \int_{0}^{t}\|\bar{\boldsymbol{f}}\|_{\boldsymbol{H}}^{2} + \int_{0}^{t}\|\bar{\boldsymbol{u}}\|_{\boldsymbol{V}^{*}}^{2}\right).$$

Then, the conclusion follows by applying the Gronwall lemma. The proof of Theorem 2.8 is thus complete.

3.4 Refined convergence

Here we prove Theorem 2.10. We show that the extra assumption (2.40) on the graphs yields additional estimates on the solutions.

First of all, since assumption (2.40) induces the analogous inequalities on the respective Yosida approximations (details are given in [13, Appendix]), the estimate (3.14) implies

$$\|\beta_{\lambda}(u_{\lambda}^{\delta})\|_{L^{2}(0,T;H)} + \|\beta_{\Gamma,\lambda}(u_{\lambda,\Gamma}^{\delta})\|_{L^{2}(0,T;H_{\Gamma})} \leq C \qquad \forall \lambda \in (0,\lambda_{0}),$$

from which, taking the limit as $\lambda \searrow 0$,

$$\|\xi^{\delta}\|_{L^{2}(0,T;H)} + \|\xi^{\delta}_{\Gamma}\|_{L^{2}(0,T;H_{\Gamma})} \le C.$$
(3.32)

Now, recalling Remark 2.4, by comparison in (2.23) and using the estimate (3.17) we have

$$\|\partial_{\nu}u^{\delta} - \delta\Delta_{\Gamma}u^{\delta}_{\Gamma}\|_{L^{2}(0,T;H_{\Gamma})} + \delta\|\Delta_{\Gamma}u^{\delta}_{\Gamma}\|_{L^{2}(0,T;Z_{\Gamma}^{*})} \leq C.$$
(3.33)

At this point, one can pass to the limit as $\delta \searrow 0$ as above by exploiting the additional estimates (3.32)–(3.33), which yield the extra regularities

$$\xi_{\Gamma} \in L^2(0,T;H_{\Gamma}), \qquad \partial_{\nu} u \in L^2(0,T;H_{\Gamma}).$$

By elliptic regularity (see [6, Thm. 3.2]) this implies that

$$u \in L^2(0, T; H^{3/2}(\Omega)),$$

hence also by the trace theory that

 $u_{\Gamma} \in L^2(0,T;V_{\Gamma}).$

Eventually, the pointwise inclusion $\xi_{\Gamma} \in \beta_{\Gamma}(u_{\Gamma})$ almost everywhere on Σ can be obtained arguing as in Remark 2.6. This concludes the proof of Theorem 2.10.

3.5 Error estimate

Here we prove Theorem 2.12. To this end, taking the difference of the variational formulations (2.14)–(2.15) and (2.18)–(2.19), we obtain, thanks to the additional regularity of ξ_{Γ} , that

$$\langle \partial_t (\boldsymbol{u}^{\delta} - \boldsymbol{u}), \boldsymbol{z} \rangle_{\boldsymbol{V}^*, \boldsymbol{V}} + \int_{\Omega} \nabla (\mu^{\delta} - \mu) \cdot \nabla z + \int_{\Gamma} \nabla_{\Gamma} (\mu^{\delta}_{\Gamma} - \mu_{\Gamma}) \cdot \nabla_{\Gamma} z_{\Gamma} = 0$$
(3.34)

and

$$(\boldsymbol{\mu}^{\delta} - \boldsymbol{\mu}, \boldsymbol{z})_{\boldsymbol{H}} = \int_{\Omega} \nabla (u^{\delta} - u) \cdot \nabla z + \delta \int_{\Gamma} \nabla_{\Gamma} u_{\Gamma}^{\delta} \cdot \nabla z_{\Gamma} + (\boldsymbol{\xi}^{\delta} - \boldsymbol{\xi} + \boldsymbol{\pi}(\boldsymbol{u}^{\delta}) - \boldsymbol{\pi}(\boldsymbol{u}) + \boldsymbol{f} - \boldsymbol{f}^{\delta}, \boldsymbol{z})_{\boldsymbol{H}}$$
(3.35)

for every $\boldsymbol{z} \in \boldsymbol{V}$, almost everywhere in (0, T). Now, since $m(\boldsymbol{u}_0^{\delta}) = m(\boldsymbol{u}_0)$ by assumption, testing equation (3.34) by $1/(|\Omega| + |\Gamma|)$ we infer that

$$m((\boldsymbol{u}^{\delta} - \boldsymbol{u})(t)) = 0 \quad \forall t \in [0, T].$$
(3.36)

Hence, one can test (3.34) by $\mathcal{L}^{-1}(\boldsymbol{u}^{\delta}-\boldsymbol{u})$, (3.35) by $-(\boldsymbol{u}^{\delta}-\boldsymbol{u})$, integrate in time, and add the respective equations: taking into account the usual cancellation of terms and (2.13), we obtain

$$\begin{aligned} &\frac{1}{2} \| (\boldsymbol{u}^{\delta} - \boldsymbol{u})(t) \|_{*}^{2} + \int_{Q_{t}} |\nabla(\boldsymbol{u}^{\delta} - \boldsymbol{u})|^{2} + \delta \int_{\Sigma_{t}} |\nabla_{\Gamma} \boldsymbol{u}_{\Gamma}^{\delta}|^{2} + \int_{0}^{t} (\boldsymbol{\xi}^{\delta} - \boldsymbol{\xi}, \boldsymbol{u}^{\delta} - \boldsymbol{u})_{\boldsymbol{H}} \\ &= \frac{1}{2} \| \boldsymbol{u}_{0}^{\delta} - \boldsymbol{u}_{0} \|_{*}^{2} + \delta \int_{\Sigma_{t}} \nabla_{\Gamma} \boldsymbol{u}_{\Gamma}^{\delta} \cdot \nabla_{\Gamma} \boldsymbol{u}_{\Gamma} + \int_{0}^{t} (\boldsymbol{f}^{\delta} - \boldsymbol{f} + \boldsymbol{\pi}(\boldsymbol{u}) - \boldsymbol{\pi}(\boldsymbol{u}^{\delta}), \boldsymbol{u}^{\delta} - \boldsymbol{u})_{\boldsymbol{H}}. \end{aligned}$$

At this point, taking condition (3.31) into account on the left-hand side together with the monotonicity of β and β_{Γ} , and using the Lipschitz continuity of π , and the Young inequality on the right-hand side, we infer that

$$\begin{aligned} \|(\boldsymbol{u}^{\delta}-\boldsymbol{u})(t)\|_{\boldsymbol{V}^{*}}^{2} &+ \int_{0}^{t} \|\boldsymbol{u}^{\delta}-\boldsymbol{u}\|_{\boldsymbol{Z}}^{2} + \delta \int_{0}^{t} \|\nabla_{\Gamma} u_{\Gamma}^{\delta}\|_{H_{\Gamma}}^{2} \\ &\leq C \left(\delta \int_{\Sigma_{t}} \nabla_{\Gamma} u_{\Gamma}^{\delta} \cdot \nabla_{\Gamma} u_{\Gamma} + \|\boldsymbol{u}_{0}^{\delta}-\boldsymbol{u}_{0}\|_{\boldsymbol{V}^{*}}^{2} + \int_{0}^{t} \|\boldsymbol{f}^{\delta}-\boldsymbol{f}\|_{\boldsymbol{H}}^{2} + \int_{0}^{t} \|\boldsymbol{u}^{\delta}-\boldsymbol{u}\|_{\boldsymbol{H}}^{2} \right). \end{aligned}$$

Now, using the Young inequality and the regularity of \boldsymbol{u} one has

$$\delta \int_{\Sigma_t} \nabla_{\Gamma} u_{\Gamma}^{\delta} \cdot \nabla_{\Gamma} u_{\Gamma} \leq \frac{\delta}{2} \int_0^t \|\nabla_{\Gamma} u_{\Gamma}^{\delta}\|_{H_{\Gamma}}^2 + \frac{\delta}{2} \|\boldsymbol{u}\|_{L^2(0,T;\boldsymbol{V})}^2$$

Consequently, using the Ehrling inequality (3.6) on the right-hand side and rearranging the terms we obtain, updating the value of C,

$$\|(\boldsymbol{u}^{\delta} - \boldsymbol{u})(t)\|_{\boldsymbol{V}^{*}}^{2} + \int_{0}^{t} \|\boldsymbol{u}^{\delta} - \boldsymbol{u}\|_{\boldsymbol{Z}}^{2} + \frac{\delta}{2} \int_{0}^{t} \|\nabla_{\Gamma} \boldsymbol{u}_{\Gamma}^{\delta}\|_{H_{\Gamma}}^{2} \\ \leq C \left(\delta + \|\boldsymbol{u}_{0}^{\delta} - \boldsymbol{u}_{0}\|_{\boldsymbol{V}^{*}}^{2} + \|\boldsymbol{f}^{\delta} - \boldsymbol{f}\|_{L^{2}(0,T;\boldsymbol{H})}^{2} + \int_{0}^{t} \|\boldsymbol{u}^{\delta} - \boldsymbol{u}\|_{\boldsymbol{V}^{*}}^{2} \right).$$
(3.37)

The Gronwall lemma yields the desired error estimate, hence also the rate of convergence. Moreover, we note that this implies also the boundedness of $\{u_{\Gamma}^{\delta}\}_{\delta \in (0,1)}$ in $L^2(0,T;V_{\Gamma})$, from which the weak convergence

$$oldsymbol{u}^{\delta}
ightarrow oldsymbol{u}$$
 in $L^2(0,T;oldsymbol{V})$

follows as $\delta \searrow 0$. This concludes the proof of Theorem 2.12.

A Appendix

Lemma A.1. In the setting of Section 2, there exists a constant $C_p > 0$ such that

$$||z||_V \le C_p ||\nabla z||_H \qquad \forall \, \boldsymbol{z} = (z, z_{\Gamma}) \in \boldsymbol{Z}_0.$$

Proof. It is enough to prove that there exists C > 0 such that

$$||z||_{H} \leq C ||\nabla z||_{H} \qquad \forall \, \boldsymbol{z} = (z, z_{\Gamma}) \in \boldsymbol{Z}_{0}.$$

By contradiction, suppose that there exists a sequence $\{\boldsymbol{z}_n\}_{n\in\mathbb{N}}\subset \boldsymbol{Z}_0$ such that

$$||z_n||_H > n ||\nabla z_n||_H \qquad \forall n \in \mathbb{N}.$$

Then, setting $\boldsymbol{w}_n := \boldsymbol{z}_n / \| \boldsymbol{z}_n \|_H$, $n \in \mathbb{N}$, it holds for every $n \in \mathbb{N}$ that

$$||w_n||_H = 1, \qquad ||\nabla w_n||_H < \frac{1}{n}, \qquad m(\boldsymbol{w}_n) = 0.$$

We deduce that there exists $w \in V$ such that

$$w_n \to w$$
 in H , $w_n \to w$ in V , $||w||_H = 1$, $\nabla w = 0$.

In particular, setting $w_{\Gamma} := w_{|\Gamma}$ it holds that $\boldsymbol{w} := (w, w_{\Gamma}) \in \boldsymbol{Z}$. Since w is constant with $\|w\|_{H} = 1$, it necessarily holds that $m(\boldsymbol{w}) \neq 0$. However, the weak convergence $w_{n} \rightharpoonup w$ in V yields in particular that

$$0 = m(\boldsymbol{w}_n) \to m(\boldsymbol{w}),$$

which is absurd. This completes the proof.

Proposition A.2. Let u_0 satisfy (2.28). Then, if (2.40) holds there exists a sequence $\{u_0^{\delta}\}_{\delta \in (0,1)}$ satisfying (2.30) and such that $u_0^{\delta} \rightarrow u$ in \mathbf{Z} as $\delta \searrow 0$.

Proof. In order to introduce a family $\{u_0^{\delta}\}_{\delta \in (0,1)}$ we consider the elliptic system

$$u_0^{\delta} - \delta \Delta u_0^{\delta} = u_0$$
 a.e. in Ω , (A.1)

$$u_{0|\Gamma}^{\delta} = u_{0,\Gamma}^{\delta}, \quad -\partial_{\nu} u_{0}^{\delta} \in \beta_{\Gamma}(u_{0,\Gamma}^{\delta}) \qquad \text{a.e. on } \Gamma.$$
(A.2)

Note that (A.1)–(A.2) admits a unique solution $\boldsymbol{u}_0^{\delta} = (u_0^{\delta}, u_{0,\Gamma}^{\delta})$ with $u_0^{\delta} \in W$, as proved, e.g., in [1, Prop. 2.9, p. 62]. Now, setting $\xi_{0,\Gamma}^{\delta} := -\partial_{\nu}u_0^{\delta} \in Z_{\Gamma}$ we have that $\xi_{0,\Gamma}^{\delta} \in \beta_{\Gamma}(u_{0,\Gamma}^{\delta})$ almost everywhere on Γ . So, testing (A.1) by $u_0^{\delta} - \Delta u_0^{\delta}$, we integrate by parts with the aid of the boundary conditions in (A.2). Thanks to the Young inequality we obtain exactly

$$\frac{1}{2} \|u_0^{\delta}\|_H^2 + \frac{1}{2} \|\nabla u_0^{\delta}\|_H^2 + (\xi_{0,\Gamma}^{\delta}, u_{0,\Gamma}^{\delta} - u_{0,\Gamma})_{H_{\Gamma}} \\
+ \delta \|\nabla u_0^{\delta}\|_H^2 + \delta \|\Delta u_0^{\delta}\|_H^2 + \delta (\xi_{0,\Gamma}^{\delta}, u_{0,\Gamma}^{\delta})_{H_{\Gamma}} \\
\leq \frac{1}{2} \|u_0\|_H^2 + \frac{1}{2} \|\nabla u_0\|_H^2.$$

At this point, recalling that $\beta_{\Gamma} = \partial \widehat{\beta}_{\Gamma}$, we have that

$$(\xi_{0,\Gamma}^{\delta}, u_{0,\Gamma}^{\delta} - u_{0,\Gamma})_{H_{\Gamma}} \ge \int_{\Gamma} \widehat{\beta}_{\Gamma}(u_{0,\Gamma}^{\delta}) - \int_{\Gamma} \widehat{\beta}_{\Gamma}(u_{0,\Gamma}),$$

while by monotonicity of β_{Γ} and the fact that $0 \in \beta_{\Gamma}(0)$ it holds that

$$\delta(\xi_{0,\Gamma}^{\delta}, u_{0,\Gamma}^{\delta})_{H_{\Gamma}} \ge 0.$$

Consequently, we infer that

$$\frac{1}{2} \|u_0^{\delta}\|_V^2 + \|\widehat{\beta}_{\Gamma}(u_{0,\Gamma}^{\delta})\|_{L^1(\Gamma)} + \delta \|\Delta u_0^{\delta}\|_H^2 \le \frac{1}{2} \|u_0\|_V^2 + \|\widehat{\beta}_{\Gamma}(u_{0,\Gamma})\|_{L^1(\Gamma)}, \tag{A.3}$$

where the right-hand side is finite due to (2.28). This readily implies that there exists $v_0 \in V$ such that, in principle along a subsequence,

$$u_0^{\delta} \rightharpoonup v_0 \quad \text{in } V, \qquad \delta \Delta u_0^{\delta} \to 0 \quad \text{in } H.$$

Passing to the limit in (A.1) we realise that $u_{0,\delta} \to u_0$ in H along the entire family $\delta \searrow 0$, hence also that $u_0 = v_0$ almost everywhere in Ω . Moreover, we recall that the system (A.1)–(A.2) can be seen as the limit as $\lambda \searrow 0$ of the corresponding one where β_{Γ} is replaced by its Yosida approximation $\beta_{\Gamma,\lambda}$. Hence, testing the respective equation approximating (A.1) by $\beta_{\lambda}(u_0^{\delta,\lambda})$, where β_{λ} is the Yosida approximation of β , we obtain

$$\int_{\Omega} \beta_{\lambda}(u_{0}^{\delta,\lambda})(u_{0}^{\delta,\lambda}-u_{0}) + \delta \int_{\Omega} \beta_{\lambda}'(u_{0}^{\delta,\lambda}) |\nabla u_{0}^{\delta,\lambda}|^{2} + \delta \int_{\Gamma} \beta_{\lambda}(u_{0,\Gamma}^{\delta,\lambda}) \beta_{\Gamma,\lambda}(u_{0,\Gamma}^{\delta,\lambda}) = 0,$$

which yields by monotonicity and the subdifferential relation for $\widehat{\beta}_{\lambda}$ that

$$\int_{\Omega} \widehat{\beta}_{\lambda}(u_0^{\delta,\lambda}) + \delta \int_{\Gamma} \beta_{\lambda}(u_{0,\Gamma}^{\delta,\lambda}) \beta_{\Gamma,\lambda}(u_{0,\Gamma}^{\delta,\lambda}) \leq \int_{\Omega} \widehat{\beta}_{\lambda}(u_0) du_{0,\Gamma}^{\delta,\lambda}$$

Hence, exploiting (2.40) on the Yosida approximations as

$$\int_{\Gamma} \beta_{\lambda}(u_{0,\Gamma}^{\delta,\lambda}) \beta_{\Gamma,\lambda}(u_{0,\Gamma}^{\delta,\lambda}) \geq \frac{1}{2M} \int_{\Gamma} |\beta_{\Gamma,\lambda}(u_{0,\Gamma}^{\delta,\lambda})|^2 - C,$$

we infer that

$$\int_{\Omega} \widehat{\beta}_{\lambda}(u_0^{\delta,\lambda}) + \frac{\delta}{2M} \int_{\Gamma} |\beta_{\Gamma,\lambda}(u_{0,\Gamma}^{\delta,\lambda})|^2 \le C + \int_{\Omega} \widehat{\beta}_{\lambda}(u_0).$$

Consequently, taking the limit as $\lambda \searrow 0$ and using assumption (2.28) it is possible to prove that

$$\int_{\Omega} \widehat{\beta}(u_0^{\delta}) + \frac{\delta}{2M} \int_{\Gamma} |\xi_{0,\Gamma}^{\delta}|^2 \le C + \int_{\Omega} \widehat{\beta}(u_0),$$

which by comparison in (A.2) implies in particular that

$$\left\|\widehat{\beta}(u_0^{\delta})\right\|_{L^1(\Omega)} + \delta \|\partial_{\nu} u_0^{\delta}\|_{H_{\Gamma}}^2 \le C.$$
(A.4)

Now, collecting the information given by (A.3) and (A.4), using the elliptic regularity theory [6, Thm. 3.2, p. 1.79] and the trace theorems [6, Thm 2.27, p. 1.64] we infer that

$$\delta \|u_{0,\Gamma}^{\delta}\|_{V_{\Gamma}}^2 \le C. \tag{A.5}$$

Then, the estimates (A.3), (A.4), and (A.5) allow us to conclude the proof.

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