NON-UNIQUENESS IN LAW FOR TWO-DIMENSIONAL NAVIER-STOKES EQUATIONS WITH DIFFUSION WEAKER THAN A FULL LAPLACIAN

KAZUO YAMAZAKI

Abstract. We study the two-dimensional Navier-Stokes equations forced by random noise with a diffusive term generalized via a fractional Laplacian that has a positive exponent strictly less than one. Because intermittent jets are inherently three-dimensional, we instead adapt the theory of intermittent form of the two-dimensional stationary flows to the stochastic approach presented by Hofmanová, Zhu & Zhu (2019, arXiv:1912.11841 [math.PR]) and prove its non-uniqueness in law.

Keywords: convex integration; fractional Laplacian; Navier-Stokes equations; non-uniqueness; random noise.

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1. Introduction

1.1. **Motivation from physics.** The study of turbulence was pioneered by Novikov [47] more than half a century ago. Motivations to investigate the two-dimensional (2-d) turbulence include applications in meteorology and atmospheric sciences, and its attraction from researchers that led to remarkable progress can be accredited to many reasons: the 2-d flows are easier to simulate than the counterpart in the three-dimensional (3-d) case; vorticity, in addition to kinetic energy, becomes a bounded quantity allowing more flexibility in directions to explore. Indeed, the 2-d turbulence has been extensively studied theoretically (e.g., [37]), numerically (e.g., [3]), as well as experimentally (e.g., [49]).

Various forms of dissipation have been introduced in the physics literature: frictional dissipation in [50]; fractional Laplacian $(-\Delta)^m$ as a Fourier operator with its Fourier symbol of $|\xi|^{2m}$ so that $\widehat{(-\Delta)^m f}(\xi) = |\xi|^{2m}\hat{f}(\xi)$ in the study of surface quasi-geostrophic equations (e.g., [16, Equation (1)]). In fact, the study of the Navier-Stokes (NS) equations with diffusive term in the latter form, to which we shall hereafter refer as the generalized NS (GNS) equations (1), can be traced back as far as [40, Equation (8.7) on pg. 263] in 1959 by Lions. The purpose of this manuscript is to prove a certain non-uniqueness for the 2-d GNS equations forced by random noise which we introduce next.

1.2. **Previous results.** Throughout this manuscript we define $\mathbb{T}^n \triangleq [-\pi,\pi]^n$ to be the principal spatial domain for $x = (x^1,\dots,x^n)$, denote $\partial_t \triangleq \frac{\partial}{\partial t}$, $\nabla \triangleq (\partial_{x^1},\dots,\partial_{x^n})$, as well as $u \triangleq (u^1,\dots,u^n)$, and π that map from $\mathbb{R}_+ \times \mathbb{T}^n$ as the n-dimensional (n-d) velocity vector and pressure scalar fields, respectively. We let $\nu \geq 0$ represent the viscosity coefficient so that the GNS equations read

$$\partial_t u + \nu(-\Delta)^m u + \operatorname{div}(u \otimes u) + \nabla \pi = 0, \quad \nabla \cdot u = 0, \quad t > 0, \tag{1}$$

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which recovers the classical NS equations when $m = 1$ and $\nu > 0$, as well as the Euler equations when $\nu = 0$. We call $u \in C_t L_x^2$ a weak solution to (1) over $[0, T]$ if $u(t, \cdot)$ is weakly divergence-free, is mean-zero; i.e., $\int_{\mathbb{T}^n} u(t, x) dx = 0$, and satisfies (1) distributionally against a smooth and divergence-free function. A Leray-Hopf weak solution, only in case $\nu > 0$, due to [35, 39] requires an additional regularity of $L_t^2 \dot{H}_x^m$ and must satisfy an energy inequality

$$\frac{1}{2}\|u(t)\|_{L_x^2}^2 + \nu \int_0^t \|u(s)\|_{\dot{H}_x^m}^2 ds \leq \frac{1}{2}\|u(0)\|_{L_x^2}^2 \tag{2}$$

for all $t \geq 0$ (see [9, Definitions 3.1, 3.5, and 3.6] for precise statements). Due to the rescaling property of the GNS equations that $(u_\lambda, \pi_\lambda)(t, x) \triangleq (\lambda^{2m-1} u, \lambda^{4m-2}\pi)(\lambda^{2m} t, \lambda x)$ solves (1) if $(u, \pi)(t, x)$ solves it, a standard classification states that (1) is sub-critical, critical and super-critical with respect to $L^2(\mathbb{T}^n)$-norm if $m > \frac{1}{2} + \frac{n}{4}, m = \frac{1}{2} + \frac{n}{4}$, and $m < \frac{1}{2} + \frac{n}{4}$, respectively.

Only a decade after [40], Lions (see [41, Remark 6.11 on pg. 96]) already claimed the uniqueness of a Leray-Hopf weak solution when $\nu > 0$ and $m \geq \frac{1}{2} + \frac{n}{4}$. It has been more than 50 years since then, and we still find this threshold to be sharp; specifically, except a logarithmic improvement by Tao [55] (and also [2] for further logarithmic improvements), it is not known whether (1) with $\nu > 0$ and $m < \frac{1}{2} + \frac{n}{4}$ for $n \geq 3$ admits a unique solution that emanates from a smooth initial data and preserves the initial regularity or not (e.g. see [56, Theorem 4.1] for such a result under a smallness constraint on initial data). The case $n = 2$ offers a strikingly different picture when initial data has high regularity. Indeed, Yudovich [60] proved that if the vorticity $\nabla \times u$ belongs initially to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then even the 2-d Euler equations admit a globally unique solution, essentially due to the fact that the nonlinear term vanishes upon an $L^p(\mathbb{R}^2)$-estimate of the vorticity for any $p \in [2, \infty]$ (e.g., [44, pg. 320]). That being said, starting from an arbitrary initial data in L_x^2, the lack of diffusion and therefore a lack of high regularity creates an obstacle in constructing a weak solution via a classical argument relying on Aubin-Lions compactness lemma (e.g. [41, 53]).

We now discuss the recent developments on Onsager's conjecture which led to a better understanding of equations of fluid and various new techniques. In 1949 a chemist and a physicist Onsager [48] conjectured the following dichotomy in any spatial dimension $n \geq 2$:

- every weak solution to the n-d Euler equations with Hölder regularity in space of exponent $\alpha > \frac{1}{3}$, i.e., C_x^α, conserves kinetic energy $\frac{1}{2}\|u(t)\|_{L_x^2}^2$;
- for any $\alpha \leq \frac{1}{3}$ there exists a weak solution in C_x^α that dissipates kinetic energy $\frac{1}{2}\|u(t)\|_{L_x^2}^2$.

The case $\alpha > \frac{1}{3}$ proved to be easier to demonstrate, settled partially by Eyink [26] and then fully by Constantin, E, and Titi [17]. Towards Onsager's conjecture in case $\alpha \leq \frac{1}{3}$, Scheffer [51] and subsequently Shnirelman [52] proved the existence of a weak solution to 2-d Euler equations with compact support in space and time so that kinetic energy is both created and destroyed; however, the solutions in [51, 52] were only in $L_T^2 L_x^2$ and thus far from the threshold of $C_x^\alpha, \alpha \leq \frac{1}{3}$. The remarkable series of breakthroughs which unfolded next were inspired by the work of Nash [46] who proved the C^1 isometric embedding by constructing a sequence of short isometric embeddings, each of which fails to be isometric by a certain error that vanishes in the limit. Gromov considered the work of Nash, as well as that of Kuiper [38], as part of h-principle ([31, pg. 3]) and initiated the theory of convex integration [31, Part 2.4]; we refer to [24] for further discussions on the h-principle.

After Müller and Šverák [45] extended the convex integration to Lipschitz maps, De Lellis and Székelyhidi Jr. [22] reformulated the Euler equations as a differential inclusion and improved the results of [51, 52] by constructing a weak solution in $L_T^\infty L_x^\infty$ with compact support in space and time in any spatial dimension $n \geq 2$ (see also [23]). Subsequently, in [25] they proved the existence of weak solutions to 3-d Euler equations in $C([0,T] \times \mathbb{T}^3)$ which dissipate the kinetic energy through a novel application of Beltrami flows. Together with Buckmaster and Isett in [6], they improved the regularity of the solution up to $C_{t,x}^\alpha$ for any $\alpha < \frac{1}{5}$, where we write $f \in C_{t,x}^\alpha$ if there exists $C \geq 0$ such that

$$|f(t+\Delta t, x+\Delta x) - f(t,x)| \leq C(|\Delta t| + |\Delta x|)^\alpha \quad \text{uniformly in } t, x, \Delta t, \text{ and } \Delta x$$

(see also (9)). At last, via a certain gluing approximation technique and Mikado flows, Isett [36] proved that for any $\alpha < \frac{1}{3}$ there exists a non-zero weak solution to n-d Euler equations for $n \geq 3$ in $C_{t,x}^\alpha$ that fails to conserve kinetic energy ([36, Theorem 1] only states the claim for $n = 3$, but [36, pg. 877] claims that it can be extended to any $n \geq 3$). Integrating ideas of intermittency from turbulence to Beltrami flows and constructing intermittent Beltrami waves, Buckmaster and Vicol [8] proved the non-uniqueness of weak solutions to the 3-d NS equations in the class $C_T H_x^\beta$ for some $\beta > 0$, which can be seen to be quite small from its proof. Relying on the intermittent Beltrami waves, Luo and Titi [43] extended the result of [8] up to Lions' exponent $m < \frac{5}{4}$ for (1) when $n = 3$. Mimicking the benefits of Mikado flows, Buckmaster, Colombo, and Vicol [7] introduced intermittent jets to prove non-uniqueness for a class of weak solutions to 3-d GNS equations with $m < \frac{5}{4}$ which have bounded kinetic energy, integrable vorticity, and are smooth outside a fractal set of singular times with Hausdorff dimension strictly less than one.

As already mentioned in Subsection 1.1, the study of NS equations forced by random noise, to which hereafter we refer as the stochastic NS (SNS) equations, has a long history since [47] (see also [4]). Our focus will be on the following stochastic GNS (SGNS) equations: for $x \in \mathbb{T}^n$,

$$du + (-\Delta)^m u dt + \operatorname{div}(u \otimes u) dt + \nabla \pi dt = F(u) dB, \quad \nabla \cdot u = 0, \quad t > 0, \tag{3}$$

where $F(u)dB$ represents the random noise, to be specified subsequently. Via a probabilistic Galerkin approximation and variations of Aubin-Lions compactness results aforementioned, Flandoli and Gatarek [28] proved the existence of a weak solution to the n-d SNS equations for $n \geq 2$ under some assumptions on the noise; their solution has the regularity of $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ but does not necessarily satisfy the energy inequality (see [28, Definition 3.1] and also [27, Definition 4.3]). Via the approach of martingale problem due to Stroock and Varadhan [54], Flandoli and Romito constructed a Leray-Hopf weak solution to the 3-d SNS equations; i.e., the solutions constructed therein have the regularity of $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ and satisfy a stochastic analogue of the energy inequality (see [29, MP3 in Definition 3.3]). Very recently, Hofmanová, Zhu, and Zhu [33] adapted the convex integration approach through intermittent jets from [9, Chapter 7] to the 3-d SNS equations and proved the non-uniqueness in law within a class of weak solutions, which also implies the lack of path-wise uniqueness by Yamada-Watanabe theorem (see also [5, 13, 34] for probabilistic convex integration on stochastic Euler equations); we must emphasize that their result does not extend to the Leray-Hopf weak solution from [29].

Remark 1.1. *It is worth pointing out that the proof of non-uniqueness in the stochastic case has a layer of complexity that is absent in the deterministic case in the following manner. For example, Buckmaster and Vicol in* [8, Theorem 1.2] *specifically proved that there exists* $\beta > 0$ *such that for any non-negative smooth function* $e(t)\colon [0,T] \mapsto \mathbb{R}_+ \cup \{0\}$, *there exists $u \in C_T H_x^{\beta}$ that is a weak solution to the NS equations and satisfies $\|u(t)\|_{L_x^2}^2 = e(t)$ for all $t \in [0,T]$. One may take e.g. $e(t) = e^t - 1$ so that $e(0) = 0$. Because $u \equiv 0$ for all $(t,x) \in [0,T] \times \mathbb{T}^3$ solves the NS equations and satisfies $\|u(0)\|_{L_x^2}^2 = 0$, this immediately deduces non-uniqueness. This approach clearly fails in the stochastic case because $u \equiv 0$ for all $(t,x) \in [0,T] \times \mathbb{T}^3$ does not solve the stochastic NS equations due to the presence of the noise (see* [27, Remark 4.16] *for a similar discussion). More precisely, particularly in the case of an additive noise, one may split* (3) *to a linear stochastic PDE solved by z and the rest of the terms solved by v as in* (37a)-(37b) *in hope to adapt the proof of* [8, Theorem 1.2] *to the equation of v; unfortunately, $v \equiv 0$ does not solve* (37b) *as aforementioned. Another major difficulty that arises in the stochastic case will be discussed in Remark 1.2.*

Similarly to our discussion in Subsection 1.1, the 2-d SNS equations have received a considerable amount of attention from researchers who have produced a wealth of results many of which remain open in the 3-d case. Path-wise uniqueness, and consequently uniqueness in law due to Yamada-Watanabe theorem, of the aforementioned weak solution with regularity $L_t^{\infty} L_x^2 \cap L_t^2 \dot{H}_x^1$ that does not necessarily satisfy the energy inequality which was constructed in [28] are well-known. In the case of an additive noise, upon considering the difference of two possible solutions, the noise cancels out and thus a deterministic approach immediately implies uniqueness (see [27, Exercise 3.1 on p. 72]); in the case of a multiplicative noise we refer to [15, Theorem 2.4]. Same uniqueness results for the Leray-Hopf weak solutions to the 2-d SNS equations directly follow. We also refer to [15, Theorem 3.2] and [32] concerning large deviation principle and ergodicity with hypo-elliptic noise, respectively. The purpose of this manuscript is to prove the non-uniqueness in law, and therefore a lack of path-wise uniqueness, for (3) when $n = 2$ and $m \in (0,1)$, which has been studied by many authors previously (e.g., [18]).

Remark 1.2. *As we remarked already, the theory of global well-posedness for* (1) *in the 2-d case is significantly richer than that in the 3-d case. Vice versa, proving non-uniqueness in the 2-d case should present considerable difficulty, not seen in the 3-d case. A natural approach to prove the non-uniqueness in law for* (3) *with $n = 2$ and $m \in (0,1)$ will be to try to follow the arguments in* [33] *on the 3-d SNS equations. Concerning the fractional Laplacian, we can follow the arguments in* [58] *in which the analogous result was proven for* (3) *when $n = 3$ and $m \in (\frac{13}{20}, \frac{5}{4})$.*

First major obstacle arises in the fact that intermittent jets, utilized in [33, 58] *following* [9, Chapter 7]*, are inherently 3-d in space and thus inapplicable to* (3) *when $n = 2$; we recall that the lack of a suitable replacement for Mikado flows in the 2-d case is precisely the reason the case $n = 2$ was left out in the resolution of Onsager's conjecture by Isett (see* [36, pg. 877]*). Fortunately, a 2-d analogue of the 3-d Beltrami flows from* [25] *was already established by Choffrut, De Lellis, and Székelyhidi Jr.* [14]*, to which we refer as 2-d stationary flows. Moreover, its intermittent form, to which we refer as 2-d intermittent stationary flows, was very recently introduced by Luo and Qu* [42]*. Thus, a good candidate for strategy now is to somehow adapt the application of 2-d intermittent stationary flows in* [42] *to the stochastic setting in* [33].

Second major obstacle that arises in this endeavor is that the arguments in [42] *follow closely those of* [8] *and not* [9, Chapter 7]*, quite naturally because the 2-d intermittent stationary flows is an extension of the intermittent Beltrami waves in* [8]*, not intermittent jets in* [9, Chapter 7]. *It turns out that some of the crucial estimates achieved in* [8, 42] *seem to be difficult in the stochastic setting. E.g., while* [8, Equation (2.4)] *and* [42, Equation (2.13)] *claim certain bounds on the $C_{t,x}^1$-norm of Reynolds stress, our Reynolds stress in*

(116) *includes* R_{com2} *defined in* (100e) *that consists of* z*, and* z *is known to be only in* $C_t^{\frac{1}{2}-2\delta}$ *for* $\delta > 0$ *from* (49). *Therefore, obtaining an analogous estimate to* [8, Equation (2.4)] *and* [42, Equation (2.13)] *seems to be completely out of reach. Thus, our task is not only to apply the theory of 2-d intermittent stationary flows from* [42] *but consider an extension of the arguments in* [42] *to that of* [9, Chapter 7] *and then adjust that in the stochastic setting of* [33]*, while simultaneously considering the approach of* [58] *to treat the fractional Laplacian. We will carefully define various parameters, all of which depend on the value of* m *(e.g.,* (65)-(70)*, and* (117)*). Our proofs are inspired by those of* [8, 9, 33, 42] *while on various occasions we need to make crucial modifications (e.g., Remarks 4.1-4.4).*

2. Statement of main results

Only for simplicity of presentations, we assume $\nu = 1$ hereafter. Following [33] we consider two types of random noises within (3): additive; linear multiplicative.

2.1. **The case of an additive noise.**

In the case of an additive noise, we consider (3) with $n = 2$, $F \equiv 1$, and B to be a GG^*-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where G is a certain Hilbert-Schmidt operator to be described in more detail subsequently (see (10)), and the asterisk denotes the adjoint operator. Finally, $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration generated by B.

Theorem 2.1. *Suppose that* $n = 2, F \equiv 1, m \in (0,1)$*,* B *is a* GG^**-Wiener process, and* $Tr((-\Delta)^{2-m+2\sigma}GG^*) < \infty$ *for some* $\sigma > 0$*. Then given* $T > 0, K > 1$*, and* $\kappa \in (0,1)$*, there exists* $\varepsilon \in (0,1)$ *and a* $\boldsymbol{P}$*-almost surely (a.s.) strictly positive stopping time* $\mathfrak{t}$ *such that* $\boldsymbol{P}(\{\mathfrak{t} \geq T\}) > \kappa$ *and the following is additionally satisfied. There exists an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* u *that is a weak solution to* (3) *starting from a deterministic initial condition* u^{in}*, satisfies*

$$esssup_{\omega\in\Omega} \sup_{s\in[0,\mathfrak{t}]} \|u(s,\omega)\|_{H_x^\varepsilon} < \infty, \tag{4}$$

and on the set $\{\mathfrak{t} \geq T\}$*,*

$$\|u(T)\|_{L_x^2} > K\|u^{in}\|_{L_x^2} + K(T Tr(GG^*))^{\frac{1}{2}}. \tag{5}$$

Remark 2.1. *For the 3-d SGNS equations* (3) *with* $m \in (\frac{13}{20}, \frac{5}{4})$*,* [58] *required a hypothesis of* $Tr((-\Delta)^{\frac{5}{2}-m+2\sigma}GG^*) < \infty$ *(see* [58, Remark 2.1]*). Here in the 2-d case, we need* $Tr((-\Delta)^{2-m+2\sigma}GG^*) < \infty$ *for the purpose of Proposition 4.4.*

Theorem 2.2. *Suppose that* $n = 2, F \equiv 1, m \in (0,1)$*,* B *is a* GG^**-Wiener process, and* $Tr((-\Delta)^{2-m+2\sigma}GG^*) < \infty$ *for some* $\sigma > 0$*. Then non-uniqueness in law holds for* (3) *on* $[0,\infty)$*. Moreover, for all* $T > 0$ *fixed, non-uniqueness in law holds for* (3) *on* $[0, T]$.

2.2. **The case of a linear multiplicative noise.**

In the case of a linear multiplicative noise, we will consider $F(u) = u$ and B to be an $\mathbb{R}$-valued Wiener process on $(\Omega, \mathcal{F}, \mathbf{P})$.

Theorem 2.3. *Suppose that* $n = 2, F(u) = u$*,* $m \in (0,1)$*, and* B *is an* $\mathbb{R}$*-valued Wiener process on* $(\Omega, \mathcal{F}, \boldsymbol{P})$*. Then given* $T > 0, K > 1$*, and* $\kappa \in (0,1)$*, there exists* $\varepsilon \in (0,1)$ *and a* $\boldsymbol{P}$*-a.s. strictly positive stopping time* $\mathfrak{t}$ *such that* $\boldsymbol{P}(\{\mathfrak{t} \geq T\}) > \kappa$ *and the following is additionally satisfied. There exists an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* u *that is a weak solution to* (3) *starting from a deterministic initial condition* u^{in}*, satisfies*

$$esssup_{\omega\in\Omega} \sup_{s\in[0,\mathfrak{t}]} \|u(s,\omega)\|_{H_x^\varepsilon} < \infty, \tag{6}$$

and on the set $\{t \geq T\}$,

$$\|u(T)\|_{L_x^2} > Ke^{\frac{T}{2}}\|u^{in}\|_{L_x^2}. \tag{7}$$

Theorem 2.4. *Suppose that* $n = 2, F(u) = u$, $m \in (0,1)$*, and* B *is an* $\mathbb{R}$*-valued Wiener process on* $(\Omega, \mathcal{F}, \boldsymbol{P})$*. Then non-uniqueness in law holds for* (3) *on* $[0,\infty)$*. Moreover, for any* $T > 0$ *fixed, non-uniqueness in law holds for* (3) *on* $[0, T]$.

Remark 2.2. *After this work was completed, Cheskidov and Luo* [12] *proved non-uniqueness for the 2-d deterministic Navier-Stokes equations in the class of* $C_t L_x^p$ *for* $p \in [1,2)$*. We point out that on one hand, they proved non-uniqueness with a full Laplacian while Theorems 2.1-2.4 are concerned with the GNS equations diffused via* $(-\Delta)^m, m \in (0,1)$*. On the other hand, the spatial regularity of the solutions constructed in* [12] *are in* L_x^p *for* $p \in [1,2)$ *while those in Theorems 2.1-2.4 are in* H_x^ϵ*, although for* $\epsilon \in (0,1)$ *very small, as can be seen from their proofs.*

The rest of this manuscript is organized as follows: Section 3 with a minimum amount of notations, assumptions, and past results; Section 4 with proofs of Theorems 2.1 and 2.2; Section 5 with proofs of Theorems 2.3 and 2.4; Appendix with additional past results and details of some proofs.

3. Preliminaries

We denote $\mathbb{N} \triangleq \{1,2,\dots,\}$ and $\mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}$. We write $A \lesssim_{a,b} B$ and $A \approx_{a,b} B$ to imply the existence of a constant $C = C(a,b) \geq 0$ such that $A \leq CB$ and $A = CB$, respectively. We write $A \overset{(\cdot)}{\lesssim} B$ to indicate that this inequality is due to an equation $(\cdot)$. For any $\mathbb{R}^2$-valued maps f and g, we denote a tensor product by $f \otimes g$ while its trace-free part by

$$f \mathring{\otimes} g \triangleq \begin{pmatrix} f^1 g^1 - \frac{1}{2} f \cdot g & f^1 g^2 \\ f^2 g^1 & f^2 g^2 - \frac{1}{2} f \cdot g \end{pmatrix}. \tag{8}$$

We write for $p \in [1,\infty]$

$$\|f\|_{L^p} \triangleq \|f\|_{L_t^\infty L_x^p}, \ \|f\|_{C^N} \triangleq \|f\|_{L_t^\infty C_x^N} \triangleq \sum_{0 \leq |\alpha| \leq N} \|D^\alpha f\|_{L^\infty}, \ \|f\|_{C_{t,x}^N} \triangleq \sum_{0 \leq n + |\alpha| \leq N} \|\partial_t^n D^\alpha f\|_{L^\infty}. \tag{9}$$

We also define $L_\sigma^2 \triangleq \{f \in L_x^2 : \nabla \cdot f = 0\}$, reserve $\mathbb{P} \triangleq \text{Id} - \nabla \Delta^{-1} \nabla \cdot$ as the Leray projection operator, and $\mathbb{P}_{\leq r}$ to be a Fourier operator with a Fourier symbol of $1_{|\xi| \leq r}(\xi)$. For any Polish space H, we write $\mathcal{B}(H)$ to represent the σ-algebra of Borel sets in H. We denote a mathematical expectation with respect to (w.r.t.) any probability measure P by $\mathbb{E}^P$. We represent an $L^2(\mathbb{T}^2)$-inner product, a cross variation of A and B, and a quadratic variation of A respectively by $\langle A, B\rangle$, $\langle\langle A, B\rangle\rangle$, and $\langle\langle A\rangle\rangle \triangleq \langle\langle A, A\rangle\rangle$. We define $\mathcal{P}(\Omega_0)$ as the set of all probability measure on $(\Omega_0, \mathcal{B})$ where $\Omega_0 \triangleq C([0,\infty); H^{-3}(\mathbb{T}^2)) \cap L_{\text{loc}}^\infty([0,\infty); L_\sigma^2)$ and $\mathcal{B}$ is the Borel σ-field of Ω_0 from the topology of locally uniform convergence on Ω_0. We define the canonical process $\xi : \Omega_0 \mapsto H^{-3}(\mathbb{T}^2)$ by $\xi_t(\omega) \triangleq \omega(t)$. Similarly, for $t \geq 0$ we define $\Omega_t \triangleq C([t,\infty); H^{-3}(\mathbb{T}^2)) \cap L_{\text{loc}}^\infty([t,\infty); L_\sigma^2)$ and the following Borel σ-algebras for $t \geq 0$: $\mathcal{B}^t \triangleq \sigma(\{\xi(s) : s \geq t\})$; $\mathcal{B}_t^0 \triangleq \sigma(\{\xi(s) : s \leq t\})$; $\mathcal{B}_t \triangleq \cap_{s>t} \mathcal{B}_s^0$. For any Hilbert space U we denote by $L_2(U, L_\sigma^2)$ the space of all Hilbert-Schmidt operators from U to L_σ^2 with the norm $\|\cdot\|_{L_2(U, L_\sigma^2)}$. We require $F : L_\sigma^2 \mapsto L_2(U, L_\sigma^2)$ to be $\mathcal{B}(L_\sigma^2)/\mathcal{B}(L_2(U, L_\sigma^2))$-measurable and that it satisfies for any $\phi \in C^\infty(\mathbb{T}^2) \cap L_\sigma^2$

$$\|F(\phi)\|_{L_2(U, L_\sigma^2)} \leq C(1 + \|\phi\|_{L_x^2}) \quad \text{and} \quad \lim_{n\to\infty} \|F(\theta_n)^* \phi - F(\theta)^* \phi\|_U = 0 \tag{10}$$

for some constant $C \geq 0$ if $\lim_{n \to \infty} \|\theta_n - \theta\|_{L_x^2} = 0$.

The following notations will be useful in the case of a linear multiplicative noise. We assume the existence of another Hilbert space U_1 such that the embedding $U \hookrightarrow U_1$ is Hilbert-Schmidt. We define $\bar{\Omega} \triangleq C([0,\infty); H^{-3}(\mathbb{T}^2) \times U_1) \cap L^\infty_{\text{loc}}([0,\infty); L^2_\sigma \times U_1)$ and $\mathcal{P}(\bar{\Omega})$ as the set of all probability measures on $(\bar{\Omega}, \bar{\mathcal{B}})$, where $\bar{\mathcal{B}}$ is the Borel σ-algebra on $\bar{\Omega}$. Analogously we define the canonical process on $\bar{\Omega}$ as $(\xi,\theta)\colon \bar{\Omega} \mapsto H^{-3}(\mathbb{T}^2) \times U_1$ by $(\xi_t(\omega), \theta_t(\omega)) \triangleq \omega(t)$. We extend the previous definitions of $\mathcal{B}^t, \mathcal{B}^0_t$ and $\mathcal{B}_t$ to $\bar{\mathcal{B}}^t \triangleq \sigma(\{(\xi,\theta)(s)\colon s \geq t\})$, $\bar{\mathcal{B}}^0_t \triangleq \sigma(\{(\xi,\theta)(s)\colon s \leq t\})$, and $\bar{\mathcal{B}}_t \triangleq \cap_{s>t} \bar{\mathcal{B}}^0_s$ for $t \geq 0$, respectively.

Next, we describe some notations and results concerning the 2-d intermittent stationary flows introduced in [14] (e.g., [14, Lemma 4]) and extended in [42]. We let

$$\Lambda^+ \triangleq \{\frac{1}{5}(3e_1 \pm 4e_2), \frac{1}{5}(4e_1 \pm 3e_2)\} \quad \text{and} \quad \Lambda^- \triangleq \{\frac{1}{5}(-3e_1 \mp 4e_2), \frac{1}{5}(-4e_1 \mp 3e_2)\}, \tag{11}$$

i.e. $\Lambda^- = -\Lambda^+$, and $\Lambda \triangleq \Lambda^+ \cup \Lambda^-$, where e_j for $j \in \{1,2\}$ is a standard basis of $\mathbb{R}^2$. It follows immediately that $\Lambda \subset \mathbb{S}^1 \cap \mathbb{Q}^2$, $5\Lambda \subset \mathbb{Z}^2$, and

$$\min_{\zeta,\zeta' \in \Lambda\colon \zeta \neq -\zeta'} |\zeta + \zeta'| \geq \frac{\sqrt{2}}{5} \tag{12}$$

(cf. [8, pg. 110], [43, Equation (9)]). For all $\zeta \in \Lambda$ and any frequency parameter $\lambda \in 5\mathbb{N}$, we define b_ζ and its potential ψ_ζ as

$$b_\zeta(x) \triangleq b_{\zeta,\lambda}(x) \triangleq i\zeta^\perp e^{i\lambda\zeta\cdot x}, \quad \psi_\zeta(x) \triangleq \psi_{\zeta,\lambda}(x) \triangleq \frac{1}{\lambda} e^{i\lambda\zeta\cdot x} \tag{13}$$

(cf. [14, Equation (14)]). It follows that for all $N \in \mathbb{N}_0$,

$$b_\zeta(x) = \nabla^\perp \psi_\zeta(x), \quad \nabla \cdot b_\zeta(x) = 0, \quad \nabla^\perp \cdot b_\zeta(x) = \Delta\psi_\zeta(x) = -\lambda^2 \psi_\zeta(x), \tag{14a}$$

$$\overline{b_\zeta}(x) = b_{-\zeta}(x), \quad \overline{\psi_\zeta}(x) = \psi_{-\zeta}(x), \quad \|b_\zeta\|_{C^N_x} \overset{(9)}{\leq} (N+1)\lambda^N, \quad \|\psi_\zeta\|_{C^N_x} \overset{(9)}{\leq} (N+1)\lambda^{N-1}. \tag{14b}$$

Lemma 3.1. (Geometric lemma from [42, Lemma 4.1]; cf. [25, Lemma 3.2], [14, Lemma 6]) Denote by $\mathcal{M}$ the linear space of 2×2 symmetric trace-free matrices. Then there exists a set of positive smooth functions $\{\gamma_\zeta \in C^\infty(\mathcal{M})\colon \zeta \in \Lambda\}$ such that for each $\mathring{R} \in \mathcal{M}$,

$$\gamma_{-\zeta}(\mathring{R}) = \gamma_\zeta(\mathring{R}), \quad \mathring{R} = \sum_{\zeta\in\Lambda} (\gamma_\zeta(\mathring{R}))^2 (\zeta \mathring{\otimes} \zeta), \quad \text{and} \quad \gamma_\zeta(\mathring{R}) \lesssim (1 + |\mathring{R}|)^{\frac{1}{2}}. \tag{15}$$

For convenience we set

$$C_\Lambda \triangleq 2\sqrt{12}(4\pi^2+1)^{\frac{1}{2}}|\Lambda| \quad \text{and} \quad M \triangleq C_\Lambda \sup_{\zeta\in\Lambda} (\|\gamma_\zeta\|_{C(B_{\frac{1}{2}}(0))} + \|\nabla\gamma_\zeta\|_{C(B_{\frac{1}{2}}(0))}). \tag{16}$$

Similarly to [8, pg. 111] we consider a 2-d Dirichlet kernel for $r \in \mathbb{N}$

$$D_r(x) \triangleq \frac{1}{2r+1} \sum_{k\in\Omega_r} e^{ik\cdot x} \text{ where } \Omega_r \triangleq \{k = \begin{pmatrix} k^1 & k^2 \end{pmatrix}^T \colon k^i \in \mathbb{Z} \cap [-r,r] \text{ for } i = 1,2\}, \tag{17}$$

where T denotes a transpose, that satisfies

$$\|D_r\|_{L^p_x} \lesssim r^{1-\frac{2}{p}}, \quad \text{and} \quad \|D_r\|_{L^2_x} = 2\pi \quad \forall\, p \in (1,\infty]. \tag{18}$$

The role of r is to parametrize the number of frequencies along edges of the cube Ω_r. We introduce σ such that $\lambda\sigma \in 5\mathbb{N}$ to parametrize the spacing between frequencies, or equivalently such that the resulting rescaled kernel is $(\mathbb{T}/\lambda\sigma)^2$-periodic. In particular, this will be needed in application of Lemma 6.2 in (89). Lastly, μ measures the amount of temporal oscillation in the building blocks. In sum, the parameters we introduced are required to satisfy

$$1 \ll r \ll \mu \ll \sigma^{-1} \ll \lambda, \quad r \in \mathbb{N}, \quad \text{and} \quad \lambda, \lambda\sigma \in 5\mathbb{N}. \tag{19}$$

Now we define the directed-rescaled Dirichlet kernel by

$$\eta_\zeta(t,x) \triangleq \eta_{\zeta,\lambda,\sigma,r,\mu}(t,x) \triangleq \begin{cases} D_r(\lambda\sigma(\zeta\cdot x+\mu t), \lambda\sigma\zeta^\perp\cdot x) & \text{if } \zeta\in\Lambda^+, \\ \eta_{-\zeta,\lambda,\sigma,r,\mu}(t,x) & \text{if } \zeta\in\Lambda^-, \end{cases} \tag{20}$$

so that

$$\frac{1}{\mu}\partial_t\eta_\zeta(t,x) = \pm(\zeta\cdot\nabla)\eta_\zeta(t,x)\ \forall\ \zeta\in\Lambda^\pm, \tag{21a}$$

$$\fint_{\mathbb{T}^2}\eta_\zeta^2(t,x)dx = 1, \quad \text{and} \quad \|\eta_\zeta\|_{L_t^\infty L_x^p} \lesssim r^{1-\frac{2}{p}}\ \forall\ p\in(1,\infty] \tag{21b}$$

(cf. [8, Equations (3.8)-(3.10)]). Finally, we define the intermittent 2-d stationary flow as

$$\mathbb{W}_\zeta(t,x) \triangleq \mathbb{W}_{\zeta,\lambda,\sigma,r,\mu}(t,x) \triangleq \eta_{\zeta,\lambda,\sigma,r,\mu}(t,x)b_{\zeta,\lambda}(x) \tag{22}$$

(cf. [8, Equation (3.11)]). Similarly to the 3-d case in [8] it follows that for all $\zeta,\zeta'\in\Lambda$ (see [42, Equations (4.16)-(4.19)])

$$\mathbb{P}_{\leq 2\lambda}\mathbb{P}_{\geq\frac{\lambda}{2}}\mathbb{W}_\zeta = \mathbb{W}_\zeta, \tag{23a}$$

$$\mathbb{P}_{\leq 4\lambda}\mathbb{P}_{\geq\frac{\lambda}{5}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}) = \mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'} \quad \text{if } \zeta+\zeta'\neq 0, \tag{23b}$$

$$\mathbb{P}_{\geq\frac{\lambda\sigma}{2}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}) = \mathbb{P}_{\neq 0}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}), \tag{23c}$$

$$\mathbb{P}_{\neq 0}\eta_\zeta = \mathbb{P}_{\geq\frac{\lambda\sigma}{2}}\eta_\zeta. \tag{23d}$$

Lemma 3.2. ([42, Lemma 4.2]; cf. [8, Proposition 3.4]) Define $\mathbb{W}_\zeta$ by (22). Then for any $\{a_\zeta\}_{\zeta\in\Lambda}\subset\mathbb{C}$ such that $a_{-\zeta}=\bar{a}_\zeta$, a function $\sum_{\zeta\in\Lambda}a_\zeta$ is $\mathbb{R}$-valued and for all $\mathring{R}\in\mathcal{M}$,

$$\sum_{\zeta\in\Lambda}(\gamma_\zeta(\mathring{R}))^2\fint_{\mathbb{T}^2}\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{-\zeta}dx = -\mathring{R}. \tag{24}$$

Lemma 3.3. ([42, Lemma 4.3]; cf. [8, Proposition 3.5]) Define η_ζ and $\mathbb{W}_\zeta$ respectively by (20) and (22), and assume (19). Then for any $p\in(1,\infty]$, $k, N\in\{0,1,2,3\}$,

$$\|\nabla^N\partial_t^k\mathbb{W}_\zeta\|_{L_t^\infty L_x^p} \lesssim_{N,k,p} \lambda^N(\lambda\sigma r\mu)^k r^{1-\frac{2}{p}}, \tag{25a}$$

$$\|\nabla^N\partial_t^k\eta_\zeta\|_{L_t^\infty L_x^p} \lesssim_{N,k,p} (\lambda\sigma r)^N(\lambda\sigma r\mu)^k r^{1-\frac{2}{p}}. \tag{25b}$$

4. Proofs of Theorems 2.1-2.2

4.1. **Proof of Theorem 2.2 assuming Theorem 2.1.** We first present general results for F defined through (10); thereafter, we apply them in case $F\equiv 1$ and B is a GG^*-Wiener process to prove Theorems 2.1-2.2. We fix $\varepsilon\in(0,1)$ for the following definitions, which are in the spirit of previous works such as [29, 30, 54].

Definition 4.1. *Let $s\geq 0$ and $\xi^{in}\in L_\sigma^2$. Then $P\in\mathcal{P}(\Omega_0)$ is a martingale solution to* (3) *with initial condition ξ^{in} at initial time s if*

(M1) $P(\{\xi(t)=\xi^{in}\ \forall\ t\in[0,s]\})=1$ and for all $n\in\mathbb{N}$

$$P(\{\xi\in\Omega_0\colon \int_0^n\|F(\xi(r))\|_{L_2(U,L_\sigma^2)}^2dr<\infty\})=1, \tag{26}$$

(M2) for every $\mathfrak{g}_i\in C^\infty(\mathbb{T}^2)\cap L_\sigma^2$ and $t\geq s$

$$M_{t,s}^i \triangleq \langle\xi(t)-\xi(s),\mathfrak{g}_i\rangle + \int_s^t\langle div(\xi(r)\otimes\xi(r))+(-\Delta)^m\xi(r),\mathfrak{g}_i\rangle dr \tag{27}$$

is a continuous, square-integrable $(\mathcal{B}_t)_{t\geq s}$-martingale under P such that $\langle\langle M_{t,s}^i\rangle\rangle = \int_s^t\|F(\xi(r))^*\mathfrak{g}_i\|_U^2dr$, *(M3) for any $q \in \mathbb{N}$ there exists a function $t \mapsto C_{t,q} \in \mathbb{R}_+$ for all $t \geq s$ such that*

$$\mathbb{E}^P[\sup_{r\in[0,t]} \|\xi(r)\|_{L_x^2}^{2q} + \int_s^t \|\xi(r)\|_{H_x^\varepsilon}^2 dr] \leq C_{t,q}(1 + \|\xi^{in}\|_{L_x^2}^{2q}). \tag{28}$$

The set of all such martingale solutions with the same constants $C_{t,q}$ in (28) *for every $q \in \mathbb{N}$ and $t \geq s$ will be denoted by $\mathcal{C}(s, \xi^{in}, \{C_{t,q}\}_{q\in\mathbb{N},t\geq s})$.*

In the case of an additive noise, if $\{\mathfrak{g}_i\}_{i=1}^\infty$ is a complete orthonormal system consisting of eigenvectors of GG^*, then $M_{t,s} \triangleq \sum_{i=1}^\infty M_{t,s}^i \mathfrak{g}_i$ becomes a GG^*-Wiener process w.r.t. the filtration $(\mathcal{B}_t)_{t\geq s}$ under P. Given any stopping time $\tau \colon \Omega_0 \mapsto [0, \infty]$ we define the space of trajectories stopped at τ by

$$\Omega_{0,\tau} \triangleq \{\omega(\cdot \wedge \tau(\omega)) \colon \omega \in \Omega_0\} \tag{29}$$

and denote the σ-field associated to τ by $\mathcal{B}_\tau$.

Definition 4.2. *Let $s \geq 0$, $\xi^{in} \in L_\sigma^2$, and $\tau \geq s$ be a stopping time of $(\mathcal{B}_t)_{t\geq s}$. Then $P \in \mathcal{P}(\Omega_{0,\tau})$ is a martingale solution to* (3) *on $[s, \tau]$ with initial condition ξ^{in} at initial time s if*

(M1) $P(\{\xi(t) = \xi^{in} \ \forall\ t \in [0, s]\}) = 1$ and for all $n \in \mathbb{N}$

$$P(\{\xi \in \Omega_0 \colon \int_0^{n\wedge\tau} \|F(\xi(r))\|_{L_2(U,L_\sigma^2)}^2 dr < \infty\}) = 1, \tag{30}$$

(M2) for every $\mathfrak{g}_i \in C^\infty(\mathbb{T}^2) \cap L_\sigma^2$ and $t \geq s$

$$M_{t\wedge\tau,s}^i \triangleq \langle \xi(t \wedge \tau) - \xi^{in}, \mathfrak{g}_i \rangle + \int_s^{t\wedge\tau} \langle div(\xi(r) \otimes \xi(r)) + (-\Delta)^m \xi(r), \mathfrak{g}_i \rangle dr \tag{31}$$

is a continuous, square-integrable $(\mathcal{B}_t)_{t\geq s}$-martingale under P such that $\langle\langle M_{t\wedge\tau,s}^i \rangle\rangle = \int_s^{t\wedge\tau} \|F(\xi(r))^* \mathfrak{g}_i\|_U^2 dr$,

(M3) for any $q \in \mathbb{N}$ there exists a function $t \mapsto C_{t,q} \in \mathbb{R}_+$ for all $t \geq s$ such that

$$\mathbb{E}^P[\sup_{r\in[0,t\wedge\tau]} \|\xi(r)\|_{L_x^2}^{2q} + \int_s^{t\wedge\tau} \|\xi(r)\|_{H_x^\varepsilon}^2 dr] \leq C_{t,q}(1 + \|\xi^{in}\|_{L_x^2}^{2q}). \tag{32}$$

Proposition 4.1. *For any $(s, \xi^{in}) \in [0, \infty) \times L_\sigma^2$, there exits $P \in \mathcal{P}(\Omega_0)$ which is a martingale solution to* (3) *with initial condition ξ^{in} at initial time s according to Definition 4.1. Additionally, if there exists a family $\{(s_n, \xi_n)\}_{n\in\mathbb{N}} \subset [0, \infty) \times L_\sigma^2$ such that* $\lim_{n\to\infty} \|(s_n, \xi_n) - (s, \xi^{in})\|_{\mathbb{R}\times L_x^2} = 0$ *and $P_n \in \mathcal{C}(s_n, \xi_n, \{C_{t,q}\}_{q\in\mathbb{N},t\geq s_n})$, then there exists a subsequence $\{P_{n_k}\}_{k\in\mathbb{N}}$ that converges weakly to some $P \in \mathcal{C}(s, \xi^{in}, \{C_{t,q}\}_{q\in\mathbb{N},t\geq s})$.*

Proof of Proposition 4.1. We omit the proof of the existence of a martingale solution because it has become very standard by now; we refer to [29, Theorem 4.1] for 3-d NS equations, [30, Theorem 6.2] for a more general case of spatial dimension, as well as [61, Theorem 4.2.4] for the case of a diffusive term in the form of a fractional Laplacian with an arbitrary small exponent (see also [57, Theorem 3.1]). The stability result can also be proven following the proof of [33, Theorem 3.1] (also [58, Proposition 4.1]); because the estimates can differ slightly due to the arbitrary weak diffusion in the current case, we leave a sketch of proof elaborating on treatments of diffusive terms in the Appendix for completeness. □

Proposition 4.1 leads to the following results; the proofs of analogous results in [33] did not depend on spatial dimension or specific form of diffusive terms and thus directly apply to our case.

Lemma 4.2. ([33, Proposition 3.2]) Let τ be a bounded stopping time of $(\mathcal{B}_t)_{t\geq 0}$. Then for every $\omega \in \Omega_0$ there exists $Q_\omega \in \mathcal{P}(\Omega_0)$ such that

$$Q_\omega(\{\omega' \in \Omega_0 : \xi(t, \omega') = \omega(t) \ \forall \ t \in [0, \tau(\omega)]\}) = 1, \tag{33a}$$

$$Q_\omega(A) = R_{\tau(\omega), \xi(\tau(\omega), \omega)}(A) \ \forall \ A \in \mathcal{B}^{\tau(\omega)}, \tag{33b}$$

where $R_{\tau(\omega), \xi(\tau(\omega), \omega)} \in \mathcal{P}(\Omega_0)$ is a martingale solution to (3) with initial condition $\xi(\tau(\omega), \omega)$ at initial time $\tau(\omega)$. Furthermore, for every $B \in \mathcal{B}$ the map $\omega \mapsto Q_\omega(B)$ is $\mathcal{B}_\tau$-measurable.

Let us mention that in the proof of Lemma 4.2, Q_ω is derived as the unique probability measure

$$Q_\omega = \delta_\omega \otimes_{\tau(\omega)} R_{\tau(\omega), \xi(\tau(\omega), \omega)} \in \mathcal{P}(\Omega_0), \tag{34}$$

where δ_ω is the Dirac mass, such that (33a)-(33b) hold.

Lemma 4.3. ([33, Proposition 3.4]) Let $\xi^{\text{in}} \in L^2_\sigma$ and P be a martingale solution to (3) on $[0, \tau]$ with initial condition ξ^{in} at initial time 0 according to Definition 4.2. Assume the hypothesis of Lemma 4.2 and additionally that there exists a Borel set $\mathcal{N} \subset \Omega_{0,\tau}$ such that $P(\mathcal{N}) = 0$ and Q_ω from Lemma 4.2 satisfies for every $\omega \in \Omega_0 \setminus \mathcal{N}$

$$Q_\omega(\{\omega' \in \Omega_0 : \tau(\omega') = \tau(\omega)\}) = 1. \tag{35}$$

Then a probability measure $P \otimes_\tau R \in \mathcal{P}(\Omega_0)$ defined by

$$P \otimes_\tau R(\cdot) \triangleq \int_{\Omega_0} Q_\omega(\cdot) P(d\omega) \tag{36}$$

satisfies $P \otimes_\tau R|_{\Omega_{0,\tau}} = P|_{\Omega_{0,\tau}}$ and it is a martingale solution to (3) on $[0, \infty)$ with initial condition ξ^{in} at initial time 0.

Now we split (3) to

$$dz + (-\Delta)^m z dt + \nabla \pi^1 dt = dB, \quad \nabla \cdot z = 0 \text{ for } t > 0, \quad z(0, x) \equiv 0, \tag{37a}$$

$$\partial_t v + (-\Delta)^m v + \operatorname{div}((v + z) \otimes (v + z)) + \nabla \pi^2 = 0, \nabla \cdot v = 0 \text{ for } t > 0, v(0, x) = u^{\text{in}}(x) \tag{37b}$$

so that $u = v + z$ solves (3) with $\pi = \pi^1 + \pi^2$ starting from u^{in} at $t = 0$. We fix a GG^*-Wiener process B on $(\Omega, \mathcal{F}, \mathbf{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ as the canonical filtration of B augmented by all the $\mathbf{P}$-negligible sets and apply Definitions 4.1-4.2, Proposition 4.1, and Lemmas 4.2-4.3 with $F \equiv 1$ and such B.

Proposition 4.4. *Suppose that $m \in (0, 1)$ and that $Tr((-\Delta)^{2-m+2\sigma} GG^*) < \infty$ for some $\sigma > 0$. Then for all $\delta \in (0, \frac{1}{2})$ and $T > 0$,*

$$\mathbb{E}^{\mathbf{P}}[\|z\|_{C_T H_x^{\frac{4+\sigma}{2}}} + \|z\|_{C_T^{\frac{1}{2}-\delta} H_x^{\frac{2+\sigma}{2}}}] < \infty. \tag{38}$$

Proof of Proposition 4.4. Similarly to [33, Proposition 3.6] and [58, Proposition 4.4], this follows from a straight-forward modification of the proof of [21, Proposition 34] and an application of Kolmogorov's test [20, Theorem 3.3]. Because our diffusion is significantly weaker than the cases in [33, 58], we require a stronger hypothesis on G. In short, one can define

$$Y(s) \triangleq \frac{\sin(\pi\alpha)}{\pi} \int_0^s e^{-(-\Delta)^m (s-r)} (s-r)^{-\alpha} \mathbb{P} dB(r) \text{ where } \alpha \in (0, \frac{3\sigma}{4m}), \tag{39}$$

show that $\mathbb{E}^{\mathbf{P}}[\|(-\Delta)^{\frac{4+\sigma}{4}} Y\|^{2k}_{L_T^{2k} L_x^2}] \lesssim_k 1$ for all $k \in \mathbb{N}$ using $\text{Tr}((-\Delta)^{2-m+2\sigma} GG^*) < \infty$ from hypothesis, use the identities of

$$z(t) = \int_0^t e^{-(t-r)(-\Delta)^m} \mathbb{P} dB(r) \quad \text{and} \quad \int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha} ds = \frac{\pi}{\sin(\alpha\pi)} \text{ for any } \alpha \in (0, 1)$$

respectively from (37a) and [20, pg. 131] to write

$$\int_0^t (t-s)^{\alpha-1} e^{-(-\Delta)^m (t-s)} Y(s) ds = z(t),$$

and conclude the first bound. This immediately gives for any $\beta < \frac{1}{2}$

$$\mathbb{E}^{\mathbf{P}}[\sup_{t,t+h\in[0,T]} \|(-\Delta)^{\frac{2+\sigma}{4}} (z(t+h) - z(t))\|_{L_x^2}^{2k}] \lesssim_{\sigma,m,\beta,k,T} |h|^{2\beta k} \tag{40}$$

(we refer to [21, Equation (55)] and [19, Proposition A.1.1]) so that applying Kolmogorov's test deduces the second bound. We refer to [21, Proposition 3.4] for complete details. □

Next, for every $\omega \in \Omega_0$ we define

$$M_{t,0}^{\omega} \triangleq \omega(t) - \omega(0) + \int_0^t \mathbb{P}\text{div}(\omega(r) \otimes \omega(r)) + (-\Delta)^m \omega(r) dr, \tag{41a}$$

$$Z^{\omega}(t) \triangleq M_{t,0}^{\omega} - \int_0^t \mathbb{P}(-\Delta)^m e^{-(t-r)(-\Delta)^m} M_{r,0}^{\omega} dr. \tag{41b}$$

If P is a martingale solution to (3), then M is a GG^*-Wiener process under P and it follows from (41a)-(41b) that we can write

$$Z(t) = \int_0^t \mathbb{P} e^{-(t-r)(-\Delta)^m} dM_{r,0}. \tag{42}$$

We can deduce from Proposition 4.4 that P-a.s. $Z \in C_T H_x^{\frac{4+\sigma}{2}} \cap C_T^{\frac{1}{2}-\delta} H_x^{\frac{2+\sigma}{2}}$. For $n \in \mathbb{N}$ and $\delta \in (0, \frac{1}{12})$ we define

$$\begin{aligned} \tau_L^n(\omega) \triangleq &\inf\{t \geq 0 \colon C_S \|Z^{\omega}(t)\|_{H_x^{\frac{4+\sigma}{2}}} > (L - \frac{1}{n})^{\frac{1}{4}}\} \\ &\wedge \inf\{t \geq 0 \colon C_S \|Z^{\omega}\|_{C_t^{\frac{1}{2}-2\delta} H_x^{\frac{2+\sigma}{2}}} > (L - \frac{1}{n})^{\frac{1}{2}}\} \wedge L, \end{aligned} \tag{43a}$$

$$\tau_L \triangleq \lim_{n\to\infty} \tau_L^n, \tag{43b}$$

where $C_S > 0$ is the Sobolev constant such that $\|f\|_{L_x^\infty} \leq C_S \|f\|_{H_x^{\frac{2+\sigma}{2}}}$ for all $f \in H^{\frac{2+\sigma}{2}}(\mathbb{T}^2)$, so that $(\tau_L^n)_{n\in\mathbb{N}}$ is non-decreasing in n. By [33, Lemma 3.5] it follows that τ_L^n is a stopping time of $(\mathcal{B}_t)_{t\geq 0}$ for all $n \in \mathbb{N}$ and hence so is τ_L.

Next, we shall assume Theorem 2.1 on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and denote by P the law of the solution u constructed from Theorem 2.1.

Proposition 4.5. *Let τ_L be defined by* (43b). *Then P, the law of u, is a martingale solution of* (3) *on* $[0, \tau_L]$ *according to Definition 4.2.*

Proof of Proposition 4.5. For $C_S > 0$ from (43a), $L > 1$, and $\delta \in (0, \frac{1}{12})$, we define

$$T_L \triangleq \inf\{t \geq 0 \colon C_S \|z(t)\|_{H_x^{\frac{4+\sigma}{2}}} \geq L^{\frac{1}{4}}\} \wedge \inf\{t \geq 0 \colon C_S \|z\|_{C_t^{\frac{1}{2}-2\delta} H_x^{\frac{2+\sigma}{2}}} \geq L^{\frac{1}{2}}\} \wedge L. \tag{44}$$

Due to Proposition 4.4 we see that $\mathbf{P}$-a.s. $T_L > 0$ and $T_L \nearrow +\infty$ as $L \nearrow +\infty$. The stopping time t in the statement of Theorem 2.1 is actually T_L for $L > 0$ sufficiently large. The rest of the proof of Proposition 4.5 follows that of [33, Proposition 3.7] (see also [58, Proposition 4.5]). □

Proposition 4.6. *Let τ_L be defined by* (43b) *and P denote the law of u constructed from Theorem 2.1. Then the probability measure $P \otimes_{\tau_L} R$ in* (36) *is a martingale solution to* (3) *on* $[0, \infty)$ *according to Definition 4.1.*

Proof of Proposition 4.6. Because τ_L is a stopping time of $(\mathcal{B}_t)_{t\geq 0}$ that is bounded by L due to (43a), and P is a martingale solution to (3) on $[0,\tau_L]$ due to Proposition 4.5, Lemma 4.3 gives us the desired result once we verify (35). The rest of the proof follows that of [33, Proposition 3.8] (see also [58, Proposition 4.6]). □

Taking Theorem 2.1 for granted we are ready to prove Theorem 2.2.

Proof of Theorem 2.2 assuming Theorem 2.1. This follows from the proof of [33, Theorem 1.2] (see also the proof of [58, Theorem 2.2]). In short, we can fix $T > 0$ arbitrarily, any $\kappa \in (0,1)$ and $K > 1$ such that $\kappa K^2 \geq 1$, rely on Theorem 2.1 and Proposition 4.6 to deduce the existence of $L > 1$ and a measure $P \otimes_{\tau_L} R$ that is a martingale solution to (3) on $[0,\infty)$ and coincides with P, the law of the solution constructed in Theorem 2.1, over a random interval $[0,\tau_L]$. Therefore, $P \otimes_{\tau_L} R$ starts with a deterministic initial condition ξ^{in} from the proof of Theorem 2.1. It follows that

$$P \otimes_{\tau_L} R(\{\tau_L \geq T\}) \overset{(36)}{=} \int_{\Omega_0} Q_\omega(\{\omega' \in \Omega_0 : \tau_L(\omega') \geq T\}) P(d\omega) = \mathbf{P}(\{T_L \geq T\}) > \kappa \tag{45}$$

where the last inequality is due to Theorem 2.1. Consequently,

$$\mathbb{E}^{P\otimes_{\tau_L} R}[\|\xi(T)\|_{L_x^2}^2] \overset{(5)(45)}{>} \kappa[K\|\xi^{\text{in}}\|_{L_x^2} + K(T\text{Tr}(GG^*))^{\frac{1}{2}}]^2 \geq \kappa K^2(\|\xi^{\text{in}}\|_{L_x^2}^2 + T\text{Tr}(GG^*)). \tag{46}$$

On the other hand, it is well known that a Galerkin approximation can give us another martingale solution Θ (e.g., [29]) which starts from the same initial condition ξ^{in} and satisfies

$$\mathbb{E}^{\Theta}[\|\xi(T)\|_{L_x^2}^2] \leq \|\xi^{\text{in}}\|_{L_x^2}^2 + T\text{Tr}(GG^*).$$

Because $\kappa K^2 \geq 1$, this implies $P \otimes_{\tau_L} R \neq \Theta$ and hence a lack uniqueness in law for (3). □

4.2. **Proof of Theorem 2.1 assuming Proposition 4.8.** Considering (37b), for $q \in \mathbb{N}_0$ we will construct a solution $(v_q, \mathring{R}_q)$ to

$$\partial_t v_q + (-\Delta)^m v_q + \text{div}((v_q + z) \otimes (v_q + z)) + \nabla \pi_q = \text{div}\mathring{R}_q, \quad \nabla \cdot v_q = 0, \quad t > 0, \tag{47}$$

where $\mathring{R}_q$ is assumed to be a trace-free symmetric matrix. For any $a \in 10\mathbb{N}, b \in \mathbb{N}$, $\beta \in (0,1)$, and $L \geq 1$, to be selected more precisely in Sub-Subsection 4.3.1, we define

$$\lambda_q \triangleq a^{b^q}, \quad \delta_q \triangleq \lambda_q^{-2\beta}, \quad M_0(t) \triangleq L^4 e^{4Lt}, \tag{48}$$

from which we see that $\lambda_{q+1} \in 10\mathbb{N} \subset 5\mathbb{N}$, as required in (19). The reason why we take $a \in 10\mathbb{N}$ rather than $a \in 5\mathbb{N}$ will be e.g. explained after (128). Due to Sobolev embedding in $\mathbb{T}^2$ we see from (44) that for any $\delta \in (0, \frac{1}{12})$ and $t \in [0, T_L]$

$$\|z(t)\|_{L_x^\infty} \leq L^{\frac{1}{4}}, \quad \|\nabla z(t)\|_{L_x^\infty} \leq L^{\frac{1}{4}}, \quad \|z\|_{C_t^{\frac{1}{2}-2\delta} L_x^\infty} \leq L^{\frac{1}{2}}. \tag{49}$$

Let us observe that if $a^{\beta b} > 3$ and $b \geq 2$, then $\sum_{1\leq \iota \leq q} \delta_\iota^{\frac{1}{2}} < \frac{1}{2}$ for any $q \in \mathbb{N}$. We set the convention that $\sum_{1\leq\iota\leq 0} \triangleq 0$, denote by $c_R > 0$ a universal small constant to be described subsequently throughout the proof of Proposition 4.8 (e.g., (84)), and assume the following bounds over $t \in [0, T_L]$ inductively:

$$\|v_q\|_{C_t L_x^2} \leq M_0(t)^{\frac{1}{2}}(1 + \sum_{1\leq\iota\leq q} \delta_\iota^{\frac{1}{2}}) \leq 2M_0(t)^{\frac{1}{2}}, \tag{50a}$$

$$\|v_q\|_{C_{t,x}^1} \leq M_0(t)^{\frac{1}{2}} \lambda_q^4, \tag{50b}$$

$$\|\mathring{R}_q\|_{C_t L_x^1} \leq M_0(t) c_R \delta_{q+1}. \tag{50c}$$

We denote an anti-divergence operator by $\mathcal{R}$ in the following proposition (see Lemma 6.1).

Proposition 4.7. *Let*

$$v_0(t,x) \triangleq \frac{L^2 e^{2Lt}}{2\pi} \begin{pmatrix} \sin(x^2) & 0 \end{pmatrix}^T. \tag{51}$$

Then together with

$$\mathring{R}_0(t,x) \triangleq \frac{2L^3 e^{2Lt}}{2\pi} \begin{pmatrix} 0 & -\cos(x^2) \\ -\cos(x^2) & 0 \end{pmatrix} + (\mathcal{R}(-\Delta)^m v_0 + v_0 \mathring{\otimes} z + z \mathring{\otimes} v_0 + z \mathring{\otimes} z)(t,x), \tag{52}$$

it satisfies (47) *at level $q = 0$. Moreover,* (50) *are satisfied at level $q = 0$ provided*

$$(50)9\pi^2 < 50\pi^2 a^{2\beta b} \leq c_R L \leq c_R (a^4 \pi - 1) \tag{53}$$

where the inequality $9 < a^{2\beta b}$ is assumed only for the justification of the second inequality in (50a)*. Furthermore, $v_0(0,x)$ and $\mathring{R}_0(0,x)$ are both deterministic.*

Proof of Proposition 4.7. Using the facts that the divergence of a matrix $(A^{ij})_{1\leq i,j\leq 2}$ is a 2-d vector, of which k-th component is $\sum_{j=1}^2 \partial_j A^{kj}$ and that $\text{div}(v_0 \otimes v_0) = 0$, one can immediately verify that v_0 and $\mathring{R}_0$ from (51)-(52) satisfy (47) at level $q = 0$ if we choose $\pi_0 = -(v_0 \cdot z + \frac{1}{2}|z|^2)$. We also point out that v_0 is divergence-free while $\mathring{R}_0$ is trace-free and symmetric due to Lemma 6.1, as required. Next, we can compute

$$\|v_0(t)\|_{L_x^2} = \frac{M_0(t)^{\frac{1}{2}}}{\sqrt{2}} \leq M_0(t)^{\frac{1}{2}}, \quad \|v_0\|_{C_{t,x}^1} = \frac{L^2 e^{2Lt}(L+1)}{\pi} \overset{(53)}{\leq} M_0(t)^{\frac{1}{2}} \lambda_0^4, \tag{54}$$

and thus (50a)-(50b) at level $q = 0$ hold. Next, we can compute

$$\|\mathring{R}_0(t)\|_{L_x^1} \overset{(49)(54)}{\leq} 16L^3 e^{2Lt} + 2\pi \|\mathcal{R}(-\Delta)^m v_0\|_{L_x^2} + 20\pi M_0(t)^{\frac{1}{2}} L^{\frac{1}{4}} + 5(2\pi)^2 L^{\frac{1}{2}}. \tag{55}$$

Using the facts that v_0 is mean-zero, divergence-free, and satisfies $\Delta v_0 = -v_0$ we can rely on (193) and interpolation to deduce

$$\|\mathcal{R}(-\Delta)^m v_0\|_{L_x^2} \leq 2(\|v_0\|_{L_x^2} + \|\Delta v_0\|_{L_x^2}) = 4\|v_0\|_{L_x^2}. \tag{56}$$

Therefore, due to the second inequality of (53), continuing from (55) we obtain

$$\|\mathring{R}_0(t)\|_{L_x^1} \overset{(48)(54)(56)}{\leq} 16LM_0(t)^{\frac{1}{2}} + 8\pi M_0(t)^{\frac{1}{2}} + 20\pi M_0(t)^{\frac{1}{2}} L^{\frac{1}{4}} + 5(2\pi)^2 L^{\frac{1}{2}} \overset{(53)}{\leq} M_0(t) c_R \delta_1. \tag{57}$$

This verifies (50c) at level $q = 0$. Finally, it is clear that $v_0(0,x)$ is deterministic, and consequently $\mathring{R}_0(0,x)$ is also deterministic because $z(0,x) \equiv 0$ from (37a). □

Proposition 4.8. *Let $L > (50)9\pi^2 c_R^{-1}$ and suppose that $(v_q, \mathring{R}_q)$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process that solves* (47) *and satisfies* (50)*. Then there exists a choice of parameters a, b, and β such that* (53) *is fulfilled and an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(v_{q+1}, \mathring{R}_{q+1})$ that satisfies* (47)*,* (50) *at level $q+1$, and*

$$\|v_{q+1}(t) - v_q(t)\|_{L_x^2} \leq M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}. \tag{58}$$

Moreover, if $v_q(0,x)$ and $\mathring{R}_q(0,x)$ are deterministic, then so are $v_{q+1}(0,x)$ and $\mathring{R}_{q+1}(0,x)$.

Taking Proposition 4.8 granted we can now prove Theorem 2.1.

Proof of Theorem 2.1 assuming Proposition 4.8. The proof is similar to that of [33, Theorem 1.1] (see also the proof of [58, Theorem 2.1]); we sketch it for completeness. Given $T > 0$, $K > 1$, and $\kappa \in (0,1)$, starting from $(v_0, \mathring{R}_0)$ in Proposition 4.7, Proposition 4.8 gives us $(v_q, \mathring{R}_q)$ for $q \geq 1$ that satisfies (50) and (58). Then, for all $\varepsilon \in (0, \frac{\beta}{4+\beta})$ and $t \in [0, T_L]$, by Gagliardo-Nirenberg's inequality, and the fact that $b^{q+1} \geq b(q+1)$ for all $q \geq 0$ and $b \geq 2$, we can deduce

$$\sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{H_x^\varepsilon} \lesssim \sum_{q\geq 0} M_0(t)^{\frac{1-\varepsilon}{2}} \delta_{q+1}^{\frac{1-\varepsilon}{2}} (M_0(t)^{\frac{1}{2}} \lambda_{q+1}^4)^\varepsilon \lesssim M_0(t)^{\frac{1}{2}}. \tag{59}$$

Therefore, we can deduce the existence of $\lim_{q\to\infty} v_q \triangleq v \in C([0, T_L]; H^\varepsilon(\mathbb{T}^2))$ for which there exists a deterministic constant $C_L > 0$ such that

$$\sup_{t\in[0,T_L]} \|v(t)\|_{H_x^\varepsilon} \leq C_L. \tag{60}$$

As each v_q is $(\mathcal{F}_t)_{t\geq 0}$-adapted, it follows that v is also $(\mathcal{F}_t)_{t\geq 0}$-adapted. Furthermore, for all $t \in [0, T_L]$, $\|\mathring{R}_q\|_{C_t L_x^1} \to 0$ as $q \to +\infty$ due to (50c). Therefore, v is a weak solution to (37b) over $[0, T_L]$; consequently, we see from (37a) that $u = v + z$ solves (3). Now for $c_R > 0$ to be determined from the proof of Proposition 4.8, we can choose $L = L(T, K, c_R, \mathrm{Tr}(GG^*)) > (50)9\pi^2 c_R^{-1}$ larger if necessary to satisfy

$$\frac{3}{2} + \frac{1}{L} < (\frac{1}{\sqrt{2}} - \frac{1}{2})e^{LT} \text{ and } L^{\frac{1}{4}} 2\pi + K(T\mathrm{Tr}(GG^*))^{\frac{1}{2}} \leq (e^{LT} - K)\|u^{\mathrm{in}}\|_{L_x^2} + Le^{LT} \tag{61}$$

where $u^{\mathrm{in}}(x) = v(0, x)$ as $z(0, x) \equiv 0$ from (37b). Because $\lim_{L\to\infty} T_L = +\infty$ **P**-a.s. due to Proposition 4.4, for the fixed $T > 0$ and $\kappa > 0$, increasing L larger if necessary allows us to obtain $\mathbf{P}(\{T_L \geq T\}) > \kappa$. Now because $z(t)$ from (37a) is $(\mathcal{F}_t)_{t\geq 0}$-adapted, we see that u is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Moreover, (60) and (49) imply (4). Next, we compute

$$\|v(t) - v_0(t)\|_{L_x^2} \overset{(58)}{\leq} M_0(t)^{\frac{1}{2}} \sum_{q\geq 0} a^{-b^{q+1}\beta} \leq M_0(t)^{\frac{1}{2}} \sum_{q\geq 0} a^{-b(q+1)\beta} \overset{(53)}{<} M_0(t)^{\frac{1}{2}} (\frac{1}{2}) \tag{62}$$

for all $t \in [0, T_L]$. We also see by utilizing (61) that

$$(\|v(0)\|_{L_x^2} + L)e^{LT} \overset{(54)(62)}{\leq} (\frac{3}{2} M_0(0)^{\frac{1}{2}} + L)e^{LT} \overset{(61)}{<} (\frac{1}{\sqrt{2}} - \frac{1}{2}) M_0(T)^{\frac{1}{2}} \overset{(54)(62)}{<} \|v(T)\|_{L_x^2}. \tag{63}$$

Therefore, on $\{T_L \geq T\}$

$$\|u(T)\|_{L_x^2} \overset{(63)}{>} (\|v(0)\|_{L_x^2} + L)e^{LT} - \|z(T)\|_{L_x^\infty} 2\pi \overset{(37a)(49)(61)}{\geq} K\|u^{\mathrm{in}}\|_{L_x^2} + K(T\mathrm{Tr}(GG^*))^{\frac{1}{2}}, \tag{64}$$

which implies (5). At last, because $v_0(0, x)$ is deterministic from Proposition 4.7, Proposition 4.8 implies that $u^{\mathrm{in}}(x) = v(0, x)$ remains deterministic. □

4.3. **Proof of Proposition 4.8.**

4.3.1. *Choice of parameters.* Let us define

$$m^* \triangleq \begin{cases} 2m - 1 & \text{if } m \in (\frac{1}{2}, 1), \\ 0 & \text{if } m \in (0, \frac{1}{2}]; \end{cases} \tag{65}$$

it follows that $m^* \in [0, 1)$. Furthermore, we fix

$$\eta \in \mathbb{Q}_+ \cap (\frac{1 - m^*}{16}, \frac{1 - m^*}{8}] \tag{66}$$

from which we see that $\eta \in (0, \frac{1}{8}]$. We also fix $L > (50)9\pi^2 c_R^{-1}$ and

$$\alpha \triangleq \frac{1 - m}{400}. \tag{67}$$

We set

$$r \triangleq \lambda_{q+1}^{1-6\eta}, \quad \mu \triangleq \lambda_{q+1}^{1-4\eta}, \quad \text{and} \quad \sigma \triangleq \lambda_{q+1}^{2\eta-1}, \tag{68}$$

from which we immediately observe that $1 \ll r \ll \mu \ll \sigma^{-1} \ll \lambda_{q+1}$ from (19) is satisfied. Moreover, for the $\alpha > 0$ fixed we can choose $b \in \{\iota \in \mathbb{N}: \iota > \frac{16}{\alpha}\}$ such that $r \in \mathbb{N}$ and $\lambda_{q+1}\sigma \in 10\mathbb{N}$ so that the conditions of $r \in \mathbb{N}$ and $\lambda_{q+1}\sigma \in 5\mathbb{N}$ from (19) are satisfied. Indeed, because $\eta \in \mathbb{Q}_+ \cap (0, \frac{1}{8}]$, we can write $1 - 6\eta = \frac{n_1}{d_1}$ and $2\eta = \frac{n_2}{d_2}$ for some $n_1, n_2, d_1, d_2 \in \mathbb{N}$, and then take $b \in \mathbb{N}$ to be a multiple of $d_1 d_2$; it follows that $r = \lambda_{q+1}^{1-6\eta} = a^{b^{q+1}(1-6\eta)} \in \mathbb{N}$ and $\lambda_{q+1}\sigma = \lambda_{q+1}^{2\eta} = a^{b^{q+1}2\eta} \in 10\mathbb{N}$ as $a \in 10\mathbb{N}$. For the α from (67) and such $b > 0$ fixed, we take $\beta > 0$ sufficiently small so that

$$\alpha > 16\beta b. \tag{69}$$

We also choose

$$l \triangleq \lambda_{q+1}^{-\frac{3\alpha}{2}} \lambda_q^{-2}. \tag{70}$$

Together with the condition that $b > \frac{16}{\alpha}$, by taking $a \in 10\mathbb{N}$ sufficiently large we obtain

$$l\lambda_q^4 \leq \lambda_{q+1}^{-\alpha} \quad \text{and} \quad l^{-1} \leq \lambda_{q+1}^{2\alpha}. \tag{71}$$

Remark 4.1. *We will have numerous requirements that $\alpha \in (0, C\eta)$ for various constants $C > 0$; e.g., the second inequality of* (89) *will require that we bound*

$$\lambda_{q+1}^{-\frac{1}{2}} \sigma^{-\frac{1}{2}} l^{-\frac{11}{2}} \overset{(71)(68)}{\leq} \lambda_{q+1}^{-\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}-\eta} \lambda_{q+1}^{11\alpha}$$

by a constant that does not depend on relevant parameters and therefore we need $\alpha \leq \frac{\eta}{11}$. Thus, to be able to fix the value of α explicitly as we did in (67)*, we decided to restrict η to have the lower bound of $\frac{1-m^*}{16}$ in* (66)*, differently from* [42, Equation (2.3)]. *It follows that α defined in* (67) *indeed satisfies $\alpha \leq \frac{\eta}{11}$ as*

$$\alpha \overset{(67)}{=} \frac{1-m}{400} \overset{(65)}{\leq} (\frac{1-m^*}{16})(\frac{1}{11}) \overset{(66)}{\leq} \frac{\eta}{11},$$

and we will see that our choice of α in (67) *will satisfy all other instances when it needs to be sufficiently smaller w.r.t. η.*

Concerning (53), taking $a \in 10\mathbb{N}$ sufficiently large gives $c_R L \leq c_R(a^4\pi - 1)$ while $\beta > 0$ sufficiently small allows $(50)9\pi^2 < 50\pi^2 a^{2\beta b} \leq c_R L$. Because we chose L such that $L > (50)9\pi^2 c_R^{-1}$, this is possible. Thus, we shall hereafter consider such m^*, η, α, b, and l fixed, preserving our freedom to take $a \in 10\mathbb{N}$ larger and $\beta > 0$ smaller as necessary.

Remark 4.2. *Let us remark on some differences in our choice of parameters and those of other works. First, the work of* [42] *did not have a parameter that is equivalent to our α (The "α" in* [42, Equation (2.3)] *is actually our η defined in* (66)*). Our α in* (67) *plays the role of defining $l = \lambda_{q+1}^{-\frac{3\alpha}{2}} \lambda_q^{-2}$ in* (70)*. Instead, the choice of $l = \lambda_q^{-20}$ is taken in* [42, Equation (3.1)]*, which has appeared in others' previous works (e.g.,* [8, Equation (4.16)]*). As we described already in Remark 4.1, parts of our proof such as* (89) *required α to be taken small w.r.t. η and because η in* (66) *depends on m^* defined in* (65) *which in turn depends on m, we chose $l = \lambda_{q+1}^{-\frac{3\alpha}{2}} \lambda_q^{-2}$ where α depends on m via* (67) *following* [33, Equation (4.17)] *and* [58, Equation (69)].

On the other had, the works of [8, 33] *did not have a parameter that is equivalent to our η in* (66) *because* [8, 33] *were concerned with the Navier-Stokes equations and hence there was no parameter m. For further references we note that after this work was completed, a parameter that is analogous to η in* (66) *continued to see utility in others' works (e.g.,* [10, Equations (2.3)] *and* [59, Equation (92)]*).*

4.3.2. *Mollification.* We let $\{\phi_\epsilon\}_{\epsilon>0}$ and $\{\varphi_\epsilon\}_{\epsilon>0}$, specifically $\phi_\epsilon(\cdot) \triangleq \frac{1}{\epsilon^2}\phi(\frac{\cdot}{\epsilon})$ and $\varphi_\epsilon(\cdot) \triangleq \frac{1}{\epsilon}\varphi(\frac{\cdot}{\epsilon})$, respectively be families of standard mollifiers on $\mathbb{R}^2$ and $\mathbb{R}$ with mass one where the latter is compactly supported on $\mathbb{R}_+$. Then we mollify $v_q, \mathring{R}_q$, and z to obtain

$$v_l \triangleq (v_q *_x \phi_l) *_t \varphi_l, \quad \mathring{R}_l \triangleq (\mathring{R}_q *_x \phi_l) *_t \varphi_l, \quad z_l \triangleq (z *_x \phi_l) *_t \varphi_l. \tag{72}$$

It follows from (47) that v_l satisfies

$$\partial_t v_l + (-\Delta)^m v_l + \operatorname{div}((v_l + z_l) \otimes (v_l + z_l)) + \nabla \pi_l = \operatorname{div}(\mathring{R}_l + R_{\text{com}1}) \tag{73}$$

if

$$\pi_l \triangleq (\pi_q *_x \phi_l) *_t \varphi_l - \frac{1}{2}(|v_l + z_l|^2 - (|v_q + z|^2 *_x \phi_l) *_t \varphi_l), \tag{74a}$$

$$R_{\text{com}1} \triangleq R_{\text{commutator}1} \triangleq (v_l + z_l)\mathring{\otimes}(v_l + z_l) - (((v_q + z)\mathring{\otimes}(v_q + z)) *_x \phi_l) *_t \varphi_l. \tag{74b}$$

We can estimate for all $t \in [0, T_L]$ and $N \geq 1$, by using the fact that $\beta \ll \alpha$ from (69) and taking $a \in 10\mathbb{N}$ sufficiently large

$$\|v_q - v_l\|_{C_t L_x^2} \overset{(50b)}{\lesssim} l M_0(t)^{\frac{1}{2}} \lambda_q^4 \overset{(71)}{\leq} \frac{1}{4} M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}, \tag{75a}$$

$$\|v_l\|_{C_t L_x^2} \leq \|v_q\|_{C_t L_x^2} \overset{(50a)}{\leq} M_0(t)^{\frac{1}{2}} (1 + \sum_{1 \leq \iota \leq q} \delta_\iota^{\frac{1}{2}}), \tag{75b}$$

$$\|v_l\|_{C_{t,x}^N} \overset{(50b)}{\lesssim} l^{-N+1} M_0(t)^{\frac{1}{2}} \lambda_q^4 \overset{(67)(70)}{\leq} l^{-N} M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{-\alpha}. \tag{75c}$$

4.3.3. *Perturbation.* We let χ be a smooth function such that

$$\chi(z) \triangleq \begin{cases} 1 & \text{if } z \in [0, 1], \\ z & \text{if } z \in [2, \infty), \end{cases} \tag{76}$$

and $z \leq 2\chi(z) \leq 4z$ for $z \in (1, 2)$. We define for $t \in [0, T_L]$ and $\omega \in \Omega$

$$\rho(\omega, t, x) \triangleq 4 c_R \delta_{q+1} M_0(t) \chi((c_R \delta_{q+1} M_0(t))^{-1} |\mathring{R}_l(\omega, t, x)|). \tag{77}$$

Then it follows that

$$\left|\frac{\mathring{R}_l(\omega, t, x)}{\rho(\omega, t, x)}\right| = \frac{|\mathring{R}_l(\omega, t, x)|}{4 c_R \delta_{q+1} M_0(t) \chi((c_R \delta_{q+1} M_0(t))^{-1} |\mathring{R}_l(\omega, t, x)|)} \leq \frac{1}{2}. \tag{78}$$

We can estimate for any $p \in [1, \infty]$ and $t \in [0, T_L]$

$$\begin{aligned} \|\rho(\omega)\|_{C_t L_x^p} &\overset{(76)}{\leq} \sup_{s \in [0,t]} 4 c_R \delta_{q+1} M_0(s) \| 1 + 3(c_R \delta_{q+1} M_0(s))^{-1} |\mathring{R}_l(\omega, s, x)| \|_{L_x^p} \\ &\leq 12((4\pi^2)^{\frac{1}{p}} c_R \delta_{q+1} M_0(t) + \|\mathring{R}_l(\omega)\|_{C_t L_x^p}). \end{aligned} \tag{79}$$

Next, for any $N \geq 0$ and $t \in [0, T_L]$, due to the embedding of $W^{3,1}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$,

$$\|\mathring{R}_l\|_{C_{t,x}^N} \overset{(9)}{\lesssim} \sum_{0 \leq n + |\alpha| \leq N} \|\partial_t^n D^\alpha (-\Delta)^{\frac{3}{2}} \mathring{R}_l\|_{L_t^\infty L_x^1} \overset{(50c)}{\lesssim} l^{-N-3} M_0(t) c_R \delta_{q+1}. \tag{80}$$

For any $N \geq 0$, $k \in \{0, 1, 2\}$, and $t \in [0, T_L]$ we can deduce by taking $a \in 10\mathbb{N}$ sufficiently large

$$\|\rho\|_{C_t C_x^N} \lesssim c_R \delta_{q+1} M_0(t) l^{-3-N} \quad \text{and} \quad \|\rho\|_{C_t^1 C_x^k} \lesssim c_R \delta_{q+1} M_0(t) l^{-4(k+1)}. \tag{81}$$

Indeed, the first inequality can be computed using (79)-(80) when $N = 0$, while (76)-(77) and [6, Equation (129)] in case $N \geq 1$; the second inequality can be computed by directly applying ∂_t and ∇ and then relying on (80). Next, we define the amplitude function by

$$a_\zeta(\omega, t, x) \triangleq a_{\zeta,q+1}(\omega, t, x) \triangleq \rho(\omega, t, x)^{\frac{1}{2}} \gamma_\zeta\left(\frac{\mathring{R}_l(\omega, t, x)}{\rho(\omega, t, x)}\right). \tag{82}$$

Remark 4.3. *We note that analogous definitions of a_ζ in previous works had "$Id - \cdot$" in their arguments; e.g.,*

$$\text{“}a_{(\xi)}(\omega, t, x) := a_{\xi,q+1}(\omega, t, x) := \rho(\omega, t, x)^{1/2}\gamma_\xi\left(Id - \frac{\mathring{R}_l(\omega, t, x)}{\rho(\omega, t, x)}\right)(2\pi)^{-\frac{3}{4}}\text{”}$$

in [33, Equation (4.26)] *(see also* [8, Equation (4.12)]*). The geometric lemma in the 3-d case that was used in* [8, 33]*, specifically* [8, Proposition 3.2] *and* [33, Lemma B.1]*, had a ball around an identity matrix in the space of* 3×3 *symmetric matrices as the domain of γ_ζ. On the other hand, the available geometric lemma in the 2-d case, specifically Lemma 3.1 from* [42, Lemma 4.1]*, requires that the argument of γ_ζ be not only symmetric but also trace-free. Because $Id - \frac{\mathring{R}_l(\omega,t,x)}{\rho(\omega,t,x)}$ would not be trace-free, we chose $\frac{\mathring{R}_l(\omega,t,x)}{\rho(\omega,t,x)}$ as the argument.*

Furthermore, our choice of the argument of a_ζ also differs from that of [42, Equation (5.1)] *because theirs includes not only $\mathring{R}_l$ but also R_{com1}. We chose to refrain from including R_{com1} within the argument of γ_ζ because in contrast to* [42, Equation (3.6)]*, our R_{com1} in* (74b) *includes z and requires separate delicate treatments (see* (133)*).*

Next, we have the following identity:

$$\sum_{\zeta,\zeta' \in \Lambda} a_\zeta(\omega, t, x) a_{\zeta'}(\omega, t, x) \fint_{\mathbb{T}^2} \mathbb{W}_\zeta \mathring{\otimes} \mathbb{W}_{\zeta'}(t, x) dx = -\mathring{R}_l(\omega, t, x). \tag{83}$$

Indeed, the fact that $b_\zeta(x) \mathring{\otimes} b_{-\zeta}(x) \overset{(13)}{=} -\zeta \mathring{\otimes} \zeta$ leads to

$$\sum_{\zeta,\zeta' \in \Lambda} \gamma_\zeta(\mathring{R})\gamma_{\zeta'}(\mathring{R}) \fint_{\mathbb{T}^2} \mathbb{W}_\zeta \mathring{\otimes} \mathbb{W}_{\zeta'}(t, x) dx \overset{(15)(20)(21b)(22)}{=} -\mathring{R}$$

which in turn gives (83) by using (82).

Remark 4.4. *Let us note that this identity* (83) *differs slightly from the analogous ones previous works, e.g.,*

$$\text{“}(2\pi)^{\frac{3}{2}} \sum_{\xi \in \Lambda} a_{(\xi)}^2 \fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)} dx = \rho Id - \mathring{R}_l\text{”}$$

in [33, Equation (4.27)] *(cf. also* [8, Equation (4.14)], [9, Equation (7.30)], [42, Equation (5.3)]*). The identity* (83) *will be necessary in deriving* (102) *and ultimately* (115a)-(115b).

Concerning a_ζ we can estimate for all $t \in [0, T_L]$ with C_Λ and M from (16)

$$\|a_\zeta\|_{C_t L_x^2} \overset{(16)(78)(79)}{\leq} [12(4\pi^2 c_R \delta_{q+1} M_0(t) + \|\mathring{R}_l(\omega)\|_{C_t L_x^1})]^{\frac{1}{2}} \frac{M}{C_\Lambda} \overset{(16)(50c)}{\leq} \frac{c_R^{\frac{1}{4}} M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}}{2|\Lambda|} \tag{84}$$

by requiring $c_R^{\frac{1}{4}} \leq \frac{1}{M}$. We also have for all $t \in [0, T_L]$, $N \in \mathbb{N}_0$, and $k \in \{0, 1, 2\}$,

$$\|a_\zeta\|_{C_t C_x^N} \leq c_R^{\frac{1}{4}} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2} - 4N} \quad \text{and} \quad \|a_\zeta\|_{C_t^1 C_x^k} \leq c_R^{\frac{1}{4}} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-(k+1)4}. \tag{85}$$

Indeed, the first inequality in case $N = 0$ follows from (16), (78), (81)-(82), while the first inequality in case $N \in \mathbb{N}$ follows from (78), (81)-(82), an application of [6, Equations

(129)-(130)], and the fact that $\rho(t) \geq 2c_R\delta_{q+1}M_0(t)$ due to (76)-(77). Finally, the second inequality can be verified by applying ∂_t and ∇, and relying on (78), (81)-(82).

Next, we recall $\psi_\zeta, \eta_\zeta, \mathbb{W}_\zeta$, and μ respectively from (13), (20), (22), and (68), and define the perturbation

$$w_{q+1} \triangleq w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} \text{ and } v_{q+1} \triangleq v_l + w_{q+1} \tag{86}$$

where

$$w_{q+1}^{(p)} \triangleq \sum_{\zeta\in\Lambda} a_\zeta \mathbb{W}_\zeta,\ w_{q+1}^{(c)} \triangleq \sum_{\zeta\in\Lambda} \nabla^\perp(a_\zeta\eta_\zeta)\psi_\zeta,\ w_{q+1}^{(t)} \triangleq \mu^{-1}(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-})\mathbb{P}\mathbb{P}_{\neq 0}(a_\zeta^2\mathbb{P}_{\neq 0}\eta_\zeta^2\zeta). \tag{87}$$

We have the identity of

$$(w_{q+1}^{(p)} + w_{q+1}^{(c)})(t,x) \overset{(14a)(22)}{=} \nabla^\perp(\sum_{\zeta\in\Lambda} a_\zeta(t,x)\eta_\zeta(t,x)\psi_\zeta(x)). \tag{88}$$

It follows that w_{q+1} is divergence-free and mean-zero. Now by (13) and (20) we see that $\mathbb{W}_\zeta$ in (22) is $(\mathbb{T}/\lambda_{q+1}\sigma)^2$-periodic. Thus, we can apply Lemma 6.2 to deduce

$$\|w_{q+1}^{(p)}\|_{C_tL_x^2} \overset{(25a)(84)}{\lesssim} \sum_{\zeta\in\Lambda} \frac{c_R^{\frac{1}{4}} M_0(t)^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}}{|\Lambda|} + \lambda_{q+1}^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}c_R^{\frac{1}{4}}\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}l^{-\frac{11}{2}} \overset{(71)}{\lesssim} c_R^{\frac{1}{4}}\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}, \tag{89}$$

where the last inequality used the fact that $11\alpha - \eta \leq 0$ due to (65)-(67); preserving $c_R^{\frac{1}{4}}$ here will be needed in deriving (92). Next, for all $p \in (1,\infty)$ and $t \in [0, T_L]$ we can estimate

$$\|w_{q+1}^{(p)}\|_{C_tL_x^p} \overset{(87)}{\leq} \sup_{s\in[0,t]} \sum_{\zeta\in\Lambda} \|a_\zeta(s)\|_{L_x^\infty}\|\mathbb{W}_\zeta(s)\|_{L_x^p} \overset{(25a)(85)}{\lesssim} \delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}l^{-\frac{3}{2}}r^{1-\frac{2}{p}}, \tag{90a}$$

$$\|w_{q+1}^{(c)}\|_{C_tL_x^p} \overset{(87)}{\lesssim} \sup_{s\in[0,t]} \sum_{\zeta\in\Lambda} \|\nabla^\perp(a_\zeta\eta_\zeta)(s)\|_{L_x^p}\|\psi_\zeta\|_{L_x^\infty} \overset{(14b)(25b)(85)}{\lesssim} \delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}l^{-\frac{11}{2}}\sigma r^{2-\frac{2}{p}}, \tag{90b}$$

$$\|w_{q+1}^{(t)}\|_{C_tL_x^p} \overset{(87)}{\lesssim} \mu^{-1}\sum_{\zeta\in\Lambda} \|a_\zeta\|_{C_tL_x^\infty}^2\|\eta_\zeta\|_{C_tL_x^{2p}}^2 \overset{(25b)(85)}{\lesssim} \mu^{-1}\delta_{q+1}M_0(t)l^{-3}r^{2-\frac{2}{p}}. \tag{90c}$$

The estimates (90b)-(90c) allow us to deduce for all $p \in (1,\infty)$ and $t \in [0, T_L]$

$$\begin{aligned} &\|w_{q+1}^{(c)}\|_{C_tL_x^p} + \|w_{q+1}^{(t)}\|_{C_tL_x^p} \\ \overset{(90b)(90c)(71)}{\lesssim}\ & \delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}l^{-3}r^{2-\frac{2}{p}}[\lambda_{q+1}^{5\alpha+2\eta-1} + \lambda_{q+1}^{4\eta-1}\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}] \lesssim \delta_{q+1}M_0(t)l^{-3}r^{2-\frac{2}{p}}\lambda_{q+1}^{4\eta-1} \end{aligned} \tag{91}$$

where the second inequality used that $5\alpha + 2\eta < 4\eta - \beta$ due to (66), (67), and (69). We deduce from the estimate (91) by taking $a \in 10\mathbb{N}$ sufficiently large that for all $t \in [0, T_L]$

$$\begin{aligned} \|w_{q+1}\|_{C_tL_x^2} &\overset{(86)(89)(91)}{\lesssim} c_R^{\frac{1}{4}}\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}} + \delta_{q+1}M_0(t)l^{-3}r\lambda_{q+1}^{4\eta-1} \\ &\overset{(68)(71)}{\leq} \delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}[\frac{3}{8} + CM_0(L)^{\frac{1}{2}}\lambda_{q+1}^{6\alpha-2\eta}] \leq \frac{3}{4}\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}} \end{aligned} \tag{92}$$

where the second inequality is by taking $c_R \ll 1$ and the third inequality used that $6\alpha - 2\eta < 0$ due to (66)-(67). We are now ready to verify (50a) at level $q+1$ and (58) as follows:

$$\|v_{q+1}\|_{C_tL_x^2} \overset{(86)}{\leq} \|v_l\|_{C_tL_x^2} + \|w_{q+1}\|_{C_tL_x^2} \overset{(75b)(92)}{\leq} M_0(t)^{\frac{1}{2}}(1 + \sum_{1\leq\iota\leq q+1} \delta_\iota^{\frac{1}{2}}), \tag{93a}$$

$$\|v_{q+1}(t) - v_q(t)\|_{L_x^2} \overset{(86)}{\leq} \|w_{q+1}(t)\|_{L_x^2} + \|v_l(t) - v_q(t)\|_{L_x^2} \overset{(75a)(92)}{\leq} M_0(t)^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}. \tag{93b}$$

Next, we estimate norms of higher order. First, for all $t \in [0, T_L]$

$$\|w_{q+1}^{(p)}\|_{C^1_{t,x}} \overset{(87)}{\lesssim} \sum_{\zeta\in\Lambda} \|a_\zeta\|_{C^1_{t,x}} \|\mathbb{W}_\zeta\|_{L^\infty_t L^\infty_x} + \|a_\zeta\|_{L^\infty_t L^\infty_x} \|\mathbb{W}_\zeta\|_{C^1_{t,x}}$$

$$\overset{(25a)(85)(71)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{1-6\eta} l^{-\frac{3}{2}} [\lambda_{q+1}^{8\alpha} + \lambda_{q+1}^{2-8\eta}] \lesssim \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{3-14\eta} l^{-\frac{3}{2}}, \tag{94a}$$

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x}} \overset{(87)}{\leq} \sum_{\zeta\in\Lambda} \|(\nabla^\perp a_\zeta \eta_\zeta + a_\zeta \nabla^\perp \eta_\zeta)\psi_\zeta\|_{C^1_{t,x}} \tag{94b}$$

$$\overset{(14b)(25b)(85)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r [l^{-\frac{19}{2}} \lambda_{q+1}^{-1} + l^{-\frac{11}{2}} \sigma\mu r + l^{-\frac{3}{2}} \lambda_{q+1} \sigma^2 r^2 \mu] \lesssim \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{3-18\eta} l^{-\frac{3}{2}},$$

where the last inequality in (94a) used the fact that $8\alpha < 2 - 8\eta$ which can be verified by (65)-(67). Next, due to $\mathbb{P}\mathbb{P}_{\neq 0}$ not being bounded in $C^1_{t,x}$, we go down to L^p space for $p \in (1, \infty)$ in the expense of λ_{q+1}^α and estimate for all $t \in [0, T_L]$

$$\begin{aligned}\|w_{q+1}^{(t)}\|_{C^1_{t,x}} &\overset{(87)}{\lesssim} \mu^{-1} \sum_{\zeta\in\Lambda} \lambda_{q+1}^\alpha [\|a_\zeta\|_{C_t C_x} \|a_\zeta\|_{C^1_t C_x} \|\eta_\zeta\|^2_{C_t C_x} + \|a_\zeta\|^2_{C_t C_x} \|\eta_\zeta\|_{C_t C_x} \|\eta_\zeta\|_{C^1_t C_x} \\ &\quad + \|a_\zeta\|_{C_t C_x} \|a_\zeta\|_{C_t C^1_x} \|\eta_\zeta\|^2_{C_t C_x} + \|a_\zeta\|^2_{C_t C_x} \|\eta_\zeta\|_{C_t C_x} \|\eta_\zeta\|_{C_t C^1_x}] \\ &\overset{(25b)(71)(85)}{\lesssim} \lambda_{q+1}^{4\eta-1} \lambda_{q+1}^\alpha \delta_{q+1} M_0(t) l^{-3} (\lambda_{q+1}^{1-6\eta})^2 [\lambda_{q+1}^{8\alpha} + \lambda_{q+1}^{2-8\eta}] \lesssim \lambda_{q+1}^{3-16\eta+\alpha} \delta_{q+1} M_0(t) l^{-3}\end{aligned} \tag{95}$$

where the last inequality used the fact that $8\alpha < 2 - 8\eta$ due to (65)-(67). Therefore, by taking $a \in 10\mathbb{N}$ sufficiently large we conclude that (50b) at level $q + 1$ holds as follows:

$$\|v_{q+1}\|_{C^1_{t,x}} \overset{(75c)(94)(95)}{\leq} M_0(t)^{\frac{1}{2}} [l^{-1} \lambda_{q+1}^{-\alpha} + C\lambda_{q+1}^{3-14\eta} l^{-\frac{3}{2}} + C\lambda_{q+1}^{3-16\eta+\alpha} M_0(t)^{\frac{1}{2}} l^{-3}] \leq M_0(t)^{\frac{1}{2}} \lambda_{q+1}^4 \tag{96}$$

where the second inequality is due to

$$l^{-1} \lambda_{q+1}^{-\alpha} \overset{(71)}{\leq} \lambda_{q+1}^\alpha \overset{(67)}{\leq} \frac{1}{4} \lambda_{q+1}^4, \tag{97a}$$

$$C\lambda_{q+1}^{3-14\eta} l^{-\frac{3}{2}} \overset{(71)}{\leq} C\lambda_{q+1}^{3-14\eta} \lambda_{q+1}^{3\alpha} \overset{(65)(66)(67)}{\leq} \frac{1}{4} \lambda_{q+1}^4, \tag{97b}$$

$$C\lambda_{q+1}^{3-16\eta+\alpha} M_0(t)^{\frac{1}{2}} l^{-3} \overset{(71)}{\leq} C\lambda_{q+1}^{3-16\eta+7\alpha} M_0(L)^{\frac{1}{2}} \overset{(65)(66)(67)}{\leq} \frac{1}{4} \lambda_{q+1}^4. \tag{97c}$$

Finally, we estimate for all $p \in (1, \infty)$ and $t \in [0, T_L]$

$$\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_t W^{1,p}_x} \overset{(88)}{=} \|\nabla^\perp (\sum_{\zeta\in\Lambda} a_\zeta \eta_\zeta \psi_\zeta)\|_{C_t W^{1,p}_x} \tag{98a}$$

$$\overset{(14b)(25b)(85)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p}} [l^{-\frac{19}{2}} \lambda_{q+1}^{-1} + l^{-\frac{3}{2}} \sigma^2 r^2 \lambda_{q+1} + l^{-\frac{3}{2}} \lambda_{q+1}] \lesssim \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p}} l^{-\frac{3}{2}} \lambda_{q+1},$$

$$\|w_{q+1}^{(t)}\|_{C_t W^{1,p}_x} \overset{(87)}{\lesssim} \mu^{-1} \sum_{\zeta\in\Lambda} \|a_\zeta\|_{C_t C_x} \|a_\zeta\|_{C_t C^1_x} \|\eta_\zeta\|^2_{C_t L^{2p}_x} + \|a_\zeta\|^2_{C_t C_x} \|\eta_\zeta\|_{C_t L^{2p}_x} \|\eta_\zeta\|_{C_t W^{1,2p}_x} \tag{98b}$$

$$\overset{(25b)(85)}{\lesssim} \mu^{-1} \delta_{q+1} M_0(t) l^{-3} r^{2-\frac{2}{p}} [l^{-4} + \lambda_{q+1} \sigma r] \overset{(71)}{\lesssim} \mu^{-1} \delta_{q+1} M_0(t) l^{-3} r^{3-\frac{2}{p}} \lambda_{q+1} \sigma.$$

4.3.4. *Reynolds stress.* We can compute from (47), (73), and (86) that

$$\begin{aligned}&\operatorname{div} \mathring{R}_{q+1} - \nabla \pi_{q+1} \\ &= \underbrace{(-\Delta)^m w_{q+1} + \partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \operatorname{div}((v_l + z_l) \otimes w_{q+1} + w_{q+1} \otimes (v_l + z_l))}_{\operatorname{div}(R_{\text{lin}}) + \nabla \pi_{\text{lin}}}\end{aligned} \tag{99}$$

$$+\underbrace{\mathrm{div}((w_{q+1}^{(c)}+w_{q+1}^{(t)})\otimes w_{q+1}+w_{q+1}^{(p)}\otimes (w_{q+1}^{(c)}+w_{q+1}^{(t)}))}_{\mathrm{div}(R_{\mathrm{cor}})+\nabla\pi_{\mathrm{cor}}}+\underbrace{\mathrm{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)}+\mathring{R}_l)+\partial_t w_{q+1}^{(t)}}_{\mathrm{div}(R_{\mathrm{osc}})+\nabla\pi_{\mathrm{osc}}}$$
$$+\underbrace{\mathrm{div}(v_{q+1}\otimes z-v_{q+1}\otimes z_l+z\otimes v_{q+1}-z_l\otimes v_{q+1}+z\otimes z-z_l\otimes z_l)}_{\mathrm{div}(R_{\mathrm{com2}})+\nabla\pi_{\mathrm{com2}}}+\mathrm{div}R_{\mathrm{com1}}-\nabla\pi_l$$

within which we specify

$$\begin{aligned}
R_{\mathrm{lin}} \triangleq& R_{\mathrm{linear}}\\
\triangleq& \mathcal{R}(-\Delta)^m w_{q+1}+\mathcal{R}\partial_t(w_{q+1}^{(p)}+w_{q+1}^{(c)})+(v_l+z_l)\mathring{\otimes}w_{q+1}+w_{q+1}\mathring{\otimes}(v_l+z_l), && (100a)\\
\pi_{\mathrm{lin}} \triangleq& \pi_{\mathrm{linear}}\triangleq (v_l+z_l)\cdot w_{q+1}, && (100b)\\
R_{\mathrm{cor}} \triangleq& R_{\mathrm{corrector}}\triangleq (w_{q+1}^{(c)}+w_{q+1}^{(t)})\mathring{\otimes}w_{q+1}+w_{q+1}^{(p)}\mathring{\otimes}(w_{q+1}^{(c)}+w_{q+1}^{(t)}), && (100c)\\
\pi_{\mathrm{cor}} \triangleq& \pi_{\mathrm{corrector}}\triangleq \frac{1}{2}[(w_{q+1}^{(c)}+w_{q+1}^{(t)})\cdot w_{q+1}+w_{q+1}^{(p)}\cdot(w_{q+1}^{(c)}+w_{q+1}^{(t)})], && (100d)\\
R_{\mathrm{com2}} \triangleq& R_{\mathrm{commutator2}}\triangleq v_{q+1}\mathring{\otimes}(z-z_l)+(z-z_l)\mathring{\otimes}v_{q+1}+(z-z_l)\mathring{\otimes}z+z_l\mathring{\otimes}(z-z_l), && (100e)\\
\pi_{\mathrm{com2}} \triangleq& \pi_{\mathrm{commutator2}}\triangleq v_{q+1}\cdot(z-z_l)+\frac{1}{2}|z|^2-\frac{1}{2}|z_l|^2. && (100f)
\end{aligned}$$

Concerning R_{osc} that is arguably the most technical, first we can write

$$\mathrm{div}(w_{q+1}^{(p)}\otimes w_{q+1}^{(p)})=\mathrm{div}(w_{q+1}^{(p)}\mathring{\otimes}w_{q+1}^{(p)})+\nabla\frac{1}{2}|w_{q+1}^{(p)}|^2, \tag{101}$$

while

$$w_{q+1}^{(p)}\mathring{\otimes}w_{q+1}^{(p)}+\mathring{R}_l\overset{(83)(87)}{=}\sum_{\zeta,\zeta'\in\Lambda}a_\zeta a_{\zeta'}\mathbb{P}_{\neq 0}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'})=\sum_{\zeta,\zeta'\in\Lambda}a_\zeta a_{\zeta'}\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}) \tag{102}$$

because the minimal separation between active frequencies of $\mathbb{W}_\zeta\otimes\mathbb{W}_{\zeta'}$ and the zero frequency is given by $\lambda_{q+1}\sigma$ for $\zeta'=-\zeta$ and by $\frac{\lambda_{q+1}}{5}\geq\lambda_{q+1}\sigma$ for $\zeta'\neq-\zeta$ due to (23b)-(23c) (cf. [8, Equation (5.12)]). This leads to

$$\begin{aligned}
\mathrm{div}(w_{q+1}^{(p)}\mathring{\otimes}w_{q+1}^{(p)}+\mathring{R}_l)\overset{(102)}{=}&\mathbb{P}_{\neq 0}\Big(\sum_{\zeta,\zeta'\in\Lambda}\nabla(a_\zeta a_{\zeta'})\cdot\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'})\\
&+a_\zeta a_{\zeta'}\nabla\cdot\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'})\Big)=\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda}\mathcal{E}_{\zeta,\zeta',1}+\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda}\mathcal{E}_{\zeta,\zeta',2},
\end{aligned}\tag{103}$$

where

$$\begin{aligned}
\mathcal{E}_{\zeta,\zeta',1}\triangleq&\mathbb{P}_{\neq 0}(\nabla(a_\zeta a_{\zeta'})\cdot\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}+\mathbb{W}_{\zeta'}\mathring{\otimes}\mathbb{W}_\zeta)), && (104a)\\
\mathcal{E}_{\zeta,\zeta',2}\triangleq&\mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'}\nabla\cdot(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}+\mathbb{W}_{\zeta'}\mathring{\otimes}\mathbb{W}_\zeta)), && (104b)
\end{aligned}$$

in which we used symmetry, and also dropped the unnecessary frequency projection $\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}$ in $\mathcal{E}_{\zeta,\zeta',2}$. Now for any $\zeta,\zeta'\in\Lambda\subset\mathbb{S}^1$, we can compute

$$\begin{aligned}
(\zeta^\perp\otimes\zeta'^\perp+\zeta'^\perp\otimes\zeta^\perp)(\zeta+\zeta')=&\begin{pmatrix}\zeta^1\zeta^2\zeta'^2+\zeta^2\zeta'^1\zeta'^2-(\zeta^2)^2\zeta'^1-\zeta^1(\zeta'^2)^2\\-(\zeta^1)^2\zeta'^2-\zeta^2(\zeta'^1)^2+\zeta^1\zeta^2\zeta'^1+\zeta^1\zeta'^1\zeta'^2\end{pmatrix}\\
=&\begin{pmatrix}\zeta^1[\zeta^2\zeta'^2+(\zeta'^1)^2-1]+\zeta'^1[\zeta^2\zeta'^2+(\zeta^1)^2-1]\\\zeta^2[(\zeta'^2)^2+\zeta^1\zeta'^1-1]+\zeta'^2[(\zeta^2)^2+\zeta^1\zeta'^1-1]\end{pmatrix}=(\zeta^\perp\cdot\zeta'^\perp-1)\mathrm{Id}(\zeta+\zeta').
\end{aligned}\tag{105}$$

It follows that

$$\nabla\cdot(b_\zeta\mathring{\otimes}b_{\zeta'}+b_{\zeta'}\mathring{\otimes}b_\zeta)(x)=\nabla\cdot(b_\zeta\otimes b_{\zeta'}+b_{\zeta'}\otimes b_\zeta-b_\zeta\cdot b_{\zeta'}\mathrm{Id})(x)$$

$$\overset{(13)(105)}{=} i\lambda_{q+1}e^{i\lambda_{q+1}(\zeta+\zeta')\cdot x}(\zeta+\zeta') \overset{(13)}{=} \nabla(\lambda_{q+1}^2\psi_\zeta\psi_{\zeta'})(x). \tag{106}$$

Consequently,

$$\nabla\cdot(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}+\mathbb{W}_{\zeta'}\mathring{\otimes}\mathbb{W}_\zeta) \overset{(106)}{=} (b_\zeta\mathring{\otimes}b_{\zeta'}+b_{\zeta'}\mathring{\otimes}b_\zeta)\cdot\nabla(\eta_\zeta\eta_{\zeta'})+(\eta_\zeta\eta_{\zeta'})\nabla(\lambda_{q+1}^2\psi_\zeta\psi_{\zeta'}). \tag{107}$$

After splitting $\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda}\mathcal{E}_{\zeta,\zeta',2} = \frac{1}{2}(\sum_{\zeta,\zeta'\in\Lambda:\,\zeta+\zeta'\neq 0}+\sum_{\zeta,\zeta'\in\Lambda:\,\zeta+\zeta'=0})\mathcal{E}_{\zeta,\zeta',2}$, this allows us to write

$$\begin{aligned}&\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda:\,\zeta+\zeta'\neq 0}\mathcal{E}_{\zeta,\zeta',2} \qquad (108)\\ &\overset{(23b)(104b)}{=}\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda:\,\zeta+\zeta'\neq 0}\mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'}\nabla\cdot\mathbb{P}_{\geq\frac{\lambda_{q+1}}{10}}(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{\zeta'}+\mathbb{W}_{\zeta'}\mathring{\otimes}\mathbb{W}_\zeta)) \overset{(107)}{=} \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda}\sum_{k=1}^4\mathcal{E}_{\zeta,\zeta',2,k}\end{aligned}$$

where

$$\mathcal{E}_{\zeta,\zeta',2,1} \triangleq \mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'}\mathbb{P}_{\geq\frac{\lambda_{q+1}}{10}}[(b_\zeta\mathring{\otimes}b_{\zeta'}+b_{\zeta'}\mathring{\otimes}b_\zeta)\cdot\nabla(\eta_\zeta\eta_{\zeta'})]1_{\zeta+\zeta'\neq 0}, \tag{109a}$$

$$\mathcal{E}_{\zeta,\zeta',2,2} \triangleq \nabla\mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'}\mathbb{P}_{\geq\frac{\lambda_{q+1}}{10}}(\eta_\zeta\eta_{\zeta'}\lambda_{q+1}^2\psi_\zeta\psi_{\zeta'}))1_{\zeta+\zeta'\neq 0}, \tag{109b}$$

$$\mathcal{E}_{\zeta,\zeta',2,3} \triangleq -\mathbb{P}_{\neq 0}(\nabla(a_\zeta a_{\zeta'})\mathbb{P}_{\geq\frac{\lambda_{q+1}}{10}}(\eta_\zeta\eta_{\zeta'}\lambda_{q+1}^2\psi_\zeta\psi_{\zeta'}))1_{\zeta+\zeta'\neq 0}, \tag{109c}$$

$$\mathcal{E}_{\zeta,\zeta',2,4} \triangleq -\mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'}\mathbb{P}_{\geq\frac{\lambda_{q+1}}{10}}(\nabla(\eta_\zeta\eta_{\zeta'})\lambda_{q+1}^2\psi_\zeta\psi_{\zeta'}))1_{\zeta+\zeta'\neq 0} \tag{109d}$$

(cf. [8, pg. 131]). On the other hand, in case $\zeta+\zeta'=0$ we have $\nabla(\lambda_{q+1}^2\psi_\zeta\psi_{-\zeta}) \overset{(13)}{=} 0$, while we can multiply (21a) by $2\eta_\zeta$ to deduce $\mu^{-1}\partial_t|\eta_\zeta|^2 = \pm(\zeta\cdot\nabla)|\eta_\zeta|^2$ for all $\zeta\in\Lambda^\pm$. Hence,

$$\begin{aligned}&\nabla\cdot(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{-\zeta}+\mathbb{W}_{-\zeta}\mathring{\otimes}\mathbb{W}_\zeta) \overset{(22)(106)}{=} [b_\zeta\mathring{\otimes}b_{-\zeta}+b_{-\zeta}\mathring{\otimes}b_\zeta]\nabla(\eta_\zeta\eta_{-\zeta})\\ &\overset{(13)}{=} 2\zeta^\perp\mathring{\otimes}\zeta^\perp\nabla\eta_\zeta^2 = [\mathrm{Id}-2\zeta\otimes\zeta]\nabla\eta_\zeta^2 = \nabla\eta_\zeta^2-2(\zeta\cdot\nabla)\eta_\zeta^2\zeta = \nabla\eta_\zeta^2\mp 2\mu^{-1}(\partial_t\eta_\zeta^2)\zeta. \qquad (110)\end{aligned}$$

This allows us to write

$$\begin{aligned}\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda:\,\zeta+\zeta'=0}\mathcal{E}_{\zeta,\zeta',2} &\overset{(15)(104b)}{=}\frac{1}{2}\sum_{\zeta\in\Lambda}\mathbb{P}_{\neq 0}(a_\zeta^2\nabla\cdot(\mathbb{W}_\zeta\mathring{\otimes}\mathbb{W}_{-\zeta}+\mathbb{W}_{-\zeta}\mathring{\otimes}\mathbb{W}_\zeta)) \qquad (111)\\ &\overset{(110)}{=}\frac{1}{2}\sum_{\zeta\in\Lambda}\nabla(a_\zeta^2\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2)-\mathbb{P}_{\neq 0}(\nabla a_\zeta^2\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2)\\ &\quad -\mu^{-1}(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-})\partial_t\mathbb{P}_{\neq 0}(a_\zeta^2\mathbb{P}_{\neq 0}(\eta_\zeta^2\zeta))-\mathbb{P}_{\neq 0}(\partial_t a_\zeta^2\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}(\eta_\zeta^2\zeta))\end{aligned}$$

where we also used that η_ζ is $(\mathbb{T}/\lambda_{q+1}\sigma)^2$-periodic and hence $\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2 = \mathbb{P}_{\neq 0}\eta_\zeta^2$. At last, we obtain by using the definition of $\mathbb{P} = \mathrm{Id}-\nabla\Delta^{-1}\nabla\cdot$

$$\begin{aligned}&\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda:\,\zeta+\zeta'=0}\mathcal{E}_{\zeta,\zeta',2}+\partial_t w_{q+1}^{(t)}\\ &\overset{(87)(111)}{=}\frac{1}{2}\sum_{\zeta\in\Lambda}\nabla(a_\zeta^2\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2)-\mathbb{P}_{\neq 0}(\nabla a_\zeta^2\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}\eta_\zeta^2)\\ &\quad -\mu^{-1}(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-})\partial_t\mathbb{P}_{\neq 0}(a_\zeta^2\mathbb{P}_{\neq 0}(\eta_\zeta^2\zeta))-\mathbb{P}_{\neq 0}(\partial_t a_\zeta^2\mathbb{P}_{\geq\frac{\lambda_{q+1}\sigma}{2}}(\eta_\zeta^2\zeta))\\ &\quad +\mu^{-1}(\sum_{\zeta\in\Lambda^+}-\sum_{\zeta\in\Lambda^-})(\mathrm{Id}-\nabla\Delta^{-1}\nabla\cdot)\partial_t\mathbb{P}_{\neq 0}(a_\zeta^2\mathbb{P}_{\neq 0}\eta_\zeta^2\zeta) = \sum_{k=1}^4 A_k \qquad (112)\end{aligned}$$

where

$$A_1 \triangleq \frac{1}{2}\sum_{\zeta\in\Lambda} \nabla(a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} \eta_\zeta^2), \tag{113a}$$

$$A_2 \triangleq -\frac{1}{2}\sum_{\zeta\in\Lambda} \mathbb{P}_{\neq 0}(\nabla a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} \eta_\zeta^2), \tag{113b}$$

$$A_3 \triangleq \mu^{-1}(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-})\mathbb{P}_{\neq 0}(\partial_t a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} (\eta_\zeta^2 \zeta)), \tag{113c}$$

$$A_4 \triangleq -\nabla\Delta^{-1}\nabla\cdot \mu^{-1}(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-})\mathbb{P}_{\neq 0}\partial_t(a_\zeta^2 \mathbb{P}_{\neq 0}\eta_\zeta^2 \zeta). \tag{113d}$$

Therefore,

$$\begin{aligned}
&\operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_l) + \partial_t w_{q+1}^{(t)} \\
&\overset{(101)(103)}{=} \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda} \mathcal{E}_{\zeta,\zeta',1} + \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda} \mathcal{E}_{\zeta,\zeta',2} + \partial_t w_{q+1}^{(t)} + \nabla \frac{1}{2}|w_{q+1}^{(p)}|^2 \\
&\overset{(108)(112)}{=} \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda} \mathcal{E}_{\zeta,\zeta',1} + \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda}\sum_{k=1,3,4} \mathcal{E}_{\zeta,\zeta',2,k} + A_2 + A_3 \\
&\quad + \nabla[\frac{1}{2}|w_{q+1}^{(p)}|^2 + \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda} \mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'} \mathbb{P}_{\geq \frac{\lambda_{q+1}}{10}}(\eta_\zeta \eta_{\zeta'} \lambda_{q+1}^2 \psi_\zeta \psi_{\zeta'})) \\
&\quad + \frac{1}{2}\sum_{\zeta\in\Lambda} a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} \eta_\zeta^2 - \Delta^{-1}\nabla\cdot \mu^{-1}(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-})\mathbb{P}_{\neq 0}\partial_t(a_\zeta^2 \mathbb{P}_{\neq 0}\eta_\zeta^2 \zeta)],
\end{aligned} \tag{114}$$

which finally leads us to define

$$R_{\text{osc}} \triangleq R_{\text{oscillation}} \triangleq \mathcal{R}(\frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda} \mathcal{E}_{\zeta,\zeta',1} + \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda}\sum_{k=1,3,4} \mathcal{E}_{\zeta,\zeta',2,k} + A_2 + A_3), \tag{115a}$$

$$\begin{aligned}
\pi_{\text{osc}} \triangleq \pi_{\text{oscillation}} \triangleq\ & \frac{1}{2}|w_{q+1}^{(p)}|^2 + \frac{1}{2}\sum_{\zeta,\zeta'\in\Lambda} \mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'} \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}}(\eta_\zeta \eta_{\zeta'} \lambda_{q+1}^2 \psi_\zeta \psi_{\zeta'}))1_{\zeta+\zeta'\neq 0} \\
& + \frac{1}{2}\sum_{\zeta\in\Lambda} a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} \eta_\zeta^2 - \Delta^{-1}\nabla\cdot \mu^{-1}(\sum_{\zeta\in\Lambda^+} - \sum_{\zeta\in\Lambda^-})\mathbb{P}_{\neq 0}\partial_t(a_\zeta^2 \mathbb{P}_{\neq 0}\eta_\zeta^2 \zeta).
\end{aligned} \tag{115b}$$

Considering (99) we define

$$\pi_{q+1} \triangleq \pi_l - \pi_{\text{lin}} - \pi_{\text{cor}} - \pi_{\text{osc}} - \pi_{\text{com2}} \text{ and } \mathring{R}_{q+1} \triangleq R_{\text{lin}} + R_{\text{cor}} + R_{\text{osc}} + R_{\text{com2}} + R_{\text{com1}}. \tag{116}$$

Now we choose

$$p^* \triangleq \frac{16(1-6\eta)}{300\alpha + 16(1-7\eta)}, \tag{117}$$

which can be readily verified to be an element in $(1,2)$ using (65)-(67). For R_{lin} we first estimate by Gagliardo-Nirenberg's inequality for all $t \in [0, T_L]$

$$\begin{aligned}
\|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_t L_x^{p^*}} \lesssim & \|w_{q+1}\|_{C_t L_x^{p^*}}^{1-m^*} (\|\nabla(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_t L_x^{p^*}} + \|\nabla w_{q+1}^{(t)}\|_{C_t L_x^{p^*}})^{m^*} \\
& \overset{(86)(90a)(91)(98)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} (l^{-\frac{3}{2}} + \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-3} r \lambda_{q+1}^{4\eta-1})^{1-m^*} \\
& \times (l^{-\frac{3}{2}} \lambda_{q+1} + \mu^{-1}\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-3} r^2 \lambda_{q+1}\sigma)^{m^*}.
\end{aligned} \tag{118}$$

Second, for all $t \in [0, T_L]$

$$\|\mathcal{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_t L_x^{p^*}} \overset{(88)}{\lesssim} \sum_{\zeta \in \Lambda} \|\partial_t(a_\zeta \eta_\zeta)\psi_\zeta\|_{C_t L_x^{p^*}}$$
$$\overset{(14b)(25b)(85)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} [l^{-4}\lambda_{q+1}^{-1} + l^{-\frac{3}{2}}\sigma\mu r] \lesssim \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{1-8\eta}. \quad (119)$$

Finally, we can estimate for all $t \in [0, T_L]$

$$\|(v_l + z_l)\mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes}(v_l + z_l)\|_{C_t L_x^{p^*}} \lesssim (\|v_q\|_{C_t C_x} + \|z\|_{C_t C_x})\|w_{q+1}\|_{C_t L_x^{p^*}}$$
$$\overset{(49)(50b)(90a)(91)}{\lesssim} M_0(t)^{\frac{1}{2}} \lambda_q^4 [\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} + \delta_{q+1} M_0(t) l^{-3} r^{2-\frac{2}{p^*}} \lambda_{q+1}^{4\eta-1}]. \quad (120)$$

Due to (118)-(120) we obtain for all $t \in [0, T_L]$

$$\begin{aligned}
\|R_{\text{lin}}\|_{C_t L_x^{p^*}} \overset{(100a)}{\leq} & \|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_t L_x^{p^*}} + \|\mathcal{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_t L_x^{p^*}} \\
& + \|(v_l + z_l)\mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes}(v_l + z_l)\|_{C_t L_x^{p^*}} \\
\overset{(118)(119)(120)}{\lesssim} & \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} (l^{-\frac{3}{2}} + \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-3} r \lambda_{q+1}^{4\eta-1})^{1-m^*} \\
& \qquad \times (l^{-\frac{3}{2}}\lambda_{q+1} + \mu^{-1}\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-3} r^2 \lambda_{q+1}\sigma)^{m^*} \\
& + \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{1-8\eta} \\
& + M_0(t)^{\frac{1}{2}} \lambda_q^4 [\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} + \delta_{q+1} M_0(t) l^{-3} r^{2-\frac{2}{p^*}} \lambda_{q+1}^{4\eta-1}] \\
\lesssim & M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{m^*} + M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{1-8\eta} + M_0(t) r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_q^4. \quad (121)
\end{aligned}$$

Now within the right hand side of (121), first we can estimate using $2\beta b < \frac{\alpha}{8}$ from (69) and taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}
& M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{m^*} \\
= & \begin{cases} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \overset{(71)}{\lesssim} M_0(t)\delta_{q+2}\lambda_{q+2}^{2\beta}\lambda_{q+1}^{(1-6\eta)(1-\frac{2}{p^*})}\lambda_{q+1}^{3\alpha} & \text{if } m \in (0, \frac{1}{2}), \\ M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{2m-1} \overset{(71)}{\lesssim} M_0(t)\delta_{q+2}\lambda_{q+2}^{2\beta}\lambda_{q+1}^{(1-6\eta)(1-\frac{2}{p^*})}\lambda_{q+1}^{3\alpha}\lambda_{q+1}^{2m-1} & \text{if } m \in [\frac{1}{2}, 1), \end{cases} \\
\overset{(117)}{\lesssim} & M_0(t)\delta_{q+2}\lambda_{q+1}^{-\frac{275\alpha}{8}} \ll (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t)c_R\delta_{q+2}}{15}. \quad (122)
\end{aligned}$$

Second within (121) we estimate using $2\beta b < \frac{\alpha}{8}$ from (69) and taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}
M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{1-8\eta} & \overset{(71)}{\lesssim} M_0(t)\delta_{q+2}\lambda_{q+1}^{\frac{\alpha}{8}} (\lambda_{q+1}^{1-6\eta})^{1-\frac{2}{p^*}} \lambda_{q+1}^{3\alpha}\lambda_{q+1}^{1-8\eta} \\
& \overset{(117)}{\approx} M_0(t)\delta_{q+2}\lambda_{q+1}^{-\frac{275\alpha}{8}} \ll (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t)c_R\delta_{q+2}}{15}. \quad (123)
\end{aligned}$$

Third within (121) we estimate also using $2\beta b < \frac{\alpha}{8}$ from (69) and taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}
M_0(t) r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_q^4 & \overset{(71)}{\lesssim} M_0(t)\delta_{q+2}\lambda_{q+1}^{\frac{\alpha}{8}+4\alpha} (\lambda_{q+1}^{1-6\eta})^{1-\frac{2}{p^*}} \\
& \overset{(117)}{\lesssim} M_0(t)\delta_{q+2}\lambda_{q+1}^{-\frac{267\alpha}{8}} \ll (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t)c_R\delta_{q+2}}{15}. \quad (124)
\end{aligned}$$

By applying (122)-(124) to (121), we obtain

$$\|R_{\text{lin}}\|_{C_t L_x^{p^*}} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t)c_R\delta_{q+2}}{5}. \quad (125)$$

Next, for all $t \in [0, T_L]$ we estimate by Hölder's inequality, utilizing $2\beta b < \frac{\alpha}{8}$ due to (69), and taking $a \in 10\mathbb{N}$ sufficiently large,

$$\begin{aligned}
\|R_{\mathrm{cor}}\|_{C_t L_x^{p^*}} &\overset{(86)(100c)}{\lesssim} (\|w_{q+1}^{(c)}\|_{C_t L_x^{2p^*}} + \|w_{q+1}^{(t)}\|_{C_t L_x^{2p^*}})(\|w_{q+1}^{(c)}\|_{C_t L_x^{2p^*}} + \|w_{q+1}^{(t)}\|_{C_t L_x^{2p^*}} + \|w_{q+1}^{(p)}\|_{C_t L_x^{2p^*}}) \\
&\overset{(71)(90)(91)}{\lesssim} [M_0(t)^{\frac{1}{2}} r^{2-\frac{1}{p^*}} l^{-3} (\lambda_{q+1}^{5\alpha} \lambda_{q+1}^{2\eta-1} + M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{4\eta-1})] \\
&\quad \times [M_0(t)^{\frac{1}{2}} r^{1-\frac{1}{p^*}} l^{-\frac{3}{2}} (\lambda_{q+1}^{-2\eta} \lambda_{q+1}^{3\alpha} M_0(t)^{\frac{1}{2}} + 1)] \\
&\overset{(117)}{\lesssim} \delta_{q+2} M_0(t) \lambda_{q+1}^{-\frac{227\alpha}{8}} M_0(t)^{\frac{1}{2}} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t) c_R \delta_{q+2}}{5}.
\end{aligned} \tag{126}$$

Next, we estimate $R_{\text{oscillation}}$ from (115a). First, we rely on Lemma 6.3, use that $2\beta b < \frac{\alpha}{8}$ due to (69), and take $a \in 10\mathbb{N}$ sufficiently large to deduce for all $t \in [0, T_L]$

$$\begin{aligned}
&\|\mathcal{R}(\frac{1}{2} \sum_{\zeta, \zeta' \in \Lambda} \mathcal{E}_{\zeta, \zeta', 1})\|_{C_t L_x^{p^*}} \overset{(104a)}{\lesssim} (\frac{\lambda_{q+1}\sigma}{2})^{-1} \sum_{\zeta, \zeta' \in \Lambda} \|\nabla (a_\zeta a_{\zeta'})\|_{C_t C_x^2} \|\mathbb{W}_\zeta \mathring{\otimes} \mathbb{W}_{\zeta'} + \mathbb{W}_{\zeta'} \mathring{\otimes} \mathbb{W}_\zeta\|_{C_t L_x^{p^*}} \\
&\overset{(25a)(85)}{\lesssim} \lambda_{q+1}^{-2\eta} \delta_{q+1} M_0(t) l^{-15} r^{2-\frac{2}{p^*}} \overset{(71)(117)}{\lesssim} \delta_{q+2} M_0(t) \lambda_{q+1}^{-\frac{59\alpha}{8}} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{c_R \delta_{q+2} M_0(t)}{25}.
\end{aligned} \tag{127}$$

Here the hypothesis of Lemma 6.3 requires that $\frac{\lambda_{q+1}\sigma}{2} \in \mathbb{N}$ which is satisfied because $\lambda_{q+1}\sigma \in 10\mathbb{N}$ by our choice; we also clearly see that $\lambda_{q+1}\sigma \in 5\mathbb{N}$ would not have been sufficient for this purpose. Similarly to (127), relying on Lemma 6.3 we can estimate for all $t \in [0, T_L]$

$$\begin{aligned}
&\|\mathcal{R}(\frac{1}{2} \sum_{\zeta, \zeta' \in \Lambda} \mathcal{E}_{\zeta, \zeta', 2, 3})\|_{C_t L_x^{p^*}} \overset{(109c)}{\lesssim} \sum_{\zeta, \zeta' \in \Lambda} (\frac{\lambda_{q+1}}{10})^{-1} \|\nabla (a_\zeta a_{\zeta'})\|_{C_t C_x^2} \|\eta_\zeta \eta_{\zeta'} \lambda_{q+1}^2 \psi_\zeta \psi_{\zeta'}\|_{C_t L_x^{p^*}} \\
&\overset{(14b)(25b)(85)}{\lesssim} \lambda_{q+1}^{-1} \delta_{q+1} M_0(t) l^{-15} r^{2-\frac{2}{p^*}} \lesssim \delta_{q+2} M_0(t) \lambda_{q+1}^{-\frac{59\alpha}{8}-1+2\eta} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{c_R \delta_{q+2} M_0(t)}{25}.
\end{aligned} \tag{128}$$

Here the hypothesis of Lemma 6.3 requires $\frac{\lambda_{q+1}}{10} \in \mathbb{N}$ and thus $\lambda_{q+1} \in 10\mathbb{N}$ instead of $\lambda_{q+1} \in 5\mathbb{N}$ was needed. Next, for all $t \in [0, T_L]$ we estimate also relying on Lemma 6.3, using that $2\beta b < \frac{\alpha}{8}$ due to (69), and taking $a \in 10\mathbb{N}$ sufficiently large,

$$\begin{aligned}
&\|\mathcal{R}(\frac{1}{2} \sum_{\zeta, \zeta' \in \Lambda} \mathcal{E}_{\zeta, \zeta', 2, 1})\|_{C_t L_x^{p^*}} \overset{(109a)}{\lesssim} \sum_{\zeta, \zeta' \in \Lambda} (\frac{\lambda_{q+1}}{10})^{-1} \|a_\zeta a_{\zeta'}\|_{C_t C_x^2} \|(b_\zeta \mathring{\otimes} b_{\zeta'} + b_{\zeta'} \mathring{\otimes} b_\zeta) \cdot \nabla (\eta_\zeta \eta_{\zeta'})\|_{C_t L_x^{p^*}} \\
&\overset{(14b)(25b)(85)}{\lesssim} M_0(t) l^{-11} \lambda_{q+1}^{-4\eta} r^{2-\frac{2}{p^*}} \overset{(71)}{\lesssim} \delta_{q+2} M_0(t) \lambda_{q+1}^{-\frac{123\alpha}{8}-2\eta} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{c_R \delta_{q+2} M_0(t)}{25}.
\end{aligned} \tag{129}$$

Next, relying also on Lemma 6.3 we can estimate for all $t \in [0, T_L]$ similarly to (129)

$$\begin{aligned}
&\|\mathcal{R}(\frac{1}{2} \sum_{\zeta, \zeta' \in \Lambda} \mathcal{E}_{\zeta, \zeta', 2, 4})\|_{C_t L_x^{p^*}} \overset{(109d)}{\lesssim} \sum_{\zeta, \zeta' \in \Lambda} (\frac{\lambda_{q+1}}{10})^{-1} \|a_\zeta a_{\zeta'}\|_{C_t C_x^2} \|\nabla (\eta_\zeta \eta_{\zeta'}) \lambda_{q+1}^2 \psi_\zeta \psi_{\zeta'}\|_{C_t L_x^{p^*}} \\
&\qquad \overset{(14b)(25b)(85)}{\lesssim} M_0(t) l^{-11} \lambda_{q+1}^{-4\eta} r^{2-\frac{2}{p^*}} \overset{(71)}{\leq} (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{c_R \delta_{q+2} M_0(t)}{25}.
\end{aligned} \tag{130}$$

Next, we estimate for all $t \in [0, T_L]$ by applying Lemma 6.3, using that $2\beta b < \frac{\alpha}{8}$ due to (69), and taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}
&\|\mathcal{R}(A_2 + A_3)\|_{C_t L_x^{p^*}} \overset{(113)}{\lesssim} (\frac{\lambda_{q+1}\sigma}{2})^{-1} \sum_{\zeta \in \Lambda} \|\nabla a_\zeta^2\|_{C_t C_x^2} \|\eta_\zeta^2\|_{C_t L_x^{p^*}} + \mu^{-1} \|\partial_t a_\zeta^2\|_{C_t C_x^2} \|\eta_\zeta^2\|_{C_t L_x^{p^*}} \\
&\overset{(85)}{\lesssim} \lambda_{q+1}^{-2\eta} [M_0(t) l^{-15} + \lambda_{q+1}^{4\eta-1} M_0(t) l^{-\frac{27}{2}}] r^{2-\frac{2}{p^*}} \overset{(71)}{\lesssim} M_0(t) \delta_{q+2} \lambda_{q+1}^{-\frac{59\alpha}{8}} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{c_R \delta_{q+2} M_0(t)}{25}.
\end{aligned} \tag{131}$$

Therefore, we conclude from (127)-(131) applied to (115a) that

$$\|R_{\text{osc}}\|_{C_tL_x^{p^*}} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})}\frac{c_R\delta_{q+2}M_0(t)}{5}. \tag{132}$$

Next, for all $t \in [0, T_L]$ we estimate using that $\delta \in (0, \frac{1}{12})$, $2\beta b < \frac{\alpha}{8}$ from (69), $\alpha b > 16$ due to our choice of b, and taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}\|R_{\text{com1}}\|_{C_tL_x^1} &\overset{(74b)}{\lesssim} l(\|v_q\|_{C^1_{t,x}} + \|z\|_{C_tC_x^1})(\|v_q\|_{C_tL_x^2} + \|z\|_{C_{t,x}}) + l^{\frac{1}{2}-2\delta}\|z\|_{C_t^{\frac{1}{2}-2\delta}C_x}(\|v_q\|_{C_tL_x^2} + \|z\|_{C_{t,x}}) \\ &\overset{(49)}{\lesssim} l^{\frac{1}{2}-2\delta}M_0(t)\lambda_q^4 \overset{(70)}{\lesssim} \delta_{q+2}M_0(t)a^{b^q[-\frac{\alpha b}{2}+\frac{10}{3}+\frac{\alpha b}{8}]} \lesssim \delta_{q+2}M_0(t)a^{b^q[-\frac{8}{3}]} \leq \frac{M_0(t)c_R\delta_{q+2}}{5}. \end{aligned} \tag{133}$$

Lastly, for all $t \in [0, T_L]$ we can estimate by using that $l^{\frac{1}{2}-2\delta}\lambda_q^4 \ll \frac{c_R\delta_{q+2}}{5}$ in (133), (50a) at level $q+1$ that we already verified, and taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}\|R_{\text{com2}}\|_{C_tL_x^1} &\overset{(100e)}{\lesssim} \sup_{s\in[0,t]}[\|v_{q+1}(s)\|_{L_x^1} + \|z(s)\|_{L_x^1}]l^{\frac{1}{2}-2\delta}(\|z\|_{C_t^{\frac{1}{2}-2\delta}L_x^\infty} + \|z\|_{C_tC_x^{\frac{1}{2}-2\delta}}) \\ &\overset{(49)}{\lesssim} M_0(t)l^{\frac{1}{2}-2\delta} \leq \frac{M_0(t)c_R\delta_{q+2}}{5}. \end{aligned} \tag{134}$$

Therefore, we can now conclude from (125), (126), (132)-(134) that

$$\begin{aligned}\|\mathring{R}_{q+1}\|_{C_tL_x^1} \overset{(116)}{\leq} & (2\pi)^{2(\frac{p^*-1}{p^*})}[\|R_{\text{lin}}\|_{C_tL_x^{p^*}} + \|R_{\text{cor}}\|_{C_tL_x^{p^*}} + \|R_{\text{osc}}\|_{C_tL_x^{p^*}}] \\ &+ \frac{2M_0(t)c_R\delta_{q+2}}{5} \leq M_0(t)c_R\delta_{q+2}\end{aligned} \tag{135}$$

due to Hölder's inequality. This verifies (50c) at level $q+1$.

At last, similarly to the argument in [33] we can conclude by commenting on how $(v_{q+1}, \mathring{R}_{q+1})$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and that $(v_{q+1}, \mathring{R}_{q+1})(0, x)$ are both deterministic if $(v_q, \mathring{R}_q)(0, x)$ are deterministic. First, we recall that z in (37a) is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Due to the compact support of φ_l in $\mathbb{R}_+$, it follows that z_l from (72) is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Similarly, because $(v_q, \mathring{R}_q)$ are both $(\mathcal{F}_t)_{t\geq 0}$-adapted by hypothesis, it follows that $(v_l, \mathring{R}_l)$ from (72) are both $(\mathcal{F}_t)_{t\geq 0}$-adapted. Because $M_0(t)$ from (48) is deterministic, it follows that ρ from (77) is also $(\mathcal{F}_t)_{t\geq 0}$-adapted. Due to ρ and $\mathring{R}_l$ being $(\mathcal{F}_t)_{t\geq 0}$-adapted, a_ζ from (82) is also $(\mathcal{F}_t)_{t\geq 0}$-adapted. Because $\mathbb{W}_\zeta, \eta_\zeta$, and ψ_ζ respectively from (22), (20), and (13) are all deterministic, it follows that all of $w_{q+1}^{(p)}, w_{q+1}^{(c)}$, and $w_{q+1}^{(t)}$ from (87) are $(\mathcal{F}_t)_{t\geq 0}$-adapted. Consequently, w_{q+1} from (86) is $(\mathcal{F}_t)_{t\geq 0}$-adapted, which in turn implies that v_{q+1} from (86) is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Moreover, it is also clear from the compact support of φ_l in $\mathbb{R}_+$ that if $v_q(0, x)$ and $\mathring{R}_q(0, x)$ are deterministic, then so are $v_l(0, x), \mathring{R}_l(0, x)$, and $\partial_t\mathring{R}_l(0, x)$. Because $z(0, x) \equiv 0$ by (37a), $R_{\text{com1}}(0, x)$ from (74b) is also deterministic. Because $M_0(t)$ is deterministic, we see that $\rho(0, x)$ and $\partial_t\rho(0, x)$ from (77) are also deterministic; this implies that $a_\zeta(0, x)$ and $\partial_t a_\zeta(0, x)$ from (82) are also deterministic. As $\mathbb{W}_\zeta, \eta_\zeta$, and ψ_ζ respectively from (22), (20), and (13) are all deterministic, we see that all of $w_{q+1}^{(p)}(0, x)$, $\partial_t w_{q+1}^{(p)}(0, x)$, $w_{q+1}^{(c)}(0, x)$, $\partial_t w_{q+1}^{(c)}(0, x)$, and $w_{q+1}^{(t)}(0, x)$ from (87) are deterministic and consequently $w_{q+1}(0, x)$ from (86) is deterministic. Because $v_l(0, x)$ is deterministic, it follows that $v_{q+1}(0, x)$ from (86) is deterministic. Moreover, we see that all of $R_{\text{lin}}(0, x)$, $R_{\text{cor}}(0, x)$, and $R_{\text{com2}}(0, x)$ from (100) are deterministic. Finally, $\sum_{\zeta,\zeta'\in\Lambda}\mathcal{E}_{\zeta,\zeta',1}|_{t=0}$, $\sum_{\zeta,\zeta'\in\Lambda}\sum_{k=1,3,4}\mathcal{E}_{\zeta,\zeta',2,k}|_{t=0}$, and $A_2 + A_3|_{t=0}$ respectively from (104a), (109), and (113) are all deterministic and hence $R_{\text{osc}}(0, x)$ from (115a) is deterministic, and consequently, so is $\mathring{R}_{q+1}(0, x)$ from (116).

5. Proofs of Theorems 2.3-2.4

5.1. Proof of Theorem 2.2 assuming Theorem 2.1. Let us recall the definitions of $U_1, \bar{\Omega}$, and $\bar{\mathcal{B}}_t$ from Section 3. We first present general results for F defined through (10) and θ; thereafter, we apply them in case $F(u) = u$ and B is an $\mathbb{R}$-valued Wiener process to prove Theorems 2.3-2.4. We fix any $\varepsilon \in (0,1)$ for the purpose of the following definitions.

Definition 5.1. *Let $s \geq 0$, $\xi^{in} \in L^2_\sigma$, and $\theta^{in} \in U_1$. A probability measure $P \in \mathcal{P}(\bar{\Omega})$ is a probabilistically weak solution to* (3) *with initial condition (ξ^{in}, θ^{in}) at initial time s if*

(M1) $P(\{\xi(t) = \xi^{in}, \theta(t) = \theta^{in} \ \forall\, t \in [0,s]\}) = 1$ and for all $n \in \mathbb{N}$

$$P(\{(\xi,\theta) \in \bar{\Omega}: \int_0^n \|F(\xi(r))\|^2_{L_2(U, L^2_\sigma)} dr < \infty\}) = 1, \tag{136}$$

(M2) under P, θ is a cylindrical $(\bar{\mathcal{B}}_t)_{t\geq s}$-Wiener process on U starting from initial condition θ^{in} at initial time s and for every $\mathfrak{g}_i \in C^\infty(\mathbb{T}^2) \cap L^2_\sigma$ and $t \geq s$,

$$\langle \xi(t) - \xi(s), \mathfrak{g}_i \rangle + \int_s^t \langle div(\xi(r) \otimes \xi(r)) + (-\Delta)^m \xi(r), \mathfrak{g}_i \rangle dr = \int_s^t \langle \mathfrak{g}_i, F(\xi(r)) d\theta(r) \rangle, \tag{137}$$

(M3) for any $q \in \mathbb{N}$ there exists a function $t \mapsto C_{t,q} \in \mathbb{R}_+$ for all $t \geq s$ such that

$$\mathbb{E}^P[\sup_{r\in[0,t]} \|\xi(r)\|^{2q}_{L^2_x} + \int_s^t \|\xi(r)\|^2_{H^\varepsilon_x} dr] \leq C_{t,q}(1 + \|\xi^{in}\|^{2q}_{L^2_x}). \tag{138}$$

The set of all such probabilistically weak solutions with the same constant $C_{t,q}$ in (138) *for every $q \in \mathbb{N}$ and $t \geq s$ is denoted by $\mathcal{W}(s, \xi^{in}, \theta^{in}, \{C_{t,q}\}_{q\in\mathbb{N}, t\geq s})$.*

For any stopping time τ we set

$$\bar{\Omega}_\tau \triangleq \{\omega(\cdot \wedge \tau(\omega)) : \omega \in \bar{\Omega}\} \tag{139}$$

and denote the σ-field associated to τ by $\bar{\mathcal{B}}_\tau$.

Definition 5.2. *Let $s \geq 0$, $\xi^{in} \in L^2_\sigma$, and $\theta^{in} \in U_1$. Let $\tau \geq s$ be a stopping time of $(\bar{\mathcal{B}}_t)_{t\geq s}$. A probability measure $P \in \mathcal{P}(\bar{\Omega}_\tau)$ is a probabilistically weak solution to* (3) *on $[s,\tau]$ with initial condition (ξ^{in}, θ^{in}) at initial time s if*

(M1) $P(\{\xi(t) = \xi^{in}, \theta(t) = \theta^{in} \ \forall\, t \in [0,s]\}) = 1$ and for all $n \in \mathbb{N}$

$$P(\{(\xi,\theta) \in \bar{\Omega}: \int_0^{n\wedge\tau} \|F(\xi(r))\|^2_{L_2(U, L^2_\sigma)} dr < \infty\}) = 1, \tag{140}$$

(M2) under P, $\langle \theta(\cdot \wedge \tau), l_i \rangle_U$, where $\{l_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of U, is a continuous, square-integrable $(\bar{\mathcal{B}}_t)_{t\geq s}$-martingale with initial condition $\langle \theta^{in}, l_i \rangle$ at initial time s with its quadratic variation process given by $(t \wedge \tau - s)\|l_i\|^2_U$ and for every $\mathfrak{g}_i \in C^\infty(\mathbb{T}^2) \cap L^2_\sigma$ and $t \geq s$

$$\langle \xi(t\wedge\tau) - \xi(s), \mathfrak{g}_i \rangle + \int_s^{t\wedge\tau} \langle div(\xi(r) \otimes \xi(r)) + (-\Delta)^m \xi(r), \mathfrak{g}_i \rangle dr = \int_s^{t\wedge\tau} \langle \mathfrak{g}_i, F(\xi(r)) d\theta(r) \rangle, \tag{141}$$

(M3) for any $q \in \mathbb{N}$ there exists a function $t \mapsto C_{t,q} \in \mathbb{R}_+$ for all $t \geq s$ such that

$$\mathbb{E}^P[\sup_{r\in[0,t\wedge\tau]} \|\xi(r)\|^{2q}_{L^2_x} + \int_s^{t\wedge\tau} \|\xi(r)\|^2_{H^\varepsilon_x} dr] \leq C_{t,q}(1 + \|\xi^{in}\|^{2q}_{L^2_x}). \tag{142}$$

The joint uniqueness in law for (3) is equivalent to the uniqueness of probabilistically weak solution in Definition 5.1, which holds if probabilistically weak solutions starting from the same initial distributions are unique.

Proposition 5.1. *For every* $(s, \xi^{in}, \theta^{in}) \in [0,\infty) \times L^2_\sigma \times U_1$, *there exists a probabilistically weak solution* $P \in \mathcal{P}(\bar{\Omega})$ *to* (3) *with initial condition* (ξ^{in}, θ^{in}) *at initial time* s *according to Definition 5.1. Moreover, if there exists a family* $(s_n, \xi_n, \theta_n) \subset [0,\infty) \times L^2_\sigma \times U_1$ *such that* $\lim_{n\to\infty} \|(s_n, \xi_n, \theta_n) - (s, \xi^{in}, \theta^{in})\|_{\mathbb{R}\times L^2_x \times U_1} = 0$ *and* $P_n \in \mathcal{W}(s_n, \xi_n, \theta_n, \{C_{t,q}\}_{q\in\mathbb{N}, t\geq s_n})$, *then there exists a subsequence* $\{P_{n_k}\}_{k\in\mathbb{N}}$ *that converges weakly to some* $P \in \mathcal{W}(s, \xi^{in}, \theta^{in}, \{C_{t,q}\}_{q\in\mathbb{N}, t\geq s})$.

Proof of Proposition 5.1. The existence of the probabilistically weak solution according to Definition 5.1 follows from Proposition 4.1 and an application of martingale representation theorem (e.g., [20, Theorem 8.2]) while the proof of stability result can follow that of [33, Theorem 5.1] with appropriate modifications concerning the differences in spatial dimension and fractional Laplacian, similarly to the proof of Proposition 4.1 (see also [58, Proposition 5.1]). □

Next, we have the following results as a consequence of Proposition 5.1; the proofs of analogous results from [33] did not rely on the specific form of the diffusive term or the spatial dimension and thus apply to our case.

Lemma 5.2. ([33, Proposition 5.2]) Let τ be a bounded stopping time of $(\bar{\mathcal{B}}_t)_{t\geq 0}$. Then for every $\omega \in \bar{\Omega}$ there exists $Q_\omega \in \mathcal{P}(\bar{\Omega})$ such that

$$Q_\omega(\{\omega' \in \bar{\Omega}\colon (\xi,\theta)(t,\omega') = (\xi,\theta)(t,\omega) \ \forall\, t \in [0,\tau(\omega)]\}) = 1, \tag{143a}$$

$$Q_\omega(A) = R_{\tau(\omega),\xi(\tau(\omega),\omega),\theta(\tau(\omega),\omega)}(A) \ \forall\, A \in \bar{\mathcal{B}}^{\tau(\omega)}, \tag{143b}$$

where $R_{\tau(\omega),\xi(\tau(\omega),\omega),\theta(\tau(\omega),\omega)} \in \mathcal{P}(\bar{\Omega})$ is a probabilistically weak solution to (3) with initial condition $(\xi(\tau(\omega),\omega), \theta(\tau(\omega),\omega))$ at initial time $\tau(\omega)$. Moreover, for every $A \in \bar{\mathcal{B}}$ the map $\omega \mapsto Q_\omega(A)$ is $\bar{\mathcal{B}}_\tau$-measurable.

Lemma 5.3. ([33, Proposition 5.3]) Let $\xi^{\text{in}} \in L^2_\sigma$ and P be a probabilistically weak solution to (3) on $[0,\tau]$ with initial condition $(\xi^{\text{in}}, 0)$ at initial time 0 according to Definition 5.2. In addition to the hypothesis of Lemma 5.2, suppose that there exists a Borel set $\mathcal{N} \subset \bar{\Omega}_\tau$ such that $P(\mathcal{N}) = 0$ and Q_ω from Lemma 5.2 satisfies for every $\omega \in \bar{\Omega}_\tau \setminus \mathcal{N}$

$$Q_\omega(\{\omega' \in \bar{\Omega}\colon \tau(\omega') = \tau(\omega)\}) = 1. \tag{144}$$

Then the probability measure $P \otimes_\tau R \in \mathcal{P}(\bar{\Omega})$ defined by

$$P \otimes_\tau R(\cdot) \triangleq \int_{\bar{\Omega}} Q_\omega(\cdot) P(d\omega) \tag{145}$$

satisfies $P \otimes_\tau R|_{\bar{\Omega}_\tau} = P|_{\bar{\Omega}_\tau}$ and it is a probabilistically weak solution to (3) on $[0,\infty)$ with initial condition $(\xi^{\text{in}}, 0)$ at initial time 0.

Now we fix an $\mathbb{R}$-valued Wiener process B on $(\Omega, \mathcal{F}, \mathbb{P})$ and apply Definitions 5.1-5.2, Proposition 5.1, and Lemmas 5.2-5.3 with $F(u) = u$ and such B. For $n \in \mathbb{N}, L > 1$, and $\delta \in (0, \frac{1}{12})$ we define similarly to (43a)-(43b)

$$\tau_L^n(\omega) \triangleq \inf\{t \geq 0\colon |\theta(t,\omega)| > (L - \frac{1}{n})^{\frac{1}{4}}\} \wedge \inf\{t > 0\colon \|\theta(\omega)\|_{C_t^{\frac{1}{2}-2\delta}} > (L - \frac{1}{n})^{\frac{1}{2}}\} \wedge L, \tag{146a}$$

$$\tau_L \triangleq \lim_{n\to\infty} \tau_L^n. \tag{146b}$$

It follows from [33, Lemma 3.5] that τ_L^n is a stopping time of $(\bar{\mathcal{B}}_t)_{t\geq 0}$ and thus so is τ_L. For the fixed $(\Omega, \mathcal{F}, \mathbf{P})$ we assume Theorem 2.3 and denote by u the solution constructed from Theorem 2.3 on $[0, \mathrm{t}]$ where $\mathrm{t} = T_L$ for L sufficiently large and

$$T_L \triangleq \inf\{t > 0\colon |B(t)| \geq L^{\frac{1}{4}}\} \wedge \inf\{t > 0\colon \|B\|_{C_t^{\frac{1}{2}-2\delta}} \geq L^{\frac{1}{2}}\} \wedge L \text{ with } \delta \in (0, \frac{1}{12}). \tag{147}$$

We observe that $T_L \nearrow +\infty$ $\mathbf{P}$-a.s. as $L \nearrow +\infty$. Let us also denote the law of (u, B) by P.

Proposition 5.4. *Let τ_L be defined by* (146b)*. Then P, the law of (u, B), is a probabilistically weak solution to* (3) *on $[0, \tau_L]$ according to Definition 5.2.*

Proof of Proposition 5.4. The proof is similar to that of Proposition 4.5 making use of the fact that

$$\theta(t,(u,B)) = B(t)\ \forall\ t \in [0,T_L]\ \mathbf{P}\text{-almost surely} \tag{148}$$

(see also the proofs of [33, Propositions 3.7 and 5.4] and [58, Proposition 4.5]). □

Next, we extend P on $[0, \tau_L]$ to $[0, \infty)$.

Proposition 5.5. *Let τ_L be defined by* (146b) *and P denote the law of (u, B) constructed from Theorem 2.3. Then the probability measure $P \otimes_{\tau_L} R$ in* (145) *is a probabilistically weak solution to* (3) *on $[0, \infty)$ according to Definition 5.1.*

Proof of Proposition 5.5. Because τ_L is a stopping time of $(\bar{\mathcal{B}}_t)_{t\geq 0}$ that is bounded by L due to (146a), the hypothesis of Lemma 5.2 is verified. By Proposition 5.4, P is a probabilistically weak solution to (3) on $[0, \tau_L]$. Therefore, Lemma 5.3 gives us the desired result once we verify the existence of a Borel set $\mathcal{N} \subset \bar{\Omega}_\tau$ such that $P(\mathcal{N}) = 0$ and (144) holds for every $\omega \in \bar{\Omega}_\tau \setminus \mathcal{N}$, and that can be achieved similarly to the proof of Proposition 4.6 (see also the proofs of [33, Propositions 3.8 and 5.5] and [58, Proposition 4.6]). □

Taking Theorem 2.3 for granted, we are now able to prove Theorem 2.4.

Proof of Theorem 2.4 assuming Theorem 2.3 . The proof is similar to that of Theorem 2.2 assuming Theorem 2.1 in Subsection 4.1; we sketch it for completeness. We fix $T > 0$ arbitrarily, any $\kappa \in (0, 1)$, and $K > 1$ such that $\kappa K^2 \geq 1$. The probability measure $P \otimes_{\tau_L} R$ from Proposition 5.5 satisfies

$$P \otimes_{\tau_L} R(\{\tau_L \geq T\}) \overset{(145)}{=} \mathbf{P}(\{\tau_L(u,B)) \geq T\}) \overset{(146)(147)(148)}{=} \mathbf{P}(\{T_L \geq T\}) > \kappa,$$

where the last inequality is due to Theorem 2.3. This leads us to $\mathbb{E}^{P\otimes_{\tau_L} R}[\|\xi(T)\|_{L_x^2}^2] > \kappa K^2 e^T \|\xi^{\text{in}}\|_{L_x^2}^2$, where ξ^{in} is the deterministic initial condition constructed through Theorem 2.3. On the other hand, via a classical Galerkin approximation scheme (e.g., [29]) one can readily construct a probabilistically weak solution Θ to (3) starting also from ξ^{in} such that $\mathbb{E}^{\Theta}[\|\xi(T)\|_{L_x^2}^2] \leq e^T \|\xi^{\text{in}}\|_{L_x^2}^2$. Because $\kappa K^2 \geq 1$, this implies the lack of uniqueness of probabilistically weak solution to (3) and equivalently the lack of joint uniqueness in law for (3), and consequently the non-uniqueness in law for (3) by [33, Theorem C.1], which is an infinite-dimensional version of [11, Theorem 3.1] due to Cherny. □

5.2. **Proof of Theorem 2.3 assuming Proposition 5.7.** We define $\Upsilon(t) \triangleq e^{B(t)}$ and $v \triangleq \Upsilon^{-1} u$ for $t \geq 0$. It follows from Ito's product formula (e.g., [1, Theorem 4.4.13]) on (3) that

$$\partial_t v + \frac{1}{2} v + (-\Delta)^m v + \Upsilon \text{div}(v \otimes v) + \Upsilon^{-1} \nabla \pi = 0, \quad \nabla \cdot v = 0, \quad t > 0. \tag{149}$$

Considering (149), for every $q \in \mathbb{N}_0$ we will construct $(v_q, \mathring{R}_q)$ that solves

$$\partial_t v_q + \frac{1}{2} v_q + (-\Delta)^m v_q + \Upsilon \text{div}(v_q \otimes v_q) + \nabla p_q = \text{div} \mathring{R}_q, \quad \nabla \cdot v_q = 0, \quad t > 0, \tag{150}$$

when $\mathring{R}_q$ is assumed to be a trace-free symmetric matrix. Similarly to (48) in the additive case, we continue to define $\lambda_q \triangleq a^{b^q}, \delta_q \triangleq \lambda_q^{-2\beta}$ for $a \in 10\mathbb{N}$, $b \in \mathbb{N}$, and $\beta \in (0, 1)$ so that the requirement of $\lambda_{q+1} \in 5\mathbb{N}$ of (19) is satisfied, while differently from (48) we define

$$M_0(t) \triangleq e^{4Lt+2L} \quad \text{and} \quad m_L \triangleq \sqrt{3}L^{\frac{1}{4}} e^{\frac{1}{2}L^{\frac{1}{4}}}. \tag{151}$$

Due to (147) we obtain for all $L > 1, \delta \in (0, \frac{1}{12})$, and $t \in [0, T_L]$

$$|B(t)| \leq L^{\frac{1}{4}} \text{ and } \|B\|_{C_t^{\frac{1}{2}-2\delta}} \leq L^{\frac{1}{2}} \tag{152}$$

which immediately implies

$$\|\Upsilon\|_{C_t^{\frac{1}{2}-2\delta}} + |\Upsilon(t)| + |\Upsilon^{-1}(t)| \leq e^{L^{\frac{1}{4}}} L^{\frac{1}{2}} + 2e^{L^{\frac{1}{4}}} \leq m_L^2. \tag{153}$$

For induction we assume that $(v_q, \mathring{R}_q)$ satisfy the following bounds on $[0, T_L]$:

$$\|v_q\|_{C_t L_x^2} \leq m_L M_0(t)^{\frac{1}{2}} (1 + \sum_{1 \leq \iota \leq q} \delta_\iota^{\frac{1}{2}}) \leq 2m_L M_0(t)^{\frac{1}{2}}, \tag{154a}$$

$$\|v_q\|_{C_{t,x}^1} \leq m_L M_0(t)^{\frac{1}{2}} \lambda_q^4, \tag{154b}$$

$$\|\mathring{R}_q\|_{C_t L_x^1} \leq M_0(t) c_R \delta_{q+1}, \tag{154c}$$

where $c_R > 0$ is again a universal constant to be determined subsequently and we assumed again $a^{\beta b} > 3$, as formally stated in (157), in order to deduce $\sum_{1 \leq \iota} \delta_\iota^{\frac{1}{2}} < \frac{1}{2}$.

Proposition 5.6. *Let $L > 1$ and define*

$$v_0(t, x) \triangleq \frac{m_L e^{2Lt+L}}{2\pi} \begin{pmatrix} \sin(x^2) & 0 \end{pmatrix}^T. \tag{155}$$

Then together with

$$\mathring{R}_0(t, x) \triangleq \frac{m_L(2L + \frac{1}{2})e^{2Lt+L}}{2\pi} \begin{pmatrix} 0 & -\cos(x^2) \\ -\cos(x^2) & 0 \end{pmatrix} + \mathcal{R}(-\Delta)^m v_0(t, x), \tag{156}$$

it satisfies (150) *at level $q = 0$. Moreover,* (154) *is satisfied at level $q = 0$ provided*

$$72\sqrt{3} < 8\sqrt{3}a^{2\beta b} \leq \frac{c_R e^{L - \frac{1}{2}L^{\frac{1}{4}}}}{L^{\frac{1}{4}}(2L + \frac{1}{2} + \pi)}, \quad L \leq a^4\pi - 1, \tag{157}$$

where the inequality $9 < a^{2\beta b}$ is assumed for the sake of second inequality in (154a)*. Furthermore, $v_0(0, x)$ and $\mathring{R}_0(0, x)$ are both deterministic.*

Proof of Proposition 5.6. The proof is similar to that of Proposition 4.7. Let us observe that v_0 is divergence-free, while $\mathring{R}_0$ is trace-free and symmetric. It may be immediately verified that $(v_0, \mathring{R}_0)$ solves (150) with $p_0 \equiv 0$ by using the fact that $(v_0 \cdot \nabla)v_0 = 0$ and Lemma 6.1. Next, for all $t \in [0, T_L]$ we can compute similarly to (54)

$$\|v_0(t)\|_{L_x^2} = \frac{m_L M_0(t)^{\frac{1}{2}}}{\sqrt{2}} \leq m_L M_0(t)^{\frac{1}{2}}, \ \|v_0\|_{C_{t,x}^1} = \frac{m_L(1 + L)M_0(t)^{\frac{1}{2}}}{\pi} \overset{(157)}{\leq} m_L M_0(t)^{\frac{1}{2}} \lambda_0^4. \tag{158}$$

Finally, using $\|\mathcal{R}(-\Delta)^m v_0\|_{L_x^2} \leq 4\|v_0\|_{L_x^2}$ due to $\Delta v_0 = -v_0$ and (56) we can compute

$$\|\mathring{R}_0(t)\|_{L_x^1} \leq m_L(2L + \frac{1}{2})M_0(t)^{\frac{1}{2}} 8 + (2\pi)4\|v_0(t)\|_{L_x^2} \overset{(157)(158)}{\leq} M_0(t) c_R \delta_1. \tag{159}$$

$\square$

We point out that

$$72\sqrt{3} < \frac{c_R e^{L-\frac{1}{2}L^{\frac{1}{4}}}}{L^{\frac{1}{4}}(2L+\frac{1}{2}+\pi)} \tag{160}$$

is not sufficient but necessary to satisfy (157).

Proposition 5.7. *Let $L > 1$ satisfy* (160) *and suppose that $(v_q, \mathring{R}_q)$ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution to* (150) *that satisfies* (154)*. Then there exists a choice of parameters $a, b,$ and β such that* (157) *is fulfilled and an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(v_{q+1}, \mathring{R}_{q+1})$ that satisfies* (150)*,* (154) *at level $q+1$, and*

$$\|v_{q+1}(t) - v_q(t)\|_{L_x^2} \leq m_L M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \quad \forall\, t \in [0, T_L]. \tag{161}$$

Furthermore, if $v_q(0,x)$ and $\mathring{R}_q(0,x)$ are deterministic, then so are $v_{q+1}(0,x)$ and $\mathring{R}_{q+1}(0,x)$.

Taking Proposition 5.7 for granted, we can now prove Theorem 2.3.

Proof of Theorem 2.3 assuming Proposition 5.7. This proof is similar to the proof of Theorem 2.1 assuming Proposition 4.8 in Subsection 4.2; we sketch it in the Appendix for completeness. ◻

5.3. Proof of Proposition 5.7.

5.3.1. *Choice of parameters.* We fix L sufficiently large so that it satisfies (160). We take the same choices of $m^*, \eta, \alpha, r, \mu,$ and σ in (65) - (68), and $b \in \{\iota \in \mathbb{N}: \iota > \frac{16}{\alpha}\}$ such that $r \in \mathbb{N}$ and $\lambda_{q+1}\sigma \in 10\mathbb{N}$ so that both requirements of $r \in \mathbb{N}$ and $\lambda_{q+1}\sigma \in 5\mathbb{N}$ from (19) are satisfied. Then we define $\beta > 0$ sufficiently small to satisfy (69) and l by (70) so that (71) remains valid. We take $a \in 10\mathbb{N}$ larger if necessary so that $a^{26} \geq \sqrt{3}L^{\frac{1}{4}}e^{\frac{1}{2}L^{\frac{1}{4}}}$; because $\alpha b > 16$ and $c_R \ll 1$ we see that this implies

$$m_L \overset{(151)}{\leq} a^{\frac{3\alpha b}{2}+2} \overset{(70)}{\leq} l^{-1} \text{ and } m_L \overset{(151)(160)}{\leq} c_R e^L \leq M_0(t)^{\frac{1}{2}}. \tag{162}$$

Lastly, taking $a \in 10\mathbb{N}$ even larger can guarantee $L \leq a^4\pi - 1$ in (157) while taking $\beta > 0$ even smaller if necessary allows the other inequalities in (157) to be satisfied, namely

$$72\sqrt{3} < 8\sqrt{3}a^{2\beta b} \leq \frac{c_R e^{L-\frac{1}{2}L^{\frac{1}{4}}}}{L^{\frac{1}{4}}(2L+\frac{1}{2}+\pi)}.$$

Thus, hereafter we consider such $m^*, \eta, \alpha, b,$ and l fixed, preserving our freedom to take $a \in 10\mathbb{N}$ larger and $\beta > 0$ smaller as needed.

5.3.2. *Mollification.* We mollify $v_q, \mathring{R}_q,$ and $\Upsilon(t) = e^{B(t)}$ by ϕ_l and φ_l again so that

$$v_l \triangleq (v_q *_x \phi_l) *_t \varphi_l, \quad \mathring{R}_l \triangleq (\mathring{R}_q *_x \phi_l) *_t \varphi_l, \quad \text{and} \quad \Upsilon_l \triangleq \Upsilon *_t \varphi_l. \tag{163}$$

By (150) we see that $v_l, \mathring{R}_l,$ and Υ_l satisfy

$$\partial_t v_l + \frac{1}{2}v_l + (-\Delta)^m v_l + \Upsilon_l \operatorname{div}(v_l \otimes v_l) + \nabla p_l = \operatorname{div}(\mathring{R}_l + R_{\text{com1}}) \tag{164}$$

where

$$p_l \triangleq (p_q *_x \phi_l) *_t \varphi_l - \frac{1}{2}(\Upsilon_l |v_l|^2 - ((\Upsilon |v_q|^2) *_x \phi_l) *_t \varphi_l), \tag{165a}$$

$$R_{\text{com1}} \triangleq R_{\text{commutator1}} \triangleq -((\Upsilon(v_q \mathring{\otimes} v_q)) *_x \phi_l) *_t \varphi_l + \Upsilon_l (v_l \mathring{\otimes} v_l). \tag{165b}$$

Next, making use of the fact that $\alpha b > 16$ and taking $a \in 10\mathbb{N}$ sufficiently large we obtain for all $t \in [0, T_L]$ and $N \geq 1$

$$\|v_q - v_l\|_{C_t L_x^2} \overset{(71)(154b)}{\lesssim} m_L M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{-\alpha} \leq \frac{m_L}{4} M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}, \tag{166a}$$

$$\|v_l\|_{C_t L_x^2} \overset{(154a)}{\leq} m_L M_0(t)^{\frac{1}{2}} (1 + \sum_{1 \leq \iota \leq q} \delta_\iota^{\frac{1}{2}}) \overset{(157)}{\leq} 2 m_L M_0(t)^{\frac{1}{2}}, \tag{166b}$$

$$\|v_l\|_{C_{t,x}^N} \overset{(154b)}{\lesssim} l^{-N+1} m_L M_0(t)^{\frac{1}{2}} \lambda_q^4 \overset{(70)}{\leq} l^{-N} m_L M_0(t)^{\frac{1}{2}} \lambda_{q+1}^{-\alpha}. \tag{166c}$$

5.3.3. *Perturbation.* We proceed with the same definition of χ in (76) and ρ in (77) identically except that $M_0(t)$ is now defined by (151) instead of (48). Although our definition of $\mathring{R}_0$ in (156) differs from that of (52), the estimates of (78) and (79) remain valid as their proofs depend only on the definitions of ρ and χ, not $M_0(t)$ or $\mathring{R}_l$. We define a modified amplitude function to be

$$\bar{a}_\zeta(\omega, t, x) \triangleq \bar{a}_{\zeta,q+1}(\omega, t, x) \triangleq \Upsilon_l^{-\frac{1}{2}} a_\zeta(\omega, t, x), \tag{167}$$

where $a_\zeta(\omega, t, x)$ is identical to that defined in (82). For convenience let us observe a simple estimate of

$$\|\Upsilon_l^{-\frac{1}{2}}\|_{C_t} \overset{(151)(152)}{\leq} m_L. \tag{168}$$

Using this estimate, for all $t \in [0, T_L]$ by taking $c_R \ll M^{-4}$ we can obtain

$$\|\bar{a}_\zeta\|_{C_t L_x^2} \overset{(78)(79)}{\leq} m_L \sqrt{12}[4\pi^2 c_R \delta_{q+1} M_0(t) + \|\mathring{R}_l(\omega)\|_{C_t L_x^1}]^{\frac{1}{2}} (\frac{M}{C_\Lambda}) \overset{(16)}{\leq} \frac{c_R^{\frac{1}{4}} m_L M_0(t)^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}}{2|\Lambda|}. \tag{169}$$

Because (50c) and (154c) are identical except the definitions of $M_0(t)$, tracing the proof of (80) we see that we still have (80) which leads us to (81) as well as (85). For all $t \in [0, T_L]$, $N \geq 0$ and $k \in \{0, 1, 2\}$, along with (168) this allows us to deduce the estimates of

$$\|\bar{a}_\zeta\|_{C_t C_x^N} \overset{(85)}{\leq} m_L c_R^{\frac{1}{4}} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2}-4N}, \quad \|\bar{a}_\zeta\|_{C_t^1 C_x^k} \overset{(85)(153)(162)}{\leq} m_L c_R^{\frac{1}{8}} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{13}{2}-4k}, \tag{170}$$

where we took $c_R \ll 1$ to eliminate implicit constants in the second inequality.

Now we define $w_{q+1}^{(p)}$ and $w_{q+1}^{(c)}$ as in (87) with a_ζ replaced by $\bar{a}_\zeta$ from (167) and $M_0(t)$ from (151) within the definition of $\rho(\omega, t, x)$, and finally $w_{q+1}^{(t)}$ identically as in (87) with a_ζ from (82), only with $M_0(t)$ from (151). Then we define the perturbation identically as in (86):

$$w_{q+1} \triangleq w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} \text{ and } v_{q+1} \triangleq v_l + w_{q+1}. \tag{171}$$

We see that as a consequence of (88)

$$(w_{q+1}^{(p)} + w_{q+1}^{(c)})(t, x) \overset{(88)(167)}{=} \Upsilon_l^{-\frac{1}{2}}(t) \nabla^\perp (\sum_{\zeta \in \Lambda} a_\zeta(t, x) \eta_\zeta(t, x) \psi_\zeta(x)). \tag{172}$$

Consequently, we see that w_{q+1} is both divergence-free and mean-zero. Next, the following estimates for all $t \in [0, T_L]$ and $p \in (1, \infty)$ are essentially immediate consequences of (89), (90a), (90b), and (168):

$$\|w_{q+1}^{(p)}\|_{C_t L_x^2} \overset{(167)}{\leq} m_L \sum_{\zeta \in \Lambda} \|a_\zeta \mathbb{W}_\zeta\|_{C_t L_x^2} \overset{(89)}{\lesssim} m_L c_R^{\frac{1}{4}} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}}, \tag{173a}$$

$$\|w_{q+1}^{(p)}\|_{C_t L_x^p} \overset{(167)}{\leq} m_L \sup_{s \in [0,t]} \sum_{\zeta \in \Lambda} \|a_\zeta(s)\|_{L_x^\infty} \|\mathbb{W}_\zeta(s)\|_{L_x^p} \overset{(90a)}{\lesssim} m_L \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p}}, \tag{173b}$$

$$\|w_{q+1}^{(c)}\|_{C_tL_x^p} \overset{(167)}{\leq} m_L \sum_{\zeta\in\Lambda} \|\nabla^\perp(a_\zeta\eta_\zeta)\|_{C_tL_x^p}\|\psi_\zeta\|_{L_x^\infty} \overset{(90b)}{\lesssim} m_L\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{11}{2}}\sigma r^{2-\frac{2}{p}}. \tag{173c}$$

Finally, the estimate of $\|w_{q+1}^{(t)}\|_{C_tL_x^p}$ in (90c) remains valid. Therefore, for all $t \in [0, T_L]$ we can estimate from (171) by taking $c_R \ll 1$ and $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}\|w_{q+1}\|_{C_tL_x^2} &\overset{(90c)(173a)(173c)}{\lesssim} m_L c_R^{\frac{1}{4}}\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} + m_L\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{11}{2}}\sigma r + \mu^{-1}\delta_{q+1}M_0(t) l^{-3} r \\ &\overset{(71)}{\leq} m_L M_0(t)^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}[\frac{3}{8} + C\lambda_{q+1}^{11\alpha-4\eta} + CM_0(L)^{\frac{1}{2}}\lambda_{q+1}^{6\alpha-2\eta}] \\ &\leq \frac{3m_L M_0(t)^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}}{4},\end{aligned} \tag{174}$$

where the last inequality used the facts that $11\alpha - 4\eta < 0$ and $6\alpha - 2\eta < 0$, both of which may be readily verified by (65)-(67). It follows from similar computations to (93) that (154a) at level $q + 1$ and (161) can now be verified as follows:

$$\|v_{q+1}\|_{C_tL_x^2} \overset{(171)}{\leq} \|v_l\|_{C_tL_x^2} + \|w_{q+1}\|_{C_tL_x^2} \overset{(166b)(174)}{\leq} m_L M_0(t)^{\frac{1}{2}}(1 + \sum_{1\leq\iota\leq q+1}\delta_\iota^{\frac{1}{2}}),$$

$$\|v_{q+1}(t) - v_q(t)\|_{L_x^2} \overset{(166a)(171)}{\leq} \|w_{q+1}\|_{C_tL_x^2} + \frac{m_L}{4}M_0(t)^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}} \overset{(174)}{\leq} m_L M_0(t)^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}.$$

Next, we estimate for all $t \in [0, T_L]$

$$\begin{aligned}\|w_{q+1}^{(p)}\|_{C_{t,x}^1} &\leq \sum_{\zeta\in\Lambda}\|\bar{a}_\zeta\|_{C_{t,x}^1}\|\mathbb{W}_\zeta\|_{C_{t,x}^1} \\ &\overset{(25a)(170)}{\lesssim} (m_L\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}} l^{-\frac{13}{2}})\lambda_{q+1}\sigma\mu r^2 \leq m_L M_0(t)^{\frac{1}{2}} l^{-\frac{13}{2}}\lambda_{q+1}\sigma\mu r^2,\end{aligned} \tag{175a}$$

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{C_{t,x}^1} &\leq \sum_{\zeta\in\Lambda}\|\nabla^\perp(\bar{a}_\zeta\eta_\zeta)\psi_\zeta\|_{C_{t,x}^1} \overset{(14b)(25b)(170)}{\lesssim} m_L\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}\lambda_{q+1}^{1-6\eta} \\ &\times [l^{-\frac{21}{2}}\lambda_{q+1}^{-1} + l^{-\frac{11}{2}}\lambda_{q+1}^{1-8\eta} + l^{-\frac{13}{2}}\lambda_{q+1}^{-4\eta} + l^{-\frac{3}{2}}\lambda_{q+1}^{2-12\eta}] \lesssim m_L\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}}\lambda_{q+1}^{3-18\eta}l^{-\frac{3}{2}},\end{aligned} \tag{175b}$$

where we used $\delta_{q+1}^{\frac{1}{2}}$ to eliminate implicit constant in (175a). On the other hand, the estimate of $\|w_{q+1}^{(t)}\|_{C_{t,x}^1}$ from (95) remains applicable for us. We may now verify (154b) at level $q + 1$ as follows. For any $t \in [0, T_L]$

$$\begin{aligned}\|v_{q+1}\|_{C_{t,x}^1} &\overset{(166c)(171)}{\leq} l^{-1}m_L M_0(t)^{\frac{1}{2}}\lambda_{q+1}^{-\alpha} + \|w_{q+1}^{(p)}\|_{C_{t,x}^1} + \|w_{q+1}^{(c)}\|_{C_{t,x}^1} + \|w_{q+1}^{(t)}\|_{C_{t,x}^1} \\ &\leq m_L M_0(t)^{\frac{1}{2}}[l^{-1}\lambda_{q+1}^{-\alpha} + C\lambda_{q+1}^{13\alpha+3-14\eta} + C\lambda_{q+1}^{3-18\eta}l^{-\frac{3}{2}} + C\lambda_{q+1}^{3-16\eta+\alpha}M_0(t)^{\frac{1}{2}}l^{-3}] \leq m_L M_0(t)^{\frac{1}{2}}\lambda_{q+1}^4\end{aligned} \tag{176}$$

where the last inequality used (97) and that $13\alpha + 3 - 14\eta < 4$ which can be readily verified by (65)-(67). Next, as a consequence of (88) we have the identity of

$$(w_{q+1}^{(p)} + w_{q+1}^{(c)})(t, x) \overset{(167)}{=} \Upsilon_l^{-\frac{1}{2}}(t)\nabla^\perp(\sum_{\zeta\in\Lambda} a_\zeta(t, x)\eta_\zeta(t, x)\psi_\zeta(x)). \tag{177}$$

This allows us to estimate for all $t \in [0, T_L]$ and $p \in (1, \infty)$, by utilizing (98a) and (168)

$$\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_tW_x^{1,p}} \leq \|\Upsilon_l^{-\frac{1}{2}}\|_{C_t}\|\nabla^\perp\sum_{\zeta\in\Lambda} a_\zeta\eta_\zeta\psi_\zeta\|_{C_tW_x^{1,p}} \lesssim m_L\delta_{q+1}^{\frac{1}{2}}M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p}} l^{-\frac{3}{2}}\lambda_{q+1}. \tag{178}$$

On the other hand, the estimate of $\|w_{q+1}^{(t)}\|_{C_tW_x^{1,p}}$ from (98b) remains applicable for us.

5.3.4. *Reynolds stress.* We can choose the same p^* from (117) and compute from (150), (164), and (171)

$$\begin{aligned}
&\operatorname{div}\mathring{R}_{q+1} - \nabla p_{q+1} \qquad (179)\\
=&\underbrace{\frac{1}{2}w_{q+1} + (-\Delta)^m w_{q+1} + \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \Upsilon_l \operatorname{div}(v_l \otimes w_{q+1} + w_{q+1} \otimes v_l)}_{\operatorname{div}(R_{\text{lin}})+\nabla p_{\text{lin}}}\\
&+ \underbrace{\Upsilon_l \operatorname{div}((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)}))}_{\operatorname{div}(R_{\text{cor}})+\nabla p_{\text{cor}}}\\
&+ \underbrace{\operatorname{div}(\Upsilon_l w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_l) + \partial_t w_{q+1}^{(t)}}_{\operatorname{div}(R_{\text{osc}})+\nabla p_{\text{osc}}} + \underbrace{(\Upsilon - \Upsilon_l)\operatorname{div}(v_{q+1} \otimes v_{q+1})}_{\operatorname{div}(R_{\text{com2}})+\nabla p_{\text{com2}}} + \operatorname{div}(R_{\text{com1}}) - \nabla p_l
\end{aligned}$$

where

$$\begin{aligned}
R_{\text{lin}} \triangleq & R_{\text{linear}}\\
\triangleq & \mathcal{R}(\frac{1}{2}w_{q+1} + (-\Delta)^m w_{q+1} + \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})) + \Upsilon_l(v_l \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_l), \qquad (180a)\\
p_{\text{lin}} \triangleq & p_{\text{linear}} \triangleq \Upsilon_l(v_l \cdot w_{q+1}), \qquad (180b)\\
R_{\text{cor}} \triangleq & R_{\text{corrector}} \triangleq \Upsilon_l((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} (w_{q+1}^{(c)} + w_{q+1}^{(t)})), \qquad (180c)\\
p_{\text{cor}} \triangleq & p_{\text{corrector}} \triangleq \frac{\Upsilon_l}{2}((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \cdot w_{q+1} + w_{q+1}^{(p)} \cdot (w_{q+1}^{(c)} + w_{q+1}^{(t)})), \qquad (180d)\\
R_{\text{com2}} \triangleq & R_{\text{commutator2}} \triangleq (\Upsilon - \Upsilon_l)(v_{q+1} \mathring{\otimes} v_{q+1}), \qquad (180e)\\
p_{\text{com2}} \triangleq & p_{\text{commutator2}} \triangleq \frac{\Upsilon - \Upsilon_l}{2}|v_{q+1}|^2. \qquad (180f)
\end{aligned}$$

Concerning R_{osc} and p_{osc} we have

$$\begin{aligned}
&\operatorname{div}(\Upsilon_l w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_l) + \partial_t w_{q+1}^{(t)} \qquad (181)\\
\overset{(167)}{=}& \operatorname{div}((\sum_{\zeta \in \Lambda} a_\zeta \mathbb{W}_\zeta) \otimes (\sum_{\zeta' \in \Lambda} a_{\zeta'} \mathbb{W}_{\zeta'}) + \mathring{R}_l) + \partial_t w_{q+1}^{(t)}\\
\overset{(114)}{=}& \frac{1}{2}\sum_{\zeta,\zeta' \in \Lambda} \mathcal{E}_{\zeta,\zeta',1} + \frac{1}{2}\sum_{\zeta,\zeta' \in \Lambda}\sum_{k=1,3,4} \mathcal{E}_{\zeta,\zeta',2,k} + A_2 + A_3\\
&+ \nabla[\frac{1}{2}|\sum_{\zeta \in \Lambda} a_\zeta \mathbb{W}_\zeta|^2 + \frac{1}{2}\sum_{\zeta,\zeta' \in \Lambda} \mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'} \mathbb{P}_{\geq \frac{\lambda_{q+1}}{10}}(\eta_\zeta \eta_{\zeta'} \lambda_{q+1}^2 \psi_\zeta \psi_{\zeta'}))\\
&+ \frac{1}{2}\sum_{\zeta \in \Lambda} a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} \eta_\zeta^2 - \Delta^{-1}\nabla \cdot \mu^{-1}(\sum_{\zeta \in \Lambda^+} - \sum_{\zeta \in \Lambda^-})\mathbb{P}_{\neq 0}\partial_t(a_\zeta^2 \mathbb{P}_{\neq 0}\eta_\zeta^2 \zeta)].
\end{aligned}$$

Therefore, we can define similarly to (115a) - (115b)

$$R_{\text{osc}} \triangleq R_{\text{oscillation}} \triangleq \mathcal{R}(\frac{1}{2}\sum_{\zeta,\zeta' \in \Lambda} \mathcal{E}_{\zeta,\zeta',1} + \frac{1}{2}\sum_{\zeta,\zeta' \in \Lambda}\sum_{k=1,3,4} \mathcal{E}_{\zeta,\zeta',2,k} + A_2 + A_3), \qquad (182a)$$

$$\begin{aligned}
p_{\text{osc}} \triangleq p_{\text{oscillation}} \triangleq & \frac{1}{2}|\sum_{\zeta \in \Lambda} a_\zeta \mathbb{W}_\zeta|^2 + \frac{1}{2}\sum_{\zeta,\zeta' \in \Lambda} \mathbb{P}_{\neq 0}(a_\zeta a_{\zeta'} \mathbb{P}_{\geq \frac{\lambda_{q+1}}{10}}(\eta_\zeta \eta_{\zeta'} \lambda_{q+1}^2 \psi_\zeta \psi_{\zeta'}))1_{\zeta+\zeta' \neq 0}\\
&+ \frac{1}{2}\sum_{\zeta \in \Lambda} a_\zeta^2 \mathbb{P}_{\geq \frac{\lambda_{q+1}\sigma}{2}} \eta_\zeta^2 - \Delta^{-1}\nabla \cdot \mu^{-1}(\sum_{\zeta \in \Lambda^+} - \sum_{\zeta \in \Lambda^-})\mathbb{P}_{\neq 0}\partial_t(a_\zeta^2 \mathbb{P}_{\neq 0}\eta_\zeta^2 \zeta) \qquad (182b)
\end{aligned}$$

and claim the same bound as in (132) for R_{osc}. Thus, let us define formally

$$p_{q+1} \triangleq -p_{\text{lin}} - p_{\text{cor}} - p_{\text{osc}} - p_{\text{com2}} + p_l \text{ and } \mathring{R}_{q+1} \triangleq R_{\text{lin}} + R_{\text{cor}} + R_{\text{osc}} + R_{\text{com2}} + R_{\text{com1}}. \quad (183)$$

Now we compute for all $t \in [0, T_L]$ from (180a)

$$\begin{aligned}\|R_{\text{lin}}\|_{C_t L_x^{p^*}} \lesssim & \|w_{q+1}\|_{C_t L_x^{p^*}} + \|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_t L_x^{p^*}} \\ & + \|\mathcal{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_t L_x^{p^*}} + \|\Upsilon_l(v_l \mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes} v_l)\|_{C_t L_x^{p^*}}. \end{aligned} \quad (184)$$

First, by the estimate of $m_L \leq M_0(t)^{\frac{1}{2}}$ from (162) we can compute from (171) for all $t \in [0, T_L]$

$$\begin{aligned}\|w_{q+1}\|_{C_t L_x^{p^*}} \overset{(90c)(173b)(173c)}{\lesssim} & m_L \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} + m_L \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{11}{2}} \sigma r^{2-\frac{2}{p^*}} \\ & + \mu^{-1}\delta_{q+1} M_0(t) l^{-3} r^{2-\frac{2}{p^*}} \overset{(65)(66)(67)(71)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} m_L l^{-\frac{3}{2}}.\end{aligned} \quad (185)$$

By Gagliardo-Nirenberg's inequality this also leads us to

$$\begin{aligned}\|\mathcal{R}(-\Delta)^m w_{q+1}\|_{C_t L_x^{p^*}} \lesssim & [\delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} m_L l^{-\frac{3}{2}}]^{1-m^*} [\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_t W_x^{1,p^*}} + \|w_{q+1}^{(t)}\|_{C_t W_x^{1,p^*}}]^{m^*} \\ & \overset{(98b)(178)}{\lesssim} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} m_L l^{-\frac{3}{2}} \lambda_{q+1}^{m^*}.\end{aligned} \quad (186)$$

Second, for all $t \in [0, T_L]$ we can make use of (119) and (168) and estimate

$$\begin{aligned}& \|\mathcal{R}\partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_t L_x^{p^*}} \\ \overset{(172)}{\lesssim} & \sum_{\zeta \in \Lambda} \|\Upsilon_l^{-\frac{1}{2}}\|_{C_t}^3 \|\partial_t \Upsilon_l\|_{C_t} \|a_\zeta\|_{C_t C_x} \|\eta_\zeta\|_{C_t L_x^{p^*}} \|\psi_\zeta\|_{C_x} + \|\Upsilon_l^{-\frac{1}{2}}\|_{C_t} \|\partial_t(a_\zeta \eta_\zeta)\psi_\zeta\|_{C_t C_x} \\ \overset{(14a)(25b)(85)(119)}{\lesssim} & m_L^3 l^{-1} \|\Upsilon\|_{C_t} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} l^{-\frac{3}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1}^{-1} \\ & + m_L \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} l^{-\frac{3}{2}} \lambda_{q+1}^{1-8\eta} \overset{(153)}{\lesssim} m_L l^{-\frac{3}{2}} \delta_{q+1}^{\frac{1}{2}} M_0(t)^{\frac{1}{2}} r^{1-\frac{2}{p^*}} \lambda_{q+1}^{1-8\eta}.\end{aligned} \quad (187)$$

Third, we can estimate for all $t \in [0, T_L]$

$$\begin{aligned}\|\Upsilon_l(v_l \mathring{\otimes} w_{q+1} + w_{q+1}\mathring{\otimes} v_l)\|_{C_t L_x^{p^*}} \lesssim & \|\Upsilon\|_{C_t} \|v_q\|_{C_{t,x}^1} \|w_{q+1}\|_{C_t L_x^{p^*}} \\ \overset{(153)(154b)(185)}{\lesssim} & m_L^4 M_0(t) \lambda_q^4 r^{1-\frac{2}{p^*}} \delta_{q+1}^{\frac{1}{2}} l^{-\frac{3}{2}}.\end{aligned} \quad (188)$$

Hence, applying (185)-(188) to (184) and taking $a \in 10\mathbb{N}$ sufficiently large give us

$$\begin{aligned}\|R_{\text{lin}}\|_{C_t L_x^{p^*}} \overset{(68)(71)}{\lesssim} & M_0(t)\delta_{q+2}[\lambda_{q+2}^{2\beta}(\lambda_{q+1}^{1-6\eta})^{1-\frac{2}{p^*}} m_L \lambda_{q+1}^{3\alpha} \lambda_{q+1}^{m^*} \\ & + \lambda_{q+2}^{2\beta} m_L \lambda_{q+1}^{3\alpha} (\lambda_{q+1}^{1-6\eta})^{1-\frac{2}{p^*}} \lambda_{q+1}^{1-8\eta} + \lambda_{q+2}^{2\beta} m_L^4 \lambda_{q+1}^{\frac{\alpha}{4}} (\lambda_{q+1}^{1-6\eta})^{1-\frac{2}{p^*}} \lambda_{q+1}^{3\alpha}] \\ \overset{(69)(117)}{\lesssim} & M_0(t)\delta_{q+2}[m_L \lambda_{q+1}^{-\frac{275\alpha}{8}} + m_L^4 \lambda_{q+1}^{\frac{-273\alpha-8+64\eta}{8}}] \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t) c_R \delta_{q+2}}{5}\end{aligned} \quad (189)$$

where we used the facts that $2\beta b < \frac{\alpha}{8}$ due to (69) and $-273\alpha - 8 + 64\eta \leq -273\alpha - 8m^* < 0$ due to (66).

Next, for all $t \in [0, T_L]$ we estimate from (180c) by taking $a \in 10\mathbb{N}$ sufficiently large

$$\begin{aligned}& \|R_{\text{cor}}\|_{C_t L_x^{p^*}} \\ \lesssim & \|\Upsilon_l\|_{C_t}(\|w_{q+1}^{(c)}\|_{C_t L_x^{2p^*}} + \|w_{q+1}^{(t)}\|_{C_t L_x^{2p^*}})(\|w_{q+1}^{(c)}\|_{C_t L_x^{2p^*}} + \|w_{q+1}^{(t)}\|_{C_t L_x^{2p^*}} + \|w_{q+1}^{(p)}\|_{C_t L_x^{2p^*}}) \\ \overset{(68)(71)(153)}{\lesssim} & m_L^2 M_0(t)[m_L \lambda_{q+1}^{-\frac{31\alpha}{4}-3\eta} + \lambda_{q+1}^{-\eta-34\alpha}][m_L \lambda_{q+1}^{-\frac{31\alpha}{4}-3\eta} + \lambda_{q+1}^{-\eta-34\alpha} + m_L \lambda_{q+1}^{\eta-\frac{63\alpha}{4}}]\end{aligned} \quad (190)$$

$$\lesssim M_0(t)\delta_{q+2}\lambda_{q+2}^{2\beta} m_L^3 \lambda_{q+1}^{-34\alpha-\frac{63\alpha}{4}} \overset{(69)}{\lesssim} M_0(t)\delta_{q+2} m_L^3 \lambda_{q+1}^{-34\alpha-\frac{125\alpha}{8}} \leq (2\pi)^{-2(\frac{p^*-1}{p^*})} \frac{M_0(t)c_R\delta_{q+2}}{5}.$$

Next, for all $t \in [0, T_L]$ we estimate using (168), $\lambda_q^4 l^{\frac{1}{2}-2\delta} \lesssim \delta_{q+2}\lambda_q^{-\frac{8}{3}}$ from (133), and taking $a \in 10\mathbb{N}$ sufficiently large

$$\|R_{\text{com1}}\|_{C_tL_x^1} \overset{(165b)(153)(154)}{\lesssim} m_L^4 M_0(t) l^{\frac{1}{2}-2\delta}\lambda_q^4 \overset{(133)}{\lesssim} M_0(t)\delta_{q+2} m_L^4 \lambda_q^{-\frac{8}{3}} \leq \frac{c_R M_0(t)\delta_{q+2}}{5}. \quad (191)$$

Finally, using $|\Upsilon_l(t) - \Upsilon(t)| \overset{(153)}{\lesssim} l^{\frac{1}{2}-2\delta} m_L^2$, and $\lambda_q^4 l^{\frac{1}{2}-2\delta} \lesssim \delta_{q+2}\lambda_q^{-\frac{8}{3}}$ from (133) again, and taking $a \in 10\mathbb{N}$ sufficiently large we obtain for all $t \in [0, T_L]$

$$\|R_{\text{com2}}\|_{C_tL_x^1} \overset{(180e)}{\leq} \|\Upsilon_l - \Upsilon\|_{C_t} \|v_{q+1}\|_{C_tL_x^2}^2 \overset{(166b)(174)}{\lesssim} l^{\frac{1}{2}-2\delta} m_L^4 M_0(t) \leq \frac{M_0(t)c_R\delta_{q+2}}{5}. \quad (192)$$

Therefore, considering (189), (190), (132), (191), and (192), we are able to conclude that $\|\mathring{R}_{q+1}\|_{C_tL_x^1} \leq M_0(t)c_R\delta_{q+2}$ identically as we did in (135). This verifies (154c) at level $q+1$.

Finally, essentially identical arguments in the proof of Proposition 4.8 shows that $(v_q, \mathring{R}_q)$ being $(\mathcal{F}_t)_{t\geq 0}$-adapted leads to $(v_{q+1}, \mathring{R}_{q+1})$ being $(\mathcal{F}_t)_{t\geq 0}$-adapted, and that $(v_q, \mathring{R}_q)(0, x)$ being deterministic implies $(v_{q+1}, \mathring{R}_{q+1})(0, x)$ being deterministic.

6. Appendix

6.1. **Past results.** We collect results from previous works which were used in the proofs of Theorems 2.1-2.4.

Lemma 6.1. ([14, Definition 9, Lemma 10], also [42, Definition 7.1, Lemmas 7.2 and 7.3]) For $f \in C(\mathbb{T}^2)$, set

$$\mathcal{R}f \triangleq \nabla g + (\nabla g)^T - (\nabla \cdot g)\text{Id}, \quad (193)$$

where $\Delta g = f - \fint_{\mathbb{T}^2} f dx$ and $\fint_{\mathbb{T}^2} g dx = 0$. Then for any $f \in C(\mathbb{T}^2)$ such that $\fint_{\mathbb{T}^2} f dx = 0$, $\mathcal{R}f(x)$ is a trace-free symmetric matrix for all $x \in \mathbb{T}^2$. Moreover, $\nabla \cdot \mathcal{R}f = f$ and $\fint_{\mathbb{T}^2} \mathcal{R}f(x)dx = 0$. When f is not mean-zero, we overload the notation and denote by $\mathcal{R}f \triangleq \mathcal{R}(f - \fint_{\mathbb{T}^2} f dx)$. Finally, for all $p \in (1, \infty)$, $\|\mathcal{R}\|_{L_x^p \mapsto W_x^{1,p}} \lesssim 1$, $\|\mathcal{R}\|_{C_x \mapsto C_x} \lesssim 1$, and $\|\mathcal{R}f\|_{L_x^p} \lesssim \|(-\Delta)^{-\frac{1}{2}} f\|_{L_x^p}$.

Lemma 6.2. ([42, Lemma 6.2]) Let $f, g \in C^\infty(\mathbb{T}^2)$ where g is also $(\mathbb{T}/\kappa)^2$-periodic for some $\kappa \in \mathbb{N}$. Then there exists a constant $C \geq 0$ such that

$$\|fg\|_{L_x^2} \leq \|f\|_{L_x^2}\|g\|_{L_x^2} + C\kappa^{-\frac{1}{2}}\|f\|_{C_x^1}\|g\|_{L_x^2}. \quad (194)$$

Lemma 6.3. ([42, Lemma 7.4]) For any given $p \in (1, \infty), \lambda \in \mathbb{N}, a \in C^2(\mathbb{T}^2)$, and $f \in L^p(\mathbb{T}^2)$,

$$\|(-\Delta)^{-\frac{1}{2}}\mathbb{P}_{\neq 0}(a\mathbb{P}_{\geq\lambda} f)\|_{L_x^p} \lesssim \lambda^{-1}\|a\|_{C_x^2}\|f\|_{L_x^p}. \quad (195)$$

6.2. **Continuation of the proof of Proposition 4.1.** First, the proof of the following result from [33] in case $x \in \mathbb{T}^3$ goes through verbatim in case $x \in \mathbb{T}^2$.

Lemma 6.4. ([33, Lemma A.1]) Let $\{(s_n, \xi_n)\}_{n\in\mathbb{N}} \subset [0, \infty) \times L_\sigma^2$ be a family such that $\lim_{n\to\infty} \|(s_n, \xi_n) - (s, \xi^{\text{in}})\|_{\mathbb{R}\times L_x^2} = 0$ and $\{P_n\}_{n\in\mathbb{N}}$ be a family of probability measures on Ω_0 satisfying for all $n \in \mathbb{N}$, $P_n(\{\xi(t) = \xi_n \ \forall\, t \in [0, s_n]\}) = 1$ and for some $\gamma, \kappa > 0$ and any $T > 0$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{P_n}[\|\xi\|_{C([0,T];L_x^2)} + \sup_{r,t\in[0,T]:\, r\neq t} \frac{\|\xi(t) - \xi(r)\|_{H_x^{-3}}}{|t - r|^\kappa} + \|\xi\|_{L^2([s_n,T];H_x^\gamma)}^2] < \infty. \quad (196)$$

Then $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{M} \triangleq C_{\text{loc}}([0, \infty); H^{-3}(\mathbb{T}^2)) \cap L_{\text{loc}}^2([0, \infty); L_\sigma^2)$.

Now we fix $\{P_n\} \subset \mathcal{C}(s_n, \xi_n, \{C_{t,q}\}_{q\in\mathbb{N}, t\geq s_n})$ and will show that it is tight in $\mathbb{M}$ by relying on Lemma 6.4. We define $J(\xi) \triangleq -\mathbb{P}\mathrm{div}(\xi\otimes\xi) - (-\Delta)^m \xi$. By definition of $\mathcal{C}(s_n, \xi_n, \{C_{t,q}\}_{q\in\mathbb{N}, t\geq s_n})$ and (M2) of Definition 4.1, we know that for all $n \in \mathbb{N}$ and $t \in [s_n, \infty)$

$$\xi(t) = \xi_n + \int_{s_n}^t J(\xi(r))dr + M_{t,s_n}^{\xi} \quad P_n\text{-a.s.}, \tag{197}$$

where the map $t \mapsto M_{t,s_n}^{\xi,i} \triangleq \langle M_{t,s_n}^{\xi}, \mathrm{g}_i \rangle$ for $\xi \in \Omega_0$ and $\mathrm{g}_i \in C^\infty(\mathbb{T}^2) \cap L_\sigma^2$ is a continuous, square-integrable martingale w.r.t. $(\mathcal{B}_t)_{t\geq s_n}$ such that $\langle\langle M_{t,s_n}^{\xi,i} \rangle\rangle = \int_{s_n}^t \|G(\xi(r))^* \mathrm{g}_i\|_U^2 dr$. We can compute for any $p \in (1, \infty)$,

$$\mathbb{E}^{P_n}[\sup_{r,t\in[s_n,T]:\, r\neq t} \frac{\|\int_r^t J(\xi(l))dl\|_{H_x^{-3}}^p}{|t-r|^{p-1}}] \leq \mathbb{E}^{P_n}[\int_{s_n}^T (\|\xi\otimes\xi\|_{H_x^{-2}} + \|\xi\|_{H_x^{2m-3}})^p dl]$$

by Hölder's inequality where $\|\xi\otimes\xi\|_{H_x^{-2}} \lesssim \|\xi\|_{L_x^2}^2$ and $\|\xi\|_{H_x^{2m-3}} \lesssim 1 + \|\xi\|_{L_x^2}^2$ because $m \in (0,1)$. Therefore,

$$\mathbb{E}^{P_n}[\sup_{r,t\in[s_n,T]:\, r\neq t} \frac{\|\int_r^t J(\xi(l))dl\|_{H_x^{-3}}^p}{|t-r|^{p-1}}] \overset{(M3)}{\lesssim}_p TC_{T,p}(1 + \|\xi_n\|_{L_x^2}^{2p}). \tag{198}$$

On the other hand, making use of (10), (M2) and (M3) of Definition 4.1 and Kolmogorov's test (e.g., [20, Theorem 3.3]) gives us for any $\alpha \in (0, \frac{p-1}{2p})$

$$\mathbb{E}^{P_n}[\sup_{r,t\in[0,T]:\, r\neq t} \frac{\|M_{t,s_n}^{\xi} - M_{r,s_n}^{\xi}\|_{L_x^2}}{|t-r|^{\alpha}}] \lesssim_p C_{t,p}(1 + \|\xi_n\|_{L_x^2}^{2p}). \tag{199}$$

Making use of (197)-(199) leads to for all $\kappa \in (0, \frac{1}{2})$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{P_n}[\sup_{r,t\in[0,T]:\, r\neq t} \frac{\|\xi(t) - \xi(r)\|_{H_x^{-3}}}{|t-r|^{\kappa}}] < \infty. \tag{200}$$

Hence, (M1), (28) with $q = 1$, and (200) together allow us to deduce that $\{P_n\}$ is tight in $\mathbb{M}$ by Lemma 6.4. By Prokhorov's theorem (e.g., [20, Theorem 2.3]) we deduce that P_n converges weakly to some $P \in \mathcal{P}(\Omega_0)$ and by Skorokhod's representation theorem (e.g., [20, Theorem 2.4]) there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\mathbb{M}$-valued random variables $\{\tilde{\xi}_n\}_{n\in\mathbb{N}}$ and $\tilde{\xi}$ such that

$$\tilde{\xi}_n \text{ has the law } P_n \ \forall\, n \in \mathbb{N}, \tilde{\xi}_n \to \tilde{\xi} \text{ in } \mathbb{M}\ \tilde{P}\text{-a.s. and } \tilde{\xi} \text{ has the law } P. \tag{201}$$

Making use of (201) and (M1) for P_n immediately leads to

$$P(\{\xi(t) = \xi^{\mathrm{in}} \ \forall\, t \in [0, s]\}) = \lim_{n\to\infty} \tilde{P}(\{\tilde{\xi}_n(t) = \xi_n \ \forall\, t \in [0, s_n]\}) = 1, \tag{202}$$

which implies (M1) for P. Next, it follows immediately that for every $\mathrm{g}_i \in C^\infty(\mathbb{T}^2)$, $\tilde{P}$-a.s.

$$\langle \tilde{\xi}_n(t), \mathrm{g}_i \rangle \to \langle \tilde{\xi}(t), \mathrm{g}_i \rangle, \quad \int_{s_n}^t \langle J(\tilde{\xi}_n(r)), \mathrm{g}_i \rangle dr \to \int_s^t \langle J(\tilde{\xi}(r)), \mathrm{g}_i \rangle dr. \tag{203}$$

In particular, to prove the second convergence we can write

$$\begin{aligned}
&\mathbb{E}^{\tilde{P}}[\int_{s_n}^t \langle J(\tilde{\xi}_n(r), \mathrm{g}_i \rangle dr - \int_s^t \langle J(\tilde{\xi}(r)), \mathrm{g}_i \rangle dr] \\
=&\mathbb{E}^{\tilde{P}}[\int_{s_n}^s \langle -\mathbb{P}\mathrm{div}(\tilde{\xi}_n \otimes \tilde{\xi}_n) - (-\Delta)^m \tilde{\xi}_n, \mathrm{g}_i \rangle dr \\
&+ \int_s^t \langle -\mathbb{P}\mathrm{div}(\tilde{\xi}_n \otimes \tilde{\xi}_n) + \mathbb{P}\mathrm{div}(\tilde{\xi} \otimes \tilde{\xi}), \mathrm{g}_i \rangle dr + \int_s^t \langle -(-\Delta)^m (\tilde{\xi}_n - \tilde{\xi}), \mathrm{g}_i \rangle dr],
\end{aligned}$$

among which we only point out that

$$\mathbb{E}^{\tilde{P}}[\int_{s_n}^{s}\langle -(-\Delta)^m\tilde{\xi}_n, \mathfrak{g}_i\rangle dr] \leq \mathbb{E}^{\tilde{P}}[\int_{s_n}^{s}\|\tilde{\xi}_n\|_{L_x^2}\|(-\Delta)^m\mathfrak{g}_i\|_{L_x^2}dr] \to 0,$$
$$\mathbb{E}^{\tilde{P}}[\int_{s}^{t}\langle (-\Delta)^m(\tilde{\xi}_n - \tilde{\xi}), \mathfrak{g}_i\rangle dr] \leq \mathbb{E}^{\tilde{P}}[\int_{s}^{t}\|\tilde{\xi}_n - \tilde{\xi}\|_{L_x^2}\|(-\Delta)^m\mathfrak{g}_i\|_{L_x^2}dr] \to 0$$

as $n \to \infty$ by (201). Next, we can compute for every $t \in [s, \infty)$ and $p \in (1, \infty)$,

$$\sup_{n\in\mathbb{N}}\mathbb{E}^{\tilde{P}}[|M_{t,s_n}^{\tilde{\xi}_n,i}|^{2p}] \overset{(M3)(197)}{\lesssim_p} 1 \quad\text{and}\quad \lim_{n\to\infty}\mathbb{E}^{\tilde{P}}[|M_{t,s_n}^{\tilde{\xi}_n,i} - M_{t,s}^{\tilde{\xi},i}|] \overset{(197)(203)}{=} 0. \tag{204}$$

Next, we let $t > r \geq s$ and g be any $\mathbb{R}$-valued, $\mathcal{B}_r$-measurable and continuous function on $\mathbb{M}$. Then we can compute

$$\mathbb{E}^P[(M_{t,s}^{\xi,i} - M_{r,s}^{\xi,i})g(\xi)] \overset{(204)}{=} \lim_{n\to\infty}\mathbb{E}^{\tilde{P}}[(M_{t,s_n}^{\tilde{\xi}_n,i} - M_{r,s_n}^{\tilde{\xi}_n,i})g(\tilde{\xi}_n)] = 0. \tag{205}$$

This implies that the map $t \mapsto M_{t,s}^i$ is a $(\mathcal{B}_t)_{t\geq s}$-martingale under P. Next, we can deduce

$$\lim_{n\to\infty}\mathbb{E}^{\tilde{P}}[|M_{t,s_n}^{\tilde{\xi}_n,i} - M_{t,s}^{\tilde{\xi},i}|^2] \overset{(204)}{=} 0. \tag{206}$$

This leads us to

$$\mathbb{E}^P[((M_{t,s}^{\xi,i})^2 - (M_{r,s}^{\xi,i})^2 - \int_r^t\|G(\xi(l))^*\mathfrak{g}_i\|_U^2 dl)g(\xi)] \overset{(201)(206)}{=} 0. \tag{207}$$

Therefore, $(M_{t,s}^{\xi,i})^2 - \int_s^t\|G(\xi(l))^*\mathfrak{g}_i\|_U^2 dl$ is a $(\mathcal{B}_t)_{t\geq s}$-martingale under P which implies $\langle\langle M_{t,s}^{\xi,i}\rangle\rangle = \int_s^t\|G(\xi(l))^*\mathfrak{g}_i\|_U^2 dl$ under P; it follows that $M_{t,s}^{\xi,i}$ is square-integrable. Therefore, (M2) for P was shown. Finally, to prove (M3) it suffices to define

$$R(t,s,\xi) \triangleq \sup_{r\in[0,t]}\|\xi(r)\|_{L_x^2}^{2q} + \int_s^t\|\xi(r)\|_{H_x^\varepsilon}^2 dr, \tag{208}$$

and observe that the map $\xi \mapsto R(t,s,\xi)$ is lower semicontinuous on $\mathbb{M}$ so that $\mathbb{E}^P[R(t,s,\xi)] \leq C_{t,q}(1 + \|\xi^{\text{in}}\|_{L_x^2}^{2q})$. Therefore, (M3) holds for P so that $P \in \mathcal{C}(s, \xi_0, \{C_{t,q}\}_{q\in\mathbb{N}, t\geq s})$.

6.3. **Continuation of the proof of Theorem 2.3 assuming Proposition 5.7.** We fix any $T > 0, K > 1$ and $\kappa \in (0,1)$, and take L sufficiently large that satisfies (160), as well as

$$(\frac{1}{\sqrt{2}} - \frac{1}{2})e^{2LT} > (\frac{1}{\sqrt{2}} + \frac{1}{2})e^{2L^{\frac{1}{2}}} \text{ and } L > [\ln(Ke^{\frac{T}{2}})]^2. \tag{209}$$

We start from $(v_0, \mathring{R}_0)$ in Proposition 5.6, and via Proposition 5.7 inductively obtain $(v_q, \mathring{R}_q)$ that satisfies (150), (154), and (161). Identically to (59) we can show that for any $\varepsilon \in (0, \frac{\beta}{4+\beta})$ and any $t \in [0, T_L]$, $\sum_{q\geq 0}\|v_{q+1}(t) - v_q(t)\|_{H_x^\varepsilon} \lesssim m_L M_0(t)^{\frac{1}{2}}$ by (161) and (154b). This allows us to deduce the limiting solution $\lim_{q\to\infty} v_q \triangleq v \in C([0,T_L]; H^\varepsilon(\mathbb{T}^2))$ that is $(\mathcal{F}_t)_{t\geq 0}$-adapted because $\lim_{q\to\infty}\|\mathring{R}_q\|_{C_{T_L}L_x^1} = 0$ due to (154c). Because $u = e^{B(t)}v$ where $|e^{B(t)}| \leq e^{L^{\frac{1}{4}}}$ for all $t \in [0, T_L]$ due to (152), we are able to deduce (6) by choosing $\mathrm{t} = T_L$ for L sufficiently large. Moreover, we can show identically to (62) that for all $t \in [0, T_L]$, $\|v(t) - v_0(t)\|_{L_x^2} \leq \frac{m_L}{2}M_0(t)^{\frac{1}{2}}$ by (157) and (161) which in turn implies

$$e^{2L^{\frac{1}{2}}}\|v(0)\|_{L_x^2} \leq e^{2L^{\frac{1}{2}}}(\|v(0) - v_0(0)\|_{L_x^2} + \|v_0(0)\|_{L_x^2}) \overset{(158)}{\leq} e^{2L^{\frac{1}{2}}}(\frac{1}{2} + \frac{1}{\sqrt{2}})m_L M_0(0)^{\frac{1}{2}}. \tag{210}$$

These lead us to, on a set $\{T_L \geq T\}$

$$\|v(T)\|_{L_x^2} \overset{(158)}{\geq} \frac{m_L M_0(T)^{\frac{1}{2}}}{\sqrt{2}} - \|v(T) - v_0(T)\|_{L_x^2} \overset{(209)(210)}{\geq} e^{2L^{\frac{1}{2}}} \|v(0)\|_{L_x^2}^2. \tag{211}$$

Moreover, for the fixed $T > 0$, $\kappa \in (0,1)$, one can take L even larger to deduce $\mathbf{P}(\{T_L \geq T\}) > \kappa$. We also see that $u^{\text{in}}(x) = \Upsilon(0)v(0,x) = v(0,x)$ which is deterministic because $v_q(0,x)$ is deterministic for all $q \in \mathbb{N}_0$ by Propositions 5.6 and 5.7. Clearly from (149), $u = \Upsilon v$ is a $(\mathcal{F}_t)_{t\geq 0}$-adapted solution to (3). Furthermore, it follows from (152), (209), and (211) that $\|u(T)\|_{L_x^2} \geq e^{L^{\frac{1}{2}}} \|u^{\text{in}}\|_{L_x^2} > Ke^{\frac{T}{2}} \|u^{\text{in}}\|_{L_x^2}$ on the set $\{\text{t} \geq T\}$ which implies (7).

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Texas Tech University, Department of Mathematics and Statistics, Lubbock, TX, 79409-1042, U.S.A.; Phone: 806-834-6112; Fax: 806-742-1112; E-mail: (kyamazak@ttu.edu)