# THE POLYHEDRAL GEOMETRY OF PIVOT RULES AND MONOTONE PATHS 

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#### Abstract

Motivated by the analysis of the performance of the simplex method we study the behavior of families of pivot rules of linear programs. We introduce normalized-weight pivot rules which are fundamental for the following reasons: First, they are memory-less, in the sense that the pivots are governed by local information encoded by an arborescence. Second, many of the most used pivot rules belong to that class, and we show this subclass is critical for understanding the complexity of all pivot rules. Finally, normalized-weight pivot rules can be parametrized in a natural continuous manner.

We show the existence of two polytopes, the pivot rule polytopes and the neighbotopes, that capture the behavior of normalized-weight pivot rules on polytopes and linear programs. We explain their face structure in terms of multi-arborescences. We compute upper bounds on the number of coherent arborescences, that is, vertices of our polytopes.

Beyond optimization, our constructions provide new perspectives on classical geometric combinatorics. We introduce a normalized-weight pivot rule, we call the max-slope pivot rule which generalizes the shadow-vertex pivot rule. The corresponding pivot rule polytopes and neighbotopes refine monotone path polytopes of Billera-Sturmfels. Moreover special cases of our polytopes yield permutahedra, associahedra, and multiplihedra. For the greatest improvement pivot rules we draw connections to sweep polytopes and polymatroids.


## 1. Introduction

For $A \in \mathbb{R}^{n \times d}, b \in \mathbb{R}^{n}, c \in \mathbb{R}^{d}$ we consider the linear program (LP)

$$
\begin{array}{ll}
\max & c^{t} x \\
\text { s.t. } & A x \leq b
\end{array}
$$

The simplex method is one of the most popular algorithms for solving linear programs (see [11, 17, 33]). The key ingredient, which is decisive for the running time on a given instance, is the choice of a pivot rule. Since the inception of the simplex algorithm, many different pivot rules have been proposed and analyzed. Starting with Klee and Minty in 1972 [27] many of the popular pivot rules have been shown to require an exponential number of steps; see [4, 6, 21, 25, 40, 41, 43] and references there. To this day, no pivot rule is known to take only polynomially many steps on every LP. In this paper we study the behavior of parametric families of pivot rules and uncover a rich polyhedral structure. We define polytopes whose geometry capture the behavior of pivot rules on given LPs. This provides a new perspective on the study of the performance of the simplex method.
Our constructions are also of interest to the (geometric) combinatorics community. A generic linear function $c$ induces an acyclic orientation on the graph of the polytope $P=\left\{x \in \mathbb{R}^{d}: A x \leq b\right\}$. The collection of $c$-monotone paths has a natural topological structure that is studied under the name Baues poset. In the seminal paper [12], Billera, Kapranov, and Sturmfels showed that the Baues poset has the homotopy type of a sphere and is represented by the boundary of the monotone path polytope from [13]. The vertices of monotone path polytopes are in bijection to special monotone paths, called coherent. Later, many important combinatorial constructions and polytopes arose as monotone path polytopes. By replacing $c$-monotone paths by $c$-monotone arborescences, our constructions provide a generalization of the theory of monotone path polytopes and many prominent combinatorial polytopes.


Figure 1. A footprint and three arborescences on the unit cube. The second arborescence from the left does not come from a NW-rule.

Formalizing the notion of a pivot rule is complicated; for example, several authors showed that pivot rules can be used to encode problems that are hard in the sense of complexity theory $[2,18,20]$. We will not try to give a precise definition of what constitutes a pivot rule because our taxonomy of pivot rules relies only on a polyhedral geometry perspective. Throughout, we will refer to $(P, c)$ as the linear program. Geometrically the simplex method finds a $c$-monotone path in the graph of $P$ from any initial vertex $v$ of $P$ to the optimal vertex $v_{o p t}$. The algorithm proceeds along directed edges. At any non-sink $v$, the pivot rule chooses a neighboring vertex $u$ of $v$ with $c^{t} u>c^{t} v$.

Definition 1.1. The footprint of a pivot rule $R$ on an LP $(P, c)$ is the directed acyclic subgraph obtained as the union of all $c$-monotone paths produced by $R$ for every starting vertex. The pivot rule $R$ is a memory-less pivot rule if its footprint for every LP is an arborescence, i.e., a directed tree with root at the optimal vertex $v_{o p t}$.

Figure 1 shows a footprint and three such arborescences on a 3 -cube. Equivalently, a pivot rule is memory-less if it chooses the neighbor of $v \neq v_{\text {opt }}$ using only local information provided by the set of neighbors $\mathrm{Nb}_{P}(v)$ of $v$. Many rules that are used in practice, including greatest improvement and steepest edge, are memory-less (c.f. Section 2). Pivot rules not in this class include Zadeh's least-entered facet rule as well as the original shadow vertex rule.
For a given LP $(P, c)$ a memory-less pivot rule is represented by its arborescence, that is, at every vertex, the choice made by that pivot rule is encoded in the outgoing arc of the arborescence. In particular, every memory-less pivot rule corresponds to a choice of an arborescence for every LP $(P, c)$. From this perspective, for every pivot-rule, there is a memory-less pivot rule, given by the shortest-path arborescence of the footprint, which takes at most the same number of steps. In consequence, if every memory-less pivot rule takes exponentially many steps, then so does every pivot rule.

The two main questions that we address in this paper are
(A) How do the arborescences vary for fixed objective function $c$ and varying pivot rule?
(B) How do the arborescences vary for fixed pivot rule and varying objective function $c$ ?

To be able to change the pivot rules in a controlled and continuous manner, we restrict to the following setup: For given $P \subset \mathbb{R}^{d}$ and $c \in \mathbb{R}^{d}$, choose a normalization $\eta: \mathbb{R}^{d} \rightarrow \mathbb{R}$ and a weight $w \in \mathbb{R}^{d}$. For $v \neq v_{o p t}$, the next vertex on the simplex-path from $v$ to $v_{o p t}$ is

$$
\begin{equation*}
u_{*}=\operatorname{argmax}\left\{\frac{w^{t}(u-v)}{\eta(u-v)}: u \text { adjacent to } v \text { and } c^{t} u>c^{t} v\right\} \tag{1}
\end{equation*}
$$

A choice of $w$ and $\eta$ for given $(P, c)$ is called a normalized-weight pivot rule, or NW-rule for short. If $R$ is a normalized-weight pivot rule, we sometimes write $\eta^{R}(P, c)$ and $w^{R}(P, c)$ to stress the dependence of $\eta$ and $w$ on the LP $(P, c)$. NW-rules are memory-less pivot rules: for a fixed LP $(P, c)$ Equation (1) determines an arborescence $\mathcal{A}$, that is, a map on the vertices of $P$ with $\mathcal{A}\left(v_{\text {opt }}\right)=v_{\text {opt }}$ and $\mathcal{A}(v)=u_{*}$ otherwise.
As we explain in the next section, several well-known pivot rules (greatest-improvement, steepestedge, etc.) as well as the max-slope pivot rule, a memory-less generalization of the shadow-vertex rule (see below), belong to that class. While NW-rules are a strict subclass of memory-less pivot rules, we show that they are universal in the following sense.

Theorem 1.2. For every simple polytope $P$ there is a perturbation $P^{\prime}$, combinatorially isomorphic to $P$, such that for any memory-less pivot rule there is a NW-rule that produces the same arborescence for $\left(P^{\prime}, c\right)$ for every $c$.

If $(P, c)$ is a non-degenerate LP, then $\left(P^{\prime}, c\right)$ has the same optimal basis and we may assume that $(P, c)$ is already sufficiently generic. Hence, NW-rules are essentially all we need to study to understand memory-less rules. Furthermore, by our earlier argument, memory-less rules are essentially all we need to study to understand all pivot rules. We may put these observations together in the following Corollary.
Corollary 1.3. If there is a pivot rule for which the simplex method takes polynomially many steps on every $L P$, then there is an $N W$-rule that takes takes polynomially many steps on every $L P$.

We can continuously change the pivot rule by varying the weight $w$. We call an arborescence that arises via (1) for a fixed weight $w$ a coherent arborescence and write $\mathcal{A}=\mathcal{A}_{P, c}^{\eta}(w)$. This terminology underlines the proximity to the theory of coherent monotone paths [12, 13] (see below).
An answer to question (A) is provided by the following theorem. For a polytope $Q \subseteq \mathbb{R}^{d}$ and $w \in \mathbb{R}^{d}$, we write $Q^{w}=\left\{x \in Q: w^{t} x \geq w^{t} y, y \in Q\right\}$ to denote the face that maximizes $x \mapsto w^{t} x$.
Theorem 1.4. Let $(P, c)$ be a linear program and $\eta$ a normalization. There is a polytope $\Pi_{P, c}^{\eta} \subset \mathbb{R}^{d}$, called the pivot rule polytope of $(P, c)$ and $\eta$, such that the following holds: For any generic weights $w, w^{\prime}$

$$
\mathcal{A}_{P, c}^{\eta}(w)=\mathcal{A}_{P, c}^{\eta}\left(w^{\prime}\right) \quad \Longleftrightarrow \quad\left(\Pi_{P, c}^{\eta}\right)^{w}=\left(\Pi_{P, c}^{\eta}\right)^{w^{\prime}} .
$$

Question (B) is strongly related to parametric linear programming. Whereas a basic question there is roughly which objective functions yield the same optimum, we will address the more subtle question which objective functions yield the same arborescence. We make two assumptions on the NW-rule $R$, namely that $\eta^{R}(P, c)$ is independent of $c$ and that $w^{R}(P, c)=c$. Thus, for a fixed normalization function $\eta$, we will write $\mathcal{B}_{P}^{\eta}(c):=\mathcal{A}_{P, c}^{\eta}(c)$ for the arborescence of $(P, c)$ obtained from (1) with respect to $\eta$ and weight $w=c$. We show that the collection of arborescences $\mathcal{B}_{P}^{\eta}(c)$ is governed by another polytope.
Theorem 1.5. Let $P \subset \mathbb{R}^{d}$ be a polytope and $\eta$ a normalization. There is a polytope $\Gamma_{P}^{\eta} \subset \mathbb{R}^{d}$, called the neighbotope of $P$ and $\eta$, such that the following holds: For any generic objective functions $c, c^{\prime} \in \mathbb{R}^{d}$

$$
\mathcal{B}_{P}^{\eta}(c)=\mathcal{B}_{P}^{\eta}\left(c^{\prime}\right) \quad \Longleftrightarrow \quad\left(\Gamma_{P}^{\eta}\right)^{c}=\left(\Gamma_{P}^{\eta}\right)^{c^{\prime}} .
$$

We prove Theorems 1.4 and 1.5 in Section 3.
We describe the face structure of pivot rule polytopes and neighbotopes in terms of multiarborescences and discuss the relation to general arborescences of LPs that were studied and enumerated by Athanasiadis et al. in [5]. In particular, we give bounds on the number of coherent arborescences; see Section 5.

As a memory-less version of the shadow-vertex pivot rule we introduce the max-slope (MS) pivot rule: For given LP $(P, c)$ choose $\eta^{\mathrm{MS}}(u-v)=c^{t}(u-v)$ and $w^{\mathrm{MS}} \in \mathbb{R}^{d}$ generic and linearly independent of $c$. Thus the resulting arborescence $\mathcal{A}$ satisfies

$$
\begin{equation*}
\mathcal{A}(v)=\operatorname{argmax}\left\{\frac{w^{t}(u-v)}{c^{t}(u-v)}: u \text { adjacent to } v \text { and } c^{t} u>c^{t} v\right\}, \tag{2}
\end{equation*}
$$

for $v \neq v_{o p t}$.
Let $r=P^{w}$ be the vertex selected by $w$, the unique path in the arborescence $\mathcal{A}$ above, starting at $r$, is precisely the path followed by the shadow-vertex pivot rule (see Proposition 6.1). Let
$v_{- \text {opt }}$ be the vertex of $P$ minimizing $c$. The unique path in the arborescence $\mathcal{A}$ starting at $v_{- \text {opt }}$ passes through $r$. It is the coherent monotone path of $(P, c)$ with respect to $w$ in the sense of [13]. For varying $w$ the resulting coherent monotone paths are parametrized by the vertices of the monotone path polytope $\Sigma_{c}(P)$. Obviously the arborescence contains more information than just the monotone path from $v_{-o p t}$. This refinement can be seen geometrically in terms of Minkowski sums (see Section 6.1).
Theorem 1.6. Let $P \subset \mathbb{R}^{d}$ be a polytope and $c$ a generic objective function. Then the monotone path polytope $\Sigma_{c}(P)$ is a weak Minkowski summand of the max-slope pivot rule polytope $\Pi_{P, c}^{\mathrm{MS}}$. If $P$ is a zonotope, then $\Sigma_{c}(P)$ is normally equivalent to $\Pi_{P, c}^{\mathrm{MS}}$.

Interestingly the construction of pivot rule polytopes is fundamentally different from that of monotone path polytopes in [13]. In particular, the result gives a new way of studying monotone path polytopes of zonotopes. In Section 4 we highlight that Stasheff's associahedra and multiplihedra can be realized as max-slope pivot rule polytopes.

The pivot rule polytopes for the greatest improvement pivot rule relate to yet another important construction from geometric combinatorics going back to classical work of Goodman and Pollack; see [22] and references therein. The sweep polytope $\operatorname{SP}\left(p_{1}, \ldots, p_{n}\right)$, introduced by Padrol and Philippe in [30], captures the orderings of a point configuration $p_{1}, \ldots, p_{n}$ induced by varying linear functions. For a polytope $P \subset \mathbb{R}^{d}$ and a normalization $\eta$, define the set of normalized edge directions $\mathrm{ED}^{\eta}(P):=\left\{\frac{u-v}{\eta(u-v)}: u v \in E(P)\right\}$. If $c$ is a generic objective function, then let $\mathrm{ED}^{\eta}(P, c):=\left\{\frac{u-v}{\eta(u-v)}: u v \in E(P), c^{t} u>c^{t} v\right\}$ the collection of normalized $c$-improving edge directions.
Theorem 1.7. Let $(P, c)$ be a linear program and $\eta$ a normalization. Then the pivot rule polytope $\Pi_{P, c}^{\eta}$ is a weak Minkowski summand of the sweep polytope of normalized c-improving edge directions $\operatorname{SP}\left(\mathrm{ED}^{\eta}(P, c)\right)$.
Furthermore, the neighbotope $\Gamma_{P}^{\eta}$ is a weak Minkowski summand of the sweep polytope of normalized edge directions $\operatorname{SP}\left(\mathrm{ED}^{\eta}(P)\right)$.

We show that in a particularly interesting case the neighbotope and the sweep polytope of edge directions are normally equivalent. Let us write $\mathrm{ED}(P)=\mathrm{ED}^{1}(P)$ for the unnormalized edge directions. If $\Phi \subset \mathbb{R}^{n}$ is an irreducible crystallographic root system, then we associate to it the Coxeter zonotope $Z_{\Phi}=\frac{1}{2} \sum_{\alpha \in \Phi}[-\alpha, \alpha]$. It is easy to see that $\operatorname{ED}\left(Z_{\Phi}\right)=\Phi$.
Theorem 1.8. Let $\Phi$ be an irreducible crystallographic root system with Coxeter zonotope $Z_{\Phi}$. Then the greatest-improvement neighbotope $\Gamma_{Z_{\Phi}}^{\mathrm{GI}}$ is normally equivalent to $\operatorname{SP}\left(\mathrm{ED}\left(Z_{\Phi}\right)\right)=\operatorname{SP}(\Phi)$.

The proof relies on a result (Theorem 6.11) on irreducible crystallographic root systems that is of independent interest: for every pair $\alpha, \beta$ of elements that are incomparable in the root poset of $\Phi$ there is a simple system $\Delta \subseteq \Phi$ whose only positive roots are $\alpha$ and $\beta$.
We give several of examples of pivot rule polytopes and neighbotopes in Section 4 but defer a detailed discussion to the forthcoming paper [15].
From Equation (1) one can see that arborescences for NW-rules in general are obtained by local greedy choices. For the greatest improvement pivot rule we show in Section 7 that its arborescences can be derived from a basic combinatorial optimization problem, we named the Max Potential Energy Branching. This problem has the structure of a polymatroid and can be solved by the greedy algorithm. We explain the associated polytope in detail, which also justifies the name "neighbotope".
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## 2. LPs, PIVOT RULES, AND ARBORESCENCES

Let $P \subseteq \mathbb{R}^{d}$ be a fixed polytope. We will denote by $V(P)$ the vertex set of $P$ and by $G(P)=(V(P), E(P))$ the graph of $P$. A linear function $c \in \mathbb{R}^{d}$ is (edge) generic if $c^{t} u \neq c^{t} v$ for all edges $u v \in E(P)$. Every generic linear function $c$ induces an acyclic orientation on $G(P)$ by orienting $v \rightarrow u$ if $c^{t} u>c^{t} v$. The directed graph has a unique sink $v_{o p t}$ and, in fact, every subgraph of a face will have a unique sink. Such an orientation is called a unique sink orientation and we call $(P, c)$ a linear program. For a vertex $v \in V(P)$, we write $\operatorname{Nb}_{P}(v):=\{u$ : $u v \in E(P)\}$ for the neighbors of $v$ in $G(P)$ and we write $\operatorname{Nb}_{P, c}(v):=\left\{u \in \operatorname{Nb}_{P}(v): c^{t} u>c^{t} v\right\}$ for the $c$-improving neighbors.
A $c$-arborescence of $P$ is a map $\mathcal{A}: V(P) \rightarrow V(P)$ satisfying $\mathcal{A}(v)=v$ if and only if $v=v_{\text {opt }}$ and $\mathcal{A}(v) \in \mathrm{Nb}_{P, c}(v)$ for all $v \in V(P) \backslash v_{o p t}$. For a memory-less pivot rule, the choice of the neighboring vertex $u_{*} \in \mathrm{Nb}_{P, c}(v)$ for $v \neq v_{o p t}$ results in a $c$-arborescence $\mathcal{A}$, which captures the behavior of the pivot rule on the linear program $(P, c)$. Arborescences of polytopes have appeared as oracles that allow geometric enumeration output-sensitive algorithms [7].

For a given normalization $\eta$ and weight $w,(1)$ determines an arborescence $\mathcal{A}=\mathcal{A}_{P, c}^{\eta}(w)$ given by

$$
\begin{equation*}
\mathcal{A}(v):=\operatorname{argmax}\left\{\frac{w^{t}(u-v)}{\eta(u-v)}: u \in \mathrm{Nb}_{P, c}(v)\right\} \tag{3}
\end{equation*}
$$

for $v \neq v_{o p t}$ and $\mathcal{A}\left(v_{o p t}\right):=v_{o p t}$.
The following well-known and important pivot rules belong to the class of NW-rules (this requires the assumption that $P$ is a simple polyhedron):

Greatest improvement (GI): choose $w^{\mathrm{GI}}=c$ and $\eta^{G I}(u-v) \equiv 1$; $p$-Steepest edge $(\mathbf{p S E})$ : choose $w^{\mathrm{pSE}}=c$ and $\eta^{\mathrm{pSE}}(u-v)=\|u-v\|_{p}$ for some fixed $p \geq 1$; Max-slope (MS): choose $\eta^{\mathrm{MS}}(u-v)=c^{t}(u-v)$ and $w^{\mathrm{MS}}$ linearly independent of $c$.

The max-slope rule is a memory-less version of the shadow-vertex rule, that we will treat in depth in Section 6.1. Figure 2 shows the six arborescences of the tetrahedron including the five arborescences obtained from the max-slope rule.


Figure 2. The arborescences of the tetrahedron.

It turns out that all $3!=6$ arborescences of the tetrahedron can be obtained from a NW-rule for a suitable choice of a normalization. However, this is not true in general.
Indeed, the arborescence in middle of Figure 1 cannot be obtained from an NW-rule. Observe that any NW-rule makes a choice based only on the set of edge directions

$$
D_{P, c}(v):=\left\{u-v: u \in \mathrm{Nb}_{P, c}(v)\right\} .
$$

Hence, if two vertices $v, v^{\prime} \in V(P)$ satisfy $D_{P, c}\left(v^{\prime}\right) \subseteq D_{P, c}(v)$ and $\mathcal{A}(v)-v \in D_{P, c}\left(v^{\prime}\right)$, then $\mathcal{A}(v)-v=\mathcal{A}\left(v^{\prime}\right)-v^{\prime}$. So, the choice of the improving neighbor for $v$ forces the improving neighbor for $v^{\prime}$ to be the same. The middle arborescence of Figure 1 violates this constraint.
We call a polytope $P$ edge-generic if $u-v \neq u^{\prime}-v^{\prime}$ for any two distinct edges $u v, u^{\prime} v^{\prime} \in E(P)$.

Proposition 2.1. Let $P$ be an edge-generic polytope and $c$ a generic objective function. For any c-arborescence $\mathcal{A}$ there is a normalization $\eta$ and a weight $w$ such that $\mathcal{A}=\mathcal{A}_{P, c}^{\eta}(w)$.

Proof. It follows from edge-genericity that $D_{P, c}(v) \cap D_{P, c}\left(v^{\prime}\right)=\varnothing$ for all $v \neq v^{\prime}$. Define the normalization $\eta: \mathbb{R}^{d} \rightarrow \mathbb{R}$ by $\eta(\mathcal{A}(v)-v):=1$ for all $v \in V(P)$ and $\eta(x):=\kappa$ for $x \notin\{\mathcal{A}(v)-v$ : $v \in V(P)\}$ and some sufficiently large constant $\kappa \gg 0$. For $w:=c$, we then get for all $v \neq v_{\text {opt }}$

$$
\mathcal{A}(v)=\operatorname{argmax}\left\{\frac{c^{t}(u-v)}{\eta(u-v)}: u \in \mathrm{Nb}_{P, c}(v)\right\}
$$

and hence $\mathcal{A}=\mathcal{A}_{P, c}^{\eta}(w)$.

Proof of Theorem 1.2. Let $P$ be a simple polytope given by $P=\{x: A x \leq b\}$ for some matrix $A$ and vector $b$. Simplicity implies that for every $A^{\prime}$ there is an $\varepsilon>0$ such that $P^{\prime}:=\{x$ : $\left.\left(A+\varepsilon A^{\prime}\right) x \leq b\right\}$ is combinatorially isomorphic to $P$. It is straightforward to verify that for a sufficiently general $A^{\prime}$, the polytope $P^{\prime}$ is edge-generic. Let $\mathcal{A}_{1}, \ldots, \mathcal{A}_{s}$ be the $c$-arborescences produced by the given memory-less pivot rule and let $c_{1}, \ldots, c_{s}$ objective functions such that $\mathcal{A}_{i}$ was produced for $\left(P^{\prime}, c_{i}\right)$. Let $\eta_{i}$ be the normalization of Proposition 2.1 and let $R$ be the NW-rule with $w^{R}\left(P^{\prime}, c_{i}\right)=c_{i}$ and $\eta^{R}\left(P^{\prime}, c_{i}\right)=\eta_{i}$. It follows from Proposition 2.1 that $\mathcal{A}_{P^{\prime}, c_{i}}^{R}=\mathcal{A}_{P^{\prime}, c_{i}}^{\eta_{i}}(c)=\mathcal{A}_{i}$, which proves the claim.

## 3. Two constructions: Proof of Existence Theorems 1.4 and 1.5

We prove the existence of the two polytopes parametrizing NW pivot rules. They correspond to Theorems 1.4 and 1.5.
3.1. Pivot rule polytopes. Let $(P, c)$ be a fixed linear program and $\eta$ a normalization. In this section we prove Theorem 1.4, which provides a complete answer to question (A) from the introduction:

How does the arborescence of a memory-less pivot rule change when the weight $w$ changes?
For an arborescence $\mathcal{A}$ of $G(P)$ we define

$$
\begin{equation*}
\psi^{\eta}(\mathcal{A}):=\sum_{v} \frac{\mathcal{A}(v)-v}{\eta(\mathcal{A}(v)-v)} \tag{4}
\end{equation*}
$$

where we tacitly declare $\frac{0}{\eta(0)}=0$. The pivot polytope of $(P, c)$ and a fixed normalization $\eta$ is defined as

$$
\begin{equation*}
\Pi_{P, c}^{\eta}:=\operatorname{conv}\left\{\psi^{\eta}(\mathcal{A}): \mathcal{A} c \text {-arborescence of }(P, c)\right\} \tag{5}
\end{equation*}
$$

We remind the reader that for $w$, the arborescence of $(P, c)$ determined by (1) is denoted by $\mathcal{A}_{P, c}^{\eta}(w)$. We can now prove Theorem 1.4. Recall that for polytopes $P_{1}=\operatorname{conv}\left(V_{1}\right)$ and $P_{2}=\operatorname{conv}\left(V_{2}\right)$, the Minkowski sum is the polytope

$$
P_{1}+P_{2}=\left\{p_{1}+p_{2}: p_{1} \in P_{1}, p_{2} \in P_{2}\right\}=\operatorname{conv}\left(v_{1}+v_{2}: v_{1} \in V_{1}, v_{2} \in V_{2}\right)
$$

Proof of Theorem 1.4. For a vertex $v \neq v_{\text {opt }}$ define

$$
\begin{equation*}
\Pi_{P, c}^{\eta}(v):=\operatorname{conv}\left\{\frac{u-v}{\eta(u-v)}: u \in \mathrm{Nb}_{P, c}(v)\right\} \tag{6}
\end{equation*}
$$

It follows from the definition of Minkowski sums that

$$
\begin{equation*}
\Pi_{P, c}^{\eta}=\sum_{v \neq v_{o p t}} \Pi_{P, c}^{\eta}(v) \tag{7}
\end{equation*}
$$

For a generic weight $w \in \mathbb{R}^{d}$ note that

$$
\left(\Pi_{P, c}^{\eta}\right)^{w}=\sum_{v \neq v_{o p t}}\left(\Pi_{P, c}^{\eta}(v)\right)^{w}
$$

Hence $\left(\Pi_{P, c}^{\eta}\right)^{w}$ is a vertex if and only if $\left(\Pi_{P, c}^{\eta}(v)\right)^{w}$ is a vertex for all $v \neq v_{\text {opt }}$. Now, $\frac{u_{*}-v}{\eta\left(u_{*}-v\right)}$ is this vertex if and only if $\frac{w^{t}\left(u_{*}-v\right)}{\eta\left(u_{*}-v\right)}>\frac{w^{t}(u-v)}{\eta(u-v)}$ for all $u \in \operatorname{Nb}_{P, c}(v) \backslash u_{*}$. Set $\mathcal{A}(v):=u_{*}$ and $\mathcal{A}\left(v_{\text {opt }}\right):=v_{\text {opt }}$. It now follows from (1) that $\psi^{\eta}(\mathcal{A})=\left(\Pi_{P, c}^{\eta}\right)^{w}$ if and only if $\mathcal{A}=\mathcal{A}_{P, c}^{\eta}(w)$, which proves the claim.
3.2. Neighbotopes. Let $P \subset \mathbb{R}^{d}$ be a polytope and $c$ a generic objective function that induces a unique sink orientation on the graph $G(P)$ with optimum $v_{\text {opt }}$. The question of basic parametric linear programming is for which objective functions $c^{\prime}$ will $v_{\text {opt }}$ be the sink. Geometrically, this is given by the interior of the normal cone $\mathcal{N}_{P}\left(v_{\text {opt }}\right)=\left\{y: y^{t} v_{\text {opt }}>y^{t} u\right.$ for all $\left.u \in \operatorname{Nb}_{P}\left(v_{\text {opt }}\right)\right\}$. The collection $\mathcal{N}_{P}=\left\{\mathcal{N}_{P}(v): v \in V(P)\right\}$ gives rise to the normal fan of $P$, whose cones give a conical subdivision of $\mathbb{R}^{d}$.
A more refined question is which $c^{\prime}$ yield the same unique sink orientation as $c$. Obviously $c^{\prime}$ has to satisfy $\left(c^{\prime}\right)^{t} u>\left(c^{\prime}\right)^{t} v$ for all edges $u v \in E(P)$ such that $c^{t} u>c^{t} v$, which defines the interior of a polyhedral cone. The collection of these cones for varying $c$ again yield a fan structure, that is the normal fan of a polytope. To be precise, we define the edge zonotope (or EZ-tope)

$$
\mathcal{E}(P):=\sum_{u v \in E(P)}[u-v, v-u]
$$

and it is straightforward to verify that the vertices of $\mathcal{E}(P)$ are in bijection to unique sink orientations induced by objective functions. The EZ-tope was introduced by Gritzmann-Sturmfels [23] under the name edgotope.
We address our second primary question of the paper:
Given a fixed $P$ and NW-rule, how does the arborescence change when $c$ is varied?
We will answer it but make the following assumption on the NW-rule $R$ :
i) The normalization function does not depend on $c: \eta^{R}(P, c)=\eta^{R}\left(P, c^{\prime}\right)$ for all $c, c^{\prime}$;
ii) The rule $R$ chooses $c$ as the weight: $w^{R}(P, c)=c$.

For example, greatest-improvement as well as $p$-steepest-edge belong to this class but max-slope with normalization $\eta^{\mathrm{MS}}(u-v)=c^{t}(u-v)$ does not. To stress the two requirements above, we write $\mathcal{B}_{P}^{\eta}(c):=\mathcal{A}_{P, c}^{\eta}(c)$ for the arborescence obtained from the linear program $(P, c)$ with respect to the NW-rule with normalization $\eta$ and weight $w=c$. If $\mathcal{B}=\mathcal{B}_{P}^{\eta}(c)$, we note that for all $v \in V(P)$

$$
\begin{equation*}
\mathcal{B}(v)=\operatorname{argmax}\left\{\frac{c^{t}(u-v)}{\eta(u-v)}: u \in N(P, v) \cup\{v\}\right\} . \tag{8}
\end{equation*}
$$

Indeed, let us denote by $v_{\text {opt }}$ the unique sink of $P$ with respect to $c$. For $v \neq v_{o p t}$ there is a neighbor $u \in \mathrm{Nb}_{P}(v)$ with $c^{t} u>c^{t} v$ and the right-hand side of (8) coincidences with (1). If $v=v_{\text {opt }}$, then the maximum is attained at $u=v$ and we get $\mathcal{B}(v)=v$.
Let $G(P)$ be the undirected graph of $P$. An arborescence on $G(P)$ is a map $\mathcal{B}: V(P) \rightarrow V(P)$ such that
(a) there is a unique $v_{\text {opt }} \in V(P)$ with $\mathcal{B}\left(v_{\text {opt }}\right)=v_{\text {opt }}$,
(b) for all $v \neq v_{\text {opt }}$ we have $\mathcal{B}(v) \in \operatorname{Nb}_{P}(v)$, and
(c) for all $v$ there is $k \geq 1$ such that $\mathcal{B}^{k}(v)=v_{\text {opt }}$.

In particular, every $c$-arborescence of $(P, c)$ is an arborescence; cf. Section 2.

Consistently, we define for an arborescence $\mathcal{B}$

$$
\psi^{\eta}(\mathcal{B}):=\sum_{v} \frac{\mathcal{B}(v)-v}{\eta(\mathcal{B}(v)-v)},
$$

and we define the neighbotope of $P$ for the normalization $\eta$

$$
\begin{equation*}
\Gamma_{P}^{\eta}:=\operatorname{conv}\left\{\psi_{P}^{\eta}(\mathcal{B}): \mathcal{B} \text { arborescence of } G(P)\right\} . \tag{9}
\end{equation*}
$$

Let us emphasize that the neighbotope is defined in terms of all arborescences of the undirected graph $G(P)$.

Proof of Theorem 1.5. The proof is along similar lines as that of Theorem 1.4. For a vertex $v \in V(P)$ we define

$$
\begin{equation*}
\Gamma_{P}^{\eta}(v):=\operatorname{conv}\left\{\frac{u-v}{\eta(u-v)}: u \in \operatorname{Nb}_{P}(v) \cup\{v\}\right\} . \tag{10}
\end{equation*}
$$

Let $c$ be a generic objective function and let $\mathcal{B}=\mathcal{B}_{P}^{\eta}(c)$. For $v \in V(P)$ it follows directly from (8) that

$$
\Gamma_{P}^{\eta}(v)^{c}=\frac{u-v}{\eta(u-v)} \quad \text { if and only if } \quad \mathcal{B}(v)=u
$$

Hence the coherent arborescences are precisely the vertices of

$$
Q:=\sum_{v \in V(P)} \Gamma_{P}^{\eta}(v),
$$

which is the convex hull over all $\psi^{\eta}(f)$ where $f$ ranges over all maps $f: V(P) \rightarrow V(P)$ with $f(v) \in \mathrm{Nb}_{P}(v)$ for all $v \in V(P)$. However, the above argument shows that we can discard those $f$ that are not arborescences of $G(P)$ and hence $Q=\Gamma_{P}^{\eta}$ as claimed.

The structural similarity between pivot polytopes and neighbotopes can be made more precise.
Corollary 3.1. Let $P \subset \mathbb{R}^{d}$ be a polytope, and let $\eta$ be a normalization. Then the neighbotope $\Gamma_{P}^{\eta}$ is given by

$$
\Gamma_{P}^{\eta}=\operatorname{conv}\left(\bigcup_{c \in \mathbb{R}^{n}} \Pi_{P, c}^{\eta}\right) .
$$

So far we have presented two constructions, which help classify and organize all pivot rules of a linear program. We will now present some examples to illustrate the construction and, at the same time, highlights the incredibly rich combinatorics that the constructions bring to light.

## 4. Examples of pivot rule polytopes and neighbotopes

Let us begin with three examples that illustrate the richness of pivot rule polytopes:
Example 4.1 (GI- and pSE-Pivot polytopes of simplices). Let $\Delta_{d-1}=\operatorname{conv}\left(e_{1}, \ldots, e_{d}\right) \subset \mathbb{R}^{d}$ be the standard $d$-simplex. An objective function $c$ is generic for $\Delta_{d-1}$ if and only if $c_{i} \neq c_{j}$ for all $i \neq j$. Up to symmetry, we may assume that $c_{1}<c_{2}<\cdots<c_{d}$. Observe that $\eta^{\mathrm{PSE}}\left(e_{i}-e_{j}\right)=\left\|e_{i}-e_{j}\right\|_{p}=2^{1 / p}$ for all $i \neq j \in[n]$, which implies that the pivot rule polytopes for the greatest-improvement and $p$-steepest-edge normalizations are the same up to scaling. Thus, it suffices to focus on the greatest-improvement normalization $\eta^{\mathrm{GI}} \equiv 1$.
An arborescence of $\left(\Delta_{d-1}, c\right)$ can be identified with a map $\mathcal{A}:[d] \rightarrow[d]$ with $\mathcal{A}(d)=d$ and $\mathcal{A}(j)>j$ for all $j<d$. There are precisely $(d-1)$ ! arborescences, since there are $d-j$ independent choices of an outgoing edge for each $j$. However, not all of these arborescences will necessarily arise from GI-rules. To characterize those that do, choose $w \in \mathbb{R}^{d}$ such that $w_{i} \neq w_{j}$ for all $i \neq j$. We can associate to $w$ the permutation $\sigma$ such that $w_{\sigma^{-1}(1)}<w_{\sigma^{-1}(2)}<\cdots<w_{\sigma^{-1}(d)}$. This permutation uniquely identifies the arborescence in the sense that $w^{\prime}$ yields the same coherent arborescence as $w$ if and only if $w_{\sigma^{-1}(1)}^{\prime}<w_{\sigma^{-1}(2)}^{\prime}<\cdots<w_{\sigma^{-1}(d)}^{\prime}$. A left-to-right maximum
of $\sigma=\left(\sigma_{1}, \sigma_{2}, \ldots, \sigma_{d}\right)$ is an index $j$ such that $\sigma_{i}<\sigma_{j}$ for all $i<j$. Let $1 \leq j_{1}<j_{2}<\cdots<j_{k} \leq d$ be the positions of left-to-right maxima. It follows from (1) that the coherent arborescence with respect to $w$ is given by $\mathcal{A}(i)=j_{s}$ if $j_{s-1} \leq i<j_{s}$, where we set $j_{0}=0$. Although all possible subsets of $[d]$ can occur as positions of left-to-right maxima, the position 1 is never relevant. Therefore, there are exactly $2^{d-2}$ coherent arborescences.
For an arborescence $\mathcal{A}$, let $\delta_{i}:=\left|\mathcal{A}^{-1}(i)\right|$, so that $\delta(\mathcal{A})=\left(\delta_{1}, \ldots, \delta_{d}\right)$ is the in-degree sequence of $\mathcal{A}$. It now follows from (4) that $\psi^{\mathrm{GI}}(\mathcal{A})=\delta-\mathbf{1}_{[d-1]}$ and hence

$$
\Pi_{\Delta_{d-1}, c}^{\mathrm{GI}}+\mathbf{1}_{[d-1]}=\operatorname{conv}\left\{\delta(\mathcal{A}): \mathcal{A} \text { arborescence of }\left(\Delta_{d-1}, c\right)\right\}
$$

From (7), we infer that

$$
\Pi_{\Delta_{d-1}, c}^{\mathrm{GI}}+\mathbf{1}_{[d-1]}=\sum_{i=1}^{d-1} \operatorname{conv}\left\{e_{i+1}, \ldots, e_{d}\right\} .
$$

Following the exposition [31, Sect. 8.5], this shows that $\Pi_{\Delta_{d-1}, c}^{G I}$ is the Pitman-Stanley polytope [38].

Example 4.2 (Pivot rule polytopes of cubes). Let $C_{d}=[0,1]^{d}$ be the $d$-dimensional standard cube. Up to symmetry, we can assume that a generic objective function $c$ satisfies $0<c_{1}<$ $\cdots<c_{d}$. We can identify vertices of $C_{d}$ with characteristic vectors $\mathbf{1}_{J} \subseteq\{0,1\}^{d}$ for $J \subseteq[d]$. In particular, $\mathrm{Nb}_{C_{d}, c}\left(\mathbf{1}_{J}\right)=\left\{\mathbf{1}_{J \cup k}: k \notin J\right\}$ and for $u \in \mathrm{Nb}_{C_{d}, c}\left(\mathbf{1}_{J}\right)$, we have $u-\mathbf{1}_{J}=e_{k}$ for some $k \notin J$. This again shows that the pivot polytopes for greatest improvement and $p$-steepestedge are identical. For the max-slope normalization, it follows from (4) that $\Pi_{C_{d}, c}^{\mathrm{MS}}$ is linearly isomorphic to $\Pi_{C_{d}, c}^{\mathrm{GI}}$ with respect to the linear transformation $x \mapsto \operatorname{diag}\left(c_{1}, \ldots, c_{d}\right) x$. Thus, we only consider the pivot polytope for greatest improvement.
An arborescence can be identified with a map $\mathcal{A}: 2^{[d]} \rightarrow 2^{[d]}$ with $\mathcal{A}([d])=[d]$ and $\mathcal{A}(J)=J \cup\{i\}$ for some $i \in[d] \backslash J$. Since all choices are independent, the total number of arborescences is

$$
\prod_{J} 2^{d-|J|}=\prod_{i=0}^{d}\left(2^{i}\right)^{\binom{d}{i}}=2^{d \cdot 2^{d-1}} .
$$

Let $w \in \mathbb{R}^{d}$ be a generic weight. We can again assume that there is a unique permutation $\sigma$ such that $w_{\sigma^{-1}(1)}<w_{\sigma^{-1}(2)}<\cdots<w_{\sigma^{-1}(d)}$. The corresponding coherent arborescence $\mathcal{A}$ then satisfies that $\mathcal{A}(J)=J \cup k$ whenever $\sigma(k)>\sigma(h)$ for all $h \notin J \cup k$. To see that every such arborescence $\mathcal{A}$ determines a unique permutation $\sigma$, we set $\sigma(d):=\mathcal{A}(\varnothing)$ and $\sigma(k):=\mathcal{A}(\{\sigma(k+$ $1), \ldots, \sigma(d)\})$ for $1 \leq k<d$. This establishes a bijection between $d$-permutations $\sigma$ and coherent arborescences $\mathcal{A}_{\sigma}$ of $\left(C_{d}, c\right)$ for the greatest improvement normalization. If $\sigma=(1,2, \ldots, d)$, then $\psi^{\mathrm{GI}}\left(\mathcal{A}_{\sigma}\right)_{k}$ is the number of proper subsets $J \subseteq[d]$ such that $\max ([d] \backslash J)=k$. Thus $\psi^{\mathrm{GI}}\left(\mathcal{A}_{\sigma}\right)=$ $\left(1,2, \ldots, 2^{d-1}\right)$. For any other permutation $\sigma^{\prime}$ one observes that $\psi^{\mathrm{GI}}\left(\mathcal{A}_{\sigma^{\prime}}\right)=\sigma^{\prime}\left(\psi^{\mathrm{GI}}\left(\mathcal{A}_{\sigma}\right)\right)$. Hence

$$
\Pi_{C_{d}, c}^{\mathrm{GI}}=\operatorname{conv}\left\{\left(2^{\sigma(1)-1}, 2^{\sigma(2)-1}, \ldots, 2^{\sigma(d)-1}\right): \sigma d \text {-permutation }\right\}
$$

is a permutahedron; cf. [31]. The pivot polytope for $C_{3}$ together with the corresponding arborescences is depicted in Figure 3. We will see a stronger relation in Section 6.1.

Now we present another rich example
Example 4.3 (Max-slope pivot polytopes of simplices). Let $P$ be $(d-1)$-dimensional simplex and $c$ a generic linear function. We briefly sketch the pivot polytope of $(P, c)$ associated to the max-slope normalization $\eta^{\mathrm{MS}}(u-v)=c^{t}(u-v)$ and defer the reader to [15] for details. Let $v_{1}, \ldots, v_{d}$ be the vertices of $P$ labeled such that $c^{t} v_{i}<c^{t} v_{j}$ if and only if $i<j$. As in Example 4.1, we identify an arborescence with a map $\mathcal{A}:[d] \rightarrow[d]$ with $\mathcal{A}(d)=d$ and $\mathcal{A}(i)>i$ for $i<d$. We call such an arborescence non-crossing if there are no $i<j$ with $j<\mathcal{A}(i)<\mathcal{A}(j)$. We show in [15] that an arborescence is coherent if and only if it is non-crossing.


Figure 3. The pivot polytope of $[0,1]^{3}$ with arborescences associated to the vertices.
It is easy to see that non-crossing arborescences are in bijection to binary trees on $d-1$ nodes. If $d=1$, then there is a unique arborescence that we map to the empty binary tree. For $d>1$, let $i$ be minimal with $\mathcal{A}(i)=d$. Let $\mathcal{A}_{L}:[i] \rightarrow[i]$ be defined by $\mathcal{A}_{L}(a)=\mathcal{A}(a)$ for $a<i$ and $\mathcal{A}_{L}(i)=i$. Further, define $\mathcal{A}_{R}:[d-i] \rightarrow[d-i]$ by $\mathcal{A}_{R}(a):=\mathcal{A}(i+a)-i$. Then $\mathcal{A}_{L}$ and $\mathcal{A}_{R}$ are non-crossing arborescences on fewer nodes that yield the left and right subtrees of the binary tree associated to $\mathcal{A}$. Binary trees can be equipped with a natural partial order and the resulting poset is called the associahedron.
It is a famous result due to Milnor (unpublished), Haiman (unpublished), and Lee [29] that the associahedron is isomorphic to the face lattice of a convex polytope. We further show in [15] that $\Pi_{P, c}^{\mathrm{MS}}$ is combinatorially isomorphic to the associahedron.
Example 4.4 (Max-slope pivot polytopes of prisms over simplices). The associahedron was originally introduced as the poset of partial bracketings of a product of $n$ elements in a nonassociative multiplicative structure. Stasheff's multiplihedron $\mathcal{J}_{n}$ extends this to the following setup; see [39]. Let $f: \mathbf{A} \rightarrow \mathbf{B}$ be a morphism between two non-associative multiplicative structures. For elements $a_{1}, a_{2}, \ldots, a_{n} \in \mathbf{A}$. What the multiplihedron roughly encodes is the possible ways of (partially) evaluating $f\left(a_{1} a_{2} \cdots a_{n}\right)$. Figure 4 gives an example for $n=3$. It turns out that the multiplihedron is combinatorially isomorphic to the max-slope pivot rule polytope for the prism over the simplex. More precisely, if $P=\Delta_{n-1} \times \Delta_{1}$ and $c=\left(c_{1}<\right.$ $c_{2}<\cdots<c_{n-1}<c_{n}$ ) is any linear function, then $\Pi_{P, c}^{\mathrm{MS}}$ is combinatorially isomorphic to the multiplihedron $\mathcal{J}_{n}$. The relation between of max-slope arborescences of products of simplices and non-associative structures will be the main subject of [15].
Example 4.5 (Greatest-improvement neighbotope of the cube). Let $C_{d}=[0,1]^{d}$ be the unit cube. As in Example 4.2, we observe that the neighbotope for $\eta$ will be homothetic to $\Gamma_{C_{d}}^{\mathrm{GI}}$ if the normalization satisfies $\eta\left( \pm e_{i}\right)=$ const for all $i$.
To get the number of arborescences of the $d$-cube, we observe that every arborescence $\mathcal{B}$ is given by a spanning tree of $G\left(C_{d}\right)$ together with the choice of a root $v_{\text {opt }} \in V\left(C_{d}\right)$. The arborescence is then obtained by directing edges of the spanning tree towards the root. The number of spanning trees $\tau\left(C_{d}\right)$ of $C_{d}$ can be found in [36, Example 5.6.10] and gives the number of arborescences

$$
2^{n} \cdot \tau\left(C_{d}\right)=\prod_{k=1}^{d}(2 k)^{\binom{d}{k}}=2^{2^{d}-1} \prod_{k=1}^{d} k^{\binom{d}{k}} .
$$

For a vertex $\mathbf{1}_{J} \in\{0,1\}^{d}$, we have

$$
\Gamma_{C_{d}}^{\mathrm{GI}}\left(\mathbf{1}_{J}\right)=\operatorname{conv}\left(\left\{-e_{i}: i \in J\right\} \cup\{0\} \cup\left\{e_{i}: i \notin J\right\}\right) .
$$

Let $S=(\mathbb{Z} / 2 \mathbb{Z})^{d} \cong\{-1,+1\}^{d}$ be the group of sign flips. Every element is of the form $s=\mathbf{1}-2 \mathbf{1}_{J}$ for some $J \subseteq[d]$ and $\Gamma_{C_{d}}^{\mathrm{GI}}\left(\mathbf{1}_{J}\right)=s \cdot \Gamma_{C_{d}}^{\mathrm{GI}}\left(\mathbf{1}_{\varnothing}\right)$. Thus

$$
\Gamma_{C_{d}}^{\mathrm{GI}}=\sum_{s \in S} s \cdot \Gamma_{C_{d}}^{\mathrm{GI}}\left(\mathbf{1}_{\varnothing}\right)
$$



Figure 4. The 2-dimensional multiplihedron $\mathcal{J}_{3}$ and the corresponding maxslope arborescences for $\Delta_{2} \times \Delta_{1}$

Let $W$ be the reflection group of type $B_{d}$, which acts on $\mathbb{R}^{d}$ by signed permutations. Since $\Gamma_{C_{d}}^{\mathrm{GI}}\left(\mathbf{1}_{\varnothing}\right)$ is invariant under permutations, $\Gamma_{C_{d}}^{\mathrm{GI}}$ is invariant with respect to $W$. Thus, for a given objective function $c$, we may assume that $0<c_{1}<\cdots<c_{d}$ and it follows from Corollary 3.1 that

$$
\left(\Gamma_{C_{d}}^{\mathrm{GII}}\right)^{c}=\left(\Pi_{P, c}^{\mathrm{GI}}\right)^{c}=\left(1,2, \ldots, 2^{d-1}\right) .
$$

This shows that $\Gamma_{C_{d}}^{\mathrm{GI}}$ is the type-B permutahedron with respect to the point $\left(1,2, \ldots, 2^{d-1}\right)$, which has $d!2^{d}$ vertices.

Example 4.6 (Neighbotopes of cross-polytopes). The $d$-dimensional cross-polytope is the non-simple polytope $C_{d}^{*}=\operatorname{conv}\left\{ \pm e_{i}: i=1, \ldots, d\right\}$. For $v=s e_{i}$ with $s \in\{-1,+1\}$

$$
\Gamma_{C_{d}^{*}}^{\mathrm{GII}}(v)=\operatorname{conv}\left(\left\{ \pm e_{j}: j \neq i\right\} \cup\left\{s e_{i}\right\}\right),
$$

which is a pyramid over $C_{d-1}^{*}$. The cross-polytope is also invariant under the group $W$ of signed permutations. Hence, we may again assume that $0<c_{1}<\cdots<c_{d}$ and the corresponding arborescence $\mathcal{B}=\mathcal{B}_{C_{d}^{*}}^{\eta}(c)$ satisfies $\mathcal{B}(v)=e_{d}$ if $v \neq-e_{d}$ and $\mathcal{B}\left(-e_{d}\right)=e_{d-1}$. It follows that $\Gamma_{C_{d}^{*}}^{\mathrm{GI}}$ is the type- $B$ permutahedron for the point $(2 d-1) e_{d}+e_{d-1}$ and has $4 d(d-1)$ vertices.

## 5. The Combinatorics of Pivot rule polytopes and Neighbotopes

We investigate the basic combinatorial questions on polyhedra for our constructions and the relation to fiber polytopes and sweep polytopes.
5.1. Faces of pivot rule polytopes. Before we discuss general faces of pivot rule polytopes, we look at vertices and their numbers.

We recall that a $d$-dimensional polytope $P$ is simple if every vertex is incident to $d$ edges. For a simple $d$-polytope $P \subset \mathbb{R}^{d}$ and generic objective function $c$, denote by $h_{i}$ the number of vertices with in-degree $i$. Since $P$ is simple, $h_{i}$ is independent of $c$ and $h(P)=\left(h_{0}, \ldots, h_{d}\right)$ is the $h$-vector of $P$; cf. [9, Ch. 6].

Proposition 5.1 ([5, Prop. 3.1]). For a simple d-polytope $P$ and a generic objective function, the total number of arborescences is given in terms of the entries of the $h$-vector by

$$
1^{h_{1}} 2^{h_{2}} \cdots d^{h_{d}}
$$

We now show that this upper bound cannot be attained for coherent arborescences, independent of the normalization.
Theorem 5.2. Let $P \subset \mathbb{R}^{d}$ be a simple d-polytope with $n>d+1 \geq 4$ vertices and $h$-vector $h(P)=\left(h_{0}, \ldots, h_{d}\right)$. For fixed objective function $c$ and arbitrary normalization $\eta$, the number of coherent arborescences is strictly less than

$$
1^{h_{1}} 2^{h_{2}} \cdots d^{h_{d}}-2(n-m-2),
$$

where $m$ is the number of facets of $P$.
Proof. We need to bound the number of vertices of $\Pi_{P, c}^{\eta}$. Recall from (7) that $\Pi_{P, c}^{\eta}$ is a Minkowski sum of polytopes $\Pi_{P, c}^{\eta}(v)$ for $v \neq v_{\text {opt }}$. Since $P$ is simple, the polytopes $\Pi_{P, c}^{\eta}(v)$ are all simplices of various dimensions. Using the interpretation of the $h$-vector given above, we see that the number of $(k-1)$-simplices is precisely $h_{k}$. In particular, we have $h_{0}=1$ and $h_{1}=m-d$ vertices with in-degree 1 , where $m$ is the number of facets. Since $P$ is not a simplex, we have $n \geq(d+1)(d-2)+m(d-1)$ by the Lower Bound Theorem (cf. [42]). Thus $\Pi_{P, c}^{\eta}$ is a Minkowski sum of $N:=n-(m-d+1) \geq(d+2)(d-2)+1+m(d-2) \geq d+2$ simplices of positive dimension. Let $v_{1}, \ldots, v_{N}$ be the corresponding vertices and set $\Pi_{i}:=\Pi_{P, c}^{\eta}\left(v_{i}\right)$.
We slightly extend the argument from [32, Sect. 6]: For $u \in V\left(\Pi_{i}\right)$, let $\mathcal{N}_{\Pi_{i}}(u)$ be set of linear functions $w$ such that $\{u\}=\Pi_{i}^{w}$. This is a non-empty open polyhedral cone. For $u_{i} \in V\left(\Pi_{i}\right)$, we have that $\sum_{i} u_{i}$ corresponds to a vertex of $\Pi_{P, c}^{\eta}$ if and only if $\bigcap_{i} \mathcal{N}_{\Pi_{i}}\left(u_{i}\right) \neq \varnothing$. Fix $u_{i} \in V\left(\Pi_{i}\right)$ for $i=d+2, \ldots, N$ and assume that for all choices of $u_{j} \in V\left(\Pi_{j}\right)$ for $j=1, \ldots, d+1, \sum_{i=1}^{N} u_{i}$ corresponds to a vertex. For $1 \leq j \leq d+1$, define $\mathcal{C}_{j}$ to be the collection of open convex sets

$$
\mathcal{N}_{\Pi_{j}}(u) \cap \mathcal{N}_{\Pi_{d+2}}\left(u_{d+2}\right) \cap \cdots \cap \mathcal{N}_{\Pi_{N}}\left(u_{N}\right)
$$

for $u \in V\left(\Pi_{j}\right)$. The sets in $\mathcal{C}_{j}$ are pairwise disjoint. Since $C_{1} \cap \cdots \cap C_{d+1} \neq \varnothing$ for all choices $C_{j} \in \mathcal{C}_{j}, j=1, \ldots, d+1$, Lovász' colorful Helly Theorem (cf. [8]) implies that $\bigcap_{C \in \mathcal{C}_{j}} C \neq \varnothing$ for some $j$, which yields a contradiction. There are at least $2(N-(d+1))$ choices of vertices $u_{d+2}, \ldots, u_{N}$, which finishes the proof.

A precise but more involved bound can be obtained from the Minkowski Upper Bound Theorem [1]. If $P$ is a $d$-dimensional simplex, then $h(P)=(1, \ldots, 1)$. Neither Theorem 5.2 nor the Minkowski Upper Bound Theorem rules out the possibility, that $P$ has $1^{h_{1}} 2^{h_{2}} \cdots d^{h_{d}}=d$ ! many coherent arborescences.
Question 5.3. For every $d \geq 1$, is there a normalization $\eta$ such that all arborescences of $\left(\Delta_{d}, c\right)$ are coherent?

In the rest of the section we discuss the facial structure of the pivot polytope $\Pi_{P, c}^{\eta}$ for the LP $(P, c)$ and a normalization $\eta$. For this we assume that $P$ is a simple polytope. A $c$-multi-arborescence is a map $\mathcal{A}: V(P) \rightarrow 2^{V(P)} \backslash\{\varnothing\}$ that satisfies $\mathcal{A}\left(v_{\text {opt }}\right)=\left\{v_{\text {opt }}\right\}$ and $\mathcal{A}(v) \subseteq \mathrm{Nb}_{P, c}(v)$ for all $v \neq v_{\text {opt }}$. We will abuse notation and identify one-element subsets of $V(P)$ with the elements themselves. Hence we can view $c$-arborescences as $c$-multi-arborescences with $|\mathcal{A}(v)|=1$ for all $v \in V(P)$. If $w \in \mathbb{R}^{d}$ is a non-generic weight, then the maximum in (1) may not be uniquely attained for all $v$ and gives rise to coherent $c$-multi-arborescence that we will also denote by $\mathcal{A}_{P, c}^{\eta}(w)$.
Given two multi-arborescences $\mathcal{A}, \mathcal{A}^{\prime}$, we say that $\mathcal{A}$ refines $\mathcal{A}^{\prime}$, written $\mathcal{A} \preceq \mathcal{A}^{\prime}$, if $\mathcal{A}(v) \subseteq \mathcal{A}^{\prime}(v)$ for all $v \in V(P)$. This is a partial order on the collection of multi-arborescences of $(P, c)$. The proof of Theorem 1.4 yields the facial structure of pivot polytopes.
Theorem 5.4. Let $P \subset \mathbb{R}^{d}$ be a simple polytope, $c$ a generic objective function, and $\eta$ a normalization. For two weights $w, w^{\prime}$ we have

$$
\mathcal{A}(w) \preceq \mathcal{A}\left(w^{\prime}\right) \quad \Longleftrightarrow \quad\left(\Pi_{P, c}^{\eta}\right)^{w} \subseteq\left(\Pi_{P, c}^{\eta}\right)^{w^{\prime}} .
$$

Thus, the poset of coherent arborescences is isomorphic to the face lattice of $\Pi_{P, c}^{\eta}$.

Two arborescences $\mathcal{A}, \mathcal{A}^{\prime}$ differ by an edge rerouting if there is a unique vertex $v \in V(P)$ with $\mathcal{A}(v) \neq \mathcal{A}^{\prime}(v)$. As a consequence, we get a necessary condition for the adjacency of two coherent arborescences.
Corollary 5.5. If the vertices of $\Pi_{P, c}^{\eta}$ corresponding to two coherent arborescences $\mathcal{A}, \mathcal{A}^{\prime}$ are adjacent, then $\mathcal{A}, \mathcal{A}^{\prime}$ differ by an edge rerouting.

Note that the definition of $\Pi_{P, c}^{\eta}$ given in (5) involves all arborescences. If $\mathcal{A}$ is a non-coherent arborescence, the geometry of $\Pi_{P, c}^{\eta}$ gives us to the finest coherent coarsening of $\mathcal{A}$.

Proposition 5.6. Let $\mathcal{A}$ be a c-arborescence of $(P, c)$ and let $F \subseteq \Pi_{P, c}^{\eta}$ be the unique face containing $\psi^{\eta}(\mathcal{A})$ in its relative interior. Then the c-multi-arborescence $\mathcal{A}^{\prime}$ associated to $F$ is the finest coherent coarsening of $\mathcal{A}$.

Proof. Let $w$ be a weight such that $F=\left(\Pi_{P, c}^{\eta}\right)^{w}$. For every $v \neq v_{o p t}$, let $F_{v}=\left(\Pi_{P, c}^{\eta}(v)\right)^{w}$. Then $F=\sum_{v} F_{v}$ and, in particular $\frac{\mathcal{A}(v)-v}{\eta(\mathcal{A}(v)-v)} \in F_{v}$ for all $v \neq v_{\text {opt }}$. The multi-arborescence associated to $F$ is given by

$$
\mathcal{A}^{\prime}(v)=\left\{u \in N_{P}(c, v): \frac{\mathcal{A}(v)-v}{\eta(\mathcal{A}(v)-v)} \in F_{v}\right\}
$$

and hence is a coarsening of $\mathcal{A}$. If $\mathcal{A}^{\prime \prime}$ is another coherent multi-arborescence coarsening $\mathcal{A}$ with corresponding face $G$, then $\psi^{\eta}(\mathcal{A}) \in G$ and hence $\psi^{\eta}(\mathcal{A}) \in G \cap F$. But since $\psi^{\eta}(\mathcal{A})$ is contained in the relative interior of $F$, it follows that $F \cap G=F$, which shows that $\mathcal{A}^{\prime}$ is the finest coherent coarsening of $\mathcal{A}$.

Let us close by remarking that the poset of all $c$-arborescences on $(P, c)$ can also be realized as the face poset of a convex polytope.
Proposition 5.7. Let $P$ a simple polytope with $h$-vector $h(P)=\left(h_{0}, \ldots, h_{d}\right)$. For a generic linear function $c$, the poset of all c-multiarborescences is isomorphic to the face poset of the polytope

$$
\Delta_{0}^{h_{1}} \times \Delta_{1}^{h_{2}} \times \cdots \times \Delta_{d}^{h_{d}} .
$$

Proof. Recall the face structure of a product of simplices $\prod_{i=1}^{n} \Delta_{d_{i}}$ for $d_{i} \in \mathbb{N}$ is given by a choice of subset $S_{i}$ from each $\left[d_{i}\right]$. Then two collections of subsets $\left\{S_{i}\right\}$ and $\left\{T_{i}\right\}$ correspond to faces that contain one another precisely when $S_{i} \subseteq T_{i}$ for all $i \in[n]$.
Let $v_{\text {opt }}$ be the unique maximizer of $c$ over $P$. A $c$-multi-arborescence is given by the choice of a non-empty subset $S_{v}$ of $c$-improving neighbors for every vertex $v \neq v_{o p t}$. This choice is made independently, so any possible collection of subsets corresponds to some $c$-multi-arborescence. The collection of all multi-arborescences then corresponds to all choices of sets of outgoing neighbors.
For a generic orientation on a simple polytope, the $h$-vector counts the number of vertices with a given out-degree. Hence, the set of all possible choices of subsets of outgoing edges is given by

$$
\Delta_{0}^{h_{1}} \times \Delta_{1}^{h_{2}} \times \cdots \times \Delta_{d}^{h_{d}}
$$

The space of all monotone paths yields a similar cell-complex called the Baues poset of cellular strings. In general, that complex is not polytopal. Proposition 5.7 shows that the analogous poset for arborescences is instead always the face lattice of product of simplices. Furthermore, the choice of simplices is independent of $c$ so long as $c$ is generic, since the h -vector is invariant.
More generally, pivot polytopes and their lattices of coherent multi-arborescences behave in analogy to fiber polytopes and their lattices of coherent subdivisions. This analogy is particularly strong in the case of monotone path polytopes and secondary polytopes. To start, given a multi-arborescence or subdivision, evaluating whether it is coherent in its respective context corresponds to solving a linear feasibility problem. For adjacency in the monotone path polytope or secondary polytope, adjacent vertices must satisfy that their corresponding coherent monotone
paths or coherent triangulations differ by a flip. However, differing by a flip is not sufficient to guarantee adjacency. The edge reroutings thus play the role of flips for pivot polytopes. Furthermore, given any monotone path or triangulation, we can assign it a canonical point the unique face containing the point in its relative interior corresponds to the finest coherent coarsening of the monotone path or triangulation respectively. This is precisely the statement of Proposition 5.6 for pivot rule polytopes.
5.2. Faces of neighbotopes. As in the previous section we start by understanding the vertices of neighbotopes. We can again use the description as a Minkowski sum to derive an upper bound on the number of coherent arborescences.

Proposition 5.8. Let $P$ be a simple d-polytope with $n$ vertices, then the number of coherent arborescences of $P$ is at most $d^{n}\left(1-\frac{1}{d^{d+1}}\right)$.

Proof. If $P$ is simple, then $\Gamma_{P}^{\eta}(v)$ is a $(d-1)$-simplex for all $v \in V(P)$. The same argument as in the proof of Theorem 1.5 applies and shows that of the $d^{n}$ possible vertices of the Minkowski sum at least $d^{n-(d+1)}$ fail to be vertices.

Note that this bound is in general far from being tight: If $P$ is a $d$-simplex, then $\Gamma_{P}^{\eta}(v)=-v+P$ for all vertices $v$ and $\Gamma_{P}^{\eta}$ is homothetic to $P$; see also Proposition 6.7. Of course, the Minkowski sum decomposition is also valid for non-simple polytopes and a more involved upper bound maybe derived in the same manner.

We now consider the face lattice of neighbotopes. A multi-arborescence of a polytope $P$ is a map $\mathcal{B}: V(P) \rightarrow 2^{V(P)}$ that satisfies
(a) for all $v$ we have $\mathcal{B}(v) \subseteq \operatorname{Nb}_{P}(v) \cup\{v\}$;
(b) there is a unique face $F_{\mathcal{B}} \subseteq P$ with $\{v: v \in \mathcal{B}(v)\}=V\left(F_{\mathcal{B}}\right)$ and
(c) $\mathcal{B}(v)=\mathrm{Nb}_{P}(v) \cup\{v\}$ for all $v \in V\left(F_{\mathcal{B}}\right)$;
(d) for all $v \in V$ there is $k \geq 1$ with $\mathcal{B}^{k}(v)=V\left(F_{\mathcal{B}}\right)$.

Proposition 5.9. Let $P$ be a polytope and $\eta$ a normalization. Every face of the neighbotope $\Gamma_{P}^{\eta}$ can be identified with a unique multi-arborescence.

Proof. Let $F=\left(\Gamma_{P}^{\eta}\right)^{c}$ be a face of the neighbotope $\Gamma_{P}^{\eta}$. It follows from the proof of Theorem 1.5 and (10) that $F=\sum_{v} F_{v}$, where $F_{v}=\Gamma_{P}^{\eta}(v)^{c}$. We define a multi-arborescence $\mathcal{B}$ by $u \in \mathcal{B}(v)$ if and only if $\frac{u-v}{\eta(u-v)} \in F_{v}$ for all $v \in V(P)$. Unless $v \in F_{\mathcal{B}}:=P^{c}$, there is always an improving neighbor $u \in \mathcal{B}(v)$ and $v \notin \mathcal{B}(v)$. Otherwise, $\mathcal{B}(v)=\operatorname{Nb}_{P}(v) \cup\{v\}$. This shows that $\mathcal{B}$ satisfies all defining properties of a multi-arborescence. Since $F$ can be recovered from $\mathcal{B}$, it also shows that every face corresponds to a unique multi-arborescence.

This injection furthermore encodes a partial order. Namely, we say that $\mathcal{B} \preceq \mathcal{B}^{\prime}$ for two multiarborescences $\mathcal{B}$ and $\mathcal{B}^{\prime}$, when $\mathcal{B}(v) \subseteq \mathcal{B}^{\prime}(v)$ for all $v \in V$. This partial order corresponds to the partial order of the face lattice. Namely, for faces $F_{1}, F_{2}$ of $\Gamma_{P}^{\eta}$ with corresponding multiarborescences $\mathcal{B}_{1}$ and $\mathcal{B}_{2}$, we have that $F_{1}$ is a face of $F_{2}$ if and only if $\mathcal{B}_{1} \preceq \mathcal{B}_{2}$.

## 6. Monotone path polytopes and sweep polytopes

Now we make connections to two famous constructions in geometric combinatorics.
6.1. Max-slope pivot rule polytopes and monotone path polytopes. The shadow-vertex rule is a well-known and well analyzed pivot rule that in its usual form does not belong to the class of memory-less rules. We show that the max-slope rule is a natural generalization that has the benefit of being a NW-rule. We also show that it is intimately related to the theory of (coherent) cellular strings on polytopes of Billera-Kapranov-Sturmfels [12].

For the setup, let $P \subset \mathbb{R}^{d}$ be a $d$-polytope and $c$ a generic objective function. Let $r$ be a vertex of $P$ together with a weight $w \in \mathbb{R}^{d}$ such that $r=P^{w}$. We seek to find the optimal vertex $v_{o p t}=P^{c}$. Define a linear projection $\pi: \mathbb{R}^{d} \rightarrow \mathbb{R}^{2}$ by $\pi(x):=\left(c^{t} x, w^{t} x\right)$. By construction $\pi(r)$ and $\pi\left(v_{o p t}\right)$ are vertices of the projection $\pi(P)$. There is a unique path from $\pi(r)$ to $\pi\left(v_{o p t}\right)$ that is increasing with respect to $c$. Since $c$ and $w$ where assumed to be generic, the pre-image of that path is a $c$-increasing vertex-edge path on $P$ from $r$ to $v_{o p t}$. This is called a shadow-vertex path from $r$ to $v_{\text {opt }}$.
Note that the path is not determined by $r$ but rather by the choice of $w$. Given $r$, $w$, the shadow path can be found with a variant of the simplex algorithm but it is clear that the procedure outlined above does not yield pivots for vertices outside the shadow-vertex path. In fact, it does not even yield a memory-less pivot rule in the sense of Section 1 for the vertices on the shadow-vertex path as they would require a choice of a weight that might lead to a different shadow-vertex path.
The vertices on the shadow-path from $r$ to $v_{\text {opt }}$ can be characterized locally. Starting from $r$, a $c$-improving neighbor $s \in \mathrm{Nb}_{P, c}(r)$ will be the next vertex on the shadow path if $[\pi(r), \pi(s)]$ is an edge of $\pi(P)$. This happens if the slope of the edge in the plane is maximal among all edges of $\pi(P)$ incident to $\pi(r)$. That is, if $\frac{w^{t}(s-r)}{c^{t}(s-r)}>\frac{w^{t}(u-r)}{c^{t}(u-r)}$ for all $u \in \mathrm{Nb}_{P, c}(r) \backslash s$. The max-slope rule now extends this condition to all vertices: if $v \neq v_{o p t}$, then max-slope chooses the neighbor

$$
\begin{equation*}
u_{*}=\operatorname{argmax}\left\{\frac{w^{t}(u-v)}{c^{t}(u-v)}: u \in \mathrm{Nb}_{P, c}(v)\right\} \tag{11}
\end{equation*}
$$

Our discussion above now proves the following.
Proposition 6.1. For $(P, c)$, let $w$ be a generic weight and $r=P^{w}$. Let $\mathcal{A}^{\mathrm{MS}}=\mathcal{A}_{P, c}^{\mathrm{MS}}(w)$ be the max-slope arborescence of $(P, c)$ with respect to $w$. The path $\left(r_{i}\right)_{i \geq 0} \in V(P)$ with $r_{0}:=r$ and $r_{i+1}:=\mathcal{A}^{\mathrm{MS}}\left(r_{i}\right)$ for $i \geq 0$ is precisely the shadow-vertex path of $(P, c)$ with respect to $w$.

Let $v_{-o p t}=P^{-c}$ be the minimizer with respect to $c$. A cellular string on $(P, c)$ is a sequence of faces $F_{0}, F_{1}, \ldots, F_{k}$ with the property that $v_{-o p t}=F_{0}^{-c}, v_{o p t}=F_{k}^{c}$ and $F_{i-1}^{c}=F_{i}^{-c}$ for all $i=1, \ldots, k$. A cellular string can be refined by replacing some of the $F_{i}$ by cellular strings of $\left(F_{i}, c\right)$. This yields a partial order on cellular strings, called the Baues poset of $(P, c)$. The minimal elements are the monotone paths from $v_{-o p t}$ to $v_{o p t}$.
Billera-Sturmfels [13] and Billera-Kapranov-Sturmfels [12] developed the theory of coherent cellular strings on polytopes. A monotone path $v_{-o p t}=v_{0}, v_{1}, \ldots, v_{k}=v_{o p t}$ is called coherent if there is a $w \in \mathbb{R}^{d}$ such that $v_{i}$ is the unique maximizer of $w$ over the slice $P \cap\left\{x: c^{t} x=c^{t} v_{i}\right\}$ for all $i$. For any monotone path $W=v_{0}, \ldots, v_{k}$, define the point

$$
\Phi_{W}:=\sum_{i=1}^{k} \frac{c^{t}\left(v_{i}-v_{i-1}\right)}{c^{t}\left(v_{k}-v_{0}\right)}\left(v_{i}-v_{i-1}\right)
$$

and with it the monotone path polytope

$$
\Sigma_{c}(P):=\operatorname{conv}\left\{\Phi_{W}: W \text { monotone path of }(P, c)\right\}
$$

The vertices of $\Sigma_{c}(P)$ are precisely the coherent monotone paths and, stronger even, the face lattice of $\Sigma_{c}(P)$ defines the subposet of the Baues poset of coherent cellular strings of $(P, c)$.
We will show next that the max-slope pivot polytope provides a refinement of the monotone path polytope in the following sense. A polytope $Q \subset \mathbb{R}^{d}$ is a weak Minkowski summand of a polytope $P \subset \mathbb{R}^{d}$ if there is $\lambda>0$ and a polytope $R \subset \mathbb{R}^{d}$ such that $\lambda P=Q+R$. This implies that there is a poset map from the face lattice of $P$ onto the face lattice of $Q$ with favorable combinatorial and topological properties. Roughly, this means that the combinatorics of $P$ refines the combinatorics of $Q$.

Proposition 6.2. Let $P \subset \mathbb{R}^{d}$ be a polytope and $c$ a generic objective function. Then the monotone path polytope $\Sigma_{c}(P)$ is a weak Minkowski summand of the max-slope pivot polytope $\Pi_{P, c}^{\mathrm{MS}}$.

Proof. We use a result of Shephard (cf. [24, Theorem 15.1.2]) that states that $Q$ is a weak Minkowski sum of $P$ if and only if $Q^{w}$ is a vertex whenever $P^{w}$ is a vertex.
Let $w \in \mathbb{R}^{d}$ be generic and $\mathcal{A}=\mathcal{A}_{P, c}^{\mathrm{MS}}(w)$ the max-slope arborescence of $(P, c)$ corresponding to the vertex $\left(\Pi_{P, c}^{\mathrm{MS}}\right)^{w}$. Let $r=P^{w}$. We can apply the same argument as before and obtain a shadow-vertex path from $r$ to $v_{- \text {opt }}$. Combining this path with the shadow-vertex path from $r$ to $v_{\text {opt }}$ yields the max-slope path from $v_{-o p t}$ to $v_{o p t}$. Verifying condition (11) along this path then shows that this max-slope path is precisely the coherent monotone path induced by $w$, which shows that $\Sigma_{c}(P)^{w}$ is a vertex.
Example 6.3 (Monotone path polytopes of simplices). Let $P$ be a $d$-simplex with vertices $v_{0}, v_{1}, \ldots, v_{d}$ ordered according to a generic objective function $c$. In [13, p. 545] it is shown that $\Sigma_{c}(P)$ is combinatorially isomorphic to a $(d-1)$-dimensional cube. Any choice $0=: i_{0}<i_{1}<$ $\cdots<i_{k-1}<i_{k}:=d$ defines a monotone path $v_{i_{0}}, v_{i_{1}}, \ldots, v_{i_{k-1}}, v_{i_{k}}$ and it is straightforward to show that every such path is coherent. Continuing Example 4.3, we see that choosing a noncrossing arborescence for every interval $\left[i_{j}, i_{j+1}\right]=\left\{i_{j}, i_{j}+1, \ldots, i_{j+1}\right\}$ yields a non-crossing arborescence of $(P, c)$ that contains the given monotone path. This shows that the set of refinements of a given monotone path to a non-crossing arborescence has the structure of a product of associahedra.

Note that we did not require $P$ to be simple in Proposition 6.2. When $P$ is simple, Proposition 6.2 yields a necessary criterion when a multi-arborescence is coherent. Let $\mathcal{A}$ be a multi-arborescence. For every $v \in V(P)$ there is a unique smallest face $F_{v} \subset P$ with $v \cup \mathcal{A}(v) \subseteq F$. We can associate to $\mathcal{A}$ a cellular string as follows: Set $u_{0}:=v_{- \text {opt }}$ and $F_{0}:=F_{u_{0}}$. For $i \geq 1$, let $u_{i}:=F_{i-1}^{c}$ and $F_{i}:=F_{u_{i}}$. We call the $c$-multi-arborescence $\mathcal{A}$ cellular if $\mathcal{A}(v) \subseteq V\left(F_{i}\right)$ for all $v \in V\left(F_{i}\right) \backslash u_{i+1}$ and all $i$. That is, if $\mathcal{A}$ restricts to a $c$-multi-arborescence of $\left(F_{i}, c\right)$ except for $u_{i+1}$. Note that every $c$-arborescence is cellular.
Corollary 6.4. If $\mathcal{A}$ is a coherent multi-arborescence, then $\mathcal{A}$ is cellular.
Proof. Let $\mathcal{A}$ be a coherent $c$-multi-arborescence with corresponding face $F=\left(\Pi_{P, c}^{\mathrm{MS}}\right)^{w}$ for some weight $w$. Consider the linear projection $\pi: \mathbb{R}^{d} \rightarrow \mathbb{R}^{2}$ given by $\pi(x)=\left(c^{t} x, w^{t} x\right)$. The polygon $\bar{P}$ has the two distinguished vertices $\bar{v}_{-o p t}=\pi\left(v_{-o p t}\right)$ and $\bar{v}_{\text {opt }}=\pi\left(v_{o p t}\right)$ that minimize and maximize the first coordinates, respectively. Let $\bar{v}_{-o p t}=\bar{v}_{0} \bar{v}_{1} \ldots \bar{v}_{k}=\bar{v}_{\text {opt }}$ be the upper path with respect to the second coordinate. It is straightforward to verify that $F_{i}=\pi^{-1}\left(\left[\bar{v}_{i}, \bar{v}_{i+1}\right]\right)$ yields the cellular string as constructed above. If $v \in V\left(F_{i}\right) \backslash u_{i+1}$ then $v^{\prime}=\mathcal{A}(v)$ if and only if the slope of the segment $\left[\bar{v}, \bar{v}^{\prime}\right]$ is maximal among all segments $\left[\bar{v}, \bar{v}^{\prime \prime}\right]$ for $\bar{v}^{\prime \prime} \in \operatorname{Nb}_{P, c}(v)$. Clearly this slope is at most that of $\left[\bar{v}_{i}, \bar{v}_{i+1}\right]$ and equal whenever $v^{\prime} \in V\left(F_{i}\right)$. This shows that $\mathcal{A}$ is cellular.

Note that there does not seem to be a natural map from general multi-arborescences of $(P, c)$ to the Baues poset.
In [13, Example 5.4] it is shown that the monotone path polytope of the $d$-cube $C_{d}=[0,1]^{d}$ with respect to $c=(1, \ldots, 1)$ is the permutahedron $\Pi_{d-1}=\operatorname{conv}\{(\sigma(1), \ldots, \sigma(d)): \sigma$ permutation $\}$. In fact for any generic $c, \Sigma_{c}(P)$ will be combinatorially isomorphic to $\Pi_{d-1}$. It is remarkable that the max-slope pivot polytope of the cube is also combinatorially isomorphic a permutahedron; cf. Example 4.2. This is not a coincidence. Recall that a polytope $P \subset \mathbb{R}^{d}$ is a zonotope if there are $t, z_{1}, \ldots, z_{n} \in \mathbb{R}^{d}$ such that

$$
-t+P=\left[-z_{1}, z_{1}\right]+\left[-z_{2}, z_{2}\right]+\cdots+\left[-z_{n}, z_{n}\right]
$$

Moreover, two polytopes $P, Q \subset \mathbb{R}^{d}$ are normally equivalent if $Q$ is a weak Minkowski summand of $P$ and $P$ is a weak Minkowski summand of $Q$.

Theorem 6.5. Let $P$ be a polytope and $c$ a generic objective function. If $P$ is a zonotope, then $\Sigma_{c}(P)$ and $\Pi_{P, c}^{\mathrm{MS}}$ are normally equivalent.

Proof. In light of Proposition 6.2, we only have to show that $\Pi_{P, c}^{\mathrm{MS}}$ is a weak Minkowski summand of $\Sigma_{c}(P)$. From (7), we see that it suffices to show that $\Pi_{P, c}^{\mathrm{MS}}(v)$ is a weak Minkowski summand of $\Sigma_{c}(P)$ for all $v \neq v_{o p t}$.
We may assume that $P=\left[-z_{1}, z_{1}\right]+\left[-z_{2}, z_{2}\right]+\cdots+\left[-z_{n}, z_{n}\right]$ and that $c^{t} z_{i}>0$ for all $i$. Since $P$ is a linear projection of $C_{d}=[0,1]^{d}$, it follows from Lemma 2.3 and Theorem 4.1 of [13] that the monotone path polytope $\Sigma_{c}(P)$ is normally equivalent to the zonotope

$$
\tilde{\Sigma}_{c}(P)=\sum_{i<j}\left[z_{i}-z_{j}, z_{j}-z_{i}\right]
$$

If $u, v$ are adjacent vertices of $P$, then $u-v= \pm z_{j}$ for some $j$. Thus for $v \neq v_{\text {opt }}$ there is $J \subseteq[n]$ such that

$$
\Pi_{P, c}^{\mathrm{MS}}(v)=\operatorname{conv}\left\{z_{j}: j \in J\right\}
$$

For a generic $w \in \mathbb{R}^{d}$ the vertex $\tilde{\Sigma}_{c}(P)^{w}$ is determined by the permutation $\sigma$ such that $w^{t} z_{\sigma(1)}>$ $w^{t} z_{\sigma(2)}>\cdots>w^{t} z_{\sigma(n)}$. However, this shows that $\left(\Pi_{P, c}^{\mathrm{MS}}(v)\right)^{w}=z_{\sigma(k)}$, where $k=\min \sigma^{-1}(J)$. Hence $\left(\Pi_{P, c}^{\mathrm{MS}}(v)\right)^{w}$ is a vertex whenever $\tilde{\Sigma}_{c}(P)^{w}$ is, which proves the claim.

Theorem 6.5 gives a new way of computing monotone path polytopes of zonotopes. In particular, it says that for every coherent monotone path there is a unique extension to a coherent arborescence.
A polytope $P$ is a belt polytope [16] if $P$ is normally equivalent to a zonotope. Equivalently, if the normal fan of $P$ is given by a hyperplane arrangement; cf. [42, Ch. 7]. The next result implies that Theorem 6.5 can actually be extended to belt polytopes.
Theorem 6.6. Let $P, P^{\prime} \subset \mathbb{R}^{d}$ be polytopes and $c$ a generic objective function. If $P$ is normally equivalent to $P^{\prime}$, then $\Pi_{P, c}^{\mathrm{MS}}=\Pi_{P^{\prime}, c}^{\mathrm{MS}}$.

Proof. Let $v \in V(P)$. If $v=P^{w}$, then $P^{w}=v^{\prime}$ is a vertex that is independent of $w$. This yields a bijection $V(P) \rightarrow V\left(P^{\prime}\right)$. Moreover, it follows from normal equivalence that if $u, v$ are adjacent vertices of $P$, then $u^{\prime}, v^{\prime}$ are adjacent in $P^{\prime}$ and $u-v=\lambda\left(u^{\prime}-v^{\prime}\right)$ for some $\lambda>0$. Thus $u \in \operatorname{Nb}_{P, c}(v)$ if and only if $u^{\prime} \in \operatorname{Nb}_{P^{\prime}, c}\left(v^{\prime}\right)$ and $\frac{u-v}{c^{t}(u-v)}=\frac{u^{\prime}-v^{\prime}}{c^{t}\left(u^{\prime}-v^{\prime}\right)}$. Now (6) yields $\Pi_{P, c}^{\mathrm{MS}}(v)=\Pi_{P^{\prime}, c}^{\mathrm{MS}}\left(v^{\prime}\right)$ and the claim follows from (7).
6.2. Neighbotopes and Sweep Polytopes. In this section, we relate the neighbotopes with respect to greatest-improvement pivot rule to another class of well-known polytopes, the sweep polytopes [30]. Let $p_{1}, \ldots, p_{n} \in \mathbb{R}^{d}$ be a configuration of points. A permutation $\sigma$ of $[n]$ is called a sweep if there is a generic linear function $c \in \mathbb{R}^{d}$ such that

$$
c^{t} p_{\sigma^{-1}(1)}<c^{t} p_{\sigma^{-1}(2)}<\cdots<c^{t} p_{\sigma^{-1}(n)}
$$

The sweep polytope, introduced by Padrol and Philippe in [30], captures the sweeps of a point configuration and is defined as

$$
\mathrm{SP}\left(p_{1}, \ldots, p_{n}\right):=\sum_{i<j}\left[p_{i}-p_{j}, p_{j}-p_{i}\right]
$$

If $p_{1}, \ldots, p_{n}$ are the vertices of a polytope $P$, then the sweep is related to line shellings of the dual to $P$. It was studied in [23] under the name shellotope.
Recall from the introduction that for a polytope $P \subset \mathbb{R}^{d}$ and a normalization $\eta$, the set of normalized edge directions is $\operatorname{ED}^{\eta}(P)=\left\{\frac{u-v}{\eta(u-v)}: u v \in E(P)\right\}$. If $c$ is a generic objective function, then $\mathrm{ED}^{\eta}(P, c)=\left\{\frac{u-v}{\eta(u-v)}: u v \in E(P), c^{t} u>c^{t} v\right\}$ is the set of $c$-improving edge
directions. If $\eta \equiv 1$, then we also write $\operatorname{ED}(P)=\operatorname{ED}^{\eta}(P)$. Note that $z \in \operatorname{ED}(P)$ if and only if $-z \in \operatorname{ED}(P)$.

Proof of Theorem 1.7. Let $v \in V(P)$ be a vertex. It follows from the definition that the vertices of $\Pi_{P, c}^{\eta}(v)$ are a subset of $\mathrm{ED}^{\eta}(P, c)$. Hence if we have a total order on $\mathrm{ED}^{\eta}(P, c)$ induced by a linear function $w$, then this determines the unique maximizer $\Pi_{P, c}^{\eta}(v)^{w}$ for all $v$ and therefore a vertex of $\Pi_{P, c}^{\eta}$. Since $w$ induces a total ordering on $\operatorname{ED}^{\eta}(P, c)$ if and only if $\operatorname{SP}\left(\operatorname{ED}^{\eta}(P, c)\right)^{w}$ is a vertex, this proves the first claim.
The second claim follows in the same manner.
Proposition 6.7. Let $\eta$ be a normalization with $\eta(x)>0$ for $x \neq 0$. Any polytope $P$ is a weak Minkowski summand of $\Gamma_{P}^{\mathrm{GI}}$. Normal equivalence holds precisely for 2 -neighborly polytopes.

Proof. Let $c$ be a generic linear function. It follows from convexity that $P^{c}=v$ if and only if $v$ has no $c$-improving neighbor. It follows from (10) that $\Gamma_{P}^{\mathrm{GI}}(v)^{c}=\{0\}$. Since $\Gamma_{P}^{\mathrm{GI}}=\sum_{v} \Gamma_{P}^{\mathrm{GI}}(v)$, we see that if $\left(\Gamma_{P}^{\mathrm{GI}}\right)^{c}$ is a vertex, then so is $P^{c}$.
A polytope $P$ is called 2-neighborly if any two vertices are adjacent. If $P$ is 2-neighborly, then $\Gamma_{P}^{\mathrm{GI}}(v)=-v+P$ and hence

$$
\frac{1}{n} \Gamma_{P}^{\mathrm{GI}}(v)=b(P)+P,
$$

where $b(P)=\frac{1}{n} \sum V(P)$ is the barycenter of $P$. Now, assume that $P$ is not two neighborly and $u-v \notin \Gamma_{P}^{\mathrm{GI}}(v)$. Then there is a linear function $c$ such that $\Gamma_{P}^{\mathrm{GI}}(v)^{c}$ is a vertex but $\operatorname{dim} P^{c}>0$.

The proof actually shows that $P$ is a weak Minkowski summand of $\Gamma_{P}^{\eta}$ for any normalization with $\eta(x)>0$ whenever $x \neq 0$.

Remark 6.8. It also follows from convexity that $P$ is a Minkowski summand of the edge zonotope $\mathcal{E}(P)$ and $\mathcal{E}(P)$ is by construction a Minkowski summand of $\operatorname{SP}(\operatorname{ED}(P))$. However, it is not true in general that $\mathcal{E}(P)$ is a weak Minkowski summand of $\Gamma_{P}^{\text {GII }}$ nor the other way around: If $P=\Delta_{d-1}$, then $\mathcal{E}(P)=\Pi_{d-1}$ is a permutahedron while $\Gamma_{\Delta_{d-1}}^{\mathrm{GI}}=d \Delta_{d-1}$ up to translation. If $P$ is a zonotope, then $\mathcal{E}(P)$ is normally equivalent to $P$ but $\Gamma_{P}^{\mathrm{GI}}$ can have more vertices than $P$.

For the $d$-cube we have $\operatorname{ED}\left(C_{d}\right)=\left\{ \pm e_{1}, \ldots, \pm e_{d}\right\}$ and $\mathrm{SP}\left(\operatorname{ED}\left(C_{d}\right)\right)$ is the type- $B$ permutahedron with respect to $(1, \ldots, d)$; see [30, Sect. 2.2.2]. In contrast to Theorem 6.5, it is in general not true that the GI-neighbotope and the sweep polytope of edge directions are normally equivalent.
Example 6.9. Consider the zonotope $Z \subset \mathbb{R}^{2}$ for the vectors $( \pm 1,1)$ and $\left(\frac{1}{2}, 1\right)$. Then $\Gamma_{Z}^{\mathrm{GI}}$ is a zonotope with 12 vertices whereas $\operatorname{SP}(\operatorname{ED}(Z))$ has 14 vertices.
For the zonotope $Z$ generated by the vectors $(1,0,0),(0,1,0),(0,0,1),(1,1,4)$, one can check that $\Gamma_{Z}^{\mathrm{GI}}$ is not even a belt polytope.

We now show that for a very interesting class of zonotopes related to crystallographic reflection reflection groups, normal equivalence nevertheless holds. We refer the reader to [14, 26] for more information about the geometry and combinatorics of root systems.
A finite nonempty set $\Phi \subset \mathbb{R}^{n} \backslash\{0\}$ is a root system if $\Phi \cap \mathbb{R} \alpha=\{-\alpha, \alpha\}$ for all $\alpha \in \Phi$, and $s_{\alpha}(\Phi)=\Phi$ where $s_{\alpha}(x)=x-2 \frac{\alpha^{t} x}{\alpha^{t} \alpha} \alpha$ is the reflection in the hyperplane $\alpha^{\perp}$. The root system is irreducible if there is no partition $\Phi=\Phi^{\prime} \uplus \Phi^{\prime \prime}$ into nonempty subsets such that $\alpha^{t} \beta=0$ for all $\alpha \in \Phi^{\prime}, \beta \in \Phi^{\prime \prime}$. The group $W \subset O\left(\mathbb{R}^{n}\right)$ generated by the reflections $s_{\alpha}$ for $\alpha \in \Phi$ is finite and called a reflection group. Define the zonotope associated to $\Phi$

$$
Z_{\Phi}:=\frac{1}{2} \sum_{\alpha \in \Phi}[-\alpha, \alpha] .
$$

By construction, $Z_{\Phi}$ is $W$-invariant and has edge directions $\operatorname{ED}\left(Z_{\Phi}\right)=\Phi$. The sweep polytope is then

$$
\operatorname{SP}\left(\operatorname{ED}\left(Z_{\Phi}\right)\right)=\mathrm{SP}(\Phi)=\sum_{\alpha, \beta \in \Phi}[\alpha-\beta, \beta-\alpha]
$$

The greatest improvement pivot rule is sensitive to the length of edges and in this generality, the lengths of roots is not a meaningful invariant. A root system is crystallographic if $\frac{2 \alpha^{t} \beta}{\alpha^{t} \alpha} \in \mathbb{Z}$ for all $\alpha \in \Phi$. Equivalently, if the group $W$ stabilizes the lattice spanned by $\Phi$. In this case $W$ is called a Weyl group. Crystallographic root systems are completely classified; see [26, Chapter 2] and Appendix A. In particular, the zonotope $Z_{\Phi}$ is unique up to rigid motion and homothety.
Theorem 6.10. Let $\Phi \subset \mathbb{R}^{n}$ be an irreducible crystallographic root system. Then $\Gamma_{Z_{\Phi}}^{\mathrm{GI}}$ is normally equivalent to $\operatorname{SP}\left(\operatorname{ED}\left(Z_{\Phi}\right)\right)$.

Let $c \in \mathbb{R}^{n}$ be generic so that $\Phi \cap\left\{x: c^{t} x=0\right\}=\varnothing$. The positive system $\Phi^{+} \subset \Phi$ associated to $c$ is $\Phi^{+}:=\Phi \cap\left\{x: c^{t} x>0\right\}$ and we can rewrite

$$
Z_{\Phi}=\sum_{\alpha \in \Phi^{+}}[-\alpha, \alpha]
$$

The sweep polytope is clearly $W$-invariant and can be rewritten as

$$
\mathrm{SP}(\mathrm{ED}(\Phi))=\sum_{\alpha, \beta \in \Phi}[\alpha-\beta, \beta-\alpha]=2 \sum_{\alpha, \beta \in \Phi^{+}}[\alpha-\beta, \beta-\alpha]+\sum_{\alpha, \beta \in \Phi^{+}}[-\alpha-\beta, \alpha+\beta]
$$

Let $\Delta \subseteq \Phi^{+}$be the unique minimal set of generators of the cone $C:=\operatorname{cone}\left(\Phi^{+}\right)$, called the simple system of $\Phi^{+}$. Let $v_{o p t}=\sum_{\alpha \in \Phi^{+}} \alpha$ be the unique maximizer of $Z_{\Phi}$ for the linear function $x \mapsto c^{t} x$. Then

$$
\Gamma_{Z_{\Phi}}^{\mathrm{GI}}\left(v_{o p t}\right):=\operatorname{conv}\left\{u-v_{o p t}: u \in \mathrm{Nb}_{Z_{\Phi}}\left(v_{o p t}\right) \cup\left\{v_{o p t}\right\}\right\}=\operatorname{conv}(-\Delta \cup\{0\})
$$

By construction $Z_{\Phi}$ is invariant under $W$ and, in fact, $W$ acts simply transitive on the vertices. It thus follows that

$$
\Gamma_{Z_{\Phi}}^{\mathrm{GI}}=\sum_{w \in W} \operatorname{conv}(-w \Delta \cup\{0\})
$$

and hence $\Gamma_{Z_{\Phi}}^{\mathrm{GI}}$ is also $W$-invariant.
The dual cone $C^{\vee}=\left\{w \in \mathbb{R}^{n}: w^{t} \alpha \geq 0\right.$ for all $\left.\alpha \in \Phi^{+}\right\}$is a fundamental domain for the action of $W$ on $\mathbb{R}^{n}$. If we want to show that $\Gamma_{Z_{\Phi}}^{\mathrm{GI}}$ is normally equivalent to $\operatorname{SP}(\operatorname{ED}(\Phi))$, then it suffices to show that for all $w \in C^{\vee}$, if $\operatorname{SP}(\operatorname{ED}(\Phi))^{w}$ is not a vertex, then $\left(\Gamma_{Z_{\Phi}}^{\mathrm{GI}}\right)^{w}$ is not a vertex. In fact, $Z_{\Phi}$ is a weak Minkowski summand of both polytopes and $Z_{\Phi}^{w}$ is not a vertex whenever $w \in \partial C^{\vee}$. Thus, we may restrict to $w \in \operatorname{int}\left(C^{\vee}\right)$. Note that $c \in \operatorname{int}\left(C^{\vee}\right)$ and since $\Phi^{+}$and hence $\Delta$ are unchanged if we replace $c$ by some other $c^{\prime} \in \operatorname{int}\left(C^{\vee}\right)$, we may as well assume $w=c$.
As a first observation, note that $[-\alpha-\beta, \alpha+\beta]^{c}=\alpha+\beta$ and hence

$$
\operatorname{SP}(\mathrm{ED}(\Phi))^{c}=2 v_{o p t}+2 \sum_{\alpha, \beta \in \Phi^{+}}[\alpha-\beta, \beta-\alpha]^{c}
$$

The cone $C$ induces a partial order on $\mathbb{R}^{n}$ by $x \preceq y$ if $y-x \in C$. If $\Phi$ is crystallographic, then every $\alpha \in \Phi^{+}$is a nonnegative integer linear combination of simple roots. Hence $\alpha \preceq \beta$ for $\alpha, \beta \in \Phi^{+}$if and only if $\beta-\alpha=\sum_{\gamma \in \Delta} c_{\gamma} \gamma$ where $c_{\gamma} \in \mathbb{Z}_{\geq 0}$. The poset $\left(\Phi^{+}, \preceq\right)$ is called the (positive) root poset of $\Phi$. Two roots $\alpha, \beta$ are incomparable if $\alpha-\beta$ as well as $\beta-\alpha$ are not contained in $C$. This implies that there is some $t \in \mathbb{R}^{n}$ such that

$$
\operatorname{SP}(\mathrm{ED}(\Phi))^{c}=t+2 \sum_{\substack{\alpha, \beta \in \Phi^{+} \\ \alpha, \beta \text { incomparable }}}[\alpha-\beta, \beta-\alpha]^{c}
$$

We note that if $\operatorname{SP}(\operatorname{ED}(\Phi))^{c}$ is not a vertex, then there is some pair of incomparable roots $\alpha, \beta \in \Phi^{+}$with $c^{t} \alpha=c^{t} \beta$.

On the other hand, we observe that for $w \in W \backslash\{e\}$ and $v=w v_{o p t}$ the corresponding vertex of $Z_{\Phi}$, we have

$$
\Gamma_{Z_{\Phi}}^{\mathrm{GI}}(v)^{c}=\operatorname{conv}(-w \Delta \cup\{0\})^{c}=\operatorname{conv}\left(-w \Delta \cap \Phi^{+}\right)^{c} .
$$

Indeed, if $w \neq e$, then $v \neq v_{o p t}$. Thus $v$ has a $c$-improving edge direction and all the improving edge directions are precisely $\Phi^{+}$. The longest element of $W$ is the unique $w_{0} \in W$ with $w_{0} \Phi^{+}=-\Phi^{+}$. In particular $w w_{0} \Delta=-w \Delta$ and the following result, whose proof we give in Appendix A, then proves Theorem 6.10.
Theorem 6.11. Let $\Phi$ be an irreducible crystallographic root system with positive and simple systems $\Phi^{+} \supseteq \Delta$ and let $W$ be the corresponding Weyl group. If $\alpha, \beta \in \Phi^{+}$are incomparable, then there is $w \in W$ with $w \Delta \cap \Phi^{+}=\{\alpha, \beta\}$.

## 7. Greatest-improvement and graphical neighbotopes

Theorem 1.7 insinuates that branchings for the greatest-improvement rule can be obtained in a greedy-like fashion. Indeed the corresponding arborescence is determined once the edge directions $\mathrm{ED}(P)$ are sorted according to the cost vector $c$. The corresponding neighbotopes can be viewed as solving a certain optimization problem for a fixed polytope $P$ and varying objective function $c$. In this section, we make the connection to greedy-like structures more precise.
Let $G=(V, E)$ be an abstract graph that throughout this section we will assume to be simple and undirected. Let $c \in \mathbb{R}^{V}$ be node potentials. For an ordered pair of adjacent nodes $(u, v)$ we call $c_{v}-c_{u}$ the potential difference. A branching on $G$ is a map $\mathcal{B}: V \rightarrow V$ such that $\mathcal{B}(v) \in \mathrm{Nb}_{G}(v) \cup\{v\}$ and for every $v \in V$ there is a $k \geq 0$ with $\mathcal{B}^{k}(v)=\mathcal{B}^{k+1}(v)$. The set $V_{\mathcal{B}}=\{v: \mathcal{B}(v)=v\}$ is the set of sinks of the branching. The potential energy of a branching is

$$
\begin{equation*}
c(\mathcal{B}):=\sum_{v \in V} c_{\mathcal{B}(v)}-c_{v} \tag{12}
\end{equation*}
$$

and the Max Potential Energy Branching is the problem of finding a branching of maximal potential energy. A scenario that we can imagine is that $V$ is a collection of sites in a mountainous region where $c_{v}$ gives the elevation. The potential difference $c_{u}-c_{v}$ is related to the energy (coming from, say, water turbines) that can be generated by setting up a flow from $v$ to $u$ and the edges $E$ encode the admissible connections. The optimization problem is now to find the energy-optimal routings from every node to a sink. This is a particular instance of the Maximum Weight Branching Problem; see [28, Chapter 6.2].
A polyhedral reformulation is apparent. Continuing Example 4.1, let $\bar{\delta}(\mathcal{B}) \in \mathbb{R}^{V}$ be the reduced in-degree sequence of $\mathcal{B}$ with $\bar{\delta}(\mathcal{B})_{v}:=\left|\mathcal{B}^{-1}(v)\right|-1$. Rewriting (12) to

$$
c(\mathcal{B})=\sum_{v \in V}\left|\mathcal{B}^{-1}(v)\right| c_{v}-\sum_{v \in V} c_{v}=\sum_{v \in V} \bar{\delta}(\mathcal{B})_{v} c_{v}
$$

shows that we are optimizing the linear function $c$ over the graphical neighbotope

$$
\Gamma_{G}=\operatorname{conv}\{\bar{\delta}(\mathcal{B}): \mathcal{B} \text { branching of } G\}
$$

We call a branching $\mathcal{B}$ a greedy branching if

$$
\mathcal{B}(v)=\operatorname{argmax}\left\{c_{u}-c_{v}: u \in \operatorname{Nb}_{G}(v) \cup\{v\}\right\} .
$$

Note that not all branchings are greedy. Indeed for vertices $v, v^{\prime}$ with $\mathrm{Nb}_{G}(v)=\mathrm{Nb}_{G}\left(v^{\prime}\right)$ the greedy condition would force $\mathcal{B}(v)=\mathcal{B}\left(v^{\prime}\right)$.
Theorem 7.1. The vertices of $\Gamma_{G}$ are in bijection to greedy branchings of $G$.
Proof. As before, we note that the graphical neighbotope can be written as a Minkowski sum $\Gamma_{G}=\sum_{v} \Gamma_{G}(v)$ for

$$
\Gamma_{G}(v):=\operatorname{conv}\left(e_{u}-e_{v}: u \in \operatorname{Nb}_{G}(v) \cup\{v\}\right)
$$

which then shows that the vertices are in one-to-one correspondence with greedy branchings.

The greatest-improvement neighbotopes can be viewed as graphical neighbotopes with certain restrictions on node potentials.

Proposition 7.2. Let $P \subset \mathbb{R}^{d}$ a polytope with graph $G=(V, E)$. Then the greatest-improvement neighbotope $\Gamma_{P}^{\mathrm{GI}}$ is the image of $\Gamma_{G}$ under the linear projection $\pi: \mathbb{R}^{V} \rightarrow \mathbb{R}^{d}$ given by $\pi\left(e_{v}\right):=v$.

So, $\Gamma_{G(P)}$ is an extended formulation of $\Gamma_{P}^{G I}$ from which an inequality description as well as the facial structure can be recovered.

Note that greedy branchings need not be arborescences, i.e., there is not necessarily a unique sink. For suitable node potentials, we obtain arborescences. In particular, Proposition 7.2 shows that any node weighting of a polytope graph coming from applying a linear objective function to each vertex will always yield an arborescence.
The structure underlying greedy branchings is that of a polymatroid (for details see [34]). Recall a set function $f_{G}: 2^{V} \rightarrow \mathbb{Z}_{\geq 0}$ is called a polymatroid if
i) $f(\varnothing)=0$,
ii) $f$ is non-decreasing: $A \subseteq B$ implies $f(A) \leq f(B)$, and
iii) $f$ is submodular: $f(A \cup B)+f(A \cap B) \leq f(A)+f(B)$,
for all $A, B \subseteq V$. The associated polymatroid polytope is given by

$$
P_{f}=\left\{x \in \mathbb{R}_{\geq 0}^{V}: x(A) \leq f(A) \text { for all } A \subseteq V\right\}
$$

where $x(A):=\sum_{v \in A} x_{v}$. The polymatroid base polytope is $B_{f}:=P_{f} \cap\{x: x(V)=f(V)\}$. Polymatroids and polymatroid (base) polytopes generalize matroids and independence polytopes. They were introduced by Edmonds [19], who also showed that linear functions on $P_{f}$ can be maximized by a greedy-type algorithm.

Proposition 7.3. The polytope $\Gamma_{G}$ is the polymatroid base polytope for the polymatroid

$$
f(S)=|S|+\left|\mathrm{Nb}_{G}(S)\right|,
$$

where $\operatorname{Nb}_{G}(S)=\{u \in V \backslash S: u v \in E$ for some $v \in S\}$. In particular, $\Gamma_{G}=B_{f}$.
Proof. It follows from the description as a Minkowski sum that $\Gamma_{G}$ is a generalized permutahedron in the sense of [31]. Thus the submodular function is given by $f(S)=\max \left\{\sum_{u \in S} x_{u}: x \in \Gamma_{G}\right\}$. It follows that $f(S)$ is the number of vertices $v \in V$ such that $S \cap\left(\mathrm{Nb}_{G}(v) \cup\{v\}\right) \neq \varnothing$ and this is precisely $|S|+\left|\mathrm{Nb}_{G}(S)\right|$.

The greedy algorithm for polymatroids [19, 34] gives us a simple combinatorial algorithm to construct greedy branchings for given $G=(V, E)$ and generic $c \in \mathbb{R}^{V}$ :
(1) Let $M \leftarrow \varnothing$ be the collection of marked vertices.
(2) Let $D \leftarrow \varnothing$ be the collection of already directed vertices.
(3) If $V=M$, STOP. Otherwise, choose $u \in V \backslash M$ with $c_{u}$ maximal and $M \leftarrow M \cup\{u\}$.
(4) if $u \notin D$, then $\mathcal{B}(u):=u$ and $D \leftarrow D \cup\{u\}$.
(5) for every $v \in \operatorname{Nb}_{G}(u) \backslash(M \cup D)$ set $\mathcal{B}(v):=u$ and $D \leftarrow D \cup\{v\}$.
(6) Repeat (3).

The algorithm also shows that if $\mathcal{B}$ is a greedy branching, then there is a vertex $u$ with $\mathcal{B}(v)=u$ for all $v \in \mathrm{Nb}_{G}(u)$. This is the key to recovering a greedy branching from its reduced indegree sequence.
(1) Let $M \leftarrow \varnothing$ be the collection of marked vertices.
(2) Let $D \leftarrow \varnothing$ be the collection of already directed vertices.
(3) If $V=M$, STOP. Otherwise, choose $u \in V \backslash M$ with $\bar{\delta}_{u}=\left|\mathrm{Nb}_{G}(u) \backslash(M \cup D)\right|$ and mark $u(M \leftarrow M \cup\{u\})$.
(4) If no $u$ exists in step (3), choose any $u \in V \backslash D$, and $M \leftarrow M \cup\{u\}$.
(5) If no $u$ exists in steps (3) and (4), then $D=V$, and we are done.
(6) Otherwise, for $v \in \mathrm{Nb}_{G}(u) \backslash(M \cup D)$, direct $v$ to $u D \leftarrow D \cup\{v\}$. If $u \notin D$, direct $u$ to itself $(D \leftarrow D \cup\{u\})$. Return to Step (3).

Via the greedy algorithm, the $u$ with highest weight will have all of its neighbors $\mathrm{Nb}_{G}(u)$ directed towards it. Furthermore, the vertex of next highest weight will have all its neighbors towards except those that have already been directed. Hence, so long as there exist vertices that have not been directed towards another vertex by the algorithm, there will always exist some vertex satisfying the conditions of step (3). After no vertex satisfying step (3) exists, all remaining vertices that are not directed must be directed to themselves. That case is accounted for by step (4), which iterates until $D=V$.

The graphical neighbotopes are instances of the hypergraphic polytopes of Benedetti et al [10]; see also [3]. A hypergraph is a collection of hyperedges $\mathcal{H} \subseteq 2^{V}$ for some finite set $V$. The associated hypergraph polytope is

$$
P_{\mathcal{H}}=\sum_{H \in \mathcal{H}} \operatorname{conv}\left\{e_{v}: v \in H\right\}
$$

For $G$, we can associate the hypergraph $\mathcal{H}_{G}=\left\{\mathrm{Nb}_{G}(v): v \in V\right\}$. It now follows from the proof of Theorem 7.1 that $\Gamma_{G}=P_{\mathcal{H}_{G}}$. The vertices and faces of $P_{\mathcal{H}}$ were interpreted in terms of acyclic orientations of $\mathcal{H}$. They can be translated directly to our greedy branchings.

Example 7.4 (Complete bipartite graphs). Consider the complete bipartite graph $K_{m, n}$ with color classes $A=[m]$ and $B=[n]$. Let $c$ be a node potential on $K_{m, n}$ with corresponding greedy branching $\mathcal{B}$. Let $a \in A$ and $b \in B$ be the nodes that attain the maximal potential on $A$ and $B$, respectively. Let us assume that $c_{a}>c_{b}$. Then $\mathcal{B}(v)=a$ for all $v \in B, \mathcal{B}(u)=u$ for all $u \in A$ with $c_{u}>c_{b}$ and $\mathcal{B}(u)=b$ otherwise. Hence the branching is completely determined by the nodes $a, b$ and the set $S=\left\{u \in A: c_{u}>c_{b}\right\}$. In particular, every such triple $(a, b, S)$ can occur. Exchanging the roles of $A$ and $B$ then yields the total number of greedy branchings as

$$
m \sum_{k=1}^{n} k\binom{n}{k}+n \sum_{k=1}^{m} k\binom{m}{k}=m n\left(2^{m-1}+2^{n-1}\right)
$$

Example 7.5 (Path graphs). For $n \geq 1$, let $P_{n}$ be the path on nodes $V=\{1, \ldots, n\}$ and edges $E=\{\{i, i+1\}: 1 \leq i<n\}$.
We may encode a branching $\mathcal{B}$ of $P_{n}$ uniquely as a word $W=W_{1} W_{2} \ldots W_{n}$ of length $n$ over the alphabet $\{L, R, S\}$ where we set $W_{i}=L$ if $\mathcal{B}(i)=i-1, w_{i}=R$ if $\mathcal{B}(i)=i+1$ and $w_{i}=S$ if $\mathcal{B}(i)=i$. Note that the only forbidden subword is $R L$. This allows us to count all branchings. The number of branchings of $P_{n}$ is the number of walks in the following directed graph $D_{1}$ that start at node 1 and end at node 2 after $n$ steps:


Using the transfer matrix method [37, Ch. 4.7], one finds that the number of branchings is the Fibonacci number $F(2 n)$.

For the greedy branchings, one further observes that the only additional forbidden subword is $S S$. Hence, the number of greedy branchings of $P_{n}$ is the number of walks in the following directed graph that start at node 1 and end at node 2 after $n$ steps:


The number of greedy branchings is given by the sequence $a(n)$ with $a(0)=0, a(1)=1, a(2)=2$ and $a(n+3)=2 a(n+2)+a(n+1)-a(n)$ for $n \geq 0$; see also [35].

Remark 7.6. There is an obvious graphical generalization of general neighbotopes. For a given graph $G=(V, E)$ let $\eta: E \rightarrow \mathbb{R}_{>0}$ be a normalization. We can define the normalized graphical neighbotope $\Gamma_{G}^{\eta}$ as the Minkowski sum of

$$
\Gamma_{G}(v):=\operatorname{conv}\left(\frac{e_{u}-e_{v}}{\eta(u v)}: u \in \mathrm{Nb}_{G}(v) \cup\{v\}\right),
$$

for $v \in V$. Branchings can still be found with a greedy-type algorithm that for every node $v$ makes the optimal choice.

## Appendix A. Proof of Theorem 6.11

In this section we give a non-uniform (that is, case-by-case) proof of Theorem 6.11. Let $\Phi \subset \mathbb{R}^{n}$ an irreducible crystallographic root system with simple and positive systems $\Delta \subseteq \Phi^{+}=\Phi \cap\{x$ : $\left.c^{t} x>0\right\}$. The simple system consists of linearly independent vectors and $r=|\Delta|$ is the rank of the root system. Crystallographic root systems are completely classified: There are four infinite families $A_{n-1}, B_{n}, C_{n}, D_{n}$ for $n \geq 2$ as well as sporadic instances $G_{2}, F_{4}, E_{6}, E_{7}, E_{8}$, where the subscript gives the rank; see [26, Ch. 2.8-2.10].
For $G_{2}, F_{4}, E_{6}, E_{7}$, and $E_{8}$, the number of incomparable pairs is $2,55,204,546$, and 1540 , respectively. The claim can be checked by a computer: Let $Z_{\Phi}$ be the zonotope associated to the root system. If $v \in V\left(Z_{\Phi}\right)$ is a vertex with improving edge directions $\Delta_{v}$, then $\mathrm{Nb}_{Z_{\Phi}, c}(v)=$ $\left\{s_{\alpha}(v): \alpha \in \Delta_{v}\right\}$ and $u=s_{\alpha}(v)$ has improving edge directions $\Delta_{u}=s_{\alpha}\left(\Delta^{\prime}\right)$. This yields a naive, yet quite fast depth-first search algorithm with starting point $v=-v_{\text {opt }}$ and $\Delta_{v}=\Delta$. Explicit coordinates for the simple roots are given in [26, Section 2.10].
In the following we check the four infinite families $A_{n-1}, B_{n}, C_{n}$, and $D_{n}$.
Type $A_{n-1}$. A realization of the root system of type $A_{n-1}$ is given by $\Phi=\left\{e_{i}-e_{j}: i, j \in\right.$ $[n], i \neq j\}$. For $c \in \mathbb{R}^{n}$ with $c_{1}>c_{2}>\cdots>c_{n}$, the positive system and simple system are

$$
\Phi^{+}=\left\{e_{i}-e_{j}: 1 \leq i<j \leq n\right\} \quad \text { and } \quad \Delta=\left\{e_{1}-e_{2}, e_{2}-e_{3}, \ldots, e_{n-1}-e_{n}\right\} .
$$

For $h=1, \ldots, n$, let $s_{h}(x)=x_{1}+\cdots+x_{h}$. The cone $C=\operatorname{cone}(\Delta)$ is then given by

$$
C=\left\{x \in \mathbb{R}^{n}: s_{1}(x) \geq 0, s_{2}(x) \geq 0, \ldots, s_{n-1}(x) \geq 0\right\}
$$

For $i, j \in[n]$, we write $[i, j]:=\{k \in[n]: i \leq k \leq j\}$.
Proposition A. 1 (Type $A_{n-1}$ incomparable pairs). Let $i<j$ and $k<l$. Then
(A1) $e_{i}-e_{j}$ and $e_{k}-e_{l}$ are incomparable if and only if $[i, j] \nsubseteq[k, l]$ and $[k, l] \nsubseteq[i, j]$.
There are precisely $2\binom{n}{4}+\binom{n}{3}$ pairs of incomparable positive roots.
Proof. We may assume $i \leq k$. If $i=k$, then $\left(e_{k}-e_{l}\right)-\left(e_{i}-e_{j}\right)=e_{j}-e_{l} \in C$ or $e_{l}-e_{j} \in C$. Hence $i<k$ and the first nonzero coordinate of $\left(e_{k}-e_{l}\right)-\left(e_{i}-e_{j}\right)$ is negative. This shows $\left(e_{k}-e_{l}\right)-\left(e_{i}-e_{j}\right) \notin C$. Now, $s_{h}\left(\left(e_{i}-e_{j}\right)-\left(e_{k}-e_{l}\right)\right)<0$ for some $h=1, \ldots, n-1$ and hence $\left(e_{i}-e_{j}\right)-\left(e_{k}-e_{l}\right) \notin C$ if and only if $i<j \leq k<l$ or $i<k<j<l$.

The reflection group $W$ associated to the $A_{n-1}$ root system acts on $\mathbb{R}^{n}$ by permuting coordinates. For a permutation $\tau$ of $[n]$ we set $\tau\left(e_{i}\right):=e_{\tau(i)}$ for $i=1, \ldots, n$. In particular, $\tau \Delta=\left\{e_{\tau(1)}-\right.$ $\left.e_{\tau(2)}, \ldots, e_{\tau(n-1)}-e_{\tau(n)}\right\}$ is a simple system and every simple system arises this way. Note that $e_{\tau(i)}-e_{\tau(i+1)}$ is a positive root if and only if $\tau(i)<\tau(i+1)$, that is, $i$ is an ascent of $\tau$.

Proof of Theorem 6.11 for $A_{n-1}$. Let $\alpha=e_{i}-e_{j}$ and $\beta=e_{k}-e_{l}$ with $i<j$ and $k<l$ be incomparable positive roots. It suffices to give a permutation $\tau$ such that $\tau \Delta \cap \Phi^{+}=\{\alpha, \beta\}$ : For $i<j<k<l$

$$
n n-1 \ldots l+1 l-1 \ldots k+1 \mathbf{k} \mathbf{l} k-1 \ldots j+1 \mathbf{i} \mathbf{j} j-1 \ldots i+1 i-1 \ldots 1
$$

For $i<k \leq j<l$

$$
n n-1 \ldots l+1 l-1 \ldots j+1 \mathbf{i} \mathbf{j} j-1 \ldots k+1 \mathbf{k} \mathbf{l} k-1 \ldots i+1 i-1 \ldots 1
$$

Type $B_{n}$ and $C_{n}$. A realization of the root system of type $B_{n}$ is given by the roots $e_{i}-e_{j}$, $e_{i}+e_{j}$ for $i, j \in[n], i \neq j$ and $\pm e_{1}, \ldots, \pm e_{n}$. For $c \in \mathbb{R}^{n}$ with $c_{1}>c_{2}>\cdots>c_{n}>0$, the positive system and simple system are
$\Phi^{+}=\left\{e_{i}-e_{j}, e_{i}+e_{j}: 1 \leq i<j \leq n\right\} \cup\left\{e_{1}, \ldots, e_{n}\right\}$ and $\Delta=\left\{e_{1}-e_{2}, e_{2}-e_{3}, \ldots, e_{n-1}-e_{n}, e_{n}\right\}$.
The cone $C=\operatorname{cone}(\Delta)$ is given by

$$
C=\left\{x \in \mathbb{R}^{n}: s_{1}(x) \geq 0, s_{2}(x) \geq 0, \ldots, s_{n-1}(x) \geq 0, s_{n}(x) \geq 0\right\}
$$

For $i, j \in[n]$, we write $(i, j):=\{k \in[n]: i<k<j\}$.
The crystallographic root system of type $C_{n}$ differs from $B_{n}$ in that the roots $\pm e_{i}$ are replaced by $\pm 2 e_{i}$. With these modifications, the positive system and simple system are obtained from type $B_{n}$. The associated reflection group is unchanged.

Proposition A. 2 (Type $B_{n}$ and $C_{n}$ incomparable pairs). Let $i<j$ and $k<l$. For type $B_{n}$
(B1) $e_{k}-e_{l}, e_{i}-e_{j}$ are incomparable if and only if $[i, j] \nsubseteq[k, l]$ and $[k, l] \nsubseteq[i, j]$;
(B2) $e_{k}+e_{l}, e_{i}+e_{j}$ are incomparable if and only if $[i, j] \subseteq(k, l)$ or $[k, l] \subseteq(i, j)$;
(B3) $e_{k}-e_{l}, e_{i}+e_{j}$ are incomparable if and only if $k<i$;
(B4) $e_{k}-e_{l}, e_{i}$ are incomparable if and only if $k<i$;
(B5) $e_{k}+e_{l}, e_{i}$ are incomparable if and only if $i<k$.
For type $C_{n}$, the cases (B4) and (B5) are replaced by
(C4) $e_{k}-e_{l}, 2 e_{i}$ are incomparable if and only if $k<i$;
(C5) $e_{k}+e_{l}, 2 e_{i}$ are incomparable if and only if $k<i<l$.
Proof.
(B1): Since $s_{n}\left(e_{s}-e_{t}\right)=0$ for all $s<t$, we have that $e_{k}-e_{l}, e_{i}-e_{j}$ are incomparable if and only if they are incomparable in type $A_{n-1}$. The claim now follows from (A1) of Proposition A.1.
(B2): We may assume $i \leq k$. If $i=k$, then $\left(e_{k}+e_{l}\right)-\left(e_{i}+e_{j}\right)=e_{l}-e_{j}$ or $e_{j}-e_{l}$ is in $C$. Hence $i<k$ and the first nonzero entry of $\left(e_{k}+e_{l}\right)-\left(e_{i}+e_{j}\right)$ is negative. Now $s_{h}\left(e_{i}+e_{j}-e_{k}-e_{l}\right)<0$ for some $h$ if and only if $i<k<l<j$.
Note that $s_{n}\left(e_{k}-e_{l}-t\left(e_{r}+e_{s}\right)\right)=-2 t<0$ for all $r, s \in[n]$ and $t>0$ and hence $e_{k}-e_{l}-t\left(e_{r}+e_{s}\right)$ is never contained in $C$.
(B3): We only need to verify $\left(e_{i}+e_{j}\right)-\left(e_{k}-e_{l}\right) \notin C$. This is the case if the first nonzero coordinate is negative, which happens if and only if $k<i$.
(B4) and (C4): Likewise, $t e_{i}-\left(e_{k}-e_{l}\right) \notin C$ for $t \in\{1,2\}$ if and only if $k<i$.
(B5): $s_{n}\left(e_{i}-\left(e_{k}+e_{l}\right)\right)=-1$ and hence $e_{i}-\left(e_{k}+e_{l}\right) \notin C .\left(e_{k}+e_{l}\right)-e_{i} \notin C$ if and only if $i<k$.
(C5): $\left(e_{k}+e_{l}\right)-2 e_{i} \notin C$ if and only if $i<l$ and $2 e_{i}-\left(e_{k}+e_{l}\right) \notin C$ if and only if $k<i$.
The reflection group $W$ associated to the $B_{n}$ root system acts on $\mathbb{R}^{n}$ by signed permutations. A signed permutation is a pair $w=(t, \tau)$, where $\tau$ is a permutation of $[n]$ and $t \in\{-1,+1\}^{n}$. Then $w$ acts on the standard basis as $w\left(e_{i}\right)=t_{i} e_{\tau(i)}$. We represent $w$ in window notation and write $w=w_{1} \ldots w_{n} \in\{1, \ldots, n, \overline{1}, \ldots, \bar{n}\}^{n}$ where $w_{i}=\overline{\tau(i)}$ if $t_{i}=-1$ and $w_{i}=\tau(i)$ otherwise. For example, $t=(1,-1,1,-1)$ and $\tau=(3,1,2,4)$ is denoted by $w=3 \overline{1} 2 \overline{3}$.

Proof of Theorem 6.11 for $B_{n}$ and $C_{n}$. Let $\alpha$ and $\beta$ be incomparable positive roots. For each case, we give a suitable signed permutation $w$ such that $w \Delta \cap \Phi^{+}=\{\alpha, \beta\}$.
(B1): Let $e_{i}-e_{j}$ and $e_{k}-e_{l}$ be incomparable with $i<j$ and $i<k<l$.
For $i<j<k<l$

$$
\overline{1} \overline{2} \ldots \overline{i-1} \overline{i+1} \ldots \overline{j-1} \overline{\mathbf{j}} \overline{\mathbf{i}} \overline{j+1} \ldots \overline{k-1} \overline{k+1} \ldots \overline{l-1} \overline{\mathbf{l}} \overline{\mathbf{k}} \overline{l+1} \ldots \overline{n-1} \bar{n}
$$

For $i<k \leq j<l$

$$
\overline{1} \overline{2} \ldots \overline{i-1} \overline{i+1} \ldots \overline{k-1} \overline{\mathbf{l}} \overline{\mathbf{k}} \overline{k+1} \ldots \overline{j-1} \overline{\mathbf{j}} \overline{\mathbf{i}} \overline{j+1} \ldots \overline{l-1} \overline{l+1} \ldots \overline{n-1} \bar{n}
$$

(B2): For $e_{i}+e_{j}$ and $e_{k}+e_{l}$ with $i<k<l<j$

$$
\overline{1} \overline{2} \ldots \overline{i-1} \overline{i+1} \ldots \overline{k-1} \mathbf{k} \overline{\mathbf{l}} j-1 j-2 \ldots l+1 l-1 \ldots k+1 \mathbf{i} \overline{\mathbf{j}} \overline{j+1} \ldots \overline{n-1} \bar{n} .
$$

(B3): For $e_{k}-e_{l}$ with $k<l$ and $e_{i}+e_{j}$ with $k<i<j$

$$
\overline{1} \overline{2} \ldots \overline{i-1} n n-1 \ldots l+1 \mathbf{k} \mathbf{l} l-1 \ldots i+1 \mathbf{i} \overline{\mathbf{j}} .
$$

(B4) and (C4): For $e_{k}-e_{l}$ with $k<l$ and $e_{i}$ with $k<i$

$$
\overline{1} \overline{2} \ldots \overline{i-1} n n-1 \ldots l+1 \mathbf{k} \mathbf{l} l-1 \ldots i+1 \mathbf{i} .
$$

(B5) and (C5): For $e_{k}+e_{l}$ with $k<l$ and $e_{i}$ with $i<l$

$$
\overline{1} \overline{2} \ldots \overline{i-1} \mathbf{k} \overline{\mathbf{1}} n n-1 \ldots l+1 l-1 \ldots i+1 \mathbf{i} .
$$

Type $D_{n}$. A realization of the root system of type $D_{n}$ is given by the roots $\pm\left(e_{i}-e_{j}\right)$ and $\pm\left(e_{i}+e_{j}\right)$ for $1 \leq i<j \leq n$. For $c \in \mathbb{R}^{n}$ with $c_{1}>c_{2}>\cdots>c_{n}>0$, the positive and simple system are
$\Phi^{+}=\left\{e_{i}-e_{j}, e_{i}+e_{j}: 1 \leq i<j \leq n\right\} \quad$ and $\quad \Delta=\left\{e_{1}-e_{2}, e_{2}-e_{3}, \ldots, e_{n-1}-e_{n}, e_{n-1}+e_{n}\right\}$.
The cone $C=\operatorname{cone}(\Delta)$ is given by

$$
C=\left\{x \in \mathbb{R}^{n}: s_{1}(x) \geq 0, \ldots, s_{n-1}(x) \geq 0, s_{n}(x) \geq 0, s_{n-1}(x) \geq x_{n}\right\} .
$$

Proposition A. 3 (Type $D_{n}$ incomparable pairs). Let $i<j$ and $k<l$.
(D1) $e_{k}-e_{l}, e_{i}-e_{j}$ are incomparable if and only if $[i, j] \nsubseteq[k, l]$ and $[k, l] \nsubseteq[i, j]$;
(D2) $e_{k}+e_{l}, e_{i}+e_{j}$ are incomparable if and only if $[i, j] \subseteq(k, l)$ or $[k, l] \subseteq(i, j)$;
(D3) $e_{k}-e_{l}, e_{i}+e_{j}$ are incomparable if and only if $k<i$ or $j=l=n$.
Proof. The cases (D1) and (D2) follow from Proposition A.2. For (D3) we again note that $s_{n}\left(\left(e_{k}-e_{l}\right)-\left(e_{i}+e_{j}\right)\right)<0$. Now, $x:=\left(e_{i}+e_{j}\right)-\left(e_{k}-e_{l}\right) \notin C$ if the first nonzero entry is negative or $s_{n-1}(x)<x_{n}$. The former happens if and only if $k<i$. The latter is true if and only if $j=l=n$.

The reflection group $W$ associated to the $D_{n}$ root system acts on $\mathbb{R}^{n}$ by signed permutations with an even number of sign changes. Thus, only those signed permutations $w=w_{1} \ldots w_{n} \in$ $\{1, \ldots, n, \overline{1}, \ldots, \bar{n}\}^{n}$ are permitted with an even number of barred positions.

Proof of Theorem 6.11 for $D_{n}$. (D1): Let $e_{i}-e_{j}$ and $e_{k}-e_{l}$ with $i<j$ and $i<k<l$. If $n$ is even, then the signed permutations (B1) have an even number of signs and should be used. Otherwise, if $n$ is odd and $l<n$, then

$$
\overline{1} \overline{2} \ldots \overline{j-1} \overline{\mathbf{j}} \overline{\mathbf{i}} \overline{j+1} \ldots \overline{l-1} \overline{\mathbf{l}} \overline{\mathbf{k}} \overline{l+1} \ldots \overline{n-1} \mathbf{n} .
$$

If $l=n$, then

$$
\overline{1} \overline{2} \ldots \overline{j-1} \overline{\mathbf{j}} \overline{\mathbf{i}} \overline{j+1} \ldots \overline{n-1} \overline{\mathbf{n}} \mathbf{k}
$$

(D2): For $e_{i}+e_{j}$ and $e_{k}+e_{l}$ with $1 \leq i<k<l<j \leq n$, if $j<n$, then, depending on the parity

$$
\begin{aligned}
& \overline{1} \overline{2} \ldots \overline{k-1} \mathbf{k} \overline{\mathbf{1}} j-1 j-2 \ldots k+1 \mathbf{i} \overline{\mathbf{j}} \overline{j+1} \ldots \overline{n-1} \bar{n} \quad \text { or } \\
& \overline{1} \overline{2} \ldots \overline{k-1} \mathbf{k} \overline{\mathbf{l}} j-1 j-2 \ldots k+1 \mathbf{i} \overline{\mathbf{j}} \overline{j+1} \ldots \overline{n-1} n .
\end{aligned}
$$

If $j=n$, then, depending on the parity,

$$
\begin{aligned}
& \overline{1} \overline{2} \ldots \overline{k-1} \mathbf{k} \overline{\mathbf{1}} j-1 j-2 \ldots k+1 \mathbf{i} \overline{\mathbf{j}} \\
& \overline{1} \overline{2} \ldots \overline{k-2} \mathbf{k} \overline{\mathbf{1}} j-1 j-2 \ldots k+1 k-1 \mathbf{i} \overline{\mathbf{j}} .
\end{aligned}
$$

(D3): For $e_{k}-e_{l}$ with $k<l$ and $e_{i}+e_{j}$ with $k<i<j<n$

$$
\begin{aligned}
& \overline{1} \overline{2} \ldots \overline{i-1} n-1 \ldots k+1 \mathbf{k} \mathbf{l} k-1 \ldots i+1 \mathbf{i} \overline{\mathbf{j}} n \\
& \overline{1} \overline{2} \ldots \overline{i-1} n-1 \ldots k+1 \mathbf{k} \mathbf{l} k-1 \ldots i+1 \mathbf{i} \overline{\mathbf{j}} \bar{n} .
\end{aligned}
$$

If $j=l=n$ and $i=k$, then, depending on whether $n$ is odd or not:

$$
\overline{1}, \overline{2}, \ldots, \overline{n-1}, \mathbf{i}, \overline{\mathbf{n}} \quad \text { or } \quad \overline{1}, \overline{2}, \ldots, \overline{n-1}, \mathbf{i}, \mathbf{n}
$$

If $j=l=n$ and $i \neq k$, then, depending on whether $n$ is odd or not:

$$
\overline{1}, \overline{2}, \ldots, \overline{n-1}, \mathbf{i}, \mathbf{n}, \mathbf{k} \quad \text { or } \quad \overline{1}, \overline{2}, \ldots, \overline{n-1}, \mathbf{i}, \mathbf{n}, \overline{\mathbf{k}}
$$

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