# The number of cliques in graphs covered by long cycles 

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#### Abstract

Let $G$ be a 2 -connected $n$-vertex graph and $N_{s}(G)$ be the total number of $s$-cliques in $G$. Let $k \geq 4$ and $s \geq 2$ be integers. In this paper, we show that if $G$ has an edge $e$ which is not on any cycle of length at least $k$, then $N_{s}(G) \leq r\binom{k-1}{s}+\binom{t+2}{s}$, where $n-2=r(k-3)+t$ and $0 \leq t \leq k-4$. This result settles a conjecture of Ma and Yuan and provides a clique version of a theorem of Fan, Wang and Lv. As a direct corollary, if $N_{s}(G)>r\binom{k-1}{s}+\binom{t+2}{s}$, every edge of $G$ is covered by a cycle of length at least $k$.


Keywords: clique, long cycle, the Erdős-Gallai theorem

## 1 Introduction

All graphs considered in this paper are simple. Let $G$ be a graph and and $N_{s}(G)$ be the total number of $s$-cliques in $G$ (a complete subgraph with $s$ vertices). Particularly, $N_{2}(G)$ is the number of edges of $G$, which is often denoted by $e(G)$. The well-known Erdős-Gallai theorem [1] states that if a graph with $n$ vertices has no cycle of length at least $k$ where $n \geq k \geq 3$, then $e(G) \leq(k-1)(n-1) / 2$, which was originally conjectured by Turán (cf. [6]). The Erdős-Gallai theorem was improved by Kopylov [9] for 2-connected graphs.

Before presenting Kopylov's result, we need some extra notations. Let $H_{n, k, a}$ be an $n$-vertex graph whose vertex set can be partitioned into three sets $A, B$ and $C$ such that $|A|=a,|B|=n-(k-a)$ and $|C|=k-2 a$, where integers $n, k$ and $a$ satisfy $n \geq k \geq 4$ and $k / 2>a \geq 1$, and whose edge set consists of all edges between $A$ and $B$, and all edges in $A \cup C$. Note that $H_{n, k, a}$ is 2 -connected if $a \geq 2$ and has no cycle longer than $k-1$. For $s \geq 2$, define

$$
f_{s}(n, k, a)=\binom{k-a}{s}+(n-k+a)\binom{a}{s-1} .
$$

Then $N_{s}\left(H_{n, k, a}\right)=f_{s}(n, k, a)$ and, particularly, $e\left(H_{n, k, a}\right)=N_{2}\left(H_{n, k, a}\right)=f_{2}(n, k, a)$. The following is Kopylov's result for 2-connected graphs.

Theorem 1.1 (Kopylov, [9]). Let $G$ be a 2-connected graph on $n$ vertices, and let $n \geq k \geq 5$ and $t=\left\lfloor\frac{k-1}{2}\right\rfloor$. If $G$ has no cycle of length at least $k$, then

$$
e(G) \leq \max \left\{f_{2}(n, k, 2), f_{2}(n, k, t)\right\}
$$

and the equality holds only if $G=H_{n, k, 2}$ or $G=H_{n, k, t}$.

[^0]It is worth mentioning that Fan, Lv and Wang [4] proved a result slightly stronger than the above theorem for $n \geq k \geq 2 n / 3$. Together with a result of a result of Woodall [13], it provided an alternative proof of Theorem 1.1 Recently, a clique version of Theorem 1.1 has been proven by Luo [10] as follows.
Theorem 1.2 (Luo, [10]). Let $G$ be a 2-connected $n$-vertex graph, and let $n \geq k \geq 5$ and $t=\left\lfloor\frac{k-1}{2}\right\rfloor$. If $G$ has circumference less than $k$, then the number of $s$-cliques of $G$ satisfies

$$
N_{s}(G) \leq \max \left\{f_{s}(n, k, 2), f_{s}(n, k, t)\right\}
$$

A stability result of the Theorem 1.2 is obtained Ma and Yuan [11], which also can be viewed as the clique version of a stability result of Theorem 1.1 given by Füredi, Kostochka and Verstraëte [5].

Another result of Erdős and Gallai in [1] shows that a graph without a path of length at least $k$ has $e(G) \leq n(k-2) / 2$. The result of Erdős and Gallai for paths was strengthened by Fan for 2-connected graphs (Theorem 5 in [2]), which states that the longest path between any pair of vertices in a 2 -connected graph with more than $(k+2)(n-2) / 2$ edges has length at least $k$. Fan's result is sharp when $n-2$ is divisible by $k-2$, which was further sharpened by Wang and Lv [12] for all possible values $n \geq 3$. The sharpness of the results of Fan [2], Wang and Lv [12] can be shown by the following constructions.

Let $X_{n, k}$ to be an $n$-vertex graph defined as follows. Assuming $n-2=r(k-3)+t$ where $0 \leq t \leq k-4$, the graph $X_{n, k}$ consists of three disjoint parts $A, B$ and $C$ such that $A$ is an edge $u v$, and $B$ is a union of $r$ vertex disjoint $(k-3)$-cliques, and $C$ is a $t$-clique, and all edges between $A$ and $B \cup C$. For $s \geq 2$, define

$$
g_{s}(n, k)= \begin{cases}r\binom{k-1}{s}+\binom{t+2}{s} & \text { if } s \geq 3 \\ r\binom{k-3}{2}+\binom{t}{2}+2(n-2)+1 & \text { if } s=2\end{cases}
$$

Then $g_{s}(n, k)=N_{s}\left(X_{n, k}\right)$, and $e\left(X_{n, k}\right)=N_{2}\left(X_{n, k}\right)=g_{2}(n, k) \leq r\binom{k-1}{2}+\binom{t+2}{2}$. These graphs $X_{n, k}$ have no cycle containing the edge $u v$ longer than $k-1$. Note that, if $k>n$, then $X_{n, k}$ is a clique and $g_{s}(n, k)=\binom{n}{s}$.

Theorem 1.3 (Fan [2], Wang and Lv [12] ). Let $G$ be a 2-connected n-vertex graph with $n \geq 3$. If $G$ has an edge uv such that $G$ has no cycle of length at least $k \geq 4$ containing uv. Then

$$
e(G) \leq g_{2}(n, k)
$$

In [11, Ma and Yuan made the following conjecture, which can be treated as a clique version of Theorem [1.3. As indicated in [11], the conjecture (if it is true) is a key tool to prove a more general stability result of of Theorem 1.2

Conjecture 1.4 (Ma and Yuan, [11). Let $G$ be a 2-connected n-vertex graph with $n \geq 3$ and let uv be an edge in $G$. Let $k \geq 4$ and $s \geq 2$ be integers, and let $n-2=r(k-3)+t$ for some $0 \leq t \leq k-4$. If

$$
N_{s}(G)>r\binom{k-1}{s}+\binom{t+2}{s}
$$

then there is a cycle on at least $k$ vertices containing the edge $u v$.
Note that, the bound of the above conjecture is not the best possible for the case $s=2$ because of Theorem 1.3 and $g_{2}(n, k)<r\binom{k-1}{2}+\binom{t+2}{2}$ for $n \geq k$. The following is our main result, which completely settles Conjecture 1.4 .

Theorem 1.5. Let $G$ be a 2-connected n-vertex graph with $n \geq 3$. If $G$ has an edge uv such that $G$ has no cycle containing uv of length at least $k \geq 4$, then the number of $s$-cliques of $G$ with $s \geq 2$ satisfies

$$
N_{s}(G) \leq g_{s}(n, k)
$$

The bound in Theorem 1.5 is sharp due to these graphs $X_{n, k}$ constructed above. A direct corollary of Theorem 1.5 shows that if the clique number of a graph $G$ is large enough, every edge of $G$ belongs to a long cycle.

Corollary 1.6. Let $G$ be a 2-connected $n$-vertex graph with $n \geq 3$. If $N_{s}(G)>g_{s}(n, k)$ where $k \geq 4$ and $s \geq 2$, then every edge of $G$ is covered by a cycle of length at least $k$.

## 2 Preliminaries

Let $X_{n, k}$ be an $n$-vertex graph defined in the previous section, and let $Q_{n, k}$ be the $n$-vertex graph with $n \geq 2$ obtained from $X_{n+1, k+1}$ by contracting the edge $u v$ of $A$ into a single vertex $w$. Then for graphs $Q_{n, k}$, it holds that $n-1=r(k-2)+t$ and $0 \leq t \leq k-3$. For $s \geq 2$, define

$$
\psi_{s}(n, k)=r\binom{k-1}{s}+\binom{t+1}{s}
$$

Then $\psi_{s}(n, k)=N_{s}\left(Q_{n, k}\right)$. Note that, if $k>n$, then $Q_{n, k}$ is a clique and $\psi_{s}(n, k)=\binom{n}{s}$. Let $f_{s}(n, k, a)$ the function defined in the previous section. By comparing the graphs $H_{n, k, 2}, H_{n, k,\left\lfloor\frac{k-1}{2}\right\rfloor}$ and the graph $Q_{n, k}$, it not hard to derive the following proposition.

Proposition 2.1. For integers $n \geq k \geq 5$ and $s \geq 2$, the functions $f_{s}(n, k, a)$ and $\psi_{s}(n, k)$ satisfy

$$
\max \left\{f_{s}(n, k, 2), f_{s}\left(n, k,\left\lfloor\frac{k-1}{2}\right\rfloor\right)\right\} \leq \psi_{s}(n, k)
$$

The following result slightly strengthens Luo's clique version of the Erdős-Gallai theorem (Corollary 1.5 in [10]), which serves as an important step toward the proof of our main result-Theorem 1.5. Note that, the bound in this result is sharp because of the graphs $Q_{n, k}$ constructed above.

Theorem 2.2. Let $G$ be a connected $n$-vertex graph with $n \geq 2$. If $G$ has no cycle of length at least $k \geq 4$, then the number of $s$-cliques with $s \geq 2$ of $G$ satisfies

$$
N_{s}(G) \leq \psi_{s}(n, k)
$$

Proof. Let $G$ be a connected $n$-vertex graph with $n \geq 2$. Use induction on $n$, the number of vertices of $G$. The result holds trivially for $n \leq 3$. So assume that $n \geq 4$ in the following, and the theorem holds for all connected graphs with the number of vertices smaller than $n$.

If $k=4$, every maximal 2-connected subgraph of $G$ is a triangle because the longest cycle of $G$ has length at most $k-1=3$. So each block of $G$ is either a triangle or a single edge. It follows that $N_{3}(G) \leq(n-1) / 2$ and equality holds if and only if $G$ is the graph with $(n-1) / 2$ triangles sharing a common vertex, and $N_{2}(G) \leq \psi_{2}(n, k)$. Hence

$$
N_{s}(G) \leq \psi_{s}(n, k)
$$

and the theorem holds. So, in the following, assume that $k \geq 5$.
First, assume that $G$ is 2 -connected. If $n<k$, then

$$
N_{s}(G) \leq\binom{ n}{s}=\psi_{s}(n, k)
$$

and hence the theorem holds. If $n \geq k \geq 5$, then it follows from Theorem 1.2 and Proposition 2.1 that

$$
N_{s}(G) \leq \max \left\{f_{s}(n, k, 2), f_{s}\left(n, k,\left\lfloor\frac{k-1}{2}\right\rfloor\right)\right\} \leq \psi_{s}(n, k)
$$

and the theorem holds.
Hence, we may assume that $G$ has a cut-vertex $v$. Let $H$ be a connected component of $G-v$, and let $G_{1}=G[H \cup\{v\}]$ and $G_{2}=G-H$. Then both $G_{1}$ and $G_{2}$ are connected and $G_{1} \cap G_{2}=\{v\}$. For each $i \in[2]$, let $n_{i}=\left|V\left(G_{i}\right)\right| \geq 2$, and assume $n_{i}-1=r_{i}(k-2)+t_{i}$ with $0 \leq t_{i} \leq k-3$. Then $n=n_{1}+n_{2}-1$. Then, for $s \geq 2$, the following inequality holds,

$$
\binom{t_{1}+1}{s}+\binom{t_{2}+1}{s} \leq\left\{\begin{array}{cl}
\binom{t_{1}+t_{2}+1}{s} & \text { if } t_{1}+t_{2} \leq k-2  \tag{1}\\
\binom{k-1}{s}+\binom{t_{1}+t_{2}-k+3}{s} & \text { if } k-1 \leq t_{1}+t_{2} \leq 2(k-3)
\end{array}\right.
$$

Applying inductive hypothesis to each $G_{i}$, we have

$$
\begin{aligned}
N_{s}(G) & =N_{s}\left(G_{1}\right)+N_{s}\left(G_{2}\right) \leq \psi_{s}\left(n_{1}, k\right)+\psi_{s}\left(n_{2}, k\right) \\
& =r_{1}\binom{k-1}{s}+\binom{t_{1}+1}{s}+r_{2}\binom{k-1}{s}+\binom{t_{2}+1}{s} \\
& \leq \psi_{s}(n, k)
\end{aligned}
$$

where the last inequality follows from Inequality (1). This completes the proof.
Another ingredient we need to prove Theorem 1.5 is edge-switching operation, which was introduced by Fan [3] to study subgraph covering. This operation appears in an earlier paper [8] of Klemans which studied the probabilities of the number of connected components under this operation.

Let $G$ be a graph and $v$ be a vertex of $G$. Let $N(v)=\{u \mid u v \in E(G)\}$ and let $N[v]=N(v) \cup\{v\}$. The degree of $v$ in $G$ is denoted by $d_{G}(v)$ which is equal to $|N(v)|$. For a given edge $u v$, an edge-switching from $v$ to $u$ is to replace each edge $v x$ by a new edge $u x$ for every $x \in N(v) \backslash N[u]$. The resulting graph is called the edge-switching graph of $G$ from $v$ to $u$, denoted by $G[v \rightarrow u]$. The following lemma is a trivial observation.

Lemma 2.3. Let $G$ be a 2-connected graph and let uv be an edge of $G$.
(i) If $N(u) \cap N(v)=\emptyset$ and $G / u v$ is not 2-connected, then $\{u, v\}$ is a vertex cut of $G$.
(ii) If $N(u) \cap N(v) \neq \emptyset$ and the edge-switching graph $G[v \rightarrow u]$ is not 2 -connected, then $\{u, v\}$ is a vertex cut of $G$.

The following lemma shows that the edge-switching operation does not increase the length of longest cycles through certain edges.

Lemma 2.4 (Ji and Chen, [7). Let $G$ be a connected graph and let uv be an edge. For any edge ux, let $k$ be the length of a longest cycle of $G$ containing ux. Then the length of a longest cycle containing ux in the edge-switching graph $G[v \rightarrow u]$ is at most $k$.

The contraction and the edge-switching operations do not reduce the number of $s$-cliques except small values of $s$ as shown in the following lemma.

Lemma 2.5. Let $G$ be a connected graph and let uv be an edge of $G$.
(i) If $N(u) \cap N(v)=\emptyset$ and $s \geq 3$, then $N_{s}(G / u v) \geq N_{s}(G)$;
(ii) For $s \geq 2$, it holds that $N_{s}(G[v \rightarrow u]) \geq N_{s}(G)$.

Proof. (i) Since $N(u) \cap N(v)=\emptyset$, the graph $G$ has no $s$-cliques containing $u v$ for $s \geq 3$. Hence, every $s$-clique of $G$ remains as an $s$-clique in $G / u v$. Hence (1) follows.
(ii) Let $S(G)$ be the set of unlabeled copies of $K_{s}$ in a graph $G$. Consider an edge-switching from $v$ to $u$, and let $G^{\prime}=G[v \rightarrow u]$. Denote $W=\{v x \mid x \in N(v) \backslash N[u]\}$. Define a map $\pi: S(G) \rightarrow S\left(G^{\prime}\right)$ as follows, for each $Q \in S(G)$,

$$
\pi(Q)= \begin{cases}Q & \text { if } E(Q) \cap W=\emptyset \\ Q^{\prime} & \text { otherwise }\end{cases}
$$

where $V\left(Q^{\prime}\right)=(V(Q) \backslash\{v\}) \cup\{u\}$ and $E\left(Q^{\prime}\right)=(E(Q) \backslash W) \cup\{u x \mid v x \in W \cap E(Q)\}$. If $v x \in W \cap E(Q)$, then $x \in N(v) \backslash N[u]$ and it follows that $u \notin Q$. All neighbors of $v$ in $Q$ are neighbors of $u$ in $Q^{\prime}$. Hence $Q^{\prime}$ is indeed an $s$-clique of $G^{\prime}$. So $\pi$ is well-defined. Note that, $\pi\left(Q_{1}\right) \neq \pi\left(Q_{2}\right)$ for two different $s$-cliques $Q_{1}$ and $Q_{2}$ of $G$. Therefore $\pi$ is an injection. So $N_{s}(G[v \rightarrow u]) \geq N_{s}(G)$ and (ii) holds.

## 3 Proof of Theorem 1.5

Now, we are ready to prove our main theorem. Note that Theorem 1.5 follows from Theorem 1.3 directly for the case $s=2$. In the following, we only need to prove it for $s \geq 3$.

Proof of Theorem 1.5. Suppose to the contrary that $G$ is a counterexample. For an edge $e$ of $G$, let $c_{e}(G)$ be the maximum length of cycles containing $e$. Then $G$ is a 2-connected $n$-vertex graph with $N_{s}(G)>g_{s}(n, k)$ but does have an edge $e$ such that $c_{e}(G)<k$. Let

$$
\ell(G)=\max \left\{d_{G}(v) \mid v \text { is an end-vertex of some edge } e \text { with } c_{e}(G)<k\right\}
$$

Among all the counterexamples, choose $G$ such that: (1) the number of vertices of $G$ is as small as possible, and (2) subject to $(1), \ell(G)$ is as large as possible.

Note that the theorem holds trivially for $n=3$. If $k=4$, then $G$ consists of $n-2$ triangles which share a common edge. Then $N_{3}(G)=n-2=g_{3}(n, 4)$ and $N_{s}(G)=0$ for $s \geq 4$, a contradiction to that $G$ is a counterexample. So in the following, assume that $n \geq 4$ and $k \geq 5$.

Claim 1. The graph $G$ does not have a 2 -vertex cut $\{x, y\}$ such that $x y$ is an edge.
Proof of Claim 1. If not, assume that $\{x, y\}$ is a vertex cut of $G$ with $x y \in E(G)$. Let $H_{1}$ a connected component of $G-\{x, y\}$. Further, let $G_{1}=G\left[V\left(H_{1}\right) \cup\{x, y\}\right]$ and $G_{2}=G-H_{1}$. Then both $G_{1}$ and $G_{2}$ are 2 -connected. For convenience, let $\left|V\left(G_{i}\right)\right|=n_{i} \geq 3$, and $n_{i}-2=r_{i}(k-3)+t_{i}$ and $0 \leq t_{i} \leq k-4$ for $i \in[2]$.

If $N_{s}\left(G_{i}\right)>g_{s}\left(n_{i}, k\right)$ for some $i \in[2]$, without loss of generality assume $N_{s}\left(G_{1}\right)>g_{s}\left(n_{1}, k\right)$. Since $G$ is a counterexample with the smallest number of vertices, the subgraph $G_{1}$ is smaller and hence not a counterexample. Therefore, $c_{e}\left(G_{1}\right) \geq k$ for any edge $e \in E\left(G_{1}\right)$. So $c_{e}(G) \geq c_{e}\left(G_{1}\right) \geq k$ for each $e \in E\left(G_{1}\right)$. For an edge $e^{\prime} \in E\left(G_{2}\right)$ and $e^{\prime} \neq x y$, it follows from 2-connectivity of $G_{2}$ that $G_{2}$ has a cycle $C^{\prime}$ containing both $e^{\prime}$ and $x y$. Let $C$ be a longest cycle of $G_{1}$ containing $x y$. Then $\left(C \cup C^{\prime}\right)-\{x y\}$ is a cycle of $G$ which contains $e^{\prime}$ and

$$
c_{e^{\prime}}(G) \geq|V(C)|+\left|V\left(C^{\prime}\right)\right|-2>|V(C)| \geq c_{x y}\left(G_{1}\right) \geq k
$$

Thus, $c_{e}(G) \geq k$ for any edge $e \in E(G)$, a contradiction to that $G$ has an edge $e$ with $c_{e}(G)<k$. Hence $N_{s}\left(G_{i}\right) \leq g_{s}\left(n_{i}, k\right)$ holds for both $i=1$ and $i=2$. Therefore,

$$
\begin{align*}
N_{s}(G) & =N_{s}\left(G_{1}\right)+N_{s}\left(G_{2}\right) \leq g_{s}\left(n_{1}, k\right)+g_{s}\left(n_{2}, k\right) \\
& =r_{1}\binom{k-1}{s}+\binom{t_{1}+2}{s}+r_{2}\binom{k-1}{s}+\binom{t_{2}+2}{s}  \tag{2}\\
& =\left(r_{1}+r_{2}\right)\binom{k-1}{s}+\binom{t_{1}+2}{s}+\binom{t_{2}+2}{s} .
\end{align*}
$$

For $s \geq 3$, the following inequality holds,

$$
\binom{t_{1}+2}{s}+\binom{t_{2}+2}{s} \leq\left\{\begin{array}{cl}
\binom{t_{1}+t_{2}+2}{s} & \text { if } t_{1}+t_{2} \leq k-4  \tag{3}\\
\binom{k-1}{s}+\binom{t_{1}+t_{2}-k+5}{s} & \text { if } k-3 \leq t_{1}+t_{2} \leq 2(k-4)
\end{array}\right.
$$

Note that, $n=n_{1}+n_{2}-2$ and hence $n-2=\left(r_{1}+r_{2}\right)(k-3)+\left(t_{1}+t_{2}\right)$, which implies $r=r_{1}+r_{2}$ and $t=t_{1}+t_{2}$ if $t_{1}+t_{2} \leq k-4$, and $r=r_{1}+r_{2}+1$ and $t=t_{1}+t_{2}-k+3$ if $k-3 \leq t_{1}+t_{2} \leq 2(k-4)$. Combining Inequalities (2) and (3), it follows that

$$
N_{s}(G) \leq g_{s}(n, k)
$$

This yields a contradiction to that $N_{s}(G)>g_{s}(n, k)$, which completes the proof of Claim 1.
Choose an edge $u v$ of $G$ such that $c_{u v}(G)<k$ and $d_{G}(v) \leq d_{G}(u)=\ell(G)$. Then the vertex $u$ satisfies the following claim.

Claim 2. The vertex $u$ is adjacent to all vertices of $G-u$.
Proof of Claim 2. Suppose to the contrary that there exists a vertex $w$ in $G$ such that $u w \notin E(G)$. Since $G$ is 2-connected, there is a $(u, w)$-path $P$ which does not contain $v$. Choose $P=u u_{1} \cdots u_{k} w$ to be a shortest $(u, w)$-path avoiding $v$. Then $u u_{i} \notin E(G)$ for $i \geq 2$. If $N(u) \cap N\left(u_{1}\right)=\emptyset$, then contract the edge $u u_{1}$ and let $G^{\prime}=G / u u_{1}$. By (i) of Lemma 2.3 and Claim 1, the graph $G^{\prime}$ is 2 -connected. Since $u u_{1} \neq u v$, it follows that

$$
c_{u v}\left(G^{\prime}\right) \leq c_{u v}(G)<k
$$

Since $G^{\prime}$ is smaller than $G$, the graph $G^{\prime}$ is not a counterexample and hence $N_{s}\left(G^{\prime}\right) \leq g_{s}(n-1, k)$. It follows from (i) of Lemma 2.5 that

$$
N_{s}(G) \leq N_{s}\left(G^{\prime}\right) \leq g_{s}(n-1, k) \leq g_{s}(n, k)
$$

which contradicts the assumption $N_{s}(G)>g_{s}(n, k)$.
So assume that $N(u) \cap N\left(u_{1}\right) \neq \emptyset$. Let $G^{\prime \prime}=G\left[u_{1} \rightarrow u\right]$. Then $d_{G^{\prime \prime}}(u)>d_{G}(u)$ because $u_{2}$ and $w$ are not adjacent to $u$. By (ii) of Lemma 2.3 and Claim 1, the graph $G^{\prime \prime}$ is 2-connected. It follows from Lemma 2.4 that $c_{u v}\left(G^{\prime \prime}\right) \leq c_{u v}(G)<k$. Further, by (ii) of Lemma 2.5, it holds that

$$
N_{s}\left(G^{\prime \prime}\right) \geq N_{s}(G)>g_{s}(n, k) .
$$

Then $\ell\left(G^{\prime \prime}\right) \geq d_{G^{\prime \prime}}(u)>d_{G}(u)=\ell(G)$ because $u u_{2} \notin E(G)$, which contradicts the maximality of $\ell(G)$. This completes the proof of Claim 2.

By Claim 2, $u$ is adjacent to all other vertices of $G$. Further, we have the following claim.
Claim 3. The graph $G-u$ has no cycle of length at least $k-1$.
Proof of Claim 3. Suppose to the contrary that $G-u$ has a cycle $C=x_{1} x_{2} \ldots x_{l} x_{1}$ with $l \geq k-1$. If $v \in V(C)$, let $v=x_{1}$ (relabelling $x_{i}$ 's if necessary). Then $u v x_{2} \ldots x_{l} u$ is a cycle of length at least $k$ containing $u v$ in $G$ because $u$ is adjacent to all other vertices of $G$, a contradiction to $c_{u v}(G)<k$. Now assume $v \notin V(C)$. Since $G$ is 2-connected, the graph $G-u$ is connected. Hence $G-u$ has a path from $v$ to $C$ which is internally disjoint from $C$. Without loss of generality, assume $x_{1}$ is the end-vertex of $P$ on $C$. Then $u v P x_{1} x_{2} \ldots x_{l} u$ is a cycle of $G$ which has length at least $k$, a contradiction to $c_{u v}(G)<k$. This completes the proof of Claim 3.

By Claim 3 and Theorem [2.2] we have

$$
N_{s}(G-u) \leq \psi_{s}(n-1, k-1)=r\binom{k-2}{s}+\binom{t+1}{s},
$$

where $(n-1)-1=r(k-3)+t$ and $0 \leq t \leq k-4$. By Claim 2, it follows that

$$
\begin{aligned}
N_{s}(G) & =N_{s}(G-u)+N_{s-1}(G-u) \\
& \leq r\binom{k-2}{s}+\binom{t+1}{s}+r\binom{k-2}{s-1}+\binom{t+1}{s-1} \\
& =r\binom{k-1}{s}+\binom{t+2}{s} \\
& =g_{s}(n, k),
\end{aligned}
$$

which yields a desired contradiction to $N_{s}(G)>g_{s}(n, k)$. This completes the proof.

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