RADIATIVE DECAY OF EDGE STATES IN FLOQUET MEDIA

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ABSTRACT. We consider the effect of time-periodic forcing on a one-dimensional Schrödinger equation with a topologically protected defect (edge) mode. The unforced system models a domain-wall or dislocation defect in a periodic structure, and it supports a defect mode which bifurcates from the Dirac point (linear band crossing) of the underlying bulk medium. We study the robustness of this state against time-periodic forcing of the type that arises in the study of Floquet Topological Insulators in condensed matter, photonics, and cold-atoms systems. Our numerical simulations demonstrate that under time-periodic forcing of sufficiently high frequency, the defect state undergoes radiative leakage of its energy away from the interface into the bulk; the time-decay is exponential on a time-scale proportional to the inverse square of the forcing amplitude. The envelope dynamics of our Floquet system are approximately governed, on long time scales, by an effective (homogenized) periodically-forced Dirac equation. Multiple scale analysis of the effective envelope dynamics yields an expansion of the radiating solution, which shows excellent agreement with our numerical simulations.

1. INTRODUCTION

Consider wave propagation in a periodic and non-dissipative medium. In such a medium, spatially localized defects or extended interfaces often give rise to *defect modes*, states in which energy can concentrate. These are described by timeharmonic solutions of the underlying wave equations. In the field of topological insulators, a class of defect modes of particular interest are those whose existence is tied to a linear or conical touching of spectral bands (*Dirac points*) of an underlying bulk periodic medium, and whose robustness against perturbations is owed to the breaking of time reversal symmetry and the associated non-triviality of a topological invariant associated with the band structure. Such robust states are of broad interest in applications to storage and transmission of information.

Recently there has been great interest in *Floquet materials*, in which timeperiodic forcing is applied to spatially periodic media in settings like those above [7,32]. As with other perturbations which break time-reversal symmetry, Floquet systems give rise to chirality (uni-directionality) of modes and topological stability against defects. However, Floquet materials have the advantage of a larger design-space, allowing one to tune a system to convert between being topologically trivial and non-trivial [9], and even be more conducive to nonlinearly localized modes [1, 2, 22, 31]. Floquet systems also display a greater variety of topological properties than their static counterparts; these systems are periodic in one additional dimension and hence have a larger family of topological invariants defined on their high dimensional Brillouin zone [36, 38].

The topological robustness of defect states in such systems has been explored in experiments [17,32,33,43], and theoretically, mainly in the context of tight-binding

(discrete) models [4, 9, 18, 37]. Simulations of one- and two-dimensional optical systems show Floquet edge modes that persist over long propagation distances, but eventually disperse and decay [19, 21, 34]. In a class of hierarchical replica models, edge currents have been shown to exist for arbitrarily long, but finite, time scales [5].

In this article we show, in a class of models, that defect states which exhibit topological robustness in the autonomous (unforced) setting are only metastable against time-periodic parametric forcing; on a sufficiently long time scale, such states resonantly couple to bulk (radiation) modes and radiate their energy away from the defect and into the bulk. We derive an asymptotic theory based on a parametrically forced effective (homogenized) Dirac operator, which gives predictions in agreement with our numerical simulations of both the Schrödinger and the effective Dirac models. Our numerical simulations cover a broader set of models than our analysis; while smoothness plays a key role in the derivation, the phenomenon is demonstrated for non-smooth domain walls, which are often more similar to experimental settings.

Our results stand in contrast to the common assertions in the topological insulators literature. There, it is common to consider a discrete (tight-binding) model with finitely many bands, and then its temporally-driven analog [4,9,18,36]. These works either compute numerically or assert the existence of Floquet edge states, i.e., point-spectrum of the Floquet Hamiltonian (or the monodromy, the period-evolution operator). But such tight-binding models are phenomenological, or derived as approximations of a PDE, e.g., the Schrödinger equation. Do Floquet edge states even exist in the generic PDE model? Our results suggest that, to the contrary, only metastable modes exist, and those eventually decay into the bulk due to the resonant effect of forcing. It is an interesting open question, beyond the scope of this current paper, whether these meta-stable resonant modes can be understood as topologically protected, or whether this phenomena is universal in PDE models.

1.1. Overview of the model. Our unforced Hamiltonian, $H_{dw}^{\varepsilon} \equiv -\partial_x^2 + U_{\varepsilon}(x)$, is an asymptotically periodic Schrödinger operator; i.e.,

$$H_{\mathrm{dw}}^{\varepsilon} = \begin{cases} H_{+}^{\varepsilon} &= -\partial_{x}^{2} + V(x) + \varepsilon W(x) & \text{ for } x \gg +1 \\ H_{-}^{\varepsilon} &= -\partial_{x}^{2} + V(x) - \varepsilon W(x) & \text{ for } x \ll -1, \end{cases}$$

Here, H^{ε}_{+} and H^{ε}_{-} are $\mathbb{Z}-$ periodic. Both H^{ε}_{+} and H^{ε}_{-} are perturbations of the underlying bulk operator $H = -\partial_x^2 + V(x)$, which has a linear crossing in its band structure at energy E_D (Dirac point). H^{ε}_{\pm} have a common spectral gap (called the bulk gap) of width $\mathcal{O}(\varepsilon)$ about $E = E_D$, which is induced by the small but spatially periodic perturbations $\pm \varepsilon W(x)$. The operator H^{ε}_{dw} interpolates between H^{ε}_{-} at $x = -\infty$ and H^{ε}_{+} at $x = +\infty$, via a domain wall. The Hamiltonian H^{ε}_{dw} has, for all ε small, a protected mid-gap eigenstate $\psi^{\varepsilon}_{\star}(x)$ of energy $E_D + \mathcal{O}(\varepsilon^2)$. Moreover, $\psi^{\varepsilon}_{\star}(x)$ has a multi-scale structure; with respect to the band structure of H, it is (to an excellent approximation) a wave-packet which is spectrally concentrated about the Dirac point with an envelope characterized by the zero energy eigenstate of an effective (homogenized) Dirac operator. A more detailed discussion of the model and its analytic properties is given in Section 3. A realization of this model in photonic waveguides has been studied in [28]. In this article we study the initial value problem for the *parametrically forced* Schrödinger equation

$$i\psi_t = (H^{\varepsilon}_{dw} + 2i\varepsilon A(\varepsilon t)\partial_x)\psi, \qquad 0 < \varepsilon \ll 1,$$
(1.1a)

where

$$A(T) \equiv 2\beta \cos(\omega T), \qquad \omega > 0.$$
(1.1b)

The perturbing operator $2i\varepsilon A(\varepsilon t)\partial_x$ arises from an *effective vector potential*, $\varepsilon A(\varepsilon t)$, induced by a time-dependent deformation of H^{ε}_{dw} . In Section 1.3, we derive (1.1) in the setting of a coupled array of optical waveguides. This model is also relevant in the context of a coupled array of trapped ultracold fermions [23]. We ask the following:

Question: Does the topologically protected edge state $\psi^{\varepsilon}_{\star}$ of H^{ε}_{dw} persist in the presence of time-period forcing $(A(t) \neq 0)$?

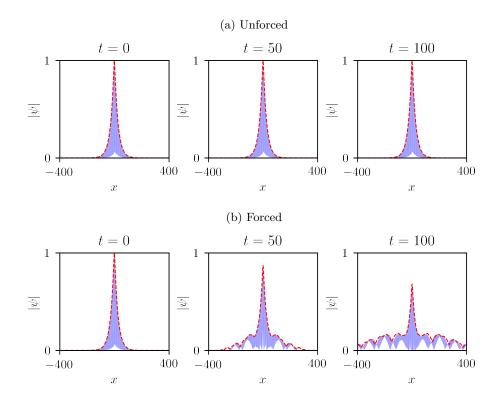


FIGURE 1. (a) Numerically computed evolution of edge state, $\psi_{\star}^{\varepsilon}$, under the unforced Schrödinger equation, (1.1) with $A(T) \equiv 0$, (blue, solid) and its envelope evolution approximated by the effective (unforced) Dirac equation (red, dashes). (b) Periodically-forced Schrödinger evolution $(A(T) \neq 0)$ and its envelope approximated by an effective periodically-forced Dirac dynamics; see Sections 4 and 6.2.

1.2. Summary of analytical and numerical results.

- (A) **Defect mode decay in the Schrödinger equation.** In Figure 1 we contrast the time-evolution (1.1) for the initial data $\psi(x,0) = \psi^{\varepsilon}_{\star}$, in the cases of unforced (A = 0) and forced $(A \neq 0)$ dynamics; a detailed discussion is given in Section 4. The top row of Figure 1 is consistent with the persistence of the edge state; indeed, for all t we have $\psi(t,x) = e^{-iE_{\varepsilon}t}\psi_{\star}(x)$. The slow time-decay of the solution is demonstrated in the bottom row of Figure 1. For a forcing $\beta A(T) = \beta \cos(\omega T)$ with β small and fixed, the envelope decays at an exponential rate $\approx \exp(-\beta^2 \varepsilon \Gamma_0 t)$, for some $\Gamma_0 = \Gamma_0(\omega) > 0$. This type of decay is observed only for driving frequencies ω above some threshold frequency, which we discuss in item (C) below.
- (B) Effective Dirac equation. In Section 5 we derive and prove the validity of an effective (spatially homogenized) time-periodically forced Dirac equation (Theorem 5.1 and [39]), as an approximation to the envelope dynamics of forced Schrödinger evolution for initial data which are spectrally localized near the Dirac point. The Dirac dynamics accurately approximate the envelope dynamics for data corresponding to the multiple scale edge state $\psi^{\varepsilon}_{\star}(x)$.

The time-dependent Schrödinger equation (1.1) excites a wide range of spatial and temporal scales. In contrast, the effective Dirac approximation is comparatively easy to solve numerically on long time-scales; its numerical solution does not require the simultaneous resolution of multiple temporal and spatial scales.

A numerical comparison between the Schrödinger equation with $\psi(0, x) = \psi_{\star}^{\varepsilon}$ and the effective forced Dirac equation, for data given by the envelope of $\psi_{\star}^{\varepsilon}$, shows excellent agreement on long time scales. In Figure 1 we plot both the numerically computed solution of the (multi-scale) Schrödinger equation (1.1) and the solution of the effective forced Dirac equation. Figure 1 demonstrates that the Schrödinger envelope is very well tracked by the (slowly varying) solution of the effective forced Dirac equation. For nontrivial forcing A(T), the envelope-decay in the effective Dirac equation, matches that of the Schrödinger equation on large time-scales.

(C) Multiscale analysis of radiation damping. Our results on the approximation of (1.1) by an effective Dirac equation, and an asymptotic solution (β small) of the effective Dirac equation, imply that on large and finite time-scales, $0 \le t \le \varepsilon^{-1}\beta^{-2}$, the wave-packet decays exponentially

$$\left| \text{Projection of } \psi^{\varepsilon}(t, \cdot) \text{ on } \psi^{\varepsilon}_{\star} \right| \approx e^{-\Gamma_0(\omega)\beta^2 \varepsilon t} \,. \tag{1.2}$$

The mechanism of decay is radiation damping; parametric forcing resonantly couples the edge state to radiation modes associated with the continuous spectrum, which acts as an energy sink; see Figure 2. The radiation rate is given by a variant of the *Fermi golden rule*, see, e.g., [40, Chapter 5]. A derivation of this radiation damping effect and, in particular, its rate is given in Section 7. Specifically, we show a threshold effect: if the driving

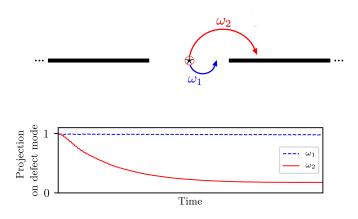


FIGURE 2. **Top:** The spectrum of the unforced Hamiltonian (star and bold lines denote point and continuous spectrum, respectively). **Bottom:** Simulations of the effective forced Dirac equation (5.1), see details in Section 6.2. When the frequency of the forcing is small (ω_1 , blue dashes), the defect mode does not couple to the continuum modes, and the mode's localization persists. When the frequency is sufficiently large to couple the point and continuous spectrum, i.e., ω_2 > bulk spectral gap, (solid red), power is transferred between the localized mode and the radiation; the defect mode decays.

frequency ω in the effective Dirac operator is larger than half the spectral gap width, then radiative decay takes place on the time scale β^{-2} .¹

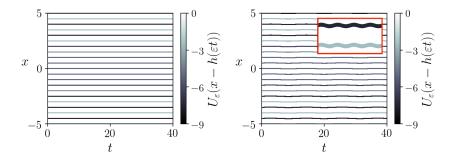


FIGURE 3. Left: The domain-wall potential $U_{\varepsilon}(x)$, see (3.4) (in particular $U_{1/2}^{(2)}$, see App. B). Right: The coiled waveguides potential $U_{\varepsilon}(x - h(t))$ with $h(t) = 2\omega^{-1}\beta\sin(\omega\varepsilon t)$ with a zoom-in in the inset.

¹If ω does not satisfy this resonance condition, then the time scale on which radiative decay takes place is expected to be $T \approx \beta^{-2r}$, for some $r \geq 2$, with a corresponding result for the Schrödinger evolution.

1.3. Derivation of the effective vector potential, A(T). We briefly explain how an effective vector potential in (3.12), A(t), arises in the context of nearly monochromatic light-propagation in a planar array of undulating waveguides; see, for example, [1, 2, 24, 33]. Consider a planar array of waveguides, whose index of refraction variations along the transverse (x) direction are described by the timeindependent potential $U_{\varepsilon}(x)$; see the left panel of Fig. 3. Illumination of the array on its input side (left) by a continuous wave (CW, nearly monochromatic) laser results in propagation along the longitudinal direction (a time-like direction, labeled t) which is approximated by the paraxial Schrödinger equation $i\psi_t = (-\partial_x^2 + U_{\varepsilon})\psi$. Consider now a planar array of undulating waveguides, where the propagation is given by

$$i\partial_t \psi(t,x) = \left[-\partial_x^2 + U_\varepsilon(x-h(t))\right]\psi$$

Here $h : \mathbb{R} \to \mathbb{R}$ describes the center-point of waveguides in the array; see right panel of Fig. 3. Switching to the undulating system of coordinates $\tilde{x} = x - h(t)$, $\tilde{t} = t$, one gets

$$\left[i\partial_{\tilde{t}} - ih'(t)\partial_{\tilde{x}}\right]\psi(\tilde{t},\tilde{x}) = \left[-\partial_{\tilde{x}}^2 + U_{\varepsilon}(\tilde{x})\right]\psi.$$

Dropping the tilde notation and setting $h'(t) \equiv 2A(t)$, we obtain (1.1).

Note that by the further change of variables: $\phi \equiv \psi \exp(-i \int_0^t |A(s)|^2 ds)$ we obtain a Schrödinger equation $i\partial_t \phi = (\partial_x + iA(t))^2 \phi + U_{\varepsilon}(s)\phi$ with a vector-potential term. By Maxwell's equations, a spatially-independent vector potential A(t) gives rise to a time-periodic *electric* field and zero magnetic field.

We comment here that, from a physical point of view, A(T) could be *any* timedependent function, or even spatially dependent (if the coiling is not uniform in x). However, in the context of demonstrating and deriving the Fermi Golden Rule, we focus on the simple periodic forcing $A(T) = \beta \cos(\omega T)$. For a study of multifrequency (almost periodic) Fermi Golden Rule radiation damping, see [25].

1.4. Future directions and open problems.

• Analytic challenges in rigorously establishing radiative decay. In our study of the time-decay of defect modes for the effective periodicallyforced Dirac Hamiltonian, $\mathcal{D}(T) = \mathcal{D}_0 + A(T)\sigma_1$, we derive a formal asymptotic solution, whose validity on the radiation damping time-scale is supported by estimating the expansion terms on the appropriate time-scale. A fully rigorous analysis of this derivation is an open question, which cannot be addressed by an application of previous methods developed in the study of radiative decay in linear and nonlinear radiation damping problems; see, e.g., the time-dependent resonance approach in [8, 30, 41].

We elaborate on the analytic issues. The first challenge arises from the matrix character of the effective Dirac equation; the forcing operator, $\beta A(T)\sigma_1$, does not commute with the unperturbed Dirac operator \not{D}_0 and hence cannot be removed via a change of phase. Even if β is assumed to be small, this operator cannot be treated perturbatively; indeed there is no spatial localization in the perturbing operator, which would have allowed for the control of this perturbation via local energy time-decay estimates for $\exp(-i\not{D}_0 t)$ (on the continuous spectral part of \not{D}_0). What could succeed is a variant of the above methods for Floquet Dirac Hamiltonians; in particular, dispersive estimates for forced Dirac Hamiltonians and the necessary spectral decomposition with which to reduce to an appropriate parametrically forced particle-field model (compare with our analysis in Section 8). Research in this direction is currently in progress.

- Resonant decay in discrete models. Our study of resonant decay of defect modes is in the context of continuum PDE models. However, the question of existence of Floquet edge modes also appears in the context of discrete (e.g., tight-binding) models in the physics literature [4,9,37]. For concreteness, consider the Su-Schrieffer-Heeger (SSH) model, which for a certain parameter regime has a localized edge mode inside a bulk spectral gap. In the numerical investigations of [9], the existence and non-existence of defect modes of time-periodically driven SSH, and the dependence on the driving frequency, is studied. We believe that our approach is applicable to the analysis of such Floquet models.
- Non-smooth potentials. Finally we believe that our smoothness assumptions on the coefficients of H^{ε}_{dw} can be relaxed. However, our discontinuoustransition domain wall / dislocation model, $U^{(3)}_{\varepsilon}$, (see Section 4.1 and Appendix B) violates the assumption of a slowly varying domain wall transition. In this case, our derivation of an effective Dirac operator and hence the asymptotic theory would seem not to apply. Nevertheless, numerical investigation shows that the midgap mode persists and undergoes radiation damping on the time-scale of the previous examples. Furthermore, and remarkably, the damped envelope dynamics are well-described by the effective Dirac operator $\mathcal{P}^{(3)}(T)$. It would be of interest to understand this robustness of radiative phenomena against large deformation, i.e., in regimes other than those allowing for a multi-scale structure.

One natural setting to consider would be Schrödinger Hamiltonians, and corresponding effective Dirac Hamiltonians, within the same "topological class"; those which can be continuously deformed into each other without closing the bulk gap. For a study of topological robustness of defect modes against such deformations within a family of dislocation Hamiltonians, see [10].

1.5. Structure of the paper. In Section 2 we introduce notation, conventions and briefly review Floquet-Bloch theory for Schrödinger operators on the line with a \mathbb{Z} - periodic potential. The autonomous Hamiltonian H_{dw}^{ε} and the parametrically forced Schrödinger equation are discussed in Section 3. Detailed numerical simulations of the parametrically forced Schrödinger equation are presented in Section 4 for concrete choices of domain-wall Schrödinger Hamiltonians. In Section 5 we derive, via a multiple scales analysis, the parametrically forced effective Dirac system and state a theorem on its validity on time-scales of interest. In Section 6 we present results on the radiation damping mechanism in terms of an effective Dirac model, and corroborate our predictions by numerical simulations. The derivation of the radiative decay phenomena is presented in Section 7, and necessary dispersive decay estimates for the unperturbed Dirac operator are derived in Section 8

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2. Preliminaries

2.1. Floquet-Bloch theory. We begin with general remarks on the Floquet-Bloch spectral theory of periodic self-adjoint differential operators; for details, see, e.g., [12, 16, 27, 35]. Let V be a sufficiently smooth real-valued \mathbb{Z} periodic potential, i.e., V(x+1) = V(x) for all $x \in \mathbb{R}$, and introduce the periodic Schrödinger operator

$$H \equiv -\partial_x^2 + V(x) \,, \tag{2.1}$$

acting in the space $L^2(\mathbb{R})$. *H* is self-adjoint and commutes with translations in the integer lattice, \mathbb{Z} .

Any $f \in L^2(\mathbb{R})$ can be represented, via the Floquet-Bloch transform, as a superposition of eigenstates of the integer translation operator $f(x) \to f(x+1)$, i.e., $L^2(\mathbb{R}) = \int_{\mathcal{B}}^{\oplus} L_k^2 dk$, where

$$L_k^2 \equiv \left\{ f \in L_{\text{loc}}^2 : f(x+1;k) = e^{ik} f(x;k) \right\},\,$$

where k varies over $\mathcal{B} \equiv [0, 2\pi]$, the fundamental cell of the dual lattice $2\pi\mathbb{Z}$ (the Brillouin zone). Since H commutes with integer translations, the spectral theory of H acting on $L^2(\mathbb{R})$ can be reduced to the study of H acting in each L_k^2 space:

$$H = \int_{\mathcal{B}}^{\oplus} H_k \, dk \,, \text{ where } \left. H_k = H \right|_{L^2_k}.$$

For each $k \in \mathcal{B}$, H_k is a self-adjoint operator and has compact resolvent. Hence, each H_k has an infinite sequence of finite multiplicity real eigenvalues, tending to infinity

$$E_1(k) \leq E_2(k) \leq \cdots \leq E_b(k) \leq \cdots$$

The corresponding eigenmodes (Bloch modes) $\Phi_b(\cdot; k) \in L^2_k$ satisfy

$$H\Phi_b(x;k) = E_b(k)\Phi_b(x;k), \qquad \Phi_b(\cdot;k) \in L_k^2.$$
(2.2)

The maps $E_b : \mathcal{B} \to \mathbb{R}$ are Lipschitz continuous, and their graphs $E_b(k)$ are called the *dispersion curves* of H. The collection of all pairs $(E_b(k), \Phi_b(x; k))$ for all $k \in \mathcal{B}$ and $b \geq 1$ is called the *band structure* of H.

3. Hamiltonians; $H_{\mathrm{dw}}^{\varepsilon}$ and its periodic forcing, (1.1)

In this section we present a concise systematic discussion (more detailed than in the introduction) of the underlying bulk Hamiltonian, H, the domain-wall Hamiltonian H_{dw}^{ε} with asymptotes H_{\pm}^{ε} , and the parametrically forced Hamiltonian $H_{\varepsilon}(t) =$ $H_{dw}^{\varepsilon} + 2i\varepsilon\beta A(\varepsilon t)\partial_x$. Figure 4 will serve as a guide to the of hierarchy of timeindependent operators and their spectra.

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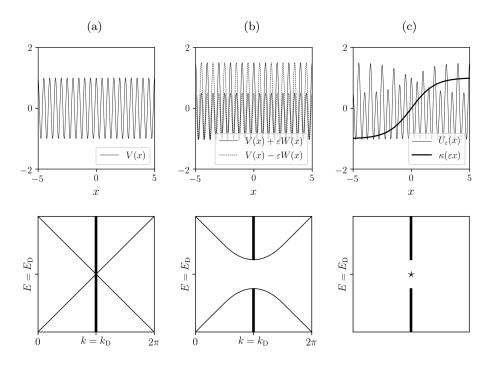


FIGURE 4. Step by step construction of multi-scale domain wall potential U_{ε} (top row) and corresponding spectra (bottom row), following Section 3. (a) \mathbb{Z} - periodic potential, V(x), and its linear band-crossing (Dirac point) due to additional $\frac{1}{2}\mathbb{Z}$ -translation symmetry. (b) Perturbed potentials $V(x) \pm \varepsilon W(x)$ (bulk structure) with minimal period lattice \mathbb{Z} , and its spectral gap about the Dirac point, due to broken symmetry. (c) Asymptotically periodic potential with domain wall defect, $U_{\varepsilon}(x)$; see (3.4). Interpolating domain wall function $\kappa(\varepsilon x)$ superimposed. Mid-gap point eigenvalue is indicated. Note that since U_{ε} is not translation invariant, k is not indicated in the lower panel, just the energy spectrum.

3.1. Schrödinger Hamiltonians with Dirac points. We begin with a Schrödinger operator $H = -\partial_x^2 + V(x)$, where V is real-valued and $\frac{1}{2}\mathbb{Z}$ -periodic, i.e., $V(x+\frac{1}{2}) = V(x)$. For simplicity, we assume that V(x) is even; for modifications required to treat the general case, see [11]. We shall embed H in a family of Hamiltonians H_{\pm}^{ε} , with $H_{\pm}^0 = H$ and H_{\pm}^{ε} of minimal period equal to one. Hence, it is natural to consider H as acting in $L^2(\mathbb{R}/\mathbb{Z})$ as having an additional symmetry; H commutes with $\frac{1}{2}\mathbb{Z}$ lattice translations.

Due to this additional symmetry, the band structure of H acting in $L^2(\mathbb{R}/\mathbb{Z})$ has *Dirac points* [14, 16]: quasi-momentum / energy pairs (k_D, E_D) , where $k_D = \pi$. Specifically, there exists $b \geq 1$ such that there are dispersion curves $k \mapsto$

 $E_b(k), E_{b+1}(k)$ curves, which cross linearly:

$$E_b(k) - E_D = -v_D |k - k_D| (1 + \mathcal{O}(|k - k_D|)), \qquad (3.1a)$$

$$E_{b+1}(k) - E_{\rm D} = +v_{\rm D}|k - k_{\rm D}|(1 + \mathcal{O}(|k - k_{\rm D}|)), \qquad (3.1b)$$

for $|k - k_{\rm D}| \ll 1$, where $v_{\rm D} > 0$ is a constant, the Dirac (or Fermi) velocity. The band structure around a Dirac point $(k, E) = (\pi, E_{\rm D})$ is shown in the first column of Figure 4.

It is convenient to introduce a basis $\{\Phi_1, \Phi_2\}$ of the two-dimensional $L^2_{k_{\rm D}}$ eigenspace of $E_{\rm D}$, such that $\Phi_2(x) = \Phi_1(-x)$. In terms of this basis

$$v_{\rm D} \equiv 2i \langle \Phi_1, \partial_x \Phi_1 \rangle_{L^2([0,1])} = -2i \langle \Phi_2, \partial_x \Phi_2 \rangle_{L^2([0,1])} \,. \tag{3.2}$$

By this appropriate choice of Φ_1 , one arranges for $v_D > 0$.

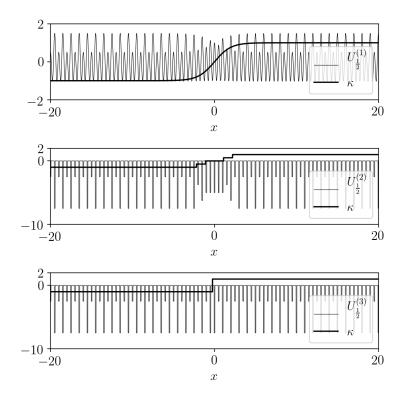


FIGURE 5. Three examples of the potential U_{ε} (3.12) **Top:** Cosine potential (B.1). **Middle:** Periodic array of square-wells (B.4). **Bottom:** Periodic array of square-wells with a sharp transition (B.5).

3.2. Bulk Hamiltonians H_{\pm}^{ε} . Let W(x) be real-valued and periodic with minimal period 1. We introduce, for $\varepsilon \neq 0$ and real, bulk operators which break the 1/2- periodicity of H

$$H_{\pm}^{\varepsilon} = H \pm \varepsilon W(x)$$
, where $W(x+1/2) = -W(x)$.

Since W(x) is a non-compact perturbation of H, it may change the essential spectrum. Indeed, the operators H_{\pm}^{ε} have a spectral gap about $E_{\rm D}$ of width $\mathcal{O}(\varepsilon)$; see [16, Appendix F]. Introducing the shift operator $S_{\frac{1}{2}}[f](x) \equiv f(x + \frac{1}{2})$, we have that $S_{\frac{1}{2}}^* = S_{-\frac{1}{2}}$ and $S_{\frac{1}{2}}H_{\pm}^{\varepsilon}S_{\frac{1}{2}}^* = H_{-}^{\varepsilon}$. Thus, H_{\pm}^{ε} and H_{-}^{ε} have the same spectrum and, in particular, have a common spectral gap about $E_{\rm D}$. We refer to this common gap as the *bulk spectral gap*.

3.3. H_{dw}^{ε} , asymptotically periodic Hamiltonian with domain wall defect. Introduce a *domain wall function*, $\kappa(X)$, which is real-valued and such that

$$\kappa(X) \to \pm \kappa_{\infty} \quad \text{as } X \to \pm \infty, \text{ where } \kappa_{\infty} > 0.$$
 (3.3)

We then define a *domain wall Hamiltonian* which interpolates between the bulk Hamiltonians H_{-}^{ε} and H_{+}^{ε} :²

$$H_{\rm dw}^{\varepsilon} \equiv -\partial_x^2 + U_{\varepsilon}(x) \,, \qquad U_{\varepsilon}(x) \equiv V(x) + \varepsilon \kappa(\varepsilon x) W(x) \,, \tag{3.4}$$

Two examples of asymptotically periodic potentials with domain wall defects are displayed in the top and middle panels of Figure 5.

The essential spectrum of H_{dw}^{ε} is equal to that of the operators H_{\pm}^{ε} , but point spectrum may arise in the spectral gaps of H_{dw}^{ε} and, in particular, in the bulk spectral gap about the Dirac energy $E_{\rm D}$. We next give a brief discussion of the bifurcation of discrete eigenvalues of H_{dw}^{ε} , from the Dirac point at $\varepsilon = 0$ into the spectral gap about $E_{\rm D}$ for ε non-zero and small.

Introduce the effective Dirac operator, D_0 , defined in terms of the domain wall function $\kappa(X)$, Dirac velocity v_D , and $\vartheta_{\sharp} \equiv \langle \Phi_1, W \Phi_2 \rangle$:

$$D_0 \equiv i v_{\rm D} \sigma_3 \partial_X + \vartheta_{\sharp} \kappa(X) \sigma_1, \qquad (3.5)$$

where the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(3.6)

Note that the parameters $v_{\rm D}$ and ϑ_{\sharp} are determined by the "Dirac eigenspace" for the energy $E_{\rm D}$. Since $\kappa(X) \to \pm \kappa_{\infty}$ as $X \to \pm \infty$, the continuum spectrum of \not{D}_0 acting in $L^2(\mathbb{R}; \mathbb{C}^2)$ is equal to $(-\infty, -|\vartheta_{\sharp}|\kappa_{\infty}] \cup [|\vartheta_{\sharp}|\kappa_{\infty}, \infty)$.

The spectrum of $\not D_0$ also contains a discrete eigenvalue at zero energy, $\alpha_{\star}(X)$, satisfying:

$$\not\!\!D_0 \alpha_\star = 0, \quad \|\alpha_\star\|_{L^2(\mathbb{R};\mathbb{C}^2)} = 1$$

It is easily verified that α_{\star} is given, up to a constant phase, by

$$\alpha_{\star}(X) = C \begin{pmatrix} 1\\ i \end{pmatrix} e^{-(\vartheta_{\sharp}/v_{\mathrm{D}}) \int_{0}^{X} \kappa(s) \, ds}, \qquad (3.7)$$

where C > 0 is a normalization constant. Since $\kappa_{\infty} > 0$, the function (3.7) is exponentially decaying at infinity. Thus it is the change in sign of $\kappa(X)$ that is responsible for the existence of a robust zero energy eigenstate; zero is a rigid eigenvalue, with respect to perturbations of $\kappa(X)$ which are spatially localized.

²We take V(x), W(x) and $\kappa(X)$ to be sufficiently smooth and $\kappa^2(X) - \kappa_{\infty}^2$ tending to zero sufficiently rapidly. Precise smoothness hypotheses on these functions, which we believe can be somewhat relaxed, are spelled out in [16].

To simplify the discussion, we make the

Assumption: 0 is the only eigenvalue of
$$\mathcal{D}_0$$
. (3.8)

Its corresponding normalized eigenstate is denoted α_{\star} :

$$\mathcal{D}_0 \alpha_\star = 0, \quad \|\alpha_\star\|_{L^2(\mathbb{R};\mathbb{C}^2)} = 1.$$

This assumption is satisfied, for example, by $\not D_0$ for $\kappa(X) = \tanh(X)$; see [29, Appendix A]. A schematic of the spectrum of $\not D_0$ is shown in Figure 6.

$$\begin{array}{c|c} & & & \\ \hline & & & \\ & -\vartheta_{\sharp}\kappa_{\infty} & 0 & \vartheta_{\sharp}\kappa_{\infty} \end{array} \end{array} \cdots$$

FIGURE 6. Spectrum of the (unforced) Dirac operator D_0 , see (3.5).

The defect mode α_{\star} of the Dirac equation (3.5) seeds a corresponding defect mode of the Schrödinger equation (3.4):

Theorem 3.1 (Theorem 5.1 in [16]; see also [11,14]). There exists $\varepsilon_0 > 0$ such that for all $0 < |\varepsilon| < \varepsilon_0$, the following is a characterization of all point spectrum in the spectral gap about E_D , bounded away from the edges of the continuous spectrum:

(1) There exists a smooth curve defined for $\varepsilon \in [0, \varepsilon_0)$:

$$\varepsilon \mapsto E^{\varepsilon} = E_{\mathrm{D}} + \mathcal{O}(\varepsilon^{2}) \in (E_{\mathrm{D}} - |\vartheta_{\sharp}| \kappa_{\infty} \varepsilon, E_{\mathrm{D}} + |\vartheta_{\sharp}| \kappa_{\infty} \varepsilon),$$

$$\varepsilon \mapsto \psi^{\varepsilon}_{\star} \in H^{1}(\mathbb{R})$$

such that $(E^{\varepsilon}, \psi^{\varepsilon}_{\star})$ is a normalized eigenpair of H^{ε}_{dw}

$$H^{\varepsilon}_{\rm dw}\psi^{\varepsilon}_{\star} = E^{\varepsilon}\psi^{\varepsilon}_{\star}\,. \tag{3.9}$$

(2) The eigenstate $\psi_{\star}^{\varepsilon}$ has a multi-scale structure; to leading order in ε , it is a Dirac wave packet, a slow modulation of the two-dimensional eigenspace span{ Φ_1, Φ_2 }, associated with the Dirac point at $(k, E) = (\pi, E_D)$ (see Section 3.1):

$$\psi_{\star}^{\varepsilon}(x) = \varepsilon^{\frac{1}{2}} \left(\alpha_{\star,1}(\varepsilon x) \Phi_{1}(x; k_{\mathrm{D}}) + \alpha_{\star,2}(\varepsilon x) \Phi_{2}(x; k_{\mathrm{D}}) \right) + \mathcal{O}_{H^{2}(\mathbb{R})}(\varepsilon)$$
$$\equiv \varepsilon^{\frac{1}{2}} \alpha_{\star}(\varepsilon x)^{\top} \Phi(x) + \mathcal{O}_{H^{2}(\mathbb{R})}(\varepsilon) .$$
(3.10)

The envelope, $\alpha_{\star}(X) = (\alpha_{1\star}(X), \alpha_{2\star}(X))^{\top}$, is given by (3.7), the zero energy eigenstate of the effective Dirac operator $\not D_0$.

In Appendix A we present a derivation of the effective Dirac equation in the more general (temporally-forced) setting of (1.1); see also [16,39]. Two examples of defect states $\psi_{\star}^{\varepsilon}(x)$ and the corresponding envelopes $\alpha_{\star}(X)$ are displayed in Figure 7.

Remark 3.2. We may relax the assumption (3.8). In general $\not D_0$ has 2N + 1 of eigenvalues in its bulk gap $(-|\vartheta_{\sharp}|\kappa_{\infty}, +|\vartheta_{\sharp}|\kappa_{\infty})$ for some $N \ge 0$: a zero energy eigenvalue, (3.7), and 2N eigenvalues located symmetrically about zero. In [11], a more general version of Theorem 3.1 is proved in which the point spectrum of H_{dw}^{ε} is given by 2N+1 eigenvalues in its $\mathcal{O}(\varepsilon)$ width spectral gap about $E_{\rm D}$, with analogous multiple scale expansions of the corresponding edge eigenstates. We believe that the results of this article (numerical and analytical) on radiation damping can be extended to this multimode setting; see [26] for relevant analysis.

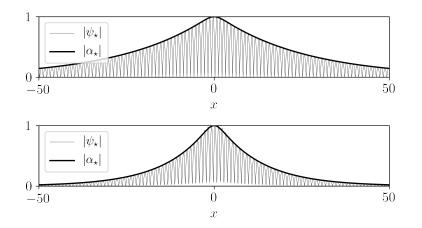


FIGURE 7. The amplitude of $|\psi_{\star}^{\varepsilon}|$, the defect mode of the Schrödinger Hamiltonian $H_{dw}^{\varepsilon}((3.9), \text{ grey})$, showing its multiscale (carrier + envelope) structure for (a) $U_{\varepsilon}^{(1)}$ (B.1), and (b) $U_{\varepsilon}^{(2)}$ (B.4), see also Figure 5. The envelope is well-approximated by α_{*} , ((3.8), black), of the corresponding Dirac operator \mathcal{D}_{0} , see (3.5).

3.4. Floquet model; parametric time-periodic forcing of H_{dw}^{ε} . Our goal in this paper is to study the effect of time-periodic (parametric) forcing on the defect state ψ_{\star} of H_{dw}^{ε} ; see (3.4) and (3.10). Denote our time-periodic Hamiltonian by

$$H_{\varepsilon}(t) \equiv H_{\rm dw}^{\varepsilon} + 2i\varepsilon\beta A(\varepsilon t)\partial_x = -\partial_x^2 + U_{\varepsilon}(x) + 2i\varepsilon A(\varepsilon t)\partial_x \qquad (3.11)$$

where $A(T) = 2\beta \cos(\omega T)$. We shall study the initial value problem:

$$i\partial_t \psi(t, x) = H_\varepsilon(t)\psi, \qquad (3.12a)$$

$$\psi(0, x) = \psi_{\star}^{\varepsilon}(x) \,. \tag{3.12b}$$

In Section 4, we present long-time numerical simulations of (3.12) showing the resonant radiation damping effect, and in In Sections 5 and 6, we provide an analytical understanding of this phenomenon on the physically interesting time-scale.

4. Long-time simulations of the forced Schrödinger equation, (3.12); slow radiative decay of the defect state

In this section we numerically investigate the long-time behavior of solutions of the initial value problem (3.12) for two Hamiltonians:

$$H_{\varepsilon}^{(\ell)}(t) = -\partial_x^2 + U_{\varepsilon}^{(\ell)}(x) + 2i \ \varepsilon\beta \cos(\varepsilon\omega t)\partial_x \,, \qquad \ell = 1, 2 \,, \tag{4.1}$$

- (1) $U_{\varepsilon}^{(1)}$ smoothly interpolates between asymptotic phase shifted cosine-potentials
- (2) $U_{\varepsilon}^{(2)}$ interpolates between asymptotic phase-shifted arrays of double square wells

for different choices of the parameters $\beta, \omega > 0$. See the top two panels of Figure 5 for illustrations of $U_{\varepsilon}^{(1)}$ and $U_{\varepsilon}^{(2)}$. Throughout our simulations, we set $\varepsilon = 1/2$.

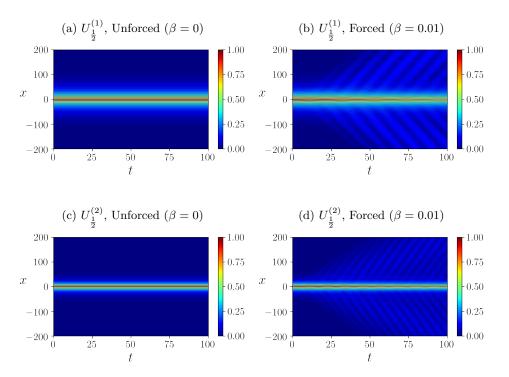


FIGURE 8. Solutions of (3.12) with $\psi(0,x) = \psi_{\star}(x)$. (a) $U_{\frac{1}{2}}^{(1)}$ with $\beta = 0$, the unforced problem, and (b) with $\beta = 0.01$ and $\omega = 0.6$. (c) and (d) Same as (a) and (b), respectively, for $U_{\frac{1}{2}}^{(2)}$ with $\omega = 1.1$.

The precise definitions of $U_{\varepsilon}^{(j)}$ are given in Appendix B. Details concerning the numerical method are given in Appendix C.

Consider first the case $\beta = 0$. Defect-mode initial data $\psi(x, 0) = \psi_{\star}(x)$ gives rise to the solution $\psi(t, x) = e^{-iE^{\varepsilon}t}\psi_{\star}^{\varepsilon}(x)$; see (3.9). Figures 8a and 8c indeed show, for both potential choices, that $|\psi(t, x)| = |\psi_{\star}(x)|$ for $0 \le t \le 100$.

We next consider the dynamics under temporal forcing. For $U_{\varepsilon}^{(1)}$ (and $U_{\varepsilon}^{(2)}$) we fix $\omega = 0.6$ (and $\omega = 1.1$, respectively) and $\beta > 0$ and observe decay of the solution with advancing time; see Figs. 8b and 8d.³

In Fig. 9, we explore the decay rate by tracking the time-dependence of projection of the solution, $\psi(\cdot, t)$, onto the defect state , i.e., $\langle \psi_{\star}^{\varepsilon}, \psi(t, \cdot) \rangle_{L^{2}(\mathbb{R})}$. Our numerical results $(U_{\varepsilon}^{(1)} \text{ with } \varepsilon = 1/2)$ are consistent with exponential decay of this projection over the very long time scale, $t \leq 1000$, $\langle \psi_{\star}, \psi(t, \cdot) \rangle_{L^{2}(\mathbb{R})} \approx \exp(-\Gamma t)$. Further, we studied the dependence of the exponential rate, Γ , on β over a range of small β . The

³The choice of $\omega = 1.1$ in Fig. 8d is not an arbitrary one. In Section 6 we show that radiation damping occurs if $|\omega|$ exceeds half the bulk spectral gap width of \mathcal{D}_0 , the effective Dirac Hamiltonian, see Fig. 2. For $\mathcal{D}_0^{(1)}$, which is derived from $U_{\varepsilon}^{(1)}$, the size of the bulk spectral gap is 1, and therefore $|\omega| > 0.5$ satisfies the resonance condition (similarly with $|\omega| > 1$ for $\mathcal{D}_0^{(2)}$).

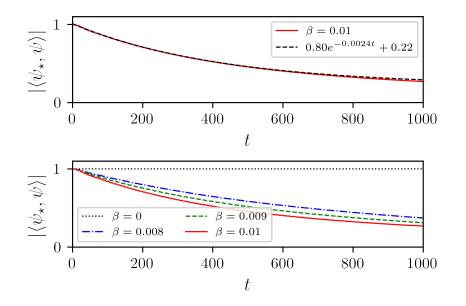


FIGURE 9. Projections of solutions of (3.12) with $U_{\varepsilon}^{(1)}$, $\omega = 0.6$, and $\psi(0, x) = \psi_{\star}(x)$ onto $\psi_{\star}(x)$. **Top:** An exponential decay fitted to the projection for $\beta = 0.01$. The projection curve and the fitted line are nearly indistinguishable. **Bottom:** The projection $|\langle \psi_{\star}, \psi(t, \cdot) \rangle_{L^{2}(\mathbb{R})}|$ for $\beta = 0, 0.008, 0.009, 0.01$.

decay is consistent with a decay ~ exp $(-\Gamma(\beta)t)$, where $\Gamma(\beta) \sim \beta^{2.07}$ and ~ $\beta^{2.10}$ for $U_{\varepsilon}^{(1)}$ and $U_{\varepsilon}^{(2)}$, respectively.

4.1. The case of a sharp transition. In the potential $U_{\varepsilon}^{(2)}$, the transition between the asymptotics at $-\infty$ and $+\infty$ is done *slowly* via a piecewise constant domain wall function $\kappa(X)$. A more common configuration in optical experiments is a sharp transition between phase-shifted waveguide arrays [6]; see lower panel of Figure 5.

Thus we consider (4.1) with $U_{\varepsilon}^{(3)}$, where the asymptotic bulk potentials, $V(x) \pm \varepsilon W(x)$, are identical to those of $U_{\varepsilon}^{(2)}$ (see Appendix B), but now $\kappa(X) = \operatorname{sgn}(X)$. Due to the very different scaling properties of $U_{\varepsilon}^{(3)}$, the analysis leading to Theorem 3.1 does not apply. However, we expect similar qualitative behavior.

We repeated our simulations for $U_{\varepsilon}^{(3)}$ and have corroborated these heuristics. We observe, as in the previous examples, that H_{dw}^{ε} has a defect mode which, under periodic forcing, radiation damps on time scales of order β^{-2} ; see Figure 10.

5. The effective Dirac equation

Our goal in this section is to approximate the multi-scale evolution of the Schrödinger equation (3.12) by simpler effective envelope dynamics. The separation of fast scales (x and t) of the underlying Bloch modes and slow scales $(X = \varepsilon x \text{ and } T = \varepsilon t)$ of the domain wall and parametric forcing, allow for the derivation of an effective homogenized equation for wave-packet envelope.

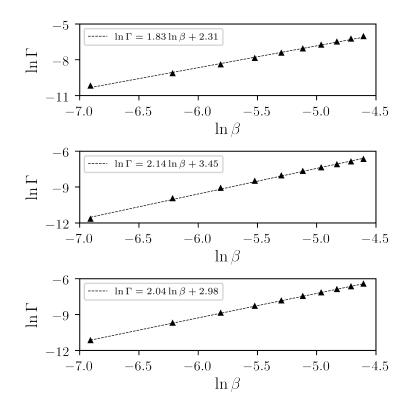


FIGURE 10. Top: The decay rate Γ against the forcing amplitude β on a log-log scale (triangles) and an exponential fit of $\Gamma \sim \beta^{1.83}$. Middle and Bottom: Same as the first row, for $U_{\varepsilon}^{(2)}$ and $U_{\varepsilon}^{(3)}$ respectively, with $\omega = 1.1$, and exponential fits of $\Gamma \sim \beta^{2.14}$ and $\Gamma \sim \beta^{2.04}$.

In Appendix A we show that the envelope of the solution of the time-periodically forced initial value problem (3.12), with initial data

$$\psi(0,x) = \psi_{\star}^{\varepsilon}(x) \approx \varepsilon^{1/2} \alpha_{\star}(\varepsilon x)^{\top} \Phi(x)$$

evolves in the form

$$\psi(t,x) \approx \varepsilon^{\frac{1}{2}} \alpha(\varepsilon t, \varepsilon x)^{\top} \Phi(x) e^{-iE_{\rm D}t}$$

where $\alpha(T, X)$ is governed by the initial value problem for an effective periodicallyforced Dirac equation:

$$i\partial_T \alpha(T, X) = D(T)\alpha, \qquad \alpha(0, X) = \alpha_0(X) \in H^4(\mathbb{R}; \mathbb{C}^2).$$
 (5.1a)

$$\mathcal{D}(T) \equiv \mathcal{D}_0 + v_{\rm D} A(T) \sigma_3 \,, \tag{5.1b}$$

Here, \not{D}_0 is given by (3.5) and the Pauli matrices σ_1 and σ_3 , see (3.6). The effective Dirac model, and its validity guaranteed in Theorem 5.1, hold for all smooth and periodic choices of A(T).

Let $\psi_0 \mapsto \mathcal{U}^{\varepsilon}[\psi_0](t,x)$ denote the solution of Schrödinger equation (3.12) with $\psi(0,\cdot) = \psi_0$, and $\alpha_0(X) \mapsto \mathcal{U}_{\text{Dir}}[\alpha_0](T,X)$ denote the solution of the effective Dirac equation (5.1) with $\alpha(0,\cdot) = \alpha_0$.

Theorem 5.1 (Effective Dirac dynamics). Suppose κ has bounded derivatives to all orders. There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the following holds: Consider (3.12), the Schrödinger equation with time periodic Hamiltonian, and initial data of the form, $\psi_0(x) = \varepsilon^{1/2} \alpha_0(\varepsilon x)^\top \Phi(x)$, with $\alpha_0 \in H^4(\mathbb{R}^2; \mathbb{C}^2)$. Fix constants $T_0 > 0$ and $0 < \rho < 1$.

Then, there is a constant, C, which depends on the Hamiltonian H, ρ and α_0 , such that for $0 \le t \le T_0 \varepsilon^{-(3/2-\rho)}$,

$$\left\| \mathcal{U}^{\varepsilon}[\psi_0](t,x) - \varepsilon^{\frac{1}{2}} \mathcal{U}_{\mathrm{Dir}}[\alpha_0](\varepsilon t, \varepsilon x)^{\top} \Phi(x; k_{\mathrm{D}}) e^{-iE_{\mathrm{D}}t} \right\|_{L^2(\mathbb{R}_x)} \le C \varepsilon^{\rho} \,. \tag{5.2}$$

The proof is presented in Appendix A. Analogous results on the effective Dirac dynamics in two-dimensional analogs of (3.12), unforced and forced, were obtained in [15, 39]. Theorem 5.1 follows from arguments which closely follow those in [39, Section 8].

Note that, of the three potentials considered in this paper (see Fig. 5), Theorem 5.1 is only applicable to (B.1). However, our numerical simulations suggest that the formally-derived Dirac equations effectively approximate the corresponding Schrödinger equation for very long times, even if slightly less so than in the smooth settings. The study of this robustness in non-smooth settings remains an interesting open problem.

6. Metastability of edge states and radiation damping

In Section 6.1 we summarize our analytical results on radiation damping for initial value problem for the periodically forced Dirac equation:

$$i\partial_T \alpha = (\not\!\!\!D_0 + \beta A(T)\sigma_3)\alpha, \qquad \alpha(0,X) = \alpha_\star(X).$$
 (6.1)

Here,

(1) A(T) is T_{per} -periodic and of mean zero

- (2) $\not{D}_0 = i\sigma_3\partial_X + \kappa(X)\sigma_1$, and
- (3) α_{\star} is the zero-energy defect mode of $\not D_0$:

$$\mathcal{D}_0 \alpha_\star = 0, \quad \|\alpha_\star\|_{L^2(\mathbb{R}:\mathbb{C}^2)} = 1.$$

The parameter β , the strength of periodic forcing, is taken to be small.

After discussing our results for the forced Dirac initial value problem (6.1), we apply them to radiation damping in the initial value problem for the forced Schrödinger equation. In Section 6.2 we present numerical simulations which corroborate our theoretical predictions. The multiscale analysis underlying the results in Section 6.1 is presented in Section 7.

6.1. Summary of analytical results on radiation damping. For $\beta = 0$, the solution of (7.1) is $\alpha(T, X) = \alpha_{\star}(X)$ for all $T \in \mathbb{R}$.

For β small, it is natural to represent the solution to the perturbed evolution equation (7.1) using the spectral theory of the operator \mathcal{D}_0 , outlined in Section 3.3. In particular, for the case that \mathcal{D}_0 has only one bound state we have:

$$\operatorname{spec}({
ot\!\!\!\!D}_0) \ = \ \{0\} \ \cup \ {\mathbb R} \setminus [-\vartheta_\sharp \kappa_\infty, \vartheta_\sharp \kappa_\infty].$$

The $L^2(\mathbb{R}; \mathbb{C}^2)$ -orthogonal projections onto the bound (defect) state subspace and the continuous (dispersive) spectral subspace of $\not D_0$ are, respectively:

$$\mathscr{P}_0 = \langle \alpha_\star, \cdot \rangle \, \alpha_\star \quad \text{and} \quad \mathscr{P}_c = I - \mathscr{P}_0 \qquad (\mathscr{P}_0 + \mathscr{P}_c = I) \,.$$
 (6.2)

Thus, we decompose the solution of (5.1) relative to these orthogonal projections:

$$\alpha(T,X) = g(T)\alpha_{\star}(X) + \alpha_d(T,X), \quad \left\langle \alpha_{\star}(\cdot), \alpha_d(T,\cdot) \right\rangle = 0, \qquad T \ge 0.$$
 (6.3)

In Section 7 we derive a coupled dynamical system for the oscillator-like degree of freedom, $g(T;\beta)$ and field-like degree of freedom $\alpha_{\rm d}(T,X;\beta)$, which we solve for small β via a multiple scale expansion. Our constructed g(T) decays on the time scale $\mathcal{O}(\beta^{-2})$. In particular, we show the following:

Leading order multiscale expansion: Fix $T_0 > 0$ and arbitrary. Then, there exist $\rho_0, \beta_0 > 0$ such that for all $0 < \beta < \beta_0$ and $0 \le T \le T_0 \beta^{-2}$

$$g(T;\beta) = e^{i(\beta^2 \Lambda_0 T - \beta \eta_A(T))} e^{-\Gamma_0 \beta^2 T} (1 + o(1))$$
(6.4a)

$$\alpha_{\rm d}(T,X) = \mathcal{O}\left(\beta e^{-\Gamma_0 \beta^2 T}\right) \quad \text{in } L^2(\langle x \rangle^{-\rho_0} dx), \quad \text{as } \beta \to 0.$$
 (6.4b)

Here, $\Gamma_0 = \Gamma_0(\omega)$ is given by

$$\Gamma_{0}(\omega) \equiv \frac{\pi}{4} \left\langle \mathscr{P}_{c} \sigma_{3} \alpha_{\star}, \left(\delta(\not{\!\!D}_{0} + \omega) + \delta(\not{\!\!D}_{0} - \omega) \right) \mathscr{P}_{c} \sigma_{3} \alpha_{\star} \right\rangle \geq 0, \qquad (6.5)$$

is generically strictly positive,

and

$$\eta_A(T) = \langle \alpha_\star, \sigma_3 \alpha_\star \rangle \int_0^T A(s) ds$$

is a real, bounded, and T_{per} -periodic.

Exponential decay on the time scale $\mathcal{O}(\beta^{-2})$: Note from (6.4) that exponential decay on the time-scale $T \sim \beta^{-2}$ holds if $\Gamma_0(\omega) > 0$. Since $\delta(\not{D}_0 \pm \omega)$ are non-negative self-adjoint operators, $\Gamma_0(\omega) \ge 0$ for any ω . The vector $\delta(\not{D}_0 \pm \omega) \mathscr{P}_c \sigma_3 \alpha_\star$ is the projection of vector $\mathscr{P}_c \sigma_3 \alpha_\star$ onto the continuous spectral subspace of \not{D}_0 at frequency $\mp \omega$. Hence, if $|\omega| < \kappa_{\infty} |\vartheta_{\sharp}|$, i.e., ω is in the spectral gap of \not{D}_0 , then $\Gamma_0(\omega) = 0$, see Fig. 11.

On the other hand, suppose $|\omega| > \kappa_{\infty} |\vartheta_{\sharp}|$, then ω is in the continuous spectrum of \not{D}_{0} . For generic data, $\delta(\not{D}_{0} \pm \omega) \mathscr{P}_{c} \sigma_{3} \alpha_{\star}$ is non-zero, as we expect generic functions to have non-zero projection on every part of the continuous spectrum [3, Section 4]. Hence, generically $\Gamma_{0} > 0$ implying exponential decay.

Since $\mathscr{P}_c \sigma_3 \alpha_{\star} \in L^2(\mathbb{R}; \mathbb{C}^2)$, the inner product (6.5) which gives the exponential rate of decrease, $\Gamma_0(\omega)$, decays to zero for $|\omega| \to \infty$. Our numerical simulations in the bottom row of Figure 11 are consistent with this behavior; we observe that $\omega \mapsto g(T; \omega)|_{T=500}$ increases for ω sufficiently large, consistent with rate of decay of $g(T; \omega)$ being slower for large $|\omega|$.

Remark 6.1. We expect that the exponential decay is a large-time transient, which applies on the time scale $T \leq \beta^{-2}$. While our analysis does not cover $|T| \gg \beta^{-2}$, we expect the time-decay to be algebraic and given by the free dispersive rate, $\mathcal{O}(t^{-1/2})$; see, e.g., [41].

Radiative time-decay in parametrically forced Schrödinger equation:

In terms of the original variables of the parametrically forced Schrödinger equation (3.12) we have, by (6.4a), for $\rho \in (0, 1)$ and $0 \le t \le T_0 \varepsilon^{-(2-\rho)}$,

$$\langle \psi_{\star}, \psi(t, \cdot) \rangle_{L^{2}(\mathbb{R})} = e^{-i\beta\eta_{A}(\varepsilon t)} e^{-iE^{\varepsilon}t} e^{-\beta^{2}\varepsilon(\Gamma_{0}+i\Lambda_{0})t} + \mathcal{O}(\beta, \varepsilon^{\rho}),$$

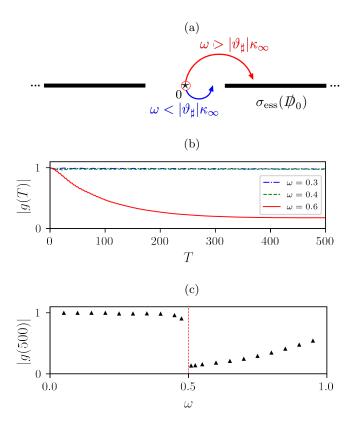


FIGURE 11. The effect of the frequency ω of the forcing A(T)on the decay of the Dirac zero mode α_{\star} , see (3.8). (a) When $|\omega| > \vartheta_{\sharp}\kappa_{\infty}$ (red), α_{\star} couples to the essential spectrum. Otherwise $|\omega| < \vartheta_{\sharp}\kappa_{\infty}$ (blue), and the coupling effect is expected to be negligible. (b) Same settings as in Fig. 1a, where $\vartheta_{\sharp}\kappa_{\infty} = 0.5$. The projection onto the zero mode $g(T) = \langle \alpha_{\star}, \alpha(\cdot, T) \rangle_{L^2(\mathbb{R};\mathbb{C}^2)}$ for $\beta = 0.01$, with $\omega \in \{0.3, 0.4, 0.6\}$. The curves for $\omega = 0.3$ and $\omega = 0.4$ are nearly indistinguishable. (c) g(500) for varying values of ω (triangles) and a dotted line for the $\omega = \vartheta_{\sharp}\kappa_{\infty}$ threshold.

and where $E^{\varepsilon} = E_D + \mathcal{O}(\varepsilon^2)$; see (3.9). This gives exponential decay on the timescale $\mathcal{O}(\varepsilon^{-1}\beta^{-2})$.

6.2. Simulations of the effective Dirac equation. For the time-periodically forced Schrödinger equations (3.12) with the potentials $U_{\varepsilon}^{(\ell)}$ ($\ell = 1, 2, 3$), introduced in Section 4, we consider the effective periodically-forced Dirac Hamiltonians

see (5.1b). The parameters which define these effective operators are given in Appendix B.

To confirm the analytical asymptotic results outlined in Section 6.1, we simulate the evolution under the effective Hamiltonians $\not D^{(1)}(T)$ (Fig. 1) and $\not D^{(2)}(T)$ (Fig.

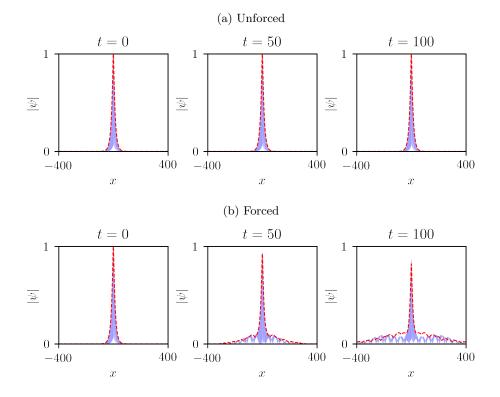


FIGURE 12. Same as Fig. 1, only for the square well Schrödinger and Dirac equations, (B.4) and (B.8), respectively, with $\omega = 1.1$.

12), and compare the solution $\alpha(T, X)$ with the wave-envelope of the solution of the corresponding Schrödinger equations (3.12) with $\varepsilon = 1/2$. In the unforced case $(\beta = 0)$, the envelope of the Schrödinger defect mode $|\psi_{\star}(x)|$ is tracked very well by $|\alpha_{\star}(\varepsilon x)|$ for $0 \le t \le 100$, as we expect from (3.10).⁴ For the forced cases, we see that the Dirac envelope decays and fits the multi-scale Schrödinger solution extremely well for t = 50, and quite well (though small discrepancies do appear) up to t = 100.

Figure 13 presents numerical results on radiation damping rate for the effective Dirac Hamiltonians $\not D^{(\ell)}(T)$, $(\ell = 1, 2, 3)$ which are consistent with the predicted exponential decay rate, $\Gamma(\beta) = \Gamma_0 \beta^2$, (see (6.4a)). These results are almost one-to-one comparable with those of the corresponding Schrödinger equations; compare with Figure 10.

It is remarkable that the effective dynamics given by $\not D^{(3)}(T)$ (with a discontinuous $\kappa(X)$) tracks the Schrödinger dynamics for large time, since the Hamiltonian violates the scaling assumptions used to derive the effective Dirac dynamics from the Schrödinger dynamics. We remarked on this as an open problem in Section 1.4.

⁴Note that, since $\varepsilon = 1/2$ in (3.12), the simulation of (5.1) runs on the slow time-scale of $0 \le T \le 50$.

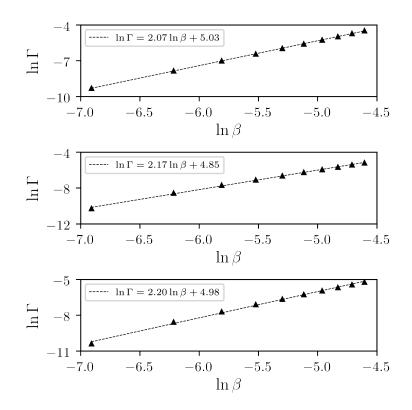


FIGURE 13. Decay rates for the effective Dirac equations as a function of β , see (5.1). Top: $\not{D}_0^{(1)}$. Middle: $\not{D}_0^{(2)}$. Bottom: $\not{D}_0^{(3)}$.

7. Multiscale analysis of radiation damping

In this section, we provide a multiple scale analysis and derivation of the radiation damping phenomena observed in our numerical simulations. Consider the initial-value-problem (IVP) for $\alpha(T, X)$, the effective parametrically forced Dirac equation (5.1) with initial data given by the zero energy defect mode, $\alpha_{\star}(X)$, of the unforced Dirac operator \mathcal{D}_0 :

$$i\partial_T \alpha = (\not\!\!D_0 + \beta A(T)\sigma_3)\alpha, \qquad \alpha(0,X) = \alpha_\star(X).$$
 (7.1)

From here on, it will be useful to make the forcing amplitude parameter β explicit, multiplying the forcing function $A(T) = \cos(\omega T)$, by a slight abuse of notation; β is real and will be taken sufficiently small. Recall that since α_{\star} is the zero energy eigenstate of \not{D}_0 , we have that if $\beta = 0$, then $\alpha(T, X) = \alpha_{\star}(X)$.

Since β is small it is natural to decompose α into its projection onto the bound state and the dispersive part of \mathcal{D}_0 , orthogonal to α_{\star}

$$\alpha(T,X) = g(T)\alpha_{\star}(X) + \alpha_d(T,X), \qquad \left\langle \alpha_{\star}(\cdot), \alpha_d(T,\cdot) \right\rangle = 0; \qquad (7.2)$$

see also (6.3).

Inserting (6.3) into (5.1), and using the relation $\mathcal{D}_0 \alpha_{\star} = 0$, we obtain⁵

Applying the projections \mathscr{P}_0 and \mathscr{P}_c , as defined in (6.2), to (7.3), we obtain the coupled system for g(T) and $\alpha_d(T)$:

$$i\partial_T g(T) = \beta \langle \alpha_\star, \sigma_3 \alpha_\star \rangle A(T)g(T) + \beta A(T) \langle \alpha_\star, \sigma_3 \alpha_d(T) \rangle, \qquad (7.4)$$

$$g(0) = 1, \qquad \alpha_d(0) = 0.$$
 (7.6)

To avoid cumbersome expressions, we have set $\vartheta_{\sharp} = v_{\rm D} = 1$. The first term on the right hand side of (7.4) contributes a rapidly varying phase, which we can remove by setting

$$G(T) \equiv e^{i\beta\eta_A(T)}g(T), \quad \text{where} \\ \eta_A(T) \equiv \left\langle \alpha_\star, \sigma_3 \alpha_\star \right\rangle \int_0^T A(s) \, ds \,. \tag{7.7}$$

Note that since A(T) has mean zero,⁶ $\eta_A(T)$ is bounded on \mathbb{R} . Equations (7.4), (7.5), (7.6) becomes a system for G(T) and $\alpha_d(T)$:

$$i\partial_T G(T) = \beta e^{i\beta\eta_A(T)} A(T) \left\langle \alpha_\star, \sigma_3 \alpha_d(T) \right\rangle,$$

$$(7.8a)
 (i\partial_T - \not\!\!\!D_0) \alpha_d(T) = \beta \mathscr{P}_c \sigma_3 \alpha_\star e^{-i\beta\eta_A(T)} A(T) G(T) + \beta \mathscr{P}_c \sigma_3 A(T) \alpha_d(T),$$

$$(7.8b)
 G(0) = 1, \quad \alpha_d(0) = 0.$$

$$(7.8c)$$

The structure of (7.8) and numerical simulations (Section 6.2) suggest that solution varies on disparate temporal scales: from the rapid time-scale of order $\beta^0 = 1$, of the forcing function, A(T), to the decay time-scale of the bound state amplitude, β^{-2} . Thus, we shall seek the solution of (7.8) in the form of a multiple scale expansion. First, introduce the hierarchy of time scales

$$T, \quad \tau_1 = \beta T, \quad \tau_2 = \beta^2 T, \quad (\beta \text{ small})$$

and view G and α_d as functions of these variables. Thus, (7.8) becomes:

Next, we expand the solution $(G(\cdot; \beta), \alpha_d(\cdot; \beta))$ of (7.9) in the small parameter β :

$$G = G^{(0)} + \beta G^{(1)} + \beta^2 G^{(2)} + \cdots, \qquad G^{(n)}(T, \tau_1, \tau_2) \in \mathbb{C},$$
(7.10a)

$$\alpha_d = \alpha^{(0)} + \beta \alpha^{(1)} + \beta^2 \alpha^{(2)} + \cdots, \qquad \alpha^{(n)}(T, \tau_1, \tau_2) \perp \alpha_\star .$$
 (7.10b)

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⁵In the calculations below, we shall frequently suppress the X-dependence of α_d . Thus, $\alpha(T) = g(T)\alpha_* + \alpha_d(T)$.

⁶Without loss of generality, we can assume A(T) has mean zero. Otherwise, the mean of A(T) can be removed by a change of variables.

We capture the initial data (7.8c) by imposing the initial conditions:

$$G^{(0)} = 1$$
, for $T = 0$, $\tau_1 = 0$, $\tau_2 = 0$, (7.11a)

$$G^{(n)} = 0$$
, for $T = 0$, $\tau_1 = 0$, $\tau_2 = 0$ and all $n \ge 1$, (7.11b)

$$\alpha^{(n)} = 0$$
, for $T = 0$, $\tau_1 = 0$, $\tau_2 = 0$ and all $n \ge 0$. (7.11c)

Substitution of (7.10) into (7.9) and equating like terms in powers of β , leads to a hierarchy of equations for the functions $\{G^{(n)}\}_{n\geq 0}$ and $\{\alpha^{(n)}\}_{n\geq 0}$. These functions are determined by the requirement that contributions from terms at one order in β are smaller or equal to terms of lower order in β on the radiation damping time-scale $T \sim \beta^{-2}$.

For any fixed $\hat{T} > 0$ and $n \ge 1$, we require that:

$$\lim_{\beta \downarrow 0} \sup_{0 < T < \hat{T}/\beta^2} \beta^n |G^{(n)}(T, \beta T, \beta^2 T)| = 0, \qquad (7.12)$$

$$\lim_{\beta \downarrow 0} \sup_{0 \le T \le \hat{T}/\beta^2} \beta^n \| \langle X \rangle^{-\rho} \alpha^{(n)}(T, \beta T, \beta^2 T, X) \|_{L^2(\mathbb{R}_X)} = 0.$$
(7.13)

Here, $\rho > 0$ is fixed and is taken sufficiently large.

The norm in (7.13) is a measure of spatially localized energy and below we see how it naturally arises. We shall construct the order β^0 , β^1 and β^2 terms of the expansion (7.9) in order to exhibit the transient exponential decay on the time-scale $T \sim \beta^{-2}$. The expansion can be carried out systematically to any finite order in β . Section 7.1 assembles some technical tools used along the way and the expansion is constructed in Section 7.2.

7.1. Technical interlude. Throughout this section, assume that $\kappa_{\infty} = 1$, i.e., that $\lim_{X \to \pm \infty} \kappa(X) = \pm 1$, and furthermore, that the convergence to ± 1 is sufficiently fast. ⁷ We shall also assume that $\kappa(X)$ is smooth. While such assumptions are stringent, our numerical simulations in Sections 4 and 6.2 suggest that the phenomena is not dependent on such smoothness properties.

For any $\epsilon > 0$, the operator $e^{-i\not{D}_0 s}/(\not{D}_0 - \lambda - i\epsilon)$ is bounded in $L^2(\mathbb{R})$. Introduce the operator

$$\frac{e^{-i\not\!\!D_0 s}}{(\not\!\!D_0 - \lambda - i0)} \equiv \lim_{\epsilon \to 0^+} \frac{e^{-i\not\!\!D_0 s}}{(\not\!\!D_0 - \lambda - i\epsilon)} \,. \tag{7.14}$$

Proposition 7.1. Assume $\lambda \in \operatorname{spec}_{\operatorname{ess}}(\mathcal{D}_0)$ but that λ is not an endpoint of $\operatorname{spec}_{\operatorname{ess}}(\mathcal{D}_0)$. Then, for any $r \geq r_0 > 0$ sufficiently large, there exist $\rho > 0$ such that the operator $e^{-i\mathcal{D}_0 s}(\mathcal{D}_0 - \lambda - i0)^{-1}$ is well-defined from $L^2(\langle x \rangle^r dx)$ to $L^2(\langle x \rangle^{-r} dx)$ and satisfies the following local energy decay bound for all s > 0:

$$\left\| \langle x \rangle^{-r} \frac{e^{-i\mathcal{\Psi}_0 s}}{(\mathcal{D}_0 - \lambda - i0)} \langle x \rangle^{-r} \right\|_{\mathcal{B}(L^2)} \le C \langle s \rangle^{-\rho}.$$

See Section 8 for the proof of Proposition 7.1.

⁷A sufficient condition for the decay rate of $\kappa(X)$ emerges from our application of wave operators in Section 8.2; see [44,45].

Proposition 7.2. (1) For $f \in S(\mathbb{R})$, the following identity holds in $L^2(\langle x \rangle^{-\rho} dx)$ for a fixed $\rho > 0$:

$$\int_{0}^{s} e^{-i\not\!\!D_{0}(s-s_{1})} A(s_{1}) \mathscr{P}_{c}f \ ds_{1} = \mathscr{G}(s)f.$$
(7.15)

Here, $A(T) = \cos(\omega T)$ and $\mathscr{G}(t)$ denotes the operator

$$+\frac{\pi}{2}\left(e^{i\omega s}\delta(\not\!\!\!D_0+\omega)+e^{-i\omega s}\delta(\not\!\!\!D_0-\omega)\right)\mathscr{P}_{\rm c}$$
(7.16b)

(2) For
$$f_1, f_2 \in \mathcal{S}(\mathbb{R}),$$

 $\left\langle \mathscr{P}_{c}f_1, \int_0^s e^{-i\mathcal{D}_0(s-s_1)}A(s_1)\mathscr{P}_{c}f_2\,ds_1 \right\rangle = \left\langle \mathscr{P}_{c}f_1, \mathscr{G}(s)\mathscr{P}_{c}f_2 \right\rangle.$ (7.17)

(3) For
$$f \in \mathcal{S}(\mathbb{R})$$
, we have $\sup_{s \ge 0} \|\langle x \rangle^{-\rho} \mathscr{G}(s) f\|_{L^2} < \infty$.

Proof of Proposition 7.2. The lemma is a consequence of the following calculation which holds in $\mathcal{S}'(\mathbb{R})$. For any fixed $s \geq 0$, we evaluate the integral via a regularization procedure. Using that $A(s) = \cos(\omega s)$ we have:

$$\begin{split} \int_{0}^{s} e^{-i\not{\mathcal{D}}_{0}(s-s_{1})}A(s_{1})\mathscr{P}_{c}\,ds_{1} &= \frac{1}{2}\int_{0}^{s} e^{-i\not{\mathcal{D}}_{0}s}\left(e^{i(\not{\mathcal{D}}_{0}+\omega)s_{1}} + e^{i(\not{\mathcal{D}}_{0}-\omega)s_{1}}\right)\mathscr{P}_{c}\,ds_{1} \\ &= \lim_{\epsilon \to 0^{+}} \frac{1}{2}\int_{0}^{s} e^{-i\not{\mathcal{D}}_{0}s}\left(e^{i(\not{\mathcal{D}}_{0}+\omega-i\epsilon)s_{1}} + e^{i(\not{\mathcal{D}}_{0}-\omega-i\epsilon)s_{1}}\right)\mathscr{P}_{c}\,ds_{1} \\ &= \frac{1}{2}\lim_{\epsilon \to 0^{+}} \left[\frac{e^{i\omega s} - e^{-i\not{\mathcal{D}}_{0}s}}{i(\not{\mathcal{D}}_{0}+\omega-i\epsilon)} + \frac{e^{-i\omega s} - e^{-i\not{\mathcal{D}}_{0}s}}{i(\not{\mathcal{D}}_{0}-\omega-i\epsilon)}\right]\mathscr{P}_{c}\,. \end{split}$$

Recall the distributional identity (Plemelj-Sokhotski relation)

$$\frac{1}{x-i0} = \lim_{\epsilon \to 0^+} \frac{1}{x-i\epsilon} = \mathrm{PV}\frac{1}{x} + i\pi\delta(x), \qquad (7.18)$$

where PV denotes the Cauchy Principal Value. Applying (7.18) we obtain

$$\int_{0}^{s} e^{-i\mathcal{D}_{0}(s-s_{1})} A(s_{1}) \mathscr{P}_{c} \, ds_{1} = \mathscr{G}(s).$$

Here, $\mathscr{G}(s)$ is displayed in (7.16). Finally, to prove the uniform bound in item (3), we return to the decomposition of $\mathscr{G}(s)$ as given in (7.16). That the operators in (7.16c) satisfy the desired bound is a direct result of Proposition 7.1. The other terms (7.16a)–(7.16b), can be written (and are obtained from) as $e^{\pm i\omega s}[\not D_0 \pm \omega]^{-1}$. The required bound can be reduced to the analogous bounds for $e^{\pm i\omega s}[Z_0 \pm \omega]^{-1}$, where Z_0 is the free Schrödinger Hamiltonian. The details of a similar argument are presented in Section 8, and in particular Section 8.2.

7.2. Implementing the expansion through order β^2 . A hierarchy of equations for the functions $G^{(n)}$ and $\alpha^{(n)}$ is obtained by substituting (7.10) into (7.9) and equating terms of like power in β . We carry this out to second order in β . The only non-algebraic term, $e^{i\beta_A(T)}$, is expanded to

$$e^{i\beta\eta_A(T)} = 1 + i\beta\eta_A(T) + \mathcal{O}(\beta^2).$$

By recalling that $\eta_A(T)$ is bounded, we obtain the hierarchy equations, the first several of which we now display and solve.

 $\mathcal{O}(\beta^0)$ equation.

$$i\partial_T G^{(0)} = 0, \qquad (i\partial_T - D_0)\alpha^{(0)} = 0.$$
 (7.19)

 $\mathcal{O}(\beta^1)$ equation.

$$i\partial_T G^{(1)} = -i\partial_{\tau_1} G^{(0)} + A(T) \left\langle \sigma_3 \alpha_\star, \alpha^{(0)} \right\rangle , \qquad (7.20a)$$
$$(i\partial_T - \not D_0) \alpha^{(1)} = -i\partial_{\tau_1} \alpha^{(0)} + \mathscr{P}_c \sigma_3 \alpha_\star A(T) G^{(0)}$$
$$+ \mathscr{P}_c \sigma_3 \alpha_\star A(T) \alpha^{(0)}. \qquad (7.20b)$$

 $\mathcal{O}(\beta^2)$ equation.

$$i\partial_T G^{(2)} = -i\partial_{\tau_1} G_1 - i\partial_{\tau_2} G^{(0)} + A(T) \left\langle \sigma_3 \alpha_\star, \alpha^{(1)} \right\rangle + i\eta_A(T) \left\langle \sigma_3 \alpha_\star, \alpha^{(0)} \right\rangle, \qquad (7.21a)$$
$$(i\partial_T - D_0) \alpha^{(2)} = -i\partial_{\tau_1} \alpha^{(1)} - i\partial_{\tau_2} \alpha^{(0)} - i\mathscr{P}_c \sigma_3 \alpha_\star \eta_A(T) A(T) G^{(0)}$$

$$+ \mathscr{P}_{c}\sigma_{3}\alpha_{\star}A(T)\alpha^{(1)} + \mathscr{P}_{c}\sigma_{3}\alpha_{\star}A(T)G^{(1)}.$$
 (7.21b)

 $\mathcal{O}(\beta^n)$:

We next solve the equations of this hierarchy through order β^2 .

 $\mathcal{O}(\beta^0)$ solution. Since $i\partial_T G^{(0)} = 0$, we have that

$$G^{(0)} = G^{(0)}(\tau_1, \tau_2).$$
(7.22)

Furthermore, $(i\partial_T - \not{D}_0)\alpha^{(0)} = 0$ with initial data $\alpha^{(0)}(0) = 0$ implies that we can take

.

$$\alpha^{(0)}(T) \equiv 0. \tag{7.23}$$

 $\mathcal{O}(\beta^1)$ solution. Using that $\alpha^{(0)}(T) \equiv 0$, the next order of equations simplifies to

$$i\partial_T G^{(1)} = -i\partial_{\tau_1} G^{(0)} ,$$
 (7.24)

$$(i\partial_T - \not\!\!D_0)\alpha^{(1)} = \mathscr{P}_{\mathbf{c}}\sigma_3\alpha_\star A(T)G^{(0)}.$$
(7.25)

Integrating (7.24) and using that $G^{(1)}$ vanishes for T = 0, we have

$$iG^{(1)}(T,\tau_1,\tau_2) = -i\partial_{\tau_1}G^{(0)}(\tau_1,\tau_2)T.$$

Condition (7.12) for n = 1 then implies $\partial_{\tau_1} G^{(0)}(\tau_1, \tau_2) = 0$ and hence $i \partial_T G^{(1)} = 0$. Therefore,

$$G^{(0)} = G^{(0)}(\tau_2)$$
 and $G^{(1)} = G^{(1)}(\tau_1, \tau_2)$.

The degrees of freedom offered by $G^{(1)}(\tau_1, \tau_2)$ are not required to remove resonances in subsequent terms of the hierarchy; we however need to satisfy (7.11b) for n = 1. We can therefore set

$$G^{(1)} \equiv 0.$$
 (7.26)

We solve (7.25) using Duhamel's principle and obtain, using Proposition 7.2, that

$$\alpha^{(1)}(T,\tau_2) = -i \int_0^T e^{-i\mathcal{D}_0(T-s)} \mathscr{P}_c \sigma_3 \alpha_\star A(s) \, ds \, G^{(0)}(\tau_2)$$
$$= -i\mathscr{G}(T) \mathscr{P}_c \sigma_3 \alpha_\star \, G^{(0)}(\tau_2) \,, \tag{7.27}$$

where $\mathscr{G}(T)$ is defined in Proposition 7.2. Hence, by item (3) of Proposition 7.2, for ρ taken sufficiently large we have

$$\|\langle X \rangle^{-\rho} \alpha^{(1)}(T, X, \tau_2)\|_{L^2(\mathbb{R}_X)} \lesssim |G^{(0)}(\tau_2)|, \quad \text{for } T \ge 0.$$
 (7.28)

Below we determine $G^{(0)}(\tau_2) = G^{(0)}(\beta^2 T)$, which we shall see is bounded on the time scale β^{-2} . Therefore, by (7.28), (7.13) is satisfied for n = 1.

 $\mathcal{O}(\beta^2)$ solution. The system (7.21) reduces to (using (7.23) and (7.26))

$$i\partial_T G^{(2)} = -i\partial_{\tau_2} G^{(0)} + A(T) \left\langle \sigma_3 \alpha_\star, \alpha^{(1)}(T, \tau_2) \right\rangle, \qquad (7.29a)$$

$$(i\partial_T - \not\!\!D_0)\alpha^{(2)} = -i\mathscr{P}_c\sigma_3\alpha_\star\eta_A(T)A(T)G^{(0)} + \mathscr{P}_c\sigma_3\alpha_\star A(T)\alpha^{(1)}.$$
 (7.29b)

Integration of (7.29a), using the initial condition $G^{(2)}(0) = 0$, implies

$$iG^{(2)}(T,\tau_2) = T \left(-i\partial_{\tau_2} G^{(0)}(\tau_2) + \frac{1}{T} \int_0^T A(s) \left\langle \sigma_3 \alpha_\star, \alpha^{(1)}(s,\tau_2) \right\rangle \, ds \right).$$
(7.30)

Since we seek an expansion where $\sup_{0 \le T \le \beta^{-2}} \beta^2 |G^{(2)}(T)| = o(1)$ as $\beta \downarrow 0$ (see (7.12)), we require that

$$i\partial_{\tau_2}G^{(0)}(\tau_2) = \lim_{T \to \infty} \frac{1}{T} \int_0^T A(s) \left\langle \sigma_3 \alpha_\star, \alpha^{(1)}(s, \tau_2) \right\rangle \, ds \,. \tag{7.31}$$

Let's next study the inner product: $\langle \sigma_3 \alpha_{\star}, \alpha^{(1)}(s, \tau_2) \rangle$ appearing in (7.31). By (7.27),

$$\left\langle \sigma_3 \alpha_\star, \alpha^{(1)}(s, \tau_2) \right\rangle = \left\langle \sigma_3 \alpha_\star, -i\mathscr{G}(T) \mathscr{P}_{\mathrm{c}} \sigma_3 \alpha_\star \right\rangle G^{(0)}(\tau_2).$$

Therefore, $G^{(0)}(\tau_2)$ satisfies:

$$i\partial_{\tau_2} G^{(0)}(\tau_2) = \lim_{T \to \infty} \frac{1}{T} \int_0^T A(s) \Upsilon(s) ds \ G^{(0)}(\tau_2), \tag{7.32}$$

where

$$\Upsilon(s) \equiv \langle \sigma_3 \alpha_\star, -i\mathscr{G}(T)\mathscr{P}_{\mathbf{c}} \sigma_3 \alpha_\star \rangle$$

In order to evaluate the limit in (7.32), we apply Proposition 7.2 with $f_1 = f_2 =$ $\sigma_3 \alpha_{\star}$ to expand the expression for $\Upsilon(s)$

$$\begin{split} \Upsilon(s) &= \langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, -i\mathscr{G}(T)\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \rangle \tag{7.33} \\ &= -\frac{e^{i\omega s}}{2} \left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \mathrm{PV}\frac{1}{\not{D}_{0}+\omega} \,\,\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle - \frac{e^{-i\omega s}}{2} \left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \mathrm{PV}\frac{1}{\not{D}_{0}-\omega} \,\,\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle \\ &- \frac{i\pi}{2}e^{i\omega s} \left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \delta(\not{D}_{0}+\omega)\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle - \frac{i\pi}{2}e^{-i\omega s} \left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \delta(\not{D}_{0}-\omega)\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle \\ &+ \Upsilon_{1}(s) \,. \end{split}$$

The first four terms in $\Upsilon(s)$ lead to the resonant decay formulas (6.5)–(6.6). The error term, $\Upsilon_1(s)$, is given by:

and will now be shown to decay with s.

Since α_{\star} is exponentially decaying (see (3.7)), $\langle x \rangle^r \alpha_* \in L^2$ for any fixed r > 0, and therefore by inserting $\langle x \rangle^r \langle x \rangle^{-r}$ and applying Cauchy-Schwartz inequality, we have

$$|\Upsilon_{1}(t)| \leq \left\| \langle x \rangle^{r} \sigma_{3} \alpha_{\star} \right\|_{L^{2}} \cdot \sum_{\pm} \left\| \langle x \rangle^{-r} \frac{e^{-i\mathcal{P}_{0}t}}{i(\mathcal{P}_{0} \pm \omega - i0)} \mathscr{P}_{c} \sigma_{3} \alpha_{\star} \right\|_{L^{2}}.$$
 (7.34)

Here again, since α_* is exponentially decaying, there exists r > 0 such that Proposition 7.1 is applicable and yields

$$|\Upsilon_1(s)| \lesssim \langle s \rangle^{-\rho}$$
, with $\rho > 0.$ (7.35)

Next, substituting $A(s) = \cos(\omega s) = \frac{1}{2}(e^{i\omega s} + e^{-i\omega s})$ into (7.33) and using the bound (7.35) we obtain

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T A(s) \Upsilon(s) ds = -i\Gamma_0 - \Lambda_0 , \qquad (7.36)$$

where (see also (6.5) and (6.6))

$$\Gamma_{0}(\omega) \equiv \frac{\pi}{4} \left(\left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \delta(\not{\!\!\!D}_{0}+\omega)\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle + \left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \delta(\not{\!\!\!D}_{0}-\omega)\mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle \right),$$

$$(7.37)$$

$$\Lambda_{0}(\omega) \equiv \left(\left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \operatorname{PV}\frac{1}{\not{\!\!D}_{0}+\omega} \,\, \mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle + \left\langle \mathscr{P}_{c}\sigma_{3}\alpha_{\star}, \operatorname{PV}\frac{1}{\not{\!\!D}_{0}-\omega} \,\, \mathscr{P}_{c}\sigma_{3}\alpha_{\star} \right\rangle \right).$$

$$(7.38)$$

Equation (7.32) and (7.36) imply

$$\partial_{\tau_2} G^{(0)}(\tau_2) = (-\Gamma_0 + i\Lambda_0) \, G^{(0)}(\tau_2) \,. \tag{7.39}$$

The operators $\delta(\not\!\!\!D_0 \pm \omega)$ are non-negative self-adjoint operators. Generically, since $\omega \in \operatorname{spec}_{ess}(\mathcal{D}_0)$, Γ is strictly positive [3, Section 4] and hence $G^{(0)}(\tau_2)$ is exponentially decaying. Since $\tau_2 = \beta^2 T$, the decay is on the time scale $T \sim \beta^{-2}$. Returning now to the expression for $G^{(2)}(\tau_2)$ in (7.30) we have

$$G^{(2)}(T,\tau_2) = -iT \left(\frac{1}{T} \int_0^T A(s)\Upsilon(s) \, ds - (-i\Gamma_0 - \Lambda_0) \right) G^{(0)}(\tau_2) \,.$$
(7.40)

Using (7.36) we see that the expression in parenthesis in (7.40) tends to zero as $T \to \infty$.

Recall from (7.10) and (7.26) that $G(T;\beta) \approx G^{(0)}(\tau_2) + \beta^2 G^{(2)}(T,\tau_2)$. By (7.40), we have

$$\beta^2 G^{(2)}(T, \tau_2) = \beta^2 \ o(T) \times G^{(0)}(\tau_2) \text{ as } T \to \infty.$$

Therefore,

$$G(T;\beta) \approx (1+\beta^2 o(T)) G^{(0)}(\beta^2 T) = (1+\beta^2 o(T)) e^{(-\Gamma_0+i\Lambda_0)\beta^2 T},$$
 (7.41)

which satisfies (7.12) through order β^2 .

Finally $\alpha^{(2)}$, which satisfies (7.29b), can be bounded using the bound on $\alpha^{(1)}$ in (7.28), Proposition 7.2 and Theorem 7.1. Together with (7.41) we have verified that our approximate solution

$$G \approx G^{(0)} + \beta G^{(1)} + \beta^2 G^{(2)},$$
(7.42)

$$\alpha_d \approx \alpha^{(0)} + \beta \alpha^{(1)} + \beta^2 \alpha^{(2)} , \qquad (7.43)$$

satisfies (7.12), (7.13). This completes our derivation of the radiation damping effect on the time-scale β^{-2} .

8. Proof of the time decay estimate of Proposition 7.1

We prove the following: There exists $\rho > 0$ and r > 0 such that for any $f \in \mathcal{S}(\mathbb{R})$ we have

$$\sup_{t\geq 0} t^{\rho} \left\| \langle x \rangle^{-r} \frac{e^{-i\mathcal{P}_0 t}}{i(\mathcal{P}_0 - \omega - i0)} \mathscr{P}_{\mathbf{c}} f \right\|_{L^2} < \infty.$$

$$(8.1)$$

Here, \mathscr{P}_{c} denotes the continuous (dispersive) spectral part of D_{0} .

Bounds similar to (8.1) are proved in [42] for scalar 3D Klein-Gordon equations with spatially varying and decaying potentials. Our strategy is to make an algebraic reduction to a problem of this type and to apply appropriate 1D dispersive estimates. In particular, to prove (8.1) (and hence the decay of $\Upsilon_1(t)$, see (7.34)), we shall re-express $e^{-i\mathcal{D}_0 t}$ in terms of a diagonal Klein-Gordon evolution operators to which we can apply known time-decay estimates.

We begin with the an algebraic observation.

Lemma 8.1. (1)

$$\mathcal{D}_0^2 = S\mathcal{D}S^* , \qquad (8.2)$$

$$S \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} , \quad \mathcal{D} \equiv \begin{pmatrix} Z_+ & 0 \\ 0 & Z_- \end{pmatrix} ,$$

and

$$Z_{\pm} \equiv -v_{\rm D}\partial_X^2 + \vartheta_{\sharp}^2 \kappa^2(X) \mp v_{\rm D}\vartheta_{\sharp}\kappa'(X) \,.$$

(2) For any continuous function ϕ on the spectrum of $\not{\mathbb{D}}_0^2$,

$$\phi(\not\!\!D_0^2)\mathscr{P}_{\mathbf{c}}(\not\!\!D_0) = S\phi(\mathcal{D})S^*$$

Note that Z_+ and Z_- are non-negative self-adjoint operators, by (8.2). It is also useful to note that Z_+ and Z_- can be expressed as spatially localized perturbations of the constant coefficient operator

$$Z_0 = -\partial_x^2 + \kappa_\infty^2; \quad \text{namely}, \tag{8.3}$$

$$Z_{\pm} = Z_0 + \vartheta_{\sharp}^2 (\kappa^2(X) - \kappa_{\infty}^2) \mp v_{\mathrm{D}} \vartheta_{\sharp} \kappa'(X) \,. \tag{8.4}$$

Proof of Lemma 8.1. Using the commutation relation $\sigma_3 \sigma_1 = -\sigma_1 \sigma_3$, we obtain

$$\begin{split} \not{D}_{0}^{2} &= (iv_{\mathrm{D}}\sigma_{3}\partial_{X} + \vartheta_{\sharp}\kappa(X)\sigma_{1})^{2} \\ &= I(-v_{\mathrm{D}}^{2}\partial_{X}^{2} + \vartheta_{\sharp}^{2}\kappa^{2}(X)) - v_{\mathrm{D}}\vartheta_{\sharp}\sigma_{2} \,\,\kappa'(X) \,. \end{split}$$

Hence, \not{D}_0^2 is diagonalizable using the eigenvectors of σ_2 .

Define $\mathscr{P}_+ \equiv \mathscr{P}_c(\not{\!\!\!D}_0 > 0)$ and $\mathscr{P}_- \equiv \mathscr{P}_c(\not{\!\!\!D}_0 < 0)$ so that $\mathscr{P}_c = \mathscr{P}_+ + \mathscr{P}_-$.

Proposition 8.1.

$$\mathcal{D}_0 \mathscr{P}_{\pm} f = \pm S \mathcal{D}^{\frac{1}{2}} S^* \mathscr{P}_{\pm} f$$

and, in particular,

$$\frac{e^{-i\mathcal{D}_0 t}}{i(\mathcal{D}_0 - \omega - i0)} P_{\pm} f = S \frac{e^{\pm i\mathcal{D}^{\frac{1}{2}} t}}{i(\pm \mathcal{D}^{\frac{1}{2}} - \omega - i0)} S^* P_{\pm} f.$$
(8.5)

Proof of Proposition 8.1. Since \mathcal{D} is similar to the positive semi-definite $\not{\!\!\!\!D}_0^2$ (positive definite on its continuous part), we have $\not{\!\!\!\!\!\!D}_0 \mathscr{P}_{\pm} f = \pm (S\mathcal{D}S^*)^{\frac{1}{2}} \mathscr{P}_{\pm} f$. For simplicity, we restrict our attention to $f \in \operatorname{Range}(\mathscr{P}_+)$. The case where $f \in \operatorname{Range}(\mathscr{P}_-)$ negative case follows analogously.

Recall that for any positive-semidefinite operator A we have the following formula 8

$$A^{\frac{1}{2}} = \pi A \int_{0}^{\infty} z^{-\frac{1}{2}} \left(zI + A \right)^{-1} dz \,.$$

Therefore

$$(S\mathcal{D}S^*)^{\frac{1}{2}} = S\mathcal{D}S^* \pi \int_{0}^{\infty} z^{-\frac{1}{2}} (zI + S\mathcal{D}S^*)^{-1} dz$$

= $S\mathcal{D}S^* \pi \int_{0}^{\infty} z^{-\frac{1}{2}} (S(zI + \mathcal{D})S^*)^{-1} dz$
= $S\mathcal{D}S^* S \left[\pi \int_{0}^{\infty} z^{-\frac{1}{2}} (zI + \mathcal{D})^{-1} dz\right] S^*$
= $S \left[\pi \mathcal{D} \int_{0}^{\infty} z^{-\frac{1}{2}} (zI + \mathcal{D})^{-1} dz\right] S^* = S\mathcal{D}^{\frac{1}{2}}S^*.$

Hence, $D_0 \mathscr{P}_{\pm} f = \pm S \mathcal{D}^{\frac{1}{2}} S^* \mathscr{P}_{\pm}$. The equation (8.5) now follows.

⁸This formula is obtained by using functional calculus and calculating the integral $\int_0^\infty z^{-1/2}(z+a)^{-1} dz$ for a scalar a > 0.

8.1. **Proof of Proposition 7.1, time-decay estimate** (8.1). Let J be a nonempty open interval in \mathbb{R} containing zero and let $\chi_J(y)$ denote a smoothed out characteristic function with support in J. The support of J will be fixed below. We write $\mathbb{R} = J + J^c$ and hence $1 = \chi_J(y) + \chi_{J^c}(y)$. Therefore,

$$\left\| \langle x \rangle^{-r} S \frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}} - \omega - i0)} S^* \mathscr{P}_+ f \right\|_{L^2} \leq \text{Term I} + \text{Term II}, \quad \text{where}$$
$$\text{Term I} \equiv \left\| \langle x \rangle^{-r} S \frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}} - \omega - i0)} \chi_J(|\mathcal{D}^{\frac{1}{2}} - \omega|) S^* \mathscr{P}_+ f \right\|_{L^2}$$
(8.6)

Term II
$$\equiv \left\| \langle x \rangle^{-r} S \frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}} - \omega - i0)} \chi_{J^c}(|\mathcal{D}^{\frac{1}{2}} - \omega|) S^* \mathscr{P}_+ f \right\|_{L^2}.$$
 (8.7)

We next estimate the expressions Term I and Term II. In particular, we show that for any N, there exists r = r(N) such that

$$\text{Term I} \le t^{-N} \| \langle x \rangle^N f \|_{L^2}, \tag{8.8}$$

Ferm II
$$\leq t^{-\frac{1}{4}} \|f\|_{W^{1,\frac{4}{3}}}$$
. (8.9)

We first bound Term II and then Term I.

8.2. Time-decay estimates for Term II, given by (8.7).

Let
$$q^{-1} + (p')^{-1} = 1/2$$
. Then,
Term II = $\left\| \langle x \rangle^{-r} S \frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}} - \omega - i0)} \chi_{J^c}(|\mathcal{D}^{\frac{1}{2}} - \omega|) S^* P_+ f \right\|_{L^2}$
 $\leq \| \langle x \rangle^{-r} \|_{L^q} \left\| S \frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}} - \omega - i0)} \chi_{J^c}(|\mathcal{D}^{\frac{1}{2}} - \omega|) S^* P_+ f \right\|_{L^{p'}}.$

Next, we prove a time-decay estimate for the latter factor.

For j = +, -, introduce wave operators W_j which intertwine each Z_j on its continuous spectral part [44, 45], i.e., $Z_j \mathscr{P}_c(Z_j) = W_j Z_0 W_j^*$, and define the block diagonal operator $W = \text{diag}(W_+, W_-)$. For any Borel measurable function, ϕ , we have

$$\phi(Z_{\pm})\mathscr{P}_{c}(Z_{\pm}) = W_{\pm}\phi(Z_{0})W_{\pm}^{*}, \qquad (8.10)$$

and hence for matrix operator-valued Borel functions

$$\phi(\mathcal{D})\mathscr{P}_{\mathbf{c}}(\mathcal{D}) = W\phi(Z_0\sigma_0)W^*$$

where σ_0 is the 2 × 2 identity matrix. Therefore,

$$\phi(\not{D}_0^2)P_c(\not{D}_0) = S\phi(\mathcal{D})\mathscr{P}_c(\mathcal{D})S^* = SW\phi(\mathcal{D}_0)(SW)^*, \tag{8.11}$$

where $\mathcal{D}_0 = \sigma_0 Z_0$. In particular, $(SW)(SW)^* = \mathscr{P}_{c}(\not{\!\!\!D}_0)$.

As shown in [44, 45] the wave operators W_j , and hence W, are bounded in $W^{k,p}(\mathbb{R}^n)$. This then allows us to reduce decay estimates for $e^{-i\mathcal{D}^{\frac{1}{2}t}}\mathscr{P}_{c}(\mathcal{D})$ to

those for $e^{-iZ_0^{\frac{1}{2}}t}$. Therefore, we have with $p^{-1} + (p')^{-1} = 1$:

$$\begin{aligned} \text{Term II} &\lesssim \left\| SW \frac{e^{-i\mathcal{D}_{0}^{\frac{1}{2}}t}}{i(\mathcal{D}_{0}^{\frac{1}{2}} - \omega - i0)} \chi_{J^{c}}(|\mathcal{D}_{0}^{\frac{1}{2}} - \omega|)(SW)^{*}f \right\|_{L^{p'}} \\ &\lesssim \left\| \frac{e^{-i\mathcal{D}_{0}^{\frac{1}{2}}t}}{i(\mathcal{D}_{0}^{\frac{1}{2}} - \omega - i0)} \chi_{J^{c}}(|\mathcal{D}_{0}^{\frac{1}{2}} - \omega|)(SW)^{*}f \right\|_{L^{p'}} \\ &\lesssim \left\| e^{-i\mathcal{D}_{0}^{\frac{1}{2}}t} \right\|_{L^{p'} \leftarrow W^{1,p}} \left\| (\mathcal{D}_{0}^{\frac{1}{2}} - \omega)^{-1}\chi_{J^{c}}(|\mathcal{D}_{0}^{\frac{1}{2}} - \omega|)(SW)^{*}f \right\|_{W^{1,p}} \\ &\lesssim \left\| e^{-i\mathcal{D}_{0}^{\frac{1}{2}}t} \right\|_{L^{p'} \leftarrow W^{1,p}} \left\| \mathcal{D}_{0}^{\frac{1}{2}}(\mathcal{D}_{0}^{\frac{1}{2}} - \omega)^{-1}\chi_{J^{c}}(|\mathcal{D}_{0}^{\frac{1}{2}} - \omega|)(SW)^{*}f \right\|_{L^{p}} \\ &\lesssim \left\| e^{-i\mathcal{D}_{0}^{\frac{1}{2}}t} \right\|_{L^{p'} \leftarrow W^{1,p}} \left\| \mathcal{D}_{0}^{\frac{1}{2}}(\mathcal{D}_{0}^{\frac{1}{2}} - \omega)^{-1}\chi_{J^{c}}(|\mathcal{D}_{0}^{\frac{1}{2}} - \omega|) \right\|_{L^{p} \leftarrow L^{p}} \left\| (SW)^{*}f \right\|_{L^{p}}. \end{aligned}$$

The second factor just above is bounded because it can be expressed in terms of a Fourier multiplier on L^p and the third factor is controlled by $||f||_{L^p}$ by boundness of wave operators. Hence,

Term II
$$\lesssim \left\| e^{-i\mathcal{D}_0^{\frac{1}{2}}t} \right\|_{W^{1,p} \to L^{p'}} \cdot \|f\|_{L^p} \lesssim \left\| e^{-iZ_0^{\frac{1}{2}}t} \right\|_{W^{1,p} \to L^{p'}} \cdot \|f\|_{L^p}.$$

Dispersive time-decay estimates for the 1D Klein-Gordon equation yield:

$$||e^{-iZ_0^{\frac{1}{2}}t}f||_{L^{\infty}} \le t^{-1/2}||(I-\partial_x^2)^{\frac{1}{4}}f||_{L^1}, \quad t \gg 1;$$

for general results, see, e.g., [13]. Together with the unitarity of the Klein-Gordon flow $\|e^{-iZ_0^{\frac{1}{2}}t}f\|_{L^2} = \|f\|_{L^2}$ we have, using interpolation, that

$$\begin{split} \|e^{-iZ_0^{\frac{1}{2}}t}f\|_{L^{p'}} &\lesssim \|e^{-iZ_0^{\frac{1}{2}}t}f\|_{L^2}^{\frac{2}{p'}} \cdot \|e^{-iZ_0^{\frac{1}{2}}t}f\|_{L^{\infty}}^{1-\frac{2}{p'}} \\ &\lesssim \|f\|_{L^2}^{\frac{2}{p'}} \cdot \left(t^{-\frac{1}{2}}\|(I-\partial_x^2)^{\frac{1}{4}}f\|_{L^1}\right)^{1-\frac{2}{p'}} \\ &\lesssim t^{-\frac{1}{2}+\frac{1}{p'}} \|f\|_{L^2}^{\frac{2}{p'}} \cdot \|f\|_{W^{\frac{1}{2},1}}^{1-\frac{2}{p'}}, \end{split}$$

where $p^{-1} + (p')^{-1} = 1$. Finally, let us fix p = 4/3 and p' = 4. Then, we have that

$$\text{Term II} \leq t^{-\frac{1}{4}} \, \|f\|_{L^{\frac{4}{3}}} \, \cdot \, \|f\|_{L^{2}}^{\frac{1}{2}} \, \cdot \, \|f\|_{W^{\frac{1}{2},1}}^{\frac{1}{2}}.$$

8.3. Time decay estimate for Term I given by (8.6). Recall that

Term I
$$\equiv \left\| \langle x \rangle^{-r} S \frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}} - \omega - i0)} \chi_J(|\mathcal{D}^{\frac{1}{2}} - \omega|) S^* P_+ f \right\|_{L^2}$$
.

Noting that, by integrating the right hand side of (??) for $\varepsilon > 0$ and taking the upper limit of the integral to $\tau \to \infty$,

$$\frac{e^{-i\mathcal{D}^{\frac{1}{2}}t}}{i(\mathcal{D}^{\frac{1}{2}}-\omega-i0)}\chi_{J}(|\mathcal{D}^{\frac{1}{2}}-\omega|)S^{*}P_{+}$$
$$=-e^{-i\omega t}\lim_{\varepsilon\to 0^{+}}\int_{t}^{\infty}e^{-i(\mathcal{D}^{\frac{1}{2}}-\omega-i\varepsilon)\tau}\chi_{J}(|\mathcal{D}^{\frac{1}{2}}-\omega|)S^{*}P_{+}\,d\tau\,,\qquad(8.12)$$

it suffices to show sufficient decay of an appropriate operator norm of the integrand of (8.12). Since we have $\mathcal{D} = \text{diag}(Z_+, Z_-)$, time-decay bounds of the integrand in (8.12) can be reduced to $e^{-iZ_{\pm}t}P_c(Z_{\pm})$. The Mourre approach to scattering estimates, as applied in [42, Section 2], yields

$$\|\langle x\rangle^{-r}Se^{-i\mathcal{D}^{\frac{1}{2}}\tau}\chi_J(|\mathcal{D}^{1/2}-\omega|)S^*P_+f\|_{L^2} \lesssim \left(\langle t\rangle^{-r}+\langle t\rangle^{-N/2}\right)\|\langle x\rangle^{N/2}f\|_{L^2}, \qquad N \in \mathbb{N}$$

$$(8.13)$$

where r and N can be taken large. The desired bound on the expression Term I (see (8.6)) now follows from estimating the integrand of (8.12) using (8.13) and integrating.

With the estimates (8.8) and (8.9) on Term I and Term II now proved, the proof of Proposition 7.1 is now complete.

Appendix A. Derivation of the effective Dirac equation

Our proof proceeds in two parts. We first formally derive the effective Dirac equation and the corrector equations using multiple scales methods. Then, using energy estimates, we bound the corrector term for large but finite times.

Introduce $X = \varepsilon x$ and $T = \varepsilon t$, the slow space and time variables, respectively. Formally, we view ψ as dependent on all four variables, i.e., we seek $\Psi(t, T, x, X)$ such that $\psi(t, x)$, the true solution of the Schrödinger equation (3.12), is well approximated by $\Psi(t, T, x, X)\Big|_{T=\varepsilon t, X=\varepsilon X}$. Hence, $\partial_x \mapsto \partial_x + \varepsilon \partial_X$ and $\partial_t \mapsto \partial_t + \varepsilon \partial_T$, and (3.12) is rewritten into

$$i(\partial_t - H_0)\Psi(t, x, T, X) = \varepsilon H_1 \Psi + \varepsilon^2 H_2 \Psi, \qquad (A.1a)$$

where $H_0 = -\partial_x^2 + V(x)$ as usual, and

$$H_1 \equiv -i\partial_T - 2\partial_x \partial_X + \kappa(X)W(x) + 2iA(T)\partial_x , \quad H_2 \equiv -\partial_X^2 + 2i\beta A(T)\partial_X .$$
(A.1b)

Since we are looking for wavepackets which are spectrally centered at the Dirac point, the boundary conditions (or function space) in which we solve (A.1) is $k_{\rm D} = \pi$ -pseudo periodicity in x and $L_X^2(\mathbb{R})$ in X.

We seek Ψ as an expansion in orders of ε ,

$$\psi = \psi^{(0)}(t, T, x, X) + \varepsilon \psi^{(1)}(t, T, x, X) + \varepsilon^2 \psi^{(2)}(t, T, x, X) + \eta^{\varepsilon}(t, x).$$
(A.2)

Substituting (A.2) into (A.1) and collecting terms by orders of ε , at the ε^0 order we obtain the initial value problem

$$(i\partial_t - H_0)\psi^{(0)} = 0,$$
 (A.3a)

$$\psi^{(0)}(0,0,x,X) = \alpha_{*,1}(X)\Phi_1(x;k_{\rm D}) + \alpha_{*,2}(X)\Phi_2(x;k_{\rm D}), \qquad (A.3b)$$

with $\alpha_{\star} \in L^2_X(\mathbb{R}; \mathbb{C}^2)$ as defined in (3.5).⁹ As noted in Section 3.1, the $L^2_{k_{\mathrm{D}}}$ -nullspace of $E_{\mathrm{D}}I - H_{\mathrm{bulk}}$ is spanned by the Bloch modes $\{\Phi_1, \Phi_2\}$, and so the solution of (A.3) is given by

$$\psi^{(0)} = e^{-iE_{\rm D}t} \sum_{j=1}^{2} \alpha_j(T, X) \Phi_j(x) , \qquad (A.4)$$

where $\alpha_1(T, X)$ and $\alpha_2(T, X)$ are rapidly-decaying functions yet to be determined with $\alpha_j(0, X) = \alpha_{*,j}(X)$ for j = 1, 2.

Proceeding to order ε , we obtain

$$(i\partial_t - H_0)\psi^{(1)} = (-i\partial_T - 2\partial_x\partial_X + \kappa(X)W(x) + 2iA(T)\partial_x)\psi^{(0)}.$$
(A.5)

Since $\psi^{(0)}$ oscillates with frequency $E_{\rm D}$, it will be convenient to extract the fast oscillatory behavior of $\psi^{(1)}$ by defining

$$\psi^{(1)}(t,T,x,X) = e^{-iE_{\rm D}t} \widetilde{\psi}^{(1)}(T,x,X) \,.$$

Substituting the above ansatz and (A.4) into (A.5), we obtain

$$(E_{\rm D}I - H_0)\widetilde{\psi}^{(1)}(T, x, X) = -\sum_{j=1}^2 i\partial_T \alpha_j(T, X) \Phi_j + i\sum_{j=1}^2 \partial_X \alpha_j \cdot 2i\partial_x \Phi_j + \sum_{j=1}^2 \kappa(X)\alpha_j W(x)\Phi_j + \sum_{j=1}^2 A(T)\alpha_j \cdot 2i\partial_x \Phi_j.$$
(A.6)

The solvability of (A.6) for $\tilde{\psi}^{(1)} \in L^2_{k_{\mathrm{D}}}$ requires the $L^2_{k_{\mathrm{D}}}$ -orthogonality of its righthand side to Φ_1 and Φ_2 . In [11, Propsition 2.2], it is shown that

$$\begin{pmatrix} \left\langle \Phi_1, 2i\partial_x \Phi_1 \right\rangle & \left\langle \Phi_2, 2i\partial_x \Phi_1 \right\rangle \\ \left\langle \Phi_1, 2i\partial_x \Phi_2 \right\rangle & \left\langle \Phi_2, 2i\partial_x \Phi_2 \right\rangle \end{pmatrix} = v_{\mathrm{D}}\sigma_3 ,$$
 (A.7a)

$$\begin{pmatrix} \langle \Phi_1, W\Phi_1 \rangle & \langle \Phi_2, W\Phi_1 \rangle \\ \langle \Phi_1, W\Phi_2 \rangle & \langle \Phi_2, W\Phi_2 \rangle \end{pmatrix} = \vartheta_{\sharp} \sigma_1 ,$$
 (A.7b)

where $v_{\rm D}, \vartheta_{\sharp} \neq 0$. Thus, the solvability conditions reduce to

$$i\partial_T \alpha_1 = \langle \Phi_1, 2i\partial_x \Phi_1 \rangle (i\partial_X + \beta A(T))\alpha_1 + \langle \Phi_1, W \Phi_2 \rangle \kappa(X)\alpha_2, i\partial_T \alpha_2 = \langle \Phi_2, 2i\partial_x \Phi_2 \rangle (i\partial_X + \beta A(T))\alpha_2 + \langle \Phi_2, W \Phi_1 \rangle \kappa(X)\alpha_1,$$

from which we obtain the Dirac equation

$$i\partial_T \alpha(T, X) = (iv_{\rm D}\sigma_3\partial_X + \vartheta_{\sharp}\kappa(X)\sigma_1 + v_{\rm D}A(T)\sigma_3)\,\alpha\,,\tag{A.8}$$

with $\alpha(0, X) = \alpha_{\star}(X)$, as first introduced in (5.1).

Let π^{\perp} denote the projection onto the orthogonal complement of span{ Φ_1, Φ_2 } in $L^2_{k_{\rm D}}$:

$$\pi^{\perp} = I - \sum_{j=1}^{2} \langle \Phi_j, \cdot \rangle \Phi_j = \sum_{j \ge 3} \langle \Phi_j, \cdot \rangle \Phi_j;$$

⁹Admittedly, the initial condition ψ_{\star} is only well-approximated by $\alpha_{\star}^{\top} \Phi$, but since we are only formally exampling the initial value problem and disregard small error term, we can take this approximation in (A.3b).

for convenience we indexed the $L_{k_{\rm D}}^2$ eigenpairs of H^0 such that span $\{\Phi_1, \Phi_2\}^{\perp} =$ span $\{\Phi_j : j \geq 3\}$. If (α_1, α_2) is constrained to satisfy (A.8), then $\pi^{\perp}(H_1\Psi_0) = H_1\psi^{(0)}$ and so:

$$\widetilde{\psi}^{(1)}(t, x, T, X) = (E_D I - H^0)^{-1} H_1 \psi^{(0)} + \varepsilon \sum_{j=1,2} \beta_j(T, X) \Phi_j(x) , \qquad (A.9)$$

where $\varepsilon \beta_j$ are decaying functions of X which are to be determined at the next order equation (order ε^2)), and finally $\psi^{(1)} = \widetilde{\psi}^{(1)} e^{-iE_D t}$.

Turning next to the ε^2 order equations, we get

$$(i\partial_t - H_0)\psi^{(2)} = H_1\psi^{(1)} + H_2\psi^{(0)}, \qquad \psi^{(2)}(0, x, 0, X) = 0, \qquad (A.10)$$

we again write $\psi^{(2)}(t, x, T, X) = \tilde{\psi}^{(2)}(x, T, X)e^{-iE_D t}$. In analogy with our first order analysis, the condition for solvability condition of (A.10) in $L^2_{k_D}$ is that $H_1(\tilde{\psi}^{(1)} + \sum_j \beta_j \Phi_j)$ is orthogonal to Φ_1 and Φ_2 . In a manner analogous to the derivation of (5.1), we obtain a system of *forced* Dirac equations for $\beta \equiv (\beta_1, \beta_2)^{\top}$:

$$i\partial_T \beta(T, X) - \not\!\!\!D_A(T)\beta(T, X) = F_2(T, X), \qquad (A.11)$$

where $F_2 = (F_{2,1}, F_{2,2})^{\top}$, and for j = 1, 2:

$$F_{2,j} = \langle \Phi_j, H_1(E_D - H_0)^{-1} H_1 \psi^{(0)} \rangle_{L^2(\Omega)}.$$
(A.12)

We note that F_2 is independent of β , and is therefore a forcing term in (A.11). Corresponding to any solution of the initial value problem for (A.11) in $C(\mathbb{R}; L^2(\mathbb{R}; \mathbb{C}^2))$, we have that

$$\psi^{(2)}(t, x, T, X) = e^{-iE_D t} (E_D - H_0)^{-1} \pi^{\perp} \widetilde{\psi}^{(1)} .$$
(A.13)

Remark A.1. In writing $H_1 \tilde{\psi}^{(1)}$ and $H_1 (E_D - H_0)^{-1} H_1 \psi^{(0)}$ above, we apply the operator ∂_X to $\kappa(X)$. This is the reason we require that the domain-wall function κ have bounded derivatives of all orders - a sufficient, but perhaps not necessary condition. Such derivatives will be applied further in the subsequent sections without further notice.

Finally, to close the multiple scales expansion (A.2), the corrector η^{ε} is required to satisfy

$$(i\partial_t - H_0 - 2i\varepsilon A(\varepsilon t)\partial_x - \kappa(\varepsilon x)W(x))\eta^{\varepsilon}(t,x) = \varepsilon^3\mathscr{F}^{\varepsilon}(t,x), \qquad (A.14)$$

where

$$\mathscr{F}^{\varepsilon} \equiv \left[H_1 \psi^{(2)} + H_2 \psi^{(1)} + \varepsilon H_2 \psi^{(2)} \right] \Big|_{T = \varepsilon t, X = \varepsilon x}.$$
 (A.15)

A.1. Bounding the corrector, $\eta^{\varepsilon}(t, x)$. So far, (A.2) is a formal multiple scale expansion. To estimate the quality of the approximation of $\psi^{\varepsilon}(t, x)$ by $\psi^{(0)}$, we need to estimate the L^2 norms of $\psi^{(1)}$, $\psi^{(2)}$, and the corrector η^{ε} .

By self-adjointness of $H_0 + 2i\varepsilon A(\varepsilon t)\partial_x + \kappa(\varepsilon x)W(x)$ on the left hand side operator in (A.14), we have that $\partial_t \|\eta^{\varepsilon}(t)\|^2 = 2\varepsilon^3 \text{Re } \langle \eta, \mathscr{F}^{\varepsilon}(t, \cdot) \rangle$. This implies, by the Cauchy-Schwarz inequality, that $\partial_t \|\eta^{\varepsilon}(t)\| \leq \varepsilon^3 \|\mathscr{F}^{\varepsilon}(t, \cdot)\|$. Therefore, for all $t \geq 0$:

$$\|\eta^{\varepsilon}(t,\cdot)\|_{L^{2}(\mathbb{R}^{2})} \leq t\varepsilon^{3} \sup_{s\in[0,t]} \|\mathscr{F}^{\varepsilon}(s,\cdot)\|_{L^{2}(\mathbb{R}^{2})}, \qquad (A.16)$$

where $\mathscr{F}^{\varepsilon}$ is given by (A.15). By bounding the norm of $\mathscr{F}^{\varepsilon}$ we obtain the following proposition:

Proposition A.1. For all t > 0, we have the following $L^2(\mathbb{R}^2)$ bounds

$$\left\|\eta^{\varepsilon}(t,\cdot)\right\|_{L^{2}(\mathbb{R}^{2}_{x})} \lesssim t\varepsilon^{3} \sup_{s\in[0,t]} \left(\left\|\alpha(\varepsilon s,\cdot)\right\|_{H^{3}(\mathbb{R}^{2})} + \left\|\beta(\varepsilon s,\cdot)\right\|_{H^{2}(\mathbb{R}^{2})}\right).$$
(A.17)

Here, $\alpha = (\alpha_1, \alpha_2)^{\top}$ and $\beta = (\beta_1, \beta_2)^{\top}$ are solutions of the homogeneous and inhomogeneous Dirac equations (A.8) and (A.11), respectively. The implicit constant in (A.17) depends only on the functions V, W, κ , and A.

Proof. We shall use the following convention. If G(t, T, x, X) is a multi-scale function, then we write $G_{\varepsilon}(t, x) \equiv G(t, T, x, X)\Big|_{T=\varepsilon t, X=\varepsilon x}$. The proof of Proposition A.1 makes use of the following bounds:

$$\|\psi_{\varepsilon}^{(1)}(t,\cdot)\|_{2} \lesssim \|\alpha(\varepsilon t,\cdot)\|_{H^{1}} + \|\beta(\varepsilon t,\cdot)\|_{2}, \qquad (A.18)$$

- $\|\psi_{\varepsilon}^{(2)}(t,\cdot)\|_{2} \lesssim \|\alpha(\varepsilon t,\cdot)\|_{H^{1}} + \|\beta(\varepsilon t,\cdot)\|_{2}, \qquad (A.19)$
- $\|H_1\psi_{\varepsilon}^{(2)}(t,\cdot)\|_2 \lesssim \|\alpha(\varepsilon t,\cdot)\|_{H^2} + \|\beta(\varepsilon t,\cdot)\|_{H^1},$ (A.20)
- $\|H_2\psi_{\varepsilon}^{(1)}(t,\cdot)\|_2 \lesssim \|\alpha(\varepsilon t,\cdot)\|_{H^3} + \|\beta(\varepsilon t,\cdot)\|_{H^2},$ (A.21)
- $\|H_2\psi_{\varepsilon}^{(2)}(t,\cdot)\|_2 \lesssim \|\alpha(\varepsilon t,\cdot)\|_{H^3} + \|\beta(\varepsilon t,\cdot)\|_{H^2}, \qquad (A.22)$
 - $\|\mathscr{F}^{\varepsilon}(\varepsilon t, \cdot)\|_{2} \lesssim \|\alpha(\varepsilon t, \cdot)\|_{H^{3}} + \|\beta(\varepsilon t, \cdot)\|_{H^{2}}.$ (A.23)

It will be useful to decompose $\widetilde{\psi}^{(1)}$ into two separate terms and bound each of these elements separately

$$\widetilde{\Psi}_{11} = (E_D - H_0)^{-1} \pi^{\perp} H_1 \psi^{(0)}, \qquad \widetilde{\Psi}_{12} = \sum_j \beta_j \Phi_j.$$

We start with bounding $\|\tilde{\Psi}_{1,1}\|_2$. By definition

$$-H_1\psi^{(0)} = \varepsilon^{\frac{1}{2}} \sum_{j=1,2} \left(i\partial_T \alpha_j(T, X)\Phi_j + 2\partial_x \Phi_j \cdot \partial_X \alpha_j(T, X) - 2i\alpha_j(T, X)A(T) \cdot \partial_x \Phi_j - \kappa(X)W(x)\alpha_j(T, X)\Phi_j \right)$$

where $\Phi = (\Phi_1, \Phi_2)^{\top}$. Since $(E_D - H_0)^{-1}$ operates on $L^2_{k_D}$, we can represent $(E_D - H_0)^{-1} \pi^{\perp} H_1 \psi^{(0)}$ in the basis of the Bloch modes $\{\Phi_b\}_{b\geq 3}$. Using that, by

the derivation of (A.8), we enforced that $\pi^{\perp}H_1\Psi_0 = H_1\Psi_0$, we have

$$\begin{split} \left[\widetilde{\Psi}_{1,1} \right]_{\varepsilon} (t,x) &= \widetilde{\Psi}_{1,1}(t,\varepsilon t,x,\varepsilon x) \\ &= (E_D - H_0)^{-1} H_1 \psi^{(0)} \Big|_{T=\varepsilon t,X=\varepsilon x} \\ &= \varepsilon^{\frac{1}{2}} \sum_{b\geq 3} \sum_{j=1,2} \frac{\Phi_b}{E_b - E_D} \Big\langle \Phi_b, \\ &\quad -i \partial_T \alpha_j(T,X) \Phi_j - 2 \partial_x \Phi_j \cdot \partial_X \alpha_j(T,X) + i \alpha_j(T,X) A(T) \cdot \partial_x \Phi_j + \kappa(X) W(x) \Phi_j \Big\rangle_{L^2(\Omega)} \\ &= \varepsilon^{\frac{1}{2}} \sum_{b\geq 3} \frac{\Phi_b}{E_b - E_D} \Big[- \langle \Phi_b, \Phi_j \rangle_{L^2(\Omega)} \cdot \mathcal{D}(T) \alpha(T,X) \\ &\quad + \sum_{j=1,2} \langle \Phi_b, \partial_x \Phi_j \rangle_{L^2(\Omega)} \cdot (2 \partial_X \alpha(T,X) - 2iA(T) \alpha_j(T,X)) \Big]_{T=\varepsilon t,X=\varepsilon x}. \end{split}$$

We estimate $[\widetilde{\Psi}_{1,1}]_{\varepsilon}$ in $L^2(\mathbb{R})$ using that, since $\Phi_b \in L^{\infty}$, then $\|\Phi_b(\cdot)f(\cdot)\|_2 \lesssim \|f\|_2$ for any $f \in L^2$. Hence,

$$\begin{split} \|[\widetilde{\Psi}_{1,1}]_{\varepsilon}(t,\cdot)\|_{L^{2}(\mathbb{R})} &\lesssim \varepsilon^{\frac{1}{2}} \sum_{b \geq 3} \sum_{j=1,2} \frac{\|\Phi_{b}\|_{L^{\infty}}}{|E_{b} - E_{D}|} |\langle \Phi_{b}, \Phi_{j} \rangle_{L^{2}(\Omega)} | \varepsilon^{-\frac{1}{2}} \|\partial_{X} \alpha_{j}(\varepsilon t, \cdot)\|_{L^{2}(\mathbb{R}_{X})} \\ &+ \varepsilon^{\frac{1}{2}} \sum_{b \geq 3} \frac{\|\Phi_{b}\|_{L^{\infty}}}{|E_{b} - E_{D}|} ||\langle \Phi_{b}, \partial_{x} \Phi_{j} \rangle_{L^{2}(\Omega)} | \varepsilon^{-\frac{1}{2}} \|\alpha_{j}(\varepsilon t, \cdot)\|_{H^{1}(\mathbb{R}_{X})} \\ &\lesssim \|\alpha(\varepsilon t, \cdot)\|_{H^{1}(\mathbb{R}_{X})} \sum_{j=1,2} \sum_{b \geq 3} \frac{\|\Phi_{b}\|_{\infty}}{|E_{b} - E_{D}|} \left(|\langle \Phi_{b}, \Phi_{j} \rangle_{L^{2}(\Omega)} | + |\langle \Phi_{b}, \partial_{x} \Phi_{j} \rangle_{L^{2}(\Omega)} | \right) . \end{split}$$

$$(A.24)$$

By the Sobolev inequality and the relation $\partial_x^2 \Phi_b = (V - E_b) \Phi_b$, we have (with $\Omega = [0, 1]$) that any $b \ge 1$:

$$\begin{split} \|\Phi_b\|_{L^{\infty}(\Omega)} &\lesssim \|\Phi_b\|_{H^2(\Omega)} \lesssim \|\Phi_b\|_{L^2(\Omega)} + \|\partial_x^2 \Phi_b\|_{L^2(\Omega)} = \|\Phi_b\|_{L^2(\Omega)} + \|(V-E_b)\Phi_b\|_{L^2(\Omega)} \,. \end{split}$$
Furthermore, since V is bounded $\|\Phi_b\|_{L^{\infty}(\Omega)} \lesssim E_b\|\Phi_b\|_{L^2(\Omega)} \lesssim |b|^2$, where we have used that $\|\Phi_b\| = 1$, and that $E_b \sim b^2$ as $b \to \infty$, by Weyl asymptotics in one dimension. Hence, the factor $\|\Phi_b\|_{\infty}/|E_b - E_D|$ in (A.24) is uniformly bounded for all b. Therefore, bounding $\|[\tilde{\Psi}_{1,1}]_{\varepsilon}\|_{L^2(\mathbb{R})}$ reduces to showing, for j = 1, 2:

$$\sum_{b\geq 3} \left| \left\langle \Phi_b, \Phi_j \right\rangle_{L^2(\Omega)} \right| + \left| \left\langle \Phi_b, \partial_x \Phi_j \right\rangle_{L^2(\Omega)} \right| < \infty$$

We claim that both summands decay rapidly with b. Indeed, by the self-adjoint edness of H^0 , for r = 0, 1:

$$\langle \Phi_b, \partial_x^r \Phi_j \rangle = E_b^{-2} \langle (H_0)^2 \Phi_b, \partial_x^r \Phi_j \rangle = E_b^{-2} \langle \Phi_b, (H_0)^2 \partial_x^r \Phi_j \rangle \,,$$

and therefore

$$\left| \langle \Phi_b, \partial_x^r \Phi_j \rangle \right| \lesssim b^{-2} \| (H_0)^2 \partial_x^r \Phi_j \| \lesssim b^{-2} \| \Phi_j \|_{H^{4+r}}, \qquad r = 0, 1,$$

which is sufficient to ensure summability. It follows that

$$\|[\Psi_{1,1}]_{\varepsilon}(t,\cdot)\|_{L^2(\mathbb{R}_x)} \lesssim \|\alpha(\varepsilon t,\cdot)\|_{H^1(\mathbb{R}_x)}.$$
(A.25)

Together with the bound $\|[\widetilde{\Psi}_{1,2}]_{\varepsilon}(t,\cdot)\|_2 \lesssim \varepsilon \|\beta(\varepsilon t,\varepsilon \cdot)\|_{L^2(\mathbb{R}^2_x)} \lesssim \|\beta(\varepsilon t,\cdot)\|_{L^2(\mathbb{R}_x)}$, we obtain (A.18).

The upper bounds (A.19)–(A.22) proceed in a similar fashion. The upper bound (A.23) follows directly from the triangle inequality and $\mathscr{F} = H_1\Psi_2 + H_2\Psi_1 + \varepsilon H_2\Psi_2$. Finally, we prove (A.17) by combining (A.15), (A.16), and (A.23).

Proposition A.1 provides upper bounds for η^{ε} , the expansion corrector, in terms of the Sobolev norms of $\alpha(T, X)$ and $\beta(T, X)$, which satisfy the Dirac equations (A.8) and (A.11), respectively. We now turn to estimating these norms.

Lemma A.2. Let α satisfy and β denote solutions of homogeneous and nonhomogeneous Dirac equations (5.1) and (A.11). As initial data we take $\alpha(0, \cdot) = \alpha_0 \in H^4(\mathbb{R}; \mathbb{C}^2)$ and $\beta(0, x) \equiv 0$. Then, for all T > 0

$$\|\alpha(T, \cdot)\|_{H^s} \lesssim T \|\alpha(0, \cdot)\|_{H^s}, \qquad s > 0, \qquad (A.26)$$

$$\|\beta(T,\cdot)\|_{H^s} \lesssim T^2 \|\alpha(0,\cdot)\|_{H^{s+2}}, \qquad s \ge 0.$$
 (A.27)

Proof. For s > 0, we apply ∂_X^s to (A.8) to get

$$i\partial_T (\partial_X^s \alpha) = D(T)\partial_X^s \alpha + \sigma_1 \kappa^{(s)}(X)\alpha.$$

Next, we this equation from the left by the raw vector $(\partial_X^s \bar{\alpha}_1, \partial_X^s \bar{\alpha}_2)$, subtract the complex conjugate equation, and integrate over \mathbb{R}_X . By the self-adjointednees of $\mathcal{D}(T)$, we get

$$i\partial_T \|\partial_X^s \alpha\|_2^2 = \int\limits_{\mathbb{R}} \left[(\partial_X^s \alpha(T, X))^* \,\sigma_1 \kappa^{(s)}(X) \alpha(T, X) \,- \,\text{c.c.} \right] \, dX \,. \tag{A.28}$$

We bound each term in the integrand on the right-hand side separately, e.g.,

$$\int_{\mathbb{R}} \partial_X^s \bar{\alpha}_1(T, X) \kappa^{(s)}(X) \alpha_2(T, X) \, dX \le \|\partial_X^s \alpha(T, \cdot\|_2 \cdot \|\kappa^{(s)}\|_\infty \|\alpha_2(T, \cdot)\|_2$$

and so, by substituting back into (A.28), we get that

$$i\partial_T \|\partial_X^s \alpha\|_2 \le \|\kappa^{(s)}\|_\infty \|\alpha(T, \cdot)\|_2.$$
(A.29)

Now, since the Dirac equation (A.8) is unitary, $\|\alpha(T, \cdot)\|_2 = \|\alpha_0\|_2$, and so by integrating the differential inequality in (A.29) we get the upper bound in (A.26). To obtain (A.27) we have from (A.11) in $L^2(\mathbb{P}; \mathbb{C}^2)$ that

To obtain (A.27), we have from (A.11), in $L^2(\mathbb{R}; \mathbb{C}^2)$, that

$$i\partial_T \|\partial_X^s \beta(T, \cdot)\|_2^2 = 2 \operatorname{Im} \langle \beta(T, \cdot), \partial_X^{2s} F_2 \rangle = (-1)^s \cdot 2 \operatorname{Im} (\langle \partial_X^s \beta(T, \cdot), \partial_X^s F_2 \rangle), \quad (A.30)$$

where F_2 was defined in (7.1) and (A.12). Therefore, by the Cauchy-Schwarz inequality, $\partial_T \|\partial_X^s \beta(T, \cdot)\|_2 \leq \|F_2(T, \cdot)\|_{H^s(\mathbb{R}^2)}$. Finally, we bound the $H^s(\mathbb{R}; \mathbb{C}^2)$ norm of F_2 . Since, by definition

$$F_{2,j} = \langle \Phi_j, H_1(E_D - H_0)^{-1} H_1 \psi^{(0)} \rangle_{L^2_x(\Omega)}$$

we, as in Lemma A.1, using (A.26) that $||F_2(T, \cdot)||_{H^s_x} \leq ||\alpha(T, \cdot)||_{H^{s+2}_x} = T ||\alpha_0||_{H^{s+2}_x}$. Thus,

$$\|\partial_X^s \beta\|_{L^2(\mathbb{R}^2)} \lesssim \int_0^\top \|F_2(T', \cdot)\|_{H^{s+2}_X} \, dT' \lesssim T^2 \|\alpha_0\|_{H^{s+2}_X},$$

which proves (A.27).

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We now complete the proof of Theorem 5.1. With the notation and definitions introduced above our solution of the Schrödinger equation (1.1) is:

$$\psi^{\varepsilon}(t,x) = \psi^{(0)}_{\varepsilon}(t,x) + \varepsilon \psi^{(1)}_{\varepsilon}(t,x) + \varepsilon^2 \psi^{(2)}_{\varepsilon}(t,x) + \eta^{\varepsilon}(t,x).$$

We shall estimate the size of the corrector to the leading order (effective Dirac) approximation:

$$\psi^{\varepsilon}(t,x) - \psi^{(0)}_{\varepsilon}(t,x) = \varepsilon \psi^{(1)}_{\varepsilon}(t,x) + \varepsilon^2 \psi^{(2)}_{\varepsilon}(t,x) + \eta^{\varepsilon}(t,x) \,.$$

Using Lemmas A.1 and A.2 we have that

$$\begin{split} \|\psi^{\varepsilon}(t,x) - \psi^{(0)}_{\varepsilon}(t,\cdot)\|_{2} &\lesssim \varepsilon \|\psi^{(1)}_{\varepsilon}(t,\cdot)\|_{2} + \varepsilon^{2} \|\psi^{(2)}_{\varepsilon}(t,\cdot)\|_{2} + \|\eta^{\varepsilon}(t,\cdot)\|_{2} \\ &\lesssim \varepsilon \underbrace{(\|\alpha(\varepsilon t,\cdot)\|_{H^{1}} + \|\beta(\varepsilon t,\cdot)\|_{2})}_{\psi^{(1)}} + \varepsilon^{2} \underbrace{(\|\alpha(\varepsilon t,\cdot)\|_{H^{1}} + \|\beta(\varepsilon t,\cdot)\|_{2})}_{\psi^{(2)}} + \underbrace{t\varepsilon^{3}(\|\alpha(\varepsilon t,\cdot)\|_{H^{3}} + \|\beta(\varepsilon t,\cdot)\|_{H^{2}})}_{\eta} \\ &\lesssim \varepsilon \left(\varepsilon t \|\alpha_{0}\|_{H^{1}} + \varepsilon^{2}t^{2}\|\alpha_{0}\|_{H^{2}}\right) + \varepsilon^{2} \left(\varepsilon t \|\alpha_{0}\|_{H^{1}} + \varepsilon^{2}t^{2}\|\alpha_{0}\|_{H^{2}}\right) + t\varepsilon^{3} \left(\varepsilon t \|\alpha_{0}\|_{H^{3}} + t^{2}\varepsilon^{2}\|\alpha_{0}\|_{H^{4}}\right) \\ &\lesssim \varepsilon^{2}t \|\alpha_{0}\|_{H^{1}} + \varepsilon^{3}t^{2}\|\alpha_{0}\|_{H^{2}} + t^{2}\varepsilon^{4}\|\alpha_{0}\|_{H^{3}} + t^{3}\varepsilon^{5}\|\alpha_{0}\|_{H^{4}} \,. \end{split}$$

Therefore, for any $\rho > 0$ and ε sufficiently small, $\sup_{0 \le t \le \varepsilon^{-(3/2-\rho)}} \|\psi^{\varepsilon}(t,x) - \Psi_0(t,\cdot)\|_2 \le \varepsilon^{\rho}$. This completes the proof of Theorem 5.1.

Appendix B. The potentials $U_{\varepsilon}^{(\ell)}$ and their effective Dirac Hamiltonians

To construct $U_{\varepsilon}^{(1)}$, we choose $V(x) = \cos(4\pi x)$, $W(x) = \cos(2\pi x)$, $\kappa(X) = \tanh(X)$, $A(T) = \cos(\omega T)$, and $\varepsilon = \frac{1}{2}$. The initial value problem (3.12) becomes

$$i\partial_t \psi = \left[-\partial_x^2 + \cos(4\pi x) + \frac{1}{2} \tanh(\frac{1}{2}x)\cos(2\pi x) + i\beta\cos(\frac{1}{2}\omega t)\partial_x \right] \psi, \quad \text{(B.1a)}$$
$$\psi(0, x) = \psi_\star(x), \quad \text{(B.1b)}$$

where $\beta, \omega > 0$. The potential $U_{\varepsilon}^{(2)}$ models a chain of isolated and dimerized square wells, a more realistic model for an array of coiled optical waveguides as in Fig. 3. Let $\Theta(x; a)$ be a square function of radius $1 \gg a > 0$

$$\Theta(x;a) \equiv \begin{cases} 1, & |x| \le a \,, \\ 0, & |x| > a \,. \end{cases}$$

and define

$$Q_{\pm}(x;a) \equiv \sum_{z \in \mathbb{Z}} \Theta(x+z;a) \pm \Theta(x+z+\frac{1}{2};a).$$
(B.2)

We choose $V(x) = -5Q_+(x; \frac{1}{20}), W(x) = -5Q_-(x; \frac{1}{20}), \kappa = \kappa_{pw}$ where

$$\kappa_{\rm pw}(y) \equiv \begin{cases} 0, & 0 \le y < y_0, \\ \frac{1}{2}, & y_0 \le y < 2y_0, \\ 1, & y \ge 2y_0, \\ -\kappa_{\rm pw}(-y), & y < 0, \end{cases} \qquad y_0 \equiv \tanh^{-1}(\frac{1}{2}) \approx 0.549. \quad (B.3)$$

Hence, the initial value problem (3.12) becomes

$$i\partial_t \psi(x,t) = \left[-\partial_x^2 - 5Q_+(x;\frac{1}{20}) - \frac{5}{2}\kappa_{\rm pw}(\frac{1}{2}x)Q_-(x;\frac{1}{20}) + i\beta\cos(\frac{1}{2}\omega t)\partial_x \right]\psi.$$
(B.4)

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The potential $U_{\varepsilon}^{(3)}$ is similar to $U_{\varepsilon}^{(2)}$, and the only change is that we now set $\kappa(X) = \operatorname{sgn}(X + 1/2)$, the signum function, where the shift is introduced so as to avoid changes in the middle of one of the potential wells.

$$i\partial_t \psi(x,t) = \left[-\partial_x^2 - 5Q_+(x;\frac{1}{20}) - \frac{5}{2} \operatorname{sgn}(x+\frac{1}{4})Q_-(x;\frac{1}{20}) + i\beta \cos(\frac{1}{2}\omega t)\partial_x \right] \psi.$$
(B.5)

The corresponding Dirac Hamiltonians are denoted by $D_0^{(\ell)}$, where for $\ell = 1$

(5.1) with
$$v_{\rm D} = 2\pi$$
, $\vartheta_{\sharp} = \frac{1}{2}$, $\kappa(X) = \tanh(X)$, (B.6)

for $\ell = 2$

(5.1) with $v_{\rm D} \approx 6.45$, $\vartheta_{\sharp} \approx 1.03$, $\kappa(X) = \kappa_{\rm pw}(X)$, see (B.3). (B.7) and for $\ell = 3$

(5.1) with $v_{\rm D} \approx 6.45$, $\vartheta_{\sharp} \approx 1.03$, $\kappa(X) = {\rm sgn}(X)$, see (B.3). (B.8)

APPENDIX C. NUMERICAL METHODS

To solve the initial value problems (3.12) and (5.1), we use three-points stencil finite-difference discretization of the Hamiltonians, and the Crank-Nicholson method for time integration [20]. As for boundary conditions, we observe that for a computational domain [-L, L] with $L \gg 1$ sufficiently large, both periodic and vanishing boundary conditions yield seemingly identical results. The defect modes ψ_{\star} and α_{\star} were both computed using direct diagonlization of the corresponding discretized Hamiltonians.

To derive the Dirac equation (5.1) from the Schrödinger equation (3.12), we compute the parameters $v_{\rm D}$ and ϑ_{\sharp} in the following way: first, we construct a second-order finite difference matrix approximation of $H_{0,k}$, the Hamiltonian with k quasi-periodic boundary conditions, see (2.2). The Bloch modes Φ_1 and Φ_2 are then the lower-most eigenvectors with $k = k_{\rm D} = \pi$. Second, the inner-products in (3.2) and (A.7b) are computed using a trapezoid quadrature rule.

Finally, whenever curve fitting is required, we use the SciPy curve_fit nonlinear least squares framework.

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