# SUM OF SQUARES DECOMPOSITIONS OF POLYNOMIALS OVER THEIR GRADIENT IDEALS WITH RATIONAL COEFFICIENTS

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**Abstract.** Assessing non-negativity of multivariate polynomials over the reals, through the computation of *certificates of non-negativity*, is a topical issue in polynomial optimization. This is usually tackled through the computation of *sums-of-squares decompositions* which rely on efficient numerical solvers for semi-definite programming.

This method faces two difficulties. The first one is that the certificates obtained this way are *approximate* and then non-exact. The second one is due to the fact that not all non-negative polynomials are sums-of-squares.

In this paper, we build on previous works by Parrilo, Nie, Demmel and Sturmfels who introduced certificates of non-negativity modulo *gradient ideals*. We prove that, actually, such certificates can be obtained *exactly*, over the rationals if the polynomial under consideration has rational coefficients and we provide *exact* algorithms to compute them. We analyze the bit complexity of these algorithms and deduce bit size bounds of such certificates.

**Key words.** Non-negative polynomials, sum of squares decomposition, gradient ideal, zero-dimensional and radical ideal, Gröbner basis, bit complexity

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**1. Introduction.** We denote by  $\mathbb{Q}$  (resp.  $\mathbb{R}$ ) the field of rational (resp. real) numbers and by  $\boldsymbol{x}$  the n-tuple of variables  $(x_1, \ldots, x_n)$ . Let  $\mathbb{K}$  be a field, we denote by  $\mathbb{K}[\boldsymbol{x}]$  the polynomial ring with base field  $\mathbb{K}$  and variables  $\boldsymbol{x}$ . For a polynomial f of degree d in  $\mathbb{Q}[\boldsymbol{x}]$ , we consider the problem of computing certificates of non-negativity of f over  $\mathbb{R}^n$ . This is a central issue in polynomial optimization.

Prior works. Computing certificates of non-negativity is usually done by decomposing f as a sum of squares (SOS) of polynomials or rational fractions. It is well-known that all non-negative univariate polynomials with real coefficients can be decomposed as a sum of squares of polynomials. Also, any non-negative univariate polynomial f with rational coefficients can be decomposed as a weighted sum of squares with rational coefficients, i.e.  $f = \sum_i c_i s_i^2$  where  $s_i$  has rational coefficients and  $c_i$  is a positive rational [20, 33]. Further, by SOS decomposition with rational coefficients, we mean weighted SOS decompositions with rational coefficients. Several algorithms already compute such SOS decomposition with rational coefficients of non-negative univariate polynomials with rational coefficients (see [40, 10]) and bit complexity and bit size estimates are given in [26].

The multivariate case is more difficult. Following the seminal works by [21, 30], hierarchies of semidefinite programs yield approximations of weighted SOS decompositions of positive polynomials. Several heuristics have been proposed to lift such approximations to exact SOS decompositions of the input polynomial, starting with [32] and followed by [17, 18, 19]. Note that algorithms in [17, 19] allow us to compute SOS decompositions on some degenerate examples or compute SOS of rational fractions. Complexity issues are studied through the prism of perturbation-compensation techniques to compute SOS decompositions in the interior of the SOS cone [23, 24, 25].

Still, computing *exact* certificates of non-negativity is especially hard because of the two following reasons. The first one is that there exist non-negative polynomials which are not SOS, for example, Motzkin's polynomial and Robinson's polynomial. Moreover, Blekherman proved in [8] that there are many more non-negative polynomials in  $\mathbb{R}[x]$  than SOS polynomials. The second one is that, even if a given

polynomial with rational coefficients is SOS, there is no guarantee that there exists an SOS decomposition involving rational coefficients, as established in [39].

Alternative certificates of non-negativity, for instance, SAGE/SONC polynomials [27, 42] can also be used but they face similar issues to the ones met by SOS techniques when it comes with generality.

Deciding non-negativity over an arbitrary semi-algebraic set of a polynomial  $f \in \mathbb{Q}[x]$  can be done exactly using computer algebra algorithms. The best complexities for such a decision procedure are achieved by algorithms making effective the so-called critical point method [16, 6], further practical developments in [2, 3, 4, 37] and their applications in polynomial optimization in [14, 15, 5]. Note that, even if these algorithms are exact (i.e. their results are exact provided that no bug has been encountered), they do not provide a certificate assessing non-negativity which can be checked a posteriori since they are root-finding algorithms. Their complexities are exponential in the dimension of the ambient space as they reduce the input problem to computing finitely many critical points of some well-chosen maps, hence considering gradient ideals.

Hence, all in all, such gradient ideals can be used to reduce the dimension of the set over which certifying non-negativity can be done. Under some assumptions, this idea is translated in [31] to an algorithm assessing the non-negativity of a given  $f \in \mathbb{R}[x]$ . Precisely, assuming the gradient ideal  $\mathcal{I}_{\text{grad}}(f)$  generated by all partial derivatives of f is zero-dimensional and radical, and that f reaches its minimum over  $\mathbb{R}^n$ , this algorithm computes an SOS decomposition of f in the quotient algebra  $\mathbb{R}[x]/\mathcal{I}_{\text{grad}}(f)$  (or, in other words, an SOS decomposition of f modulo  $\mathcal{I}_{\text{grad}}(f)$ ), i.e., f is written as

$$c_1 s_1^2 + \dots + c_k s_k^2 + \sum_{i=1}^n q_i \frac{\partial f}{\partial x_i}$$

where the  $s_i$ 's and the  $q_i$ 's lie in  $\mathbb{R}[x]$  and the  $c_i$ 's are positive in  $\mathbb{R}$ . A similar result slightly relaxing the above assumptions is given in [29]. Note that when f has coefficients in  $\mathbb{Q}$ , there is no given guarantee that an SOS decomposition of it in  $\mathbb{Q}[x]/\mathcal{I}_{\text{grad}}(f)$  will have rational coefficients too (i.e., the  $s_i$ 's and the  $q_i$ 's have coefficients in  $\mathbb{Q}$  and the  $c_i$ 's lie in  $\mathbb{Q}$ ).

Contributions. We build on the results of [31, 29], to investigate this issue when  $f \in \mathbb{Q}[x]$ . We assume in the whole paper that the gradient ideal associated to f is radical and zero-dimensional and that f reaches its infimum over  $\mathbb{R}^n$ . We summarize our contributions as follows:

• Under the above assumptions, we prove (Theorem 3.1) that f is non-negative over  $\mathbb{R}^n$  if and only if f is an SOS of polynomials with rational coefficients over the quotient ring  $\mathbb{Q}[x]/\mathcal{I}_{grad}(f)$ . Interestingly, Theorem 3.1 can be applied to Robinson's polynomial [35], which is not an SOS of polynomials (see Example 3.5), as well as Scheiderer's polynomial [39], which is an SOS of polynomials with real coefficients but not an SOS of polynomials with rational coefficients (see Example 3.6).

The next problem we tackle is to design algorithms computing such certificates of non-negativity, estimate their bit complexity and the bit size of such certificates.

To measure the *bitsize* of a polynomial with rational coefficients, we will use its height, defined as follows. The bitsize of an integer b is denoted by  $\operatorname{ht}(b) := \lfloor \log_2(|b|) \rfloor + 1$  with  $\operatorname{ht}(0) := 1$ , where  $\log_2$  is the logarithm in base 2. Given  $a \in \mathbb{Z}$  and  $b \in \mathbb{Z}$  with  $b \neq 0$  and  $\gcd(a, b) = 1$ , we define  $\operatorname{ht}\left(\frac{a}{b}\right) = \operatorname{ht}(a) + \operatorname{ht}(b)$ . For a non-zero polynomial f with rational coefficients, the bitsize  $\operatorname{ht}(f)$  is defined as the

maximal bitsize of the non-zero coefficients of f. For two mappings  $p, q : \mathbb{N}^m \to \mathbb{R}$ , the expression "p(v) = O(q(v))" means that there exists  $b \in \mathbb{N}$  such that  $p(v) \leq bq(v)$ , for all  $v \in \mathbb{N}^m$ . We use the notation  $p(v) = \widetilde{O}(q(v))$  in order to indicate that  $p(v) = O(q(v) \log^k q(v))$  for some  $k \in \mathbb{N}$ .

• From the proof of Theorem 3.1, we derive an algorithm (Algorithm 3.1), named sosgradientshape, to compute an SOS decomposition of polynomials modulo the gradient ideal of f. This algorithm can certify non-negativity of polynomials which cannot be tackled with a direct SOS approach. We also investigate the bit complexity of sosgradientshape. We prove that, given as input an n-variate polynomial  $f \in \mathbb{Q}[x]$  of degree d with maximal bitsize of its coefficients  $\tau$ , sosgradientshape uses

$$\widetilde{O}((\tau + n + d)^2 d^{6n} + (\tau + n + d)d^{6n+4})$$

boolean operations. This is better than the complexity estimates given in [25, Theorem 12], where the reported number of boolean operations is:  $\tilde{O}(\tau^2(4d+2)^{15n+6})$ .

• We design a variant of Algorithm sosgradientshape, named sosgradient, and which, on input  $f \in \mathbb{Q}[x]$  as above, decomposes it as a sum of rational fractions modulo the gradient ideal associated to f. We prove that this variant uses

$$\widetilde{O}\left((\tau+n+d)d^{4n+4}\right)$$
.

boolean operations and, consequently, has better complexity than Algorithm sosgradientshape.

Both algorithms have been implemented using the MAPLE computer algebra system. We report on practical experiments showing that it can already assess the non-negativity of numerous polynomials which are out of reach of, e.g., hybrid methods computing sums of squares decompositions such as [23].

Structure of the paper. In the next section, we recall basic notions and fundamental results used in the paper. In Section 3, we prove the existence of an SOS decomposition of polynomials modulo the gradient ideal of f, introduce Algorithm sosgradientshape and analyze its bit complexity. The results for decomposing f as an SOS of rational fractions modulo the gradient ideal are presented in Section 4. Practical experiments are given in the last section.

2. Preliminaries. This section recalls basic notions and results from algebraic geometry, computational commutative algebra, and complexity analysis. Further details can be found in [11].

Let  $\mathbb{K}$  be a field. An additive subgroup  $\mathcal{I}$  of  $\mathbb{K}[\boldsymbol{x}]$  is said to be an *ideal* of  $\mathbb{K}[\boldsymbol{x}]$  if  $hg \in \mathcal{I}$  for any  $h \in \mathcal{I}$  and  $g \in \mathbb{K}[\boldsymbol{x}]$ . Given  $g_1, \ldots, g_r$  in  $\mathbb{K}[\boldsymbol{x}]$ , we denote by  $\langle g_1, \ldots, g_r \rangle$  the ideal generated by  $g_1, \ldots, g_r$ . If  $\mathcal{I}$  is an ideal of  $\mathbb{K}[\boldsymbol{x}]$  then, according to Hilbert's basis theorem (see, e.g., [11, Theorem 4]), there exist  $g_1, \ldots, g_r \in \mathbb{K}[\boldsymbol{x}]$  such that  $\mathcal{I} = \langle g_1, \ldots, g_r \rangle$ .

Let  $\mathcal{I}$  be an ideal of  $\mathbb{R}[x]$ . The algebraic variety associated to  $\mathcal{I}$  is defined as

$$V(\mathcal{I}) := \{ x \in \mathbb{C}^n : \forall g \in \mathcal{I}, g(\boldsymbol{x}) = 0 \}.$$

The real algebraic variety associated to  $\mathcal{I}$  is  $V^{\mathbb{R}}(\mathcal{I}) = V(\mathcal{I}) \cap \mathbb{R}^n$ . Recall that the ideal  $\mathcal{I}$  is zero-dimensional if the cardinality of  $V(\mathcal{I})$  is finite, and that  $\mathcal{I}$  is radical if

$$g^k \in \mathcal{I}$$
 for some  $k \in \mathbb{N} \Longrightarrow g \in \mathcal{I}$ .

Let f be a polynomial in  $\mathbb{R}[x]$ . Recall that the gradient ideal  $\mathcal{I}_{grad}(f)$  of f is the ideal generated by all partial derivatives of f in  $\mathbb{R}[x]$ , i.e.,

$$\mathcal{I}_{\mathrm{grad}}(f) := \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle.$$

The (resp. real) gradient variety associated to f is respectively the (resp. real) algebraic variety associated to  $\mathcal{I}_{\mathrm{grad}}(f)$ . We denote them respectively by  $V_{\mathrm{grad}}(f)$  and  $V_{\mathrm{grad}}^{\mathbb{R}}(f)$ . Let  $\mathbb{K}$  be a real field contained in  $\mathbb{R}$ . One says that f is a (weighted) sum of squares (SOS) of polynomials in  $\mathbb{K}[x]$  if there exist polynomials  $q_1, \ldots, q_s$  in  $\mathbb{K}[x]$  and positive numbers  $c_1, \ldots, c_s$  in  $\mathbb{K}$  such that  $f = \sum_{j=1}^s c_j q_j^2$ . Furthermore, f is an SOS of polynomials over the quotient ring  $\mathbb{K}[x]/\mathcal{I}_{\mathrm{grad}}(f)$  if there exists  $g \in \mathcal{I}_{\mathrm{grad}}(f)$  such that f - g is SOS in  $\mathbb{K}[x]$ , i.e., f can be decomposed as follows:

$$f = \sum_{j=1}^{s} c_j q_j^2 + \sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i},$$

for some polynomials  $q_1, \ldots, q_s, \phi_1, \ldots, \phi_s$  in  $\mathbb{K}[x]$  and positive numbers  $c_1, \ldots, c_s$  in  $\mathbb{K}$ .

Clearly, if f is SOS over  $\mathbb{R}[x]/\mathcal{I}_{\text{grad}}(f)$  then f is non-negative over  $V_{\text{grad}}^{\mathbb{R}}(f)$ . We recall below [31, Theorem 1].

Let f be a polynomial in  $\mathbb{R}[x]$ . Suppose that the gradient ideal  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical. Then, f is non-negative over  $V_{grad}^{\mathbb{R}}(f)$  if and only if f is SOS over the quotient ring  $\mathbb{R}[x]/\mathcal{I}_{grad}(f)$ .

We now recall useful results in the univariate case. It is well-known that  $f \in \mathbb{R}[t]$  is non-negative over  $\mathbb{R}$  if and only if f is SOS. This property holds also for polynomials with coefficients in a subfield  $\mathbb{K}$  of  $\mathbb{R}$ . More precisely, we have the following theorem:

THEOREM 2.1 ([20, 33]). Let  $\mathbb{K}$  be a subfield of  $\mathbb{R}$  and  $f \in \mathbb{K}[t]$ . Then, f is non-negative over  $\mathbb{R}$  if and only if f admits a weighted SOS decomposition of polynomials in  $\mathbb{K}[t]$ , i.e., there exists a positive integer s, non-negative numbers  $c_1, \ldots, c_s \in \mathbb{K}$  and polynomials  $g_1, \ldots, g_s \in \mathbb{K}[t]$ , such that  $f = \sum_{j=1}^s c_j g_j^2$ .

Let  $\mathbb{K}$  be a commutative field and < be a monomial ordering on  $\mathbb{K}[x]$  and  $\mathcal{I} \neq \{0\}$  be an ideal. We denote by  $\mathrm{LT}_{<}(\mathcal{I})$  the set of all leading terms  $\mathrm{LT}_{<}(g)$  of  $g \in \mathcal{I}$ , and by  $\langle \mathrm{LT}_{<}(\mathcal{I}) \rangle$  the ideal generated by the elements of  $\mathrm{LT}_{<}(\mathcal{I})$ .

A subset  $G = \{g_1, \ldots, g_r\}$  of  $\mathcal{I}$  is said to be a *Gröbner basis* of  $\mathcal{I}$  w.r.t. some monomial order < if

$$\langle LT_{<}(g_1), \ldots, LT_{<}(g_r) \rangle = \langle LT_{<}(\mathcal{I}) \rangle.$$

Note that every ideal in  $\mathbb{K}[x]$  has a Gröbner basis. A Gröbner basis G is reduced if the two following conditions hold: the leading coefficient of g is 1, for all  $g \in G$ ; there are no monomials of g lying in  $\langle \mathrm{LT}_{<}(G) \setminus \{g\} \rangle$ . Every ideal  $\mathcal{I}$  has a unique reduced Gröbner basis. We refer the reader to [11] for more details. Further, when the monomial order < is clear from the context, we omit as a subscript in the above notation.

Assume that  $\mathcal{I}$  is a zero-dimensional and radical ideal in  $\mathbb{Q}[x]$  and that G is the reduced Gröbner basis of  $\mathcal{I}$  with respect to the lexicographical order  $x_1 <_{lex} \cdots <_{lex} x_n$ . One says that  $\mathcal{I}$  is in *shape position* if G has the following shape:

(2.1) 
$$G = [w, x_2 - v_2, \dots, x_n - v_n],$$

where  $w, v_2, \ldots, v_n$  are polynomials in  $\mathbb{K}[x_1]$  and  $\deg w = \sharp V(\mathcal{I})$ .

The following lemma, named Shape Lemma, gives us a criteria for the shape position of an ideal.

LEMMA 2.2 (Shape Lemma, [13]). Let  $\mathcal{I}$  be a zero-dimensional and radical ideal and  $<_{lex}$  be a lexicographic monomial order in  $\mathbb{Q}[\mathbf{x}]$ . If  $V(\mathcal{I})$  is the union of  $\delta$  points in  $\mathbb{C}^n$  with distinct  $x_1$ -coordinates, then  $\mathcal{I}$  is in shape position as in (2.1), where  $v_2, \ldots, v_n$  are polynomials in  $\mathbb{Q}[x_1]$  of degrees at most  $\delta - 1$ .

Let V be a zero-dimensional algebraic subset of  $\mathbb{C}^n$ ,  $\delta := \sharp V$ . A zero-dimensional rational parametrization  $\mathcal{Q} = ((w, \kappa_1, \dots, \kappa_n), \lambda)$  of V consists in n+1 univariate polynomials  $w, \kappa_1, \dots, \kappa_n$  in  $\mathbb{Q}[t]$ , where w' is the derivative of w, such that w is monic and squarefree,  $\deg \kappa_i < \deg w$ , for  $i = 1, \dots, n$ , and a  $\mathbb{Q}$ -linear form  $\lambda$  in n variables satisfying  $\lambda(\kappa_1, \dots, \kappa_n) = tw' \mod w$ , such that

$$V = \left\{ \left( \frac{\kappa_1(t)}{w'(t)}, \dots, \frac{\kappa_n(t)}{w'(t)} \right) : w(t) = 0 \right\}.$$

The condition on the linear form  $\lambda$  states that the roots of w are precisely the values taken by  $\lambda$  on V, and that  $\lambda$  separates V, i.e.,  $\lambda(x) \neq \lambda(y)$  for any distinct pair x, y in V.

Let f be in  $\mathbb{Q}[x]$  of degree d and bitzise  $\tau$ . Assume that  $V_{\text{grad}}(f)$  is finite. By applying [38, Corollary 2] to the system of partial derivatives, we obtain the following corollary (Corollary 2.3) which states that there exists an algorithm computing a zero-dimensional rational parametrization of  $V_{\text{grad}}(f)$  and provides bit complexity estimates for when applying the algorithm in [38] to gradient ideals. The proof of Corollary 2.3 is straightforward from [38, Corollary 2] and is then postponed to Appendix A.

COROLLARY 2.3. Assume that  $V_{\text{grad}}(f)$  is finite. There exists an algorithm that takes f as in input, and that produces one of the following outputs:

- a) either a zero-dimensional rational parametrization of  $V_{\text{grad}}(f)$ ;
- b) or a zero-dimensional rational parametrization of degree less than that of  $V_{\rm grad}(f)$ ;
- c) or fails.

In any case, the algorithm uses

(2.2) 
$$\widetilde{O}\left(n^2(d+\tau)d^{2n+1}\binom{n+d}{d}\right),$$

boolean operations. Moreover, the polynomials  $w, \kappa_1, \ldots, \kappa_n$  involved in the zero-dimensional rational parametrization output have degree at most  $(d-1)^n$  and bitsize  $\widetilde{O}((d+\tau+n)(d-1)^n)$ .

Assume that  $\mathcal{Q} = ((w, \kappa_1, \dots, \kappa_n), x_1)$  is a zero-dimensional rational parametrization of  $V_{\text{grad}}(f)$  given by the algorithm from Corollary 2.3. The following lemma (Lemma 2.4) and its proof point out the explicit shape position of  $\mathcal{I}_{\text{grad}}(f)$ . Moreover, the degree and the bit complexity of the involved polynomials are estimated.

LEMMA 2.4. There exist polynomials  $w, v_2, \ldots, v_n$  in  $\mathbb{Q}[x_1]$  satisfying  $\deg v_i < \deg w$ , for  $i = 2, \ldots, n$ , such that  $\mathcal{I}_{\text{grad}}(f) = \langle w, x_2 - v_2, \ldots, x_n - v_n \rangle$ . Furthermore, to compute  $w, v_2, \ldots, v_n$ , we use

(2.3) 
$$\widetilde{O}\left((\tau+n+d)^2d^{6n}\right)$$

boolean operations. Their degrees are at most  $(d-1)^n$  and their maximal bitsizes are bounded from above by  $\widetilde{O}((\tau + n + d)d^{3n})$ .

*Proof.* Here we give only the proof of the degree estimate. The proof of the bit complexity is routine but rather technical and then postponed to Appendix B.

Because w is squarefree and w' is the derivative of w, one sees that the gcd of w and w' is 1. From the extended Euclidean algorithm [41, Algorithm 3.14], there exist two Bézout coefficients of w, w', namely a, b in  $\mathbb{Q}[x_1]$ , such that aw + bw' = 1. For  $i = 2, \ldots, n$ , we see that  $w'x_i(t) = \kappa_i(t)$  for any t satisfying w(t) = 0. As  $\deg \kappa_i \leq \deg w$  and the linear form  $\lambda = x_1$  separates V, we have  $w'x_i = \kappa_i$ . This yields  $bw'x_i = b\kappa_i$ . Since bw' = 1 - aw, we observe that  $x_i - awx_i = b\kappa_i$  and, hence,  $x_i = b\kappa_i \mod w$ . By denoting  $v_i := b\kappa_i \mod w$ , we obtain  $w, v_2, \ldots, v_n$  which are the desired polynomials.

The two following lemmas establish the bit complexity of Euclidean division algorithm and the extended Euclidean algorithm for univariate polynomials over  $\mathbb{Z}$  which will be used later on (in Proposition 3.11) to investigate the bit complexity of our algorithms.

LEMMA 2.5. Let a, b be polynomials in  $\mathbb{Z}[t]$ , with  $\deg b = m \leq \deg a = d$ , and  $\tau$  an upper bound of  $\operatorname{ht}(a)$  and  $\operatorname{ht}(b)$ . To compute the quotient q and the remainder r of the division of a by b, we use the Euclidean division algorithm [41, Algorithm 2.5]. Then, this algorithm uses  $\widetilde{O}(m\tau(d-m)^2)$  boolean operations. Furthermore, both bitsizes of q and r are bounded from above by  $\widetilde{O}(\operatorname{ht}(d-m))$ .

Again, the proof of Lemma 2.5 is routine but rather technical. We postpone it to Appendix C.

Denote by  $\mathbb{Q}(x_1)$  the field of rational fractions in variable  $x_1$  with coefficients in  $\mathbb{Q}$ . With the lexicographic monomial order  $x_2 < \cdots < x_n$ , we consider the standard (multivariate) division [11, Ch. 2, Sec 3.] of  $g \in \mathbb{Q}[x_1][x_2,\ldots,x_n]$  by the list  $[x_2-\frac{a_2}{a_0},\ldots,x_n-\frac{a_n}{a_0}]$ , with  $a_0,a_2,\ldots,a_n\in\mathbb{Q}[x_1]$ . To compute the quotients  $\phi_2,\ldots,\phi_n\in\mathbb{Q}(x_1)[x_2,\ldots,x_n]$  and remainder  $r\in\mathbb{Q}(x_1)$  such that  $g=\sum_{i=2}^n\phi_i(x_i-\frac{a_i}{a_0})+r$ , we iterate classical univariate divisions by  $x_i-\frac{a_i}{a_0}$  for  $2\leq i\leq n$  considering them as univariate in  $x_i$  so that we eliminate step by step the variables  $x_2,\ldots,x_n$  in g. The details of this algorithm, which we name Eliminate, are given in Appendix D (Algorithm D.1). The inputs of Eliminate are  $g,a_0,a_2,\ldots,a_n$  and the output is the list  $[\phi_2,\ldots,\phi_n]$  and the remainder r.

The bit complexity of Eliminate is given in the following lemma whose proof (which is quite routine) is given in Appendix D.

LEMMA 2.6. Assume that  $g \in \mathbb{Q}[x_1][x_2,\ldots,x_n]$  has degree d in  $x_2,\ldots,x_n$  and bitsize  $\tau_g$ , and that the polynomials  $a_0,a_2,\ldots,a_n \in \mathbb{Q}[x_1]$  have bitsizes at most  $\tau_a$ . Then, Algorithm Eliminate runs in

$$\widetilde{O}\left(n\tau_g + n^2d\tau_a\right)$$

boolean operations and the bitsizes of the outputs  $\phi_2, \ldots, \phi_n$  are in  $\widetilde{O}(\tau_g + nd\tau_a)$ .

- 3. SOS of polynomials modulo gradient ideals.
- **3.1.** The existence of an SOS decomposition over the rationals. The main result of this section is stated below.

Theorem 3.1. Let  $f \in \mathbb{Q}[x]$  such that the following conditions hold:

a) The infimum  $f^* = \inf\{f(x) : x \in \mathbb{R}^n\}$  is attained.

b) The gradient ideal  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical. Then, f is non-negative over  $\mathbb{R}^n$  if and only if f is an SOS of polynomials over the quotient ring  $\mathbb{Q}[\mathbf{x}]/\mathcal{I}_{grad}(f)$ .

*Proof.* Suppose that f is non-negative over  $\mathbb{R}^n$  and  $\sharp V_{\operatorname{grad}}(f) = \delta$ . We prove that f is an SOS of polynomials over the quotient ring  $\mathbb{Q}[x]/\mathcal{I}_{\operatorname{grad}}(f)$ . We consider the two following cases:

CASE 1. Distinct points in  $V_{\text{grad}}(f)$  have distinct  $x_1$ -coordinates. Consider the lexicographic monomial order  $x_1 < x_2 < \cdots < x_n$  on  $\mathbb{Q}[\boldsymbol{x}]$ . Since the gradient ideal is zero-dimensional and radical, according to Shape Lemma (Lemma 2.2), the reduced Gröbner basis of  $\mathcal{I}_{\text{grad}}(f)$  has the following shape:

$$[w, x_2 - v_2, \dots, x_n - v_n],$$

where  $v_2, \ldots, v_n$  are polynomials in  $\mathbb{Q}[x_1]$  of degree at most  $\delta - 1$ . We denote

$$(3.2) h(x_1) := f(x_1, v_2, \dots, v_n),$$

where  $x_i$  is replaced by  $v_i$  in f for i = 2, ..., n. With the order <, we divide f - h by the system in (3.1) by using the division algorithm in [11, Ch. 2, Sec 3.]. Then, there exist  $\phi_1, ..., \phi_n$  in  $\mathbb{Q}[\mathbf{x}]$ , and r in  $\mathbb{Q}[x_1]$  such that

(3.3) 
$$f - h = \phi_1 w + \sum_{i=2}^n \phi_i(x_i - v_i) + r,$$

with deg  $r < \delta$ . Let x be in  $V_{\rm grad}(f)$ . From (3.2) and (3.3), one sees that f(x) = h(x). Hence, f - h vanishes on  $V_{\rm grad}(f)$ . Clearly, the value of  $\phi_1 w + \sum_{i=2}^n \phi_i(x_i - v_i)$  is zero on  $V_{\rm grad}(f)$ . This implies that r also vanishes on the image set  $\pi(V_{\rm grad}(f))$ , where  $\pi(x_1, \ldots, x_n) = x_1$ . Since distinct points in  $V_{\rm grad}(f)$  have distinct  $x_1$ -coordinates, it holds that  $\sharp \pi(V_{\rm grad}(f)) = \sharp V_{\rm grad}(f) = \delta$ . As deg  $r < \delta$ , we conclude that  $r \equiv 0$ . Hence, from (3.3), we obtain the following representation:

(3.4) 
$$f = h + \phi_1 w + \sum_{i=2}^n \phi_i (x_i - v_i).$$

The set  $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_2 = v_2, \ldots, x_n = v_n\}$  defines a curve which is parametrized by  $x_1$ . Recall that f is non-negative over  $\mathbb{R}^n$ . Hence f is non-negative over this curve. Since f takes the same values over this curve as h takes over  $x_1$  when  $x_1$  ranges in  $\mathbb{R}$ , one can conclude that the univariate polynomial h is also non-negative over  $\mathbb{R}$ . According to the results on SOS decompositions of univariate polynomials with rational coefficients in Theorem 2.1 h is a sum of s squares in  $\mathbb{Q}[x_1]$ , i.e., there exist  $q_1, \ldots, q_s \in \mathbb{Q}[x_1]$  and  $c_1, \ldots, c_s$  in  $\mathbb{Q}_+$  such that  $h = c_1q_1^2 + \cdots + c_sq_s^2$ . Therefore, from (3.4), we assert that f is an SOS of polynomials over  $\mathbb{Q}[x]/\mathcal{I}_{\text{grad}}(f)$ .

CASE 2. There are two distinct points in  $V_{\text{grad}}(f)$  such that their  $x_1$ 's-coordinates are equal. According to [36, Lemma 2.1], there is  $j \in \{1, \ldots, (n-1)\delta(\delta-1)/2\}$  such that the linear function  $u := x_1 + jx_2 + \cdots + j^{n-1}x_n$  separates  $V_{\text{grad}}(f)$ , i.e.,  $u(x) \neq u(y)$  for any distinct points x, y in  $V_{\text{grad}}(f)$ . We consider the change of variables  $\mathbf{y} = T\mathbf{x}$ , where

(3.5) 
$$T = \begin{bmatrix} 1 & j & j^2 & \cdots & j^{n-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

We see that T is an invertible matrix. Then we obtain a polynomial  $g(\mathbf{y}) = f(T^{-1}\mathbf{y})$  in variables  $y_1, y_2, \dots, y_n$  having the following property: the infimum  $g^* = \inf\{g(y): y \in \mathbb{R}^n\}$  is attained. Because of the chain rule  $\nabla g = \nabla f \circ T^{-1}$ , we have

$$V_{\text{grad}}(g) = \{ y \in \mathbb{C}^n : y = Tx, x \in V_{\text{grad}}(f) \};$$

Thus, the gradient ideal  $\mathcal{I}_{\text{grad}}(g)$  is zero-dimensional and radical. Moreover, since  $y_1 = u(\boldsymbol{x})$  separates  $V_{\text{grad}}(f)$ , distinct points in  $V_{\text{grad}}(g)$  have distinct  $y_1$ -coordinates. We observe that  $g \in \mathbb{Q}[\boldsymbol{y}]$  is non-negative and satisfies the conditions of the theorem; Case 1 happens to  $V_{\text{grad}}(g)$  as well. Hence, there exists an SOS decomposition of g modulo  $\mathcal{I}_{\text{grad}}(g)$ 

(3.6) 
$$g(\mathbf{y}) = \sum_{j=1}^{s} c_j \bar{q}_j^2(\mathbf{y}) + \sum_{i=1}^{n} \bar{\phi}_i(\mathbf{y}) \frac{\partial g}{\partial y_i},$$

where  $\bar{\phi} = (\bar{\phi}_1, \dots, \bar{\phi}_n)$  and  $\bar{\phi}_i \in \mathbb{Q}[y]$ . In (3.6), we replace y by Tx and  $\frac{\partial g}{\partial y_i}$  by  $\frac{\partial f}{\partial x_i} \circ T^{-1}$ , we obtain an decomposition of f as follows:

(3.7) 
$$f(\boldsymbol{x}) = g(T\boldsymbol{x}) = \sum_{j=1}^{s} c_j \bar{q}_j^2(T\boldsymbol{x}) + \sum_{i=1}^{n} \bar{\phi}_i(T\boldsymbol{x}) \frac{\partial f}{\partial x_i} \circ T^{-1}.$$

Because of  $(\frac{\partial f}{\partial x_i} \circ T^{-1})(Tx) = \frac{\partial f}{\partial x_i}(x)$ , (3.7) is an SOS decomposition of f modulo  $\mathcal{I}_{\text{grad}}(f)$  of f.

We now prove the reverse conclusion. Suppose that f is SOS over the quotient ring  $\mathbb{Q}[x]/\mathcal{I}_{\text{grad}}(f)$ , i.e., f can be decomposed as follows:

(3.8) 
$$f = \sum_{j=1}^{s} c_j q_j^2 + \sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i},$$

for some polynomials  $q_1, \ldots, q_s, \phi_1, \ldots, \phi_n \in \mathbb{Q}[x]$ , and  $c_1, \ldots, c_s$  in  $\mathbb{Q}_+$ . Let  $x^* \in \mathbb{R}^n$  be such that  $f(x^*) = f^*$ . Then  $x^*$  is a critical point of f over  $\mathbb{R}^n$ , i.e.,  $x^*$  belongs to the variety  $V_{\text{grad}}(f)$ ; thus, we have

$$\sum_{i=1}^{n} \phi_i(x^*) \frac{\partial f}{\partial x_i}(x^*) = 0.$$

From (3.8), we see that  $f(x^*) = \sum_{j=1}^s c_j q_j^2(x^*)$  and so this value is non-negative. By assumption, for all x in  $\mathbb{R}^n$ ,  $f(x) \ge f(x^*)$ . Hence, f is non-negative over  $\mathbb{R}^n$ .

REMARK 3.2. Assume that Q is a real field and R is the real closure of Q. All arguments in the proof of Theorem 3.1 can be applied for f in Q[x]. Hence, the conclusion of Theorem 3.1 holds for the case Q[x], i.e., f is non-negative over  $R^n$  if and only if f is an SOS of polynomials over the quotient ring  $Q[x]/\mathcal{I}_{grad}(f)$  provided that the two following conditions hold: the infimum  $f^* = \inf\{f(x) : x \in R^n\}$  is attained; the gradient ideal  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical.

REMARK 3.3. In the proof of Theorem 3.1, one can see that f - h vanishes not only on  $V_{\text{grad}}(f)$  but also on the variety defined by  $\langle x_2 - v_2, \dots, x_n - v_n \rangle$ . Hence,  $\phi_1$  in (3.4) is zero and (3.4) becomes  $f = c_1 q_1^2 + \dots + c_s q_s^2 + \sum_{i=2}^n \phi_i(x_i - v_i)$ .

REMARK 3.4. The condition a) in Theorem 3.1 is used only to prove the reverse conclusion. Therefore, even without this condition, the following assertion still holds: if  $\mathcal{I}_{\text{grad}}(f)$  is zero-dimensional radical and f is non-negative over  $\mathbb{R}^n$ , then f is SOS modulo  $\mathcal{I}_{\text{grad}}(f)$ .

Theorem 3.1 provides certificates of non-negativity for polynomials in  $\mathbb{Q}[x]$  which satisfy its assumptions and which are not SOS of polynomials with real (or rational) coefficients. We illustrate this with two examples.

EXAMPLE 3.5. We recall a polynomial of Robinson [35] that is non-negative but cannot be represented as an SOS of polynomials,

$$\bar{f}_R = x_1^6 + x_2^6 + x_3^6 - x_1^4 x_2^2 - x_1^4 x_3^2 - x_2^4 x_1^2 - x_2^4 x_3^2 - x_3^4 x_1^2 - x_3^4 x_2^2 + 3x_1^2 x_2^2 x_3^2.$$

By substituting the third variable  $x_3$  by 1 in  $\bar{f}_R$ , we get the following non-negative polynomial:

$$f_R = x_1^6 + x_2^6 - x_1^4 x_2^2 + 3x_1^2 x_2^2 - x_1^2 x_2^4 - x_1^4 - x_2^4 - x_1^2 - x_2^2 + 1.$$

Because  $\bar{f}_R$  is the homogenization of  $f_R$ ,  $f_R$  cannot be represented as an SOS of polynomials [28, Proposition 1.2.4]. The gradient ideal  $\mathcal{I}_{grad}(f_R)$  is zero-dimensional and radical. Therefore, Theorem 3.1 tells us that  $f_R$  is an SOS of polynomials over the quotient ring  $\mathbb{Q}[x]/\mathcal{I}_{grad}(f_R)$ .

EXAMPLE 3.6. In [39], Scheiderer introduced the following homogeneous polynomial:

$$\bar{f}_S = x_1^4 + x_1 x_2^3 + x_2^4 - 3x_1^2 x_2 x_3 - 4x_1 x_2^2 x_3 + 2x_1^2 x_3^2 + x_1 x_3^3 + x_2 x_3^3 + x_3^4,$$

that can be decomposed as an SOS of polynomials with algebraic coefficients but cannot be decomposed as an SOS of polynomials with rational coefficients. By replacing the third variable  $x_3$  by -1, we obtain the non-negative polynomial

$$f_S = x_1^4 + x_1 x_2^3 + x_2^4 + 3x_1^2 x_2 + 4x_1 x_2^2 + 2x_1^2 - x_1 - x_2 + 1.$$

Note that the conclusion in [28, Proposition 1.2.4] holds for polynomials with rational coefficients, i.e.,  $g \in \mathbb{Q}[x]$  is SOS in  $\mathbb{Q}[x]$  if and only if its homogenization so is in  $\mathbb{Q}[x]$ . Hence, the polynomial  $f_S$  is also SOS with algebraic coefficients but not SOS with rational ones. The gradient ideal  $\mathcal{I}_{\text{grad}}(f_S)$  satisfies the zero-dimensional and radical condition. Hence, according to Theorem 3.1,  $f_S$  is an SOS of polynomials over the quotient ring  $\mathbb{Q}[x]/\mathcal{I}_{\text{grad}}(f_S)$ .

An explicit SOS decomposition of  $f_S$  will be given in the next section.

**3.2. Description of the algorithm.** Based on the proof of Theorem 3.1, we design an algorithm to compute an SOS decomposition of polynomials modulo the gradient ideal of a non-negative polynomial with rational coefficients.

The input of sosgradientshape is a non-negative polynomial  $f \in \mathbb{Q}[x]$  whose gradient ideal  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical and satisfies the Shape Lemma assumption, i.e., all points in  $V_{grad}(f)$  have distinct  $x_1$ -coordinates.

The output includes the cardinality  $\delta$  of  $V_{\rm grad}(f)$ , the lists of polynomials and numbers

$$[w, v_2, \ldots, v_n], [q_1, \ldots, q_s], [\phi_2, \ldots, \phi_n] \subset \mathbb{Q}[\boldsymbol{x}], \text{ and } [c_1, \ldots, c_s] \subset \mathbb{Q}_+$$

satisfying the relation

$$f = \sum_{j=1}^{s} c_j q_j^2 + \sum_{i=2}^{n} \phi_i (x_i - v_i)$$

In Step 1, we compute the reduced Gröbner basis G for  $\mathcal{I}_{grad}(f)$  by relying on a zero-dimensional rational parametrization of  $V_{grad}(f)$  mentioned in Lemma 2.4. In Step 2, we compute the quotients  $\phi_2, \ldots, \phi_n$  and the remainder r of the division of f by G. In Step 3, we compute a rational weighted SOS decomposition of the non-negative univariate polynomial h by using Algorithm univsos1 or Algorithm univsos2 described in [26, Fig. 1] or [26, Fig. 2], respectively.

### Algorithm 3.1 Computing SOS of polynomials modulo the gradient ideal

sosgradientshape := proc(f)

**Input:**  $f \in \mathbb{Q}[x]$  non-negative over  $\mathbb{R}^n$  such that  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical and all points in  $V_{grad}(f)$  have distinct  $x_1$ -coordinates

**Output:**  $\delta$  in  $\mathbb{N}$ ,  $[q_1, \ldots, q_s]$ ,  $[w, v_2, \ldots, v_n] \subset \mathbb{Q}[x_1]$ ,  $[\phi_2, \ldots, \phi_n] \subset \mathbb{Q}[x]$ , and  $[c_1, \ldots, c_s] \subset \mathbb{Q}_+$  satisfying

(3.9) 
$$f = \sum_{i=1}^{s} c_j q_j^2 + \sum_{i=2}^{n} \phi_i (x_i - v_i).$$

- 1: Compute the reduced Gröbner basis  $G = [w, x_2 v_2, \dots, x_n v_n]$  of  $\mathcal{I}_{grad}(f)$ , with the lexicographical ordering  $x_1 < x_2 < \dots < x_n$
- 2: Compute the quotients  $[\phi_2, \ldots, \phi_n]$  and remainder h of the division of f by G by performing  $\texttt{Eliminate}(f, 1, v_2, \ldots, v_n)$
- 3: Compute a rational weighted SOS decomposition  $h = c_1 q_1^2 + \cdots + c_s q_s^2$
- 4: Return  $\delta$ ,  $[q_1, \ldots, q_s]$ ,  $[\phi_2, \ldots, \phi_n]$ ,  $[w, v_2, \ldots, v_n]$ , and  $[c_1, \ldots, c_s]$

REMARK 3.7. Suppose that the Shape Lemma assumption does not hold for  $\mathcal{I}_{\mathrm{grad}}(f)$ , i.e., there are two distinct points in  $V_{\mathrm{grad}}(f)$  such that their  $x_1$ 's-coordinates are equal. As mentioned in the proof of Theorem 3.1, we can find an invertible matrix T given by (3.5), change of variables  $\mathbf{y} = T\mathbf{x}$ , and assign  $g(\mathbf{y}) := f(T^{-1}\mathbf{y})$ . Here, we have  $y_1 = x_1 + jx_2 + \cdots + j^{n-1}x_n$  for some j > 0 and  $y_i = x_i$  for i = 2, ..., n. We get a new non-negative polynomial in n new variables with rational coefficients  $g(\mathbf{y})$  whose gradient ideal satisfies Shape Lemma assumption. Now we can apply Algorithm sosgradientshape for  $g(\mathbf{y})$  and obtain the output: the number  $\bar{\delta}$ , two lists  $[\bar{q}_1, \ldots, \bar{q}_s], [\bar{w}, \bar{v}_2, \ldots, \bar{v}_n]$  of polynomials in  $\mathbb{Q}[y_1]$ , a list  $[\bar{\phi}_1, \ldots, \bar{\phi}_n]$  of polynomials in  $\mathbb{Q}[\mathbf{y}]$ , and a list  $[c_1, \ldots, c_s] \subset \mathbb{Q}_+$ . Since  $\sharp V_{\mathrm{grad}}(f) = \sharp V_{\mathrm{grad}}(g)$ , one has  $\bar{\delta} = \delta$ . The new polynomial g can be decomposed as follows:

$$g(\mathbf{y}) = \sum_{j=1}^{s} c_j \bar{q}_j^2(y_1) + \bar{\phi}_1(\mathbf{y}) \bar{w}(y_1) + \sum_{i=2}^{n} \bar{\phi}_i(\mathbf{y}) (y_i - \bar{v}_i(y_1)).$$

Hence, f can be decomposed as:

(3.10) 
$$f(\mathbf{x}) = \sum_{j=1}^{s} c_j \bar{q}_j^2(u(\mathbf{x})) + \bar{\phi}_1(T\mathbf{x})\bar{w}(u(\mathbf{x})) + \sum_{i=2}^{n} \bar{\phi}_i(T\mathbf{x})(x_i - \bar{v}_i(u(\mathbf{x}))),$$

where  $u(\mathbf{x}) = x_1 + jx_2 + \dots + j^{n-1}x_n$ . Clearly,  $[w(u), x_2 - \bar{v}_2(u), \dots, x_n - \bar{v}_n(u)]$  is also a basis for  $V_{\text{grad}}(f)$ . Hence, (3.10) provides us an SOS decomposition of f modulo the gradient ideal of f.

THEOREM 3.8. Let f be a non-negative polynomial in  $\mathbb{Q}[\mathbf{x}]$ . Suppose that f is non-negative over  $\mathbb{R}^n$ ,  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical, and all points in  $V_{grad}(f)$  have distinct  $x_1$ -coordinates. On input f, Algorithm sosgradientshape terminates and computes an SOS decomposition of f modulo  $\mathcal{I}_{grad}(f)$  with rational coefficients.

Proof. Assume that  $f \in \mathbb{Q}[x]$  is non-negative over  $\mathbb{R}^n$  and its gradient ideal is zero-dimensional and radical. Here, we use the lexicographic monomial order  $x_1 < x_2 < \cdots < x_n$ . Because the Shape Lemma's assumption holds, the reduced Gröbner basis of  $\mathcal{I}_{\operatorname{grad}}(f)$  in Step 1 has the form  $G = [w, x_2 - v_2, \dots, x_n - v_n]$ , and can be computed by using a zero-dimensional rational parametrization of  $V_{\operatorname{grad}}(f)$  as in Lemma 2.4. In Step 2, we compute the quotients  $[\phi_2, \dots, \phi_n]$  and the remainder r of the division of f by G by performing  $\operatorname{Eliminate}(f, 1, v_2, \dots, v_n)$  (as in Algorithm D.1). Here, we see that r coincides with h, where  $h = f(x_1, v_2, \dots, x_n)$  as in the proof of Theorem 3.1, because of

$$r = f - \sum_{i=2}^{n} \phi_i(x_i - v_i) = h.$$

In Step 3, the univariate polynomial h is non-negative with rational coefficients, so by using univsos1 or univsos2 [26], we can compute an SOS decomposition of  $h = c_1q_1^2 + \cdots + c_sq_s^2$ . Hence, according to the proof of Theorem 3.1, we get (3.9) which is an SOS decomposition modulo the gradient ideal of f.

To illustrate how the algorithm works, we consider the following simple example.

EXAMPLE 3.9. Consider the polynomial  $f(x_1,x_2)=2x_1^4+2x_1x_2+x_2^2+10$ . This polynomial is non-negative over  $\mathbb{R}^n$ . Firstly, the gradient ideal  $\mathcal{I}_{\mathrm{grad}}(f)$  is given by  $\mathcal{I}_{\mathrm{grad}}(f)=\left\langle 8x_1^3+2x_2,2x_1+2x_2\right\rangle$  which is zero-dimensional and radical with  $\delta=3$ . We compute the reduced Gröbner basis of  $\mathcal{I}_{\mathrm{grad}}(f)$ , namely  $\left\langle x_1^3-\frac{1}{4}x_1,x_2+x_1\right\rangle$ , here  $v_2(x_1)=-x_1$ . Secondly, with the order  $x_1< x_2$ , the quotients of the division of f by the Gröbner basis are  $\phi_1=0$  and  $\phi_2=x_1+x_2$ , and the remainder is given by  $h(x_1)=f(x_1,v_2)=2x_1^4-x_1^2+10$ . Thirdly, by using Algorithm univsos2 in [26], one gets an SOS decomposition of  $h=\frac{1}{2}x_1^4+\frac{3}{2}(x_1^2-\frac{5}{2})^2+\frac{13}{2}x_1^2+\frac{5}{8}$ . Finally, we obtain the following SOS decomposition of f modulo its gradient ideal:

$$f = \frac{1}{2}x_1^4 + \frac{3}{2}\left(x_1^2 - \frac{5}{2}\right)^2 + \frac{13}{2}x_1^2 + \frac{5}{8} + (x_1 + x_2) \times (x_2 + x_1).$$

**3.3. Bit complexity analysis.** This subsection investigates the bit complexity of sosgradientshape. Assume that d and  $\tau$  are respectively the degree and an upper bound of the bitsize of the coefficients of  $f \in \mathbb{Q}[x]$ . We provide estimates for the bitsizes of polynomials in the output of sosgradientshape(f) as well as for the number of required boolean operations required to execute it.

We use Algorithm univsos1 in [26, Fig. 1] or Algorithm univsos2 in [26, Fig. 2] to compute an SOS decomposition of the non-negative univariate polynomial h. The corresponding bit complexities are given as follows:

PROPOSITION 3.10. Let  $v_2, \ldots, v_n$  be as in Lemma 2.4 and  $h(x_1) = f(x_1, v_2, \ldots, v_n)$ . To compute an SOS decomposition of h, Algorithm univsos1 and Algorithm univsos2

run in

(3.11) 
$$\widetilde{O}\left((d^{n+1}/2)^{3d^{n+1}/2}(\tau+n+d)d^{3n+1}\right)$$

and

$$\widetilde{O}\left((\tau+n+d)d^{6n+4}\right)$$

boolean operations, respectively.

*Proof.* Let  $\tau_v = \max_i \{ \text{ht}(v_i) \}$ . Lemma 2.4 states that the bitsize of  $\tau_v$  is bounded from above by  $\widetilde{O}\left((\tau + n + d)d^{3n}\right)$ , and that the polynomials  $w, v_2, \ldots, v_n$  have degree at most  $(d-1)^n$ . Since deg f = d and  $h(x_1) = f(x_1, v_2, \ldots, v_n)$ , the degree of h is at most  $d(d-1)^n$ .

Let  $\beta$  be the minimal common denominator of all non-zero coefficients of h. Computing an SOS decomposition of h boils down to computing an SOS decomposition of  $\beta h$ . In particular, the execution time of univsos1 (resp., univsos2) on h is the same as for  $\beta h$ . Now we estimate the bitsize of the polynomial  $\beta h \in \mathbb{Z}[x_1]$ . By the definition of h, we observe that  $\operatorname{ht}(h) \leq \tau + d\tau_v$ . It follows that  $\operatorname{ht}(\beta h) \leq \operatorname{ht}(\beta) + \tau + d\tau_v$ . By definition we have  $\operatorname{ht}(\beta) \leq \tau + d\tau_v$ . This yields

From (3.13) and above results, we obtain the following bitsize estimate for  $\beta h$ :

$$\widetilde{O}\left(2(\tau+d(\tau+n+d)d^{3n})\right) = \widetilde{O}\left((\tau+n+d)d^{3n+1}\right).$$

To compute an SOS decomposition of  $\beta h$ , we rely on univsos1 or univsos2. From [26, Theorem 17] and [26, Theorem 24], the boolean running time of univsos1 corresponds to the quantity given by (3.11). If we use univsos2, then the number of boolean operations will be bounded from above by

$$\widetilde{O}\left(d^4(d-1)^{4n}+d^4(\tau+n+d)(d-1)^{6n}\right)$$

which can be further reduced to (3.12).

PROPOSITION 3.11. Let  $v_2, \ldots, v_n$  be as in Proposition 3.10. To compute the list  $\phi_2, \ldots, \phi_n$  in the output of Algorithm sosgradientshape, Algorithm Eliminate runs in  $\widetilde{O}(n^2(\tau+n+d)d^{3n+1})$  boolean operations and the bitsizes of  $\phi_2, \ldots, \phi_n$  are  $\widetilde{O}(n(\tau+n+d)d^{3n+1})$ .

Proof. From Lemma 2.4, the polynomial  $v_i$  has bitsize at most  $\widetilde{O}((\tau+n+d)d^{3n})$ . We divide f by  $[x_2-v_2,\ldots,x_n-v_n]$  while performing  $\text{Eliminate}(f,1,v_2,\ldots,v_n)$  as in Algorithm D.1 to obtain the list of quotients  $[\phi_2,\ldots,\phi_n]$  and the remainder  $h=h(x_1,v_2,\ldots,v_n)$ . Applying Lemma 2.6 for this division, we conclude that Algorithm  $\widetilde{O}(n^2(\tau+n+d)d^{3n+1})$  boolean operations, the estimate for the bitsize of  $\phi_i$  is  $\widetilde{O}(n(\tau+n+d)d^{3n+1})$  as claimed.

We are now ready to analyze the bit complexity of Algorithm 3.1.

THEOREM 3.12. Let  $f \in \mathbb{Q}[x]$  of degree d and let  $\tau$  be the maximum bitsize of its coefficients. Assume that the two conditions in Theorem 3.1 hold. Then, on input f, Algorithm sosgradientshape runs in

(3.14) 
$$\widetilde{O}\left((\tau+n+d)^2d^{6n} + (\tau+n+d)d^{3n+1}(d^{n+1}/2)^{3d^{n+1}/2}\right)$$

or

(3.15) 
$$\widetilde{O}\left((\tau + n + d)^2 d^{6n} + (\tau + n + d) d^{6n+4}\right)$$

boolean operations if in Step 3 we use Algorithm univsos1 or Algorithm univsos2, respectively.

*Proof.* Assume that in Step 3 we use univsos1 to compute an SOS decomposition of h. Then, the number of boolean operations that sosgradientshape uses to compute the SOS decomposition of f is the sum of the four following ones:

- 1. the number of boolean operations required to compute the zero-dimensional rational parametrization Q of  $V_{\text{grad}}(f)$  as in (2.2);
- 2. the number of boolean operations required to compute  $w, v_2, \ldots, v_n \in \mathbb{Q}[x_1]$ , defined in Lemma 2.4 as in (2.3);
- 3. the number of boolean operations required to compute an SOS decomposition of h by using Algorithm univsos1 as in (3.11);
- 4. the number of boolean operations required to compute  $\phi_1, \ldots, \phi_n$  in the output of sosgradientshape by using Algorithm Eliminate (mentioned in Proposition 3.11).

This sum equals

$$\widetilde{O}\left(n^{2}(d+\tau)d^{2n+1}\binom{n+d}{d} + (\tau+n+d)^{2}d^{6n} + (\tau+n+d)d^{3n+1}\left(\frac{d^{n+1}}{2}\right)^{3d^{n+1}/2} + (\tau+n+d)n^{2}d^{3n+2}\right).$$

In this sum, the third term is larger than the first and last terms for large enough d and n, yielding the estimate (3.14).

If in Step 3 we use univsos2, the number of boolean operations of the algorithm is

$$\widetilde{O}\Big(n^2(d+\tau)d^{2n+1}\binom{n+d}{d} + (\tau+n+d)^2d^{6n} + (\tau+n+d)d^{6n+4} + n^2(\tau+n+d)d^{3n+2}\Big).$$

Noting that  $\binom{n+d}{d} \leq (d+1)^n \leq d^{2n}$  for large enough d and n, we obtain (3.15).  $\square$ 

THEOREM 3.13. Assume that  $f \in \mathbb{Q}[x]$  satisfies the conditions of Theorem 3.12. Let  $w, v_2, \ldots, v_n$ , h be as in Proposition 3.10. Then, the maximum bitsize of the coefficients involved in the SOS decomposition of h obtained by using Algorithm univsos1 and Algorithm univsos2 are bounded from above, respectively, by

(3.16) 
$$O\left((\tau+n+d)(d^{n+1}/2)^{3d^{n+1}/2}d^{3n+1}\right),$$

and

$$(3.17) O\left((\tau+n+d)d^{5n+2}\right).$$

*Proof.* From the proof of Proposition 3.10, the estimates for degree and bitsize of  $\beta h$  are  $d(d-1)^n$  and  $\widetilde{O}\left((\tau+n+d)d^{3n+1}\right)$ , respectively. According to [26, Theorem 16] and [26, Theorem 23], the maximum bitsize of the coefficients involved in the SOS decomposition of  $\beta h$  obtained by using univsos1 and univsos2 are bounded from above by (3.16) and (3.17), respectively.

4. SOS of rational fractions modulo gradient ideals. Artin's Theorem [1] states that if  $f \in \mathbb{R}[x]$  is non-negative then there exists a nonzero  $g \in \mathbb{R}[x]$  such that  $g^2f$  is SOS, yielding a decomposition of f as an SOS of rational fractions. In this section, we explain how to decompose  $f \in \mathbb{Q}[x]$  as an SOS of rational fractions modulo its gradient ideal. One says that  $f \in \mathbb{Q}[x]$  is an SOS of rational fractions in  $\mathbb{Q}(x)$ , where  $\mathbb{Q}(x)$  is the field of rational fractions in the variable x over  $\mathbb{Q}$ , if there exist rational fractions  $f_1, \ldots, f_s$  in  $\mathbb{Q}(x)$  and  $[c_1, \ldots, c_s] \subset \mathbb{Q}_+$  such that  $f = \sum_{j=1}^s c_j f_j^2$ . Furthermore, f is an SOS of rational fractions over the quotient ring  $\mathbb{Q}(x)/\mathcal{I}_{\text{grad}}(f)$  if there exists  $g \in \mathcal{I}_{\text{grad}}(f)$  such that f - g is an SOS of rational fractions in  $\mathbb{Q}(x)$ , i.e., f can be decomposed as follows:

$$f = \sum_{j=1}^{s} c_j f_j^2 + \sum_{i=1}^{n} \phi_i \frac{\partial f}{\partial x_i},$$

for some rational fractions  $f_1, \ldots, f_s, \phi_1, \ldots, \phi_s$  in  $\mathbb{Q}(\boldsymbol{x})$  and  $[c_1, \ldots, c_s] \subset \mathbb{Q}_+$ .

**4.1. The existence of an SOS decomposition over the rationals.** Denote by  $\mathbb{Q}(x_1)[x_2,\ldots,x_n]$  the vector space of polynomials in n-1 variables  $(x_2,\ldots,x_n)$  with coefficients in  $\mathbb{Q}(x_1)$ .

In the following theorem, we prove the existence of an SOS decomposition of rational fractions modulo the gradient ideal for f.

THEOREM 4.1. Assume that  $f \in \mathbb{Q}[x]$  is a non-negative polynomial of degree d and that  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical. Let  $\mathcal{Q} = ((w, \kappa_1, \ldots, \kappa_n), x_1)$  be a zero-dimensional rational parametrization of  $V_{grad}(f)$ . Then, f can be decomposed as an SOS of rational fractions modulo the gradient ideal, namely

(4.1) 
$$f = \frac{1}{(w')^d} \sum_{j=1}^s c_j q_j^2 + \sum_{i=1}^n \phi_i (x_i - \frac{\kappa_i}{w'}),$$

for some rational fractions  $q_1, \ldots, q_s \in \mathbb{Q}(x_1), \ \phi_1, \ldots, \phi_n \in \mathbb{Q}(x_1)[x_2, \ldots, x_n], \ and [c_1, \ldots, c_s] \subset \mathbb{Q}_+.$ 

*Proof.* By substituting  $x_i = \kappa_i/w'$  in f, for i = 2, ..., n, one has

$$(4.2) f\left(x_1, \frac{\kappa_2}{w'}, \dots, \frac{\kappa_n}{w'}\right) = \frac{1}{(w')^d} \bar{h},$$

where  $\bar{h}(x_1)$  is a univariate polynomial. Since f is non-negative with even degree d,  $\bar{h}$  is also non-negative. In addition, the coefficients of  $w', \kappa_1, \ldots, \kappa_n$  and f are rational numbers, so the coefficients of  $\bar{h}$  are also rational numbers. Applying Theorem 2.1 for  $\bar{h}$ , we conclude that there are  $q_1, \ldots, q_s \in \mathbb{Q}[x_1]$  and  $[c_1, \ldots, c_s] \subset \mathbb{Q}_+$  such that

(4.3) 
$$\bar{h} = \sum_{j=1}^{s} c_j q_j^2.$$

Next, one considers the division of  $(w')^d f - \bar{h}$  by  $[w'x_1 - \kappa_1, \dots, w'x_n - \kappa_n]$  with the lexicographic order  $x_1 < \dots < x_n$ . Based on Buchberger's Criterion [9], we can show that this system is a Gröbner basis of the ideal generated by this system w.r.t the order < in  $\mathbb{Q}[\mathbf{x}]$ . Hence, there exist a (unique) list of quotients  $u_1, \dots, u_n$  in  $\mathbb{Q}[\mathbf{x}]$ , and r in  $\mathbb{Q}[x_1]$  such that

(4.4) 
$$(w')^d f - \bar{h} = \sum_{i=1}^n u_i (w' x_i - \kappa_i) + r,$$

with r of smaller degree than the cardinality  $\delta$  of  $V_{\text{grad}}(f)$ . The gradient variety of f can be represented as follows:

$$V_{\text{grad}}(f) = \{ \boldsymbol{x} \in \mathbb{C}^n : w = 0, w'x_1 - \kappa_1 = \dots = w'x_n - \kappa_n = 0 \}.$$

From (4.2), one sees that  $(w')^d f - \bar{h}$  vanishes on  $V_{\text{grad}}(f)$ . With the same arguments as in the proof of Theorem 3.1, we conclude that  $r \equiv 0$ . Hence, from (4.2), (4.3), and (4.4), we obtain a representation of f as in (4.1), where  $\phi_i = u_i/(w')^{d-1}$ .

In Theorem 4.5, we assume that  $Q = ((w, \kappa_1, \dots, \kappa_n), x_1)$  is a zero-dimensional rational parametrization of  $V_{\text{grad}}(f)$  which is a generic assumption. In this assumption, the linear form  $\lambda$  is given by  $\lambda(x) = x_1$ . If the assumption does not hold, we can change the coordinate system such that the obtained polynomial (with new variables) satisfies the assumption as in Case 2 of the proof of Theorem 3.1.

REMARK 4.2. From (4.2), we see that  $\deg \bar{h}$  does not exceed  $\deg_{x_1} f + d \deg(w')$ , where  $\deg_{x_1} f$  is the degree of f in the variable  $x_1$  and  $\deg w' = \deg w - 1$ . Thus, the degree of the univariate polynomial  $\bar{h}$  is at most  $d(d-1)^n$ .

4.2. Algorithm to compute an SOS of rational fractions. From the proof of Theorem 4.1, we design an algorithm named sosgradient to compute the SOS decomposition of rational fractions for f. Algorithm sosgradient is obtained by a modification of Step 1 in sosgradientshape to get a zero-dimensional rational parametrization of the gradient variety of f.

ALGORITHM 4.1 Computing SOS of rational fractions modulo the gradient ideal

sosgradient := proc(f)

**Input:**  $f \in \mathbb{Q}[x]$  of degree d such that f is non-negative over  $\mathbb{R}^n$  and  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical

**Output:**  $[w, \kappa_1, \ldots, \kappa_n], [q_1, \ldots, q_s] \subset \mathbb{Q}[x_1], [\phi_2, \ldots, \phi_n] \subset \mathbb{Q}[\boldsymbol{x}], \text{ and } [c_1, \ldots, c_s] \subset \mathbb{Q}_+$  satisfying

(4.5) 
$$f = \frac{1}{(w')^d} \sum_{j=1}^s c_j q_j^2 + \sum_{i=1}^n \frac{\phi_i}{(w')^d} \left( x_i - \frac{\kappa_i}{w'} \right).$$

- 1: Compute a zero-dimensional rational parametrization  $[w, \kappa_1, \dots, \kappa_n]$  of  $V_{\rm grad}(f)$
- 2: Compute the quotients  $[\phi_2,\ldots,\phi_n]$  and the remainder  $\bar{h}$  of the division of  $(w')^d f$  by  $[x_2-\frac{\kappa_2}{w'},\ldots,x_n-\frac{\kappa_n}{w'}]$  by performing  $\texttt{Eliminate}((w')^d f,w',\kappa_2,\ldots,\kappa_n)$
- 3: Compute a rational weighted SOS decomposition of  $\bar{h} = c_1 q_1^2 + \cdots + c_s q_s^2$
- 4: Return  $[w, \kappa_1, \ldots, \kappa_n], [q_1, \ldots, q_s], [\phi_2, \ldots, \phi_n], \text{ and } [c_1, \ldots, c_s]$

The input of sosgradient is a non-negative polynomial f in  $\mathbb{Q}[x]$  whose gradient ideal  $\mathcal{I}_{grad}(f)$  is zero-dimensional. The outputs are a zero-dimensional rational parametrization of  $V_{grad}(f)$ , a list of polynomials  $[q_1, \ldots, q_s] \subset \mathbb{Q}[x_1]$ , and a list of rational fractions  $[\phi_2, \ldots, \phi_n] \subset \mathbb{Q}(x_1)[x_2, \ldots, x_n]$  satisfying (4.5).

In Step 1, we compute a zero-dimensional rational parametrization  $[w, \kappa_1, \ldots, \kappa_n]$  of  $V_{\text{grad}}(f)$ . In Step 2, we compute the quotients  $[\phi_2, \ldots, \phi_n]$  of the division of  $(w')^d f$  by  $[x_2 - \frac{\kappa_2}{w'}, \ldots, x_n - \frac{\kappa_n}{w'}]$  while using Algorithm Eliminate. Note that the remainder of this division coincides with  $\bar{h}$  given in (4.2). In Step 3, we compute a rational

weighted SOS decomposition of the univariate polynomial  $\bar{h}$  by relying on univsos1 or univsos2.

The correctness of sosgradient is proved in a similar way as for sosgradientshape in Theorem 3.8.

THEOREM 4.3. Suppose that  $f \in \mathbb{Q}[x]$  is non-negative over  $\mathbb{R}^n$  and  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical. On input f, Algorithm sosgradient terminates and the outputs provide us an SOS decomposition of f as in (4.5).

**4.3.** Bit complexity analysis. We now estimate the bitsizes of polynomials in the output as well as the number of boolean operations required to perform Algorithm sosgradient.

PROPOSITION 4.4. Assume that  $\tau$  is the maximum bitsize of the coefficients of f in the input of sosgradient. To compute the list  $[\phi_2, \ldots, \phi_n]$  in the output, Algorithm Eliminate runs in  $\widetilde{O}\left(n^2(\tau+n+d)d^{n+1}\right)$  boolean operations. Furthermore, the bitsize of  $\phi_i$  is  $\widetilde{O}\left(n(\tau+n+d)d^{n+1}\right)$ ,  $i=2,\ldots,n$ .

Proof. We compute the division of  $(w')^d f$  by  $[x_2 - \frac{\kappa_2}{w'}, \dots, x_n - \frac{\kappa_n}{w'}]$  by performing Eliminate $((w')^d f, w', \kappa_2, \dots, \kappa_n)$ . We obtain the list of quotients  $[\phi_2, \dots, \phi_n]$  and the remainder  $\bar{h}$ . The degree of  $(w')^d f$  in  $x_2, \dots, x_n$  is d, and  $\operatorname{ht}((w')^d f) = \widetilde{O}\left((\tau + n + d)d^{n+1}\right)$ . The conclusions are obtained by applying Lemma 2.6 with  $\operatorname{ht}(\kappa_i) = \widetilde{O}\left((\tau + n + d)(d - 1)^n\right)$ .

THEOREM 4.5. Let  $f \in \mathbb{Q}[x]$  of degree d and let  $\tau$  be the maximum bitsize of its coefficients. Assume that f is non-negative over  $\mathbb{R}^n$  and  $\mathcal{I}_{grad}(f)$  is zero-dimensional and radical. Then, on input f, Algorithm sosgradient uses

(4.6) 
$$\widetilde{O}\Big((d^{n+1}/2)^{3d^{n+1}/2}(\tau+n+d)d^{n+1}\Big),$$

or

$$(4.7) \widetilde{O}((\tau + n + d)d^{4n+4})$$

boolean operations if in Step 3 we use Algorithm univsos1 or Algorithm univsos2, respectively.

*Proof.* From Corollary 2.3, the polynomials  $w, \kappa_1, \ldots, \kappa_n$  in the zero-dimensional parametrization of the gradient variety  $V_{\text{grad}}(f)$  have degree at most  $(d-1)^n$  and bitsize  $\widetilde{O}((\tau+n+d)(d-1)^n)$ . We can see that the degree of the remainder  $\bar{h}$  (as defined in (4.2)) in Step 2 of sosgradient is at most  $d(d-1)^n+d$  and its bitsize is  $\widetilde{O}((\tau+n+d)d^{n+1})$ . To compute an SOS decomposition of  $\bar{h}$ , by applying [26, Theorem 17] and [26, Theorem 24], Algorithm univsos1 and Algorithm univsos2 use

(4.8) 
$$\widetilde{O}\left((d^{n+1}/2)^{3d^{n+1}/2}(\tau+n+d)d^{n+1}\right)$$

and

$$(4.9) \widetilde{O}\left((\tau+n+d)d^{4n+4}\right)$$

boolean operations, respectively.

The estimates (4.6) and (4.7) are obtained from Corollary 2.3, Proposition 4.4, and the estimates (4.8) and (4.9) with the same line of reasoning as in the proof of Theorem 3.12.

THEOREM 4.6. Assume that  $f \in \mathbb{Q}[x]$  satisfies the conditions of Theorem 4.5. Then, the maximum bitsizes of the coefficients involved in the SOS decomposition of  $\bar{h}$ , obtained by using Algorithm univsos1 and Algorithm univsos1, are bounded from above respectively by

(4.10) 
$$\widetilde{O}\left((d^{n+1}/2)^{3d^{n+1}/2}(\tau+n+d)d^{n+1}\right)$$

and

(4.11) 
$$\widetilde{O}\left((\tau+n+d)d^{3n+3}\right).$$

*Proof.* From the proof of Theorem 4.5, the degree of  $\bar{h}$  is at most  $d(d-1)^n$  and the bitsize of  $\bar{h}$  is  $\widetilde{O}\left((\tau+n+d)d^{n+1}\right)$ . The conclusions follow from [26, Theorem 16] and [26, Theorem 23].

REMARK 4.7. In general, sosgradient is faster than sosgradientshape to certify non-negativity of polynomials with rational coefficients. When relying on univsos2, by comparing the estimates in (3.15) and (4.7), we conclude that the number of boolean operations to run sosgradientshape is  $O(d^{2n})$  times larger than the one of sosgradientshape. The underlying reason is that the maximal bitsizes of  $w, v_2, \ldots, v_n$  are  $(d-1)^{2n}$  times bigger than the ones of  $\kappa_1, \ldots, \kappa_n$  that are obtained by a zero-dimensional rational parametrization of the gradient variety.

To finish the section, we present an explicit SOS decomposition for the polynomial  $f_S$  obtained from Scheiderer's polynomial given in Example 3.6. Here, we rely on sosgradient to get the SOS decomposition.

EXAMPLE 4.8. We first compute a zero-dimensional rational parametrization Q of the gradient variety  $V_{\text{grad}}(f_S)$ :

$$w = 4x_1^9 + x_1^6 - 16x_1^5 - 4x_1^3 - 4x_1^2 - 1,$$
  

$$\kappa_1 = 15x_1^7 - 32x_1^6 - 9x_1^4 - 36x_1^3 - 6x_1 - 4,$$
  

$$\kappa_2 = -3x_1^6 + 64x_1^5 + 24x_1^3 + 28x_1^2 + 9.$$

In  $f_S$ , by substituting  $x_2=\kappa_2/w'$  as in (4.2), we get the non-negative univariate polynomial  $\bar{h}=1679616x_1^{36}+3359232x_1^{34}-559872x_1^{33}-13670208x_1^{32}+11197440x_1^{31}-32799168x_1^{30}+7301664x_1^{29}+40124160x_1^{28}-56581740x_1^{27}+118393488x_1^{26}-29030400x_1^{25}-11429649x_1^{24}+91968984x_1^{23}-162286560x_1^{22}+52664472x_1^{21}-95470992x_1^{20}-51948224x_1^{19}+37314854x_1^{18}-36173624x_1^{17}+103156448x_1^{16}+27660704x_1^{15}+94133752x_1^{14}+56849248x_1^{13}+51186288x_1^{12}+42348048x_1^{11}+20765728x_1^{10}+17391200x_1^{9}+7273168x_1^{8}+4607744x_1^{7}+1946186x_1^{6}+880960x_1^{5}+413632x_1^{4}+86580x_1^{3}+75816x_1^{2}+6561.$ 

Based on Algorithm Eliminate, we obtain the quotients of the division in Step 3 of sosgradient:  $\phi_1 = 0$  and  $\phi_2$  given at polsys.lip6.fr/~hieu/phisos.mm.

By using univsos2 to compute an SOS decomposition of  $\bar{h}$ , we obtain the list sos given at above link such that  $\bar{h} = \sum_{i=1}^{m} \cos[2i-1] * \cos[2i]^2$ , where  $\cos[i]$  stands for the *i*-th entry of sos, m is the half length of sos.

Combining the above results, we obtain an SOS of rational fractions modulo the gradient of  $f_S$  as in (4.5).

5. Practical experiments. This section is dedicated to show experimental results obtained by using the algorithms sosgradientshape (Algorithm 3.1 from Section 3) and sosgradient (Algorithm 4.1 from Section 4). Both algorithms are implemented in MAPLE, and the results are obtained on an Intel Xeon E7-4820 CPU (2GHz) with 1.5 TB of RAM.

In practice, Algorithm univsos2 runs faster than Algorithm univsos1, which is consistent with the theoretical results stated in [26, Theorem 17] and [26, Theorem 24]. In addition, as mentioned in Remark 4.7, it is practically faster to compute SOS decompositions involving rational fractions than polynomials. We compare timings of the slowest algorithm, namely sosgradientshape using univsos1 with the fastest algorithm, namely sosgradient using univsos2. For each algorithm, the first step consists of obtaining h by computing either the shape position (using the procedure Basis in Maple) in sosgradientshape or the zero-dimensional rational parametrization (using the procedure RationalUnivariateRepresentation in Maple) in sosgradient. The runtime of this step is denoted by  $t_h$ . The degree and the bitsize of h are denoted by  $d_h$  and  $\tau_h$ , respectively. The second step outputs an SOS decomposition of the non-negative univariate polynomial h by using either Algorithm univsos1 in sosgradientshape or Algorithm univsos2 in sosgradient. Here,  $t_{sos}$  is the runtime of the second step and  $\tau_{sos}$  is the maximal bitsize of the output polynomials.

				sosgradientshape					sosgradient				
					bitsize 10 <sup>6</sup> -bits		time (s)			bitsize 10 <sup>4</sup> -bits		time (s)	
	n	au	δ	$d_h$	$\tau_h$	$ au_{ t sos}$	$t_h$	$t_{\mathtt{sos}}$	$d_h$	$\tau_h$	$ au_{ exttt{sos}}$	$t_h$	$t_{\mathtt{sos}}$
	2	74	9	32	0.3	8.1	0.1	2.6	36	0.5	1.6	0.1	1.8
	3	149	27	104	2.4	153	1.1	781	108	6.6	13.4	0.2	13.3
İ	4	312	81	320	117	_	399	_	324	88	169	3.9	505
	5	590	243				_		972	940	1306	169	4965

Table 1. Comparison results of output size and performance between Algorithm sosgradientshape and Algorithm sosgradient

In Table 1, we consider random polynomials of fixed degree d=4 with number of variables n being between 2 and 5, generated as follows:  $a^4+b_1^2+\cdots+b_n^2+c+10^6$ , where a (resp.,  $b_i$ , c) is a dense linear (resp., quadratic, cubic) polynomial in n variables. Coefficients of a (resp.,  $b_i$ , c) are chosen randomly in  $\{-1,1\}$  (resp.,  $\{-3,\ldots,3\}$ ,  $\{-1,0,1\}$ ) with respect to the uniform distribution. For  $n \geq 4$ , sosgradientshape failed to provide an SOS decomposition as the execution of univsos1 did not finish after 12 hours of computation, as indicated by the symbol – in the corresponding lines. The underlying reason is that  $\tau_h$  and  $d_h$  are both very large and that the complexity of univsos1 is exponential in the degree of h [26, Theorem 17]. Note that the intermediate polynomials correspond to worst cases, i.e., the maximal possible degree of w is attained, namely deg  $w=(d-1)^n$ , so the degree of h is also maximal, i.e., deg  $h=d(d-1)^n-d$  (resp.  $d(d-1)^n$ ) in sosgradientshape (resp. in sosgradient). For such cases, sosgradient cannot compute decompositions for  $n \geq 4$  (corresponding to deg  $h \geq 324$ ) within 12 hours.

Next, we compare the performance of sosgradient (using univsos2) and Algorithm multivsos [24]. Recall that multivsos is designed to compute SOS decompositions of polynomials lying in the interior of the SOS cone. We report our experimental results in Table 2, obtained with seven classes of 50 randomly generated polynomials. The random polynomials corresponding to the four first rows, with d=4 and  $n=2,\ldots,5$ , are obtained a similar way:  $a^4+b_1^2+b_2^2+c+10^6$ , where a (resp.,  $b_i$ , c) is a dense linear (resp., quadratic, cubic) polynomial in n variables. Coefficients of a (resp.,  $b_i$ , c) are chosen randomly in  $\{\pm 1, \pm 2\}$  (resp.,  $\{-3, \ldots, 3\}$ ,  $\{-1, \ldots, 1\}$ ) with respect to the uniform distribution. The polynomials from the three last rows, with d=6 and n=2,3,4, are constructed in a similar way:  $a^6+b^2+c+10^6$ , where a (resp., b, c) is a dense linear (resp., cubic, cubic) polynomial in n variables. Coefficients of a (resp.,  $b_i$ , c) are chosen randomly in  $\{\pm 1, \pm 2\}$  (resp.,  $\{-3, \ldots, 3\}$ ,  $\{-3, \ldots, 3\}$ 

 $\{-1,\ldots,1\}$ ) with respect to the uniform distribution. Note that here the univariate polynomials generated when running the algorithm do not correspond to the worst case scenario in terms of degree and bitsize. For both algorithms, we denote by  $\tau$  (10<sup>4</sup>-bits) the average bitsize of the output and by t the average runtime in seconds.

	mu]	ltivsos	sosgradient			
d, n	success	au	t	$\tau$	t	
4,2	100%	1.3	0.16	2	2	
4,3	94%	3.7	0.26	18	22	
4,4	38%	8.9	0.18	78	153	
4,5	8%	12.5	0.32	234	630	
6,2	82%	3.5	0.24	45	142	
6,3	0%			160	500	
6,4	0%			744	4662	

Table 2. Comparison of performance between Algorithm sosgradientshape and Algorithm multivsos

From this table, we deduce that when the number of variables n increases, then the rate of success of multivsos decreases. This fact is compatible with Blekherman's theorem [8] which says that if the degree  $d \geq 4$  is fixed then, as the number of variables n grows, the cone of non-negative polynomials is significantly bigger than the cone of SOS polynomials. When multivsos succeeds in computing SOS decompositions, then it provides more concise certificates than sosgradient while being more efficient. However, when d=4 and n=5, multivsos can only decompose four polynomials out of 50 while sosgradient succeeds for all of them. This demonstrates the need of alternative procedures such as sosgradient for polynomials which presumably do not lie in the interior of the SOS cone.

Conclusions and perspectives. We designed and analyzed two algorithms to decompose a non-negative polynomial as an SOS of polynomials/rational fractions modulo the gradient ideal with rational coefficients. The correctness of our framework relies on a generic condition, namely that the gradient ideal of the input polynomial is zero-dimensional and radical. We shall improve the scalability of our algorithms by exploiting the specific structure of the input polynomial, such as correlative [22] or term sparsity [43], symmetries [34] or by using recent improvements on the computation of critical sets when the related system is invariant under group actions [12]. Furthermore, we also plan to extend our algorithms to the constrained case by relying on polar varieties as in [14].

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## Appendix.

Appendix A. Proof of Corollary 2.3. Assume that the system of partial derivatives  $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$  is given by a straight-line program  $\Gamma$  of size L, i.e., the program uses L elementary operations  $+, -, \times$  to evaluate the system from variables  $x_1, \ldots, x_n$  and integers with bitsizes at most  $\max_{i=1}^n \{ \operatorname{ht}(\frac{\partial f}{\partial x_i}) \}$ .

We claim that L is  $O(d\binom{n+d}{d})$ . Indeed, f has at most  $\binom{n+d}{d}$  terms and each term in f is defined by at most d+1 multiplications. Hence, the size of a straight-line program  $\Gamma_f$  which defines f does not exceed  $(d+1)\binom{n+d}{d}$ . By applying Baur-Strassen Theorem [7, Theorem 1], the size L is  $O(d\binom{n+d}{d})$ .

Recall that  $\operatorname{ht}(\frac{\partial f}{\partial x_i}) \leq \log d + \operatorname{ht}(f) = \log d + \tau$ , for  $i = 1, \ldots, n$ . By applying [38, Corollary 2] for the system and a single group of variables, there exists an algorithm that takes the system as in input, and that produces one of the outputs given as in items a)– c) of Corollary 2.3. The number of boolean operations of the algorithm is  $\widetilde{O}(n^2d^{2n}(\log d + \tau + (d-1))(d\binom{n+d}{d}) + n(d-1) + n^2)$ . Reduce this formula, we get (2.2). Furthermore, the polynomials in the output have degree at most  $(d-1)^n$  and bitsize  $\widetilde{O}((d-1)^n(\log d + \tau + n + (d-1))) = \widetilde{O}((d+\tau + n)d^n)$  as claimed.

Appendix B. Proof of the bit complexity in Lemma 2.4. From Corollary 2.3, the degree of w is at most  $(d-1)^n$ , and then deg w' is at most  $(d-1)^n - 1$ . Assume that  $\beta$  is the positive minimal common denominator of all non-zero coefficients of w. Then,  $\beta w$  and  $\beta w'$  belong to  $\mathbb{Z}[t]$ . Clearly,  $\deg(\beta w') = \deg(\beta w) - 1$ ,  $\deg(\beta w) \le (d-1)^n$ , and the bitsize of  $\beta w$  and  $\beta w'$  are bounded by  $\widetilde{O}((d+\tau+n)(d-1)^n)$ . We can apply [41, Theorem 6.52] to  $\beta w$  and  $\beta w'$ . The extended Euclidean algorithm computes the Bézout coefficient, denoted by b, of  $\beta w'$  using

(B.1) 
$$\widetilde{O}(\tau + n + d)^2 (d - 1)^{6n}$$

boolean operations. The bitzise of b is bounded by

(B.2) 
$$O((\tau + n + d)(d-1)^{2n}).$$

Furthermore, one sees that the degree of b satisfies

(B.3) 
$$\deg b \le \deg w - \deg \gcd(w, w') = \deg w \le (d-1)^n.$$

For every  $i=2,\ldots,n$ , we will estimate the bitsize of the polynomial  $b\kappa_i$ . Recall from Corollary 2.3 that  $\deg \kappa_i \leq (d-1)^n$ , hence from (B.3) one has  $\deg b\kappa_i \leq 2(d-1)^n$ . From (B.2), we obtain

$$\operatorname{ht}(b\kappa_i) \le \operatorname{ht}(b) + \operatorname{ht}(\kappa_i) = \widetilde{O}((\tau + n + d)(d - 1)^{2n}) + \widetilde{O}((\tau + n + d)(d - 1)^n).$$

After simplifying the last estimate, the bitsize of  $b\kappa_i$  is bounded from above by  $\tilde{O}((\tau+n+d)(d-1)^{2n})$ . Hence, the bitsize of  $\eta b\kappa_i$ , where  $\eta$  is the minimal common denominator of all non-zero coefficients of  $b\kappa_i$ , can be estimated as follows

$$\operatorname{ht}(\eta b\kappa_i) \le 2\operatorname{ht}(b\kappa_i) \le \widetilde{O}((\tau + n + d)(d - 1)^{2n}).$$

In the proof of Lemma 2.4, we considered the division of  $b\kappa_i$  by w and defined  $v_i = b\kappa_i \mod w$ . Thus, the degree of  $v_i$  is at most deg  $w \leq (d-1)^n$ . From Lemma 2.5, the Euclidean division algorithm computes  $v_i$  using at most

(B.4) 
$$\widetilde{O}((\tau + n + d)(d - 1)^{5n})$$

boolean operations. Thus, the bitsize of  $v_i$  is  $\widetilde{O}((\tau+n+d)(d-1)^{3n})$ , for  $i=2,\ldots,n$ . Therefore, computing  $[w,v_2,\ldots,v_n]$  from the zero-dimensional rational parametrization  $\mathcal{Q}$  of  $V_{\text{grad}}(f)$ , requires

$$\widetilde{O}((\tau+n+d)^2(d-1)^{6n}+(n-1)(\tau+n+d)(d-1)^{5n})$$

boolean operations, as a consequence of (B.1) and (B.4). By applying further simplification, we obtain the desired result (2.3).

The bit complexity results of the two division algorithms used in Lemma 2.5 and Lemma 2.6 are basic but we could not find their proofs in the literature. Here we state these two algorithms and prove estimates for their bit complexities.

**Appendix C. Proof of Lemma 2.5.** Assume that a, b are polynomials in  $\mathbb{Z}[t]$  with deg  $a = d \ge \deg b = m$  and that ht(a), ht(b) are bounded from above by  $\tau$ .

We recall the Euclidean division algorithm in Algorithm C.1 [41, Algorithm 2.5] to compute the quotient q and the remainder r of the division of a by b, i.e., a = qb + r with deg  $r < \deg b$ .

#### Algorithm C.1 Euclidean division algorithm

**Input:** polynomials  $a, b \in \mathbb{Z}[t]$ 

**Output:** polynomials  $q, r \in \mathbb{Q}[t]$  such that a = qb + r and  $\deg r < \deg b$ 

1: Let q := 0 and r := a

2: While  $\deg r > \deg b$  do

3: Let  $h := \operatorname{lc}(r)/\operatorname{lc}(b)t^{\deg r - \deg b}$ 

4: Let q := q + h

5: Let r := r - hb

6: Return q and r

We denote by  $r_i$  (resp.  $q_i, h_i$ ) the value of r (resp. q, h) at the i-th iteration of the while loop from Step 2. The initial values are  $q_1 = 0$  and  $r_1 = a$ . After each iteration of the while loop, the degree of r is strictly decreasing. Hence, the while loop will terminate after at most d-m iterations. When the while loop terminates, the values are  $q = q_{d-m}$  and  $r = r_{d-m}$ . From Step 3, we observe that  $\operatorname{ht}(h_i) \leq \operatorname{ht}(b) + \operatorname{ht}(r_i) = \tau + \operatorname{ht}(r_i)$ , and the number of boolean operations to perform this operation is bounded by  $\tau + \operatorname{ht}(r_i)$ . Since  $q_{i+1} = q_i + h_i$  and  $h_i$  is a monomial satisfying  $\operatorname{deg} h_i > \operatorname{deg} q_i$ , one has

(C.1) 
$$\operatorname{ht}(q_{i+1}) \le \max\{\operatorname{ht}(q_i), \tau + \operatorname{ht}(r_i)\},\$$

and the number of boolean operations to perform the operation in Step 4 is bounded by O(1). For the operation in Step 5, since  $\operatorname{ht}(hb) \leq 2\tau + \operatorname{ht}(r_i)$ , the bitsize of  $r_{i+1}$  is bounded by  $2\tau + \operatorname{ht}(r_i)$ . Moreover, the number of boolean operations to compute  $h_ib$ is  $\widetilde{O}(m(\tau + \operatorname{ht}(r_i)))$ , so  $r_{i+1}$  is also computed in  $\widetilde{O}(m(\tau + \operatorname{ht}(r_i)))$  boolean operations.

We get the recurrence formula  $\operatorname{ht}(r_{i+1}) \leq \operatorname{ht}(r_i) + 2\tau$ , for each  $i = 0, \ldots, d - m$ , with  $\operatorname{ht}(r_0) = \tau$ . It follows that  $\operatorname{ht}(r_i) \leq 2i\tau + \tau$ , for each  $i = 0, \ldots, d - m$ . This yields

$$ht(r) = ht(r_{d-m}) \le 2(d-m)\tau + \tau = O(ht(d-m)).$$

From (C.1), the bitsize of  $q = q_{d-m}$  is also bounded by  $O(\operatorname{ht}(d-m))$ . Furthermore, the number of boolean operations to perform the algorithm is

$$\sum_{i=0}^{d-m} \widetilde{O}((i+1)2m\tau) = \widetilde{O}(m\operatorname{ht}(d-m)^2).$$

This yields the desired estimates.

Appendix D. Algorithm Eliminate and the proof of Lemma 2.6.

ALGORITHM Eliminate. Let us consider  $g \in \mathbb{Q}[x_1][x_2,\ldots,x_n]$ , with deg g=d (in variables  $x_2,\ldots,x_n$ ) and ht $(g)=\tau_g$ , and the list of rational fractions:

$$G = [x_2 - \frac{a_2}{a_0}, \dots, x_n - \frac{a_n}{a_0}],$$

where  $a_0, a_2, \ldots, a_n$  are polynomials in  $\mathbb{Q}[x_1]$ ,  $a_0 \neq 0$ , and  $\operatorname{ht}(a_i) \leq \tau_a$  for  $i = 0, 2, \ldots, n$ . Recall that  $\mathbb{Q}(x_1)$  is the field of rational fractions in variable  $x_1$  with coefficients in  $\mathbb{Q}$ . Let  $x_2 < \cdots < x_n$  be a lexicographic monomial order on  $\mathbb{Q}(x_1)[x_2, \ldots, x_n]$ .

Algorithm Eliminate outputs the quotients  $\phi_2, \ldots, \phi_n \in \mathbb{Q}(x_1)[x_2, \ldots, x_n]$  and the remainder  $r \in \mathbb{Q}(x_1)$  of the multivariate division of g by the list G satisfying

(D.1) 
$$g = \sum_{i=2}^{n} \phi_i (x_i - \frac{a_i}{a_0}) + r.$$

## Algorithm D.1 Elimination algorithm

```
Eliminate := \operatorname{proc}(g, a_0, a_2, \dots, a_n)

Input: n+1 polynomials g \in \mathbb{Q}[x_1][x_2, \dots, x_n], a_0, a_2, \dots, a_n \in \mathbb{Q}[x_1]

Output: \phi_2, \dots, \phi_n in \mathbb{Q}(x_1)[x_2, \dots, x_n] and r \in \mathbb{Q}(x_1) satisfying (D.1)

1: Set r_{n+1} := g

2: For i = n to 2 do

3: Compute \phi_i := \operatorname{quo}(r_i, x_i - \frac{a_i}{a_0}, x_i)

4: Substitute x_i by \frac{a_i}{a_0} in r_{i+1} to define r_i := r_{i+1}(x_1, \dots, x_{i-1}, \frac{a_i}{a_0})

5: Set r := r_2

6: Return \phi_2, \dots, \phi_n, and r
```

In Step 3,  $\phi_i$  is the quotient of the univariate division (in the variable  $x_i$ ) of  $r_i$  by  $x_i - \frac{a_i}{a_0}$ . Since the degree of  $x_i$  in  $x_i - \frac{a_i}{a_0}$  is 1,  $\phi_i$  belongs to  $\mathbb{Q}(x_1)[x_2, \dots, x_i]$ . The remainder  $r_i$  of the division in Step 3 is given in Step 4 after replacing  $x_i$  by  $\frac{a_i}{a_0}$  in  $r_{i+1}$ ; hence one has  $r_i \in \mathbb{Q}(x_1)[x_2, \dots, x_{i-1}]$ . After Steps 3-4, we obtain

(D.2) 
$$r_i = \phi_i \left( x_i - \frac{a_i}{a_0} \right) + r_{i-1}.$$

Therefore, after Step 5, we get  $g = \sum_{i=2}^{n} \phi_i(x_i - \frac{a_i}{a_0}) + r$ , with  $r \in \mathbb{Q}(x_1)$ . Based on Buchberger's Criterion [9], we can show that the system of n-1 polynomials  $[x_2 - \frac{a_2}{a_0}, \dots, x_n - \frac{a_n}{a_0}]$  is a Gröbner basis of the ideal generated by this system w.r.t. the order  $<_{lex}$  in  $\mathbb{Q}(x_1)[x_2, \dots, x_n]$ . Hence,  $\phi_2, \dots, \phi_n$  are defined uniquely. The correctness of the algorithm is proved.

THE PROOF OF LEMMA 2.6. Now we estimate the bitsizes of  $\phi_i$ , for  $i=2,\ldots,n$ . From the definition of  $r_i$  in Step 4, one sees that  $\operatorname{ht}(r_i) \leq \operatorname{ht}(r_{i+1}) + 2d\tau_a$ . Since  $\operatorname{ht}(r_{n+1}) = \tau_g$ , the bitsize of  $r_i$  is bounded from above by  $\tau_g + 2(n-1)d\tau_a$ . The relation (D.2) leads to  $\operatorname{ht}(\phi_i) \leq \operatorname{ht}(r_{i+1} - r_i) + \operatorname{ht}(x_i - \frac{a_i}{a_0})$ . Because of  $\operatorname{ht}(r_{i+1} - r_i) \leq \max\{\operatorname{ht}(r_{i+1}),\operatorname{ht}(r_i)\}$ , and  $\operatorname{ht}(\frac{a_i}{a_0}) \leq 2\tau_a$ , we get  $\operatorname{ht}(\phi_i) \leq \tau_g + 2(nd - d + 1)\tau_a$ . It follows that  $\operatorname{ht}(\phi_i) = \widetilde{O}(\tau_g + nd\tau_a)$ .

We see that the number of boolean operations to perform Steps 3 and 4 are  $\widetilde{O}(\tau_g + nd\tau_a)$  and O(1), respectively. The for loop in Step 2 has n-1 steps. Therefore, the number of boolean operations to perform the loop is  $\widetilde{O}(n\tau_g + n^2d\tau_a)$ . This is also the number of boolean operations that Algorithm Eliminate uses.

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