

STATE-DEPENDENT TEMPERATURE CONTROL FOR LANGEVIN DIFFUSIONS*

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Abstract. We study the temperature control problem for Langevin diffusions in the context of nonconvex optimization. The classical optimal control of such a problem is of the bang-bang type, which is overly sensitive to errors. A remedy is to allow the diffusions to explore other temperature values and hence smooth out the bang-bang control. We accomplish this by a stochastic relaxed control formulation incorporating randomization of the temperature control and regularizing its entropy. We derive a state-dependent, truncated exponential distribution, which can be used to sample temperatures in a Langevin algorithm, in terms of the solution to an Hamilton–Jacobi–Bellman partial differential equation. We carry out a numerical experiment on a one-dimensional baseline example, in which the Hamilton–Jacobi–Bellman equation can be easily solved, to compare the performance of the algorithm with three other available algorithms in search of a global optimum.

Key words. Langevin diffusion, nonconvex optimization, stochastic relaxed control, entropy regularization, Boltzmann exploration, HJB equation

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1. Introduction. Consider the problem of finding the global minimizer of a *nonconvex* function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, where f is assumed to be differentiable. Traditional algorithms such as gradient descent often end up at a local optimum. The simulated annealing (SA) technique [19] has been developed to resolve the problem of algorithms being trapped in local optima. The main thrust of an SA algorithm is *exploration via randomization*: At each iteration, the algorithm *randomly* samples a solution close to the current one and moves to it according to a (possibly time-dependent) probability distribution, which in the literature is mostly exogenous as a predefined schedule. This scheme allows for a more extensive search or exploration for the global optima with the risk of moving to worse solutions at some iterations. The risk is, however, controlled by slowly cooling down over time the “temperature” which is used to characterize the level of exploration or randomization.

The *Langevin algorithm* applies the SA technique by adding an independent series of Gaussian noises to the classical gradient descent algorithm, where the variance of the noises is linearly scaled by a sequence of temperature parameters $\{\beta_k\}$ which control the level of exploration/randomization. The continuous-time version of the

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Langevin algorithm is the so-called *overdamped* (or *first-order Langevin diffusion*) governed by a stochastic differential equation (SDE), where the temperature is now a stochastic process $\{\beta(t) : t \geq 0\}$. There is a large volume of literature on Langevin diffusions for studying nonconvex optimization; see, to name just a few, [8, 14, 22]. The advantage of studying a diffusion process instead of a discrete-time iteration lies in the simplicity and tractability of the former thanks to the many available analytical tools such as stochastic calculus, stochastic control, and partial differential equations (PDEs).

If we fix the temperature process in a Langevin diffusion to be a constant $\beta > 0$, then one can prove that, under some mild assumptions on f , the Langevin diffusion process converges to a unique stationary distribution whose density is a *Gibbs measure*, $\frac{1}{Z(\beta)} e^{-\frac{f(x)}{\beta}}$, where $Z(\beta)$ is the normalizing factor [8]. As the temperature cools down (i.e., as $\beta \rightarrow 0$), this stationary distribution increasingly concentrates around the global minimizers of f which in turn justifies the use of the Langevin diffusion for nonconvex optimization. Indeed, Langevin diffusions and their variants have recently found many applications in data science including large-scale Bayesian inference and stochastic nonconvex optimization arising from machine learning; see, e.g., [9, 15, 26, 34] and the references therein.

A critical question for solving global nonconvex optimization using the Langevin diffusions is the design of the temperature schedule $\{\beta(t) : t \geq 0\}$. In the literature, the temperature $\beta(\cdot)$ is typically taken either as a constant (a hyperparameter) or various functions of time t , mostly *exogenously* given; see, e.g., [8, 13, 14, 18]. When $\beta(t) \equiv \beta > 0$, it is well known that the expected transition time between different local minima for the overdamped Langevin diffusion is exponential in $1/\beta$, a phenomenon known as metastability [3, 4]. [26] and [35] further upper bound the time of Langevin dynamics converging to an approximate global minimum of nonconvex functions, and [37] and [7] analyze the hitting time of Langevin dynamics to a local minimum, where the temperature is all assumed to be a constant. On the other hand, [24] formulates a deterministic optimal control problem to investigate optimal temperature schedules and derive an ordinary differential equation (ODE) for the time-dependent temperature $\beta(\cdot)$.

Ideally, in implementing an algorithm for finding the global optimizers of a function, the temperature should be fine tuned based on where the *current* iterate is; namely, it should be a function of the state. For instance, in order to quickly escape a deeper local minimum, one should impose a higher temperature on the process. On the other hand, only lower temperatures are needed when the current process is not trapped in local minima in order to stay near a good solution. As a result of the need for this state-dependence, the temperature should be formulated as a *stochastic process* because the state itself follows a stochastic process. [12] considers a temperature process as a specific, *exogenously* given increasing function of the current function value $f(x)$, so that the trajectory can perform larger steps in the search space when the current solution is seen to be far from optimal. They show numerically that this scheme yields more rapid convergence to global optimizers.

The goal of this paper is to develop a *theoretical* framework (instead of a heuristic approach) for designing *endogenous* state-dependent temperature schedule for Langevin diffusions in the context of nonconvex optimization. It is natural to formulate an optimal stochastic control problem in which the controlled dynamic is the Langevin diffusion with the temperature process taken as the control, while the objective is to minimize a cost functional related to the original function f to be minimized.

However, because the temperature appears linearly in the variance term of the diffusion, optimal control is of a “bang-bang” nature; i.e., the temperature ought to repeatedly switch between two extreme points which are the lower and upper bounds of the temperature parameter. Such a bang-bang solution is not desirable in practice because it is *extremely* sensitive to errors: a tiny error may cause the control to take the *wrong* extreme point; see, e.g., [2, 27]. To address this issue, we take the so-called *exploratory control* formulation, developed by [33], in which classical controls are randomized and replaced by *distributions* of controls. This allows the control to take other values than just the extreme ones. Moreover, to encourage certain level of randomization, we explicitly incorporate the entropy of the distributions—which measures the extent of randomization—into the cost functional as a *regularization* term. This formulation smoothes out the temperature control and motivates the solution to deviate, if discreetly, from the overly rigid bang-bang strategy.

Entropy regularization is a widely used heuristic in reinforcement learning (RL) in discrete time and space; see, e.g., [16]. [33] extends it to continuous time/space using a stochastic relaxed control formulation. The motivation behind [33]’s formulation is *repeated learning* in unknown environments. Policy evaluation is achieved by repeatedly sampling controls from the same distribution and applying the law of large numbers. In the present paper there is not really an issue of *learning* (as in RL) because we can assume that f is a known function. The commonality that prompts us to use the same formulation of [33] is *exploration* aiming at escaping from possible “traps.” Exploration is used to broaden the search in order to get rid of over-fitted solutions in RL, to jump out of local optima in SA, and to remove too rigid schedules in temperature control.

Under the infinite time horizon setting, we show that the optimal exploratory control is a truncated exponential distribution whose domain is the prespecified range of the temperature. The parameter of this distribution is state dependent, which is determined by a nonlinear elliptic PDE in the general multidimensional case and an ODE in the one-dimensional case. The distribution is a continuous version of the *Boltzmann* (or *softmax*) *exploration*, a widely used heuristic strategy in the discrete-time RL literature [5, 29] that applies exponential weighting of the state-action value function (also known as *Q-function*) to balance exploration and exploitation. In our setting, the *Q-function* needs to be replaced by the (generalized) *Hamiltonian*. This, however, is not surprising because in classical stochastic control theory optimal feedback control is to *deterministically* maximize the Hamiltonian [36].

As discretization of the optimal state process in our exploratory framework, which satisfies an SDE, naturally leads to a Langevin algorithm with state-dependent noise, our results have algorithmic implications whenever the Hamilton–Jacobi–Bellman equations involved can be easily solved. In particular, the HJB equation in our setting is an ODE when the state space is one-dimensional, which can be efficiently solved. For numerical demonstration, we compare the performance of this new algorithm with three other existing algorithms for the global minimization of a one-dimensional baseline nonconvex function. The first method is a Langevin algorithm with a constant temperature [26], the second one is a Langevin algorithm where the temperature decays with time in a prescribed power-law form [25, 34], and the last one is a replica exchange algorithm [10]. The experiment indicates that, at least in the one-dimensional case, our Langevin algorithm with state-dependent noise finds the global minimizer faster and outperforms the other three methods.

We must, however, emphasize that the main contribution of this paper is *theoretical* rather than *algorithmic*. It establishes and develops a theoretical framework

for studying state-dependent temperature control in SA, in which a nonconvex optimization problem is connected to an HJB PDE. Among other theoretical interests in this connection, one observation is that solving the HJB requires *global* information on the underlying function f , whereas most existing SA algorithms use only *local* information of f at any given iteration. Intuitively, the former is more advantageous other things being equal. This is actually demonstrated by the outperformance of our algorithm in the one-dimensional case when solving the HJB equation poses no numerical challenge. The insight is that one should try to make use of the global information of f as much as possible. That said, for high-dimensional nonconvex optimization, we are not advocating our theory for actual algorithmic implementation before the curse of dimensionality for PDEs has been resolved. Recently there has been some encouraging progress using deep neural networks to numerically and efficiently solve high-dimensional PDEs [1, 17]; so hopefully our results could also contribute to devising SA algorithms in the future when that line of research has come to full fruition.

The rest of the paper proceeds as follows. In section 2, we describe the problem motivation and formulation. Section 3 presents the optimal temperature control. In section 4, we report numerical results comparing the performance of our algorithm with three other methods. Finally, we conclude in section 5.

2. Problem background and formulation.

2.1. Nonconvex optimization and Langevin algorithm. Consider a nonconvex optimization problem

$$(2.1) \quad \min_{x \in \mathbb{R}^d} f(x),$$

where $f: \mathbb{R}^d \rightarrow [0, \infty)$ is a continuously differentiable, *nonconvex* function. The Langevin algorithm aims to have global convergence guarantees and has the following iterative scheme:

$$(2.2) \quad X_{k+1} = X_k - \eta f_x(X_k) + \sqrt{2\eta\beta_k} \xi_k, \quad k = 0, 1, 2, \dots,$$

where f_x is the gradient of f , $\eta > 0$ is the step size, $\{\xi_k\}$ is independent and identically distributed (i.i.d) Gaussian noise, and $\{\beta_k\}$ is a sequence of the temperature parameters (also referred to as a cooling or annealing schedule) that typically decays over time to zero. This algorithm is based on the discretization of the overdamped Langevin diffusion:

$$(2.3) \quad dX(t) = -f_x(X(t))dt + \sqrt{2\beta(t)}dW(t), \quad X(0) = x \in \mathbb{R}^d,$$

where x is an initialization, $\{W(t) : t \geq 0\}$ is a standard d -dimensional Brownian motion with $W(0) = 0$, and $\{\beta(t) : t \geq 0\}$ is an adapted, nonnegative stochastic process, both defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \geq 0})$ satisfying the usual conditions.

When $\beta(t) \equiv \beta > 0$, under some mild assumptions on f , the solution of (2.3) admits a unique stationary distribution with the density $\pi(x) \propto e^{-\frac{1}{\beta}f(x)}$. Moreover, Raginsky, Rakhlin, and Telgarsky [26] show that for a finite $\beta > 0$,

$$(2.4) \quad \mathbb{E}_{X \sim \pi} f(X) - \min_{x \in \mathbb{R}^d} f(x) \leq \mathcal{I}(\beta) := \frac{d\beta}{2} \log \left(\frac{eM}{m} \left(\frac{b}{d\beta} + 1 \right) \right),$$

where $\mathbb{E}_{X \sim \pi} f(X) := \int_{\mathbb{R}^d} f(x) \pi(dx)$, M, m, b are constants associated with f . It is clear that $\mathcal{I}(\beta) \rightarrow 0$ when $\beta \rightarrow 0$.

Our problem is to control the temperature process $\{\beta(t) : t \geq 0\}$ so that the performance of the continuous Langevin algorithm (2.3) is optimized. We measure the performance using the expected total discounted values of the iterate $\{X(t) : t \geq 0\}$, which is $\mathbb{E} \int_0^\infty e^{-\rho t} f(X(t)) dt$, where $\rho > 0$ is a discount factor. The discounting puts smaller weights on function values that happen in later iterations; hence in effect it dictates the computational budget (i.e., the number of iterations budgeted) to run the algorithm (2.3). Clearly, if this performance or cost functional value (which is always nonnegative) is small, then it implies that, on average, the algorithm strives to achieve smaller values of f over iterations and terminates at a budgeted number of iterations, T , with an iterate $X(T)$ that is close to the global optimum.

Mathematically, given $\rho > 0$, an arbitrary initialization $X(0) = x \in \mathbb{R}^d$ and the range of the temperature $U = [a, 1]$, where $0 < a < 1$, we aim to solve the following stochastic control problem where the temperature process is taken as the control:

$$(2.5) \quad \begin{array}{ll} \text{Minimize} & \mathbb{E} \int_0^\infty e^{-\rho t} f(X(t)) dt, \\ \text{subject to} & \begin{cases} (2.3), \\ \beta := \{\beta(t) : t \geq 0\} \text{ is adapted, and } \beta(t) \in U \text{ a.e. } t \geq 0 \text{ a.s.} \end{cases} \end{array}$$

Remark 2.1 (choice of the range of the temperature). Naturally the temperature has a nonnegative lower bound. We suppose that it also has an upper bound, assumed to be 1 without loss of generality. In the Langevin algorithm and SA literature, one usually uses a determinist temperature schedule $\beta(t)$ that is either a constant or decays with time t . Hence there is a natural upper bound which is the initial temperature. This quantity is generally problem dependent and can be a hyperparameter chosen by the user. The upper bound of the temperature should be tuned to be reasonably large for the Langevin algorithm to overcome all the potential barriers and to avoid early trapping into a bad local minimum.

2.2. Solving (2.5) classically. Define the optimal value function of (2.5):

$$(2.6) \quad V_0(x) = \inf_{\beta \in \mathcal{A}_0(x)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} f(X(t)) dt \mid X(0) = x \right],$$

where $x \in \mathbb{R}^d$, and $\mathcal{A}_0(x)$ is the set of admissible controls β satisfying the constraint in (2.5). A standard dynamic programming argument [36] yields that V_0 satisfies the following HJB equation:

$$(2.7) \quad -\rho v(x) + f(x) + \min_{\beta \in [a, 1]} [\beta \text{tr}(v_{xx}(x)) - f_x(x) \cdot v_x(x)] = 0, \quad x \in \mathbb{R}^d,$$

where $\text{tr}(A)$ denotes the trace of a square matrix A , and “ $x \cdot y$ ” is the inner product between two vectors x and y .

The standard verification theorem in stochastic control theory [36, Chapter 5] yields that an optimal *feedback* control policy should achieve the minimum in the above equation. However, the term inside the min operator is linear in the control variable β ; hence the optimal policy has the following bang-bang form: $\beta^*(x) = 1$ if $\text{tr}(v_{xx}(x)) < 0$, and $\beta^*(x) = a$ otherwise. In economics terms, intuitively, the value $v(x)$ can be regarded as the disutility of the resource x (here, it is *disutility*, instead of utility, because the objective is to *minimize*). When $\text{tr}(v_{xx}(x)) < 0$, v is locally concave around x suggesting a risk-seeking preference. Hence one should randomize at

the maximum possible level. A symmetric intuition applies to the opposite case when $\text{tr}(v_{xx}(x)) > 0$. This temperature control scheme shows that one should in some states heat at the highest possible temperature, while in other states cool down completely, depending on the sign of $\text{tr}(v_{xx}(x))$. This is *theoretically* the optimal scheme; but practically it is just too *rigid* to achieve good performance as it concentrates on two actions only, thereby leaving no room for errors which are inevitable in computation. This motivates the introduction of relaxed controls and entropy regularization in order to *smooth out* the temperature process in the next subsection.

Remark 2.2 (bang-bang control and parallel/simulated tempering). The bang-bang control policy bears some resemblance to the so-called parallel tempering (or replica exchange) and simulated tempering in the Markov chain Monte Carlo literature [11, 21, 32], although there are important differences. Both tempering algorithms swap processes with different temperatures based on certain mechanisms in order to sample the desired target distribution; so there is switching at work between different prespecified temperatures like the bang-bang control. However, these algorithms are mostly designed for probabilistic *sampling* with the Metropolis–Hastings rule used to accept or reject a swap between processes, while our focus is (nonconvex) *optimization*. Moreover, the tempering algorithms use state space augmentation. Specifically, parallel tempering runs $n \geq 2$ copies of Langevin dynamics with different constant temperature parameters; hence the state space is \mathbb{R}^{dn} . Simulated tempering treats the temperature $\{\beta(t) : t \geq 0\}$ as a stochastic process taking only finite values and augments it to the original dynamics so that the state space is $(d+1)$ -dimensional. In contrast, if we apply the bang-bang control, we obtain only one copy of the Langevin dynamics where the temperature parameter is state-dependent which is not a constant. Recently, [10] proposes to use the replica exchange algorithm for nonconvex optimization. In the numerical experiments reported in section 4, we will compare the performance of our algorithm with that of [10].

2.3. Relaxed control and entropy regularization. We now present our entropy-regularized relaxed control formulation of the problem. Instead of a classical control $\{\beta(t) : t \geq 0\}$, where $\beta(t) \in U = [a, 1]$ for $t \geq 0$, we consider a *relaxed* control $\pi = \{\pi(t, \cdot) : t \geq 0\}$, which is a *distribution* (or randomization) of classical controls over the control space U where a temperature $\beta(t) \in U$ can be sampled from this distribution whose probability density function is $\pi(t, \cdot)$ at time t .

Specifically, we study the following *entropy-regularized stochastic relaxed control problem*, also termed as *exploratory control problem*, following [33]:

$$(2.8) \quad V_\lambda(x) = \inf_{\pi \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} f(X^\pi(t)) dt - \lambda \int_0^\infty e^{-\rho t} \int_U -\pi(t, u) \ln \pi(t, u) du dt \middle| X^\pi(0) = x \right],$$

where $x \in \mathbb{R}^d$, the term $\int_U -\pi(t, u) \ln \pi(t, u) du$ is the differential entropy of the probability density function $\pi(t, \cdot)$ of the randomized temperature at time t , $\lambda > 0$ is a weighting parameter for entropy regularization, $\mathcal{A}(x)$ is the set of admissible distributional controls to be precisely defined below, and the state process X^π is governed by

$$(2.9) \quad dX^\pi(t) = -f_x(X^\pi(t))dt + \tilde{\sigma}(\pi(t))dW(t), \quad X^\pi(0) = x \in \mathbb{R}^d,$$

where

$$(2.10) \quad \tilde{\sigma}(\mu) := \sqrt{\int_U 2u\mu(u)du}, \quad \mu \in \mathcal{P}(U),$$

with $\mathcal{P}(U)$ being the set of probability density functions over U .

The form of $\tilde{\sigma}(\mu)$ follows from the general formulation presented in [33]. The function V_λ is called the (optimal) *value function*.

Denote by $\mathfrak{B}(U)$ the Borel algebra on U .

DEFINITION 2.3. We say a density-function-valued stochastic process $\pi = \{\pi(t, \cdot) : t \geq 0\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t \geq 0})$ along with a standard d -dimensional $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted Brownian motion $\{W(t) : t \geq 0\}$, is an *admissible (distributional) control*, denoted simply by $\pi \in \mathcal{A}(x)$, if

- (i) For almost every $t \geq 0$, $\pi(t, \cdot) \in \mathcal{P}(U)$ a.s.;
- (ii) For any $A \in \mathfrak{B}(U)$, the process $(t, \omega) \mapsto \int_A \pi(t, u, \omega) du$ is \mathcal{F}_t -progressively measurable;
- (iii) The SDE (2.9) admits solutions on the same filtered probability space, whose distributions are all identical.

So an admissible distributional control is defined in the *weak* sense, namely, the filtered probability space and the Brownian motion are also *part* of the control. Condition (iii) ensures that the performance measure in our problem (2.8) is well defined even under the weak formulation. For notational simplicity, however, we will henceforth only write $\pi \in \mathcal{A}(x)$.

Remark 2.4 (choice of the weighting parameter λ). The weighting parameter λ dictates a trade-off between optimization (exploitation) and randomization (exploration), and is also called a temperature parameter in [33]. [31] proves through a PDE argument that V_λ converges to V_0 as $\lambda \rightarrow 0$ and provides a convergence rate. Note that this temperature λ is different from the temperature β in the original Langevin diffusion. These are temperatures at *different* levels: β is at a lower, local level related to injecting Gaussian noises to the gradient descent algorithm, whereas λ is at a higher, global level associated with adding (not necessarily Gaussian) noises to the bang-bang control. It is certainly an open interesting question to endogenize λ , which is yet another temperature control problem at a different level. In implementing the Langevin algorithm, however, λ can be fine tuned as a hyperparameter.

3. Optimal temperature control.

3.1. A formal derivation. To solve the control problem (2.8), we apply Bellman's principle of optimality to derive the following HJB equation:

$$(3.1) \quad -\rho v(x) - f_x(x) \cdot v_x(x) + f(x) + \inf_{\pi \in \mathcal{P}(U)} \int_U [\text{tr}(v_{xx}(x))u + \lambda \ln \pi(u)] \pi(u) du = 0.$$

The verification approach then yields that the optimal distributional control can be obtained by solving the minimization problem in π in the HJB equation. This problem has a constraint $\pi \in \mathcal{P}(U)$, which amounts to

$$(3.2) \quad \int_U \pi(u) du = 1, \quad \pi(u) \geq 0 \quad \text{a.e. on } U.$$

For any $k \in \mathbb{R}$,

$$\int_U [\text{tr}(v_{xx}(x))u + \lambda \ln \pi(u)] \pi(u) du = \int_U [\text{tr}(v_{xx}(x))u + \lambda \ln \pi(u) + k] \pi(u) du - k.$$

The integrand on the right-hand side is a convex function of $\pi(u)$; so the first-order condition gives its unique global minimum at

$$\begin{aligned} \bar{\pi}(u; x) &= \exp\left(-\frac{1}{\lambda}[\text{tr}(v_{xx}(x))u] - \frac{k}{\lambda} - 1\right) \\ (3.3) \quad &:= \frac{1}{Z(\lambda, v_{xx}(x))} \exp\left(-\frac{1}{\lambda}[\text{tr}(v_{xx}(x))u]\right), \quad u \in U, \quad x \in \mathbb{R}^d. \end{aligned}$$

Since $\bar{\pi}(\cdot; x)$ must satisfy (3.2), we deduce

$$Z(\lambda, v_{xx}(x)) = \int_U \exp\left(-\frac{1}{\lambda}[\text{tr}(v_{xx}(x))u]\right) du > 0.$$

This gives the optimal *feedback law* $\bar{\pi}(\cdot; x)$, which is a *truncated* (in U) *exponential distribution* with the (state-dependent) parameter $c(x) := \frac{\text{tr}(v_{xx}(x))}{\lambda}$. Note we do not require $\text{tr}(v_{xx}(x)) > 0$ (i.e., v is in general nonconvex) or $c(x) > 0$ here.

Substituting (3.3) back to the (3.1) and noting

$$\begin{aligned} &\int_U [\text{tr}(v_{xx}(x))u + \lambda \ln \bar{\pi}(u; x)] \bar{\pi}(u; x) du \\ &= \int_U [\text{tr}(v_{xx}(x))u - \text{tr}(v_{xx}(x))u - \lambda \ln(Z(\lambda, v_{xx}(x)))] \bar{\pi}(u; x) du \\ &= -\lambda \ln(Z(\lambda, v_{xx}(x))), \end{aligned}$$

we obtain the following equivalent form of the HJB equation, a nonlinear elliptic PDE:

$$(3.4) \quad -\rho v(x) - f_x(x) \cdot v_x(x) + f(x) - \lambda \ln(Z(\lambda, v_{xx}(x))) = 0, \quad x \in \mathbb{R}^d.$$

Remark 3.1 (Gibbs measure and Boltzmann exploration). In reinforcement learning there is a widely used *heuristic* exploration strategy called the *Boltzmann exploration*. In the setting of *maximizing* cumulative rewards, Boltzmann exploration uses the Boltzmann distribution (or the Gibbs measure) to assign a probability $p(s_t, a)$ to action a when in state s_t at time t :

$$(3.5) \quad p(s_t, a) = \frac{e^{Q_t(s_t, a)/\lambda}}{\sum_{a=1}^m e^{Q_t(s_t, a)/\lambda}}, \quad a = 1, 2, \dots, m,$$

where $Q_t(s, a)$ is the Q-function value of a state-action pair (s, a) , and $\lambda > 0$ is a parameter that controls the level of exploration; see, e.g., [5, 6, 29]. When λ is high, the actions are chosen in almost equal probabilities. When λ is low, the highest-valued actions are more likely to be chosen, giving rise to something resembling the so-called *epsilon-greedy policies* in multiarmed bandit problems. When $\lambda = 0$, (3.5) degenerates into the Dirac measure concentrating on the action that maximizes the Q-function; namely, the best action is always chosen. In the setting of continuous time and continuous state/control, the Q-function $Q_t(s, a)$ is not well defined and can not be used to rank and select actions [30]. However, we can use the (generalized) *Hamiltonian* [36], which in the special case of the classical problem (2.5) is

$$H(x, u, v_x, v_{xx}) := -f_x(x) \cdot v_x(x) + f(x) + \text{tr}(v_{xx}(x))u.$$

Then it is easy to see that (3.3) is equivalent to the following:

$$(3.6) \quad \bar{\pi}(u; x) = \frac{\exp(-\frac{1}{\lambda}[H(x, u, v_x, v_{xx})])}{\int_{u \in U} \exp(-\frac{1}{\lambda}[H(x, u, v_x, v_{xx})])du}, \quad u \in U, \quad x \in \mathbb{R}^d,$$

which is analogous to (3.5) with x being the state and u being the action. The negative sign in the above expression is due to the minimization problem instead of a maximization one. The importance of this observation is that here we lay a *theoretical underpinning* of the Boltzmann exploration via entropy regularization, thereby providing interpretability/explainability of a largely heuristic approach.¹

3.2. Admissibility and optimality. The derivation above is formal and is legitimate only after we *rigorously* establish the verification theorem and check the admissibility of the distributional controls induced by the feedback law (3.3).

To this end we need to impose some assumptions.

Assumption 3.2. We assume

- (i) $U = [a, 1]$ for some $0 < a < 1$;
- (ii) $f \in C^1(\mathbb{R}^d)$ and $|f_x(x)| < C$ for some constant $C > 0$.

Assumption (i) imposes some minimal heating, $a > 0$, to the Langevin diffusion, capturing the fact that in practice we would generally not know whether we have reached the global minimum at any given time; therefore we always carry out exploration, however small it might be, until some prescribed stopping criterion is met. On the other hand, the difficulty of the original nonconvex optimization lies in the possibility of many *local* optima, while it is relatively easier to know (sometime from the specific context of an applied problem under consideration) the approximate range of the possible location of a global optimum. Therefore, the assumption of the gradient of f in Assumption (ii) is reasonable as the function value outside of the above range is irrelevant.

Apply the feedback law (3.3) to the controlled system (2.9) to obtain the following SDE:

$$(3.7) \quad dX^*(t) = -f_x(X^*(t))dt + h(X^*(t))dW_t, \quad X^*(0) = x,$$

where

$$(3.8) \quad h(x) := \sqrt{\frac{2}{Z(\lambda, v_{xx}(x))} \int_a^1 u \exp\left(-\frac{1}{\lambda}[\text{tr}(v_{xx}(x))u]\right) du}.$$

It is clear that $0 < \sqrt{2a} \leq h(x) \leq \sqrt{2}$. Thus the drift and diffusion coefficients of (3.7) are both uniformly bounded, while the latter satisfies the uniform elliptic condition. It follows from [20, Page 87, Theorem 1] that (3.7) has a weak solution. Moreover, [28, Theorem 6.2] asserts that the weak solution is unique.

Now define for $t \geq 0, u \in [a, 1]$,

$$(3.9) \quad \pi^*(t, u) := \bar{\pi}(u; X^*(t)) \equiv \frac{1}{Z(\lambda, v_{xx}(X^*(t)))} \exp\left(-\frac{1}{\lambda}[\text{tr}(v_{xx}(X^*(t)))u]\right),$$

where $\{X^*(t) : t \geq 0\}$ is a weak solution to (3.7). Let $\pi^* := \{\pi^*(t, \cdot) : t \geq 0\}$. Then clearly $\pi^* \in \mathcal{A}(x)$.

¹The formula (3.6) was derived in [33, equation (17)] for a more general setting, but the connection with Boltzmann exploration and Gibbs measure was not noted there.

THEOREM 3.3. *Suppose $v \in C^2$ is a solution to the HJB equation (3.1), and $|v(x)| < C(1 + |x|^k) \forall x \in \mathbb{R}^d$ for some constants $C > 0$ and $k > 0$. Then $v = V_\lambda$. Moreover, in this case the distributional control π^* defined by (3.9) is optimal.*

We need a series of technical preliminaries to prove this theorem.

LEMMA 3.4. *For any $\pi \in \mathcal{A}(x)$, we have*

$$(3.10) \quad \int_a^1 \pi(t, u) \ln \pi(t, u) du \geq a \text{ a.s., a.e. } t \geq 0.$$

Proof. Applying the general inequality $x \ln x \geq x - 1$ for $x > 0$, we have

$$(3.11) \quad \int_a^1 \pi(t, u) \ln \pi(t, u) du \geq \int_a^1 (\pi(t, u) - 1) du = a. \quad \square$$

LEMMA 3.5. *Suppose $\{X(t) : t \geq 0\}$ follows*

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad t \geq 0,$$

with

$$C_b := \text{ess sup}_{(t,\omega)} |b(t)| < \infty, \quad C_\sigma := \text{ess sup}_{(t,\omega)} |\sigma(t)| < \infty.$$

Then, for any $k \geq 1$, there exists a constant $C > 0$, which is independent of T and $X(0)$, such that

$$(3.12) \quad \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^k \right] < C(1 + T^k + |X(0)|^k) \quad \forall T \geq 0.$$

Proof. By the elementary inequality

$$(a + b + c)^k \leq (3 \max\{a, b, c\})^k \leq 3^k a^k + 3^k b^k + 3^k c^k, \quad a, b, c \geq 0,$$

we have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t)|^k \right] &\leq \mathbb{E} \left[\left(|X(0)| + \sup_{0 \leq t \leq T} \int_0^t |b(s)| ds + \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s) dW_s \right| \right)^k \right] \\ &\leq \mathbb{E} \left[\left(|X(0)| + C_b T + \sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s) dW_s \right| \right)^k \right] \\ &\leq 3^k |X(0)|^k + 3^k C_b^k T^k + 3^k \mathbb{E} \left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s) dW_s \right|^k \right] \\ &\leq 3^k |X(0)|^k + 3^k C_b^k T^k + 3^k C_k \left(\mathbb{E} \left[\int_0^T |\sigma(s)|^2 ds \right] \right)^{k/2} \\ &\leq 3^k |X(0)|^k + 3^k C_b^k T^k + 3^k C_k C_\sigma^k T^{k/2}, \end{aligned} \quad \square$$

where the second to last inequality is due to the Burkholder–Davis–Gundy inequality. This proves (3.12).

PROPOSITION 3.6. *There exists a constant $C_1 > 0$ such that the value function V satisfies*

$$(3.13) \quad |V_\lambda(x)| \leq C_1(1 + |x|) \quad \forall x \in \mathbb{R}^d.$$

Proof. It follows from Assumption 3.2(ii) that $|f(x)| \leq C|x| + |f(0)|$. Let π be the uniform distribution on $[a, 1]$. It then follows from Lemma 3.5 that

$$V_\lambda(x) \leq \mathbb{E} \left[\int_0^\infty e^{-\rho t} (C|X^\pi(t)| + |f(0)|) dt + \lambda \int_0^\infty e^{-\rho t} \int_a^1 \pi(u) \ln \pi(u) du dt \right] < C'(|x| + 1)$$

for some constant C' independent of x . On the other hand, for any $\pi \in \mathcal{A}(x)$, by virtue of Lemma 3.4 we have

$$\begin{aligned} & \mathbb{E} \left[\int_0^\infty e^{-\rho t} f(X^\pi(t)) dt - \lambda \int_0^\infty e^{-\rho t} \int_a^1 -\pi_t(u) \ln \pi_t(u) du dt \right] \\ & \geq \mathbb{E} \left[\int_0^\infty -e^{-\rho t} (C|X^\pi(t)| + |f(0)|) dt + \lambda \int_0^\infty e^{-\rho t} a dt \right] > -C''(|x| + 1) \end{aligned}$$

for some constant C'' independent of x . The proof is complete. □

We have indeed established in the above that

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} |f(X^\pi(t))| dt \right] < \infty.$$

To solve (2.8), we only need to consider those admissible controls π such that

$$(3.14) \quad \mathbb{E} \left[\int_0^\infty e^{-\rho t} \int_U \pi(t, u) \ln \pi(t, u) du dt \right] < \infty,$$

because any control violating (3.14) renders the infinite value of the minimization problem (2.8) and hence can be excluded from consideration. In view of this, (3.14) is henceforth assumed for any *admissible* distributional control.

Now we prove Theorem 3.3.

For any $\pi \in \mathcal{A}(x)$, let $\{X^\pi(t) : t \geq 0\}$ be the solution to (2.9). Define $\tau_n := \inf\{s \geq 0 : |X^\pi(s)| > n\}$. For any $T > 0$, by Itô's lemma,

$$\begin{aligned} & e^{-\rho(\tau_n \wedge T)} v(X^\pi(\tau_n \wedge T)) \\ &= v(X^\pi(0)) + \int_0^{\tau_n \wedge T} e^{-\rho t} v_x(X^\pi(t)) \cdot \tilde{\sigma}(\pi(t)) dW(t) \\ &+ \int_0^{\tau_n \wedge T} e^{-\rho t} \left(-\rho v(X^\pi(t)) - f_x(X^\pi(t)) \cdot v_x(X^\pi(t)) + \frac{1}{2} \tilde{\sigma}^2(\pi(t)) \text{tr}(v_{xx}(X^\pi(t))) \right) dt \\ &= \int_0^{\tau_n \wedge T} e^{-\rho t} v_x(X^\pi(t)) \cdot \tilde{\sigma}(\pi(t)) dW(t) + \int_0^{\tau_n \wedge T} e^{-\rho t} \left(\int_a^1 \text{tr}(v_{xx}(X^\pi(t))) u \pi(t, u) du \right. \\ &\quad \left. - f(X^\pi(t)) - \inf_{\pi \in \mathcal{P}([a,1])} \int_a^1 [\text{tr}(v_{xx}(X^\pi(t))) u + \lambda \ln \pi'(u)] \pi'(u) du \right) dt + v(x) \\ &\geq v(x) + \int_0^{\tau_n \wedge T} e^{-\rho t} v_x(X^\pi(t)) \cdot \tilde{\sigma}(\pi(t)) dW(t) \\ &+ \int_0^{\tau_n \wedge T} e^{-\rho t} \left(-f(X^\pi(t)) - \lambda \int_a^1 \pi(t, u) \ln \pi(t, u) du \right) dt. \end{aligned}$$

It follows

$$\begin{aligned} v(x) &\leq \mathbb{E} \left[e^{-\rho(\tau_n \wedge T)} v(X^\pi(\tau_n \wedge T)) \right] \\ &\quad + \mathbb{E} \left[\int_0^{\tau_n \wedge T} e^{-\rho t} \left(f(X^\pi(t)) + \lambda \int_a^1 \pi(t, u) \ln \pi(t, u) du \right) dt \right] \\ &= \mathbb{E} \left[e^{-\rho(\tau_n \wedge T)} v(X^\pi(\tau_n \wedge T)) \right] + \mathbb{E} \left[\int_0^{\tau_n \wedge T} e^{-\rho t} f(X^\pi(t)) dt \right] \\ &\quad + \lambda \mathbb{E} \left[\int_0^{\tau_n \wedge T} e^{-\rho t} \int_a^1 \pi(t, u) \ln \pi(t, u) dudt \right]. \end{aligned}$$

Sending $n \rightarrow \infty$, we have by Lemma 3.5 and the dominated convergence theorem that the sum of the first two terms converges to

$$\mathbb{E} \left[e^{-\rho T} v(X^\pi(T)) \right] + \mathbb{E} \left[\int_0^T e^{-\rho t} f(X^\pi(t)) dt \right],$$

while it follows from (3.14) along with the monotone convergence theorem (noticing $\pi(t, u) \ln \pi(t, u) \geq \pi(t, u) - 1$) that the last term converges to

$$\lambda \mathbb{E} \left[\int_0^T e^{-\rho t} \int_a^1 \pi(t, u) \ln \pi(t, u) dudt \right].$$

So

$$\begin{aligned} v(x) &\leq \mathbb{E} \left[e^{-\rho T} v(X^\pi(T)) \right] + \mathbb{E} \left[\int_0^T e^{-\rho t} f(X^\pi(t)) dt \right] \\ &\quad + \lambda \mathbb{E} \left[\int_0^T e^{-\rho t} \int_a^1 \pi(t, u) \ln \pi(t, u) dudt \right]. \end{aligned}$$

Noting Lemma 3.5 along with the assumption that v is of polynomial growth, we have

$$(3.15) \quad \lim_{T \rightarrow \infty} \mathbb{E} \left[e^{-\rho T} v(X^\pi(T)) \right] = 0.$$

Applying the same limiting argument as above we let $T \rightarrow \infty$ to get

$$v(x) \leq \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(f(X^\pi(t)) + \lambda \int_U \pi(t, u) \ln \pi(t, u) du \right) dt \right].$$

Since π is arbitrarily chosen, it follows $v \leq V_\lambda$; in other words v is a lower bound of V_λ .

All the inequalities in the above analysis become equalities when we take $\pi = \pi^* \in \mathcal{A}(x)$ defined by (3.9), because π^* attains the infimum in the HJB equation (3.1). This yields that the aforementioned lower bound is achieved by π^* ; hence π^* is optimal and $v = V_\lambda$.

Remark 3.7 (existence and uniqueness of solution to HJB). Due to the linear growth of the value function V_λ (Proposition 3.6), we can prove by a standard argument (part of it is similar to the argument used in the above proof) that V_λ is a

solution to the HJB equation (3.1) as long as $V_\lambda \in C^2$. Moreover, Theorem 3.3 yields that the solution is unique among C^2 functions of polynomial growth. The uniqueness follows from (3.15) which is essentially a boundary condition of (3.1). When V_λ is not C^2 , we can apply viscosity solution theory to study the well-posedness of the HJB equation. However, we will not pursue this direction as it is not directly related to the objective of this paper.

4. Algorithm and numerical example. As discussed in the introduction, the main contribution of this paper is to develop a theory that connects the state-dependent temperature control problem for SA to an HJB equation, the latter concerning the global information about the underlying function f . The theory also naturally gives rise to an algorithm that can be employed to solve the original nonconvex optimization problem to the extent that the HJB equation can be efficiently solved. Specifically, the optimal state process $\{X^*(t) : t \geq 0\}$ is determined by the SDE (3.7). Discretization of this equation then leads to a Langevin algorithm with state-dependent noise. One can use the Euler–Maruyama discretization (see, e.g., [23]) due to its simplicity and popularity in practice. We now report a numerical experiment for a baseline nonconvex optimization example in a one-dimensional state space for which the HJB equation is an ODE.

Consider the problem of finding the global minima of the following asymmetric double-well function:

$$f(x) = \begin{cases} 4x - 20 & \text{for } x > 6, \\ (x - 4)^2 & \text{for } 2 < x \leq 6, \\ 8 - x^2 & \text{for } -2 < x \leq 2, \\ 2(x + 3)^2 + 2 & \text{for } -6 < x \leq -2, \\ -12x - 52 & \text{for } x \leq -6. \end{cases}$$

Note that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function with two local minima at $x_1 = -3$ and $x_2 = 4$, where the latter is the (unique) global minimum. See Figure 1 for an illustration.

We now present four algorithms, described below, for minimizing f , and we compare their performances. All the algorithms are initialized at $x = -3$, the

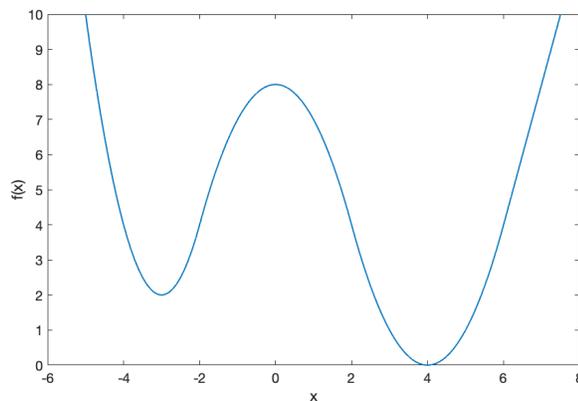


FIG. 1. A double-well function f .

suboptimal local minimum of f . We examine the convergence of the performance quantity $\mathbb{E}(f(X_k))$, where X_k is the location of the iterate at iteration k , $k = 1, 2, \dots$. In each of the algorithms, $\{\xi_k\}$ is a sequence of i.i.d. standard normal random variable, and η is the constant step size which we tune respectively for best performance. In implementation, we change the upper bound of the allowed temperature to be 500, instead of 1 in our theoretical analysis, without loss of validity. This allows the iterates of the algorithms to climb over the barrier between the two local minimizers quickly.

The four algorithms are

- (1) Langevin algorithm with constant temperature β [26]:

$$(4.1) \quad X_{k+1} = X_k - \eta f'(X_k) + \sqrt{2\eta\beta}\xi_k.$$

We choose the step size $\eta \in \{(1/2)^n : n = 0, 1, 2, \dots, 10\}$ and the temperature $\beta \in \{500 \cdot (1/2)^n : n = 0, 1, 2, \dots, 15\}$. The best-tuned step size $\eta = (1/2)^1 = 0.5$ and temperature $\beta = 500 \cdot (1/2)^{10} = 0.4883$.

- (2) Langevin algorithm with the power-law temperature schedule [25]:

$$(4.2) \quad X_{k+1} = X_k - \eta f'(X_k) + \sqrt{2\eta\beta_k}\xi_k,$$

where we choose $\beta_k = (\frac{d}{1+k})^b$ with $b \in \{0.5, 0.6, \dots, 1\}$, and $d \in \{500 \cdot (1/2)^n : n = 0, 1, 2, \dots, 7\}$; we also choose the step size $\eta \in \{(1/2)^n : n = 0, 1, 2, \dots, 10\}$. The best-tuned parameters are given by: $\eta = (1/2)^1 = 0.5$, $b = 0.9$, $d = 500 \cdot (1/2)^4 = 31.25$.

- (3) Replica exchange (GDxLD in Algorithm 1 of [10]): one runs a copy of the gradient descent $\{X_k\}$ and a copy of the Langevin algorithm $\{Y_k\}$ with a constant temperature $\gamma > 0$. If $f(X_k) > f(Y_k)$, then the positions of X_k and Y_k are swapped. Output X_N as an optimizer of f when the algorithm terminates at iteration N . We choose $X_0 = Y_0 = -3$ and tune parameters where step size $\eta \in \{(1/2)^n : n = 0, 1, 2, \dots, 10\}$ and constant temperature $\gamma \in \{500 \cdot (1/2)^n : n = 0, 1, 2, \dots, 15\}$. The best-tuned parameters are given by $\eta = (1/2)^1 = 0.5$, $\gamma = 500 \cdot (1/2)^1 = 250$.
- (4) Our Langevin algorithm based on the Euler–Maruyama discretization of the SDE (3.7):

$$(4.3) \quad X_{k+1}^* = X_k^* - \eta f'(X_k^*) + \sqrt{\eta}h(X_k^*)\xi_k.$$

In dimension one we can deduce from (3.8) that $h(x) = g(\frac{v_{xx}(x)}{\lambda})$, where v satisfies the ODE in (3.4) with

$$(4.4) \quad Z(\lambda, v_{xx}(x)) = \frac{e^{-a\frac{v_{xx}(x)}{\lambda}} - e^{-c\frac{v_{xx}(x)}{\lambda}}}{\frac{v_{xx}(x)}{\lambda}},$$

and

$$(4.5) \quad g(y) := \sqrt{2\frac{(c+1/y) \cdot e^{-cy} - (a+1/y) \cdot e^{-ya}}{e^{-cy} - e^{-ya}}}, \quad y \in (-\infty, \infty).$$

Here, $[a, c]$ is the range of the allowed temperature. The hyperparameter a in (4.4) is fixed at 0.0001 for simplicity, and the hyperparameter c , which is the upper bound on the allowed temperature, is fixed at 500. We choose $\rho, \lambda \in$

$\{5 \cdot (1/2)^n : n = 0, 1, 2, \dots, 8\}$ and step size $\eta \in \{(1/2)^n : n = 0, 1, 2, \dots, 10\}$. To implement the algorithm (4.3), we need to solve the second-order nonlinear ODE in (3.4). We use random initial conditions $[v(0), v'(0)]$ where both $v(0)$ and $v'(0)$ are independently sampled from a standard normal distribution. We select 20 random initializations for the ODE. The best-tuned parameters are given by $\eta = (1/2)^3 = 0.125$, $\lambda = 5 \cdot (1/2)^4 = 0.3125$, $\rho = 5 \cdot (1/2)^2 = 1.25$, with the ODE initialization $[v(0), v'(0)] = [-0.2853, 1.1575]$.

Figure 2 shows the performance of the four algorithms, where the expectation $\mathbb{E}(f(X_k))$ is approximated by its sample average with the sample size 500. Each algorithm is terminated if the number of iterations exceeds the allowed number of iterations, which we set to be 1000 in our experiment. In the first 100 iterations, the expected function values $\mathbb{E}(f(X_k))$ from our algorithm are very large, primarily because large noises are injected into the iterates during these initial iterations so as to escape the local minimum at -3 ; see Figure 3 for a zoomed-in version. Hence,

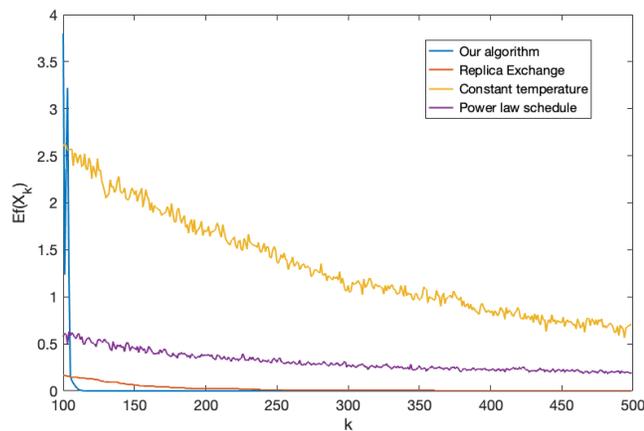


FIG. 2. Performance comparison of the four algorithms, all initialized at $X_1 = -3$.

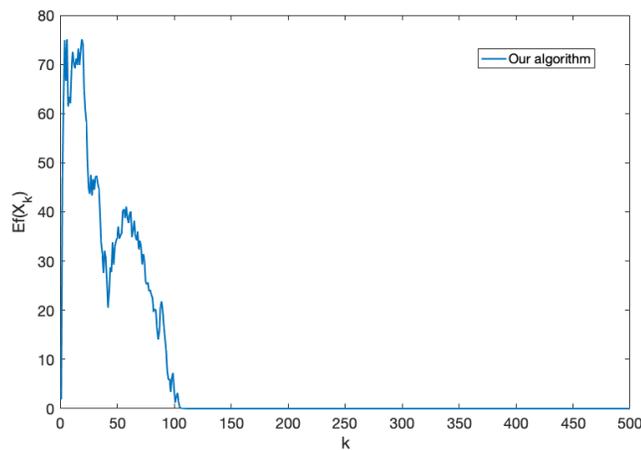


FIG. 3. Performance of our algorithm initialized at $X_1 = -3$.

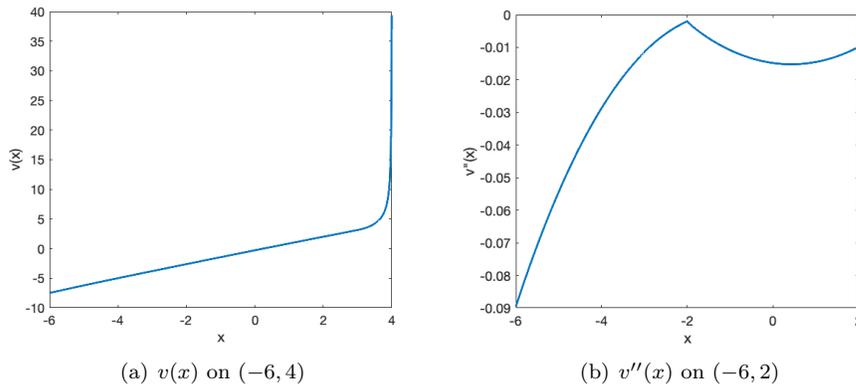


FIG. 4. The HJB solution v and its second-order derivative v'' .

for better visualization, we plot in Figure 2 the expected function values $\mathbb{E}(f(X_k))$ from $k = 100$ up to only $k = 500$ iterations. As we can see, Langevin algorithms with constant temperature and power-law temperature schedule have difficulties in locating the global minima within 500 iterations. By its very definition, replica exchange is expected to perform better than the Langevin algorithm with a constant temperature. This is confirmed by Figure 2 in which the replica exchange algorithm finds the global minimum quickly, although it needs to run two algorithms (the gradient descent and a Langevin algorithm) instead of only one algorithm. Our Langevin algorithm with state-dependent noises can find the global minimum faster than the other three algorithms. It is, however, computationally more expensive compared with other three methods, needing to solve a nonlinear ODE.

The temperature process for our algorithm depends on the solution v to the ODE (3.4). We plot in Figure 4(a) the function values $v(x)$ where $x \in (-6, 4)$. We observe that when x is close to 4, the global minimum of f , v grows very quickly. This is because much less exploration is needed near the global minimum, in which case the subtraction of the entropy value from the overall objective value in (2.8) is much less, boosting the value of v . We also plot in Figure 4(b) the second-order derivative of v on the interval $(-6, 2)$.² However, v or v'' affects the temperature process of our algorithm only indirectly, through the function h given by (4.3). So the plots of v and v'' are less informative than that of the state-dependent temperature function $\frac{h^2}{2}$ which is proportional to the variance of the noise in our algorithm; see Figure 5. We see that the temperature is close to zero for $x > 3$ and is mostly large elsewhere. This indicates that our state-dependent algorithm is “intelligent”: it uses the lowest temperature when close to the global minimum and uses a large temperature (recall the largest temperature allowed is 500) to escape from traps such as suboptimal local minima or saddle points. We also observe that there is a prominent kink at $x = -2$ in Figure 5. This is primarily due to the spike of v'' at -2 in Figure 4(b) and the fact that $h(x) = g(\frac{v_{xx}(x)}{\lambda})$ with g given in (4.5).

5. Conclusion. This paper aims to develop an endogenous temperature control scheme for applying Langevin diffusions to find non-convex global minima. We take the exploratory stochastic control framework, originally proposed by [33] for RL, to

²We do not plot $v''(x)$ for $x > 2$ in Figure 4(b) for better visualization because $v''(x)$ becomes very large when $x > 2$.

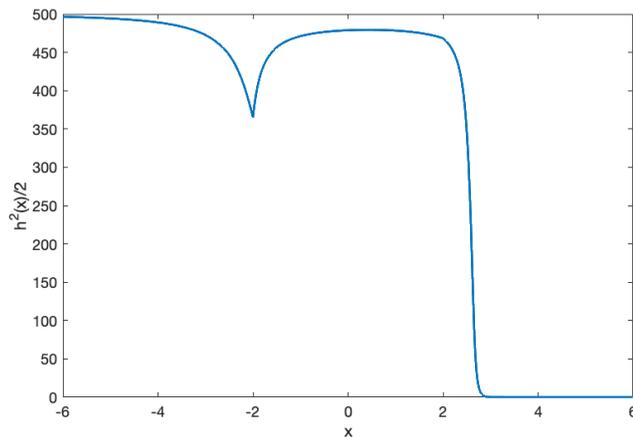


FIG. 5. State-dependent temperature function $h^2/2$.

account for the need of smoothing out the temperature process. We derive a state-dependent, Boltzmann-exploration type distributional control, which can be used to sample temperatures in a Langevin algorithm. Numerical analysis shows that our algorithm outperforms three alternative ones based on a constant temperature, a power decay schedule, and a replica exchange method, respectively. However, the function used in the numerical example is one-dimensional, for which the HJB equation is an ODE and hence easy to solve. For high-dimensional problems, the HJB equation is a PDE whose numerical solutions may suffer from the curse of dimensionality. Therefore, at least for now, the main contribution of this paper is *not* algorithmic. Rather, it is, generally, to lay a *theoretical* underpinning for smoothing out often overly rigid classical optimal controls (such as bang-bang controls) and, specifically, to provide an interpretable state-dependent temperature control scheme for Langevin diffusions via HJB equations.

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