The number of cliques in graphs covered by long cycles

Naidan Ji^{*} and Dong Ye[†]

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Abstract

Let G be a 2-connected n-vertex graph and $N_s(G)$ be the total number of s-cliques in G. Let $k \ge 4$ and $s \ge 2$ be integers. In this paper, we show that if G has an edge e which is not on any cycle of length at least k, then $N_s(G) \le r\binom{k-1}{s} + \binom{t+2}{s}$, where n-2 = r(k-3) + t and $0 \le t \le k-4$. This result settles a conjecture of Ma and Yuan and provides a clique version of a theorem of Fan, Wang and Lv. As a direct corollary, if $N_s(G) > r\binom{k-1}{s} + \binom{t+2}{s}$, every edge of G is covered by a cycle of length at least k.

Keywords: clique, long cycle, the Erdős-Gallai theorem

1 Introduction

All graphs considered in this paper are simple. Let G be a graph and $N_s(G)$ be the total number of s-cliques in G (a complete subgraph with s vertices). Particularly, $N_2(G)$ is the number of edges of G, which is often denoted by e(G). The well-known Erdős-Gallai theorem [1] states that if a graph with n vertices has no cycle of length at least k where $n \ge k \ge 3$, then $e(G) \le (k-1)(n-1)/2$, which was originally conjectured by Turán (cf. [6]). The Erdős-Gallai theorem was improved by Kopylov [9] for 2-connected graphs.

Before presenting Kopylov's result, we need some extra notations. Let $H_{n,k,a}$ be an *n*-vertex graph whose vertex set can be partitioned into three sets A, B and C such that |A| = a, |B| = n - (k-a) and |C| = k - 2a, where integers n, k and a satisfy $n \ge k \ge 4$ and $k/2 > a \ge 1$, and whose edge set consists of all edges between A and B, and all edges in $A \cup C$. Note that $H_{n,k,a}$ is 2-connected if $a \ge 2$ and has no cycle longer than k-1. For $s \ge 2$, define

$$f_s(n,k,a) = \binom{k-a}{s} + (n-k+a)\binom{a}{s-1}.$$

Then $N_s(H_{n,k,a}) = f_s(n,k,a)$ and, particularly, $e(H_{n,k,a}) = N_2(H_{n,k,a}) = f_2(n,k,a)$. The following is Kopylov's result for 2-connected graphs.

Theorem 1.1 (Kopylov, [9]). Let G be a 2-connected graph on n vertices, and let $n \ge k \ge 5$ and $t = \lfloor \frac{k-1}{2} \rfloor$. If G has no cycle of length at least k, then

$$e(G) \leq \max\{f_2(n,k,2), f_2(n,k,t)\},\$$

and the equality holds only if $G = H_{n,k,2}$ or $G = H_{n,k,t}$.

^{*}School of Mathematics and Statistics, Ningxia University, Yinchuan, Ningxia 750021. Email: jind@nxu.edu.cn.

[†]Department of Mathematical Sciences and Center for Computational Science, Middle Tennessee State University, Murfreesboro, TN 37132. Email: dong.ye@mtsu.edu. Partially supported by Simons Foundation award no. 359516.

It is worth mentioning that Fan, Lv and Wang [4] proved a result slightly stronger than the above theorem for $n \ge k \ge 2n/3$. Together with a result of a result of Woodall [13], it provided an alternative proof of Theorem 1.1. Recently, a clique version of Theorem 1.1 has been proven by Luo [10] as follows.

Theorem 1.2 (Luo, [10]). Let G be a 2-connected n-vertex graph, and let $n \ge k \ge 5$ and $t = \lfloor \frac{k-1}{2} \rfloor$. If G has circumference less than k, then the number of s-cliques of G satisfies

$$N_s(G) \le \max\{f_s(n,k,2), f_s(n,k,t)\}.$$

A stability result of the Theorem 1.2 is obtained Ma and Yuan [11], which also can be viewed as the clique version of a stability result of Theorem 1.1 given by Füredi, Kostochka and Verstraëte [5].

Another result of Erdős and Gallai in [1] shows that a graph without a path of length at least k has $e(G) \leq n(k-2)/2$. The result of Erdős and Gallai for paths was strengthened by Fan for 2-connected graphs (Theorem 5 in [2]), which states that the longest path between any pair of vertices in a 2-connected graph with more than (k+2)(n-2)/2 edges has length at least k. Fan's result is sharp when n-2 is divisible by k-2, which was further sharpened by Wang and Lv [12] for all possible values $n \geq 3$. The sharpness of the results of Fan [2], Wang and Lv [12] can be shown by the following constructions.

Let $X_{n,k}$ to be an *n*-vertex graph defined as follows. Assuming n-2 = r(k-3) + t where $0 \le t \le k-4$, the graph $X_{n,k}$ consists of three disjoint parts A, B and C such that A is an edge uv, and B is a union of rvertex disjoint (k-3)-cliques, and C is a *t*-clique, and all edges between A and $B \cup C$. For $s \ge 2$, define

$$g_s(n,k) = \begin{cases} r\binom{k-1}{s} + \binom{t+2}{s} & \text{if } s \ge 3; \\ r\binom{k-3}{2} + \binom{t}{2} + 2(n-2) + 1 & \text{if } s = 2. \end{cases}$$

Then $g_s(n,k) = N_s(X_{n,k})$, and $e(X_{n,k}) = N_2(X_{n,k}) = g_2(n,k) \le r\binom{k-1}{2} + \binom{t+2}{2}$. These graphs $X_{n,k}$ have no cycle containing the edge uv longer than k-1. Note that, if k > n, then $X_{n,k}$ is a clique and $g_s(n,k) = \binom{n}{s}$.

Theorem 1.3 (Fan [2], Wang and Lv [12]). Let G be a 2-connected n-vertex graph with $n \ge 3$. If G has an edge uv such that G has no cycle of length at least $k \ge 4$ containing uv. Then

$$e(G) \le g_2(n,k).$$

In [11], Ma and Yuan made the following conjecture, which can be treated as a clique version of Theorem 1.3. As indicated in [11], the conjecture (if it is true) is a key tool to prove a more general stability result of of Theorem 1.2.

Conjecture 1.4 (Ma and Yuan, [11]). Let G be a 2-connected n-vertex graph with $n \ge 3$ and let uv be an edge in G. Let $k \ge 4$ and $s \ge 2$ be integers, and let n - 2 = r(k - 3) + t for some $0 \le t \le k - 4$. If

$$N_s(G) > r\binom{k-1}{s} + \binom{t+2}{s},$$

then there is a cycle on at least k vertices containing the edge uv.

Note that, the bound of the above conjecture is not the best possible for the case s = 2 because of Theorem 1.3 and $g_2(n,k) < r\binom{k-1}{2} + \binom{t+2}{2}$ for $n \ge k$. The following is our main result, which completely settles Conjecture 1.4.

Theorem 1.5. Let G be a 2-connected n-vertex graph with $n \ge 3$. If G has an edge uv such that G has no cycle containing uv of length at least $k \ge 4$, then the number of s-cliques of G with $s \ge 2$ satisfies

$$N_s(G) \le g_s(n,k).$$

The bound in Theorem 1.5 is sharp due to these graphs $X_{n,k}$ constructed above. A direct corollary of Theorem 1.5 shows that if the clique number of a graph G is large enough, every edge of G belongs to a long cycle.

Corollary 1.6. Let G be a 2-connected n-vertex graph with $n \ge 3$. If $N_s(G) > g_s(n,k)$ where $k \ge 4$ and $s \ge 2$, then every edge of G is covered by a cycle of length at least k.

2 Preliminaries

Let $X_{n,k}$ be an *n*-vertex graph defined in the previous section, and let $Q_{n,k}$ be the *n*-vertex graph with $n \ge 2$ obtained from $X_{n+1,k+1}$ by contracting the edge uv of A into a single vertex w. Then for graphs $Q_{n,k}$, it holds that n-1 = r(k-2) + t and $0 \le t \le k-3$. For $s \ge 2$, define

$$\psi_s(n,k) = r\binom{k-1}{s} + \binom{t+1}{s}.$$

Then $\psi_s(n,k) = N_s(Q_{n,k})$. Note that, if k > n, then $Q_{n,k}$ is a clique and $\psi_s(n,k) = \binom{n}{s}$. Let $f_s(n,k,a)$ the function defined in the previous section. By comparing the graphs $H_{n,k,2}$, $H_{n,k,\lfloor\frac{k-1}{2}\rfloor}$ and the graph $Q_{n,k}$, it not hard to derive the following proposition.

Proposition 2.1. For integers $n \ge k \ge 5$ and $s \ge 2$, the functions $f_s(n, k, a)$ and $\psi_s(n, k)$ satisfy

$$\max\{f_s(n,k,2), f_s(n,k,\lfloor\frac{k-1}{2}\rfloor)\} \le \psi_s(n,k).$$

The following result slightly strengthens Luo's clique version of the Erdős-Gallai theorem (Corollary 1.5 in [10]), which serves as an important step toward the proof of our main result—Theorem 1.5. Note that, the bound in this result is sharp because of the graphs $Q_{n,k}$ constructed above.

Theorem 2.2. Let G be a connected n-vertex graph with $n \ge 2$. If G has no cycle of length at least $k \ge 4$, then the number of s-cliques with $s \ge 2$ of G satisfies

$$N_s(G) \le \psi_s(n,k)$$

Proof. Let G be a connected n-vertex graph with $n \ge 2$. Use induction on n, the number of vertices of G. The result holds trivially for $n \le 3$. So assume that $n \ge 4$ in the following, and the theorem holds for all connected graphs with the number of vertices smaller than n.

If k = 4, every maximal 2-connected subgraph of G is a triangle because the longest cycle of G has length at most k - 1 = 3. So each block of G is either a triangle or a single edge. It follows that $N_3(G) \le (n - 1)/2$ and equality holds if and only if G is the graph with (n - 1)/2 triangles sharing a common vertex, and $N_2(G) \le \psi_2(n, k)$. Hence

$$N_s(G) \le \psi_s(n,k)$$

and the theorem holds. So, in the following, assume that $k \geq 5$.

First, assume that G is 2-connected. If n < k, then

$$N_s(G) \le \binom{n}{s} = \psi_s(n,k)$$

and hence the theorem holds. If $n \ge k \ge 5$, then it follows from Theorem 1.2 and Proposition 2.1 that

$$N_s(G) \le \max\{f_s(n,k,2), f_s(n,k,\lfloor\frac{k-1}{2}\rfloor)\} \le \psi_s(n,k),$$

and the theorem holds.

Hence, we may assume that G has a cut-vertex v. Let H be a connected component of G - v, and let $G_1 = G[H \cup \{v\}]$ and $G_2 = G - H$. Then both G_1 and G_2 are connected and $G_1 \cap G_2 = \{v\}$. For each $i \in [2]$, let $n_i = |V(G_i)| \ge 2$, and assume $n_i - 1 = r_i(k-2) + t_i$ with $0 \le t_i \le k-3$. Then $n = n_1 + n_2 - 1$. Then, for $s \ge 2$, the following inequality holds,

$$\binom{t_1+1}{s} + \binom{t_2+1}{s} \leq \begin{cases} \binom{t_1+t_2+1}{s} & \text{if } t_1+t_2 \leq k-2; \\ \binom{k-1}{s} + \binom{t_1+t_2-k+3}{s} & \text{if } k-1 \leq t_1+t_2 \leq 2(k-3). \end{cases}$$
(1)

Applying inductive hypothesis to each G_i , we have

$$N_{s}(G) = N_{s}(G_{1}) + N_{s}(G_{2}) \leq \psi_{s}(n_{1}, k) + \psi_{s}(n_{2}, k)$$

= $r_{1}\binom{k-1}{s} + \binom{t_{1}+1}{s} + r_{2}\binom{k-1}{s} + \binom{t_{2}+1}{s}$
 $\leq \psi_{s}(n, k),$

where the last inequality follows from Inequality (1). This completes the proof.

Another ingredient we need to prove Theorem 1.5 is *edge-switching* operation, which was introduced by Fan [3] to study subgraph covering. This operation appears in an earlier paper [8] of Klemans which studied the probabilities of the number of connected components under this operation.

Let G be a graph and v be a vertex of G. Let $N(v) = \{u | uv \in E(G)\}$ and let $N[v] = N(v) \cup \{v\}$. The degree of v in G is denoted by $d_G(v)$ which is equal to |N(v)|. For a given edge uv, an *edge-switching* from v to u is to replace each edge vx by a new edge ux for every $x \in N(v) \setminus N[u]$. The resulting graph is called the *edge-switching graph* of G from v to u, denoted by $G[v \to u]$. The following lemma is a trivial observation.

Lemma 2.3. Let G be a 2-connected graph and let uv be an edge of G.

(i) If $N(u) \cap N(v) = \emptyset$ and G/uv is not 2-connected, then $\{u, v\}$ is a vertex cut of G.

(ii) If $N(u) \cap N(v) \neq \emptyset$ and the edge-switching graph $G[v \to u]$ is not 2-connected, then $\{u, v\}$ is a vertex cut of G.

The following lemma shows that the edge-switching operation does not increase the length of longest cycles through certain edges.

Lemma 2.4 (Ji and Chen, [7]). Let G be a connected graph and let uv be an edge. For any edge ux, let k be the length of a longest cycle of G containing ux. Then the length of a longest cycle containing ux in the edge-switching graph $G[v \to u]$ is at most k.

The contraction and the edge-switching operations do not reduce the number of s-cliques except small values of s as shown in the following lemma.

Lemma 2.5. Let G be a connected graph and let uv be an edge of G. (i) If $N(u) \cap N(v) = \emptyset$ and $s \ge 3$, then $N_s(G/uv) \ge N_s(G)$; (ii) For $s \ge 2$, it holds that $N_s(G[v \to u]) \ge N_s(G)$.

Proof. (i) Since $N(u) \cap N(v) = \emptyset$, the graph G has no s-cliques containing uv for $s \ge 3$. Hence, every s-clique of G remains as an s-clique in G/uv. Hence (1) follows.

(ii) Let S(G) be the set of unlabeled copies of K_s in a graph G. Consider an edge-switching from v to u, and let $G' = G[v \to u]$. Denote $W = \{vx | x \in N(v) \setminus N[u]\}$. Define a map $\pi : S(G) \to S(G')$ as follows, for each $Q \in S(G)$,

$$\pi(Q) = \begin{cases} Q & \text{if } E(Q) \cap W = \emptyset, \\ Q' & \text{otherwise,} \end{cases}$$

where $V(Q') = (V(Q) \setminus \{v\}) \cup \{u\}$ and $E(Q') = (E(Q) \setminus W) \cup \{ux \mid vx \in W \cap E(Q)\}$. If $vx \in W \cap E(Q)$, then $x \in N(v) \setminus N[u]$ and it follows that $u \notin Q$. All neighbors of v in Q are neighbors of u in Q'. Hence Q' is indeed an s-clique of G'. So π is well-defined. Note that, $\pi(Q_1) \neq \pi(Q_2)$ for two different s-cliques Q_1 and Q_2 of G. Therefore π is an injection. So $N_s(G[v \to u]) \ge N_s(G)$ and (ii) holds.

3 Proof of Theorem 1.5

Now, we are ready to prove our main theorem. Note that Theorem 1.5 follows from Theorem 1.3 directly for the case s = 2. In the following, we only need to prove it for $s \ge 3$.

Proof of Theorem 1.5. Suppose to the contrary that G is a counterexample. For an edge e of G, let $c_e(G)$ be the maximum length of cycles containing e. Then G is a 2-connected n-vertex graph with $N_s(G) > g_s(n,k)$ but does have an edge e such that $c_e(G) < k$. Let

 $\ell(G) = \max\{d_G(v) | v \text{ is an end-vertex of some edge } e \text{ with } c_e(G) < k\}.$

Among all the counterexamples, choose G such that: (1) the number of vertices of G is as small as possible, and (2) subject to (1), $\ell(G)$ is as large as possible.

Note that the theorem holds trivially for n = 3. If k = 4, then G consists of n - 2 triangles which share a common edge. Then $N_3(G) = n - 2 = g_3(n, 4)$ and $N_s(G) = 0$ for $s \ge 4$, a contradiction to that G is a counterexample. So in the following, assume that $n \ge 4$ and $k \ge 5$.

Claim 1. The graph G does not have a 2-vertex cut $\{x, y\}$ such that xy is an edge.

Proof of Claim 1. If not, assume that $\{x, y\}$ is a vertex cut of G with $xy \in E(G)$. Let H_1 a connected component of $G - \{x, y\}$. Further, let $G_1 = G[V(H_1) \cup \{x, y\}]$ and $G_2 = G - H_1$. Then both G_1 and G_2 are 2-connected. For convenience, let $|V(G_i)| = n_i \ge 3$, and $n_i - 2 = r_i(k-3) + t_i$ and $0 \le t_i \le k - 4$ for $i \in [2]$.

If $N_s(G_i) > g_s(n_i, k)$ for some $i \in [2]$, without loss of generality assume $N_s(G_1) > g_s(n_1, k)$. Since G is a counterexample with the smallest number of vertices, the subgraph G_1 is smaller and hence not a counterexample. Therefore, $c_e(G_1) \ge k$ for any edge $e \in E(G_1)$. So $c_e(G) \ge c_e(G_1) \ge k$ for each $e \in E(G_1)$. For an edge $e' \in E(G_2)$ and $e' \ne xy$, it follows from 2-connectivity of G_2 that G_2 has a cycle C' containing both e' and xy. Let C be a longest cycle of G_1 containing xy. Then $(C \cup C') - \{xy\}$ is a cycle of G which contains e' and

$$c_{e'}(G) \ge |V(C)| + |V(C')| - 2 > |V(C)| \ge c_{xy}(G_1) \ge k.$$

Thus, $c_e(G) \ge k$ for any edge $e \in E(G)$, a contradiction to that G has an edge e with $c_e(G) < k$. Hence $N_s(G_i) \le g_s(n_i, k)$ holds for both i = 1 and i = 2. Therefore,

$$N_{s}(G) = N_{s}(G_{1}) + N_{s}(G_{2}) \leq g_{s}(n_{1}, k) + g_{s}(n_{2}, k)$$

$$= r_{1}\binom{k-1}{s} + \binom{t_{1}+2}{s} + r_{2}\binom{k-1}{s} + \binom{t_{2}+2}{s}$$

$$= (r_{1}+r_{2})\binom{k-1}{s} + \binom{t_{1}+2}{s} + \binom{t_{2}+2}{s}.$$
(2)

For $s \geq 3$, the following inequality holds,

$$\binom{t_1+2}{s} + \binom{t_2+2}{s} \leq \begin{cases} \binom{t_1+t_2+2}{s} & \text{if } t_1+t_2 \leq k-4; \\ \binom{k-1}{s} + \binom{t_1+t_2-k+5}{s} & \text{if } k-3 \leq t_1+t_2 \leq 2(k-4). \end{cases}$$
(3)

Note that, $n = n_1 + n_2 - 2$ and hence $n - 2 = (r_1 + r_2)(k - 3) + (t_1 + t_2)$, which implies $r = r_1 + r_2$ and $t = t_1 + t_2$ if $t_1 + t_2 \le k - 4$, and $r = r_1 + r_2 + 1$ and $t = t_1 + t_2 - k + 3$ if $k - 3 \le t_1 + t_2 \le 2(k - 4)$. Combining Inequalities (2) and (3), it follows that

$$N_s(G) \le g_s(n,k).$$

This yields a contradiction to that $N_s(G) > g_s(n,k)$, which completes the proof of Claim 1.

Choose an edge uv of G such that $c_{uv}(G) < k$ and $d_G(v) \leq d_G(u) = \ell(G)$. Then the vertex u satisfies the following claim.

Claim 2. The vertex u is adjacent to all vertices of G - u.

Proof of Claim 2. Suppose to the contrary that there exists a vertex w in G such that $uw \notin E(G)$. Since G is 2-connected, there is a (u, w)-path P which does not contain v. Choose $P = uu_1 \cdots u_k w$ to be a shortest (u, w)-path avoiding v. Then $uu_i \notin E(G)$ for $i \geq 2$. If $N(u) \cap N(u_1) = \emptyset$, then contract the edge uu_1 and let $G' = G/uu_1$. By (i) of Lemma 2.3 and Claim 1, the graph G' is 2-connected. Since $uu_1 \neq uv$, it follows that

$$c_{uv}(G') \le c_{uv}(G) < k.$$

Since G' is smaller than G, the graph G' is not a counterexample and hence $N_s(G') \leq g_s(n-1,k)$. It follows from (i) of Lemma 2.5 that

$$N_s(G) \le N_s(G') \le g_s(n-1,k) \le g_s(n,k),$$

which contradicts the assumption $N_s(G) > g_s(n,k)$.

So assume that $N(u) \cap N(u_1) \neq \emptyset$. Let $G'' = G[u_1 \to u]$. Then $d_{G''}(u) > d_G(u)$ because u_2 and w are not adjacent to u. By (ii) of Lemma 2.3 and Claim 1, the graph G'' is 2-connected. It follows from Lemma 2.4 that $c_{uv}(G'') \leq c_{uv}(G) < k$. Further, by (ii) of Lemma 2.5, it holds that

$$N_s(G'') \ge N_s(G) > g_s(n,k).$$

Then $\ell(G'') \ge d_{G''}(u) > d_G(u) = \ell(G)$ because $uu_2 \notin E(G)$, which contradicts the maximality of $\ell(G)$. This completes the proof of Claim 2.

By Claim 2, u is adjacent to all other vertices of G. Further, we have the following claim.

Claim 3. The graph G - u has no cycle of length at least k - 1.

Proof of Claim 3. Suppose to the contrary that G - u has a cycle $C = x_1 x_2 \dots x_l x_1$ with $l \ge k - 1$. If $v \in V(C)$, let $v = x_1$ (relabelling x_i 's if necessary). Then $uvx_2 \dots x_l u$ is a cycle of length at least k containing uv in G because u is adjacent to all other vertices of G, a contradiction to $c_{uv}(G) < k$. Now assume $v \notin V(C)$. Since G is 2-connected, the graph G - u is connected. Hence G - u has a path from v to C which is internally disjoint from C. Without loss of generality, assume x_1 is the end-vertex of P on C. Then $uvPx_1x_2 \dots x_lu$ is a cycle of G which has length at least k, a contradiction to $c_{uv}(G) < k$. This completes the proof of Claim 3. By Claim 3 and Theorem 2.2, we have

$$N_s(G-u) \le \psi_s(n-1,k-1) = r\binom{k-2}{s} + \binom{t+1}{s}$$

where (n-1) - 1 = r(k-3) + t and $0 \le t \le k-4$. By Claim 2, it follows that

$$N_s(G) = N_s(G-u) + N_{s-1}(G-u)$$

$$\leq r\binom{k-2}{s} + \binom{t+1}{s} + r\binom{k-2}{s-1} + \binom{t+1}{s-1}$$

$$= r\binom{k-1}{s} + \binom{t+2}{s}$$

$$= g_s(n,k),$$

which yields a desired contradiction to $N_s(G) > g_s(n,k)$. This completes the proof.

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