# A MASS CONSERVING MIXED $h p$-FEM SCHEME FOR STOKES FLOW. PART III: IMPLEMENTATION AND PRECONDITIONING * 

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#### Abstract

This is the third part in a series on a mass conserving, high order, mixed finite element method for Stokes flow. In this part, we study a block-diagonal preconditioner for the indefinite Schur complement system arising from the discretization of the Stokes equations using these elements. The underlying finite element method is uniformly stable in both the mesh size $h$ and polynomial order $p$, and we prove bounds on the eigenvalues of the preconditioned system which are independent of $h$ and grow modestly in $p$. The analysis relates the Schur complement system to an appropriate variational setting with subspaces for which exact sequence properties and inf-sup stability hold. Several numerical examples demonstrate agreement with the theoretical results.


Key words. preconditioning mixed $h p$-finite elements, Stokes flow, domain decomposition
AMS subject classifications. 65N30, 65N55, 76M10

1. Introduction. This paper is the third part in a series discussing a mass conserving, high order, mixed finite element method for Stokes flow on a simply connected polygon $\Omega$ with boundary $\Gamma=\partial \Omega$ : Find $(\boldsymbol{u}, p) \in \boldsymbol{H}_{0}^{1}(\Omega) \times L_{0}^{2}(\Omega)$ such that

$$
\begin{align*}
a(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p) & =(\boldsymbol{f}, \boldsymbol{v}) & & \forall \boldsymbol{v} \in \boldsymbol{H}_{0}^{1}(\Omega)  \tag{1.1a}\\
b(\boldsymbol{u}, q) & =0 & & \forall q \in L_{0}^{2}(\Omega)
\end{align*}
$$

where $\boldsymbol{u}=\left(u_{1}, u_{2}\right)$ is the fluid velocity, $p$ the pressure, $\boldsymbol{f} \in \boldsymbol{L}^{2}(\Omega)$ the body force, $a(\boldsymbol{u}, \boldsymbol{v}):=\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$ and $b(\boldsymbol{v}, p):=-(\operatorname{div} \boldsymbol{v}, p)$. Without loss of generality, by rescaling, we may reduce (1.1) to the case where the kinematic viscosity $\nu=1$. Here, $H^{s}(\Omega)$ and $H_{0}^{s}(\Omega)$ denote the usual Sobolev spaces $[1], \boldsymbol{H}^{s}(\Omega), \boldsymbol{H}_{0}^{s}(\Omega)$ the vector valued Sobolev spaces, i.e. $\boldsymbol{H}^{s}(\Omega):=\left[H^{s}(\Omega)\right]^{2}$, and $L_{0}^{2}(\Omega)$ denotes the (closed) subspace of square integrable functions with vanishing average value:

$$
L_{0}^{2}(\Omega):=\left\{q \in L^{2}(\Omega): \int_{\Omega} q d \boldsymbol{x}=0\right\}
$$

Problem (1.1) is approximated a using mixed, high order, finite element scheme on a mesh $\mathcal{T}$ as follows: Find $\left(\boldsymbol{u}_{h k}, p_{h k}\right) \in \boldsymbol{V}_{0} \times Q_{0}$ such that

$$
\begin{align*}
a\left(\boldsymbol{u}_{h k}, \boldsymbol{v}\right)+b\left(\boldsymbol{v}, p_{h k}\right) & =(\boldsymbol{f}, \boldsymbol{v}) & & \forall \boldsymbol{v} \in \boldsymbol{V}_{0}  \tag{1.2a}\\
b\left(\boldsymbol{u}_{h k}, q\right) & =0 & & \forall q \in Q_{0} \tag{1.2b}
\end{align*}
$$

where the finite element spaces are chosen to be $[6,7,16]$ :

$$
\begin{aligned}
V & :=\left\{v \in H^{1}(\Omega):\left.v\right|_{K} \in \mathcal{P}_{k}(K) \forall K \in \mathcal{T}, v \text { is } C^{1} \text { at noncorner vertices }\right\} \\
Q & :=\left\{q \in L^{2}(\Omega):\left.q\right|_{K} \in \mathcal{P}_{k-1}(K) \forall K \in \mathcal{T}, q \text { is } C^{0} \text { at noncorner vertices }\right\},
\end{aligned}
$$

$V_{0}:=V \cap H_{0}^{1}(\Omega), V_{0}=V_{0} \times V_{0}, Q_{0}=Q \cap L_{0}^{2}(\Omega), \mathcal{P}_{k}$ denotes the space of all polynomials of degree at most $k$, and a corner vertex is a vertex of the physical domain $\Omega$. The local degrees of freedom of the spaces $V$ and $Q$ are illustrated in Figure 1.

[^0]

Fig. 1: Local degrees of freedom for the finite element spaces (a) $V$ and (b) $Q$ in the case $k=5$. Dots indicate degrees of freedom corresponding to evaluation at the point located at the dot while circles indicate gradient evaluation.

In Part I [6], it was shown that that these elements are uniformly inf-sup stable in the mesh size $h$ and polynomial order $k$ if the mesh $\mathcal{T}$ is corner-split which, roughly speaking, means that every element $K \in \mathcal{T}$ has at most one edge lying on the domain boundary $\Gamma$; for a precise definition, see $[6, \mathrm{p} .12]$.

Theorem 1.1 (Theorem $3.1 \&$ Corollary 3.2 [6]). If the mesh $\mathcal{T}$ is corner-split, then for every $q \in Q_{0}$, there exists a $\boldsymbol{v} \in \boldsymbol{V}_{0}$ such that $\operatorname{div} \boldsymbol{v}=q$ and

$$
\|\boldsymbol{v}\|_{\boldsymbol{H}^{1}(\Omega)} \leq \beta^{-1}\|q\|_{L^{2}(\Omega)}
$$

where $0<\beta<1$ is independent of $k$ and $h$. Thus, the spaces $\boldsymbol{V}_{0} \times Q_{0}$ are uniformly inf-sup stable:

$$
\begin{equation*}
\inf _{0 \neq q \in Q_{0}} \sup _{\mathbf{0} \neq \boldsymbol{v} \in \boldsymbol{V}_{0}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{H}^{1}(\Omega)}\|q\|_{L^{2}(\Omega)}} \geq \beta \tag{1.3}
\end{equation*}
$$

Strictly speaking, [6, Corollary 3.2$]$ shows that $\beta$ depends on the mesh-dependent quantity $\Theta(\mathcal{T})$ defined in [6, eq. (3.2)], but is nevertheless bounded independently of the mesh size $h$ and polynomial degree $k$. Moreover, the finite element solution $\boldsymbol{u}_{h k}$ will be pointwise divergence free [6, §1 and Theorem 2.6]. In Part II [7], it was shown that these elements have optimal approximation properties in both the mesh size $h$ and the polynomial order $k$. On locally quasi-uniform meshes, the finite element solution to (1.2) converges at the optimal algebraic rate to the solution to (1.1) [7, Theorem 2.2]. Moreover, if the data $\boldsymbol{f}$ belongs to a particular countably normed space, then the finite element method with properly geometrically graded meshes converges exponentially fast as both the mesh is refined and the polynomial degree is increased [7, Corollary 2.5]. The spaces $\boldsymbol{V}_{0} \times Q_{0}$ are currently the only known triangular finite element spaces that are uniformly inf-sup stable in $h$ and $k$, give pointwise divergence free velocities, and posses optimal approximation properties.

In the current work, we turn to issues relating to the practical application of the method. In particular, we give explicit bases for the spaces $\boldsymbol{V}$ and $Q$ that result in an efficient preconditioner for the solution of the resulting linear system for (1.2), which may be used in conjunction with an iterative solver for indefinite systems, such as MINRES [30]. The preconditioner consists of a standard static condensation, or elimination of the interior degrees of freedom, along with an Additive Schwarz preconditioner (ASM) [35, 37] for the resulting Schur complement system associated
with the interface degrees of freedom. Thanks to a judicious choice of basis, the condition number grow at most as $\log ^{3} k$ as $k$ is increased, and is uniform in the mesh size.

The current work finds inspiration in the early works of $[12,19,34,38]$ for $h$ version methods, [8, 22] for $h p$-version finite element methods, and [20, 24, 31, 32] for spectral element methods, each of which developed block diagonal and/or block triangular preconditioners in terms of existing preconditioners for second order elliptic problems. Unfortunately, these types of approaches do not readily extend to the mixed finite element scheme (1.2) owing to the additional smoothness requirements imposed at element vertices for both the velocity and pressure spaces. Our treatment of these degrees of freedom is similar to the treatment of the second order derivative degrees of freedom in preconditioning the stiffness matrix for $H^{2}(\Omega)$-conforming methods [5] and the treatment of the vertex degrees of freedom in preconditioning the mass matrix for $H^{1}(\Omega)$ problems [4].
2. General Form of a Block-Diagonal Preconditioner. By fixing bases for the spaces $\boldsymbol{V}_{0}$ and $Q$, we may express $\boldsymbol{u} \in \boldsymbol{V}_{0}$ and $p \in Q$ as

$$
\boldsymbol{u}=\vec{u}_{E}^{T} \vec{\Phi}_{E}+\vec{u}_{I}^{T} \vec{\Phi}_{I} \quad \text { and } \quad p=\vec{p}_{e}^{T} \vec{\psi}_{e}+\vec{p}_{\iota}^{T} \vec{\psi}_{\iota}
$$

for suitable $\vec{u}_{E}, \vec{u}_{I}, \vec{p}_{e}, \vec{p}_{\iota}$, where $\vec{\Phi}_{E}$ is the vector of exterior velocity basis functions (vertex and edge functions), $\vec{\Phi}_{I}$ the vector of interior velocity basis functions, $\vec{\psi}_{e}$ the vector of exterior pressure basis functions, and $\vec{\psi}_{\iota}$ the vector of interior pressure basis functions. Here, the exterior pressure functions consist of vertex functions and a function corresponding to the average value over each element. The variational problem (1.2) in matrix form then reads

$$
\left[\begin{array}{cc|cc}
\boldsymbol{A}_{E E} & \boldsymbol{B}_{E e} & \boldsymbol{A}_{E I} & \boldsymbol{B}_{E \iota}  \tag{2.1}\\
\boldsymbol{B}_{e E} & \mathbf{0} & \boldsymbol{B}_{e I} & \mathbf{0} \\
\hline \boldsymbol{A}_{I E} & \boldsymbol{B}_{I e} & \boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\
\boldsymbol{B}_{\iota E} & \mathbf{0} & \boldsymbol{B}_{\iota I} & \mathbf{0}
\end{array}\right]\left[\begin{array}{c}
\vec{u}_{E} \\
\vec{p}_{e} \\
\hline \vec{u}_{I} \\
\vec{p}_{\iota}
\end{array}\right]=\left[\begin{array}{c}
\vec{f}_{E} \\
\overrightarrow{0} \\
\hline \vec{f}_{I} \\
\overrightarrow{0}
\end{array}\right] .
$$

The matrix appearing in (2.1) is symmetric but indefinite, owing to the zero subblocks. The pressure variable in problem (1.2) is unique up to a constant, meaning that the matrix in (2.1) has a one-dimensional null space. Nevertheless, the system (2.1) is consistent since the components of the load vector corresponding to pressure basis functions vanish identically and, a fortiori, are orthogonal to constant pressure modes. Consequently, the system (2.1) is uniquely solvable up to the addition of a constant in the pressure thanks to the inf-sup condition (1.3) and the uniform ellipticity of $a(\cdot, \cdot)$.

The conditioning of the matrix, in common with standard $h p$-finite elements, degenerates rapidly with both the mesh size $h$ and the polynomial order $k$ of the elements. Indeed, almost every practical choice of basis function results in a rapid deterioration of the condition number $k$, even for symmetric, positive definite systems [3, 29]. We seek a preconditioner for the symmetric, indefinite system (2.1) which controls the growth of the conditioning in both $h$ and $k$.

The first step towards preconditioning is to eliminate, or statically condense, the interior degrees of freedom to arrive at the Schur complement system

$$
\boldsymbol{S}\left[\begin{array}{c}
\vec{u}_{E}  \tag{2.2}\\
\vec{p}_{e}
\end{array}\right]=\left[\begin{array}{c}
\vec{f}_{E}^{*} \\
\vec{g}_{e}^{*}
\end{array}\right]:=\left[\begin{array}{c}
\vec{f}_{E} \\
\overrightarrow{0}
\end{array}\right]-\left[\begin{array}{cc}
\boldsymbol{A}_{E I} & \boldsymbol{B}_{E \iota} \\
\boldsymbol{B}_{e I} & \mathbf{0}
\end{array}\right]\left[\begin{array}{cc}
\boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\
\boldsymbol{B}_{\iota I} & \mathbf{0}
\end{array}\right]^{-1}\left[\begin{array}{c}
\vec{f}_{I} \\
\overrightarrow{0}
\end{array}\right],
$$

where

$$
\boldsymbol{S}=\left[\begin{array}{cc}
\widetilde{\boldsymbol{A}} & \widetilde{\boldsymbol{B}}^{T}  \tag{2.3}\\
\widetilde{\boldsymbol{B}} & \mathbf{0}
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{A}_{E E} & \boldsymbol{B}_{E e} \\
\boldsymbol{B}_{e E} & \mathbf{0}
\end{array}\right]-\left[\begin{array}{cc}
\boldsymbol{A}_{E I} & \boldsymbol{B}_{E \iota} \\
\boldsymbol{B}_{e I} & \mathbf{0}
\end{array}\right]\left[\begin{array}{cc}
\boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\
\boldsymbol{B}_{\iota I} & \mathbf{0}
\end{array}\right]^{-1}\left[\begin{array}{cc}
\boldsymbol{A}_{I E} & \boldsymbol{B}_{I e} \\
\boldsymbol{B}_{\iota E} & \mathbf{0}
\end{array}\right]
$$

and we have used the fact (see Lemma 5.1) that the (2,2) block of Schur complement matrix $\boldsymbol{S}$ reduces to the zero matrix. The inverse of the matrix appearing in (2.2) and (2.3) is well-defined by Theorem 3.3. After the degrees of freedom on the element interfaces are in hand, the interior degrees of freedom can be recovered by back substitution using the relation

$$
\left[\begin{array}{c}
\vec{u}_{I} \\
\vec{p}_{\iota}
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\
\boldsymbol{B}_{\iota I} & \mathbf{0}
\end{array}\right]^{-1}\left(\left[\begin{array}{c}
\vec{f}_{I} \\
\overrightarrow{0}
\end{array}\right]-\left[\begin{array}{cc}
\boldsymbol{A}_{I E} & \boldsymbol{B}_{I e} \\
\boldsymbol{B}_{\iota E} & \mathbf{0}
\end{array}\right]\left[\begin{array}{c}
\vec{u}_{E} \\
\vec{p}_{e}
\end{array}\right]\right) .
$$

The element interface degrees of freedom are obtained by solving the Schur complement system (2.2). The matrix $\boldsymbol{S}$ defined in (2.3) is symmetric and indefinite, and inherits the one dimensional null space from the full system matrix (2.1), again corresponding to the constant pressure mode. Similarly, the right hand side in (2.2) inherits the consistency of the load vector meaning that (2.2) is uniquely solvable up to a constant pressure mode. The indefiniteness of the problem coupled with the presence of a low dimensional null space suggests using a MINRES iterative solver [30] in conjunction with a suitable preconditioner.

We seek a block diagonal matrix of the form

$$
P=\left[\begin{array}{cc}
\bar{A} & 0  \tag{2.4}\\
0 & \bar{M}
\end{array}\right]
$$

to precondition $\boldsymbol{S}$, where $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{M}}$ are symmetric positive definite matrices. The convergence of the MINRES algorithm with preconditioner $\boldsymbol{P}^{-1}$ depends on the location of the nonzero eigenvalues of $\boldsymbol{P}^{-1} \boldsymbol{S}$ [14, Remark 4.13 and $\left.\S 4.2 .4\right]$. In particular, let $\delta, \Delta, \theta$, and $\Theta$ be nonnegative constants such that

$$
\delta \leq \frac{\vec{u}_{E}^{T} \widetilde{\boldsymbol{A}} \vec{u}_{E}}{\vec{u}_{E}^{T} \overline{\boldsymbol{A}} \vec{u}_{E}} \leq \Delta \quad \forall \boldsymbol{u} \in \boldsymbol{V}_{0} \quad \text { and } \quad \theta \leq \frac{\vec{q}_{e}^{T} \widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{A}}^{-1} \widetilde{\boldsymbol{B}}^{T} \vec{q}_{e}}{\vec{q}_{e}^{T} \overline{\boldsymbol{M}} \vec{q}_{e}} \leq \Theta \quad \forall q \in Q_{0}
$$

Then, by [14, Theorem 4.7 and eq. (4.37)], the eigenvalues of $\boldsymbol{P}^{-1} \boldsymbol{S}$ lie in the set

$$
\begin{equation*}
\left[-\Theta^{2}, \frac{1}{2}\left(\delta-\sqrt{\delta^{2}+4 \delta \theta^{2}}\right)\right] \cup\{0\} \cup\left[\delta, \frac{1}{2}\left(\Delta+\sqrt{\Delta^{2}+4 \Delta \Theta^{2}}\right)\right] \tag{2.5}
\end{equation*}
$$

In order to use variational techniques like Additive Schwarz Methods to construct $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{M}}$, we must first identify the appropriate variational setting of the Schur complement system (2.2). In particular, the Schur complement is posed over the subspaces spanned by the external degrees of freedom of $\boldsymbol{V}_{0} \times Q_{0}$, which are rather nonstandard owing to the additional continuity imposed at noncorner vertices. Section 3 gives a precise characterization of these spaces including new results showing that they form a discrete exact sequence property (Theorem 3.5) and that they, like the spaces $\boldsymbol{V}_{0} \times Q_{0}$, are uniformly inf-sup stable in both $h$ and $k$ (Theorem 3.7).

Section 4 defines the Stokes extension operator and its relation to the subspace splittings. Section 5 uses the results of the previous two sections to relate the matrix form of the Schur complement system to a variational problem. In section 6, we
present an explicit set of basis functions on the reference element for the spaces $\boldsymbol{V}$ and $Q$ and then detail how these are used in the construction of the global basis functions. We develop the additive Schwarz theory and construct the matrices $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{M}}$ in section 7 , which is then applied to two numerical examples demonstrating in section 8. Appendix A contains technical lemmas related to the additive Schwarz theory.
3. Subspace Splittings, Exact Sequences, and Stability. A key property of the mixed finite element pair $\boldsymbol{V}_{0} \times Q_{0}$ is the exactness of the sequence [6, Theorem $2.6]$ and $[16, \S 3.2]$ :

$$
\begin{equation*}
0 \xrightarrow{\subset} \Sigma_{0} \xrightarrow{\text { curl }} \boldsymbol{V}_{0} \xrightarrow{\text { div }} Q_{0} \longrightarrow 0 \tag{3.1}
\end{equation*}
$$

where curl $=\left(\partial_{y},-\partial_{x}\right)^{T}$ and $\Sigma_{0}$ is the space of $H^{2}(\Omega)$-conforming piecewise polynomials (see $[6, \S 2]$ ) given by

$$
\Sigma_{0}=\left\{\phi \in H_{0}^{2}(\Omega):\left.\phi\right|_{K} \in \mathcal{P}_{k+1}(K) \forall K \in \mathcal{T}, \phi \text { is } C^{2} \text { at noncorner vertices }\right\}
$$

The exact sequence property (3.1) was used in [7, Theorem 2.2] to obtain optimal error estimates for the velocity that were independent of the pressure error. In the remainder of this section, we seek exact sequences analogous to (3.1) that respect the separation of interior and exterior degrees of freedom. Such sequences will be used to both identify the variational problem associated with the Schur complement system (2.2) and prove its uniform stability.

Before we begin, we introduce some notation. Let $\mathcal{V}$ denote the set of all element vertices, and partition $\mathcal{V}$ into: $\mathcal{V}_{C}$, the set of element vertices located at a vertex of the polygonal domain $\Omega ; \mathcal{V}_{B}$, the set of remaining element vertices on the domain boundary $\Gamma$ which are not corner vertices; and $\mathcal{V}_{I}$, the set of element vertices in the interior of domain $\Omega$. Let $\mathcal{E}$ be the set of all element edges. Given an element $K \in \mathcal{T}$, $\mathcal{E}_{K}$ denotes the edges of $K$ and $\mathcal{V}_{K}$ denotes the vertices of $K$. Likewise, given a vertex $\boldsymbol{a} \in \mathcal{V}, \mathcal{E}_{\boldsymbol{a}}$ denotes the set of edges having $\boldsymbol{a}$ as an endpoint and $\mathcal{T}_{\boldsymbol{a}}$ the set of elements having $\boldsymbol{a}$ as a vertex. We assume that $\mathcal{T}$ is a partition of the domain $\Omega$ into triangles such that the nonempty intersection of any two distinct elements from $\mathcal{T}$ is either a single common vertex or a single common edge of both elements, and there exists $\kappa>0$ independent of $\mathcal{T}$ such that

$$
\begin{equation*}
\rho_{K} \geq \kappa h_{K} \quad \forall K \in \mathcal{T} \tag{3.2}
\end{equation*}
$$

where $h_{K}:=\operatorname{diam}(K)$ and $\rho_{K}$ is the diameter of the largest inscribed circle of $K$. The mesh size $h$ denotes the diameter of the largest element, i.e. $h:=\max _{K \in \mathcal{T}} h_{K}$.
3.1. Interior Subspaces. We first examine the subspaces associated with the interior degrees of freedom given by

$$
\begin{aligned}
\Sigma_{I} & =\left\{\phi \in \Sigma_{0}:\left.\phi\right|_{\partial K}=\left.\partial_{n} \phi\right|_{\partial K}=0, \forall K \in \mathcal{T}\right\} \\
\boldsymbol{V}_{I} & =\left\{\boldsymbol{v} \in \boldsymbol{V}_{0}:\left.\boldsymbol{v}\right|_{\partial K}=\mathbf{0}, \forall K \in \mathcal{T}\right\} \\
Q_{I} & =\left\{q \in Q_{0}: \int_{K} q d \boldsymbol{x}=0,\left.q\right|_{K}(\boldsymbol{a})=0, \forall \boldsymbol{a} \in \mathcal{V}_{K}, \forall K \in \mathcal{T}\right\}
\end{aligned}
$$

which, in turn, may be decomposed into contributions from individual elements:

$$
\begin{equation*}
\Sigma_{I}=\bigoplus_{K \in \mathcal{T}} \Sigma_{I}(K), \quad \boldsymbol{V}_{I}=\bigoplus_{K \in \mathcal{T}} \boldsymbol{V}_{I}(K), \quad \text { and } \quad Q_{I}=\bigoplus_{K \in \mathcal{T}} Q_{I}(K) \tag{3.3}
\end{equation*}
$$

where

$$
\begin{aligned}
& \Sigma_{I}(K):=\mathcal{P}_{k+1}(K) \cap H_{0}^{2}(K), \quad \boldsymbol{V}_{I}(K):=\boldsymbol{\mathcal { P }}_{k}(K) \cap \boldsymbol{H}_{0}^{1}(K) \\
& Q_{I}(K):=\left\{q \in \mathcal{P}_{k-1}(K) \cap L_{0}^{2}(K): q(\boldsymbol{a})=0, \forall \boldsymbol{a} \in \mathcal{V}_{K}\right\}
\end{aligned}
$$

Both the element-level interior spaces and the corresponding interior spaces on a mesh form exact sequences:

Theorem 3.1. The following sequences are exact:

$$
\begin{equation*}
0 \xrightarrow{\subset} \Sigma_{I}(K) \xrightarrow{\text { curl }} \boldsymbol{V}_{I}(K) \xrightarrow{\text { div }} Q_{I}(K) \longrightarrow 0, \quad \forall K \in \mathcal{T}, \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
0 \xrightarrow{\subset} \Sigma_{I} \xrightarrow{\text { curl }} \boldsymbol{V}_{I} \xrightarrow{\text { div }} Q_{I} \longrightarrow 0 \tag{3.5}
\end{equation*}
$$

Proof. In view of curl $H_{0}^{2}(K) \subset \boldsymbol{H}_{0}^{1}(K)$, we have the relations curl $\Sigma_{I}(K) \subset$ $\boldsymbol{V}_{I}(K), \operatorname{curl} \Sigma_{I}(K) \subseteq$ ker div, and by $\left[6\right.$, Theorem 3.4], div $\boldsymbol{V}_{I}(K)=Q_{I}(K)$. Here, we consider div as a linear operator $\boldsymbol{V}_{I}(K) \rightarrow Q_{I}(K)$. Moreover, for $\phi \in \Sigma_{I}(K)$, $\operatorname{curl} \phi \equiv 0$ if and only if $\phi \equiv 0$ and so $\operatorname{dim} \operatorname{curl} \Sigma_{I}(K)=\operatorname{dim} \Sigma_{I}(K)$. The dimension counts $\operatorname{dim} \Sigma_{I}(K)=\frac{k^{2}-7 k+12}{2}, \operatorname{dim} \boldsymbol{V}_{I}(K)=k^{2}-3 k+2$, and $\operatorname{dim} Q_{I}(K)=\frac{k^{2}+k-8}{2}$ reveal that $\operatorname{dim} \Sigma_{I}(K)+\operatorname{dim} Q_{I}(K)-\operatorname{dim} \boldsymbol{V}_{I}(K)=0$. By the rank-nullity theorem, we have

$$
\operatorname{dim} \boldsymbol{V}_{I}(K)=\operatorname{dim} \operatorname{Im} \operatorname{div}+\operatorname{dim} \text { ker } \operatorname{div} \geq \operatorname{dim} Q_{I}(K)+\operatorname{dim} \Sigma_{I}(K)=\operatorname{dim} \boldsymbol{V}_{I}(K)
$$

and so ker div $=\operatorname{curl} \Sigma_{I}(K)$. Thus, the element-level sequence (3.4) is exact. The exactness of the global spaces (3.5) may be proved along similar lines using the exactness of the element level sequence (3.4).
Theorem 3.1 gives a useful decomposition of the interior spaces in terms of the curl operator:

Corollary 3.2. The spaces $\boldsymbol{V}_{I}(K)$ and $\boldsymbol{V}_{I}$ admit the following decompositions:

$$
\boldsymbol{V}_{I}(K)=\operatorname{curl} \Sigma_{I}(K) \oplus\left\{\boldsymbol{v} \in \mathcal{P}_{k-1}(K) \cap \boldsymbol{H}_{0}^{1}(K), a(\boldsymbol{v}, \operatorname{curl} \phi)=0, \forall \phi \in \Sigma_{I}(K)\right\}
$$

and

$$
\begin{equation*}
\boldsymbol{V}_{I}=\operatorname{curl} \Sigma_{I} \oplus\left\{\boldsymbol{v} \in \boldsymbol{V}_{0}:\left.\boldsymbol{v}\right|_{\partial K}=\mathbf{0}, \forall K \in \mathcal{T}, a(\boldsymbol{v}, \boldsymbol{\operatorname { c u r l }} \phi)=0, \forall \phi \in \Sigma_{I}\right\} \tag{3.6}
\end{equation*}
$$

The next result concerns the stability of the interior mixed finite element pair $\boldsymbol{V}_{I} \times Q_{I}$.

Theorem 3.3. Let $K \in \mathcal{T}$ and let $\beta$ be the discrete inf-sup constant appearing in (1.3). If $q \in Q_{I}(K)$, then there exists $\boldsymbol{v} \in \boldsymbol{V}_{I}(K)$ such that $\operatorname{div} \boldsymbol{v}=q$ and

$$
\begin{equation*}
h_{K}^{-1}\|\boldsymbol{v}\|_{\boldsymbol{L}^{2}(K)}+|\boldsymbol{v}|_{\boldsymbol{H}^{1}(K)} \leq \beta^{-1}\|q\|_{L^{2}(K)} \tag{3.7}
\end{equation*}
$$

Consequently, (i) the spaces $\boldsymbol{V}_{I} \times Q_{I}$ are uniformly inf-sup stable:

$$
\begin{equation*}
\inf _{0 \neq q \in Q_{I}} \sup _{\mathbf{0} \neq \boldsymbol{v} \in \boldsymbol{V}_{I}} \frac{b(\boldsymbol{v}, q)}{|\boldsymbol{v}|_{\boldsymbol{H}^{1}(\Omega)}\|q\|_{L^{2}(\Omega)}} \geq \beta \tag{3.8}
\end{equation*}
$$

and (ii) the matrix $\left[\begin{array}{cc}\boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\ \boldsymbol{B}_{\iota I} & \mathbf{0}\end{array}\right]$ is invertible, and hence the Schur complement $\boldsymbol{S}$ appearing in (2.2) is well-defined by the formula (2.3).

Proof. Let $q \in Q_{I}(K)$. Then, [6, Theorem 3.5] gives the existence of $\boldsymbol{v} \in \boldsymbol{V}_{I}(K)$ with $\operatorname{div} \boldsymbol{v}=q$ satisfying the estimate (3.7).

Now let $q \in Q_{I}$ be decomposed as in (3.3) so that $q=\sum_{K \in \mathcal{T}} q_{K}$ with $q_{K} \in$ $Q_{I}(K)$. By the first statement in the theorem, there exists $\boldsymbol{v}_{K} \in \boldsymbol{V}_{I}(K)$ with $\operatorname{div} \boldsymbol{v}_{K}=$ $-q_{K}$ satisfying (3.7). Hence, $\boldsymbol{v}:=\sum_{K \in \mathcal{T}} \boldsymbol{v}_{K}$ satisfies $\operatorname{div} \boldsymbol{v}=-q$ and

$$
|\boldsymbol{v}|_{\boldsymbol{H}^{1}(\Omega)}^{2}=\sum_{K \in \mathcal{T}}\left|\boldsymbol{v}_{K}\right|_{\boldsymbol{H}^{1}(\Omega)}^{2} \leq \beta^{-2} \sum_{K \in \mathcal{T}}\left\|q_{K}\right\|_{L^{2}(\Omega)}^{2}=\beta^{-2}\|q\|_{L^{2}(\Omega)}^{2}
$$

from which (3.8) immediately follows. (ii) follows at once thanks to the ellipticity of $a(\cdot, \cdot)$ and the inf-sup condition (3.8).
3.2. Boundary Subspaces. The subspaces $\tilde{\Sigma}_{E}, \tilde{\boldsymbol{V}}_{E}$, and $\tilde{Q}_{E}$ are defined as follows

$$
\begin{align*}
& \tilde{\Sigma}_{E}:=\left\{\phi \in \Sigma_{0}: a(\operatorname{curl} \phi, \operatorname{curl} \psi)=0, \forall \psi \in \Sigma_{I}\right\}  \tag{3.9a}\\
& \tilde{\boldsymbol{V}}_{E}:=\left\{\boldsymbol{v} \in \boldsymbol{V}_{0}: \operatorname{div} \boldsymbol{v} \in \tilde{Q}_{E}, a(\boldsymbol{v}, \operatorname{curl} \psi)=0, \forall \psi \in \Sigma_{I}\right\}  \tag{3.9b}\\
& \tilde{Q}_{E}:=\left\{q \in Q_{0}:(q, r)=0, \forall r \in Q_{I}\right\}, \tag{3.9c}
\end{align*}
$$

and correspond to degrees of freedom on the element boundaries. More precisely, we have:

Theorem 3.4. The following decompositions hold:

$$
\begin{equation*}
\Sigma_{0}=\Sigma_{I} \oplus \tilde{\Sigma}_{E}, \quad \boldsymbol{V}_{0}=\boldsymbol{V}_{I} \oplus \tilde{\boldsymbol{V}}_{E}, \quad \text { and } \quad Q_{0}=Q_{I} \oplus \tilde{Q}_{E} \tag{3.10}
\end{equation*}
$$

Proof. The decompositions of $\Sigma_{0}$ and $Q_{0}$ follow immediately using the orthogonality conditions in the definition of the spaces (3.9a) and (3.9c). The decomposition of the velocity space $\boldsymbol{V}_{0}$ is more involved. We first use the exact sequence (3.1) to write:

$$
\begin{equation*}
\boldsymbol{V}_{0}=\operatorname{curl} \Sigma_{0} \oplus\left(\operatorname{curl} \Sigma_{0}\right)^{\perp} \tag{3.11}
\end{equation*}
$$

where $\left(\boldsymbol{\operatorname { c u r l }} \Sigma_{0}\right)^{\perp}:=\left\{\boldsymbol{u} \in \boldsymbol{V}_{0}: a(\boldsymbol{u}, \boldsymbol{\operatorname { c u r l }} \psi)=0, \forall \psi \in \Sigma_{0}\right\}$. Now, let $\boldsymbol{v} \in \boldsymbol{V}_{0}$ be given. By the decomposition (3.11) and the decomposition of $\Sigma_{0}$ in (3.10), there exists $\phi_{I} \in$ $\Sigma_{I}, \tilde{\phi}_{E} \in \tilde{\Sigma}_{E}$, and $\boldsymbol{v}_{\perp} \in\left(\operatorname{curl} \Sigma_{0}\right)^{\perp} \operatorname{such}$ that $\boldsymbol{v}=\boldsymbol{\operatorname { c u r l }}\left(\phi_{I}+\tilde{\phi}_{E}\right)+\boldsymbol{v}_{\perp}$. We decompose the divergence analogously: $\operatorname{div} \boldsymbol{v}=q_{I}+\tilde{q}_{E}$ with $q_{I} \in Q_{I}$ and $\tilde{q}_{E} \in \tilde{Q}_{E}$. Thanks to the exact sequence (3.5) and the decomposition (3.6), there exists $\boldsymbol{w} \in \boldsymbol{V}_{I}$ such that $\operatorname{div} \boldsymbol{w}=q_{I}$ and $\boldsymbol{w} \in\left\{\boldsymbol{v} \in \boldsymbol{V}_{I}: a(\boldsymbol{v}, \boldsymbol{\operatorname { c u r l }} \psi)=0, \forall \psi \in \Sigma_{I}\right\}$. Then, $\boldsymbol{v}_{I}:=\boldsymbol{\operatorname { c u r l }} \phi_{I}+\boldsymbol{w}$ satisfies $\boldsymbol{v}_{I} \in \boldsymbol{V}_{I}$ and $\operatorname{div} \boldsymbol{v}_{I}=q_{I}$. Consequently, $\tilde{\boldsymbol{v}}_{E}:=\boldsymbol{v}-\boldsymbol{v}_{I}=\boldsymbol{\operatorname { c u r l }} \tilde{\phi}_{E}+\boldsymbol{v}_{\perp}-\boldsymbol{w}$ satisfies $\operatorname{div} \tilde{\boldsymbol{v}}_{E}=\operatorname{div}\left(\boldsymbol{v}-\boldsymbol{v}_{I}\right)=\tilde{q}_{E} \in \tilde{Q}_{E}$ and

$$
a\left(\tilde{\boldsymbol{v}}_{E}, \operatorname{curl} \psi\right)=a\left(\operatorname{curl} \tilde{\phi}_{E}, \operatorname{curl} \psi\right)+a\left(\boldsymbol{v}_{\perp}, \operatorname{curl} \psi\right)+a(\boldsymbol{w}, \operatorname{curl} \psi)=0, \forall \psi \in \Sigma_{I}
$$

by construction. Thus, $\tilde{\boldsymbol{v}}_{E} \in \tilde{\boldsymbol{V}}_{E}$, which completes the proof.
Theorem 3.4 means that the decompositions in the columns of the following complex (3.12) are valid. The next result shows that the rows form exact sequences:

Theorem 3.5. Each row the of the following complex is an exact sequence

where the exterior spaces $\tilde{\Sigma}_{E}, \tilde{\boldsymbol{V}}_{E}$, and $\tilde{Q}_{E}$ are given by (3.9).
Proof. [6, Theorem 2.6] gives the exactness of (3.12a) while Theorem 3.1 gives the exactness of (3.12b). Moreover, the decomposition (3.10) and the exactness the sequences (3.12a) and (3.12b) imply that $\operatorname{dim} \tilde{\Sigma}_{E}+\operatorname{dim} \tilde{Q}_{E}-\operatorname{dim} \tilde{\boldsymbol{V}}_{E}=0$. Since $\operatorname{curl} \tilde{\Sigma}_{E} \subset \tilde{\boldsymbol{V}}_{E}$ and $\operatorname{div} \tilde{\boldsymbol{V}}_{E} \subseteq \tilde{Q}_{E}$, we conclude that the sequence (3.12c) is exact using analogous arguments to those used in Theorem 3.1.
The exactness of the final row in (3.12) gives the following analogue of Corollary 3.2 for the exterior velocity space:

Corollary 3.6. The exterior velocity space $\tilde{\boldsymbol{V}}_{E}$ admits the following decomposition: $\tilde{\boldsymbol{V}}_{E}=\operatorname{curl} \tilde{\Sigma}_{E} \oplus\left\{\boldsymbol{v} \in \boldsymbol{V}_{0}: \operatorname{div} \boldsymbol{v} \in \tilde{Q}_{E}, a(\boldsymbol{v}, \operatorname{curl} \phi)=0, \forall \phi \in \Sigma_{0}\right\}$.

Theorems 1.1 and 3.3 show that the mixed finite element pairs appearing in the first two rows of (3.12) are uniformly inf-sup stable. The next result shows that the boundary spaces are also stable with the same inf-sup constant as for the full velocity and pressure spaces:

Theorem 3.7. Let $\beta$ be the discrete inf-sup constant defined in (1.3). If $q \in \tilde{Q}_{E}$, then there exists a $\boldsymbol{v} \in \tilde{\boldsymbol{V}}_{E}$ such that $\operatorname{div} \boldsymbol{v}=q$ and

$$
\begin{equation*}
|\boldsymbol{v}|_{\boldsymbol{H}^{1}(\Omega)} \leq \beta^{-1}\|q\|_{L^{2}(\Omega)} \tag{3.13}
\end{equation*}
$$

Consequently, the spaces $\tilde{\boldsymbol{V}}_{E} \times \tilde{Q}_{E}$ are uniformly inf-sup stable:

$$
\begin{equation*}
\inf _{0 \neq q \in \tilde{Q}_{E}} \sup _{\mathbf{0} \neq \boldsymbol{v} \in \tilde{\boldsymbol{V}}_{E}} \frac{b(\boldsymbol{v}, q)}{|\boldsymbol{v}|_{\boldsymbol{H}^{1}(\Omega)}\|q\|_{L^{2}(\Omega)}} \geq \beta \tag{3.14}
\end{equation*}
$$

Proof. Let $q \in \tilde{Q}_{E}$ be given. By Theorem 1.1, there exists a $\boldsymbol{w} \in \boldsymbol{V}_{0}$ such that $\operatorname{div} \boldsymbol{w}=q$ and $\|\boldsymbol{w}\|_{\boldsymbol{H}^{1}(\Omega)} \leq \beta^{-1}\|q\|_{L^{2}(\Omega)}$, where $\beta$ is independent of $h$ and $k$. According to Theorem 3.4 and (3.9b), there exists functions $\boldsymbol{w}_{I} \in \boldsymbol{V}_{I}, \tilde{\boldsymbol{w}}_{E} \in \tilde{\boldsymbol{V}}_{E}$ such that $\boldsymbol{w}=\boldsymbol{w}_{I}+\tilde{\boldsymbol{w}}_{E} . Q_{I} \ni \operatorname{div} \boldsymbol{w}_{I}=\operatorname{div}\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}_{E}\right) \in \tilde{Q}_{E}$ since $\operatorname{div} \boldsymbol{w}=q \in \tilde{Q}_{E}$, and so $\operatorname{div} \boldsymbol{w}_{I}=0$. By the exact sequence property (3.5), $\boldsymbol{w}_{I}=\boldsymbol{\operatorname { c u r l }} \phi_{I}$ for some $\phi \in \Sigma_{I}$, and thus $\boldsymbol{w}=\operatorname{curl} \phi_{I}+\tilde{\boldsymbol{w}}_{E}$. Note that this decomposition of $\boldsymbol{w}$ is $a(\cdot, \cdot)$ orthogonal by definition: $a\left(\operatorname{curl} \phi_{I}, \tilde{\boldsymbol{w}}_{E}\right)=0$ and

$$
|\boldsymbol{w}|_{H^{1}(\Omega)}^{2}=a\left(\operatorname{curl} \phi_{I}, \operatorname{curl} \phi_{I}\right)+a\left(\tilde{\boldsymbol{w}}_{E}, \tilde{\boldsymbol{w}}_{E}\right)=\left|\boldsymbol{\operatorname { c u r l }} \phi_{I}\right|_{H^{1}(\Omega)}^{2}+\left|\tilde{\boldsymbol{w}}_{E}\right|_{H^{1}(\Omega)}^{2} .
$$

Define $\boldsymbol{v}:=\tilde{\boldsymbol{w}}_{E}$. Then, $\operatorname{div} \boldsymbol{v}=\operatorname{div} \tilde{\boldsymbol{w}}_{E}=\operatorname{div}\left(\tilde{\boldsymbol{w}}_{E}+\operatorname{curl} \phi_{I}\right)=\operatorname{div} \boldsymbol{w}=q$, and $|\boldsymbol{v}|_{H^{1}(\Omega)} \leq|\boldsymbol{w}|_{H^{1}(\Omega)} \leq \beta^{-1}\|q\|_{L^{2}(\Omega)}$. (3.13) and (3.14) follow at once.
4. Stokes Extension Operator. Let $\boldsymbol{V}:=V \times V$ denote the discrete velocity space in the absence of essential boundary conditions and $Q_{I}^{\perp}$ be the orthogonal complement of $Q_{I}$ in $Q$ with the corresponding projection $\tilde{\Pi}: Q \rightarrow Q_{I}^{\perp}$,

$$
\begin{equation*}
(\tilde{\Pi} q, r)=(q, r), \forall r \in Q_{I}^{\perp}:=\left\{q \in Q:(q, r)=0 \forall r \in Q_{I}\right\} \tag{4.1}
\end{equation*}
$$

It is worthwhile noting that (4.1) means that $\tilde{Q}_{E}=Q_{I}^{\perp} \cap L_{0}^{2}(\Omega)$, so that the space $Q_{I}^{\perp}$ corresponds to boundary degrees of freedom. Let $K \in \mathcal{T}$. Then, thanks to Theorem 3.3, there exist $\boldsymbol{u}_{S, K} \in \mathcal{P}_{k}(K)$ and $p_{S, K} \in \mathcal{P}_{k-1}(K)$ satisfying

$$
\begin{align*}
a_{K}\left(\boldsymbol{u}_{S, K}, \boldsymbol{v}\right)+b_{K}\left(\boldsymbol{v}, p_{S, K}\right) & =0 & & \forall \boldsymbol{v} \in \boldsymbol{V}_{I}(K)  \tag{4.2a}\\
b_{K}\left(\boldsymbol{u}_{S, K}, q\right) & =0 & & \forall q \in Q_{I}(K)  \tag{4.2b}\\
\boldsymbol{u}_{S, K} & =\boldsymbol{u} & & \text { on } \partial K  \tag{4.2c}\\
p_{S, K}(\boldsymbol{a}) & =\left.p\right|_{K}(\boldsymbol{a}) & & \boldsymbol{a} \in \mathcal{V}_{K}  \tag{4.2~d}\\
\int_{K} p_{S, K} d \boldsymbol{x} & =\int_{K} p d \boldsymbol{x}, & & \tag{4.2e}
\end{align*}
$$

where $a_{K}(\cdot, \cdot)$ and $b_{K}(\cdot, \cdot)$ denote the restrictions of the bilinear forms to element $K$. We define the Stokes extension map $\boldsymbol{V} \times Q \ni(\boldsymbol{u}, p) \mapsto \mathscr{E}(\boldsymbol{u}, p)=:\left(\boldsymbol{u}_{S}, p_{S}\right)$ by the rule $\boldsymbol{u}_{S}=\boldsymbol{u}_{S, K}$ and $p_{S}=p_{S, K}$ on each element $K \in \mathcal{T}$.

The first result deals with the Stokes extension of a given velocity field paired a zero pressure:

Theorem 4.1. Let $\boldsymbol{\Pi}_{\boldsymbol{V}}: \boldsymbol{V} \rightarrow \boldsymbol{V}, \Pi_{Q}: \boldsymbol{V} \rightarrow Q$ be defined by the rule

$$
\begin{equation*}
\boldsymbol{V} \ni \boldsymbol{u} \mapsto\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \Pi_{Q} \boldsymbol{u}\right):=\mathscr{E}(\boldsymbol{u}, 0) \tag{4.3}
\end{equation*}
$$

Then, $\Pi_{Q} \boldsymbol{u} \in Q_{I}$ and

$$
\begin{equation*}
\left\|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}\right\|_{\boldsymbol{H}^{1}(K)}+\left\|\Pi_{Q} \boldsymbol{u}\right\|_{L^{2}(K)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1 / 2}(\partial K)}, \quad \forall K \in \mathcal{T} \tag{4.4}
\end{equation*}
$$

where $\|\cdot\|_{\boldsymbol{H}^{1 / 2}(\partial K)}$ is the usual trace norm and $C$ is independent of $k$ and $\boldsymbol{u}$. In particular, if $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$, then $\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}=\boldsymbol{u}$. Moreover, the following equivalence of seminorms holds:

$$
\begin{equation*}
|\boldsymbol{u}|_{\boldsymbol{H}^{1 / 2}(\partial K)} \leq\left|\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}\right|_{\boldsymbol{H}^{1}(K)} \leq C \beta^{-1}|\boldsymbol{u}|_{\boldsymbol{H}^{1 / 2}(\partial K)}, \quad \forall K \in \mathcal{T} \tag{4.5}
\end{equation*}
$$

where $C$ is independent of $k, h_{K}, \beta$, and $\boldsymbol{u}$.
Proof. Let $K \in \mathcal{T}$ and $\boldsymbol{u} \in \boldsymbol{V}$ be given. Conditions (4.2d) and (4.2e) imply that $\Pi_{Q} \boldsymbol{u} \in Q_{I}$. Thanks to [9, Theorem 7.4], there exists $\boldsymbol{w} \in \mathcal{P}_{k}(K)$ such that

$$
\begin{equation*}
\left.\boldsymbol{w}\right|_{\partial K}=\left.\boldsymbol{u}\right|_{\partial K} \quad \text { and } \quad\|\boldsymbol{w}\|_{\boldsymbol{H}^{1}(K)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1 / 2}(\partial K)} \tag{4.6}
\end{equation*}
$$

with $C$ independent of $k$. In particular, $\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}-\boldsymbol{w}=\boldsymbol{u}_{I}$ where $\boldsymbol{u}_{I} \in \boldsymbol{V}_{I}(K)$ satisfies

$$
\begin{aligned}
a_{K}\left(\boldsymbol{u}_{I}, \boldsymbol{v}\right)+b_{K}\left(\boldsymbol{v}, \Pi_{Q} \boldsymbol{u}\right) & =-a_{K}(\boldsymbol{w}, \boldsymbol{v}) & & \forall \boldsymbol{v} \in \boldsymbol{V}_{I}(K) \\
b_{K}\left(\boldsymbol{u}_{I}, q\right) & =-b_{K}(\boldsymbol{w}, q) & & \forall q \in Q_{I}(K)
\end{aligned}
$$

Using [18, Corollary 4.1] and Theorem 3.3, we conclude that

$$
\left\|\boldsymbol{u}_{I}\right\|_{\boldsymbol{H}^{1}(K)}+\left\|\Pi_{Q} \boldsymbol{u}\right\|_{L^{2}(K)} \leq C\|\boldsymbol{w}\|_{\boldsymbol{H}^{1}(K)}
$$

Equation (4.4) now follows from the triangle inequality and (4.6).
Now let $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$. Then, $\operatorname{div} \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u} \in \tilde{Q}_{E}$ by (4.2b) and $\boldsymbol{w}:=\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u} \in \boldsymbol{V}_{I}$ by (4.2c). Moreover, $\operatorname{div} \boldsymbol{w} \in Q_{I} \cap \tilde{Q}_{E}$ since $\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u} \in \tilde{Q}_{E}$. Thus, $\operatorname{div} \boldsymbol{w}=0$ and $\boldsymbol{w}=\operatorname{curl} \phi$ with $\phi \in \Sigma_{I}$ by the exact sequence property (3.5). For any $\psi \in \Sigma_{I}$,

$$
a(\operatorname{curl} \phi, \operatorname{curl} \psi)=a(\boldsymbol{u}, \operatorname{curl} \psi)-a\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \operatorname{curl} \psi\right)=0
$$

by the definition of $\tilde{\boldsymbol{V}}_{E}$ and choosing $\boldsymbol{v}=\operatorname{curl} \psi$ for $\psi \in \Sigma_{I}$ in (4.2a). Since $a($ curl $\cdot, \operatorname{curl} \cdot)$ is coercive on $\Sigma_{I}, \phi \equiv 0$, and $\boldsymbol{u}=\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}$.

The equivalence (4.5) is proved by arguing as in [12, Theorem 4.1].

The next result complements Theorem 4.1:
Theorem 4.2. For $p \in Q, \mathscr{E}(\mathbf{0}, p)=(\mathbf{0}, \tilde{\Pi} p)$ where $\tilde{\Pi}$ is defined in (4.1), and

$$
\begin{equation*}
\|\tilde{\Pi} p\|_{L^{2}(K)} \leq\|p\|_{L^{2}(K)} \tag{4.7}
\end{equation*}
$$

In particular, if $p \in Q_{I}^{\perp}$, then $\mathscr{E}(\mathbf{0}, p)=(\mathbf{0}, p)$.
Proof. Let $p \in Q$ and consider the Stokes extension $(\tilde{\boldsymbol{u}}, \tilde{p}):=\mathscr{E}(\mathbf{0}, p)$. Since $Q=Q_{I} \oplus Q_{I}^{\perp}$, the pressure $\tilde{p}$ may be written in the form $\tilde{p}=p_{I}+\tilde{\Pi} p$. In particular, $p_{I} \in Q_{I}$ satisfies

$$
b_{K}\left(\boldsymbol{v}, p_{I}\right)=b_{K}(\boldsymbol{v}, \tilde{p})-b_{K}(\boldsymbol{v}, \tilde{\Pi} p)=b_{K}\left(\boldsymbol{v}_{K}, \tilde{p}\right), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_{I}(K), \forall K \in \mathcal{T}
$$

where we used the fact that $b_{K}(\boldsymbol{v}, \tilde{\Pi} p)=0$ since div $\boldsymbol{V}_{I}(K)=Q_{I}(K) \perp Q_{I}^{\perp}$. Hence,

$$
\begin{aligned}
a_{K}(\tilde{\boldsymbol{u}}, \boldsymbol{v})+b_{K}\left(\boldsymbol{v}, p_{I}\right)=0 & \forall \boldsymbol{v} \in \boldsymbol{V}_{I}(K) \\
b_{K}(\tilde{\boldsymbol{u}}, q)=0 & \forall q \in Q_{I}(K),
\end{aligned}
$$

Equation (4.4) then gives $\left(\tilde{\boldsymbol{u}}, p_{I}\right)=\mathscr{E}(\mathbf{0}, 0)$; or, equally well, $\tilde{\boldsymbol{u}}=\mathbf{0}$ and $\tilde{p}=\tilde{\Pi} p$. The estimate (4.8) immediately follows since $\tilde{\Pi}$ is a projection. If $p \in Q_{I}^{\perp}$, then $\mathscr{E}(\mathbf{0}, p)=(\mathbf{0}, \tilde{\Pi} p)=(\mathbf{0}, p)$.
Combining Theorems 4.1 and 4.2 leads to the following result:
Corollary 4.3. The Stokes extension operator $\mathscr{E}(\cdot, \cdot)$ is linear and continuous: For $K \in \mathcal{T}$,

$$
\begin{equation*}
\|\mathscr{E}(\boldsymbol{u}, p)\|_{\boldsymbol{H}^{1}(K) \times L^{2}(K)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^{1 / 2}(\partial K)}+\|\tilde{\Pi} p\|_{L^{2}(K)} \quad \forall(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q \tag{4.8}
\end{equation*}
$$

where $C$ is independent of $k$. Moreover, ker $\mathscr{E}=\boldsymbol{V}_{I} \times Q_{I}$ and $\mathscr{E}(\boldsymbol{u}, p)=\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \Pi_{Q} \boldsymbol{u}+\right.$ $\tilde{\Pi} p)$.

Proof. The linearity of $\mathscr{E}(\cdot, \cdot)$ is immediate from the definition (4.2), and (4.8) then follows from (4.4) and (4.7) using the triangle inequality. A simple consequence of (4.2c)-(4.2e) is that $\operatorname{ker} \mathscr{E} \subseteq \boldsymbol{V}_{I} \times Q_{I}$. Moreover, (4.8) gives that $\boldsymbol{V}_{I} \times Q_{I} \subseteq \operatorname{ker} \mathscr{E}$. Thus, $\operatorname{ker} \mathscr{E}=\boldsymbol{V}_{I} \times Q_{I}$.
5. Variational Form of the Schur Complement System. The results of the previous two sections are used to study the Schur complement system (2.2). The first result relates the Schur complement matrix $\boldsymbol{S}$ to the discrete Stokes extension map:

Lemma 5.1. For all $(\boldsymbol{u}, p),(\boldsymbol{v}, q) \in \boldsymbol{V}_{0} \times Q$, the Stokes extension satisfies

$$
a\left(\boldsymbol{u}_{S}, \boldsymbol{v}_{S}\right)+b\left(\boldsymbol{v}_{S}, p_{S}\right)+b\left(\boldsymbol{u}_{S}, q_{S}\right)=\left[\begin{array}{c}
\vec{v}_{E}  \tag{5.1}\\
\vec{q}_{e}
\end{array}\right]^{T} \boldsymbol{S}\left[\begin{array}{c}
\vec{u}_{E} \\
\vec{p}_{e}
\end{array}\right]
$$

where $\boldsymbol{S}=\left[\begin{array}{cc}\widetilde{\boldsymbol{A}} & \widetilde{\boldsymbol{B}}^{T} \\ \widetilde{\boldsymbol{B}} & \widetilde{\boldsymbol{C}}\end{array}\right]$. Consequently, the following identities hold:

$$
\begin{align*}
\vec{u}_{E}^{T} \widetilde{\boldsymbol{A}} \vec{v}_{E} & =a\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{v}\right) & & \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{0}  \tag{5.2a}\\
\vec{p}_{e}^{T} \widetilde{\boldsymbol{B}} \vec{u}_{E} & =b\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \tilde{\Pi} p\right) & & \forall p \in Q, \boldsymbol{u} \in \boldsymbol{V}_{0}  \tag{5.2b}\\
\widetilde{\boldsymbol{C}} & =\mathbf{0} \quad \text { and } \quad \vec{g}_{e}^{*}=\overrightarrow{0}, & & \tag{5.2c}
\end{align*}
$$

where $\boldsymbol{\Pi}_{\boldsymbol{V}}$ is defined in (4.3) and $\tilde{\Pi}$ is defined in (4.1).

Proof. Let $(\boldsymbol{u}, p) \in \boldsymbol{V}_{0} \times Q$. The Stokes extension $\left(\boldsymbol{u}_{S}, p_{S}\right)=\mathcal{E}(\boldsymbol{u}, p)$ may be written as $\boldsymbol{u}_{S}=\vec{\Phi}_{E}^{T} \vec{u}_{E}+\vec{\Phi}_{I}^{T} \vec{u}_{I}^{*}$ and $p_{S}=\vec{\psi}_{e}^{T} \vec{p}_{e}+\vec{\psi}_{\iota}^{T} \vec{p}_{\iota}^{*}$ so that $\boldsymbol{u}=\vec{\Phi}_{E}^{T} \vec{u}_{E}+\vec{\Phi}_{I}^{T} \vec{u}_{I}$ and $p=\vec{\psi}_{e}^{T} \vec{p}_{e}+\vec{\psi}_{\iota}^{T} \vec{p}_{\iota}$ for suitable $\vec{u}_{E}, \vec{u}_{I}, \vec{p}_{e}$, and $\vec{p}_{\iota}$. Thanks to (4.2a) and (4.2b), $\vec{u}_{I}^{*}$ and $\vec{p}_{\iota}^{*}$ are given by

$$
\left[\begin{array}{c}
\vec{u}_{I}^{*}  \tag{5.3}\\
\vec{p}_{\iota}^{*}
\end{array}\right]=-\left[\begin{array}{cc}
\boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\
\boldsymbol{B}_{\iota I} & \mathbf{0}
\end{array}\right]^{-1}\left[\begin{array}{cc}
\boldsymbol{A}_{I E} & \boldsymbol{B}_{I e} \\
\boldsymbol{B}_{\iota E} & \mathbf{0}
\end{array}\right]\left[\begin{array}{c}
\vec{u}_{E} \\
\vec{p}_{e}
\end{array}\right] .
$$

Analogous relations hold replacing $(\boldsymbol{u}, p)$ by $(\boldsymbol{v}, q) \in \boldsymbol{V}_{0} \times Q$. Now,

$$
\begin{aligned}
& a\left(\boldsymbol{u}_{S}, \boldsymbol{v}_{S}\right)+b\left(\boldsymbol{v}_{S}, p_{S}\right)+b\left(\boldsymbol{u}_{S}, q_{S}\right) \\
& \quad=\left[\begin{array}{c}
\vec{v}_{E} \\
\vec{q}_{e} \\
\hline \vec{v}_{I}^{*} \\
\vec{q}_{\iota}^{*}
\end{array}\right]^{T}\left[\begin{array}{cc|cc}
\boldsymbol{A}_{E E} & \boldsymbol{B}_{E e} & \boldsymbol{A}_{E I} & \boldsymbol{B}_{E \iota} \\
\boldsymbol{B}_{e E} & \mathbf{0} & \boldsymbol{B}_{e I} & \mathbf{0} \\
\hline \boldsymbol{A}_{I E} & \boldsymbol{B}_{I e} & \boldsymbol{A}_{I I} & \boldsymbol{B}_{I \iota} \\
\boldsymbol{B}_{\iota E} & \mathbf{0} & \boldsymbol{B}_{\iota I} & \mathbf{0}
\end{array}\right]\left[\begin{array}{c}
\vec{u}_{E} \\
\vec{p}_{e} \\
\hline \vec{u}_{I}^{*} \\
\vec{p}_{\iota}^{*}
\end{array}\right] .
\end{aligned}
$$

and then (5.3), we obtain (5.1). Identities (5.2) are then obtained from (5.1) as follows:
(a) Choose $p=q=0$. Then, $\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}^{\dagger}, \Pi_{Q} \boldsymbol{u}\right),\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{v}, \Pi_{Q} \boldsymbol{v}\right) \in \boldsymbol{V} \times Q_{I}$ by Theorem 4.1, so $b\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{v}, \Pi_{Q} \boldsymbol{u}\right)+b\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \Pi_{Q} \boldsymbol{v}\right)=0$ by (4.2b); (5.2a) follows.
(b) Choose $p=0$ and $\boldsymbol{v}=\mathbf{0}$. By Theorem 4.2, $\mathscr{E}(\mathbf{0}, q)=(\mathbf{0}, \tilde{\Pi} q)$, and (5.2b) follows.
(c) Choose $\boldsymbol{u}=\boldsymbol{v}=\mathbf{0}$. By Theorem $4.2, \mathscr{E}(\mathbf{0}, p)=(\mathbf{0}, \tilde{\Pi} p), \mathscr{E}(\mathbf{0}, q)=(\mathbf{0}, \tilde{\Pi} q)$, and so $\vec{q}_{e}^{T} \widetilde{\boldsymbol{C}} \vec{p}_{e}=0$. Furthermore,

$$
\vec{q}_{e}^{T} \vec{g}_{e}^{*}=\vec{q}_{e}^{T} \widetilde{\boldsymbol{B}} \vec{u}_{E}=b\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \tilde{\Pi} q\right)=-b\left(\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \tilde{\Pi} q\right), \quad \forall q \in Q
$$

by $(1.2 \mathrm{~b})$. Since $\operatorname{div}\left(\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}\right) \in \operatorname{div} \boldsymbol{V}_{I}=Q_{I} \perp Q_{I}^{\perp}, b\left(\boldsymbol{u}-\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \tilde{q}\right)=0$, which completes (5.2c).
The main result of this section relates the Schur complement problem (2.2) to a Stokes problem posed on the boundary spaces $\tilde{\boldsymbol{V}}_{E} \times \tilde{Q}_{E}$ :

Theorem 5.2. The Schur complement system (2.2), is equivalent to the following variational problem: Find $(\boldsymbol{u}, p) \in \tilde{\boldsymbol{V}}_{E} \times \tilde{Q}_{E}$ such that

$$
\begin{align*}
a(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p) & =(\boldsymbol{f}, \boldsymbol{v}) & & \forall \boldsymbol{v} \in \tilde{\boldsymbol{V}}_{E}  \tag{5.4a}\\
b(\boldsymbol{u}, q) & =0 & & \forall q \in \tilde{Q}_{E} \tag{5.4b}
\end{align*}
$$

Moreover, the nonzero eigenvalues of the generalized eigenvalue problem $\widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{A}}^{-1} \widetilde{\boldsymbol{B}}^{T} \vec{q}_{e}$ $=\lambda \widetilde{\boldsymbol{M}} \vec{q}_{e}$ are contained in the interval $\left[\beta^{2}, 1\right]$, where $\widetilde{\boldsymbol{M}}$ is the matrix associated with the $L^{2}(\Omega)$-inner product on $Q_{I}^{\perp}$ and $\beta$ is the inf-sup constant in (1.3). In particular, the nonzero eigenvalues $\lambda$ are uniformly bounded away from zero in $h$ and $k$.

Proof. Let $(\boldsymbol{u}, p),(\boldsymbol{v}, q) \in \tilde{\boldsymbol{V}}_{E} \times Q_{I}^{\perp}$. Substituting the identities in Theorems 4.1 and 4.2 into (5.2) gives

$$
\begin{aligned}
{\left[\begin{array}{c}
\vec{v}_{E} \\
\vec{q}_{e}
\end{array}\right]^{T} \boldsymbol{S}\left[\begin{array}{c}
\vec{u}_{E} \\
\vec{p}_{e}
\end{array}\right] } & =\vec{v}_{E}^{T} \widetilde{\boldsymbol{A}} \vec{u}_{E}^{T}+\vec{q}_{e}^{T} \widetilde{\boldsymbol{B}} \vec{u}_{E}+\vec{v}_{E} \widetilde{\boldsymbol{B}}^{T} \vec{p}_{e} \\
& =a\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{v}\right)+b\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{v}, \tilde{\Pi} p\right)+b\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \tilde{\Pi} q\right) \\
& =a(\boldsymbol{u}, \boldsymbol{v})+b(\boldsymbol{v}, p)+b(\boldsymbol{u}, q)
\end{aligned}
$$

(5.4) now follows from (5.2c) on noting that $\tilde{Q}_{E}=Q_{I}^{\perp} \cap L_{0}^{2}(\Omega)$. Arguing as in [14, Theorem 3.22], the eigenvalue bound follows from the inf-sup condition (3.14).
6. Basis Functions. We first define a basis $\left\{\phi_{i}\right\}$ for the space of scalar-valued functions $V$. The basis is constructed so that the exclusion of particular functions gives a basis for $V_{0}=V \cap H_{0}^{1}(\Omega)$, which simplifies both the enforcement of homogeneous boundary conditions and the implementation of the preconditioner. A basis $\left\{\boldsymbol{\Phi}_{i}\right\}$ for the velocity space $\boldsymbol{V}_{0}=V_{0} \times V_{0}$ is then obtained using functions of the form $\phi_{j} \hat{\boldsymbol{e}}_{1}$ and $\phi_{j} \hat{e}_{2}$. For the pressure space, we only give a basis for $Q$ since the space $Q_{0}$ is not used in the actual implementation.
6.1. Basis Functions on a Reference Triangle. We begin by defining basis functions for the pressure and velocity spaces on the reference triangle $\hat{T}$ shown in Figure 2b.
6.1.1. Pressure Basis Functions. Let $\left\{B_{\alpha}^{k}\right\}_{\alpha \in \mathcal{I}}$ denote the Bernstein polynomials [21]:

$$
\begin{equation*}
B_{\alpha}^{k}=\frac{k!}{\alpha_{1}!\alpha_{2}!\alpha_{3}!} \lambda_{1}^{\alpha_{1}} \lambda_{2}^{\alpha_{2}} \lambda_{3}^{\alpha_{3}} \tag{6.1}
\end{equation*}
$$

where $\mathcal{I}=\left\{\alpha \in \mathbb{Z}_{+}^{3}:|\alpha|=k\right\}$ and $\left\{\lambda_{i}, 1 \leq i \leq 3\right\}$, are the barycentric coordinates on the reference triangle $\hat{T}$. The set $\left\{B_{\alpha}^{k}\right\}_{\alpha \in \mathcal{I}}$ forms a basis for $\mathcal{P}_{k}(\hat{T})$ [21]. Each Bernstein polynomial $B_{\alpha}^{k}$ can be identified with the domain point $\boldsymbol{x}_{\alpha}=\frac{\alpha_{1}}{k} \hat{\boldsymbol{a}}_{1}+$ $\frac{\alpha_{2}}{k} \hat{\boldsymbol{a}}_{2}+\frac{\alpha_{3}}{k} \hat{\boldsymbol{a}}_{3}$ on the reference triangle. Let $\mathcal{I}_{0}=\left\{\alpha \in \mathcal{I}: \alpha_{i}<k\right\}$ denote the subset corresponding to interior (non-vertex) points. Fix any $\beta \in \mathcal{I}_{0}$; since all the Bernstein polynomials (6.1) share the same average value, the set $\left\{B_{\alpha}^{k}-B_{\beta}^{k}\right\}_{\alpha \in \mathcal{I} \backslash\{\beta\}}$ is a basis for $\mathcal{P}_{k}(\hat{T}) \cap L_{0}^{2}(\hat{T})$. This set can be partitioned into:
(i) Vertex functions: $\hat{\psi}_{i}:=B_{k e_{i}}^{k}-B_{\beta}^{k}, 1 \leq i \leq 3$, satisfying $\int_{\hat{T}} \hat{\psi}_{i} d \boldsymbol{x}=0$ and $\hat{\psi}_{i}\left(\hat{\boldsymbol{a}}_{j}\right)=\delta_{i j}$ for $1 \leq i, j \leq 3$.
(ii) Interior functions: $\hat{\psi}_{\iota, \alpha}:=B_{\alpha}^{k}-B_{\beta}^{k}, \alpha \in \mathcal{I}_{0} \backslash\{\beta\}$, satisfying $\int_{\hat{T}} \hat{\psi}_{\iota, \alpha} d \boldsymbol{x}=0$ and $\hat{\psi}_{\iota, \alpha}\left(\hat{\boldsymbol{a}}_{i}\right)=0,1 \leq i \leq 3$.
In order to obtain a basis for $\mathcal{P}_{k}(\hat{T})$, we supplement this set with one additional function:
(iii) Average value function

$$
\begin{equation*}
\hat{\psi}_{\hat{T}}:=1-\sum_{i=1}^{3} \hat{\psi}_{i} \tag{6.2}
\end{equation*}
$$

satisfying $|\hat{T}|^{-1} \int_{\hat{T}} \hat{\psi}_{\hat{T}} d \boldsymbol{x}=1$ and $\hat{\psi}_{\hat{T}}\left(\boldsymbol{a}_{i}\right)=0,1 \leq i \leq 3$.
In summary, there are 3 vertex functions, one average value function and $\frac{1}{2}(k+1)(k+$ 2) -4 interior functions which total $\frac{1}{2}(k+1)(k+2)=\operatorname{dim} \mathcal{P}_{k}(\hat{T})$, and form a basis for the pressure space $Q=\mathcal{P}_{k}(\hat{T})$ on the reference element.
6.1.2. Velocity Basis Functions. The construction of the basis functions for the velocity space $V$ is more complicated owing to the higher continuity requirement. In particular, the basis functions $\left\{\hat{\phi}_{k}^{\beta}\right\},|\beta|=1, k \in\{1,2,3\}$, associated with the derivative degrees of freedom at the vertices should satisfy $D^{\alpha} \hat{\phi}_{k}^{\beta}\left(\hat{\boldsymbol{a}}_{l}\right)=\delta_{\alpha \beta} \delta_{k l},|\alpha|=$ $1, l \in\{1,2,3\}$. In order to construct these functions, we begin by considering the vector valued function given by

$$
\vec{J}_{1}:=\frac{\lambda_{1}^{2}}{P_{k-3}^{(3,3)}(-1)}\left[\begin{array}{l}
\lambda_{2} P_{k-3}^{(3,3)}\left(\lambda_{2}-\lambda_{1}\right)  \tag{6.3}\\
\lambda_{3} P_{k-3}^{(3,3)}\left(\lambda_{3}-\lambda_{1}\right)
\end{array}\right],
$$



Fig. 2: Notation for (a) general triangle $K$ and (b) reference triangle $\hat{T}$.
where $P_{k}^{(3,3)}$ is the Jacobi polynomial of degree $k$ [36]. The first component of $\vec{J}_{1}$ vanishes on edge $\gamma_{2}$ and the gradient at $\hat{\boldsymbol{a}}_{1}$ is given by $\left[\begin{array}{ll}1 & 0\end{array}\right]^{T}$, while the second component vanishes on edge $\hat{\gamma}_{3}$ and has gradient $\left[\begin{array}{ll}0 & 1\end{array}\right]^{T}$ at $\hat{\boldsymbol{a}}_{1}$. The factor $\lambda_{1}^{2}$ means that both components of $\vec{J}_{1}$ and their gradients vanish on the edge $\hat{\gamma}_{1}$. In summary, since $x=\lambda_{2}$ and $y=\lambda_{3}$, we have

$$
\vec{J}_{1}\left(\hat{\boldsymbol{a}}_{k}\right)=\overrightarrow{0} \quad \text { and } \quad\left[\begin{array}{c}
\frac{\partial}{\partial x}  \tag{6.4}\\
\frac{\partial}{\partial y}
\end{array}\right] \vec{J}_{1}^{T}\left(\hat{\boldsymbol{a}}_{k}\right)=\left[\begin{array}{c}
\frac{\partial}{\partial \lambda_{2}} \\
\frac{\partial}{\partial \lambda_{3}}
\end{array}\right] \vec{J}_{1}^{T}\left(\hat{\boldsymbol{a}}_{k}\right)=\delta_{k l}\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] .
$$

Defining $\vec{J}_{2}$ and $\vec{J}_{3}$ by cyclic permutations of the indices, we conclude that $\vec{J}_{2}$ and $\vec{J}_{3}$ vanish at the vertices and that, for $k \in\{1,2,3\}$,

$$
\left[\begin{array}{c}
\frac{\partial}{\partial \lambda_{3}}  \tag{6.5}\\
\frac{\partial}{\partial \lambda_{1}}
\end{array}\right] \vec{J}_{2}^{T}\left(\hat{\boldsymbol{a}}_{k}\right)=\delta_{k 2}\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right] \quad \text { and }\left[\begin{array}{c}
\frac{\partial}{\partial \lambda_{1}} \\
\frac{\partial^{\prime}}{\partial \lambda_{2}}
\end{array}\right] \vec{J}_{3}^{T}\left(\hat{\boldsymbol{a}}_{k}\right)=\delta_{k 3}\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]
$$

Substituting the identities

$$
\left[\begin{array}{c}
\frac{\partial}{\partial \lambda_{3}} \\
\frac{\partial}{\partial \lambda_{1}}
\end{array}\right]=\left[\begin{array}{ll}
-1 & 1 \\
-1 & 0
\end{array}\right]\left[\begin{array}{l}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array}\right] \quad \text { and } \quad\left[\begin{array}{c}
\frac{\partial}{\partial \lambda_{1}} \\
\frac{\partial}{\partial \lambda_{2}}
\end{array}\right]=\left[\begin{array}{ll}
0 & -1 \\
1 & -1
\end{array}\right]\left[\begin{array}{c}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array}\right]
$$

in (6.5) and rearranging gives

$$
\begin{align*}
& {\left[\begin{array}{c}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array}\right]\left(\left[\begin{array}{cc}
-1 & -1 \\
1 & 0
\end{array}\right] \vec{J}_{2}\right)^{T}\left(\hat{\boldsymbol{a}}_{k}\right)=\delta_{k 2}\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]}  \tag{6.6}\\
& {\left[\begin{array}{c}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y}
\end{array}\right]\left(\left[\begin{array}{cc}
0 & 1 \\
-1 & -1
\end{array}\right] \vec{J}_{3}\right)^{T}\left(\hat{\boldsymbol{a}}_{k}\right)=\delta_{k 3}\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] .} \tag{6.7}
\end{align*}
$$

Armed with (6.4), (6.6), and (6.7), we define the basis functions for the velocity space as follows:
(i) $C^{0}$ vertex functions: $\hat{\phi}_{i}=\lambda_{i}^{2}\left(3-2 \lambda_{i}\right), \leq i \leq 3$, satisfying $\hat{\phi}_{i}\left(\hat{\boldsymbol{a}}_{j}\right)=\delta_{i j}$, $D \hat{\phi}_{i}\left(\hat{\boldsymbol{a}}_{j}\right)=\mathbf{0}$, and $\left.\hat{\phi}_{i}\right|_{\hat{\gamma}_{i}}=0$ for $1 \leq i, j \leq 3$.
(ii) $C^{1}$ vertex functions

$$
\left[\begin{array}{l}
\hat{\phi}_{1}^{(1,0)} \\
\hat{\phi}_{1}^{(0,1)}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \vec{J}_{1} ; \quad\left[\begin{array}{l}
\hat{\phi}_{2}^{(1,0)} \\
\hat{\phi}_{2}^{(0,1)}
\end{array}\right]=\left[\begin{array}{cc}
-1 & -1 \\
1 & 0
\end{array}\right] \vec{J}_{2} ; \quad\left[\begin{array}{l}
\hat{\phi}_{3}^{(1,0)} \\
\hat{\phi}_{3}^{(0,1)}
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-1 & -1
\end{array}\right] \vec{J}_{3}
$$

which, thanks to (6.4), (6.6), and (6.7), satisfy $D^{\beta} \hat{\phi}_{i}^{\alpha}\left(\hat{\boldsymbol{a}}_{j}\right)=\delta_{\alpha \beta} \delta_{i j}$ and $\left.\hat{\phi}_{i}\right|_{\hat{\gamma}_{i}}=$ 0 for $1 \leq i, j \leq 3,|\alpha|=1,|\beta| \leq 1$.
(iii) Edge functions: Let $\hat{\gamma}$ be the edge connecting vertices $\hat{\boldsymbol{a}}_{i}$ and $\hat{\boldsymbol{a}}_{j}$; then the basis functions associated with the edge are defined by $\hat{\phi}_{\hat{\gamma}, l}=\lambda_{i}^{2} \lambda_{j}^{2} r_{l}\left(\lambda_{j}-\lambda_{i}\right)$ where $\left\{r_{l}\right\}$ is any basis for $\mathcal{P}_{k-4}((-1,1))$. These functions satisfy $D^{\alpha} \hat{\phi}_{\hat{\gamma}, l}\left(\hat{\boldsymbol{a}}_{k}\right)$ $=0$ for $|\alpha| \leq 1,1 \leq k \leq 3$ and $\left.\hat{\phi}_{\hat{\gamma}, l}\right|_{\hat{\gamma}^{\prime}}=\delta_{\hat{\gamma} \hat{\gamma}^{\prime}}$ for $\hat{\gamma}^{\prime} \in \mathcal{E}_{\hat{T}}$.
(iv) Interior functions: The basis functions associated with the element interior are defined by $\hat{\phi}_{I, l}=\lambda_{1} \lambda_{2} \lambda_{3} s_{l}$ where $\left\{s_{l}\right\}$ is any basis for $\mathcal{P}_{k-3}(\hat{T})$. These functions satisfy $\left.\hat{\phi}_{I, l}\right|_{\partial \hat{T}}=0$ and $D \hat{\phi}_{I, l}\left(\hat{\boldsymbol{a}}_{k}\right)=\mathbf{0}$ for $1 \leq k \leq 3$.
It is easily seen that the above functions are linearly independent. Furthermore, there are 3 functions per vertex, $\operatorname{dim} \mathcal{P}_{k-4}((-1,1))=k-3$ functions per edge, and $\operatorname{dim} \mathcal{P}_{k-3}(\hat{T})=\frac{1}{2}(k-2)(k-1)$ interior functions which total $\frac{1}{2}(k+1)(k+2)=$ $\operatorname{dim} \mathcal{P}_{k}(\hat{T})$. Hence, the above functions also form a basis for $\mathcal{P}_{k}(\hat{T})$.
6.2. Basis Functions on a Mesh. We now define the global basis functions for the spaces $Q$ and $V$. The lower continuity requirements imposed at corner vertices $\mathcal{V}_{C}$ means that extra care must be taken when defining the global basis functions associated with $\mathcal{V}_{C}$.
6.2.1. Pressure Basis Functions. The pressure space $Q$ requires $C^{0}$ continuity at all vertices except at corner vertices, where the functions are allowed to be discontinuous. This means that each element has its own degree of freedom at vertices $\boldsymbol{a} \in \mathcal{V}_{C}$, whilst at the remaining vertices $\boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{C}$, all elements share a single degree of freedom at the common vertex as shown in Figure 3a. Consequently, any given vertex $\boldsymbol{a} \in \mathcal{V}$ is associated with either (a) a single basis function supported on the patch $\mathcal{T}_{\boldsymbol{a}}$ if $\boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{C}$, or (b) a collection of basis functions, each of which is supported on a single element $K \in \mathcal{T}_{\boldsymbol{a}}$ if $\boldsymbol{a} \in \mathcal{V}_{C}$. The set of supports of the pressure functions associated with a vertex $\boldsymbol{a} \in \mathcal{V}$ is defined by

$$
\Omega_{\boldsymbol{a}}= \begin{cases}\left\{\mathcal{T}_{a}\right\} & \boldsymbol{a} \in \mathcal{V}_{C} \\ \left\{K \in \mathcal{T}_{a}\right\} & \boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{C}\end{cases}
$$

That is, the cardinality of these sets is $\left|\Omega_{\boldsymbol{a}}\right|=1$ for noncorner vertices (since there is only one vertex basis function associated to $\boldsymbol{a}$ ) whilst $\left|\Omega_{\boldsymbol{a}}\right| \geq 2$ for corner vertices, thanks to the assumption that the mesh is corner-split into at least two elements. The corresponding global vertex functions $\left\{\psi_{\boldsymbol{a}}^{\omega}: \boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}\right\}$ are defined to be pull-backs in the usual way:

$$
\psi_{\boldsymbol{a}}^{\omega}=\left\{\begin{array}{ll}
\hat{\psi}_{i} \circ \boldsymbol{F}_{K}^{-1} & \text { on } K \subseteq \omega,  \tag{6.8}\\
0 & \text { otherwise },
\end{array} \quad \psi_{K}= \begin{cases}\hat{\psi}_{\hat{T}} \circ \boldsymbol{F}_{K}^{-1} & \text { on } K \\
0 & \text { otherwise }\end{cases}\right.
$$

where $\hat{\boldsymbol{a}}_{i}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$.
The average value functions and interior functions are simpler. Each element $K \in \mathcal{T}$ has a single function $\psi_{K}$, corresponding to the average value over $K$, defined by (6.8). Similarly, each element $K \in \mathcal{T}$ has $\frac{k}{2}(k+1)-4$ interior functions also defined to be pull-backs.


Fig. 3: (a) The global pressure vertex degrees of freedom and (b) the global velocity vertex degrees of freedom on a mesh of an example domain. Observe that in (a) there are multiple pressure vertex degrees of freedom at corner vertices but only a single degree of freedom at interior vertices and in (b) there are three or more derivative degrees of freedom at corner vertices (red), two derivative degrees of freedom aligned with the domain boundary at noncorner boundary vertices (green), and two derivative degrees of freedom aligned with the coordinate axes at interior vertices (blue).
6.2.2. Velocity Basis Functions. The velocity space $V$ imposes $C^{1}$ continuity at all vertices except at corner vertices, where only $C^{0}$-continuity is required to ensure $V \subset H^{1}(\Omega)$. This means that at corner vertices $\boldsymbol{a} \in \mathcal{V}_{C}$, each element $K \in \mathcal{T}_{\boldsymbol{a}}$ has two degrees of freedom for the gradient corresponding to the two tangential derivatives corresponding to the two edges of $K$ that meet at $\boldsymbol{a}$. To enforce continuity between two neighboring elements in $\mathcal{T}_{\boldsymbol{a}}$, the tangential derivative corresponding to the common edge must be shared between the two elements. In other words, each corner vertex $\boldsymbol{a} \in \mathcal{V}_{C}$ has one derivative degree of freedom for each edge $\gamma \in \mathcal{E}_{\boldsymbol{a}}$. For the remaining noncorner vertices $\boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{C}$, all elements in $\mathcal{T}_{\boldsymbol{a}}$ share two degrees of freedom at the common vertex, corresponding to any two linearly independent directional derivatives as in Figure 3b. Consequently, a given vertex $\boldsymbol{a} \in \mathcal{V}$ is associated with either (a) two basis functions supported on the patch $\mathcal{T}_{\boldsymbol{a}}$ if $\boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{C}$, or (b) a collection of basis functions, each of which is associated to an edge $\gamma \in \mathcal{E}_{\boldsymbol{a}}$ and supported on the pair of elements sharing the common edge $\gamma$ if $\boldsymbol{a} \in \mathcal{V}_{C}$.

The set of unit vectors defining the directional derivative degrees of freedom at a vertex $\boldsymbol{a} \in \mathcal{V}$ are chosen as follows:

$$
D_{\boldsymbol{a}}= \begin{cases}\left\{\hat{\boldsymbol{e}}_{1}, \hat{\boldsymbol{e}}_{2}\right\} & \boldsymbol{a} \in \mathcal{V}_{I}  \tag{6.9}\\ \{\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}\} & \boldsymbol{a} \in \mathcal{V}_{B} \\ \left\{\hat{\boldsymbol{t}}, \gamma \in \mathcal{E}_{\boldsymbol{a}}\right\} & \boldsymbol{a} \in \mathcal{V}_{C}\end{cases}
$$

where $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{n}}$ are the unit tangent and normal vectors at a noncorner boundary vertex $\boldsymbol{a} \in \mathcal{V}_{B}$ and $\hat{\boldsymbol{t}}_{\gamma}$ denotes a unit tangent vector on an edge $\gamma \in \mathcal{E}$ as illustrated in Figure 3b. For a given vertex $\boldsymbol{a} \in \mathcal{V}$ and unit vector $\hat{\boldsymbol{\mu}} \in D_{\boldsymbol{a}}$, the global basis function $\phi_{\boldsymbol{a}}^{\mu}$ has support

$$
\operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}= \begin{cases}\left\{K \in \mathcal{T}_{\boldsymbol{a}}: \exists \gamma \in \mathcal{E}_{K} \text { with } \hat{\boldsymbol{t}}_{\gamma}= \pm \hat{\boldsymbol{\mu}}\right\} & \boldsymbol{a} \in \mathcal{V}_{C} \\ \mathcal{T}_{\boldsymbol{a}} & \boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{C}\end{cases}
$$

The global $C^{1}$ vertex functions come in pairs as follows: Given a noncorner vertex
$\boldsymbol{a} \in \mathcal{V}_{I} \cup \mathcal{V}_{B}$, let $\hat{\boldsymbol{\mu}}_{1}, \hat{\boldsymbol{\mu}}_{2}$ be unit vectors such that $D_{\boldsymbol{a}}=\left\{\hat{\boldsymbol{\mu}}_{1}, \hat{\boldsymbol{\mu}}_{2}\right\}$ as in (6.9), and define the basis functions by

$$
\left[\begin{array}{l}
\phi_{\boldsymbol{a}}^{\mu_{1}}  \tag{6.10}\\
\phi_{\boldsymbol{a}}^{\mu_{2}}
\end{array}\right]=\left[\begin{array}{ll}
\hat{\boldsymbol{\mu}}_{1} & \hat{\boldsymbol{\mu}}_{2}
\end{array}\right]^{-1} D \boldsymbol{F}_{K}\left[\begin{array}{c}
\hat{\phi}_{i}^{(1,0)} \circ \boldsymbol{F}_{K}^{-1} \\
\hat{\phi}_{i}^{(0,1)} \circ \boldsymbol{F}_{K}^{-1}
\end{array}\right] \quad \text { on } K \in \mathcal{T}_{\boldsymbol{a}}
$$

where $\hat{\boldsymbol{a}}_{i}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$. The above construction ensures that the basis functions are $C^{1}$ continuous at the vertex $\boldsymbol{a}$ : i.e. $\hat{\boldsymbol{\mu}}_{i} \cdot \nabla \phi_{\boldsymbol{a}}^{\mu_{j}}(\boldsymbol{a})=\delta_{i j}$. The case of a corner vertex $\boldsymbol{a} \in \mathcal{V}_{C}$ is more complicated since, as mentioned above, each edge $\gamma \in \mathcal{E}_{\boldsymbol{a}}$ contributes one independent basis function at the vertex, also defined by the expression (6.10), which is supported on the edge patch $\left\{K \in \mathcal{T}_{\boldsymbol{a}}: \gamma \in \mathcal{E}_{K}\right\}$. The unit vectors $\hat{\boldsymbol{\mu}}_{1}, \hat{\boldsymbol{\mu}}_{2}$ in (6.10) associated with such an element $K \in \mathcal{T}_{\boldsymbol{a}}$ are taken to be the pair of unit tangent vectors on the two edges of $K$ having an endpoint at $\boldsymbol{a}$. This means that the basis functions $\phi_{\boldsymbol{a}}^{\mu_{1}}$ and $\phi_{\boldsymbol{a}}^{\mu_{2}}$ are the only $C^{1}$ vertex functions supported on $K$.

The remaining $C^{0}$ vertex functions, edge functions, and interior functions are again defined to be pull-backs of the corresponding functions on the reference element in the usual way: i.e. $\phi_{\boldsymbol{a}}=\hat{\phi}_{i} \circ \boldsymbol{F}_{K}^{-1}$ on $K \in \mathcal{T}_{\boldsymbol{a}}$ and $\phi_{\boldsymbol{a}}=0$ otherwise, where $\hat{\boldsymbol{a}}_{i}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$. Similarly, there are $k-3$ edge functions per edge $\gamma \in \mathcal{E}$, supported on the patch of elements containing that edge $\left\{K \in \mathcal{T}: \gamma \in \mathcal{E}_{K}\right\}$, and there are $\frac{1}{2}(k-1)(k-2)$ interior functions per element $K \in \mathcal{T}$.
6.2.3. Velocity Basis Functions with Homogeneous Boundary Conditions. The above construction gives a basis for $V$ in the absence of essential boundary conditions. If nonhomogeneous essential boundary conditions are imposed, then the values of the following basis functions will be constrained by the boundary data:

- the $C^{0}$ vertex function $\phi_{\boldsymbol{a}}, \boldsymbol{a} \in \mathcal{V} \backslash \mathcal{V}_{I}$ at each vertex on the domain boundary;
- the $C^{1}$ vertex function at each noncorner boundary vertex corresponding to the tangential derivative degree of freedom, i.e. $\phi_{\boldsymbol{a}}^{t}$ for $\boldsymbol{a} \in \mathcal{V}_{B}$;
- the pair of $C^{1}$ vertex functions at each corner boundary vertex corresponding to the tangential derivatives along the domain boundary edges: $\phi_{\boldsymbol{a}}^{t_{\gamma}}, \phi_{\boldsymbol{a}}^{\boldsymbol{\tau}^{\prime}}$ for $\boldsymbol{a} \in \mathcal{V}_{C}$ where $\gamma, \gamma^{\prime} \in \mathcal{E}_{\boldsymbol{a}} \cap \Gamma$; and
- all $k-3$ edge functions for each edge on the domain boundary.

If homogeneous essential boundary conditions are imposed, then a basis for $V_{0}=$ $V \cap H_{0}^{1}(\Omega)$ is obtained by taking the following functions:

- the $C^{0}$ vertex function at each interior vertex, i.e. $\phi_{\boldsymbol{a}}, \boldsymbol{a} \in \mathcal{V}_{I}$;
- the following $C^{1}$ vertex functions: $\phi_{\boldsymbol{a}}^{\mu}, \boldsymbol{a} \in \mathcal{V}, \hat{\boldsymbol{\mu}} \in \stackrel{\circ}{D}_{\boldsymbol{a}}$ where

$$
\stackrel{\circ}{D}_{\boldsymbol{a}}= \begin{cases}\left\{\hat{\boldsymbol{e}}_{1}, \hat{\boldsymbol{e}}_{2}\right\} & \boldsymbol{a} \in \mathcal{V}_{I}  \tag{6.11}\\ \{\hat{\boldsymbol{n}}\} & \boldsymbol{a} \in \mathcal{V}_{B} \\ \left\{\hat{\boldsymbol{t}}_{\gamma}: \gamma \in \mathcal{E}_{\boldsymbol{a}} \cap \mathcal{E}_{I}\right\} & \boldsymbol{a} \in \mathcal{V}_{C}\end{cases}
$$

and $\mathcal{E}_{I}$ denotes the set of interior edges;

- all $k-3$ edge functions on each interior edge; and
- all interior functions on each element.

Condition (6.11) means that we keep both $C^{1}$ vertex functions for each interior vertex, the $C^{1}$ vertex function associated with the the outward normal of $\Gamma$ for each noncorner boundary vertex, and each $C^{1}$ vertex function corresponding to an interior edge unit tangent vector at corner vertices.
7. Constructing the Preconditioner Using Additive Schwarz Theory. In section 2, we constructed the stiffness matrix for the Stokes problem (2.1) using
bases for the spaces $\boldsymbol{V}_{0}$ and $Q$ and performed static condensation to arrive at the Schur complement system (2.2). In section 5, it was shown that the algebraic Schur complement system (2.2) was related to the mixed finite element problem (5.4) posed on the spaces $\tilde{\boldsymbol{V}}_{E} \times \tilde{Q}_{E}$. The alert reader will have noticed a slight discrepancy in the treatment of the average pressure mode over the domain $\Omega$ : in subsection 6.2 and section 2 , the average pressure modes were included in the discretization (and it was pointed out that these modes span the kernel of the Schur complement) whereas in section 5 , the pressure space $\tilde{Q}_{E}=Q_{I}^{\perp} \cap L_{0}^{2}(\Omega)$ was used, which factors out the singular mode. In order to construct a preconditioner in the form (2.4), we formulate an Additive Schwarz Method (ASM) over the spaces $\tilde{\boldsymbol{V}}_{E} \times Q_{I}^{\perp}$ rather than the seemingly more natural choice $\tilde{\boldsymbol{V}}_{E} \times \tilde{Q}_{E}$ suggested by Theorem 5.2.
7.1. Pressure ASM. We decompose the pressure space $Q_{I}^{\perp}$ as follows:

$$
\begin{equation*}
Q_{I}^{\perp}=\bigoplus_{\substack{a \in \mathcal{V} \\ \omega \in \Omega_{a}}} \tilde{Q}_{\boldsymbol{a}, \omega} \oplus \bigoplus_{K \in \mathcal{T}} \tilde{Q}_{K} \tag{7.1}
\end{equation*}
$$

where (i) the vertex spaces $\tilde{Q}_{\boldsymbol{a}, \omega}:=\operatorname{span}\left\{\tilde{\psi}_{\boldsymbol{a}}^{\omega}\right\}, \boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}$ with $\tilde{\psi}_{\boldsymbol{a}}^{\omega}:=\tilde{\Pi} \psi_{\boldsymbol{a}}^{\omega}$ are equipped with the inner product $m_{\boldsymbol{a}, \omega}(p, q):=|\omega| k^{-4} p(\boldsymbol{a}) q(\boldsymbol{a})$, and (ii) the element average spaces $\tilde{Q}_{K}:=\operatorname{span} \tilde{\psi}_{K}, K \in \mathcal{T}$, with $\tilde{\psi}_{K}:=\tilde{\Pi} \psi_{K}$ are equipped with the inner product

$$
m_{K}(p, q):=\frac{1}{|K|}\left(\int_{K} p d \boldsymbol{x}\right)\left(\int_{K} q d \boldsymbol{x}\right), \quad \forall p, q \in \tilde{Q}_{K}
$$

Applying the projection $\tilde{\Pi}$ to the formulae for $\psi_{K}(6.2)$ and (6.8) gives

$$
\tilde{\psi}_{K}= \begin{cases}1-\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \omega \supseteq K}} \tilde{\psi}_{\boldsymbol{a}}^{\omega} & \text { on } K \\ 0 & \text { otherwise }\end{cases}
$$

where the functions $\tilde{\psi}_{\boldsymbol{a}}^{\omega}$ are defined in (i) and we use the fact that $\tilde{\Pi}$ preserves constants. The direct sum decomposition (7.1) means that any $q \in Q_{I}^{\perp}$ may be uniquely expressed in the form

$$
q=\sum_{\substack{a \in \mathcal{V} \\ \omega \in \Omega_{a}}} q_{\boldsymbol{a}, \omega}+\sum_{K \in \mathcal{T}} q_{K}
$$

where

$$
q_{\boldsymbol{a}, \omega}=\left.q\right|_{\omega}(\boldsymbol{a}) \tilde{\psi}_{\boldsymbol{a}}^{\omega}, \quad \boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}, \quad q_{K}=\left(\frac{1}{|K|} \int_{K} q d \boldsymbol{x}\right) \tilde{\psi}_{K}, \quad K \in \mathcal{T}
$$

The action of the associated ASM preconditioner on a residual $g \in L^{2}(\Omega)$ is given by the solution $p \in Q_{I}^{\perp}$ of the variational problem $\bar{m}(p, q)=(g, q) \forall q \in Q_{I}^{\perp}$, where

$$
\begin{equation*}
\bar{m}(p, q):=\sum_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \omega \in \Omega_{\boldsymbol{a}}}} m_{\boldsymbol{a}, \omega}\left(p_{\boldsymbol{a}, \omega}, q_{\boldsymbol{a}, \omega}\right)+\sum_{K \in \mathcal{T}} m_{K}\left(p_{K}, q_{K}\right) \tag{7.2}
\end{equation*}
$$

The bilinear form $\bar{m}(\cdot, \cdot)$ gives rise to a matrix preconditioner $\overline{\boldsymbol{M}}$ for the pressure space defined by

$$
\begin{equation*}
\vec{p}_{e}^{T} \overline{\boldsymbol{M}} \vec{q}_{e}=\bar{m}(\tilde{\Pi} p, \tilde{\Pi} q) \quad \forall p, q \in Q \tag{7.3}
\end{equation*}
$$

7.2. Velocity ASM. We decompose the velocity space $\tilde{\boldsymbol{V}}_{E}$ as follows:

$$
\begin{equation*}
\tilde{\boldsymbol{V}}_{E}=\tilde{\boldsymbol{V}}_{c} \oplus \bigoplus_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \boldsymbol{\mu} \in D_{\boldsymbol{a}}}} \tilde{\boldsymbol{V}}_{\boldsymbol{a}, \mu} \oplus \bigoplus_{\gamma \in \mathcal{E}_{I}} \tilde{\boldsymbol{V}}_{\gamma} \tag{7.4}
\end{equation*}
$$

where (i) the global $C^{0}$ vertex space $\tilde{\boldsymbol{V}}_{\boldsymbol{c}}:=\operatorname{span}\left\{\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{1}\right), \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{2}\right): \boldsymbol{a} \in \mathcal{V}_{I}\right\}$, (ii) the $C^{1}$ vertex spaces $\tilde{\boldsymbol{V}}_{\boldsymbol{a}, \mu}:=\operatorname{span}\left\{\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}}^{\mu} \boldsymbol{e}_{1}\right), \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}}^{\mu} \boldsymbol{e}_{2}\right)\right\}, \boldsymbol{a} \in \mathcal{V}, \hat{\boldsymbol{\mu}} \in \stackrel{\circ}{D}_{\boldsymbol{a}}$, and (iii) the edge spaces $\tilde{\boldsymbol{V}}_{\gamma}:=\operatorname{span}\left\{\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\gamma, j} \boldsymbol{e}_{1}\right), \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\gamma, j} \boldsymbol{e}_{2}\right): 1 \leq j \leq k-3\right\}, \gamma \in \mathcal{E}_{I}$. Each of the velocity subspaces is equipped with the inner product $a(\cdot, \cdot)$ restricted to the appropriate space. The direct sum decomposition (7.4) means that any any $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$ may be uniquely expressed in the form

$$
\begin{equation*}
\boldsymbol{u}=\boldsymbol{u}_{c}+\sum_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \boldsymbol{\mu} \in D_{\boldsymbol{a}}}} \boldsymbol{u}_{\boldsymbol{a}, \mu}+\sum_{\gamma \in \mathcal{E}_{I}} \boldsymbol{u}_{\gamma} \tag{7.5}
\end{equation*}
$$

where

$$
\begin{aligned}
\boldsymbol{u}_{c} & =\sum_{\boldsymbol{a} \in \mathcal{V}_{I}} \sum_{i=1}^{2}\left(\boldsymbol{u} \cdot \hat{\boldsymbol{e}}_{i}\right)(\boldsymbol{a}) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right), \\
\boldsymbol{u}_{\boldsymbol{a}, \mu} & =\sum_{i=1}^{2} \partial_{\mu}\left(\left.\boldsymbol{u}\right|_{K_{\mu}} \cdot \hat{\boldsymbol{e}}_{i}\right)(\boldsymbol{a}) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}}^{\mu} \hat{\boldsymbol{e}}_{i}\right), \quad \boldsymbol{a} \in \mathcal{V}, \quad \hat{\boldsymbol{\mu}} \in \stackrel{\circ}{D}_{\boldsymbol{a}}, \quad K_{\mu} \subseteq \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}
\end{aligned}
$$

and for each $\gamma \in \mathcal{E}_{I}$,

$$
\boldsymbol{u}_{\gamma}= \begin{cases}\boldsymbol{u}-\boldsymbol{u}_{c}-\sum_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \hat{\boldsymbol{\mu}} \in D_{a}}} \boldsymbol{u}_{\boldsymbol{a}, \mu} & \text { on } \gamma \\ \mathbf{0} & \text { on the remaining edges in } \mathcal{E}\end{cases}
$$

The action of the associated ASM preconditioner on a residual $\boldsymbol{f} \in \boldsymbol{L}^{2}(\Omega)$ is given by the solution $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$ of the variational problem $\bar{a}(\boldsymbol{u}, \boldsymbol{v})=(\boldsymbol{f}, \boldsymbol{v}) \forall \boldsymbol{v} \in \tilde{\boldsymbol{V}}_{E}$, where

$$
\begin{equation*}
\bar{a}(\boldsymbol{u}, \boldsymbol{v})=a\left(\boldsymbol{u}_{c}, \boldsymbol{v}_{c}\right)+\sum_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \hat{\boldsymbol{\mu}} \in D_{\boldsymbol{a}}}} a\left(\boldsymbol{u}_{\boldsymbol{a}, \mu}, \boldsymbol{v}_{\boldsymbol{a}, \mu}\right)+\sum_{\gamma \in \mathcal{E}_{I}} a\left(\boldsymbol{u}_{\gamma}, \boldsymbol{v}_{\gamma}\right) \tag{7.6}
\end{equation*}
$$

The bilinear form $\bar{a}(\cdot, \cdot)$ gives rise to a matrix preconditioner $\overline{\boldsymbol{A}}$ for the velocity space defined by

$$
\begin{equation*}
\vec{u}_{E}^{T} \overline{\boldsymbol{A}} \vec{v}_{E}=\bar{a}\left(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{u}, \boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{v}\right), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_{0} \tag{7.7}
\end{equation*}
$$

### 7.3. The Preconditioner and Main Result.

Theorem 7.1. Let $\boldsymbol{P}$ be defined as in (2.4) with $\overline{\boldsymbol{A}}$ given by (7.7) and $\overline{\boldsymbol{M}}$ given by (7.3). Then, using $\boldsymbol{P}^{-1}$ as a preconditioner for the MINRES method reduces the norm of the residual of MINRES by a factor of at least $\frac{\sqrt{\sigma}-1}{\sqrt{\sigma}+1}$ every two iterations, where $\sqrt{\sigma} \leq C\left(1+\log ^{3} k\right)$ with $C$ independent of $k$ and $h$.

Proof. Thanks to Theorem A. 1 and the matrix correspondences (7.3) and (7.7), there holds

$$
\left[C_{2} \beta^{-2}\left(1+\log ^{3} k\right)\right]^{-1} \overline{\boldsymbol{A}} \leq \widetilde{\boldsymbol{A}} \leq C_{2} \overline{\boldsymbol{A}} \quad \text { and } \quad C_{1}^{-1} \overline{\boldsymbol{M}} \leq \widetilde{\boldsymbol{M}} \leq C_{1} \overline{\boldsymbol{M}}
$$

where $\widetilde{\boldsymbol{M}}$ is the pressure mass matrix for the space $Q_{I}^{\perp}$ and $\boldsymbol{A} \leq \boldsymbol{B}$ means that $\boldsymbol{B}-\boldsymbol{A}$ is positive semidefinite. Additionally, the inf-sup condition for the spaces $\tilde{\boldsymbol{V}}_{E} \times \tilde{Q}_{E}$ (3.14) and the boundedness of the bilinear form $b(\cdot, \cdot)$ can be expressed in matrix form using (5.2b) and the same arguments in [14, Theorem 3.22] to arrive at

$$
\beta^{2} \leq \frac{\vec{q}_{e}^{T} \widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{A}}^{-1} \widetilde{\boldsymbol{B}}^{T} \vec{q}_{e}}{\vec{q}_{e}^{T} \widetilde{\boldsymbol{M}} \vec{q}_{e}} \leq 1, \quad \forall \tilde{Q}_{E} \ni q=\vec{q}_{e}^{T} \vec{\psi}_{e}
$$

where $\beta$ is the discrete inf-sup constant in (1.3). Thus, (2.5) holds with $\delta=\beta^{2}\left[C_{2}(1+\right.$ $\left.\left.\log ^{3} k\right)\right]^{-1}, \Delta=C_{2}, \theta=\beta^{2} C_{1}^{-1}$, and $\Delta=C_{1}$.

Let $\vec{r}_{n}$ denote the residual on the $n$-th iteration of MINRES with the preconditioner $\boldsymbol{P}^{-1}$. Applying [14, Theorem 4.14] and using the fact that the inf-sup constant $\beta$ is bounded below uniformly in $k$ and $h$ gives

$$
\begin{equation*}
\left\|\vec{r}_{2 n}\right\|_{\boldsymbol{P}^{-1}} \leq 2\left(\frac{\sqrt{\sigma}-1}{\sqrt{\sigma}+1}\right)^{n}\left\|\vec{r}_{0}\right\|_{\boldsymbol{P}^{-1}} \tag{7.8}
\end{equation*}
$$

where $\|\vec{r}\|_{\boldsymbol{P}^{-1}}^{2}:=\vec{r}^{T} \boldsymbol{P}^{-1} \vec{r}$ and $\sqrt{\sigma} \leq C\left(1+\log ^{3} k\right)$ with $C$ independent of $k$ and $h$. Since all norms on finite dimension vector spaces are equivalent, (7.8) holds for any choice of norm at the expense of replacing " 2 " by an appropriate constant depending on the choice of norm, which completes the proof of Theorem 7.1.
Theorem 7.1 shows that the performance of the preconditioner deteriorates at most as $\log ^{3} k$ as the polynomial order is increased, but remains bounded as the mesh is refined provided the shape regularity assumption (3.2) is satisfied.
7.4. Implementation and Cost Analysis of the Preconditioner. To aid in the implementation and cost analysis of computing the actions of $\overline{\boldsymbol{A}}^{-1}$ and $\overline{\boldsymbol{M}}^{-1}$, we assume, for convenience, the interface degrees of freedom are ordered as follows:
(i) velocity $C^{0}$ vertex degrees of freedom,
(ii) velocity $C^{1}$ vertex degrees of freedom,
(iii) velocity edge degrees of freedom, grouped according to edge,
(iv) pressure vertex degrees of freedom,
(v) pressure average value degrees of freedom.

This ordering induces a block structure in the matrix $\widetilde{\boldsymbol{A}}$ in which the diagonal subblocks are: $\widetilde{\boldsymbol{A}}_{c}$, corresponding to the global interaction among all the global $C^{0}$ vertex functions; $\widetilde{\boldsymbol{A}}_{\boldsymbol{a}, \mu}$, the block-diagonal entry corresponding to the $C^{1}$ vertex functions $\left\{\phi_{\boldsymbol{a}}^{\mu} \hat{\boldsymbol{e}}_{1}, \phi_{\boldsymbol{a}}^{\mu} \hat{\boldsymbol{e}}_{2}\right\}$; whilst $\widetilde{\boldsymbol{A}}_{\gamma}$ corresponds to the interactions among the edge functions associated to $\gamma$. The load vectors can be similarly split into subvectors corresponding to the same groupings of degrees of freedom. The block diagonal structure of $\boldsymbol{P}$ is then exploited to compute the action of $\boldsymbol{P}^{-1}$ on a pair of vectors $\vec{f}, \vec{g}$ efficiently or in parallel, as described in Algorithm 7.1.

The cost of computing the action of $\boldsymbol{P}^{-1}$ using Algorithm 7.1 comprises of two parts: one-time setup costs and recurring costs associated with each application of Algorithm 7.1. The setup cost is dominated by eliminating the interior degrees of freedom on each element, which takes $\mathcal{O}\left(|\mathcal{T}| k^{6}\right)$ operations needed for the subassembly of the Schur complement. The matrices $\boldsymbol{A}_{c}, \boldsymbol{A}_{\boldsymbol{a}, \mu}, \boldsymbol{a} \in \mathcal{V}, \hat{\boldsymbol{\mu}} \in \stackrel{\circ}{D}_{\boldsymbol{a}}$, and $\boldsymbol{A}_{\gamma}, \gamma \in \mathcal{E}$, need only be factored once at a cost of $\mathcal{O}\left(|\mathcal{V}|^{3}+|\mathcal{E}| k^{3}\right)$ operations, giving an overall setup cost of $\mathcal{O}\left(|\mathcal{T}| k^{6}+|\mathcal{E}| k^{3}+|\mathcal{V}|^{3}\right)$.

We now turn to the cost associated with each application of Algorithm 7.1. Line 2 of Algorithm 7.1 entails the solution of the linear system $\boldsymbol{A}_{\boldsymbol{c}}$ involving all of the

```
Algorithm 7.1 Action of Preconditioner
Require: \(\widetilde{A}, \vec{f}, \vec{g}\)
    function
        \(\vec{u}_{c}=\widetilde{\boldsymbol{A}}_{c}^{-1} \vec{f}_{c} \quad \triangleright\) Global velocity \(C^{0}\) vertex function solve
        for \(\boldsymbol{a} \in \mathcal{V}, \hat{\boldsymbol{\mu}} \in \stackrel{\circ}{D}_{\boldsymbol{a}}\), do \(\quad \triangleright\) Block diagonal velocity \(C^{1}\) vertex function solve
                \(u_{\boldsymbol{a}, \mu}=\widetilde{\boldsymbol{A}}_{\boldsymbol{a}, \mu}^{-1} \vec{f}_{\boldsymbol{a}, \mu}\)
        end for
        for \(\gamma \in \mathcal{E}_{I}\) do \(\quad \triangleright\) Block diagonal velocity edge solve
                \(\vec{u}_{\gamma}=\widetilde{\boldsymbol{A}}_{\gamma}^{-1} \vec{f}_{\gamma}\)
        end for
        for \(\boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}\) do \(\quad \triangleright\) Diagonal pressure \(C^{0}\) vertex function solve
                \(p_{a, \omega}=|\omega|^{-1} k^{4} g_{a, \omega}\)
        end for
        for \(K \in \mathcal{T}\) do \(\quad \triangleright\) Diagonal pressure average value solve
            \(p_{K}=|K|^{-1} g_{K}\)
        end for
        return \(\vec{u}_{c},\left(\vec{u}_{\boldsymbol{a}, \mu}\right)_{\boldsymbol{a}, \mu},\left(\vec{u}_{\gamma}\right)_{\gamma},\left(p_{\boldsymbol{a}, \omega}\right)_{\boldsymbol{a}, \omega},\left(p_{K}\right)_{K} \quad \triangleright\) Return degrees of freedom
    end function
```

$C^{0}$ vertex functions which, thanks to the prefactorisation of $\boldsymbol{A}_{c}$, costs $\mathcal{O}\left(|\mathcal{V}|^{2}\right)$ operations per solve. Lines $3-5$ require the solution of a 2 x 2 matrix on the velocity $C^{1}$ vertex functions for each vertex $\boldsymbol{a} \in \mathcal{V}$ and derivative degree of freedom $\hat{\boldsymbol{\mu}} \in \stackrel{\Sigma}{D}_{\boldsymbol{a}}$ at a cost of $\mathcal{O}(|\mathcal{V}|)$ operations. Lines 6-8 entail a block diagonal solve over each of the edges which, again thanks to the prefactorisation of $\boldsymbol{A}_{\gamma}, \gamma \in \mathcal{E}$, can be applied using $\mathcal{O}\left(|\mathcal{E}| k^{2}\right)$ operations. Lines 9-11 require $\mathcal{O}(|\mathcal{V}|)$ operations and lines 12-14 require $\mathcal{O}(|\mathcal{T}|)$ operations by analogous arguments. In summary, the overall cost per application of Algorithm 7.1 is $\mathcal{O}\left(|\mathcal{E}| k^{2}+|\mathcal{V}|^{2}+|\mathcal{T}|\right)$, which is comparable to nonoverlapping domain decomposition methods for second order elliptic problems [37].
8. Numerical Examples. We illustrate the performance of the preconditioner described in section 7 in two numerical examples.
8.1. Moffatt Eddies. In the first example, we revisit the Moffatt problem [27] considered in [7], in which the domain $\Omega$ is the wedge with a prescribed parabolic flow profile on the top part of the boundary and no flow on the remainder of the boundary:

$$
\boldsymbol{u}(x, 0)=\left[\begin{array}{c}
1-x^{2} \\
0
\end{array}\right],-1 \leq x \leq 1, \quad \text { and } \quad \boldsymbol{u}=\mathbf{0} \text { on } \Gamma \backslash(-1,1) \times\{0\}
$$

The problem is approximated using a pure $p$-version finite element scheme on the fixed mesh shown in Figure 4a. The results in [7] show that the $k=13$ solution resolves four to five eddies, equivalent to a $10^{13}$ range of scales.

Let $\lambda_{\text {min }}^{ \pm}$and $\lambda_{\text {max }}^{ \pm}$denote the extremal eigenvalues of $\boldsymbol{P}^{-1} \boldsymbol{S}$ so that

$$
\begin{equation*}
\sigma\left(\boldsymbol{P}^{-1} \boldsymbol{S}\right) \subseteq\left[-\lambda_{\max }^{-},-\lambda_{\min }^{-}\right] \cup\{0\} \cup\left[\lambda_{\min }^{+}, \lambda_{\max }^{+}\right] \tag{8.1}
\end{equation*}
$$

According to Theorem 7.1, $\lambda_{\max }^{ \pm} \leq C$ and $\lambda_{\text {min }}^{ \pm} \geq C\left(1+\log ^{3} k\right)^{-1}$ with constant $C$ independent of $k$ and $h$. Figure 4b displays the actual values of the extreme eigenvalues. In agreement with theory, $\lambda_{\max }^{ \pm}$is uniformly bounded in $k$ and $\lambda_{\text {min }}^{+} \geq$ $C\left(1+\log ^{3} k\right)^{-1}$. However, $\lambda_{\text {min }}^{-}$appears to remain uniformly bounded in $k$, which
would mean that, in practice, the contraction factor in Theorem 7.1 is pessimistic. The residual history for $k \in\{4,7,10,13\}$ for the preconditioned MINRES solver are displayed in Figure 4c. The starting vector is taken to be $\vec{x}+\vec{\epsilon}$, where $\vec{x}$ is the true solution of the Schur complement system (2.2) and $\vec{\epsilon}$ is a random perturbation with entries uniformly distributed in $(-1,1)$. Here, and in the remaining examples, the relative residual is given by $\sqrt{\left(\vec{r}^{T} \boldsymbol{P}^{-1} \vec{r}\right) /\left(\vec{r}_{0}^{T} \boldsymbol{P}^{-1} \vec{r}_{0}\right)}$, where $\vec{r}_{0}$ is the initial residual vector, and MINRES is terminated when the relative residual is smaller than $10^{-8}$. It is observed that, as the polynomial order is raised, the iteration counts grow modestly consistent with the results in Theorem 7.1.


Fig. 4: (a) 18 element mesh, (b) extremal eigenvalues of $\boldsymbol{P}^{-1} \boldsymbol{S}$, and (c) MINRES convergence history with $k \in\{4,7,10,13\}$, stopping tolerance $10^{-8}$ for a sequence of random initial iterates for Moffatt eddies problem. $1 / \lambda_{\text {min }}^{+}$grows as $\log ^{3} k$ while the other extreme eigenvalues remain bounded as the polynomial degree $k$ is increased.
8.2. T-shaped Domain. In the next example, we consider the T-shaped domain example [2] where $\boldsymbol{f} \equiv \mathbf{0}$ and boundary conditions are parabolic flow profile on the leftmost and rightmost boundaries of the domain and no flow on the remainder of the boundary:

$$
\boldsymbol{u}\left( \pm \frac{3}{2}, y\right)=\left[\begin{array}{c}
y(1-y) \\
0
\end{array}\right], 0 \leq y \leq 1, \quad \text { and } \quad \boldsymbol{u}=\mathbf{0} \text { on } \Gamma \backslash\left\{ \pm \frac{3}{2}\right\} \times(0,1)
$$

The sequence of meshes is shown in Figure 5, in which the elements are geometrically graded and which were proved to give exponential convergence of the finite element solution [7, §7.2]. The mesh in Figure 5a consists of one layer of elements around the re-entrant corner and the most bottom corners, with a grading factor of $\sigma=0.08$. We then refine the mesh by successively adding layers of elements such that the innermost layer of elements has a diameter proportional to $\sigma^{n}$, where $n$ is the number of refinements. For example, the mesh corresponding to three levels is shown in Figures 5b and 5c. Observe that, once a mesh contains two or more layers, the shape regularity constant $\kappa(3.2)$ changes from 0.1695 to 0.0829 due to the presence of "needle" elements near the corners. In particular, several estimates in the analysis depend on $\kappa$, and thus we would expect the performance of the preconditioner to be worse for $n \geq 2$ than for $n=1$.

As with the previous example, the extremal eigenvalues (8.1), displayed in Figure 6 , remain bounded independently of the number of levels of geometric refinement, whilst $1 / \lambda_{\text {min }}^{+}$increases by a factor of roughly 10 after one level of refinement due to the change in shape regularity mentioned above. This value is an order of magnitude greater than the value of $1 / \lambda_{\text {min }}^{+}$observed for the Moffatt ( $\kappa=0.1508$ ) example and accounts for the increase of the resulting iteration counts observed in the residual histories for $k \in\{4,7,10,13\}$ in Figure 7. Thus, as one might expect, the preconditioner $\boldsymbol{P}^{-1}$ is less effective on meshes containing high aspect ratio elements owing to the fact that the inf-sup constants and inequalities employed in Appendix A all depend on the shape regularity constant appearing in (3.2). Nevertheless, similar to the behavior observed in the previous example, for each fixed $n$, the iteration counts grow modestly in $k$. For each fixed $k$, the iteration counts are bounded in $n$, and remain virtually unchanged for $n \geq 3$.


Fig. 5: (a) Mesh with $n=1$ layer of elements, (b) Mesh with $n=3$ layers of elements, and (c) Zoom on re-entrant corner of mesh with $n=3$ layers of elements.


Fig. 6: Extremal eigenvalues of $\boldsymbol{P}^{-1} \boldsymbol{S}$ for the T-shape problem with (a) $n=1$, (b) $n=2$, and (c) $n=3$ layers of geometrically graded elements at the corners. All of the extreme eigenvalues are uniformly bounded in $n$ for each fixed $k$. In addition, the introduction of small-angle "needle" elements for $n \geq 2$ greatly increases $1 / \lambda_{\text {min }}^{+}$.


Fig. 7: MINRES convergence history with $k \in\{4,7,10,13\}$, stopping tolerance $10^{-8}$ for a sequence of random initial iterates for the T-shape problem with (a) $n=1$, (b) $n=2$, and (c) $n=3$ layers of geometrically graded elements at the corners.

## Appendix A. Technical Lemmas.

In this section, we establish a spectral equivalence of the ASM preconditioners given in section 7 to the inner products appearing in the Stokes equations. The main result is the following theorem, which is an immediate consequence of Lemmas A. 4 to A. 6 and A. 9 proved later in this section:

Theorem A.1. There exists positive constants $C_{1}$ and $C_{2}$, independent of $k$ and $h$, such that

$$
\begin{equation*}
C_{1}^{-1}(p, p) \leq \bar{m}(p, p) \leq C_{1}(p, p) \quad \forall p \in Q_{I}^{\perp} \tag{A.1}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{2}^{-1} a(\boldsymbol{u}, \boldsymbol{u}) \leq \bar{a}(\boldsymbol{u}, \boldsymbol{u}) \leq C_{2} \beta^{-2}\left(1+\log ^{3} k\right) a(\boldsymbol{u}, \boldsymbol{u}) \quad \forall \boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E} \tag{A.2}
\end{equation*}
$$

where $\bar{m}(\cdot, \cdot)$ is defined in (7.2), $\bar{a}(\cdot, \cdot)$ is defined in (7.6), and $\beta$ is the inf-sup constant defined in (1.3).
A.1. Pressure ASM. We begin with the pressure ASM. The first lemma establishes a key estimate for the norm of the pressure vertex functions:

Lemma A.2. The pressure vertex functions functions $\tilde{\psi}_{\boldsymbol{a}}^{\omega}, \boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}$, satisfy

$$
\begin{equation*}
\left\|\tilde{\psi}_{\boldsymbol{a}}^{\omega}\right\|_{L^{2}(K)} \leq C h_{K} k^{-2} \quad \forall K \in \mathcal{T}, \tag{A.3}
\end{equation*}
$$

with $C$ independent of $k, h_{K}$, and $\boldsymbol{a}$.
Proof. Let $\boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}$. Define the function $\chi \in Q$ by the rule $\chi=\chi_{K}$ on each element $K \in \mathcal{T}$ where $\chi_{K}$ is chosen as in [7, Lemma 4.1]. In particular, $\chi_{K} \in \mathcal{P}_{k-1}(K) \cap L_{0}^{2}(K)$ satisfies (i) $\chi_{K} \equiv 0$ if $K \nsubseteq \omega$, (ii) $\chi_{K}(\boldsymbol{b})=\delta_{\boldsymbol{a} \boldsymbol{b}}$ for $\boldsymbol{b} \in \mathcal{V}_{K}$, and (iii) $\left\|\chi_{K}\right\|_{L^{2}(K)} \leq C h_{K} k^{-2}$ with $C$ independent of $h_{K}$, $k$, and $\boldsymbol{a}$. By (4.2d), (4.2e), and Theorem 4.2, $\tilde{\psi}_{\boldsymbol{a}}=\tilde{\Pi} \chi$, and $\left\|\tilde{\psi}_{\boldsymbol{a}}^{\omega}\right\|_{L^{2}(K)} \leq\|\chi\|_{L^{2}(K)} \leq C h_{K} k^{-2} \forall K \in \mathcal{T} . \square$

We now show that the inner products on the subspaces are coercive:

Lemma A.3. There exists a positive constant $C$ independent of $k$ and $h$ such that

$$
\begin{array}{ll}
(p, p) \leq C m_{\boldsymbol{a}, \omega}(p, p) & \forall p \in \tilde{Q}_{\boldsymbol{a}, \omega}, \quad \boldsymbol{a} \in \mathcal{V}, \quad \omega \in \Omega_{\boldsymbol{a}} \\
(p, p) \leq C m_{K}(p, p) & \forall p \in \tilde{Q}_{K}, \quad K \in \mathcal{T}
\end{array}
$$

Proof. Let $p \in \tilde{Q}_{\boldsymbol{a}, \omega}, \boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}$. Then, $p=p(\boldsymbol{a}) \tilde{\psi}_{\boldsymbol{a}}^{\omega}$, and by (A.3) and shape regularity (3.2), there holds

$$
(p, p) \leq C \sum_{K \in \mathcal{T}: K \subseteq \omega} h_{K}^{2} k^{-4}|p(\boldsymbol{a})|^{2} \leq C|\omega| k^{-4}=m_{\boldsymbol{a}, \omega}(p, p)
$$

Now let $p \in Q_{K}, K \in \mathcal{T}$. Then, $p=\left(|K|^{-1} \int_{K} p d \boldsymbol{x}\right) \tilde{\psi}_{K}$ and since $\tilde{\psi}_{\boldsymbol{a}}^{\omega} \in L_{0}^{2}(K)$,

$$
\int_{K} \tilde{\psi}_{K}^{2} d \boldsymbol{x}=\int_{K}\left\{1+\left(\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \omega \supseteq K}} \tilde{\psi}_{\boldsymbol{a}}^{\omega}\right)^{2}\right\} d \boldsymbol{x} \leq|K|+C h_{K}^{2} k^{-4} \leq C|K|
$$

Thus, $(p, p) \leq C m_{K}(p, p)$.
We are now able to establish the left-hand side of the equivalence (A.1):
Lemma A.4. There exists a constant $C$ independent of $k$ and $h$ such that

$$
\begin{equation*}
(p, p) \leq C \bar{m}(p, p) \quad \forall p \in Q_{I}^{\perp} \tag{A.4}
\end{equation*}
$$

Proof. Let $p \in Q_{I}^{\perp}$. By Cauchy-Schwarz, there holds

$$
(p, p)_{K} \leq 4\left[\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \omega \supseteq K}}\left(p_{\boldsymbol{a}, \omega}, p_{\boldsymbol{a}, \omega}\right)_{K}+\left(p_{K}, p_{K}\right)_{K}\right]
$$

where $(p, q)_{K}:=\int_{K} p q d \boldsymbol{x}$. (A.4) now follows from Lemma A. 3 and summing over the elements.
The right-hand side of the equivalence (A.1) is covered by the next result:
Lemma A.5. There exists a positive constant $C$ independent of $k$ and $h$ such that

$$
\begin{equation*}
\bar{m}(p, p) \leq C(p, p) \quad \forall p \in Q_{I}^{\perp} \tag{A.5}
\end{equation*}
$$

Proof. By [4, Lemma 6.1], there holds

$$
\left.|p|_{K}(\boldsymbol{a})\right|^{2} k^{-4}=\left|\left(\left.p\right|_{K} \circ \boldsymbol{F}_{K}\right)(\hat{\boldsymbol{a}})\right|^{2} k^{-4} \leq C\left\|p \circ \boldsymbol{F}_{K}\right\|_{L^{2}(\hat{T})}^{2} \leq C h_{K}^{2}\|p\|_{L^{2}(K)}^{2}
$$

with $\hat{\boldsymbol{a}}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$, and by shape regularity,

$$
m_{\boldsymbol{a}, \omega}\left(p_{\boldsymbol{a}, \omega}, p_{\boldsymbol{a}, \omega}\right)=\left.|\omega| k^{-4}|p|_{\omega}(\boldsymbol{a})\right|^{2} \leq C \sum_{K \in \mathcal{T}: K \subseteq \omega}\|p\|_{L^{2}(K)}^{2}
$$

Summing over $\boldsymbol{a} \in \mathcal{V}, \omega \in \Omega_{\boldsymbol{a}}$ and again using shape regularity to bound the overlap $\left|\left\{\omega: \exists \boldsymbol{a} \in \mathcal{V}: K \subseteq \omega \in \Omega_{\boldsymbol{a}}\right\}\right|$ gives

$$
\sum_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \omega \in \Omega_{\boldsymbol{a}}}} m_{\boldsymbol{a}, \omega}\left(p_{\boldsymbol{a}, \omega}, p_{\boldsymbol{a}, \omega}\right) \leq C \sum_{\substack{\boldsymbol{a} \in \mathcal{V} \\ \omega \in \Omega_{\boldsymbol{a}}}} \sum_{K \in \mathcal{T}: K \subseteq \omega}\|p\|_{L^{2}(K)}^{2} \leq C\|p\|_{L^{2}(\Omega)}^{2}
$$

To bound the remaining $m_{K}(\cdot, \cdot)$ terms, we use Cauchy-Schwarz:

$$
\sum_{K \in \mathcal{T}} m_{K}\left(p_{K}, p_{K}\right)=\sum_{K \in \mathcal{T}} \frac{1}{|K|}\left(\int_{K} p d \boldsymbol{x}\right)^{2} \leq C \sum_{K \in \mathcal{T}}\|p\|_{L^{2}(K)}^{2} \leq C\|p\|_{L^{2}(\Omega)}^{2}
$$

which completes the proof of (A.5).
A.2. Velocity ASM. We now turn to the velocity space, and start by extending the decomposition (7.5) as follows. For $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$, we define $\boldsymbol{u}_{\gamma} \equiv \mathbf{0}$ for $\gamma \in \mathcal{E} \backslash \mathcal{E}_{I}$ and $\boldsymbol{u}_{\boldsymbol{a}, \mu} \equiv \mathbf{0}$ for $\boldsymbol{a} \in \mathcal{V}, \hat{\boldsymbol{\mu}} \in D_{\boldsymbol{a}} \backslash \stackrel{\circ}{D}_{\boldsymbol{a}}$. Since the inner product on each of the subspace was taken to be $a(\cdot, \cdot)$, we immediately obtain the left-hand side of the equivalence (A.2):

Lemma A.6. For all $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$, there holds

$$
\begin{equation*}
a(\boldsymbol{u}, \boldsymbol{u}) \leq 10 \bar{a}(\boldsymbol{u}, \boldsymbol{u}) \tag{A.6}
\end{equation*}
$$

Proof. First recall that there are exactly 2 directional derivative degrees of freedom per velocity component per vertex on any given element, i.e. for $K \in \mathcal{T}$, $\left|\left\{\hat{\boldsymbol{\mu}}: K \subseteq \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}\right\}\right|=2$. By Cauchy-Schwarz, there holds

$$
|\boldsymbol{u}|_{\boldsymbol{H}^{1}(K)}^{2} \leq 10\left\{\left|\boldsymbol{u}_{c^{\prime}}\right|_{\boldsymbol{H}^{1}(K)}^{2}+\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \hat{\mu}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}}\left|\boldsymbol{u}_{\boldsymbol{a}, \mu}\right|_{\boldsymbol{H}^{1}(K)}^{2}+\sum_{\gamma \in \mathcal{E}_{K}}\left|\boldsymbol{u}_{\gamma}\right|_{\boldsymbol{H}^{1}(K)}^{2}\right\}, \quad \forall K \in \mathcal{T} .
$$

Equation (A.6) now follows by summing over the elements.
To prove the right-hand side of (A.2), we need to establish some properties of the velocity vertex functions:

Lemma A.7. The $C^{0}$ velocity vertex functions satisfy the following: For $K \in \mathcal{T}$,

$$
\begin{equation*}
\mathbb{R}^{2} \ni \boldsymbol{c}=\sum_{\boldsymbol{a} \in \mathcal{V}_{K}} \sum_{i=1}^{2}\left(\boldsymbol{c} \cdot \hat{\boldsymbol{e}}_{i}\right) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \quad \text { on } K \tag{A.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}\right\|_{\boldsymbol{H}^{1}(\hat{T})} \leq C \quad \forall \boldsymbol{a} \in \mathcal{V}, i=1,2 \tag{A.8}
\end{equation*}
$$

where $C$ depends only on the shape regularity parameter.
Moreover, the $C^{1}$ velocity vertex functions satisfy

$$
\begin{equation*}
\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}}^{\mu} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}\right\|_{\boldsymbol{H}^{1}(\hat{T})} \leq C\left\|D \boldsymbol{F}_{K}\right\|_{L^{\infty}(\hat{T})} k^{-2} \quad \boldsymbol{a} \in \mathcal{V}, \hat{\boldsymbol{\mu}} \in D_{\boldsymbol{a}}, i=1,2 \tag{A.9}
\end{equation*}
$$

where $C$ depends only on the shape regularity parameter.
Proof. Let $K \in \mathcal{T}$. A simple computation reveals that $1=\left(\lambda_{1}+\lambda_{2}+\lambda_{3}\right)^{3}=$ $\sum_{\boldsymbol{a} \in \mathcal{V}_{K}} \phi_{\boldsymbol{a}}+6 \lambda_{1} \lambda_{2} \lambda_{3}$ on $K$, where $\left\{\lambda_{i}: 1 \leq i \leq 3\right\}$ are the barycentric coordinates on $K$, and hence, for any $\boldsymbol{c} \in \mathbb{R}^{2}$,

$$
\boldsymbol{c}=\sum_{\boldsymbol{a} \in \mathcal{V}} \sum_{i=1}^{2}\left(\boldsymbol{c} \cdot \hat{\boldsymbol{e}}_{i}\right)\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right)+\underbrace{6 \boldsymbol{c} \sum_{K \in \mathcal{T}} \lambda_{1} \lambda_{2} \lambda_{3}}_{\in \boldsymbol{V}_{I}}
$$

Applying $\boldsymbol{\Pi}_{\boldsymbol{V}}$ to both sides of this identity and noting that Theorem 4.1 gives $\boldsymbol{\Pi}_{\boldsymbol{V}}$ : $\boldsymbol{V}_{I} \rightarrow\{\mathbf{0}\}$, we obtain

$$
\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{c}=\sum_{\boldsymbol{a} \in \mathcal{V}} \sum_{i=1}^{2}\left(\boldsymbol{c} \cdot \hat{\boldsymbol{e}}_{i}\right) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right)
$$

Finally, Theorem 4.1 implies that $\mathscr{E}(\boldsymbol{c}, 0)=(\boldsymbol{c}, 0)$ and (A.7) follows at once.
Now let $K \in \mathcal{T}$ and $i \in\{1,2\}$. Clearly (A.8) holds if $\boldsymbol{a} \notin \mathcal{V}_{K}$ since $\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}=\mathbf{0}$. Otherwise, if $\boldsymbol{a} \in \mathcal{V}_{K}$, we apply a scaling argument in conjunction with (4.4) to arrive at
$\frac{1}{|K|^{2}}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right)\right\|_{\boldsymbol{L}^{2}(K)}^{2}+\left|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right)\right|_{\boldsymbol{H}^{1}(K)}^{2} \leq C\left\{\left|\phi_{\boldsymbol{a}}\right|_{H^{1 / 2}(\partial K)}^{2}+\frac{1}{|\partial K|}\left\|\phi_{\boldsymbol{a}}\right\|_{L^{2}(\partial K)}^{2}\right\}$,
where $C$ is a positive constant independent of $k$ and $h_{K}$. Thus,

$$
\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}\right\|_{\boldsymbol{H}^{1}(\hat{T})} \leq C\left\|\hat{\phi}_{j}\right\|_{H^{1 / 2}(\partial \hat{T})} \leq C\left\|\hat{\phi}_{j}\right\|_{H^{1}(\hat{T})} \leq C
$$

where $\hat{\boldsymbol{a}}_{j}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$. For $K \subseteq \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}$, we argue similarly and use (6.10) to obtain

$$
\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}}^{\mu} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}\right\|_{\boldsymbol{H}^{1}(\hat{T})} \leq C\left\|\left[\begin{array}{ll}
\hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\xi}}
\end{array}\right]^{-1}\right\|\left\|D \boldsymbol{F}_{K}\right\|_{L^{\infty}(\hat{T})}\left\|\left[\begin{array}{l}
\hat{\phi}_{j}^{(1,0)} \\
\hat{\phi}_{j}^{(0,1)}
\end{array}\right]\right\|_{\boldsymbol{H}^{1 / 2}(\partial \hat{T})}
$$

where $\hat{\boldsymbol{\mu}} \neq \hat{\boldsymbol{\xi}} \in D_{\boldsymbol{a}}$ is chosen such that $\operatorname{supp} \phi_{\boldsymbol{a}}^{\xi} \supseteq K, \hat{\boldsymbol{a}}_{j}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$, and $\|\cdot\|$ is any matrix norm. By the definition of $D_{\boldsymbol{a}}(6.9)$ and shape regularity, $\left\|\left[\begin{array}{ll}\hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\xi}}\end{array}\right]^{-1}\right\|$ is uniformly bounded by a constant depending only on $\kappa(3.2)$. Since $\hat{\phi}_{j}^{(1,0)}(\hat{\boldsymbol{a}})=$ $\hat{\phi}_{j}^{(0,1)}(\hat{\boldsymbol{a}})=0$ for $\hat{\boldsymbol{a}} \in \mathcal{V}_{\hat{T}}$ by the construction (6.3), there holds

$$
\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}}^{\mu} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}\right\|_{\boldsymbol{H}^{1}(\hat{T})} \leq C\left\|D \boldsymbol{F}_{K}\right\|_{L^{\infty}(\hat{T})}\|J\|_{H_{00}^{1 / 2}(I)}
$$

where $I=(-1,1), H_{00}^{1 / 2}(I)$ is the usual Sobolev space (defined as, e.g. [23]), and

$$
J(t)=\frac{1}{P_{k-3}^{(3,3)}(-1)}\left(\frac{1+t}{2}\right)\left(\frac{1-t}{2}\right)^{2} P_{k-3}^{(3,3)}(t)
$$

Thanks to [5, Lemma B.1], $\|J\|_{L^{2}(I)} \leq C k^{-3}$ with $C$ independent of $k$. Using interpolation, and the inverse estimate $\left\|J^{\prime}\right\|_{L^{2}(I)} \leq C k^{2}\|J\|_{L^{2}(I)}$ [10, Lemma 5.4], we obtain $\|J\|_{H_{00}^{1 / 2}(I)} \leq C\|J\|_{L^{2}(I)}^{1 / 2}\|J\|_{H^{1}(I)}^{1 / 2} \leq C k^{-2}$, which completes the proof of (A.9).

We now use the properties of the vertex functions to prove element-wise stability of the subspace decomposition (7.5):

Lemma A.8. For $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$ and $K \in \mathcal{T}$, there holds

$$
\begin{equation*}
\left|\boldsymbol{u}_{c}\right|_{\boldsymbol{H}^{1}(K)}^{2}+\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \hat{\boldsymbol{\mu}}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}}\left|\boldsymbol{u}_{\boldsymbol{a}, \mu}\right|_{\boldsymbol{H}^{1}(K)}^{2}+\sum_{\gamma \in \mathcal{E}_{K}}\left|\boldsymbol{u}_{\gamma}\right|_{\boldsymbol{H}^{1}(K)}^{2} \leq C \beta^{-2}\left(1+\log ^{3} k\right)|\boldsymbol{u}|_{\boldsymbol{H}^{1}(K)}^{2}, \tag{A.10}
\end{equation*}
$$

where $C$ is independent of $k, h_{K}$ and $\boldsymbol{u}$.

Proof. Let $\boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E}$ and $K \in \mathcal{T}$. For any $\boldsymbol{c} \in \mathbb{R}^{2}$, we have the decomposition

$$
\boldsymbol{u}-\boldsymbol{c}=\boldsymbol{u}_{c}-\sum_{\boldsymbol{a} \in \mathcal{V}_{K}} \sum_{i=1}^{2}\left(\boldsymbol{c} \cdot \hat{\boldsymbol{e}}_{i}\right) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right)+\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \boldsymbol{\mu}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}} \boldsymbol{u}_{\boldsymbol{a}, \mu}+\sum_{\gamma \in \mathcal{E}_{K}} \boldsymbol{u}_{\gamma} \text { on } K
$$

thanks to (A.7). Thus,

$$
\hat{\boldsymbol{u}}-\boldsymbol{c}=\hat{\boldsymbol{u}}_{c}-\sum_{\boldsymbol{a} \in \mathcal{V}_{K}} \sum_{i=1}^{2}\left(\boldsymbol{c} \cdot \hat{\boldsymbol{e}}_{i}\right) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}+\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \hat{\boldsymbol{\mu}}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}} \hat{\boldsymbol{u}}_{\boldsymbol{a}, \mu}+\sum_{\gamma \in \mathcal{E}_{K}} \hat{\boldsymbol{u}}_{\gamma} \quad \text { on } \hat{T},
$$

where $\hat{\boldsymbol{u}}=\boldsymbol{u} \circ \boldsymbol{F}_{K}, \hat{\boldsymbol{u}}_{c}=\boldsymbol{u}_{c} \circ \boldsymbol{F}_{K}$, etc. We first bound the energy of $\hat{\boldsymbol{u}}_{c}$. Since

$$
\begin{aligned}
\hat{\boldsymbol{u}}_{c} & =\sum_{\boldsymbol{a} \in \mathcal{V}_{K}} \sum_{i=1}^{2}\left(\boldsymbol{u}(\boldsymbol{a}) \cdot \hat{\boldsymbol{e}}_{i}\right) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K} \\
& =\sum_{\hat{\boldsymbol{a}} \in \mathcal{V}_{\hat{T}}} \sum_{i=1}^{2}\left(\hat{\boldsymbol{u}}(\hat{\boldsymbol{a}}) \cdot \hat{\boldsymbol{e}}_{i}\right) \boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K} \quad \text { on } \hat{T}
\end{aligned}
$$

where $\boldsymbol{a}=\boldsymbol{F}_{K}(\hat{\boldsymbol{a}})$, we use [9, Corollary 6.3] and (A.8) to obtain

$$
\begin{align*}
\left\|\hat{\boldsymbol{u}}_{c}-\boldsymbol{c}\right\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} & \leq \sum_{\hat{\boldsymbol{a}} \in \mathcal{V}_{\hat{T}}}|\hat{\boldsymbol{u}}(\hat{\boldsymbol{a}})-\boldsymbol{c}|^{2} \sum_{i=1}^{2}\left\|\boldsymbol{\Pi}_{\boldsymbol{V}}\left(\phi_{\boldsymbol{a}} \hat{\boldsymbol{e}}_{i}\right) \circ \boldsymbol{F}_{K}\right\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \\
& \leq C(1+\log k)\|\hat{\boldsymbol{u}}-\boldsymbol{c}\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \tag{A.11}
\end{align*}
$$

We now bound the vertex derivative contribution. For $\boldsymbol{a} \in \mathcal{V}_{K}$, we note that $\partial_{\mu}\left(\boldsymbol{u} \cdot \hat{\boldsymbol{e}}_{i}\right)(\boldsymbol{a})=\hat{\boldsymbol{\mu}}^{T} D \boldsymbol{F}_{K}^{-T} D\left(\boldsymbol{u} \cdot \hat{\boldsymbol{e}}_{i} \circ \boldsymbol{F}_{K}\right)(\hat{\boldsymbol{a}}), i=1,2$, where $\hat{\boldsymbol{a}}=\boldsymbol{F}_{K}^{-1}(\boldsymbol{a})$. Applying [4, Lemma 6.1] to $D\left(\boldsymbol{u} \circ \boldsymbol{F}_{K}\right)$, and using (A.9) and shape regularity gives

$$
\begin{equation*}
\left\|\hat{\boldsymbol{u}}_{\boldsymbol{a}, \mu}\right\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \leq C\left\|D \boldsymbol{F}_{K}^{-T}\right\|_{L^{\infty}(K)}^{2} \cdot\left\|D \boldsymbol{F}_{K}\right\|_{L^{\infty}(\hat{T})}^{2} \cdot|D \hat{\boldsymbol{u}}(\hat{\boldsymbol{a}})|^{2} k^{-4} \leq C|\hat{\boldsymbol{u}}|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \tag{A.12}
\end{equation*}
$$

Now, we define

$$
\boldsymbol{u}^{\#}:=(\boldsymbol{u}-\boldsymbol{c})-\left(\boldsymbol{u}_{c}-\boldsymbol{c}\right)-\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \hat{\mu}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}} \boldsymbol{u}_{\boldsymbol{a}, \mu} \quad \text { on } K .
$$

Then, $D^{\alpha} \boldsymbol{u}^{\#}(\boldsymbol{a})=\mathbf{0}$ for $\boldsymbol{a} \in \mathcal{V}_{K},|\alpha| \leq 1$, and thanks to (A.11) and (A.12), $\hat{\boldsymbol{u}}^{\#}:=$ $\boldsymbol{u}^{\#} \circ \boldsymbol{F}_{K}$ may be estimated as follows:

$$
\begin{equation*}
\left\|\hat{\boldsymbol{u}}^{\#}\right\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \leq C(1+\log k)\|\hat{\boldsymbol{u}}-\boldsymbol{c}\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \tag{A.13}
\end{equation*}
$$

Let $\gamma \in \mathcal{E}_{K}$. Equation (4.5), shape regularity (3.2), and the trace theorem give

$$
\begin{equation*}
\left|\hat{\boldsymbol{u}}_{\gamma}\right|_{\boldsymbol{H}^{1}(\hat{T})} \leq C \beta^{-1}\left|\boldsymbol{u}_{\gamma}\right|_{\boldsymbol{H}^{1 / 2}(\partial K)} \leq C \beta^{-1}\left|\hat{\boldsymbol{u}}_{\gamma}\right|_{\boldsymbol{H}^{1 / 2}(\partial \hat{T})} \leq C \beta^{-1}\left\|\hat{\boldsymbol{u}}^{\#}\right\|_{\boldsymbol{H}_{00}^{1 / 2}(\hat{\gamma})} \tag{A.14}
\end{equation*}
$$

where $\hat{\gamma}=\boldsymbol{F}_{K}^{-1}(\gamma)$. Thanks to [9, Theorem 6.5] and the trace theorem, we have the estimate

$$
\begin{equation*}
\left\|\hat{\boldsymbol{u}}^{\#}\right\|_{\boldsymbol{H}_{00}^{1 / 2}(\hat{\gamma})} \leq C(1+\log k)\left\|\hat{\boldsymbol{u}}^{\#}\right\|_{\boldsymbol{H}^{1 / 2}(\hat{\gamma})} \leq C(1+\log k)\left\|\hat{\boldsymbol{u}}^{\#}\right\|_{\boldsymbol{H}^{1}(\hat{T})} \tag{A.15}
\end{equation*}
$$

Using (A.13)-(A.15) gives

$$
\begin{equation*}
\left|\hat{\boldsymbol{u}}_{\gamma}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \leq C \beta^{-2}\left(1+\log ^{2} k\right)\left\|\hat{\boldsymbol{u}}^{\#}\right\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \leq C \beta^{-2}\left(1+\log ^{3} k\right)\|\hat{\boldsymbol{u}}-\boldsymbol{c}\|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \tag{A.16}
\end{equation*}
$$

Combining (A.11), (A.12), and (A.16) leads to

$$
\begin{aligned}
\left|\hat{\boldsymbol{u}}_{c}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2}+\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\
\boldsymbol{\mu}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}}\left|\hat{\boldsymbol{u}}_{\boldsymbol{a}, \mu}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2}+ & \sum_{\gamma \in \mathcal{E}_{K}}\left|\hat{\boldsymbol{u}}_{\gamma}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \\
& \leq C \beta^{-2}\left(1+\log ^{3} k\right)\|\hat{\boldsymbol{u}}-\boldsymbol{c}\|_{\boldsymbol{H}^{1}(\hat{T})}^{2}
\end{aligned}
$$

where we used that $\left|\hat{\boldsymbol{u}}_{c}\right|_{\boldsymbol{H}^{1}(\hat{T})}=\left|\hat{\boldsymbol{u}}_{c}-\boldsymbol{c}\right|_{\boldsymbol{H}^{1}(\hat{T})}$. Taking the infimum over all $\boldsymbol{c} \in \mathbb{R}^{2}$ and applying the quotient norm equivalence [28, Theorem 7.2] gives

$$
\left|\hat{\boldsymbol{u}}_{c}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2}+\sum_{\substack{\boldsymbol{a} \in \mathcal{V}_{K} \\ \boldsymbol{\mu}: K \in \operatorname{supp} \phi_{\boldsymbol{a}}^{\mu}}}\left|\hat{\boldsymbol{u}}_{\boldsymbol{a}, \mu}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2}+\sum_{\gamma \in \mathcal{E}_{K}}\left|\hat{\boldsymbol{u}}_{\gamma^{\prime}}\right|_{\boldsymbol{H}^{1}(\hat{T})}^{2} \leq C \beta^{-2}\left(1+\log ^{3} k\right)|\hat{\boldsymbol{u}}|_{\boldsymbol{H}^{1}(\hat{T})}^{2}
$$

Equation (A.10) now follows from shape regularity (3.2).
Summing (A.10) over the elements leads to the following:
Lemma A.9. There exists a constant $C$ independent of $k$ and $h$ such that

$$
\begin{equation*}
\bar{a}(\boldsymbol{u}, \boldsymbol{u}) \leq C \beta^{-2}\left(1+\log ^{3} k\right) a(\boldsymbol{u}, \boldsymbol{u}) \quad \forall \boldsymbol{u} \in \tilde{\boldsymbol{V}}_{E} \tag{A.17}
\end{equation*}
$$

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