MATCHING OF GIVEN SIZES IN HYPERGRAPHS

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ABSTRACT. For all integers k,d such that $k \geq 3$ and $k/2 \leq d \leq k-1$, let n be a sufficiently large integer (which may not be divisible by k) and let $s \leq \lfloor n/k \rfloor - 1$. We show that if H is a k-uniform hypergraph on n vertices with $\delta_d(H) > \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d}$, then H contains a matching of size s. This improves a recent result of Lu, Yu, and Yuan and also answers a question of Kühn, Osthus, and Townsend. In many cases, our result can be strengthened to $s \leq \lfloor n/k \rfloor$, which then covers the entire possible range of s. On the other hand, there are examples showing that the result does not hold for certain n, k, d and $s = \lfloor n/k \rfloor$.

1. Introduction

Given $k \geq 2$, a k-uniform hypergraph (for short, k-graph) H consists of a vertex set V(H) and an edge set $E(H) \subseteq \binom{V(H)}{k}$, where every edge is a k-subset of V(H). A matching (or integer matching) in H is a collection of vertex-disjoint edges of H. A perfect matching in H is a matching that covers all vertices of H. Let |V(H)| = n. Clearly, a perfect matching in H exists only if k divides k. When k does not divide k, we call a matching k in k a near perfect matching if k.

Let H be a k-graph. For a d-subset S of V(H), where $1 \leq d \leq k-1$, we define $\deg_H(S)$ to be the number of edges in H containing S. The $minimum\ d$ -degree $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-subsets S of V(H). We refer to $\delta_1(H)$ as the $minimum\ vertex\ degree$ of H and $\delta_{k-1}(H)$ as the $minimum\ codegree$ of H.

The study of perfect matchings is one of the fundamental problems in combinatorics. In the case of graphs, that is, k = 2, a theorem of Tutte [33] gives necessary and sufficient conditions for H to contain a perfect matching, and Edmonds' Algorithm [4] finds such a matching in polynomial time. However, for the case $k \geq 3$, the decision problem whether a k-graph contains a perfect matching is famously NP-complete (see [6, 14]).

1.1. **Perfect matchings.** The following conjecture from [7, 20] gives a minimum d-degree condition that ensures a perfect matching in a k-graph.

Conjecture 1.1. Let $k, d \in \mathbb{N}$ and $1 \le d \le k-1$. Then there is an $n_0 \in \mathbb{N}$ such that the following holds for all $n \ge n_0$. Suppose H is a k-graph on $n \in k\mathbb{N}$ vertices with

$$\delta_d(H) \ge \left(\max\left\{\frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-d}\right\} + o(1)\right) \binom{n-d}{k-d},$$

then H contains a perfect matching.

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There are two types of extremal examples, namely, the so-called *divisibility barrier* and *space barrier* which show, if true, the minimum degree conditions in Conjecture 1.1 are asymptotically best possible.

Construction 1.2 ([30], Divisibility Barrier). Fix integers j, k, n such that $j \in \{0, 1\}$ and $n \in k\mathbb{N}$. Let V be a set of size n with a partition $U \cup W$ such that $|U| \not\equiv jn/k \mod 2$. Let H^j be the k-graph on V whose edges are k-sets e such that $|e \cap U| \equiv j \mod 2$.

To see why there is no perfect matching in Construction 1.2, it is not hard to see that $\delta_d(H^j) \leq (1/2 + o(1))\binom{n-d}{k-d}$ and the equality is attained when $|U| \approx |V| \approx n/2$. For the case j=0, note that any matching in H^0 covers an even number of vertices in U. Then, due to the parity of |U|, H^0 does not contain a perfect matching. For the case j=1, suppose that there exists a perfect matching M in H^1 . Note that M has n/k edges, and each edge e in M satisfies $|e \cap U| \equiv 1 \mod 2$. Summing over all edges in M, we obtain that $|U| \equiv n/k \mod 2$, contradicting to our assumption on |U|. So there cannot exist a perfect matching in H^1 . Moreover, it is known that one can construct such divisibility barriers with any finite number of parts (but with smaller minimum d-degrees).

Construction 1.3 (Space Barrier). Fix integers k, n such that $n \in k\mathbb{N}$. Let V be a vertex set of size n with a partition $U \cup W$, and |W| = s < n/k. Let $H_k^k(U, W)$ be the k-graph on V whose edges are all k-sets that intersect W.

Note that any matching in Construction 1.3 has at most s < n/k edges, so there cannot exist a perfect matching in $H_k^k(U, W)$. For $1 \le d \le k-1$, it is easy to see that $\delta_d(H_k^k(U, W)) = \binom{n-d}{k-d} - \binom{n-d-s}{k-d} = \left(1 - (1-s/n)^{k-d} + o(1)\right)\binom{n-d}{k-d}$. Moreover, the maximum value of $\delta_d(H_k^k(U, W))$ is attained by $s = \lceil n/k \rceil - 1$, which gives the second term in Conjecture 1.1.

Conjecture 1.1 has attracted a great deal of attention in recent years, see results, e.g., [19, 25, 28]. It has been confirmed for all $0.375k \le d \le k-1$ and for $1 \le k-d \le 4$ and for $(k,d) \in \{(12,5),(17,7)\}$. In particular, Rödl, Ruciński, and Szemerédi [29] determined that the minimum codegree threshold is n/2-k+C for sufficiently large $n \in k\mathbb{N}$, where $C \in \{3/2,2,5/2,3\}$ depends on the values of n and k. Furthermore, the exact thresholds for sufficiently large n are obtained for these cases except $(k,d) \in \{(5,1),(6,2)\}$ in [3,5,11,17,18,22,24,30,31,32]. For more results we refer the reader to the excellent surveys [26,34].

1.2. Matchings of other sizes. When $k \nmid n$, Rödl, Ruciński, and Szemerédi [29] showed that a minimum codegree roughly n/k guarantees the existence of a matching of size $\lfloor n/k \rfloor$. This stands as a steep contrast to the threshold for perfect matchings, which is roughly n/2. Since then the study of near perfect matchings and matchings of general size s < n/k has also attracted a lot of attention. Indeed, Rödl, Ruciński, and Szemerédi [29] determined the codegree threshold for the existence of a matching of size s in a k-graph H for all $s \leq \lfloor n/k \rfloor - k + 2$. Han [9] extended this result to all s < n/k, verifying a conjecture of Rödl, Ruciński, and Szemerédi. Moreover, Han [10] gave a divisibility barrier construction that prevents the existence of near perfect matchings in H which generalizes Construction 1.2. He also proposed a conjecture on the minimum d-degree threshold forcing a (near) perfect matching in H which generalizes Conjecture 1.1. In the same paper, He determined the minimum (k-2)-degree threshold and gave an upper bound and lower bound for general d-degree threshold. Kühn, Osthus, and Treglown [22] determined the vertex degree threshold for the existence of a matching of size s in a 3-graph H for all $s \leq n/3$, which can be seen as a strengthening of an old result of Bollobás, Daykin, and Erdős [2] for 3-graphs.

Kühn, Osthus, and Townsend [21] determined the d-degree threshold for the existence of a matching of size s in a k-graph H asymptotically for $1 \le d \le k-2$ and $s \le \min\{n/(2k-2d), (1-o(1))n/k\}$ (when $k/2 < d \le k-2$, this is equivalent to $s \le (1-o(1))n/k$). They asked whether the (1-o(1))n/k can be replaced by n/k-C for some constant C depending only on d and k. A

recent result of Lu, Yu, and Yuan [23] shows that one can take

$$C = (1 - d/k) \left\lceil \frac{k - d}{2d - k} \right\rceil,$$

which answers the aforementioned question. However, the term $(1 - d/k) \lceil \frac{k - d}{2d - k} \rceil$ can be arbitrarily large when d is close to k/2. In this paper we reduce this gap by showing that one can take C = 1 (see Corollary 1.5), and in many cases, one can actually take C = 0.

In fact, Lu, Yu, and Yuan [23] determined the exact minimum d-degree threshold for the existence of a matching of size s in a k-graph H for $k/2 < d \le k-1$ and $n/k-o(n) \le s \le n/k-(1-d/k)\lceil \frac{k-d}{2d-k} \rceil$. Our main improvements can be summarized in the following two results. The first one says that the largest s we can take is either $\lfloor n/k \rfloor$ (clearly the best possible) or $\lfloor n/k \rfloor - 1$ (best possible for certain values of k, d and n).

Theorem 1.4. For all integers k,d such that $k \geq 3$ and $k/2 \leq d \leq k-1$, there exists an $n_0 \in \mathbb{N}$ such that the following holds for $n \geq n_0$. Let $n \equiv r \mod k$ be an integer, where $0 \leq r \leq k-1$. Suppose H is an n-vertex k-graph with $\delta_d(H) > \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d}$, where

$$s = \begin{cases} \lfloor n/k \rfloor, & \text{if } k/2 \le d < \lceil 2k/3 \rceil \text{ and } r \ge 2, \text{ or if } \lceil 2k/3 \rceil \le d \le k-1 \text{ and } \\ & r \ge k-d; \\ \lfloor n/k \rfloor -1, & \text{otherwise.} \end{cases}$$

Then H contains a matching of size s. In particular, H contains a matching covering all but at most 2k - d - 1 vertices.

Remark. The theorem also holds for $s = \lfloor n/k \rfloor$ in the case $k/2 \le d \le 0.59k$ and r = 1 by a result of Han [10], which we did not include above. On the other hand, one can not hope for $s = \lfloor n/k \rfloor$ in the case $0.59k < d \le k - 1$ and r = 1 due to a construction [10, Proposition 1.11]. However, for other cases it is not clear whether the theorem holds for $s = \lfloor n/k \rfloor$.

The second result is on matchings of any given size s in H for all $s \leq \lfloor n/k \rfloor -1$, which can be easily deduced from Theorem 1.4. In particular, it says that one can take C = 1 in the aforementioned question of Kühn, Osthus, and Townsend [21].

Corollary 1.5. For all integers k,d such that $k \geq 3$ and $k/2 \leq d \leq k-1$, there exists an $n_0 \in \mathbb{N}$ such that the following holds for $n \geq n_0$. Let n be an integer (which may not be divisible by k) and let s be an integer such that $s \leq \lfloor n/k \rfloor - 1$. Suppose H is an n-vertex k-graph with $\delta_d(H) > \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d}$. Then H contains a matching of size s.

Proof. For $s \leq \lfloor n/k \rfloor - 1$, let t be an integer such that $t = \lfloor (n+t)/k \rfloor - s - 1$. Then $s+t = \lfloor (n+t)/k \rfloor - 1$. Consider the auxiliary k-graph H' by adding t vertices to H such that H' contains all edges of H and all k-sets containing any of these new vertices. Note that H' has n+t vertices and

$$\delta_d(H') = \delta_d(H) + \binom{n+t-d}{k-d} - \binom{n-d}{k-d}$$

$$> \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d} + \binom{n+t-d}{k-d} - \binom{n-d}{k-d}$$

$$= \binom{(n+t)-d}{k-d} - \binom{(n+t)-d-(s+t-1)}{k-d}.$$

Thus, we apply Theorem 1.4 on H' and conclude that H' contains a matching M' of size $s+t=\lfloor (n+t)/k\rfloor-1$. By deleting at most t edges from M' that contain the new vertices, we can get a matching in H of size s and we are done.

¹Note that one can choose such a t by trying t = 0, 1, 2, ...

Combining Theorem 1.4 and Corollary 1.5, we obtain the exact minimum d-degree threshold for a matching of size s for all $s \le n/k$ if $k/2 \le d \le 0.59k$ and r = 1, or if $k/2 \le d < \lceil 2k/3 \rceil$ and $2 \le r \le k - 1$, or if $\lceil 2k/3 \rceil \le d \le k - 1$ and $k - d \le r \le k - 1$.

At last, we remark that if we only target on the asymptotical minimum d-degree thresholds the problem would be much easier. To formulate such a result, let us introduce the notion of fractional matchings. Given a k-graph H = (V, E), a fractional matching in H is a function $\omega : E \to [0, 1]$ such that for each $v \in V(H)$ we have that $\sum_{e \ni v} w(e) \le 1$. Then $\sum_{e \in E(H)} w(e)$ is the size of w. If the size of w in H is n/k then we say that w is a perfect fractional matching. Given $k, d \in \mathbb{N}$ such that $d \le k - 1$, define $c_{k,d}^*$ to be the smallest number c such that every k-graph H on n vertices with $\delta_d(H) \ge (c + o(1)) \binom{n-d}{k-d}$ contains a perfect fractional matching. Alon et al. [1] conjectured that $c_{k,d}^* = 1 - (1 - 1/k)^{k-d}$ for all $d, k \in \mathbb{N}$ and verified the case $k - d \le 4$. So far the conjecture was verified when $d \ge 0.4k$ and when $k - d \le 4$ [21, 5].

Theorem 1.6. Let k,d be integers such that $1 \le d \le k-1$ and $\gamma > 0$, then there exists an $n_0 \in \mathbb{N}$ such that the following holds for $n \ge n_0$. Suppose H is an n-vertex k-graph with $\delta_d(H) \ge (c_{k,d}^* + \gamma)\binom{n-d}{k-d}$, then H contains a matching M that covers all but at most 2k - d - 1 vertices. In particular, when $n \in k\mathbb{N}$, M is a perfect matching or covers all but exactly k vertices.

Theorem 1.6 has been proved by Han and Treglown [12, Theorem 7.2] under the assumption $\delta_d(H) \geq (\max\{c_{k,d}^*, 1/3\} + \gamma)\binom{n-d}{k-d}$. Using the regularity method, under the assumption of Theorem 1.6, it is easy to construct a matching that leaves o(n) vertices uncovered. Here we apply the absorbing method and thus reduce the number of uncovered vertices to at most 2k-d-1.

As a typical approach in this area, we employ the absorbing method, popularized by Rödl, Ruciński, and Szemerédi [27], and distinguish the "extremal" and "non-extremal" cases, which is also the approach used by Lu, Yu, and Yuan [23]. For example, the extremal case has been resolved by Lu, Yu, and Yuan [23]. The main difference is in the part of absorption in the non-extremal case, where we deal with large and small values of d with different strategies: we use the lattice-based absorbing method of Han [10] when $k/2 \le d < \lceil 2k/3 \rceil$, and use an argument of Rödl, Ruciński, and Szemerédi [27] when $\lceil 2k/3 \rceil \le d \le k-1$. The first method was initially used in [8] for solving a complexity problem of Karpiński, Ruciński, and Szymańska [15].

Organization. Throughout the rest of the paper, k denotes an integer with $k \geq 3$. As usual, for an integer b, let $[b] = \{1, \ldots, b\}$. The rest of the paper is organized as follows. In Section 2, we first give some useful lemmas and then prove Theorem 1.4 and Theorem 1.6. In Sections 3 and 4, we give the proofs of our absorbing lemmas (Lemma 2.3 and Lemma 2.4).

2. Proofs of Theorems 1.4 and 1.6

We first extend the definition of $H_k^k(U,W)$ in Construction 1.3 to $H_k^\ell(U,W)$ for all $\ell \in [k]$. Again, let V be a vertex set with a partition $U \cup W$. Let $H_k^\ell(U,W)$ be the k-graph on V whose edges are all k-sets e such that $1 \leq |e \cap W| \leq \ell$ (see Figure 1). Given two k-graphs H_1, H_2 and a real number $\varepsilon > 0$, we say that H_2 is ε -close to H_1 if $V(H_1) = V(H_2)$ and $|E(H_1) \setminus E(H_2)| \leq \varepsilon |V(H_1)|^k$.

We write $x \ll y \ll z$ to mean that we can choose constants from right to left, that is, there exist functions f and g such that, for any z > 0, whenever $y \le f(z)$ and $x \le g(y)$, the subsequent statement holds. Statements with more variables are defined similarly. The following result by Lu, Yu, and Yuan [23] is needed for the "extremal" case.

Lemma 2.1 ([23, Lemma 2.3]). Given integers d and k such that $d \in [k-1]$, suppose $0 < 1/n \ll \varepsilon \ll 1/k$ and let $s \in \{\lfloor n/k \rfloor - 1, \lfloor n/k \rfloor\}$. Suppose H is an n-vertex k-graph with $\delta_d(H) > \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d}$, and H is ε -close to $H_k^{k-d}(U,W)$, where |W| = s-1 and |U| = n-s+1. Then H contains a matching of size s.

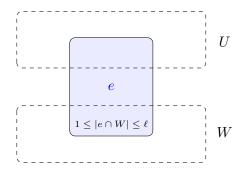


FIGURE 1. An illustration of the k-graph $H_k^{\ell}(U, W)$.

We also need the following lemma in the "non-extremal" case, which guarantees that the resulting k-graph has an almost perfect matching after taking away an absorbing matching (from Lemma 2.3 and Lemma 2.4).

Lemma 2.2 ([23, Lemma 5.7]). Given $\varepsilon, \sigma > 0$ and integers k, d such that $k/2 \le d \le k-1$, suppose $0 < 1/n \ll \rho \ll \varepsilon, 1/k$ and let $s \in \{\lfloor n/k \rfloor - 1, \lfloor n/k \rfloor\}$. Suppose H is an n-vertex k-graph with $\delta_d(H) \ge \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d} - \rho n^{k-d}$, and H is not ε -close to $H_k^{k-d}(U,W)$ for any partition U,W of V(H) with |W| = s-1. Then H contains a matching covering all but at most σn vertices.

Now we state our absorbing lemmas, which are the new ingredients in the proof of Theorem 1.4. We postpone their proofs to Sections 3 and 4. Indeed, a known method of Rödl, Ruciński, and Szemerédi [27] can be used to prove the absorbing lemma for the case $\lceil 2k/3 \rceil \le d \le k-1$ (Lemma 2.4), but does not apply for smaller d. Fortunately, for smaller values of d the quantity $\delta_d(H)$ is larger, in particular, $\delta_d(H) \ge (1/4 + o(1)) \binom{n-d}{k-d}$. This allows us to use the lattice-based absorption method, developed by Han [10], and prove the absorbing lemma for the case $\min\{3, k/2\} \le d < \lceil 2k/3 \rceil$ (Lemma 2.3).

Lemma 2.3. For $\min\{3, k/2\} \leq d < \lceil 2k/3 \rceil$ and $\gamma > 0$, suppose $0 < 1/n \ll \alpha \ll \gamma, 1/k$. Let H be an n-vertex k-graph with $\delta_d(H) \geq (1/4+\gamma)\binom{n-d}{k-d}$. Then there exists a matching M in H of size $|M| \leq \gamma n/k$ such that for any subset $R \subseteq V(H) \setminus V(M)$ with $|R| \leq \alpha^2 n$, $H[R \cup V(M)]$ contains a matching covering all but at most k+1 vertices.

Lemma 2.4. For $\lceil 2k/3 \rceil \leq d \leq k-1$ and $\gamma' > 0$, suppose $0 < 1/n \ll \beta \ll \gamma', 1/k$. Let H be an n-vertex k-graph with $\delta_d(H) \geq \gamma' n^{k-d}$, then there exists a matching M' in H of size $|M'| \leq \beta n/k$ and such that for any subset $R \subseteq V(H) \setminus V(M')$ with $|R| \leq \beta^2 n$, $H[V(M') \cup R]$ contains a matching covering all but at most 2k - d - 1 vertices.

Now we combine Lemmas 2.1, 2.2, 2.3, 2.4 and prove Theorem 1.4.

Proof of Theorem 1.4. Given $k \geq 3$ and $k/2 \leq d \leq k-1$. Let $\varepsilon > 0$ be as in Lemma 2.1, and we choose additional constants satisfying the following hierarchy depending on the value of d:

$$0<\sigma\ll\eta=\gamma\ll\varepsilon\ll1/k \text{ if } k/2\leq d<\lceil 2k/3\rceil;$$

and

$$0 < \sigma \ll \eta \ll \gamma', \varepsilon \ll 1/k \text{ if } \lceil 2k/3 \rceil \leq d \leq k-1.$$

More precisely, take $\eta := \gamma$ and $\sigma := \alpha^2 \ll \gamma$ in the case $\min\{3, k/2\} \leq d < \lceil 2k/3 \rceil$, where α is given by Lemma 2.3; and take $\eta := \beta \ll \gamma'$ and $\sigma := \beta^2 \ll \eta$ in the case $\lceil 2k/3 \rceil \leq d \leq k-2$, where β is given by Lemma 2.4. We also assume that n is sufficiently large. Let H = (V, E) be an n-vertex k-graph satisfying the degree condition as in Theorem 1.4. We may assume $d \leq k-2$ since the case d = k-1 has been proved by Han [10, Theorem 1.1]. The extremal case immediately

follows from Lemma 2.1. Note that this is the only place where the exact d-degree condition is needed. Thus, we may assume that H is not ε -close to $H_k^{k-d}(U,W)$ for any partition U,W of V with |W| = s - 1.

Note that when $\min\{3, k/2\} \le d < \lceil 2k/3 \rceil$, we have

$$\binom{n-d-s+1}{k-d} \le \left(\frac{n-s+1}{n-k+1}\right)^{k-d} \binom{n-d}{k-d} \le \left(1 - \frac{s-k}{n-k+1}\right)^{k/3} \binom{n-d}{k-d}$$

Using $1-x \le e^{-x}$ and $sk \ge n-2k$ (as $s \ge n/k-2$), we infer

$$\binom{n-d-s+1}{k-d} \le e^{-\frac{s-k}{n-k+1} \cdot \frac{k}{3}} \binom{n-d}{k-d} \le e^{-\frac{n-2k-k^2}{n-k+1} \cdot \frac{1}{3}} \binom{n-d}{k-d} \le e^{-0.33} \binom{n-d}{k-d},$$

where we used that $\frac{n-2k-k^2}{n-k+1} \ge 0.99$ as n is large. We conclude that $\binom{n-d}{k-d} - \binom{n-d-s+1}{k-d} \ge (1/4 + \gamma)\binom{n-d}{k-d}$ as $e^{-0.33} \approx 0.71$ and $\gamma \ll 1/k$. On the other hand, when $\lceil 2k/3 \rceil \le d \le k-2$, we have $\binom{n-d}{k-d} - \binom{n-d-s+1}{k-d} \ge \gamma' n^{k-d}$ by the choice of γ' .

First we find an absorbing set in H. Note that $2k-d-1 \ge k+1$ since $d \le k-2$. By Lemma 2.3 and Lemma 2.4, there exists a matching M in H of size $|M| \le \eta n/k$ such that, for every subset $R \subseteq V \setminus V(M)$ with $|R| \le \sigma n$, the induced k-graph $H[R \cup V(M)]$ contains a matching covering all but at most $\max\{k+1, 2k-d-1\} = 2k-d-1$ vertices.

Let $H_1 = H[V \setminus V(M)]$ and $n_1 = n - k|M| \ge (1 - \eta)n$. Next we find an almost perfect matching in H_1 . Recall that $\delta_d(H) > \binom{n-d}{k-d} - \binom{n-d-s+1}{k-d}$, then we have that

$$\delta_d(H_1) \ge \delta_d(H) - (\eta n)n^{k-d-1} \ge \binom{n_1 - d}{k - d} - \binom{n_1 - d - s + 1}{k - d} - 2\eta n_1^{k-d}$$

and H_1 is not $(\varepsilon/2)$ -close to $H_k^{k-d}(U,W)$ for any partition U,W of $V(H_1)$ with |W| = s-1, since n is sufficiently large and $\eta \ll \varepsilon$. We now apply Lemma 2.2 with input σ and $\varepsilon/2$ in place of ε , H_1 has a matching M_1 covering all but at most σn vertices of $V(H_1)$. Denote by U the set of these remaining vertices of $V(H_1)$. Then $|U| \leq \sigma n$.

Let $\ell k + r = n$, where $0 \le r \le k - 1$. Recall that M is an absorbing matching in H, which implies that $H[V(M) \cup U]$ contains a matching M_2 covering all but at most 2k - d - 1 vertices. Let m be the number of unmatched vertices by $M_1 \cup M_2$. Then for the case $k/2 \le d < \lceil 2k/3 \rceil$ we have $m \le k + 1$ by Lemma 2.3; more precisely, in this case m = k + r when $r \in \{0, 1\}$ and m = r when $2 \le r \le k - 1$. Moreover, for the case $\lceil 2k/3 \rceil \le d \le k - 1$ we have $m \le 2k - d - 1$ by Lemma 2.4; more precisely, in this case m = k + r when $0 \le r < k - d$ and m = r when $k - d \le r \le k - 1$. These imply that $M_1 \cup M_2$ is the desired matching.

The following result is a weaker version of Lemma 5.6 in [21]. It allows us to convert the fractional matchings into integer ones, and was proved by the weak hypergraph regularity lemma (and alternative proofs avoiding the regularity method are also known, see e.g. [1]).

Lemma 2.5 ([21]). Let $k \geq 2$ and $1 \leq d \leq k-1$ be integers, and let $\varepsilon > 0$. Suppose that for some $b, c \in (0,1)$ and some $n_0 \in \mathbb{N}$, every k-graph on $n \geq n_0$ vertices with $\delta_d(H) \geq c \binom{n-d}{k-d}$ has a fractional matching of size $(b+\varepsilon)n$. Then there exists an $n'_0 \in \mathbb{N}$ such that every k-graph H on $n \geq n'_0$ vertices with $\delta_d(H) \geq (c+\varepsilon) \binom{n-d}{k-d}$ has an integer matching of size at least bn.

As a consequence we obtain the following result by the definition of $c_{k,d}^*$.

Lemma 2.6. Let k,d be integers such that $1 \le d \le k-1$ and $\gamma,\sigma > 0$, the following holds for sufficiently large n. Suppose H is an n-vertex k-graph with $\delta_d(H) \ge (c_{k,d}^* + \gamma) \binom{n-d}{k-d}$, then H contains a matching M that covers all but at most σn vertices.

Proof. Let k,d and γ be given, and let n_0 and n'_0 be given by Lemma 2.5. Without loss of generality we may assume that $1/n \ll \sigma \ll \gamma, 1/k$. Note that every k-graph G with $\delta_d(G) \geq (c_{k,d}^* + \sigma/k) \binom{n-d}{k-d}$ has a perfect fractional matching of size n/k by the definition of $c_{k,d}^*$; and then we use Lemma 2.5 to transform this into an almost perfect integer matching. More precisely, since $\delta_d(H) \geq (c_{k,d}^* + \sigma) \binom{n-d}{k-d}$ by the choice of σ , applying Lemma 2.5 on H with $b = (1-\sigma)/k$, $c = c_{k,d}^* + \sigma/k$ and $\varepsilon = \sigma/k$, we conclude that H contains an integer matching of size at least $(1-\sigma)n/k$, that is, a matching covering all but at most σn vertices, as required.

We are now in a position to prove Theorem 1.6. Its proof is very similar to that of Theorem 1.4, where we replace Lemma 2.2 by Lemma 2.6, and Lemma 2.1 is no longer needed.

Proof of Theorem 1.6. Let $\gamma > 0$ and integers $1 \le d \le k-1$ be given. Note that $c_{k,d}^* \ge 1 - (1-1/k)^{k-d}$ (see e.g. [1]), and recall that Theorem 1.6 has been proved by Han and Treglown when $c_{k,d}^* \ge 1/3$ [12, Theorem 7.2]. For the case $1 \le d \le k/2$, we have $c_{k,d}^* \ge 1/3$. Indeed, since $(1-1/k)^k < 1/e$, we have

$$c_{k,d}^* \ge 1 - (1 - 1/k)^{k-d} > 1 - (1/e)^{1-d/k} \ge 1 - (1/e)^{1/2} \ge 1/3.$$

These remarks imply that Theorem 1.6 holds for the case $1 \le d \le k/2$. Furthermore, we may assume $d \le k-2$ since the case d=k-1 has been proved by Han [10, Theorem 1.1]. Next we deal with the case $k/2 \le d \le k-2$.

Suppose $0 < \sigma \ll \eta \ll \gamma_1, \gamma_2 \ll \gamma \ll 1/k$. Let H = (V, E) be an n-vertex k-graph with $\delta_d(H) \ge (c_{k,d}^* + \gamma) \binom{n-d}{k-d}$. Recall that $c_{k,d}^* \ge 1 - (1-1/k)^{k-d} > 1 - (1/e)^{1-d/k}$. Thus, $c_{k,d}^* > 1/4$ when $k/2 \le d < \lceil 2k/3 \rceil$. Consequently we have $\delta_d(H) \ge (1/4 + \gamma_1) \binom{n-d}{k-d}$ when $k/2 \le d < \lceil 2k/3 \rceil$ by the choice of γ_1 , and $\delta_d(H) \ge \gamma_2 n^{k-d}$ when $\lceil 2k/3 \rceil \le d \le k-2$ by the choice of γ_2 .

The proof is almost identical to the proof of Theorem 1.4 except that we use Lemma 2.6 instead of Lemma 2.2. First we find an absorbing set in H. Note that $2k-d-1 \ge k+1$ since $d \le k-2$. By Lemma 2.3 and Lemma 2.4, there exists a matching M in H of size $|M| \le \eta n/k$ such that, for every subset $R \subseteq V \setminus V(M)$ with $|R| \le \sigma n$, the induced k-graph $H[R \cup V(M)]$ contains a matching covering all but at most $\max\{k+1, 2k-d-1\} = 2k-d-1$ vertices.

Let $H_1 = H[V \setminus V(M)]$ and $n_1 = n - k|M| \ge (1 - \eta)n$. Next we find an almost perfect matching in H_1 . Recall that $\delta_d(H) \ge (c_{k,d}^* + \gamma) \binom{n-d}{k-d}$, then we have that

$$\delta_d(H_1) \ge \delta_d(H) - (\eta n) n^{k-d-1} \ge (c_{k,d}^* + \gamma/2) \binom{n_1 - d}{k - d},$$

since n is sufficiently large and $\eta \ll \gamma$. Applying Lemma 2.6 on H_1 with input σ and $\gamma/2$ in place of γ , H_1 has a matching M_1 covering all but at most σn vertices of $V(H_1)$. Denote by U the set of these remaining vertices of $V(H_1)$. Then $|U| \leq \sigma n$. Recall that M is an absorbing matching in H, which implies that $H[V(M) \cup U]$ contains a matching M_2 covering all but at most 2k - d - 1 vertices. Thus, $M_1 \cup M_2$ is the desired matching.

3. Proof of Lemma 2.3

In this section we prove Lemma 2.3. Because of the existence of the divisibility barrier (Construction 1.2), the absorbing lemma for perfect matching requires a minimum d-degree $\delta_d(H) \geq (1/2 + o(1))\binom{n-d}{k-d}$. However, we only have the minimum d-degree $\delta_d(H) \geq (1/4 + o(1))\binom{n-d}{k-d}$. Inspired by the divisibility barrier, we consider vertex partitions of H and analyze them via the distribution of edges (robust edge-lattice). Such approach was first used by Keevash and Mycroft [16].

3.1. Notation and preliminaries. In order to state our results, we first introduce some notation. Let H be a k-graph on n vertices. We say that two vertices u and v are (β, i) -reachable in H if there are at least βn^{ik-1} (ik-1)-sets $S \subseteq V(H)$ such that both $H[S \cup \{u\}]$ and $H[S \cup \{v\}]$ have perfect matchings. We say that a vertex set U is (β, i) -closed in H if every two vertices in U are (β, i) -reachable in H.

We will work on a vertex partition $\mathcal{P} = \{V_0, V_1, \dots, V_r\}$ of V(H) for some integer $r \geq 1$. We consider the r-dimensional vectors on the parts of \mathcal{P} except V_0 . Formally, the index vector $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^r$ of a subset $S \subseteq V(H)$ with respect to \mathcal{P} is the vector whose coordinates are the sizes of the intersections of S with each part of \mathcal{P} except V_0 , namely, $\mathbf{i}_{\mathcal{P}}(S)_{V_i} = |S \cap V_i|$ for $i \in [r]$. We call a vector $\mathbf{i} \in \mathbb{Z}^r$ an s-vector if all its coordinates are nonnegative and their sum equals s, and we use I_s^r to denote the set of all s-vectors. Given $\mu > 0$, a k-vector \mathbf{v} is called a μ -robust edge-vector if there are at least μn^k edges $e \in E(H)$ satisfying $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{v}$. Let $I_{\mathcal{P}}^{\mu}(H) \subseteq I_k^r$ be the set of all μ -robust edge-vectors and let $L_{\mathcal{P}}^{\mu}(H)$ be the lattice (additive subgroup) generated by the vectors in $I_{\mathcal{P}}^{\mu}(H)$. For $i \in [r]$, let $\mathbf{u}_i \in \mathbb{Z}^r$ be the i-th unit vector, namely, \mathbf{u}_i has 1 on the i-th coordinate and 0 on other coordinates. A transferral is the vector $\mathbf{u}_i - \mathbf{u}_i$ for some $i \neq j$.

Suppose I is a set of k-vectors in \mathbb{Z}^r and J is a set of vectors in \mathbb{Z}^r such that any $\mathbf{i} \in J$ can be written as a linear combination of vectors in I, namely, $\mathbf{i} = \sum_{\mathbf{v} \in I} a_{\mathbf{v}} \mathbf{v}$. We denote C(r, k, I, J) as the maximum of $|a_{\mathbf{v}}|$, $\mathbf{v} \in I$, over all $\mathbf{i} \in J$ and $C(k, J) := \max_{r \leq k, I \subseteq I_k^r} C(r, k, I, J)$.

For a k-graph H, we first establish a partition P of V(H) and then study the so-called robust edge-lattice with respect to this partition. The main tool for establishing the partition of V(H) is the following lemma from [10, Lemma 3.3].

Lemma 3.1 ([10, Lemma 3.3]). Suppose $0 < 1/n \ll \beta \ll \varepsilon \ll \delta$. Let H be an n-vertex k-graph such that $\delta_1(H) \geq (\delta + k^2 \varepsilon) \binom{n-1}{k-1}$. Then there is a partition \mathcal{P} of V(H) into V_0, V_1, \ldots, V_r with $r \leq \lfloor 1/\delta \rfloor$ such that $|V_0| \leq \sqrt{\varepsilon}n$ and for any $i \in [r], |V_i| \geq \varepsilon^2 n$ and V_i is $(\beta, 2^{\lfloor 1/\delta \rfloor - 1})$ -closed in H.

The next result from [10, Lemma 3.4] shows that $V_i \cup V_j$ is closed in H if $\mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}}(H)$, which means that we can merge V_i and V_j and keep the closedness.

Lemma 3.2 ([10, Lemma 3.4]). Let $0 < \mu, \beta \ll \varepsilon \ll 1/i_0$, then there exist $0 < \beta' \ll \mu, \beta$ and an integer $t \geq i_0$ such that the following holds for sufficiently large n. Suppose H is an n-vertex k-graph, and $\mathcal{P} = \{V_0, V_1, \ldots, V_r\}$ is a partition with $r \leq i_0$ such that $|V_0| \leq \sqrt{\varepsilon}n$ and for any $i \in [r], |V_i| \geq \varepsilon^2 n$ and V_i is (β, i_0) -closed in H. If $\mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}}(H)$, then $V_i \cup V_j$ is (β', t) -closed in H.

The following absorbing lemma from [10, Lemma 3.5] will be applied to prove Lemma 2.3. Let H be a k-graph and let $i \in k\mathbb{N}$. For a k-vertex set S, we say a vertex set T is an absorbing i-set for S if |T| = i and both H[T] and $H[T \cup S]$ contain perfect matchings.

Lemma 3.3 ([10, Lemma 3.5]). Suppose $r \leq k$ and

$$1/n \ll \alpha \ll \mu, \beta \ll 1/k, 1/t.$$

Suppose that $\mathcal{P}_0 = \{V_0, V_1, \dots, V_r\}$ is a partition of V(H) such that for $i \in [r]$, V_i is (β, t) -closed. Then there is a family \mathcal{F}_{abs} of disjoint tk^2 -sets with size at most βn such that $H[V(\mathcal{F}_{abs})]$ contains a perfect matching and every k-vertex set S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I^{\mu}_{\mathcal{P}_0}(H)$ has at least αn absorbing tk^2 -sets in \mathcal{F}_{abs} .

Let us give the following simple and useful proposition for considering d-degree together with d'-degree for some $d \neq d'$.

Proposition 3.4. Let $0 \le d' \le d \le k-1$ and H be a k-graph on n vertices. If $\delta_d(H) \ge c \binom{n-d}{k-d}$ for some $0 \le c \le 1$, then $\delta_{d'}(H) \ge c \binom{n-d'}{k-d'}$.

This proposition is straightforward since $\delta_{d'}(H) \geq \binom{n-d'}{d-d'} \delta_d(H) / \binom{k-d'}{d-d'}$.

3.2. **Proof of Lemma 2.3.** We now finish the proof of Lemma 2.3, which we restate below for convenience.

Lemma 2.3. For $\min\{3, k/2\} \leq d < \lceil 2k/3 \rceil$ and $\gamma > 0$, suppose $0 < 1/n \ll \alpha \ll \gamma, 1/k$. Let H be an n-vertex k-graph with $\delta_d(H) \geq (1/4 + \gamma) \binom{n-d}{k-d}$. Then there exists a matching M in H of size $|M| \leq \gamma n/k$ such that for any subset $R \subseteq V(H) \setminus V(M)$ with $|R| \leq \alpha^2 n$, $H[R \cup V(M)]$ contains a matching covering all but at most k+1 vertices.

One of the key steps in the proof of Lemma 2.3 is the following proposition. We postpone its proof to the end of this section.

Proposition 3.5. Given $\min\{3, k/2\} \leq d < \lceil 2k/3 \rceil$, suppose $0 < 1/n \ll \mu \ll \varepsilon \ll \gamma$. Let H be an n-vertex k-graph with $\delta_d(H) \geq (1/4 + \gamma)\binom{n-d}{k-d}$, and let $\mathcal{P} = \{V_0, V_1, \dots, V_r\}$ be a partition of V(H) with $r \leq 3$ such that $|V_0| \leq \sqrt{\varepsilon}n$ and for each $i \in [r], |V_i| \geq \varepsilon^2 n$, and $L^{\mu}_{\mathcal{P}}(H)$ contains no transferral. Then for every $U \subseteq V(H) \setminus V_0$ with |U| = k + 2, there exist $i, j \in [r]$ such that $\mathbf{i}_{\mathcal{P}}(U) - \mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}}(H)$.

Now we are ready to prove Lemma 2.3. Here is a brief outline of the proof. We apply Lemma 3.1 to H and obtain a partition \mathcal{P} of V(H) such that each part is closed and not too small. We then merge two parts V_i and V_j if the transferral $\mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}}(H)$. So we obtain a transferral-free partition $\mathcal{P}_0 = \{V_0, V_1, \ldots, V_{r'}\}$, where $r' \leq 3$. Lemma 3.3 implies the existence of a family \mathcal{F}_{abs} of disjoint absorbing tk^2 -sets, which can be used to absorb a small collection of k-sets each with index vector in $L^{\mu}_{\mathcal{P}_0}(H)$. The key point is that as long as there are at least k+2 vertices uncovered, Proposition 3.5 implies that we can "absorb" k vertices, and our (quantitative) choice of the family \mathcal{F}_{abs} allows us to proceed the absorption to reduce the number of uncovered vertices in a greedy manner. The absorption terminates when there are at most k+1 vertices left uncovered.

Proof of Lemma 2.3. Fix $\gamma > 0$ and let $C := C(k, I_{2k}^r)$. We define additional constants such that

$$0<1/n\ll\alpha\ll\beta'\ll\beta,\mu\ll\varepsilon\ll\gamma,1/k,1/t,1/C.$$

Let H=(V,E) be a k-graph with $\delta_d(H)\geq (1/4+\gamma)\binom{n-d}{k-d}$. Note that we have $\delta_1(H)\geq (1/4+\gamma)\binom{n-1}{k-1}$ by Proposition 3.4. We first apply Lemma 3.1 on H with $\delta=1/4+\gamma/2$. Then we get a partition $\mathcal{P}=\{V_0,V_1',\ldots,V_{r'}'\}$ with $r'\leq 3$ such that $|V_0|\leq \sqrt{\varepsilon}n$ and for any $i\in [r'], |V_i'|\geq \varepsilon^2n$ and V_i' is $(\beta,4)$ -closed in H. If $\mathbf{u}_i-\mathbf{u}_j\in L^\mu_\mathcal{P}(H)$ for some $i,j\in [r'], i\neq j$, then we merge V_i' and V_j' to one part, and by Lemma 3.2, $V_i'\cup V_j'$ is (β'',t') -closed for some $\beta''>0$ and $t'\geq 4$. We greedily merge the parts until there is no transferral in the μ -robust edge-lattice. Let $\mathcal{P}_0=\{V_0,V_1,\ldots,V_r\}$ be the resulting partition for some $1\leq r\leq 3$. Note that we may apply Lemma 3.2 at most twice, and we see that V_i is (β',t) -closed for each $i\in [r]$ by the choice of β' . We apply Lemma 3.3 on H and get a family \mathcal{F}_{abs} of disjoint tk^2 -sets, and we conclude that $|V(\mathcal{F}_{abs})|\leq tk^2\beta'n$, $H[V(\mathcal{F}_{abs})]$ contains a perfect matching M_1 , and every k-vertex set S with $\mathbf{i}_{\mathcal{P}_0}(S)\in I^\mu_{\mathcal{P}_0}(H)$ has at least αn absorbing tk^2 -set in \mathcal{F}_{abs} .

Let $V' = V \setminus V(\mathcal{F}_{abs})$ and H' = H[V']. Now we find a matching M_2 in H' as follows. For each $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$, we greedily pick a matching M_v of size $C\alpha^2 n$ such that $\mathbf{i}_{\mathcal{P}_0}(e) = \mathbf{v}$ for every $e \in M_v$. Then let M_2 be the union of M_v for all $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$, and we have $V_0 \cap V(M_2) = \emptyset$. It is possible to pick M_2 because there are at least μn^k edges e with $\mathbf{i}_{\mathcal{P}_0}(e) = \mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$. To be more precise, since $|I^{\mu}_{\mathcal{P}_0}(H)| \leq {k+r-1 \choose r-1} \leq {k+2 \choose 2}$ and $\alpha \ll \beta' \ll \mu \ll 1/t, 1/C$, we obtain

$$|V(M_2) \cup V(\mathcal{F}_{abs})| \le k|I^{\mu}_{\mathcal{P}_0}(H)|C\alpha^2 n + tk^2\beta' n < \mu n,$$

which yields that the number of edges intersecting these vertices is less than μn^k , as required. Next we build a matching M_3 to cover all vertices in $V_0 \setminus V(\mathcal{F}_{abs})$. Note that $|M_3| \leq |V_0| \leq \sqrt{\varepsilon}n$. Specifically, when we greedily match a vertex $v \in V_0 \setminus V(\mathcal{F}_{abs})$, we need to avoid at most $k|M_3|+|V(M_2)\cup V(\mathcal{F}_{abs})| \leq k\sqrt{\varepsilon}n+\mu n \leq 2k\sqrt{\varepsilon}n$ vertices, and thus at most $2k\sqrt{\varepsilon}n^{k-1}$ (k-1)-sets. Since $\delta_1(H) > \gamma\binom{n-1}{k-1} > 2k\sqrt{\varepsilon}n^{k-1}$, we can always find a desired edge containing v and put it to M_3 as needed.

Let $M = M_1 \cup M_2 \cup M_3$. It is easy to see that M is a matching because M_1, M_2 and M_3 are pairwise vertex disjoint. Now we prove that M is the desired matching satisfying the conclusion of Lemma 2.3. Note that $|M| \leq |M_3| + |V(M_2) \cup V(\mathcal{F}_{abs})|/k \leq 2\sqrt{\varepsilon}n \leq \gamma n/k$ since $\varepsilon \ll \gamma, 1/k$. Consider any subset $R \subseteq V \setminus V(M)$ with $|R| \leq \alpha^2 n$. Fix any set $U \subseteq R$ of k+2 vertices, there exist $i, j \in [r]$ such that $\mathbf{i}_{\mathcal{P}_0}(U) - \mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}_0}(H)$ by Proposition 3.5. Note that this does not guarantee that we can delete one vertex u from $U \cap V_i$ and delete another vertex v from $U \cap V_j$ such that $\mathbf{i}_{\mathcal{P}_0}(U \setminus \{u,v\}) \in L^{\mu}_{\mathcal{P}_0}(H)$, because it is possible that $U \cap V_i = \emptyset$ or $U \cap V_j = \emptyset$ for the i, j returned by the proposition. By $d \geq 2$ and the degree condition, there is a vector $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$ such that $\mathbf{v}_{V_i} \geq 1$ and $\mathbf{v}_{V_j} \geq 1$. Notice that M_2 contains $C\alpha^2 n$ edges with index vector \mathbf{v} . Fix one such edge $e \in E_{\mathbf{v}}$ and two vertices $v_1 \in e \cap V_i$, $v_2 \in e \cap V_j$. We delete e from M_2 and let $U' = U \cup (e \setminus \{v_1, v_2\})$. Clearly, $\mathbf{i}_{\mathcal{P}_0}(U') \in L^{\mu}_{\mathcal{P}_0}(H)$ and |U'| = 2k. Hence, by the definition of $L^{\mu}_{\mathcal{P}_0}(H)$, there exist nonnegative integers $b_{\mathbf{v}}, c_{\mathbf{v}}$ for all $\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)$ such that

$$\mathbf{i}_{\mathcal{P}_0}(U') = \sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} b_{\mathbf{v}} \mathbf{v} - \sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} c_{\mathbf{v}} \mathbf{v},$$

which implies that

$$\mathbf{i}_{\mathcal{P}_0}(U') + \sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} c_{\mathbf{v}} \mathbf{v} = \sum_{\mathbf{v} \in I^{\mu}_{\mathcal{P}_0}(H)} b_{\mathbf{v}} \mathbf{v}.$$

We have that $b_{\mathbf{v}}, c_{\mathbf{v}} \leq C$ from the definition of C. For each $\mathbf{v} \in I_{\mathcal{P}_0}^{\mu}(H)$, we pick $c_{\mathbf{v}}$ edges in M_2 with index vector \mathbf{v} . By the equation above, the union of these edges and U' can be partitioned as a collection of k-sets, which contains exactly $b_{\mathbf{v}}$ k-sets F with $\mathbf{i}_{\mathcal{P}_0}(F) = \mathbf{v}$ for each $\mathbf{v} \in I_{\mathcal{P}_0}^{\mu}(H)$. We repeat the process at most $\alpha^2 n/k$ times until there are at most k+1 vertices left. Note that for each $\mathbf{v} \in I_{\mathcal{P}_0}^{\mu}(H)$, our algorithm consumes at most $(1+C)\alpha^2 n/k < C\alpha^2 n$ edges from M_2 with index vector \mathbf{v} , which is possible by the definition of M_2 . Furthermore, after the process, we obtain at most $(2+C|I_{\mathcal{P}_0}^{\mu}(H)|)\alpha^2 n/k \leq (2+C\binom{k+2}{2})\alpha^2 n/k < \alpha n$ k-sets S with $\mathbf{i}_{\mathcal{P}_0}(S) \in I_{\mathcal{P}_0}^{\mu}(H)$ since $\alpha \ll 1/k, 1/C$. By the absorbing property of \mathcal{F}_{abs} , we can greedily absorb them by \mathcal{F}_{abs} and get a matching M_4 . Thus, $H[R \cup V(M)]$ contains a matching covering all but at most k+1 vertices. \square

3.3. The transferral-free lattices. In this subsection we prove Proposition 3.5. We study the lattice structure $L^{\mu}_{\mathcal{D}}(H)$ when it contains no transferral.

Fix $1 \leq p \leq k-1$ and any p-vector \mathbf{v} , we define its neighborhood to be $N(\mathbf{v}) := \{\mathbf{v}' \colon \mathbf{v} + \mathbf{v}' \in L^{\mu}_{\mathcal{P}}(H)\}$. Note that the vectors in $N(\mathbf{v})$ may contain negative coordinates. Moreover, assuming r=2, we claim that $N(\mathbf{v}) \cap I^2_{k-p} \neq \emptyset$ for any $1 \leq p \leq d$ and any p-vector $\mathbf{v} = (i, p-i)$. Indeed, otherwise, let \mathbf{v} be a p-vector such that $N(\mathbf{v}) \cap I^2_{k-p} = \emptyset$. This implies that the number of edges in $H[V \setminus V_0]$ with index vector \mathbf{i} such that $\mathbf{i} - \mathbf{v} \in I^2_{k-p}$ is at most $|I^2_{k-p}| \mu n^k \leq 2^{k-p} \mu n^k$. Let $A_{\mathbf{v}}$ be the set of all p-sets S with $\mathbf{i}_{\mathcal{P}}(S) = \mathbf{v}$, and thus $|A_{\mathbf{v}}| = \binom{|V_1|}{i} \binom{|V_2|}{p-i} \geq \binom{\varepsilon^2 n}{p}$. By averaging, there is a p-set $S \in A_{\mathbf{v}}$ such that

$$\deg_H(S) \leq |I_{k-p}^2|\mu n^k/|A_{\mathbf{v}}| + |V_0|n^{k-p-1} \leq 2^{k-p}\mu n^k/\binom{\varepsilon^2 n}{p} + \sqrt{\varepsilon}n^{k-p} < \gamma \binom{n-p}{k-p}$$

by $\mu \ll \varepsilon \ll \gamma$. Since $p \leq d$ and by Proposition 3.4, this contradicts that $\delta_d(H) \geq (1/4 + \gamma) \binom{n-d}{k-d}$. Note that a similar argument works for r = 3; namely, for any p-vector $\mathbf{v} = (i, i', p - i - i')$ with $1 \leq p \leq d$, $N(\mathbf{v}) \cap I_{k-p}^3 \neq \emptyset$.

Claim 3.6. Given $\min\{3, k/2\} \le d < \lceil 2k/3 \rceil$, suppose $0 < \mu \ll \varepsilon \ll \gamma$. Let H and \mathcal{P} be as defined in Proposition 3.5. If r = 2, then $(2, -2) \in L^{\mu}_{\mathcal{P}}(H)$ or $(3, -3) \in L^{\mu}_{\mathcal{P}}(H)$. If r = 3, then $(-2, 1, 1), (1, -2, 1), (1, 1, -2) \in L^{\mu}_{\mathcal{P}}(H)$.

Proof. Our proof is adapted from the proof of [10, Claim 3.7]. First assume that r=2. Fix $(a_0,b_0)\in I^\mu_{\mathcal{P}}(H)$. For the sake of a contradiction, assume that $(2,-2),(3,-3)\notin L^\mu_{\mathcal{P}}(H)$. Let L_0 be the sublattice (subgroup) of $L^\mu_{\mathcal{P}}(H)$ such that $(a,b)\in L_0$ if a+b=0, and let $L_{k-d}=\{(a,b)\colon a+b=k-d\}$. Let t be the smallest positive integer such that $(t,-t)\in L_0$, then it is easy to see that L_0 is generated by (t,-t). By our assumption, $(1,-1),(2,-2),(3,-3)\notin L^\mu_{\mathcal{P}}(H)$, and thus $t\geq 4$. Let $t_0=\min\{t,k-d+1\}$. It is easy to see that L_0 partitions L_{k-d} into t_0 cosets C_0,C_1,\ldots,C_{t_0-1} such that $C_i=(k-d-i,i)+L_0^2$ for all $0\leq i\leq t_0-1$. For any $0\leq j\leq d$ and $\mathbf{v}_j:=(d-j,j)$, by $a_0+b_0=k$, we have

$$N(\mathbf{v}_j) = (a_0, b_0) - (d - j, j) + L_0 = (k - d - (b_0 - j), b_0 - j) + L_0.$$

This means that $N(\mathbf{v}_j) = C_{i_j}$, where $i_j \equiv b_0 - j \mod t_0$. We consider the following two cases depending on the value of d.

Case 1. $d \geq 3$. Since $(1,-1),(2,-2),(3,-3) \notin L^{\mu}_{\mathcal{P}}(H)$, we claim that $N(\mathbf{v}_0),\ldots,N(\mathbf{v}_3)$ are pairwise disjoint. Indeed, if $N(\mathbf{v}_0) \cap N(\mathbf{v}_3) \neq \emptyset$, say $\mathbf{i} \in N(\mathbf{v}_0) \cap N(\mathbf{v}_3)$, then we have $\mathbf{i} + \mathbf{v}_0, \mathbf{i} + \mathbf{v}_3 \in L^{\mu}_{\mathcal{P}}(H)$ and thus $(3,-3) = \mathbf{v}_0 - \mathbf{v}_3 \in L^{\mu}_{\mathcal{P}}(H)$, a contradiction. Other cases can be dealt with similarly. Note that for any $0 \leq j \leq d$ and $\mathbf{v}_j := (d-j,j)$, we have $N(\mathbf{v}_j) = C_{i_j}$, where $i_j \equiv b_0 - j \mod t_0$. So we have $t_0 \geq 4$. For $j = 0, \ldots, 3$, consider the following sums

$$\sum_{(k-d-i_j,i_j)\in C_{i_j},\, 0\leq i_j\leq k-d} \binom{|V_1|}{k-d-i_j} \binom{|V_2|}{i_j},$$

and note that their sum is at most $\binom{n-|V_0|}{k-d}$. By the pigeonhole principle, there exists j' such that the j'-th sum is at most $\frac{1}{4}\binom{n-|V_0|}{k-d}$. This implies that

$$\delta_d(H) \leq \frac{1}{4} \binom{n - |V_0|}{k - d} + |I_{k - d}^2| \mu n^k / \binom{\varepsilon^2 n}{d} + |V_0| n^{k - d - 1} < (1/4 + \gamma) \binom{n - d}{k - d}$$

by $\mu \ll \varepsilon \ll \gamma$, a contradiction.

Case 2. d=2. In this case we have k=4 by $\min\{3,k/2\} \leq d < \lceil 2k/3 \rceil$. Then k-d=2, $t_0=3$, and for i=0,1,2, $C_i \cap I_2^2 = \{(2-i,i)\}$. Since $(1,-1),(2,-2) \notin L^\mu_{\mathcal{P}}(H)$, we know that for j=0,1,2, $N(\mathbf{v}_j) \cap I_2^2 = C_{i_j} \cap I_2^2 = \{(2-i_j,i_j)\}$ are pairwise distinct. So $\{N(\mathbf{v}_j) \cap I_2^2\}_{j=0,1,2} = \{\{(2,0)\},\{(1,1)\},\{(0,2)\}\}$. Note that

$$\min \left\{ \binom{|V_1|}{2}, |V_1| |V_2|, \binom{|V_2|}{2} \right\} \le \max_{x \in (0,1)} \min \left\{ \binom{xn}{2}, xn(1-x)n, \binom{(1-x)n}{2} \right\}
\le \max_{x \in (0,1)} \left(\min \left\{ x^2, 2x(1-x), (1-x)^2 \right\} + \gamma/4 \right) \binom{n-2}{2}
= (1/4 + \gamma/4) \binom{n-2}{2}$$

since n is large enough. By averaging, we get that

$$\delta_2(H) \le (1/4 + \gamma/4) \binom{n-2}{2} + 2^2 \mu n^4 / \binom{\varepsilon^2 n}{2} + |V_0| n < (1/4 + \gamma) \binom{n-2}{2}$$

by $\mu \ll \varepsilon \ll \gamma$, a contradiction.

²As usual, for a subgroup H of a group G and an element x of G, x + H denotes a coset of H.

Second assume that r=3. Indeed, in this case, by $\min\{3, k/2\} \le d < \lceil 2k/3 \rceil$ and Proposition 3.4, we have $\delta_2(H) \ge (1/4 + \gamma) \binom{n-2}{k-2}$. Consider the set of 2-vectors

$$I_2^3 = \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}.$$

Note that N((1,1,0)), N((1,0,1)), and N((0,1,1)) are pairwise disjoint because $L^{\mu}_{\mathcal{P}}(H)$ contains no transferral. Similarly, $N((2,0,0)) \cap N((1,1,0)) = \emptyset$ and $N((2,0,0)) \cap N((1,0,1)) = \emptyset$ (and similar equations hold for other vectors). Moreover, recall that $N(\mathbf{v}) \cap I^3_{k-2} \neq \emptyset$ for all $\mathbf{v} \in I^3_2$. Thus, I^3_{k-2} are partitioned into classes C'_1, \dots, C'_m for $m \geq 3$ where each class has the form $N(\mathbf{v}) \cap I^3_{k-2}$ for some (not necessarily unique) $\mathbf{v} \in I^3_2$. If $m \geq 4$, then consider the sums $\sum_{(j_1, j_2, j_3) \in C'_i} \binom{|V_1|}{j_1} \binom{|V_2|}{j_2} \binom{|V_3|}{j_3}$ for $i = 1, \dots, m$ and note that their sum is at most $\binom{n-|V_0|}{k-2}$. Similar to the previous cases, by averaging, we have

$$\delta_2(H) \le \frac{1}{m} \binom{n - |V_0|}{k - 2} + |I_{k-2}^3| \mu n^k / \binom{\epsilon^2 n}{2} + |V_0| n^{k-3} < (1/4 + \gamma) \binom{n - 2}{k - 2}$$

by $\mu \ll \epsilon \ll \gamma$, a contradiction. Otherwise, m=3. Since N((1,1,0)), N((1,0,1)) and N((0,1,1)) must be in different classes, we know that

$$N((1,1,0)) = N((0,0,2)), N((1,0,1)) = N((0,2,0)), \text{ and } N((0,1,1)) = N((2,0,0)),$$
 which implies that $(-2,1,1), (1,-2,1), (1,1,-2) \in L^{\mu}_{\mathcal{P}}(H)$.

Proof of Proposition 3.5. Given such a k-graph H and a partition \mathcal{P} , the conclusion is trivial for r=1. So we may assume that $r\in\{2,3\}$. Applying Claim 3.6, we conclude that (2,-2) or $(3,-3)\in L^{\mu}_{\mathcal{P}}(H)$ (for r=2), or $(-2,1,1),(1,-2,1),(1,1,-2)\in L^{\mu}_{\mathcal{P}}(H)$ (for r=3).

If r=2, then fix any $U\subseteq V(H)\backslash V_0$ with $\mathbf{i}_{\mathcal{P}}(U)=(a,k+2-a)$ for some $0\leq a\leq k+2$ and pick any $(a_0,b_0)\in I^\mu_{\mathcal{P}}(H)$. We distinguish two cases. First we assume that $(2,-2)\in L^\mu_{\mathcal{P}}(H)$, then we have $(a_0+2i,b_0-2i)\in L^\mu_{\mathcal{P}}(H)$ for any integer i. Our goal is to show that there exist $i,j\in [2]$ such that $\mathbf{i}_{\mathcal{P}}(U)-\mathbf{u}_i-\mathbf{u}_j\in L^\mu_{\mathcal{P}}(H)$. It suffices to prove that $(a-1,k+1-a)\in L^\mu_{\mathcal{P}}(H)$ or $(a,k-a)\in L^\mu_{\mathcal{P}}(H)$. Note that a-1 and a have different parities, so exactly one of (a-1,k+1-a) and (a,k-a) is in $L^\mu_{\mathcal{P}}(H)$. Second we assume that $(3,-3)\in L^\mu_{\mathcal{P}}(H)$, then we have $(a_0+3i,b_0-3i)\in L^\mu_{\mathcal{P}}(H)$ for any integer i. Our goal is to show that there exist $i,j\in [2]$ such that $\mathbf{i}_{\mathcal{P}}(U)-\mathbf{u}_i-\mathbf{u}_j\in L^\mu_{\mathcal{P}}(H)$. It suffices to prove that $(a-2,k+2-a), (a-1,k+1-a), \text{ or } (a,k-a) \text{ is in } L^\mu_{\mathcal{P}}(H)$. Note that exactly one of the three consecutive integers a-2, a-1, and a is congruent with a_0 modulo 3. Thus, exactly one of (a-2,k+2-a), (a-1,k+1-a), and <math>(a,k-a) is in $L^\mu_{\mathcal{P}}(H)$.

Next we assume r=3 and fix any $U\subseteq V(H)\setminus V_0$ with $\mathbf{i}_{\mathcal{P}}(U)=(x_1,x_2,x_3)$ for some nonnegative integers $x_1+x_2+x_3=k+2$. Pick any $(y_1,y_2,y_3)\in I^{\mu}_{\mathcal{P}}(H)$ and let $z_j=x_j-y_j$ for $j\in[3]$. Note that exactly one of the three consecutive integers $z_2-z_1,\,z_2-z_1+1$, and z_2-z_1+2 is divisible by 3. Let $i,j\in\{1,3\}$ such that $\mathbf{v}:=(z'_1,z'_2,z'_3)=(z_1,z_2,z_3)-\mathbf{u}_i-\mathbf{u}_j$ satisfies that $z'_2-z'_1$ is divisible by 3. Let $m'=(z'_2-z'_1)/3$ and $m=m'-z'_2$. Note that $z'_1+z'_2+z'_3=0$; then it is easy to see that

$$\mathbf{i}_{\mathcal{P}}(U) - \mathbf{u}_i - \mathbf{u}_j - (y_1, y_2, y_3) = \mathbf{v} = m'(-2, 1, 1) - m(1, 1, -2) \in L^{\mu}_{\mathcal{P}}(H).$$

Thus, $\mathbf{i}_{\mathcal{P}}(U) - \mathbf{u}_i - \mathbf{u}_j \in L^{\mu}_{\mathcal{P}}(H)$, and the proof is complete.

4. Proof of Lemma 2.4

We start with the following definition. Given a set S of 2k-d vertices, an edge $e \in E(H)$ that is disjoint from S is called S-absorbing if there are two disjoint edges e_1 and e_2 in E(H) such that $|e_1 \cap S| = k - \lfloor d/2 \rfloor$, $|e_1 \cap e| = \lfloor d/2 \rfloor$, $|e_2 \cap S| = k - \lceil d/2 \rceil$, and $|e_2 \cap e| = \lceil d/2 \rceil$ (see Figure 2). Note that this is not the absorbing structure in the usual sense because $e_1 \cup e_2$ misses k-d vertices of $S \cup e$. Let us explain how such absorbing structure works. Consider a matching M and a (2k-d)-set S, $V(M) \cap S = \emptyset$. If M contains an S-absorbing edge e, then one can "absorb" S into M by swapping e for e_1 and e_2 (k-d vertices of e become uncovered).

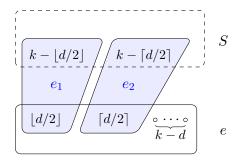


FIGURE 2. An S-absorbing edge e.

Below we restate and prove Lemma 2.4.

Lemma 2.4. For $\lceil 2k/3 \rceil \leq d \leq k-1$ and $\gamma' > 0$, suppose $0 < 1/n \ll \beta \ll \gamma', 1/k$. Let H be an n-vertex k-graph with $\delta_d(H) \geq \gamma' n^{k-d}$, then there exists a matching M' in H of size $|M'| \leq \beta n/k$ and such that for any subset $R \subseteq V(H) \setminus V(M')$ with $|R| \leq \beta^2 n$, $H[V(M') \cup R]$ contains a matching covering all but at most 2k - d - 1 vertices.

The proof of Lemma 2.4 follows an idea of Rödl, Ruciński, and Szemerédi [29, Fact 2.2, Fact 2.3], which was used for the case d = k - 1.

Proof of Lemma 2.4. Let k, d, γ' be given and let β be a constant such that $\beta \leq \beta_0 = (\gamma')^3/(12(k+1)!)$. Denote $\ell_1 := \lfloor d/2 \rfloor$ and $\ell_2 := \lceil d/2 \rceil$, then $\ell_1 + \ell_2 = d$. Let H be an n-vertex k-graph with $\delta_d(H) \geq \gamma' n^{k-d}$. Note that we have $\delta_{k-\ell_1}(H) \geq \gamma' n^{\ell_1}$ and $\delta_{k-\ell_2}(H) \geq \gamma' n^{\ell_2}$ from Proposition 3.4 and by $k - \ell_2 \leq k - \ell_1 \leq d$.

We first show that there are many S-absorbing edges in H for any (2k-d)-set of vertices S.

Claim 4.1. For every $S = \{u_1, \dots, u_{2k-d}\} \in \binom{V(H)}{2k-d}$, there are at least $\frac{1}{2}(\gamma')^3 n^k/k!$ S-absorbing edges.

Proof. Fix $k-\ell_1$ vertices $u_1,\ldots,u_{k-\ell_1}$ in S and let e_1,e_2 be as in the definition of S-absorbing edges. Let us count only those S-absorbing edges e for which the corresponding edge e_1 contains $u_1,\ldots,u_{k-\ell_1}$. We count the ordered k-tuples of distinct vertices (v_1,\ldots,v_k) such that $e=\{v_1,\ldots,v_k\}$ is disjoint from $S,\ e_2\cap e=\{v_{\ell_1+1},\ldots,v_d\}$, and $e_1=\{u_1,\ldots,u_{k-\ell_1},v_1,\ldots,v_{\ell_1}\}$, and divide the result by k!.

Note that $\{v_1,\ldots,v_{\ell_1}\}$ must be a neighbor of an already fix $(k-\ell_1)$ -tuple of vertices, so there are at least $\delta_{k-\ell_1}(H)-2kn^{\ell_1-1}$ choices for the ℓ_1 -tuple. Recall that $\ell_2=d-\ell_1$. Having selected v_1,\ldots,v_{ℓ_1} , $\{v_{\ell_1+1},\ldots,v_d\}$ must be a neighbor of an already fixed $(k-\ell_2)$ -tuple of vertices, so there are at least $\delta_{k-\ell_2}(H)-2kn^{\ell_2-1}$ choices for the ℓ_2 -tuple. Having selected v_1,\ldots,v_d , $\{v_{d+1},\ldots,v_k\}$ must be a neighbor of an already fixed d-tuple of vertices, so there are at least $\delta_d(H)-2kn^{k-d-1}$ choices for the (k-d)-tuple. Altogether since n is large enough, there are at least $(\delta_{k-\ell_1}(H)-2kn^{\ell_1-1})$ $(\delta_{k-\ell_2}(H)-2kn^{\ell_2-1})$ $(\delta_d(H)-2kn^{k-d-1})$ $\geq \frac{1}{2}(\gamma')^3n^{\ell_1+\ell_2+k-d}=\frac{1}{2}(\gamma')^3n^k$ choices of the desired ordered k-tuples. So there are at least $\frac{1}{2}(\gamma')^3n^k/k!$ S-absorbing edges in H.

Now we pick the absorbing matching M'. Select a random subset M of E(H), where each edge is chosen independently with probability $p = \beta n^{1-k}/(k+1)$. Then, the expected size of M is at most $\binom{n}{k}p < \beta n/(k+1)!$, and the expected number of intersecting pairs of edges in M is at most $n^{2k-1}p^2 < \beta^2n$. Hence, by Markov's inequality (see, e.g., [13, inequality (1.3)]), with probability at least 1 - 1/2 - 1/k!, $|M| \leq \beta n/k$ and M contains at most $2\beta^2n$ intersecting pairs of edges. Moreover, for every (2k-d)-set of vertices S, let X_S be the number of S-absorbing edges in M.

Then we have

$$\mathbb{E}(X_S) \ge p \cdot \frac{1}{2} (\gamma')^3 n^k / k! = \frac{\beta(\gamma')^3 n}{2(k+1)!}.$$

By Chernoff's bound (see, e.g., [13, Theorem 2.1]), with probability 1 - o(1), we have that $X_S \ge \frac{1}{2}\mathbb{E}(X_S) \ge \frac{\beta(\gamma')^3 n}{4(k+1)!}$ for all (2k-d)-sets S in H.

Thus, there is an $M \subseteq E(H)$ satisfying all the properties above. We delete one edge from each intersecting pairs of edges and denote the resulting subset by M', which is a matching. So $|M'| \le \beta n/k$, and for every (2k-d)-set of vertices S, M' contains at least $\frac{\beta(\gamma')^3 n}{4(k+1)!} - 2\beta^2 n \ge \beta^2 n$ S-absorbing edges by the definition of β .

It remains to show that, for any $R \subseteq V(H) \setminus V(M')$ with $|R| \leq \beta^2 n$, $H[V(M') \cup R]$ contains a matching covering all but at most 2k - d - 1 vertices. Fix $R \subseteq V(H) \setminus V(M')$ with $|R| \leq \beta^2 n$ and any (2k-d)-tuple S of R, then there are at least $\beta^2 n$ S-absorbing edges in M'. Take an S-absorbing edge e, we replace M' by $M'_S := (M' \setminus \{e\}) \cup \{e_1, e_2\}$, decreasing the number of uncovered vertices of R by k. Since we have at most $\beta^2 n/k$ iterations, there will always be an S-absorbing edge available in M'. In the end, we have at most 2k - d - 1 vertices left uncovered in $H[V(M') \cup R]$ and we are done.

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