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BOUNDARY TRACING FOR LAPLACE'S EQUATION WITH CONFORMAL MAPPING*

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Abstract. The conformal mapping technique has long been used to obtain exact solutions to Laplace's equation in two-dimensional domains with awkward geometries. However, a major limitation of the technique is that it is only directly compatible with Dirichlet and zero-flux Neumann boundary conditions. It would be useful to have a means of adapting the technique to handle more general boundary conditions, for example, Robin or nonlinear flux conditions. Boundary tracing is an unconventional method for tackling boundary value problems with generic flux boundary conditions, where one takes a known solution to the field equation and seeks new boundaries satisfying the prescribed boundary condition. In this paper, we adapt boundary tracing for compatibility with conformal mapping to produce a new prescription for studying Laplace's equation coupled with general flux boundary conditions. We illustrate the procedure via two simple examples involving heat transfer. In both cases, we demonstrate how to construct infinite families of nontrivial domains in which the solution to the chosen flux boundary value problem is exactly equal to a selected harmonic function.

Key words. boundary tracing, flux boundary condition, Laplace's equation, conformal mapping, complex plane

MSC codes. 30C35, 31A25, 35A24

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1. Introduction.

Conformal mapping techniques have enabled mathematicians to obtain exact solutions to Laplace's equation in a vast array of two-dimensional (2D) geometries. This procedure works because harmonicity is preserved under a conformal transformation. The Riemann mapping theorem guarantees that any simply-connected 2D domain can be mapped onto any other simply-connected 2D domain (via the unit disc) so that the method powerfully handles the most awkward of geometries. There exists a large catalogue of such transformations, which map between complicated domains of practical interest and simple domains (the unit disc, the half plane, etc.) [6]. Of particular note is the Schwarz–Christoffel transformation, which is useful for handling polygonal geometry.

Now a boundary value problem consists of not only a field equation but also boundary conditions. Unlike Laplace's equation, these boundary conditions may not be formally preserved under a conformal transformation (thus restricting the applicability of the method). Indeed, whilst the potential is mapped across pointwise, displacements will in general be distorted so that the potential *gradient* is not preserved (unless it vanishes). The implication is that only Dirichlet conditions and zero-flux Neumann boundary conditions are automatically compatible with conformal transformation.

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In many practical problems Laplace's equation is coupled with a flux boundary condition that is not preserved under a conformal transformation; examples include steady conduction–convection problems (which have a Robin condition), conduction–radiation (where boundary flux is proportional to the fourth power of temperature) [9], and corrosion modeling [5, 8]. What we would like to have is a simple means of adapting conformal mapping that is compatible with generic flux boundary conditions.

Taking a step back, the perennial approach to solving boundary value problems has been to fix the domain shape and then seek a solution that simultaneously satisfies the field equation and the boundary conditions. An alternative strategy for flux boundary conditions is the method of *boundary tracing*, where one chooses a known solution to the field equation and seeks boundary curves that are consistent with the boundary conditions. These curves are then used to construct new domains that also admit the chosen known solution. In this way, a vast array of nontrivial domains and results of practical interest may be obtained. In the present paper, we unify the methods of conformal mapping and boundary tracing, producing a new approach for investigating boundary value problems involving Laplace's equation and general flux boundary conditions.

In section 2, we first present the boundary tracing technique in the general context, and in particular we determine the ordinary differential equation needed for boundary tracing in the real plane. We then focus on the special case where the field equation is Laplace's equation and derive an equivalent ordinary differential equation for boundary tracing in the complex plane; this lends itself to a unification with the method of conformal mapping. In section 3 we illustrate the application of the procedure by examining two very simple steady-state heat conduction problems involving awkward geometry and flux boundary conditions. Finally we conclude in section 4 with an assessment of the usefulness of coupling the approaches of boundary tracing and conformal mapping.

2. Boundary tracing.

The method of boundary tracing was first recognized as a formal technique by Anderson [1] and was then explored in a series of articles which presented examples of its use for classical partial differential equations, both linear and nonlinear [2, 3, 4]. Most notably new, exact, and practically useful results were obtained for the Laplace–Young equation coupled with a contact condition, which is an important yet difficult problem in the field of capillarity. The boundary tracing procedure is elementary and can be applied without extensive knowledge of the underlying theoretical framework developed by Anderson, Bassom, and Fowkes [3]. We begin with a summary description of boundary tracing in the general context (section 2.1) before tying it together with conformal mapping in the special case where the field equation is Laplace's equation (section 2.2).

2.1. Boundary tracing in the general context.

Suppose we have a boundary value problem consisting of some field equation together with a flux boundary condition of the very general form

$$\mathbf{n} \cdot \boldsymbol{\nabla} T = F(x, y, T, \|\boldsymbol{\nabla} T\|). \tag{2.1}$$

To perform boundary tracing, we choose some known solution $T = T(x, y)$ of the field equation and seek *traced boundaries*, which are curves along which the boundary condition (2.1) is satisfied. Once the traced boundaries have been found, they may be patched together to construct new domains which also admit the chosen known solution.

The determination of the traced boundaries turns out to be straightforward. Using the parametrization $y = y(x)$, we have

$$\mathbf{n} = \frac{-\,\mathrm{d}y\,\mathbf{a}_x + \mathrm{d}x\,\mathbf{a}_y}{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}} \quad \text{and} \quad \boldsymbol{\nabla} T = T_x\,\mathbf{a}_x + T_y\,\mathbf{a}_y, \tag{2.2}$$

where $\mathbf{a}_x$ and $\mathbf{a}_y$ are the Cartesian basis vectors, and the subscripts on T denote partial differentiation. (The precise sign of the normal vector is irrelevant here; flipping it does not affect the traced boundaries and merely swaps the sides that are designated as interior and exterior.) The boundary condition (2.1) becomes a quadratic in $\mathrm{d}y/\mathrm{d}x$, which solves to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{T_x T_y \pm F\sqrt{{T_x}^2 + {T_y}^2 - F^2}}{{T_x}^2 - F^2}. \tag{2.3}$$

The right-hand side is a known function of x and y (for T is chosen and F is prescribed); thus boundary tracing simply amounts to solving an ordinary differential equation. The quantity under the square root sign is of significance, and we refer to it as the *viability function*:

$$\varPhi = (\boldsymbol{\nabla} T)^2 - F^2. \tag{2.4}$$

Note that real traced boundaries can only exist in the region $\varPhi \geq 0$, which we call the *viable domain*; in physical terms this is the region in which the known solution T is steep enough to satisfy the flux condition (2.1).

2.2. Boundary tracing for Laplace's equation. If the field equation happens to be Laplace's equation

$$\boldsymbol{\nabla}^2 T = 0, \tag{2.5}$$

then it is possible to combine the methods of boundary tracing and conformal mapping by first recasting the traced boundaries as solutions to an ordinary differential equation in the complex variable $z = x + \mathrm{i}y$. In this context, the arc-length parametrization $x = x(s)$, $y = y(s)$ best describes the traced boundaries. The flux boundary condition (2.1) becomes

$$T_x \frac{\mathrm{d}y}{\mathrm{d}s} - T_y \frac{\mathrm{d}x}{\mathrm{d}s} = F, \tag{2.6}$$

and using the arc length definition

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 1, \tag{2.7}$$

we obtain the system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{-T_y F \pm T_x\sqrt{\varPhi}}{(\boldsymbol{\nabla} T)^2}, \tag{2.8}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{+T_x F \pm T_y\sqrt{\varPhi}}{(\boldsymbol{\nabla} T)^2} \tag{2.9}$$

for the traced boundaries. Combining these we get

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\mathrm{d}(x+\mathrm{i}y)}{\mathrm{d}s} = \frac{(-T_y + \mathrm{i}T_x)F \pm (T_x + \mathrm{i}T_y)\sqrt{\Phi}}{(\boldsymbol{\nabla} T)^2}. \tag{2.10}$$

Complex function theory tells us that T is the real part of some analytic function, i.e.,

$$W(z) = T(x,y) + \mathrm{i}V(x,y). \tag{2.11}$$

From the Cauchy–Riemann equations, we obtain

$$-T_y + \mathrm{i}T_x = \mathrm{i}\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^*, \tag{2.12}$$

$$T_x + \mathrm{i}T_y = \left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^*, \tag{2.13}$$

and

$$(\boldsymbol{\nabla} T)^2 = \left|\frac{\mathrm{d}W}{\mathrm{d}z}\right|^2 = \left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)\left(\frac{\mathrm{d}W}{\mathrm{d}z}\right)^*, \tag{2.14}$$

where asterisks denote complex conjugation. Using these, we may eliminate $T(x,y)$ in favor of $W(z)$ in (2.10), obtaining the highly terse

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\mathrm{i}F \pm \sqrt{\Phi}}{\mathrm{d}W/\mathrm{d}z} \tag{2.15}$$

for traced boundaries in the complex plane.

At this point we are ready to bring in the method of conformal mapping. Suppose that we have $z = z(\zeta)$ according to some conformal mapping between virtual ζ-space and physical z-space. From the chain rule, we see that the traced boundaries are simply determined by

$$\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \frac{\mathrm{i}F \pm \sqrt{\Phi}}{\mathrm{d}W/\mathrm{d}\zeta} \tag{2.16}$$

in ζ-space. This form is strikingly similar to (2.15), and it is remarkable that the encoding of a traced boundary (which represents a flux boundary condition) in the complex plane is so compatible with conformal transformation. This completes the unification of boundary tracing with conformal mapping.

Now suppose that we wish to study a boundary value problem (in z-space) consisting of Laplace's equation in the interior, the flux boundary condition (2.1) on part of the boundary, and certain nonflux boundary conditions on the remainder of the boundary. We apply the following prescription:

1. Choose an analytic function $W = W(\zeta)$ and a conformal mapping $z = z(\zeta)$ corresponding to some $T = T(x,y)$ that satisfies the nonflux boundary conditions.
2. Write the flux function F and the viability function Φ in terms of ζ.
3. Compute traced boundaries in virtual ζ-space using (2.16).

4. Map the traced boundaries to physical z-space.
5. Use the traced boundaries to construct new domains which admit the solution $T(x, y)$ to the boundary value problem.

As in the usual conformal mapping approach, the idea is to make a transformation that takes complicated geometry in physical z-space to simple geometry in virtual ζ-space; only now, we have an elementary formulation that is compatible with flux boundary conditions.

For the purposes of step 1 of the prescription, we have an abundance of known analytic functions and conformal mappings to choose from; a good choice will lead to a good outcome. After the traced boundaries have been determined, it is usually the case that an infinite number of new domains can then be constructed. However, given the very nature of boundary tracing, we have limited control over the shape of the traced boundaries that will result. The real question is therefore "Are the results obtained interesting and useful?" The reader will be better able to judge by examining the examples below.

3. Examples. The first example is a problem involving mixed boundary conditions (zero value and unit flux). This problem cannot be addressed via the usual conformal mapping approach because the unit-flux boundary condition is not preserved under conformal transformation. Our prescription yields an exact result that might otherwise be obtained by inspection; it also yields a multitude of new domain shapes which admit exactly the same solution to the boundary value problem.

The second example demonstrates the application of our prescription to a heat transfer problem with heat supplied internally along a triangular boundary (represented by a Dirichlet condition) and lost externally via convection (represented by a Robin condition). As in the first example, the flux condition defeats the standard conformal mapping approach.

3.1. Example 1: Isoflux. Suppose we have a boundary value problem consisting of Laplace's equation in the interior, the Dirichlet condition $T = 0$ along $y = \pm x$, and the isoflux condition $\mathbf{n} \cdot \boldsymbol{\nabla} T = 1$ along some (as yet unspecified) boundary, as shown in Figure 1. Physically, this might represent a steady conduction problem for the temperature T in a region bounded by ice along $y = \pm x$ and by a heat-generating electrical wire along the remaining (unspecified) boundary.

Choosing the analytic function $W(\zeta) = \zeta$ together with the transformation

$$z = \sqrt{\zeta} \tag{3.1}$$

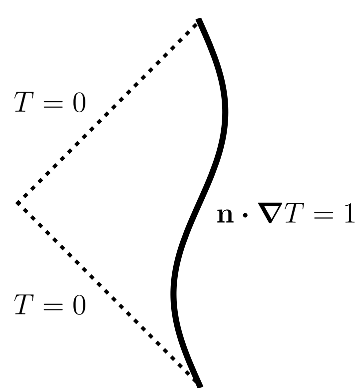
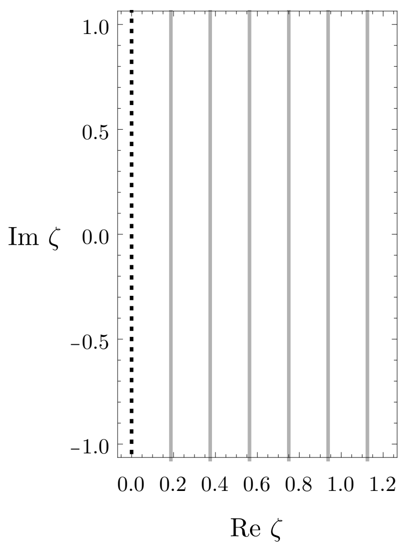
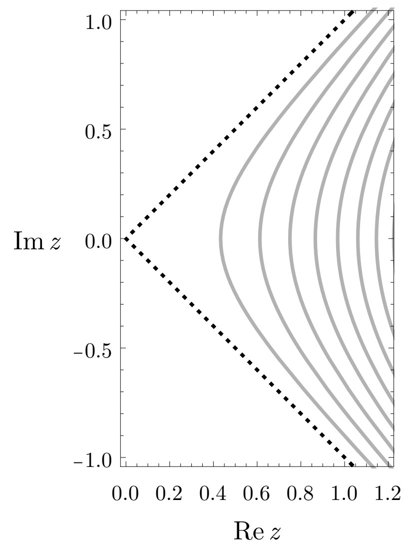


FIG. 1. *A simple isoflux boundary value problem.*



(a) ζ-space



(b) z-space

FIG. 2. *Conformal transformation* (3.1) *applied to lines of constant real part in* ζ*-space.*

fixes our known solution to Laplace's equation as

$$T = x^2 - y^2 \tag{3.2}$$

and takes care of the Dirichlet boundaries $y = \pm x$ (Figure 2). Of course, there are infinitely many other choices of $W(\zeta)$ and $z(\zeta)$ that also satisfy the Dirichlet boundaries; our choice here is a relatively simple one that allows us to illustrate the method.

The remaining isoflux boundary we determine using boundary tracing, and since it will lie in the region $y < |x|$, we restrict our attention to the half plane $\mathrm{Re}\{\zeta\} \geq 0$ in ζ-space.

For the purposes of boundary tracing, the flux function is simply $F = 1$. Since

$$(\boldsymbol{\nabla} T)^2 = \left|\frac{\mathrm{d}W}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right|^2 = \left|1 \cdot 2\sqrt{\zeta}\right|^2 = 4|\zeta|, \tag{3.3}$$

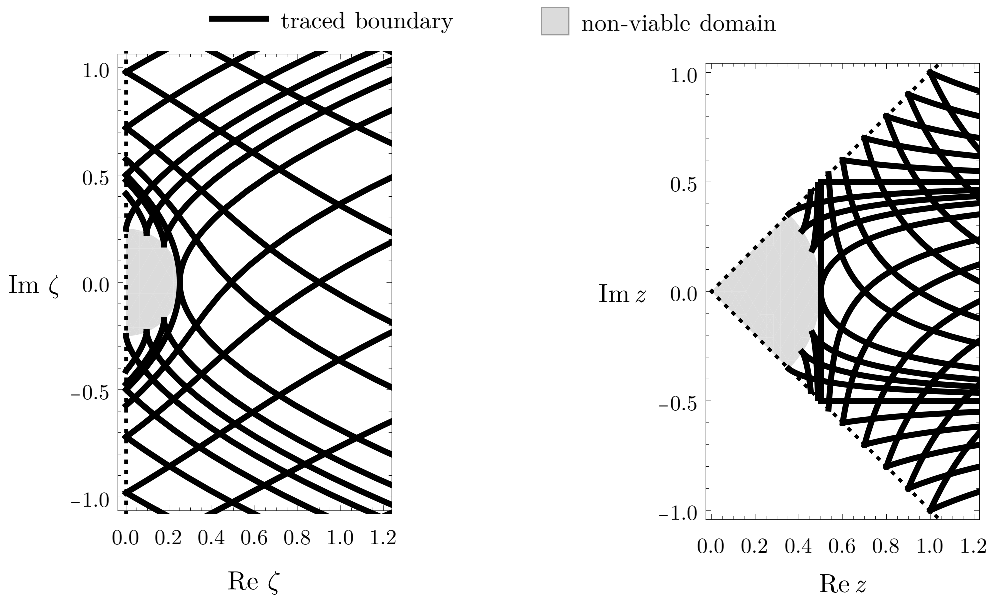
we have the viability function

$$\varPhi = 4|\zeta| - 1. \tag{3.4}$$

The ordinary differential equation (2.16) for traced boundaries in the ζ-plane becomes

$$\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \mathrm{i} \pm \sqrt{4|\zeta| - 1}. \tag{3.5}$$

This may be integrated numerically to any desired accuracy. Recalling that real traced boundaries only exist in the viable domain $\varPhi \geq 0$, we only consider $|\zeta| \geq 1/4$. The resulting traced boundaries in ζ-space along with their images in physical z-space are shown in Figure 3. There are two branches of curves corresponding to the two possible signs for the square root.



(a) ζ-space (b) z-space

FIG. 3. *Traced boundaries (isoflux) obtained by integrating* (3.5) *in ζ-space, then mapping to z-space.*

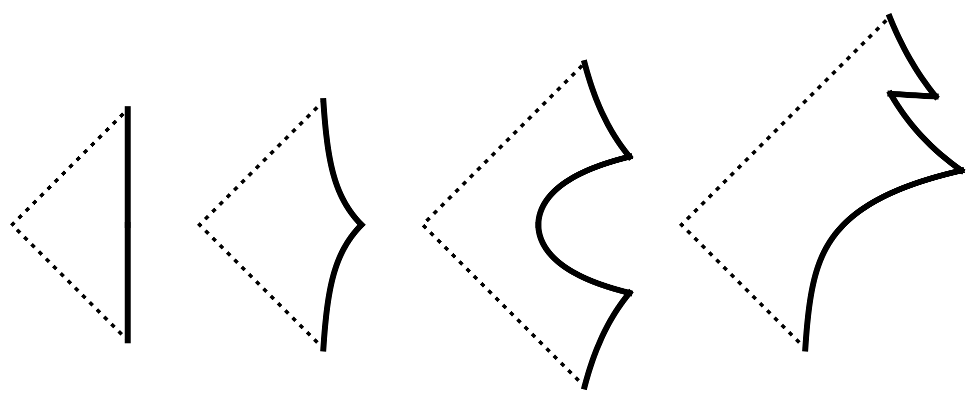


FIG. 4. *Constructed domains (in physical z-space) with zero value along the diagonal boundaries (dotted) and unit flux along the remaining boundary (solid).*

The traced boundaries can be patched together to form isoflux bridges linking the Dirichlet boundaries ($y = \pm x$). The patching may be done in an almost arbitrary manner; the only precaution we must take is to ensure that there is consistent orientation of the outward boundary normal. Given the isoflux condition, this is achieved by identifying as exterior the side on which T is greater. Thus we may construct an *infinite* number of domains (Figure 4) that admit the same exact solution $T = x^2 - y^2$ to our boundary value problem.

Taking another look at Figure 3(b), specifically at the traced boundaries through $(x, y) = (1/2, 0)$, we see what looks like a vertical line. A local asymptotic analysis at $\zeta = 1/4$ shows that we have

$$\zeta(s) = \begin{cases} (1/4 - s^2) + \mathrm{i}s, & s < 0, \\ (1/4 + 2s^2 - 24/7s^4 + \cdots) + \mathrm{i}s, & s > 0 \end{cases} \tag{3.6}$$

for the positive-square-root branch of (3.5), and a reflected version of this for the negative-square-root branch. Of course $\zeta = (1/4 - s^2) + \mathrm{i}s$ corresponds to $z = 1/2 + \mathrm{i}s$, and we indeed have a vertical line $x = 1/2$ as suspected. Thus, we have a somewhat surprising result (perhaps obvious to the astute reader): the boundary value problem with unit flux along $x = 1/2$, zero value along $y = \pm x$, and Laplace's equation in the interior of the resulting triangle admits the *exact* solution $x^2 - y^2$.

While it is certainly interesting that boundary tracing can produce infinite families of new domains (Figure 4), one might question the usefulness of having a fixed solution (here $T = x^2 - y^2$). However, as noted earlier, we could have made different choices for $W(\zeta)$ and $z(\zeta)$ in keeping with the zero-value requirement along the Dirichlet boundaries $y = \pm x$. Any sensible choice would lead to a further infinite family of new domains, all admitting a different solution $T(x, y)$ to the boundary value problem.

3.2. Example 2: Convection. Consider a simple conduction problem, where heat is lost to the environment by convection along some external boundary and supplied along an internal triangular boundary held at constant temperature (Figure 5). Suppose after scaling that these boundary conditions are given by the Robin condition

$$\mathbf{n} \cdot \boldsymbol{\nabla} T = -\frac{T}{A_0} \tag{3.7}$$

and the Dirichlet condition

$$T = B_0, \tag{3.8}$$

respectively, where $A_0 = 1.5$, $B_0 = 1.6$, and the vertices of the triangular boundary are the three cubic roots of unity in the complex plane.

We choose the Schwarz–Christoffel transformation $z = z(\zeta)$ given by

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = \frac{C_0}{\zeta^2}\left(1 - \zeta^3\right)^{2/3}, \tag{3.9}$$

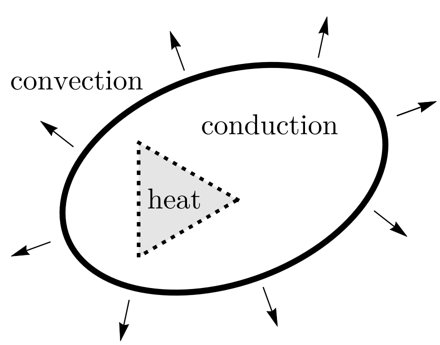


FIG. 5. *Conduction–convection problem with internal heat generation from a triangular boundary held at constant temperature.*

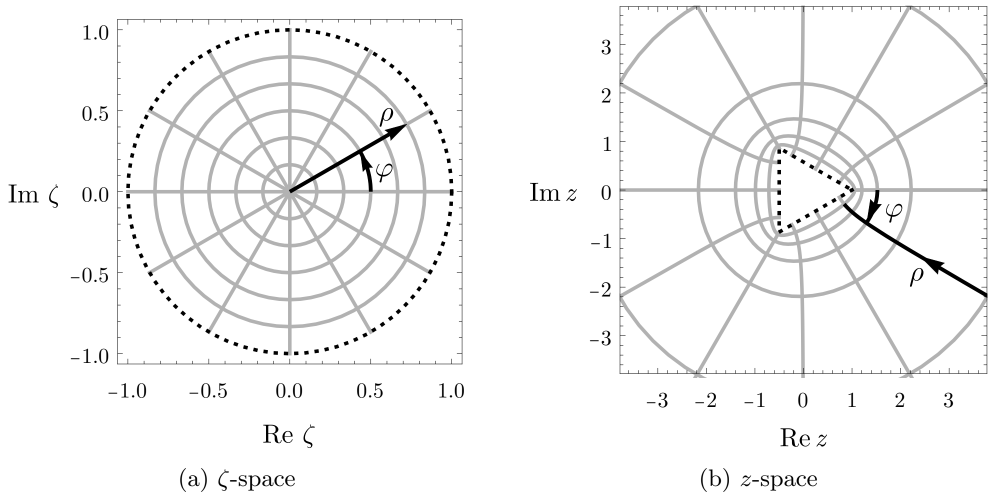


Fig. 6. *Schwarz–Christoffel transformation* (3.9) *applied to a polar grid in ζ-space (curves of constant ρ and constant φ). Note the different scales.*

which maps the interior of the unit disc in ζ-space to the exterior of an equilateral triangle in z-space (Figure 6). Integration yields

$$z(\zeta) = -\frac{C_0}{\zeta}\,{}_2\mathrm{F}_1\big(-\tfrac{2}{3}, -\tfrac{1}{3}; \tfrac{2}{3}, \zeta^3\big), \tag{3.10}$$

and we choose

$$C_0 = \frac{-1}{{}_2\mathrm{F}_1\big(-\tfrac{2}{3}, -\tfrac{1}{3}; \tfrac{2}{3}, 1\big)} = -0.730499 \tag{3.11}$$

to ensure that $\zeta = 1$ is mapped to $z = 1$. Writing (ρ, φ) for polar coordinates in ζ-space (so that $\zeta = \rho \mathrm{e}^{\mathrm{i}\varphi}$), we see that the unit circle $|\zeta| = \rho = 1$ is mapped to the desired triangular boundary in z-space. Selecting the analytic function

$$W(\zeta) = \log\left(\frac{\zeta}{\rho_0}\right), \tag{3.12}$$

where

$$\rho_0 = \exp(-B_0) = 0.201897, \tag{3.13}$$

we obtain

$$T = \mathrm{Re}\{W\} = \log\left(\frac{|\zeta|}{\rho_0}\right) = \log\left(\frac{\rho}{\rho_0}\right), \tag{3.14}$$

a harmonic function that satisfies the Dirichlet condition (3.8) along the triangular boundary in z-space (Figure 7). Note that the T-contours coincide with the curves of constant ρ.

The right-hand side of the boundary condition (3.7) is our flux function,

$$F = -\frac{T}{A_0} = -\frac{1}{A_0}\log\left(\frac{|\zeta|}{\rho_0}\right), \tag{3.15}$$

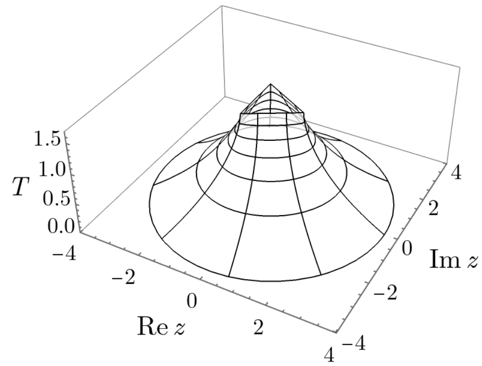


FIG. 7. *Solution* (3.14)*, mapped to* zT*-space.*

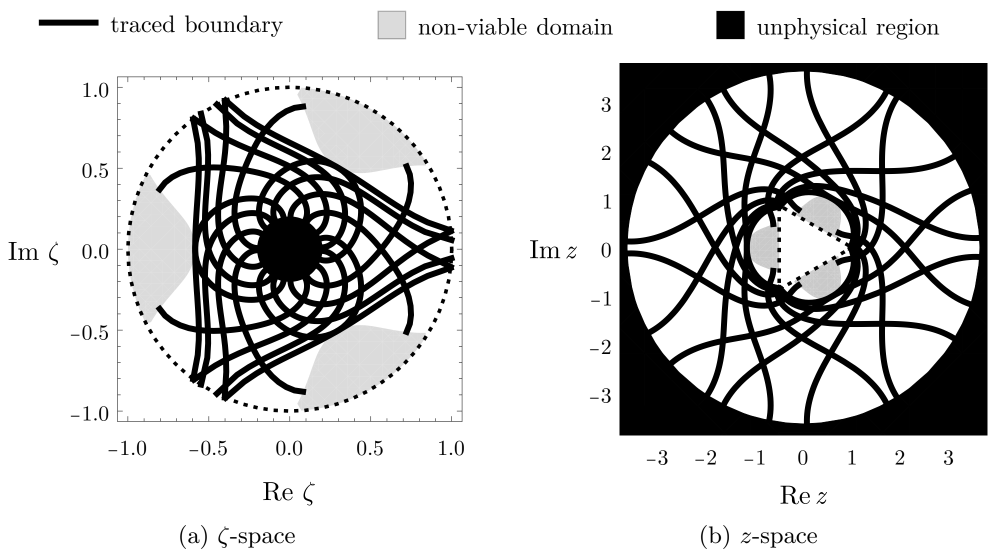


FIG. 8. *Traced boundaries (convection) obtained by integrating* (2.16) *in* ζ*-space, then mapping to* z*-space. The region* $|\zeta| < \rho_0$ *is unphysical, since* T *is negative within it.*

and from (2.14) we see that

$$(\nabla T)^2 = \left|\frac{\mathrm{d}W}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right|^2 = \left|\frac{\zeta}{C_0(1-\zeta^3)^{2/3}}\right|^2. \tag{3.16}$$

Having written F and $(\nabla T)^2$ (and hence $\Phi = (\nabla T)^2 - F^2$) as functions of ζ, we may then integrate (2.16) to obtain traced boundaries in ζ-space (Figure 8(a)). Applying the transformation (3.10) maps them to physical z-space (Figure 8(b)), where domains can be constructed by patching boundaries together as before.

Some interesting examples are shown in Figure 9. Since the temperature field in each constructed domain is the harmonic function (3.14), the power (per unit length out of the page) supplied along the inner triangular boundary is the same in all cases. Thus boundary tracing lets us generate different convective configurations

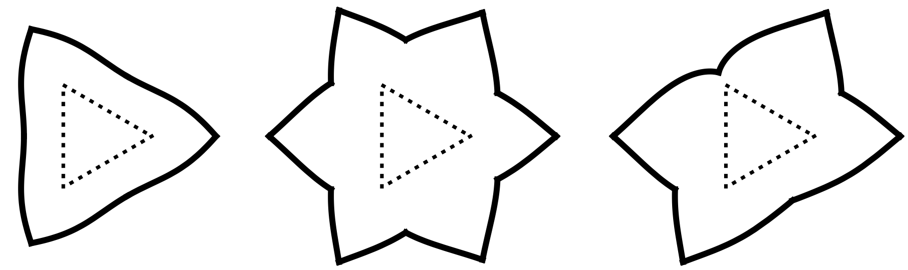


Fig. 9. *Constructed domains with constant temperature along the internal triangular boundary (dotted) and convection along the external boundary (solid).*

that all provide the same amount of heat dissipation. Moreover, the configurations in Figure 9 are not merely perturbed versions of each other, but they exhibit a genuine structural diversity; we have (roughly) a triangle, a star, and a butterfly. Note that these domains are doubly connected and that a possible strategy for extending to multiply-connected domains (corresponding to multiple internal heat sources) is as follows:

1. Choose a conformal mapping that makes T constant along multiple closed curves, which will be the multiple internal heat sources.
2. Determine the traced boundaries.
3. Use the traced boundaries to construct a curve (with correct orientation of the outward normal) that encloses the multiple internal heat sources.

4. Conclusions. The above examples show how boundary tracing can be used to extend the range of boundary conditions that can be addressed in a Laplace equation context using conformal mapping. Whilst the flux boundary conditions considered were relatively simple (namely, a unit-flux condition in Example 1 and a linear Robin condition in Example 2), nothing in the theory of boundary tracing precludes nonlinear flux conditions, and (2.1) remains the most general flux boundary condition that can be used.

Although the very nature of the boundary tracing approach means that domain shape cannot be specified a priori, the strength of the method lies in its simplicity. Boundaries are determined by straight integration of a first-order ordinary differential equation, and one obtains two branches of curves that can be patched together to form an infinite family of new domains (which all admit the same field solution). In particular, we have seen in the convection example how the resulting domains can be greatly varied in shape. Given this, one might envisage the use of such an approach in modeling of diffusive biological processes, where structural features such as folds and sacs are important.

The standard conformal mapping approach powerfully handles complicated geometry, and we may think of it as providing an abundant supply of known exact solutions to Laplace's equation. The limiting feature of the method of conformal mapping is its inability to deal with generic flux boundary conditions. Boundary tracing complements this, being a method designed for flux boundary conditions that also utilizes known exact solutions to a field equation. Our prescription unifies the two methods and allows us to use conformal mapping for a wider variety of boundary conditions than before.

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REFERENCES

[1] M. L. Anderson, *Boundary Tracing Methods for Partial Differential Equations*, Ph.D. thesis, University of Western Australia, 2002, https://doi.org/10.26182/5d4788bbf2416.

[2] M. L. Anderson, A. P. Bassom, and N. Fowkes, *Exact solutions of the Laplace–Young equation*, Proc. Roy. Soc. A, 462 (2006), pp. 3645–3656, https://doi.org/10.1098/rspa.2006.1744.

[3] M. L. Anderson, A. P. Bassom, and N. Fowkes, *Boundary tracing and boundary value problems:* I. *Theory*, Proc. Roy. Soc. A, 463 (2007), pp. 1909–1924, https://doi.org/10.1098/rspa.2007.1858.

[4] M. L. Anderson, A. P. Bassom, and N. Fowkes, *Boundary tracing and boundary value problems:* II. *Applications*, Proc. Roy. Soc. A, 463 (2007), pp. 1925–1938, https://doi.org/10.1098/rspa.2007.1859.

[5] K. Bryan and M. Vogelius, *Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling*, Quart. Appl. Math., 60 (2002), pp. 675–694, https://doi.org/10.1090/qam/1939006.

[6] H. Kober, *Dictionary of Conformal Representations*, Dover, New York, 1957.

[7] C. Li, *Applications of Boundary Tracing in Thermal Radiation and Capillarity*, Ph.D. thesis, University of Western Australia, 2021, https://doi.org/10.26182/f2wk-c635.

[8] M. Vogelius and J.-M. Xu, *A nonlinear elliptic boundary value problem related to corrosion modeling*, Quart. Appl. Math., 56 (1998), pp. 479–505, https://doi.org/10.1090/qam/1637048.

[9] J. E. Wilkins, Jr., *Minimum-mass thin fins which transfer heat only by radiation to surroundings at absolute zero*, J. Soc. Indust. Appl. Math., 8 (1960), pp. 630–639, https://doi.org/10.1137/0108047.