# ON VIETORIS-RIPS COMPLEXES (WITH SCALE 3) OF HYPERCUBE GRAPHS 

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#### Abstract

For a metric space $(X, d)$ and a scale parameter $r \geq 0$, the Vietoris-Rips complex $\mathcal{V} \mathcal{R}(X ; r)$ is a simplicial complex on vertex set $X$, where a finite set $\sigma \subseteq X$ is a simplex if and only if the diameter of $\sigma$ is at most $r$. For $n \geq 1$, let $\mathbb{I}_{n}$ denote the $n$-dimensional hypercube graph. In this paper, we show that $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; r\right)$ has non trivial reduced homology only in dimensions 4 and 7 . Therefore, we answer a question posed by Adamaszek and Adams recently.

A (finite) simplicial complex $\Delta$ is $d$-collapsible if it can be reduced to the void complex by repeatedly removing a face of size at most $d$ that is contained in a unique maximal face of $\Delta$. The collapsibility number of $\Delta$ is the minimum integer $d$ such that $\Delta$ is $d$-collapsible. We show that the collapsibility number of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; r\right)$ is $2^{r}$ for $r \in\{2,3\}$.


## 1. Introduction

For a metric space $(X, d)$ and a scale parameter $r \geq 0$, the Vietoris-Rips complex $\mathcal{V} \mathcal{R}(X ; r)$ is a simplicial complex on vertex set $X$, where a finite set $\sigma \subseteq X$ is a simplex if and only if the diameter of $\sigma$ is at most $r$, i.e., $\mathcal{V} \mathcal{R}(X ; r)=\{\sigma \subseteq X:|\sigma|<\infty$ and $d(x, y) \leq r \forall x, y \in \sigma\}$; here $|\cdot|$ denotes the cardinality of a set. The Vietoris-Rips complex was first introduced by Vietoris [38] to define a homology theory for metric spaces and independently re-introduced by E. Rips for studying hyperbolic groups, where it has been popularised as Rips-complex [26, 27]. The idea behind introducing these complexes was to create a finite simplicial model for metric spaces. The Vietoris-Rips complex and its homology have become an important tools in the applications of algebraic topology. In topological data analysis, it has been used to analyse data with persistent homology [10, 14, 42, 43]. These complexes have been used heavily in computational topology, as a simplicial model for point-cloud data [15, 16, 17, 20] and as simplicial completions of communication links in sensor networks [21, 22, 37]. For more on these complexes, the interested reader is referred to $[1,2,4,5,6,7,18,25,34,39,40]$.

Consider any graph $G$ as a metric space, where the distance between any two vertices is the length of a shortest path between them. The study of Vietoris-Rips complexes of hypercube graphs was initiated by Adamaszek and Adams in [3]. These questions on hypercubes arose from work by Kevin Emmett, Raúl Rabadán, and Daniel Rosenbloom related to the persistent homology formed from genetic trees, reticulate evolution, and medial recombination [23, 24].

For a positive integer $n$, let $\mathbb{I}_{n}$ denote the $n$-dimensional hypercube graph (see Definition 2.2). In [3], Adamaszek and Adams proved that $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is homotopy equivalent to a wedge sum of spheres of dimension 3. By using a computer calculation they proved the following.
Proposition 1.1. [3] Let $5 \leq n \leq 7$ and $0 \leq i \leq 7$. Then $\tilde{H}_{i}\left(\mathcal{V R}\left(\mathbb{I}_{n} ; 3\right) ; \mathbb{Z}\right) \neq 0$ if and only if $i \in\{4,7\}$.

Further, they asked, in what homological dimensions do the Vietoris-Rips complexes $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$ have nontrivial reduced homology? It is easy to check that the complexes $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$ are contractible for $1 \leq n \leq 3$ and $\mathcal{V R}\left(\mathbb{I}_{4} ; 3\right) \simeq S^{7}$. In this paper we prove the following.
Theorem A. Let $n \geq 5$. Then $\tilde{H}_{i}\left(\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right) ; \mathbb{Z}\right) \neq 0$ if and only if $i \in\{4,7\}$.
Let $\Delta$ be a (finite) simplicial complex. Let $\gamma \in \Delta$ such that $|\gamma| \leq d$ and $\sigma \in \Delta$ is the only maximal simplex that contains $\gamma$. An elementary $d$-collapse of $\Delta$ is the simplicial complex $\Delta^{\prime}$

[^0]obtained from $\Delta$ by removing all those simplices $\tau$ of $\Delta$ such that $\gamma \subseteq \tau \subseteq \sigma$, and we denote this elementary $d$-collapse by $\Delta \xrightarrow{\gamma} \Delta^{\prime}$.

The complex $\Delta$ is called $d$-collapsible if there exists a sequence of elementary $d$-collapses

$$
\Delta=\Delta_{1} \xrightarrow{\gamma_{1}} \Delta_{2} \xrightarrow{\gamma_{2}} \cdots \xrightarrow{\gamma_{k-1}} \Delta_{k}=\emptyset
$$

from $\Delta$ to the void complex $\emptyset$. Clearly, if $\Delta$ is $d$-collapsible and $d<c$, then $\Delta$ is $c$-collapsible. The collapsibility number of $\Delta$ is the minimal integer $d$ such that $\Delta$ is $d$-collapsible.

The notion of $d$-collapsibility of simplicial complexes was introduced by Wegner [41]. In combinatorial topology it is an important problem to determine the collapsibility number or bounds for the collapsibility number of a simplicial complex and it has been widely studied (see $[12,19,31,35,36])$. A simple consequence of $d$-collapsibility is the following:
Proposition 1.2. [41] If $X$ is d-collapsible then it is homotopy equivalent to a simplicial complex of dimension smaller than $d$.

Recently, Bigdeli and Faridi gave a connection between $d$-collapsibility and the chordal complexes; and proved that $d$-collapsibility is equivalent to the chordality of the Stanley-Reisner complexes of certain ideals [11]. For applications regarding Helly-type theorems, see [9, 29, 30]. One of the consequences of the topological colorful Helly theorem [30, Theorem 2.1] is the following.

Proposition 1.3. [31, Theorem 1.1] Let $X$ be a d-collapsible simplicial complex on vertex set $V$, and let $X^{c}=\{\sigma \subseteq V: \sigma \notin X\}$. Then, every collection of $d+1$ sets in $X^{c}$ has a rainbow set belonging to $X^{c}$.
In this paper we prove the following.
Theorem B. For $n \geq 3$, the collapsibility number of $\mathcal{V R}\left(\mathbb{I}_{n} ; 2\right)$ is 4 .
Theorem C. For $n \geq 4$, the collapsibility number of $\mathcal{V R}\left(\mathbb{I}_{n} ; 3\right)$ is 8 .
Flow of the paper: In the following Section, we list out various definitions on graph theory and simplicial complexes that are used in this paper. We also fix a few notations, which we use throughout this paper. In Section 3, we consider the complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ and prove Theorem B. Section 4 is devoted to the complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$ and divided into three subsections. In Section 4.1, we give a characterization of maximal simplices of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$. In Section 4.2, we prove Theorem C. Finally in Section 4.3, we prove Theorem A. In the last section, we posed a few conjectures and a question that arise naturally from the work done in this paper.

## 2. Preliminaries and Notations

A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of $G$ and $E(G) \subseteq\binom{V(G)}{2}$ denotes the set of edges. If $(x, y) \in E(G)$, it is also denoted by $x \sim y$ and we say that $x$ is adjacent to $y$. A subgraph $H$ of $G$ is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $U \subseteq V(G)$, the induced subgraph $G[U]$ is the subgraph whose set of vertices is $V(G[U])=U$ and whose set of edges is $E(G[U])=\{(a, b) \in E(G) \mid a, b \in U\}$.

A graph homomorphism from $G$ to $H$ is a function $\phi: V(G) \rightarrow V(H)$ such that, $(v, w) \in$ $E(G) \Longrightarrow(\phi(v), \phi(w)) \in E(H)$. A graph homomorphism $f$ is called an isomorphism if $f$ is bijective and $f^{-1}$ is also a graph homomorphism. Two graphs are called isomorphic, if there exists an isomorphism between them. If $G$ and $H$ are isomorphic, we write $G \cong H$.

Let $G$ be a graph and $v$ be a vertex of $G$. The neighbourhood of $v$ is defined as $N_{G}(v)=$ $\{w \in V(G) \mid(v, w) \in E(G)\}$ and the closed neighbourhood $N_{G}[v]=N_{G}(v) \cup\{v\}$.
Let $x$ and $y$ be two distinct vertices of $G$. A $x y$-path is a sequence $x v_{0} \ldots v_{n} y$ of vertices of $G$ such that $x \sim v_{0}, v_{n} \sim y$ and $v_{i} \sim v_{i+1}$ for all $0 \leq i \leq n-1$. The length of a $x y$-path is the number of edges appearing in the path. The distance between $x$ and $y$ is the length of a shortest path (with respect to length) among all $x y$-paths and it is denoted by $d(x, y)$. Clearly, if $(x, y) \in E(G)$, then $d(x, y)=1$. By convention, $d(v, v)=0$ for all $v \in V(G)$.

Definition 2.1. A (finite) abstract simplicial complex $X$ is a collection of finite sets such that if $\tau \in X$ and $\sigma \subset \tau$, then $\sigma \in X$.

The elements of $X$ are called simplices or faces of $X$. The dimension of a simplex $\sigma$ is equal to $|\sigma|-1$. The dimension of an abstract simplicial complex is the maximum of the dimensions of its simplices. The 0 -dimensional simplices are called vertices of $X$. If $\sigma \subset \tau$, we say that $\sigma$ is a face of $\tau$. If a simplex has dimension $k$, it is said to be $k$-dimensional or $k$-simplex. The boundary of a $k$-simplex $\sigma$ is the simplicial complex, consisting of all faces of $\sigma$ of dimension $\leq k-1$ and it is denoted by $B d(\sigma)$. A simplex which is not a face of any other simplex is called a maximal simplex or facet. The set of maximal simplices of $X$ is denoted by $M(X)$.

The join of two simplicial complexes $\mathcal{K}_{1}$ and $\mathcal{K}_{2}$, denoted as $\mathcal{K}_{1} * \mathcal{K}_{2}$, is a simplicial complex whose simplices are disjoint union of simplices of $\mathcal{K}_{1}$ and $\mathcal{K}_{2}$. Let $\Delta^{S}$ denotes a $(|S|-1)$ dimensional simplex with vertex set $S$. The cone on $\mathcal{K}$ with apex $a$, denoted as $C_{a}(\mathcal{K})$, is defined as

$$
C_{a}(\mathcal{K}):=\mathcal{K} * \Delta^{\{a\}} .
$$

In this article, we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to book [33] by Kozlov. For terminologies of algebraic topology used in this article, we refer to [28].

Let $X$ be a simplicial complex and $\tau, \sigma \in X$ such that $\sigma \subsetneq \tau$ and $\tau$ is the only maximal simplex in $X$ that contains $\sigma$. A simplicial collapse of $X$ is the simplicial complex $Y$ obtained from $X$ by removing all those simplices $\gamma$ of $X$ such that $\sigma \subseteq \gamma \subseteq \tau$. Here, $\sigma$ is called a free face of $\tau$ and $(\sigma, \tau)$ is called a collapsible pair. We denote this collapse by $X \searrow Y$. In particular, if $X \searrow Y$, then $X \simeq Y$.

Definition 2.2. For a positive integer $n$, the $n$-dimensional Hypercube graph, denoted by $\mathbb{I}_{n}$, is a graph whose vertex set $V\left(\mathbb{I}_{n}\right)=\left\{x_{1} \ldots x_{n}: x_{i} \in\{0,1\} \forall 1 \leq i \leq n\right\}$ and any two vertices $x_{1} \ldots x_{n}$ and $y_{1} \ldots y_{n}$ are adjacent if and only if $\sum_{i=1}^{n}\left|x_{i}-y_{i}\right|=1$, i.e., they are differ at exactly in one position (see Figure 2.1).

(a) $\mathbb{I}_{2}$

(b) $\mathbb{I}_{3}$

Figure 2.1
We now fix a few notations, which we use throughout this paper. For a positive integer $n$, we denote the set $\{1, \ldots, n\}$ by $[n]$. Let $v=v_{1} \ldots v_{n} \in V\left(\mathbb{I}_{n}\right)$. For any $i \in[n]$, we let $v(i)=v_{i}$. For $\left\{i_{1}, i_{2}, \ldots, i_{k}\right\} \subseteq[n]$, we let $v^{i_{1}, \ldots, i_{k}} \in V\left(\mathbb{I}_{n}\right)$ be defined by

$$
v^{i_{1}, \ldots, i_{k}}(j)= \begin{cases}v(j) & \text { if } j \notin\left\{i_{1}, \ldots, i_{k}\right\}, \\ \{0,1\} \backslash\{v(j)\} & \text { if } j \in\left\{i_{1}, \ldots, i_{k}\right\} .\end{cases}
$$

Observe that for any two vertices $v, w \in V\left(\mathbb{I}_{n}\right), d(v, w)=\sum_{i=1}^{n}|v(i)-w(i)|$ and $d(v, w)=k$ if and only if $w=v^{i_{1}, \ldots, i_{k}}$ for some $i_{1}, \ldots, i_{k} \in[n]$. Clearly, $N_{\mathbb{I}_{n}}(v)=\left\{v^{i}: i \in[n]\right\}$. For $i, j, k \in[n]$, we let $K_{v}^{i, j, k}:=\left\{v, v^{i, j}, v^{j, k}, v^{i, k}\right\}$. For the simplicity of notation, we write $N(v)$ and $N[v]$ for the sets $N_{\mathbb{I}_{n}}(v)$ and $N_{\mathbb{I}_{n}}[v]$ respectively.

Remark 1. The vertices of $\mathbb{I}_{n}$ can be consider as subsets of $[n]$, where the element $v_{1} \ldots v_{n} \in$ $V\left(\mathbb{I}_{n}\right)$ correspond to the set $\left\{i: v_{i}=1\right\}$. Hence the distance between any two vertices of $\mathbb{I}_{n}$ is same as the cardinality of the symmetric difference of corresponding sets.

Using this notations, any $\sigma \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; r\right)$ is a set where the symmetric difference between any two elements in $\sigma$ is at most $r$.

## 3. The complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$

In this section, we prove Theorem B. We first characterise the maximal simplices of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$.
Lemma 3.1. Let $n \geq 3$ and $\sigma$ be a maximal simplex of $\mathcal{V R}\left(\mathbb{I}_{n} ; 2\right)$. Then one of the following is true:
(i) $\sigma=N[v]$ for some $v \in V\left(\mathbb{I}_{n}\right)$.
(ii) $\sigma=\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}}\right\}$ for some $v \in V\left(\mathbb{I}_{n}\right)$ and $i_{0}, j_{0} \in[n]$.
(ii) $\sigma=K_{v}^{i_{0}, j_{0}, k_{0}}$ for some $v \in V\left(\mathbb{I}_{n}\right)$ and $i_{0}, j_{0}, k_{0} \in[n]$.

Proof. We consider the following cases.
Case 1. There exists a $w \in \sigma$ such that $N(w) \cap \sigma \neq \emptyset$.
Let us first assume that there exists a vertex $w \in \sigma$ such that $N(w) \cap \sigma=\{v\}$. Since $w \in N(v), w=v^{p_{0}}$ for some $p_{0} \in[n]$. We show that $\sigma=N[v]$. Suppose there exists $l_{0} \in[n]$ such that $v^{l_{0}} \notin \sigma$. Since $\sigma$ is maximal, there exists $x \in \sigma$ such that $d\left(x, v^{l_{0}}\right) \geq 3$. Further, since $v \in \sigma, d(v, x) \leq 2$. For any $t \in[n] \backslash\left\{l_{0}\right\}$, since $d\left(v^{l_{0}}, v^{t}\right)=2$, we see that $x \neq v^{t}$ and therefore $d(v, x)=2$. Hence $x=v^{i, j}$ for some $i, j \in[n]$. Since $N(w) \cap \sigma=\{v\}, p_{0} \notin\{i, j\}$. But then $d(x, w)=3$, a contradiction. Thus $N(v) \subseteq \sigma$. Since $v \in \sigma$, we see that $N[v] \subseteq \sigma$. Suppose there exists $y \in \sigma$ such that $y \notin N[v]$. Then $d(v, y)=2$ and therefore $y=v^{s, t}$ for some $s, t \in[n]$. Choose $k \in[n] \backslash\{s, t\}$. Then $d\left(v^{k}, y\right)=3$, a contradiction as $v^{k} \in \sigma$. Hence $\sigma=N[v]$. Thus $\sigma$ is of the type ( $i$ ).

Now assume that $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$. Let $u \in \sigma$. There exists $i_{0}, j_{0} \in[n]$ such that $u^{i_{0}}, u^{j_{0}} \in \sigma$. Thus $\left\{u, u^{i_{0}}, u^{j_{0}}\right\} \subseteq \sigma$. Since $\left|N\left(u^{i_{0}}\right) \cap \sigma\right| \geq 2$ and $u^{i_{0}} \nsim u^{j_{0}}$, there exists $z \in \sigma \backslash\{u\}$ such that $z \sim u^{i_{0}}$. Then $z=u^{i_{0}, k}$ for some $k \in[n] \backslash\left\{i_{0}\right\}$. Since $u^{j_{0}} \in \sigma$, $d\left(u^{j_{0}}, y\right) \leq 2$, thereby implying that $k=j_{0}$. Thus $\left\{u, u^{i_{0}}, u^{j_{0}}, u^{i_{0}, j_{0}}\right\} \subseteq \sigma$. Suppose there exists $q \in \sigma \backslash\left\{u, u^{i_{0}}, u^{j_{0}}, u^{i_{0}, j_{0}}\right\}$. If $q \sim u$, then $q=u^{i}$ for some $i \in[n] \backslash\left\{i_{0}, j_{0}\right\}$. Here $d\left(q, u^{i_{0}, j_{0}}\right)=3$, a contradiction. Hence $q \nsim u$, i.e., $d(u, q)=2$. Then $q=v^{j, k}$ for some $j, k \in[n]$. If $\left\{i_{0}, j_{0}\right\} \cap\{j, k\}=\emptyset$, then $d\left(u^{i_{0}, j_{0}}, q\right)=4$, a contradiction. Hence $\left\{i_{0}, j_{0}\right\} \cap\{j, k\} \neq \emptyset$. Without loss of generality we assume that $i_{0} \in\{j, k\}$. In this case $d\left(q, u^{j_{0}}\right)=3$, a contradiction. Thus $\sigma=\left\{u, u^{i_{0}}, u^{j_{0}}, u^{i_{0}, j_{0}}\right\}$. Hence $\sigma$ is of the type (ii).

Case 2. $N(v) \cap \sigma=\emptyset$ for all $v \in \sigma$.
Let $v \in \sigma$. Clearly, $\{v\}$ is not a maximal simplex and therefore there exists $x \in \sigma, x \neq v$. Since $N(v) \cap \sigma=\emptyset$ and $d(v, x) \leq 2$, we see that $d(v, x)=2$. There exist $i_{0}, j_{0} \in[n]$ such that $x=v^{i_{0}, j_{0}}$. Hence $\left\{v, v^{i_{0}, j_{0}}\right\} \subseteq \sigma$. For any $t \in[n] \backslash\left\{i_{0}, j_{0}\right\}$, since $d\left(v^{i_{0}, t}, v\right)=2=d\left(v^{i_{0}, t}, v^{i_{0}, j_{0}}\right)$, we see that $\left\{v, v^{i_{0}, j_{0}}, v^{i_{0}, t}\right\} \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$. Thus $\left\{v, v^{i_{0}, j_{0}}\right\}$ is not a maximal simplex and therefore there exists $y \in \sigma \backslash\left\{v, v^{i_{0}, j_{0}}\right\}$. Clearly, $d(v, y)=2$. There exist $i, j \in[n]$ such that $y=v^{i, j}$. If $\{i, j\} \cap\left\{i_{0}, j_{0}\right\}=\emptyset$, then $d\left(y, v^{i_{0}, j_{0}}\right) \geq 3$, a contradiction. Hence $\{i, j\} \cap\left\{i_{0}, j_{0}\right\} \neq \emptyset$. Without loss of generality assume that $i=i_{0}$. Thus $\left\{v, v^{i_{0}, j_{0}}, v^{i_{0}, j}\right\} \subseteq \sigma$. Since $N(v) \cap \sigma=\emptyset, v^{i_{0}} \notin \sigma$. Further, since $\sigma$ is maximal, there exists $z \in \sigma$ such that $d\left(z, v^{i_{0}}\right) \geq 3$. Clearly $d(v, z)=2$ and therefore $z=v^{k, l}$ for some $k, l \in[n]$. Since $d\left(z, v^{i_{0}}\right) \geq 3, i_{0} \notin\{k, l\}$. Using the fact that $d\left(z, v^{i_{0}, j_{0}}\right)=2=d\left(z, v^{i_{0}, j}\right)$, we conclude that $\{k, l\}=\left\{j_{0}, j\right\}$. Thus $\left\{v, v^{i_{0}, j_{0}}, v^{i_{0}, j}, v^{j_{0}, j}\right\} \subseteq \sigma$. Suppose there exists a $w \in \sigma$ such that $w \notin\left\{v, v^{i_{0}, j_{0}}, v^{i_{0}, j}, v^{j_{0}, j}\right\}$. Here, $d(v, w)=2$ and therefore $w=v^{s, t}$ for some $s, t \in[n]$. Since $d\left(w, v^{i_{0}, j_{0}}\right)=2,\left\{i_{0}, j_{0}\right\} \cap\{s, t\} \neq \emptyset$. Further, $d\left(w, v^{i_{0}, j}\right)=2$ implies that $\left\{i_{0}, j\right\} \cap\{s, t\} \neq \emptyset$ and $d\left(w, v^{j 0, j}\right)=2$ implies that $\left\{j_{0}, j\right\} \cap\{s, t\} \neq \emptyset$, which is not possible. Hence $\sigma=\left\{v, v^{i_{0}, j_{0}}, v^{i_{0}, j}, v^{j, j}\right\}=K_{v}^{i_{0}, j_{0}, j}$. Thus $\sigma$ is of the type (iii).
Lemma 3.2. Let $n \geq 3$ and $\sigma \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ be a maximal simplex. If for some $v,|N(v) \cap \sigma| \geq 3$, then either $\sigma=K_{v}^{i_{0}, j_{0}, k_{0}}$ for some $i_{0}, j_{0}, k_{0} \in[n]$ or $N(v) \subseteq \sigma$.

Proof. Let $|N(v) \cap \sigma| \geq 3$. If $n=3$, then $|N(v)|=3$ and therefore $N(v) \subseteq \sigma$. So assume that $n \geq 4$. Suppose $N(v) \nsubseteq \sigma$. Then there exists $l_{0} \in[n]$ such that $v^{l_{0}} \notin \sigma$. Since $|N(v) \cap \sigma| \geq 3$, there exist $i_{0}, j_{0}, k_{0} \in[n]$ such that $\left\{v^{i_{0}}, v^{j_{0}}, v^{k_{0}}\right\} \subseteq \sigma$. Clearly, $l_{0} \notin\left\{i_{0}, j_{0}, k_{0}\right\}$. Since $v^{l_{0}} \notin \sigma$ and $\sigma$ is a maximal simplex, there exists $x \in \sigma$ such that $d\left(x, v^{l_{0}}\right) \geq 3$. Observe that for any vertex $u$, if $d(v, u)=1$, then $d\left(u, v^{l_{0}}\right) \leq 2$. Hence $d(v, x) \geq 2$. If $d(v, x) \geq 4$, then $d\left(v^{i_{0}}, x\right) \geq 3$, a contradiction as $v^{i_{0}} \in \sigma$. Hence $d(v, x) \leq 3$. If $d(v, x)=3$, then $x=v^{i, j, k}$ for some $i, j, k \in[n]$. Since $d\left(v^{i_{0}}, x\right) \leq 2, i_{0} \in\{i, j, k\}$. Similarly $j_{0}, k_{0} \in\{i, j, k\}$. Hence $\{i, j, k\}=\left\{i_{0}, j_{0}, k_{0}\right\}$. Thus $\sigma=\left\{v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{i_{0}, j_{0}, k_{0}}\right\}=K_{v}^{i_{0}, j_{0}, k_{0}}$.

Suppose $d(v, x)=2$. Here, $x=v^{i, j}$ for some $i, j \in[n]$. If $i_{0} \notin\{i, j\}$, then $d\left(x, v^{i_{0}}\right)=3$, a contradiction as $v^{i_{0}} \in \sigma$. Hence $i_{0} \in\{i, j\}$. By similar argument, we can show that $j_{0}, k_{0} \in$ $\{i, j\}$. Hence $\left\{i_{0}, j_{0}, k_{0}\right\} \subseteq\{i, j\}$, which is not possible. Thus, $N(v) \subseteq \sigma$.

We now review a result, which will play a key role in the proof of Theorem B.
Let $X$ be a simplicial complex on vertex set $[n]$ and let $\prec: \sigma_{1}, \ldots, \sigma_{m}$ be a linear ordering of the maximal simplices of $X$. Given a $\sigma \in X$, the minimal exclusion sequence mes $_{\prec}(\sigma)$ is defined as follows. Let $i$ denote the smallest index such that $\sigma \subseteq \sigma_{i}$. If $i=1$, then $\operatorname{mes}_{\prec}(\sigma)$ is the null sequence. If $i \geq 2$, then $\operatorname{mes}_{\prec}(\sigma)=\left(v_{1}, \ldots, v_{i-1}\right)$ is a finite sequence of length $i-1$ such that $v_{1}=\min \left(\sigma \backslash \sigma_{1}\right)$ and for each $k \in\{2, \ldots, i-1\}$,

$$
v_{k}= \begin{cases}\min \left(\left\{v_{1}, \ldots, v_{k-1}\right\} \cap\left(\sigma \backslash \sigma_{k}\right)\right) & \text { if }\left\{v_{1}, \ldots, v_{k-1}\right\} \cap\left(\sigma \backslash \sigma_{k}\right) \neq \emptyset \\ \min \left(\sigma \backslash \sigma_{k}\right) & \text { otherwise }\end{cases}
$$

Let $M_{\prec}(\sigma)$ denote the set of vertices appearing in $\operatorname{mes}_{\prec}(\sigma)$. Define

$$
d_{\prec}(X):=\max _{\sigma \in X}\left|M_{\prec}(\sigma)\right| .
$$

The following result was stated and proved in [36, Proposition 1.3] as a special case where $X$ is the nerve of a finite family of sets and then generalized by Lew for arbitrary simplical complex.

Proposition 3.3. [35, Theorem 6] If $\prec$ is a linear ordering of the maximal simplices of $X$, then $X$ is $d_{\prec}(X)$-collapsible.

We are now ready to prove main result of this section.
Proof of Theorem B. We must show that for $n \geq 3$, the collapsibility number of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is 4. Since $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is homotopy equivalent to a wedge sum of spheres of dimension 3 [3], $\tilde{H}_{3}\left(\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)\right) \neq 0$ and therefore by using Proposition 1.2 we conclude that collapsibility of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is $\geq 4$. It is enough to show that $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is 4-collapsible. From Lemma 3.1, each maximal simplex of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is of the form either $(i) N[v]$ or $(i i)\left\{v, v^{i}, v^{j}, v^{i, j}\right\}$ or $(i i i) K_{v}^{i_{0}, j_{0}, k_{0}}$. It is easy to check that these three sets of maximal simplices are pairwise disjoint sets. Choose a linear order $\prec_{1}$ on maximal simplices of the type $(i)$. Extend $\prec_{1}$ to a linear order $\prec$ on maximal simplices of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$, where maximal simplices of the type (i) are ordered first, i.e., for any two maximal simplices $\sigma_{1}$ and $\sigma_{2}$, if $\sigma_{1}=N[v]$ for some $v$ and $\sigma_{2}$ is of the type (ii) or (iii), then $\sigma_{1} \prec \sigma_{2}$. Let $\tau \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$. Let $\sigma$ be the smallest (with respect to $\prec$ ) maximal simplex of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ such that $\tau \subseteq \sigma$. If $\sigma \neq N[v]$ for all $v \in V\left(\mathbb{I}_{n}\right)$, then $|\sigma|=4$ and therefore by definition $\left|M_{\prec}(\tau)\right| \leq 4$. So, assume that $\sigma=N[v]$ for some $v \in V\left(\mathbb{I}_{n}\right)$. We first prove that $\left|M_{\prec}(\tau) \cap N(v)\right| \leq 3$.

Let $\operatorname{mes}_{\prec}(\tau)=\left(x_{1}, \ldots, x_{t}\right)$. Suppose $\left|M_{\prec}(\tau) \cap N(v)\right| \geq 4$. Let $k$ be the least integer such that $\left|\left\{x_{1}, \ldots, x_{k}\right\} \cap N(v)\right|=3$. Clearly, $k<t$. Let $\left\{x_{1}, \ldots, x_{k}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$. Observe that $x_{k} \in\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$. We show that $\left\{x_{1}, \ldots, x_{k+1}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$. Let $\gamma$ be a maximal simplex such that $\gamma \prec \sigma$. If $\left\{x_{1}, \ldots, x_{k}\right\} \cap(\sigma \backslash \gamma) \neq \emptyset$, then $x_{k+1} \in\left\{x_{1}, \ldots, x_{k}\right\}$. Hence $\left\{x_{1}, \ldots, x_{k+1}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$. If $\left\{x_{1}, \ldots, x_{k}\right\} \cap(\sigma \backslash \gamma)=\emptyset$, then $\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\} \subseteq \gamma$. From Lemma 3.2, either $N(v) \subseteq \gamma$ or $\gamma=K_{v}^{i_{0}, j_{0}, k_{0}}$. Since $\gamma \prec \sigma, \gamma \neq K_{v}^{i_{0}, j_{0}, k_{0}}$. Hence $N(v) \subseteq \gamma$. Thus $x_{k+1} \notin N(v)$, thereby implying that $\left\{x_{1}, \ldots, x_{k+1}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$. If $k+1=t$, then we get a contradiction to the assumption that $\left|M_{\prec}(\tau) \cap N(v)\right| \geq 4$. Inductively assume that for all $k \leq l<t,\left\{x_{1}, \ldots, x_{l}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$. By the argument similar as above we
can show that $\left\{x_{1}, \ldots, x_{t}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}\right\}$, a contradiction. Thus $\left|M_{\prec}(\tau) \cap N(v)\right| \leq 3$. Since $\sigma=N[v]$, we conclude that $\left|M_{\prec}(\tau)\right| \leq 4$.

From Proposition 3.3, $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is 4 -collapsible. This completes the proof.

## 4. The complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$

In this section, we prove Theorem A and Theorem C. This section is divided into three subsections. In the next subsection, we characterise the maximal simplices of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$. In subsection 4.2, using the minimal exclusion sequence, we prove Theorem C. Finally, in subsection 4.3, using the Mayer-Vietoris sequence for homology, we prove Theorem A.

We first fix some notations, which we use throughout this section. For any $n \geq 1$, let $\Delta_{n}=\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$. We say that a simplex $\sigma \in \Delta_{n}$ covers all places, if for each $i \in[n]$ there exist $v, w \in \sigma$ such that $v(i)=1$ and $w(i)=0$. For each $i \in\{n\}$ and $\epsilon \in\{0,1\}$, let $\mathbb{I}_{n}^{(i, \epsilon)}$ be the induced subgraph of $\mathbb{I}_{n}$ on the vertex set $\left\{v \in V\left(\mathbb{I}_{n}\right): v(i)=\epsilon\right\}$. Observe that $\mathbb{I}_{n}^{(i, \epsilon)} \cong \mathbb{I}_{n-1}$.
4.1. Maximal simplices. We give a characterisation of the maximal simplices of $\Delta_{n}$ in Lemma 4.7. We first establish a few lemmas, which we need to prove Lemma 4.7.

Lemma 4.1. Let $n \geq 5$ and $\sigma \in \Delta_{n}$ be a maximal simplex such that $\sigma$ covers all places. Then $N(v) \cap \sigma \neq \emptyset$ for all $v \in \sigma$.
Proof. Suppose there exists $v \in \sigma$ such that $N(v) \cap \sigma=\emptyset$. Since $v^{1} \notin \sigma$ and $\sigma$ is a maximal simplex, there exists $x \in \sigma$ such that $d\left(x, v^{1}\right) \geq 4$. It is easy to see that if $d(x, v) \leq 2$, then $d\left(x, v^{1}\right) \leq 3$. Hence $d(v, x)=3$. Here, $x=v^{i_{0}, j_{0}, k_{0}}$ for some $i_{0}, j_{0}, k_{0} \in[n]$. If $1 \in\left\{i_{0}, j_{0}, k_{0}\right\}$, then $d\left(v^{1}, x\right)=2$. Hence $1 \notin\left\{i_{0}, j_{0}, k_{0}\right\}$. Since $v^{i_{0}} \notin \sigma$, there exists $y \in \sigma$ such that $d\left(y, v^{i_{0}}\right) \geq 4$. Here $d(v, y)=3$. Hence $y=v^{i, j, k}$ for some $i, j, k \in[n]$. If $\left|\left\{i_{0}, j_{0}, k_{0}\right\} \cap\{i, j, k\}\right| \leq 1$, then $d\left(y, v^{i_{0}, j_{0}, k_{0}}\right) \geq 4$, which contradict the fact that $v^{i_{0}, j_{0}, k_{0}} \in \sigma$. If $i_{0} \in\{i, j, k\}$, then $d\left(v^{i_{0}}, y\right)=2$, a contradiction. So $i_{0} \notin\{i, j, k\}$ and therefore $\left\{i_{0}, j_{0}, k_{0}\right\} \cap\{i, j, k\}=\left\{j_{0}, k_{0}\right\}$. Hence $y=v^{j 0, k_{0}, l_{0}}$ for some $l_{0} \in[n] \backslash\left\{i_{0}\right\}$. So, $\left\{v, v^{i_{0}, j_{0}, k_{0}}, v^{j_{0}, k_{0}, l_{0}}\right\} \subseteq \sigma$. Since $v^{j_{0}} \notin \sigma$, there exists $z \in \sigma$ such that $d\left(z, v^{j_{0}}\right) \geq 4$. Here, $d(v, z)=3$. Since $d\left(z, v^{i_{0}, j_{0}, k_{0}}\right) \leq 3$ and $d\left(z, v^{j_{0}, k_{0}, l_{0}}\right) \leq 3$, we conclude that $z=v^{i_{0}, k_{0}, l_{0}}$. Further, since $v^{k_{0}} \notin \sigma$, there exists $w \in \sigma$ such that $d\left(w, v^{k_{0}}\right) \geq 4$. Here $d(v, w)=3$. Since $d\left(w, v^{i_{0}, j_{0}, k_{0}}\right) \leq 3, d\left(w, v^{j_{0}, k_{0}, l_{0}}\right) \leq 3$ and $d\left(w, v^{i_{0}, k_{0}, l_{0}}\right) \leq 3$, we conclude that $w=v^{i_{0}, j_{0}, l_{0}}$. So, $\left\{v, v^{i_{0}, j_{0}, k_{0}}, v^{j_{0}, k_{0}, l_{0}}, v^{i_{0}, k_{0}, l_{0}}, v^{i_{0}, j_{0}, l_{0}}\right\} \subseteq \sigma$. Since $n \geq 5$, there exists $p \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}, l_{0}\right\}$. Observe that $v^{i_{0}, j_{0}, k_{0}}(p)=v^{j_{0}, k_{0}, l_{0}}(p)=v^{i_{0}, k_{0}, l_{0}}(p)=v^{i_{0}, j_{0}, l_{0}}(p)=v(p)$. Since $\sigma$ covers all places, there exists $u \in \sigma$ such that $u(p)=\{0,1\} \backslash\{v(p)\}$. Since $N(v) \cap \sigma=\emptyset$, $u \neq v^{p}$. Thus, either $d(v, u)=2$ or $d(v, u)=3$. If $d(v, u)=2$, then $u=v^{p, r}$ for some $r \in[n]$. If $r \notin\left\{i_{0}, j_{0}, k_{0}\right\}$, then $d\left(u, v^{i_{0}, j_{0}, k_{0}}\right)=4$, a contradiction. Hence $r \in\left\{i_{0}, j_{0}, k_{0}\right\}$. Without loss of generality we assume that $r=i_{0}$. In this case $d\left(u, v^{j_{0}, k_{0}, l_{0}}\right)=4$, a contradiction. Hence $d(v, u)=3$. Here $u=v^{p, s, t}$ for some $s, t \in[n]$. If $\left|\{s, t\} \cap\left\{i_{0}, j_{0}, k_{0}\right\}\right| \leq 1$, then $d\left(u, v^{i_{0}, j_{0}, k_{0}}\right) \geq 4$. Hence $\{s, t\} \subseteq\left\{i_{0}, j_{0}, k_{0}\right\}$. Without loss of generality we assume that $\{s, t\}=\left\{i_{0}, j_{0}\right\}$. Then $d\left(u, v^{i_{0}, k_{0}, l_{0}}\right)=4$, a contradiction. Thus there exists no $u \in \sigma$ such that $u(p)=\{0,1\} \backslash\{v(p)\}$, which is a contradiction to the hypothesis that $\sigma$ covers all places. Hence $N(v) \cap \sigma \neq \emptyset$.

Lemma 4.2. Let $n \geq 5$ and let $\sigma \in \Delta_{n}$ be a maximal simplex such that $\sigma$ covers all places. If there exists a $w \in \sigma$ such that $N(w) \cap \sigma=\{v\}$, then $N(v) \subseteq \sigma$.

Proof. Since $w \in N(v), w=v^{s}$ for some $s \in[n]$. Without loss of generality we assume that $v=v_{1} \ldots v_{n}$, where $v_{i}=0$ for each $i \in[n]$ and $s=n$, i.e., $w=v^{n}$. Suppose $N(v) \nsubseteq \sigma$. There exists $l_{0} \in[n]$ such that $v^{l_{0}} \notin \sigma$. Clearly, $l_{0} \neq n$. Since $\sigma$ covers all places, there exists $x \in \sigma$ such that $x\left(l_{0}\right)=1$. Further, since $x \neq v^{l_{0}}, d(x, v) \geq 2$. Thus, either $d(x, v)=3$ or $d(x, v)=2$. We consider the following two cases:

Case 1. $d(x, v)=3$.
Here, $x=v^{l_{0}, i_{0}, j_{0}}$ for some $i_{0}, j_{0} \in[n]$. If $n \notin\left\{i_{0}, j_{0}\right\}$, then $d\left(x, v^{n}\right)=4$. Since $v^{n} \in \sigma$, $d\left(x, v^{n}\right) \leq 3$ and thereby implying that $n \in\left\{i_{0}, j_{0}\right\}$. Without loss of generality we assume that $n=j_{0}$, i.e., $x=v^{l_{0}, i_{0}, n}$.

From Lemma 4.1, there exists $y \in \sigma$ such that $y \sim x$. Clearly $y \neq v, v^{n}$. There exists $j \in[n]$ such that $y=x^{j}$. If $j \notin\left\{l_{0}, i_{0}, n\right\}$, then $d(y, v) \geq 4$. Hence $j \in\left\{l_{0}, i_{0}, n\right\}$ and thereby implying that $d(y, v)=2$. If $j \neq n$, then $y=v^{l_{0, n}}$ or $y=v^{i, n}$. In both the cases $y \sim w=v^{n}$, which is not possible since $N(w) \cap \sigma=\{v\}$. Hence $j=n$ and $y=v^{l_{0}, i_{0}}$. So, $\left\{v, v^{n}, v^{l_{0}, i_{0}, n}, v^{l_{0}, i_{0}}\right\} \subseteq \sigma$.

Since $v^{i_{0}, n} \sim v^{n}=w$ and $N(w) \cap \sigma=\{v\}$, we see that $v^{i_{0}, n} \notin \sigma$. Further, since $\sigma$ is a maximal simplex, there exists $z \in \sigma$ such that $d\left(z, v^{i, n}\right) \geq 4$. Observe that for any vertex $t$, if $t \sim v$, then $d\left(t, v^{i}, n\right) \leq 3$ and therefore we see that $z \nsim v$. Since $z, v \in \sigma, d(z, v) \leq 3$. Thus, either $d(z, v)=3$ or $d(z, v)=2$. If $d(z, v)=3$, then $z=v^{i, j, k}$ for some $i, j, k \in[n]$. Observe that if $n \notin\{i, j, k\}$, then $d\left(z, v^{n}\right)=4$, a contradiction as $v^{n} \in \sigma$. Hence $n \in\{i, j, k\}$. Without loss of generality we assume that $i=n$, i.e., $z=v^{n, j, k}$. But, then $d\left(z, v^{i_{0}, n}\right) \leq 3$, which is a contradiction as $d\left(z, v^{i_{0}, n}\right) \geq 4$. Thus $d(z, v)=2$. So, $z=v^{i, j}$ for some $i, j \in[n]$. If $\{i, j\} \cap\left\{i_{0}, n\right\} \neq \phi$, then $d\left(v^{i, j}, v^{i, n}\right) \leq 3$. Hence $d\left(z, v^{i_{0}, n}\right) \geq 4$ implies that $\{i, j\} \cap\left\{i_{0}, n\right\}=\emptyset$. If $\{i, j\} \cap\left\{l_{0}, n, i_{0}\right\}=\emptyset$, then $d\left(v^{i, j}, v^{l_{0}, i_{0}, n}\right) \geq 4$. Since $v^{l_{0}, i_{0}, n} \in \sigma,\{i, j\} \cap\left\{l_{0}, i_{0}, n\right\} \neq \emptyset$. Thus, we conclude that $\{i, j\} \cap\left\{l_{0}, i_{0}, n\right\}=\left\{l_{0}\right\}$. Hence $z=v^{l_{0}, k_{0}}$ for some $k_{0} \neq i_{0}, n$. So, $\left\{v, v^{n}, v^{l_{0}, i_{0}, n}, v^{l_{0}, i_{0}}, v^{l_{0}, k_{0}}\right\} \subseteq \sigma$.
Since $v^{l_{0}, n} \sim v^{n}$ and $N\left(v^{n}\right) \cap \sigma=\{v\}, v^{l_{0}, n} \notin \sigma$. Hence there exists $p \in \sigma$ such that $d\left(p, v^{l_{0}, n}\right) \geq 4$. Observe that for any $u \sim v, d\left(u, v^{l_{0, n}}\right) \leq 3$ and therefore $p \nsim v$. Thus, $d(p, v) \geq 2$. Suppose $d(p, v)=3$. Then $p=v^{i, j, k}$ for some $i, j, k \in[n]$. If $n \notin\{i, j, k\}$, then $d\left(p, v^{n}\right)=4$, a contradiction as $v^{n} \in \sigma$. Hence $n \in\{i, j, k\}$. Without loss of generality we assume that $i=n$, i.e., $p=v^{n, j, k}$. But, then $d\left(p, v^{l_{0}, n}\right) \leq 3$, which is a contradiction. Thus $d(p, v)=2$. So, $p=v^{i, j}$ for some $i, j \in[n]$. If $\{i, j\} \cap\left\{l_{0}, n\right\} \neq \phi$, then $d\left(v^{i, j}, v^{l_{0}, n}\right) \leq 3$. Hence $\{i, j\} \cap\left\{l_{0}, n\right\}=\emptyset$. If $\{i, j\} \cap\left\{l_{0}, i_{0}, n\right\}=\emptyset$, then $d\left(v^{i, j}, v^{l_{0}, i_{0}, n}\right) \geq 4$. Since $v^{l_{0}, i_{0}, n} \in \sigma$, $\{i, j\} \cap\left\{l_{0}, i_{0}, n\right\} \neq \emptyset$. Hence $\{i, j\} \cap\left\{l_{0}, i_{0}, n\right\}=\left\{i_{0}\right\}$. Thus $p=v^{i_{0}, j}$ for some $j \neq n, l_{0}$. If $j \neq k_{0}$, then $d\left(p, v^{l_{0}, k_{0}}\right)=4$, which is a contradiction as $v^{l_{0}, k_{0}} \in \sigma$. Hence $p=v^{i_{0}, k_{0}}$. So, $\left\{v, v^{n}, v^{l_{0}, i_{0}, n}, v^{l_{0}, i_{0}}, v^{l_{0}, k_{0}}, v^{i_{0}, k_{0}}\right\} \subseteq \sigma$.

Since $n \geq 5$, there exists $j_{0} \in[n] \backslash\left\{l_{0}, i_{0}, k_{0}, n\right\}$. Further, since $\sigma$ covers all places, there exits $q \in \sigma$ such that $q\left(j_{0}\right)=1$. Observe that $0=v\left(j_{0}\right)=v^{n}\left(j_{0}\right)=v^{l_{0}, n, i_{0}}\left(j_{0}\right)=v^{l_{0}, i_{0}}\left(j_{0}\right)=$ $v^{l_{0}, k_{0}}\left(j_{0}\right)=v^{i_{0}, k_{0}}\left(j_{0}\right)$ and therefore $q \notin\left\{v, v^{n}, v^{l_{0}, i_{0}, n}, v^{l_{0}, i_{0}}, v^{l_{0}, k_{0}}, v^{i_{0}, k_{0}}\right\}$. Since $d\left(v^{j_{0}}, v^{l_{0}, i_{0}, n}\right)=$ $4, q \neq v^{j_{0}}$. Hence $d(q, v) \geq 2$.
(1.1) Suppose $d(q, v)=2$.

Here, $q=v^{j_{0}, i}$ for some $i \in[n]$. If $i \notin\left\{l_{0}, i_{0}, n\right\}$, then $d\left(q, v^{l_{0}, i_{0}, n}\right) \geq 4$. Hence $v^{l_{0}, i_{0}, n} \in$ $\sigma$ implies that $i \in\left\{l_{0}, i_{0}, n\right\}$. However, since $d\left(v^{j_{0}, l_{0}}, v^{i_{0}, k_{0}}\right)=4=d\left(v^{j_{0}, i_{0}}, v^{l_{0}, k_{0}}\right)=$ $d\left(v^{j_{0}, n}, v^{i_{0}, k_{0}}\right)$, we conclude that $i \notin\left\{l_{0}, i_{0}, n\right\}$. Thus, there exist no $q \in \sigma$ such that $q\left(j_{0}\right)=1$, which is a contradiction to the assumption that $\sigma$ covers all places.
(1.2) Suppose $d(q, v)=3$.

Here, $q=v^{j_{0}, i, j}$ for some $i, j \in[n]$. Observe that if $\left|\{i, j\} \cap\left\{l_{0}, i_{0}, n\right\}\right| \leq 1$, then $d\left(q, v^{l_{0}, i_{0}, n}\right) \geq 4$, which is not possible since $v^{l_{0}, i_{0}, n} \in \sigma$. Hence $\{i, j\} \subset\left\{l_{0}, i_{0}, n\right\}$. If $n \notin\{i, j\}$, then $d\left(q, v^{n}\right)=4$, a contradiction as $v^{n} \in \sigma$. Hence $n \in\{i, j\}$. Without loss of generality we assume that $i=n$, i.e., $q=v^{j, n, j}$, where $j \in\left\{l_{0}, i_{0}\right\}$. It is easy to see that $d\left(v^{j_{0}, n, l_{0}}, v^{i_{0}, k_{0}}\right) \geq 4$ and $d\left(v^{j_{0}, n, i_{0}}, v^{l_{0}, k_{0}}\right) \geq 4$. Since $v^{i_{0}, k_{0}}, v^{l_{0}, k_{0}} \in \sigma$, we see that $q \notin\left\{v^{j_{0}, l_{0}, n}, v^{j_{0}, i_{0}, n}\right\}$. Thus, there exists no $q \in \sigma$ such that $q\left(j_{0}\right)=1$, which is a contradiction.

Case 2. $d(x, v)=2$.
Here, $x=v^{l_{0}, s_{0}}$ for some $s_{0} \in[n]$. So, $\left\{v, v^{n}, v^{l_{0}, s_{0}}\right\} \subseteq \sigma$. Since $v^{l_{0}, n} \sim v^{n}=w$ and $N(w) \cap \sigma=\{v\}, v^{l_{0}, n} \notin \sigma$. Hence there exists $y_{0} \in \sigma$ such that $d\left(y_{0}, v^{l_{0}, n}\right) \geq 4$. Observe that for any $t \sim v, d\left(t, v^{l_{0}, n}\right) \leq 3$ and therefore $y_{0} \nsim v$. Thus, $d\left(y_{0}, v\right) \geq 2$. Suppose $d\left(y_{0}, v\right)=3$. Then $y_{0}=v^{i, j, k}$ for some $i, j, k \in[n]$. If $n \notin\{i, j, k\}$, then $d\left(y_{0}, v^{n}\right) \geq 4$, a contradiction as $y_{0}, v^{n} \in \sigma$. Hence $n \in\{i, j, k\}$. But, then $d\left(y_{0}, v^{l_{0}, n}\right) \leq 3$, again a contradiction. Thus, $d\left(y_{0}, v\right)=2$. Here, $y_{0}=v^{i, j}$ for some $i, j \in[n]$. If $\{i, j\} \cap\left\{l_{0}, n\right\} \neq \emptyset$, then $d\left(y_{0}, v^{l_{0}, n}\right) \leq 3$. Hence $\{i, j\} \cap\left\{l_{0}, n\right\}=\emptyset$. Further, if $s_{0} \notin\{i, j\}$, then $d\left(y_{0}, v^{l_{0}, s_{0}}\right) \geq 4$. Hence $s_{0} \in\{i, j\}$. Therefore $y_{0}=v^{s_{0}, t_{0}}$ for some $t_{0} \in[n], t_{0} \neq l_{0}, n$. So, $\left\{v, v^{n}, v^{l_{0}, s_{0}}, v^{s_{0}, t_{0}}\right\} \subseteq \sigma$.

Since $v^{s_{0}, n} \sim v^{n}$ and $N\left(v^{n}\right) \cap \sigma=\{v\}, v^{s_{0}, n} \notin \sigma$. Hence there exists $z_{0} \in \sigma$ such that $d\left(z_{0}, v^{s_{0}, n}\right) \geq 4$. By the argument similar as above for the $y_{0}$, we see that $d\left(z_{0}, v\right)=2$. Therefore $z_{0}=v^{i, j}$ for some $i, j \in[n]$. If $\{i, j\} \cap\left\{s_{0}, n\right\} \neq \emptyset$, then $d\left(z_{0}, v^{s_{0}, n}\right) \leq 3$. Hence $\{i, j\} \cap\left\{s_{0}, n\right\}=\emptyset$. If $l_{0} \notin\{i, j\}$, then $d\left(z_{0}, v^{l_{0}, s_{0}}\right) \geq 4$ and if $t_{0} \notin\{i, j\}$, then $d\left(z_{0}, v^{s_{0}, t_{0}}\right) \geq 4$. Since $v^{l_{0}, s_{0}}, v^{s_{0}, t_{0}} \in \sigma$, we conclude that $\{i, j\}=\left\{l_{0}, s_{0}\right\}$, i.e., $z_{0}=v^{l_{0}, t_{0}}$. So, $\left\{v, v^{n}, v^{l_{0}, s_{0}}, v^{s_{0}, t_{0}}, v^{l_{0}, t_{0}}\right\} \subseteq \sigma$.

Since $n \geq 5$, there exists $m_{0} \in[n] \backslash\left\{n, l_{0}, s_{0}, t_{0}\right\}$. Further, since $\sigma$ covers all places, there exists $p_{0} \in \sigma$ such that $p_{0}\left(m_{0}\right)=1$. Clearly, $u\left(m_{0}\right)=0$ for all $u \in\left\{v, v^{n}, v^{l_{0}, s_{0}}, v^{s_{0}, t_{0}}, v^{l_{0}, t_{0}}\right\}$. Hence $p_{0} \notin\left\{v, v^{n}, v^{l_{0}, s_{0}}, v^{s_{0}, t_{0}}, v^{l_{0}, t_{0}}\right\}$.

Claim 1. $p_{0}=v^{m_{0}}$.
Proof of Claim 1. Since $p_{0}, v \in \sigma, d\left(p_{0}, v\right) \leq 3$. Suppose $d\left(p_{0}, v\right)=3$. Then $p_{0}=v^{m_{0}, i, j}$ for some $i, j \in[n]$. If $n \notin\{i, j\}$, then $d\left(p_{0}, v^{n}\right)=4$ and therefore $n \in\{i, j\}$. Further, if $\left\{s_{0}, l_{0}\right\} \cap\{i, j\}=\emptyset$, then $d\left(p_{0}, v^{l_{0}, s_{0}}\right) \geq 4$, which is not possible since $v^{l_{0}, s_{0}} \in \sigma$. Hence $\{i, j\} \cap$ $\left\{l_{0}, s_{0}\right\} \neq \emptyset$ and therefore we see that $p_{0}$ is either $v^{m_{0}, n, s_{0}}$ or $v^{m_{0}, n, l_{0}}$. But $d\left(v^{m_{0}, n, s_{0}}, v^{l_{0}, t_{0}}\right)=5$ and $d\left(v^{m_{0}, n, l_{0}}, v^{s_{0}, t_{0}}\right)=5$. Hence $p_{0} \notin\left\{v^{m_{0}, n, s_{0}}, v^{m_{0}, n, l_{0}}\right\}$. Therefore $d\left(p_{0}, v\right) \leq 2$.

If $d\left(p_{0}, v\right)=2$, then $p_{0}=v^{m_{0}, i}$ for some $i \in[n]$. Since $v^{l_{0}, t_{0}}, v^{l_{0}, s_{0}} \in \sigma$ and $d\left(v^{m_{0}, s_{0}}, v^{l_{0}, t_{0}}\right)=$ $4=d\left(v^{m_{0}, t_{0}}, v^{l_{0}, s_{0}}\right)$, we conclude that $i \notin\left\{s_{0}, t_{0}\right\}$. But, then $d\left(p_{0}, v^{s_{0}, t_{0}}\right)=4$, a contradiction as $v^{s_{0}, t_{0}} \in \sigma$. Hence $d\left(p_{0}, v\right)=1$. Therefore $p_{0}=v^{m_{0}}$. This completes the proof of Claim 1 .

So, $\left\{v, v^{n}, v^{l_{0}, s_{0}}, v^{s_{0}, t_{0}}, v^{l_{0}, t_{0}}, v^{m_{0}}\right\} \subseteq \sigma$. Since $\sigma$ is a maximal simplex and $v^{l_{0}} \notin \sigma$, there exists $q_{0} \in \sigma$ such that $d\left(q_{0}, v^{l_{0}}\right) \geq 4$. Observe that for any $t$, if $d(v, t) \leq 2$, then $d\left(t, v^{l_{0}}\right) \leq 3$. Hence $d\left(v, q_{0}\right)=3$. Here, $q_{0}=v^{i, j, k}$ for some $i, j, k \in[n]$. If $n \notin\{i, j, k\}$, then $d\left(q_{0}, v^{n}\right) \geq 4$. Hence $n \in\{i, j, k\}$. If $l_{0} \in\{i, j, k\}$, then $d\left(q_{0}, v^{l_{0}}\right) \leq 3$. Hence $l_{0} \notin\{i, j, k\}$. Further, if $s_{0} \notin\{i, j, k\}$, then $d\left(q_{0}, v^{l_{0}, s_{0}}\right)=5$ and therefore we see that $s_{0} \in\{i, j, k\}$. Without loss of generality we assume that $i=n$ and $j=s_{0}$, i.e., $q_{0}=v^{n, s_{0}, k}$. If $t_{0} \neq k$, then $d\left(q_{0}, v^{l_{0}, t_{0}}\right) \geq 4$, a contradiction as $v^{l_{0}, t_{0}} \in \sigma$. Hence $k=t_{0}$, i.e., $q_{0}=v^{n, s_{0}, t_{0}}$. But, then $d\left(q_{0}, v^{m_{0}}\right)=4$, a contradiction. Thus, there exists no $q_{0}$ such that $d\left(q_{0}, v^{l_{0}}\right) \geq 4$, a contradiction.

Therefore we conclude that $N(v) \subseteq \sigma$. This completes the proof.
Recall that for a $v \in V\left(\mathbb{I}_{n}\right)$ and $i_{0}, j_{0}, k_{0} \in[n], K_{v}^{i_{0}, j_{0}, k_{0}}=\left\{v, v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}}\right\}$.
Lemma 4.3. Let $n \geq 5$ and let $\sigma \in \Delta_{n}$ be a maximal simplex such that $\sigma$ covers all places. If there exists a $w \in \sigma$ such that $N(w) \cap \sigma=\{v\}$, then $\sigma=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$ for some $i_{0}, j_{0}, k_{0} \in[n]$.

Proof. From Lemma 4.2, $N(v) \subseteq \sigma$. Since $w \sim v, w=v^{l_{0}}$ for some $l_{0} \in[n]$. Suppose there exists $x \in \sigma$ such that $d(x, v)=3$. Then $x=v^{i, j, k}$ for some $i, j, k \in[n]$. Choose $t \in[n] \backslash\{i, j, k\}$. Then $d\left(x, v^{t}\right)=4$, a contradiction as $v^{t} \in N(v) \subseteq \sigma$. Hence $d(v, x) \leq 2$ for all $x \in \sigma$. Since $N(w) \cap \sigma=\{v\}, v^{i, l_{0}} \notin \sigma$ for all $i \in[n], i \neq l_{0}$. Further, since $\sigma$ is a maximal simplex and $v^{1, l_{0}} \notin \sigma$, there exists $x_{0} \in \sigma$ such that $d\left(x_{0}, v^{1, l_{0}}\right) \geq 4$. For any $p \sim v, d\left(p, v^{1, l_{0}}\right) \leq 3$ and therefore $d\left(x_{0}, v\right)=2$. Hence $x_{0}=v^{i_{0}, j_{0}}$ for some $i_{0}, j_{0} \in[n]$. If $\left\{i_{0}, j_{0}\right\} \cap\left\{1, l_{0}\right\} \neq \emptyset$, then $d\left(x_{0}, v^{1, l_{0}}\right) \leq 3$. Hence $\left\{i_{0}, j_{0}\right\} \cap\left\{1, l_{0}\right\}=\emptyset$. Thus $\left\{v, v^{1}, \ldots, v^{n}, v^{i_{0}, j_{0}}\right\} \subseteq \sigma$. Since $v^{i_{0}, l_{0}} \notin \sigma$, there exists $y_{0} \in \sigma$ such that $d\left(y_{0}, v^{i_{0}, l_{0}}\right) \geq 4$. For any $q \in N(v), d\left(q, v^{i_{0}, l_{0}}\right) \leq 3$ and therefore $d\left(y_{0}, v\right) \geq 2$. Since $d(x, v) \leq 2$ for all $x \in \sigma$, we see that $d\left(y_{0}, v\right)=2$. Hence $y_{0}=v^{i, j}$ for some $i, j$. If $\{i, j\} \cap\left\{i_{0}, j_{0}\right\}=\emptyset$, then $d\left(y_{0}, v^{i_{0}, j_{0}}\right) \geq 4$. Hence $\{i, j\} \cap\left\{i_{0}, j_{0}\right\} \neq \emptyset$. If $i_{0} \in\{i, j\}$, then $d\left(y_{0}, v^{i_{0}, l_{0}}\right) \leq 3$. Hence $i_{0} \notin\{i, j\}$. Thus $y_{0}=v^{j_{0}, k_{0}}$ for some $k_{0} \neq$ $i_{0}, l_{0}$. So, $\left\{v, v^{1}, \ldots, v^{n}, v^{i_{0}, j_{0}}, v^{j_{0}}, k_{0}\right\} \subseteq \sigma$. Since $v^{j_{0}, l_{0}} \notin \sigma$, there exists $z_{0} \in \sigma$ such that $d\left(z_{0}, v^{j_{0}, l_{0}}\right) \geq 4$. By an argument similar as above for $y_{0}$, we can see that $z_{0}=v^{i_{0}, k_{0}}$. So, $\left\{v, v^{1}, \ldots, v^{n}, v^{i_{0}, j_{0}}, v^{j_{0}, k_{0}}, v^{i_{0}, k_{0}}\right\} \subseteq \sigma$.
Suppose there exists $p \in \sigma$ such that $p \notin\left\{v, v^{1}, \ldots, v^{n}, v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}}\right\}$. Since $d(v, x) \leq 2$ for all $x \in \sigma$ and $p \notin N(v)$, we see that $d(v, p)=2$. Here, $p=v^{i, j}$ for some $i, j \in[n]$. Since $d\left(p, v^{i_{0}, j_{0}}\right) \leq 3, d\left(p, v^{i_{0}, k_{0}}\right) \leq 3$ and $d\left(p, v^{j_{0}, k_{0}}\right) \leq 3$, we see that $\{i, j\} \cap\left\{i_{0}, j_{0}\right\} \neq \emptyset,\{i, j\} \cap$ $\left\{i_{0}, k_{0}\right\} \neq \emptyset$ and $\{i, j\} \cap\left\{j_{0}, k_{0}\right\} \neq \emptyset$. But this is possible only if $\{i, j\}=\left\{i_{0}, j_{0}\right\}$, or $\{i, j\}=$ $\left\{i_{0}, k_{0}\right\}$ or $\{i, j\}=\left\{j_{0}, k_{0}\right\}$. Thus $\sigma=\left\{v, v^{1}, \ldots, v^{n}, v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}}\right\}=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$.

Lemma 4.4. Let $n \geq 5$ and $\sigma \in \Delta_{n}$ be a maximal simplex. If $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$, then there exists $\tilde{v} \in \sigma$ such that $|N(\tilde{v}) \cap \sigma| \geq 3$.
Proof. Let $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$. If $|N(w) \cap \sigma| \geq 3$ for all $w \in \sigma$, then we are done. So assume that there exists $v \in \sigma$ such that $|N(v) \cap \sigma|=2$. There exist $i_{0}, j_{0} \in[n]$ such that $\left\{v, v^{i_{0}}, v^{j_{0}}\right\} \subseteq \sigma$. Since $|N(v) \cap \sigma|=2, v^{i} \notin \sigma$ for all $i \neq i_{0}, j_{0}$. Choose $p \in[n] \backslash\left\{i_{0}, j_{0}\right\}$. Since $v^{p} \notin \sigma$ and $\sigma$ is maximal, there exists $x_{0} \in \sigma$ such that $d\left(x_{0}, v^{p}\right) \geq 4$. Observe that for any $u \in V\left(\mathbb{I}_{n}\right)$, if $d(v, u) \leq 2$, then $d\left(v^{p}, u\right) \leq 3$. Hence $d\left(v, x_{0}\right)=3$. Here, $x_{0}=v^{i, j, k}$ for some $i, j, k \in[n]$. If $i_{0} \notin\{i, j, k\}$, then $d\left(x_{0}, v^{i_{0}}\right)=4$, a contradiction as $v^{i_{0}} \in \sigma$. Hence $i_{0} \in\{i, j, k\}$. By similar argument, $j_{0} \in\{i, j, k\}$. Thus $x_{0}=v^{i_{0}, j_{0}, k_{0}}$ for some $k_{0} \in[n]$. If $k_{0}=p$, then $d\left(x_{0}, v^{p}\right)=2$, a contradiction. Hence $k_{0} \neq p$. So, $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}\right\} \subseteq \sigma$. Since $v^{k_{0}} \notin \sigma$, there exists $y_{0} \in \sigma$ such that $d\left(y_{0}, v^{k_{0}}\right) \geq 4$. By an argument similar as above, $d\left(v, y_{0}\right)=3$ and $y_{0}=v^{i_{0}, j_{0}, l_{0}}$ for some $l_{0} \in[n]$. If $l_{0}=k_{0}$, then $d\left(y_{0}, v^{k_{0}}\right)=2$, a contradiction. Hence $l_{0} \neq k_{0}$. So, $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, j_{0}, l_{0}}\right\} \subseteq \sigma$. Observe that $N\left(v^{i_{0}, j_{0}, l_{0}}\right) \cap\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}\right\}=\emptyset$. Since $\left|N\left(v^{i_{0}, j_{0}, l_{0}}\right) \cap \sigma\right| \geq 2$, there exists $z_{0} \in \sigma$ such that $z_{0} \sim v^{i_{0}, j_{0}, l_{0}}$. Further, $d\left(z_{0}, v\right) \leq 3$ implies that $z_{0} \in\left\{v^{i_{0}, j_{0}}, v^{i_{0}, l_{0}}, v^{j_{0}, l_{0}}\right\}$. We consider the following cases.
(1) $z_{0}=v^{i_{0}, j_{0}}$.

In this case, $\left\{v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, l_{0}}\right\} \subseteq N\left(v^{i_{0}, j_{0}}\right) \cap \sigma$. We take $\tilde{v}=v^{i_{0}, j_{0}}$.
(2) $z_{0}=v^{i_{0}, l_{0}}$.

In this case, $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, j_{0}, l_{0}}, v^{i_{0}, l_{0}}\right\} \subseteq \sigma$. Since $\left|N\left(v^{i_{0}, j_{0}, k_{0}}\right) \cap \sigma\right| \geq 2$ and $N\left(v^{i_{0}, j_{0}, k_{0}}\right) \cap\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, l_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, l_{0}}\right\}=\emptyset$, there exists $u_{0} \in \sigma$ such that $u_{0} \sim$ $v^{i_{0}, j_{0}, k_{0}}$. Now $d\left(v, u_{0}\right) \leq 3$ implies that $u_{0} \in\left\{v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}}\right\}$. Since $d\left(v^{j_{0}, k_{0}}, v^{i_{0}, l_{0}}\right)=$ 4 and $v^{i_{0}, l_{0}} \in \sigma$, we see that $u_{0} \neq v^{j_{0}, k_{0}}$. If $u_{0}=v^{i_{0}, j_{0}}$, then $\left\{v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, l_{0}}\right\} \subseteq N\left(u_{0}\right) \cap \sigma$ and we take $\tilde{v}=u_{0}$. If $u_{0}=v^{i_{0}, k_{0}}$, then $\left\{v, v^{i_{0}, l_{0}}, v^{i_{0}, k_{0}}\right\} \subseteq N\left(v^{i_{0}}\right) \cap \sigma$ and we take $\tilde{v}=v^{i_{0}}$.
(3) $z_{0}=v^{j_{0}, l_{0}}$.

In this case, $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, j_{0}, l_{0}}, v^{j_{0}, l_{0}}\right\} \subseteq \sigma$. Since $\left|N\left(v^{i_{0}}\right) \cap \sigma\right| \geq 2$ and $N\left(v^{i_{0}}\right) \cap\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, j_{0}, l_{0}}, v^{j_{0}, l_{0}}\right\}=\{v\}$, there exists $w_{0} \in \sigma, w \neq v$ such that $w_{0} \sim v^{i_{0}}$. Since $w_{0} \neq v$, we see that $w_{0}=v^{i_{0}, i}$ for some $i \in[n]$. If $i \notin\left\{j_{0}, l_{0}\right\}$, then $d\left(w_{0}, v^{j_{0}, l_{0}}\right)=4$, a contradiction as $v^{j_{0}, l_{0}} \in \sigma$. Hence $i \in\left\{j_{0}, l_{0}\right\}$. If $i=j_{0}$, then $w_{0}=v^{i_{0}, j_{0}}$ and $\left\{v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, l_{0}}\right\} \subseteq N\left(w_{0}\right) \cap \sigma$. We take $\tilde{v}=w_{0}$. So, assume that $i=l_{0}$, i.e., $w_{0}=v^{i_{0}, l_{0}}$.

Here $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, j_{0}, l_{0}}, v^{j_{0}, l_{0}}, v^{i_{0}, l_{0}}\right\} \subseteq \sigma$. Since $\left|N\left(v^{i_{0}, j_{0}, k_{0}}\right) \cap \sigma\right| \geq 2$ and $N\left(v^{i_{0}, j_{0}, k_{0}}\right) \cap\left\{v, v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{i_{0}, j_{0}, l_{0}}, v^{j_{0}, l_{0}}, v^{i_{0}, l_{0}}\right\}=\emptyset$, there exists $q_{0} \in \sigma$ such that $q_{0} \sim v^{i_{0}, j_{0}, k_{0}}$. Since $d\left(v, q_{0}\right) \leq 3$, we see that $q_{0} \in\left\{v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}}\right\}$. Further, since $d\left(v^{j_{0}, k_{0}}, v^{i_{0}, l_{0}}\right)=4, q_{0} \neq v^{j_{0}, k_{0}}$. If $q_{0}=v^{i_{0}, j_{0}}$, then $\left\{v^{i_{0}}, v^{j_{0}}, v^{i_{0}, j_{0}, l_{0}}\right\} \subseteq N\left(q_{0}\right) \cap \sigma$ and we take $\tilde{v}=q_{0}$. If $q_{0}=v^{i_{0}, k_{0}}$, then $\left\{v, v^{i_{0}, l_{0}}, v^{i_{0}, k_{0}}\right\} \subseteq N\left(v^{i_{0}}\right) \cap \sigma$ and we take $\tilde{v}=v^{i_{0}}$.
This completes the proof.
Lemma 4.5. Let $n \geq 5$ and $\sigma \in \Delta_{n}$ be a maximal simplex such that $\sigma$ covers all places. Let $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$. If there exists a $v \in \sigma$ such that $|N(v) \cap \sigma| \geq 3$, then $N(v) \subseteq \sigma$.
Proof. Without loss of generality, we assume that $v=v_{1} \ldots v_{n}$, where $v_{i}=0$ for all $i \in[n]$. Suppose $N(v) \nsubseteq \sigma$. Then there exists $l_{0} \in[n]$ such that $v^{l_{0}} \notin \sigma$. Since $|N(v) \cap \sigma| \geq 3$, there exist $i_{0}, j_{0}, k_{0} \in[n] \backslash\left\{l_{0}\right\}$ such that $\left\{v^{i_{0}}, v^{j_{0}}, v^{k_{0}}\right\} \subseteq \sigma$. Further, since $\sigma$ is maximal and $v^{l_{0}} \notin \sigma$, there exists $x_{0} \in \sigma$ such that $d\left(x_{0}, v^{l_{0}}\right) \geq 4$. Observe that $d\left(v, x_{0}\right)=3$ and therefore $x_{0}=v^{i, j, k}$ for some $i, j, k \in[n]$. Since $d\left(x_{0}, v^{l_{0}}\right) \geq 4, l_{0} \notin\{i, j, k\}$. If $i_{0} \notin\{i, j, k\}$, then $d\left(x_{0}, v^{i_{0}}\right)=4$, a contradiction as $v^{i_{0}} \in \sigma$. Hence $i_{0} \in\{i, j, k\}$. By similar arguments, $j_{0}, k_{0} \in\{i, j, k\}$ and therefore $x_{0}=v^{i_{0}, j_{0}, k_{0}}$. So, $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{i_{0}, j_{0}, k_{0}}\right\} \subseteq \sigma$.

Observe that for any $u \in\left\{v, v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{i_{0}, j_{0}, k_{0}}\right\}, u\left(l_{0}\right)=0$. Since $\sigma$ covers all places, there exists $y_{0} \in \sigma$ such that $y_{0}\left(l_{0}\right)=1$. Since $v^{l_{0}} \notin \sigma, y_{0} \neq v^{l_{0}}$. Hence $d\left(v, y_{0}\right) \geq 2$. Suppose $d\left(v, y_{0}\right)=3$. Then $y_{0}=v^{l_{0}, i, j}$ for some $i, j$. If $k \in\left\{i_{0}, j_{0}, k_{0}\right\} \backslash\{i, j\}$, then $d\left(y_{0}, v^{k}\right) \geq 4$, a contradiction as $v^{k} \in \sigma$. Hence $d\left(v, y_{0}\right)=2$. So, $y_{0}=v^{l_{0}, i}$ for some $i \in[n]$. If $i \notin\left\{i_{0}, j_{0}, k_{0}\right\}$, then $d\left(y_{0}, v^{i_{0}, j_{0}, k_{0}}\right) \geq 4$. Since $v^{i_{0}, j_{0}, k_{0}} \in \sigma$, we see that $i \in\left\{i_{0}, j_{0}, k_{0}\right\}$. Without loss of generality we assume that $i=i_{0}$, i.e., $y_{0}=v^{l_{0}, i_{0}}$. So, $\left\{v, v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{l_{0}, i_{0}}\right\} \subseteq \sigma$.

Observe that $N\left(v^{l_{0}, i_{0}}\right) \cap\left\{v, v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{i_{0}, j_{0}, k_{0}}, v^{l_{0}, i_{0}}\right\}=\left\{v^{i_{0}}\right\}$. Since $\left|N\left(v^{l_{0}, i_{0}}\right) \cap \sigma\right| \geq 2$, there exists $z_{0} \in \sigma, z_{0} \neq v^{i_{0}}$ such that $z_{0} \sim v^{l_{0}, i_{0}}$. Further, since $z_{0} \neq v^{i_{0}}$ and $v^{l_{0}} \notin \sigma, z_{0}=v^{l_{0}, i_{0}, i}$ for some $i \in[n]$. If $i \neq j_{0}$, then $d\left(z_{0}, v^{j_{0}}\right)=4$, a contradiction as $v^{j_{0}} \in \sigma$. Hence $z_{0}=v^{l_{0}, i_{0}, j_{0}}$. But then $d\left(z_{0}, v^{k_{0}}\right)=4$, a contradiction. Hence $N\left(v^{l_{0}, i_{0}}\right) \cap \sigma=\left\{v^{i_{0}}\right\}$, which is a contradiction.

Thus, we conclude that $N(v) \subseteq \sigma$.
Lemma 4.6. Let $n \geq 5$ and $\sigma \in \Delta_{n}$ be a maximal simplex such that $\sigma$ covers all places. If $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$, then there exist $v, w \in \sigma$ such that $v \sim w$ and $\sigma=N(v) \cup N(w)$.

Proof. Using Lemma 4.4 and Lemma 4.5, we conclude that there exists $v \in \sigma$ such that $N(v) \subseteq$ $\sigma$. Hence $\left\{v, v^{1}, \ldots, v^{n}\right\} \subseteq \sigma$. Observe that $N\left(v^{1}\right) \cap\left\{v, v^{1}, \ldots, v^{n}\right\}=\{v\}$. Since $\left|N\left(v^{1}\right) \cap \sigma\right| \geq 2$, there exists $x_{0} \in \sigma, x_{0} \neq v$ such that $x_{0} \sim v^{1}$. Then $x_{0}=v^{1, i_{1}}$ for some $i_{1} \in[n]$. So, $\left\{v, v^{1}, \ldots, v^{n}, v^{1, i_{1}}\right\} \subseteq \sigma$. Choose $i_{2} \in[n] \backslash\left\{1, i_{1}\right\}$.

Observe that $v^{i_{2}} \in \sigma$ and $N\left(v^{i_{2}}\right) \cap\left\{v, v^{1}, \ldots, v^{n}, v^{1, i_{1}}\right\}=\{v\}$. Since $\left|N\left(v^{i_{2}}\right) \cap \sigma\right| \geq 2$, there exists $y_{0} \in \sigma, y_{0} \neq v$ such that $y_{0} \sim v^{i_{2}}$. Further, since $y_{0} \neq v$, we see that $y_{0}=v^{i_{2}, i}$ for some $i \in[n]$. If $i \notin\left\{1, i_{1}\right\}$, then $d\left(y_{0}, v^{1, i_{1}}\right)=4$, a contradiction as $v^{1, i_{1}} \in \sigma$. Hence either $y_{0}=v^{i_{2}, 1}$ or $y_{0}=v^{i_{2}, i_{1}}$. If $y_{0}=v^{i_{2}, 1}$, then $\left\{v, v^{1, i_{1}}, v^{i_{2}, 1}\right\} \subseteq N\left(v^{1}\right) \cap \sigma$. Hence from Lemma 4.5, $N\left(v_{1}\right) \subseteq \sigma$. Thus $N(v) \cup N\left(v^{1}\right) \subseteq \sigma$. If $y_{0}=v^{i_{2}, i_{1}}$, then $\left\{v, v^{1, i_{1}}, v^{i_{2}, i_{1}}\right\} \subseteq N\left(v^{i_{1}}\right) \cap \sigma$ and therefore Lemma 4.5 implies that $N\left(v^{i_{1}}\right) \subseteq \sigma$. Hence $N(v) \cup N\left(v^{i_{1}}\right) \subseteq \sigma$.

Thus, we have shown that there exist vertices $v, w \in \sigma$ such that $v \sim w$ and $N(v) \cup N(w) \subseteq \sigma$. We now show that $\sigma \subseteq N(v) \cup N(w)$. Suppose there exists $z_{0} \in \sigma$ such that $z_{0} \notin N(v) \cup N(w)$. Since $w \sim v, w=v^{l_{0}}$ for some $l_{0} \in[n]$. Clearly, $d\left(z_{0}, v\right) \geq 2$. Suppose $d\left(z_{0}, v\right)=2$. Then $z_{0}=v^{i, j}$ for some $i, j \in[n]$. If $l_{0} \in\{i, j\}$, then $z_{0} \sim v^{l_{0}}$, which is a contradiction as $z_{0} \notin N(w)$. Hence $l_{0} \notin\{i, j\}$. Choose $k_{0} \in[n] \backslash\left\{l_{0}, i, j\right\}$. Since $N\left(v^{l_{0}}\right) \subseteq \sigma$ and $v^{l_{0}} \sim v^{l_{0}, k_{0}}$, we see that $v^{l_{0}, k_{0}} \in \sigma$. But then $d\left(z_{0}, v^{l_{0}, k_{0}}\right)=4$, a contradiction. Now let $d\left(z_{0}, v\right)=3$. Then $z_{0}=v^{i, j, k}$ for some $i, j, k$. Choose $p \in[n] \backslash\{i, j, k\}$. Then $N(v) \subseteq \sigma$ implies that $v^{p} \in \sigma$. But $d\left(z_{0}, v^{p}\right)=4$, a contradiction. Thus, we conclude that $N(v) \cup N(w)=\sigma$.

We are now ready to give a characterization of maximal simplices of $\Delta_{n}$. Recall that for $i \in[n]$ and $\epsilon \in\{0,1\}, \mathbb{I}_{n}^{(i, \epsilon)}$ is the induced subgraph of $\mathbb{I}_{n}$ on the vertex set $\left\{v \in V\left(\mathbb{I}_{n}\right): v(i)=\epsilon\right\}$.
Lemma 4.7. Let $n \geq 4$ and let $\sigma \in \Delta_{n}$ be a maximal simplex. Then $\operatorname{dim}(\sigma) \in\{7, n+3,2 n-1\}$. Moreover, if $\operatorname{dim}(\sigma) \neq 7$, then either $\sigma=N(v) \cup N(w)$ for some $v \sim w$, or $\sigma=N(u) \cup K_{u}^{i, j, k}$ for some $u$ and $i, j, k \in[n]$.
Proof. Proof is by induction on $n$. Let $n=4$. For any two vertices $v, w \in V\left(\mathbb{I}_{4}\right)$, let $\overline{\{v, w\}}$ denote a simplicial complex on two vertices, i.e., $\overline{\{v, w\}} \cong S^{0}$. Let $v=0000$. It is easy to check that
$\Delta_{n}=\overline{\left\{v, v^{1,2,3,4}\right\}} * \overline{\left\{v^{1}, v^{2,3,4}\right\}} * \overline{\left\{v^{2}, v^{1,3,4}\right\}} * \overline{\left.v^{3}, v^{1,2,4}\right\}} *\left\{\overline{\left.v^{4}, v^{1,2,3}\right\}} * \overline{\left.v^{1,2}, v^{3,4}\right\}} *\left\{\overline{\left.v^{1,3}, v^{2,4}\right\}} * \overline{\left.v^{1,4}, v^{2,3}\right\}}\right.\right.$,
the join of 8 -copies of $S^{0}$. Therefore each maximal simplex of $\Delta_{4}$ is of dimension 7. So assume that $n \geq 5$. Inductively assume that result is true for each $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{r} ; 3\right)$, where $4 \leq r<n$.

Let $\sigma \in \Delta_{n}$ be a maximal simplex. Suppose $\sigma$ covers all places. Then from Lemma 4.1, $|N(v) \cap \sigma| \geq 1$ for all $v \in \sigma$. If there exists a vertex $w \in \sigma$ such that $N(w) \cap \sigma=\{v\}$, then from Lemma 4.3, $\sigma=N(v) \cup K_{v}^{i 0, j_{0}, k_{0}}$ for some $i_{0}, j_{0}, k_{0} \in[n]$. Clearly $\operatorname{dim}(\sigma)=n+3$. If $|N(v) \cap \sigma| \geq 2$ for all $v \in \sigma$, then from Lemma 4.6, there exist $v, w \in \sigma$ such that $v \sim w$ and $\sigma=N(v) \cup N(w)$. It is easy to check that $\operatorname{dim}(\sigma)=2 n-1$.

So, assume that $\sigma$ does not covers all places. There exists $l \in[n]$ such that $v(l)=w(l)$ for all $v, w \in \sigma$. Without loss of generality we assume that $v(l)=0$ for all $v \in \sigma$. Observe that $\sigma \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n}^{(l, 0)} ; 3\right)$. Since $\sigma$ is a maximal simplex in $\Delta_{n}, \sigma$ is maximal in $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n}^{(l, 0)} ; 3\right)$. Since $\mathbb{I}_{n}^{(l, 0)} \cong \mathbb{I}_{n-1}$, by induction hypothesis, either $\operatorname{dim}(\sigma)=7$ or; $\sigma=N_{\mathbb{I}_{n}^{l, 0}}(v) \cup N_{\mathbb{I}_{n}^{(l, 0)}}(w)$ for some $v, w \in V\left(\mathbb{I}_{n}^{(l, 0)}\right), v \in N_{\mathbb{I}_{n}^{(l, 0)}}(w)$ or $\sigma=N_{\mathbb{I}_{n}^{(l, 0)}}(u) \cup K_{u}^{i, j, k}$ for some $i, j, k \in[n] \backslash\{l\}$. Suppose $\operatorname{dim}(\sigma) \neq 7$. Then either $\sigma=N_{\mathbb{I}_{n}^{(l, 0)}}(v) \cup N_{\mathbb{I}_{n}^{(l, 0)}}(w)$ for some $v, w \in V\left(\mathbb{I}_{n}^{(l, 0)}\right), v \in N_{\mathbb{I}_{n}^{l(, 0)}}(w)$ or
$\sigma=N_{\mathbb{I}_{n}^{(l, 0)}}(v) \cup K_{v}^{i, j, k}$ for some $i, j, k \in[n] \backslash\{l\}$. In either case $v^{l} \notin \sigma$ and $\sigma \cup\left\{v^{l}\right\}$ is a simplex in $\Delta_{n}$, which contradicts the maximality of $\sigma$. Hence $\operatorname{dim}(\sigma)=7$. This completes the proof.
Remark 2. In [32], Kleitmann proved that for $n \geq 2 l+1$, the largest set family of subsets of $[n]$ with pairwise symmetric difference at most $2 l$ contains no more than $\sum_{t=0}^{l}\binom{n}{t}$ elements. Hence it gives the maximum possible size of a maximal simplex of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)(l=1)$ in Lemma 3.1 and the maximum possible size of a maximal simplex of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)(l=2)$ in Lemma 4.7
4.2. Collapsibility. In this section, we prove Theorem C. We first establish a few lemmas, which we need to prove Theorem C.

Let $X$ be a topological space and $A$ be a subspace of $X$. Recall that a retraction of $X$ onto $A$ is a map $r: X \rightarrow A$ such that $r(a)=a$ for all $a \in A$.
Lemma 4.8. Let $n>m$ and let $H$ be an $m$-dimensional cube subgraph of $\mathbb{I}_{n}$. Then there exists a retraction $r: \Delta_{n} \rightarrow \mathcal{V} \mathcal{R}(H ; 3)$.
Proof. Observe that, there exist sequences $\left(i_{1}, \ldots, i_{n-m}\right)$ and $\left(\epsilon_{1}, \ldots, \epsilon_{n-m}\right)$, where $i_{1}, \ldots, i_{n-m}$ $\in[n], \epsilon_{1}, \ldots, \epsilon_{n-m} \in\{0,1\}$ such that $H$ is the induced subgraph of $\mathbb{I}_{n}$ on the vertex set $\{v \in$ $\left.V\left(\mathbb{I}_{n}\right): v\left(i_{j}\right)=\epsilon_{j} \forall 1 \leq j \leq n-m\right\}$. Define $r_{1}: V\left(I_{n}\right) \rightarrow V\left(I_{n}^{i_{1}, \epsilon_{1}}\right)$ as follows: for $v \in V\left(\mathbb{I}_{n}\right)$ and $t \in[n]$,

$$
r_{1}(v)(t)= \begin{cases}v(t) & \text { if } t \neq i_{1} \\ \epsilon_{1} & \text { if } t=i_{1}\end{cases}
$$

We extend the map $r_{1}$ to $\tilde{r}_{1}: \Delta_{n} \rightarrow \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n}^{i_{1}, \epsilon_{1}} ; 3\right)$ by $\tilde{r}_{1}(\sigma):=\left\{r_{1}(v): v \in \sigma\right\}$ for all $\sigma \in \Delta_{n}$. Let $\sigma \in \Delta_{n}$ and let $v, w \in \sigma$. Then $d(v, w) \leq 3$. If $v\left(i_{1}\right)=w\left(i_{1}\right)$, then $r_{1}(v)=r_{1}(w)$ and therefore $d\left(r_{1}(v), r_{1}(w)\right)=d(v, w)$. If $v\left(i_{1}\right) \neq w\left(i_{1}\right)$, then $d\left(r_{1}(v), r_{1}(w)\right)=d(v, w)-1$. So, $d\left(r_{1}(v), r_{1}(w)\right) \leq d(v, w) \leq 3$. Thus, $\tilde{r}_{1}(\sigma) \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n}^{i_{1}, \epsilon_{1}} ; 3\right)$. Hence $\tilde{r}_{1}$ is well defined. Clearly $\tilde{r}_{1}$ is onto and for any $\sigma \in \mathcal{V} \mathcal{R}\left(\mathbb{I}_{n}^{i_{1}, \epsilon_{1}} ; 3\right), \tilde{r}_{1}(\sigma)=\sigma$. Hence $\tilde{r}_{1}$ is a retraction. If $m=n-1$, then we take $r=\tilde{r}_{1}$. Suppose $m<n-1$. Let $n-m=k$. Assume that we have a retraction $\tilde{r}_{k-1}: \Delta_{n} \rightarrow \mathcal{V} \mathcal{R}\left(H_{k-1} ; 3\right)$, where $H_{k-1}$ is the induced subgraph of $\mathbb{I}_{n}$ on the vertex set $\left\{v \in V\left(\mathbb{I}_{n}\right): v\left(i_{j}\right)=\epsilon_{j} \forall 1 \leq j \leq k-1\right\}$. Define $r_{k}: V\left(H_{k-1}\right) \rightarrow V(H)$ as follows: for $v \in V\left(H_{k-1}\right)$ and $t \in[n]$,

$$
r_{k}(v)(t)= \begin{cases}v(t) & \text { if } t \neq i_{k} \\ \epsilon_{k} & \text { if } t=i_{k}\end{cases}
$$

Extend the map $r_{k}$ to $\tilde{r}_{k}: \mathcal{V} \mathcal{R}\left(H_{k-1} ; 3\right) \rightarrow \mathcal{V} \mathcal{R}(H ; 3)$ by $\tilde{r}_{k}(\sigma):=\left\{r_{k}(v): v \in \sigma\right\}$ for all $\sigma \in \Delta_{n}$. Clearly, $\tilde{r}_{k}$ is a retraction. We take $r$ as the composition of the maps $\tilde{r}_{k}$ and $\tilde{r}_{k-1}$. This completes the proof.

Lemma 4.9. Let $n \geq 5$ and $\sigma \in \Delta_{n}$ be a maximal simplex. If for some $v,|N(v) \cap \sigma| \geq 4$, then $N[v] \subseteq \sigma$.

Proof. Let $|N(v) \cap \sigma| \geq 4$. Suppose $N(v) \nsubseteq \sigma$. Then there exists a $l_{0} \in[n]$ such that $v^{l_{0}} \notin \sigma$. Since $|N(v) \cap \sigma| \geq 4$, there exist $i_{0}, j_{0}, k_{0}, p_{0} \in[n]$ such that $\left\{v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{p_{0}}\right\} \subseteq \sigma$. Clearly $l_{0} \notin\left\{i_{0}, j_{0}, k_{0}, p_{0}\right\}$. Since $v^{l_{0}} \notin \sigma$ and $\sigma$ is a maximal simplex, there exists $x_{0} \in \sigma$ such that $d\left(x_{0}, v^{l_{0}}\right) \geq 4$. Observe that for any vertex $u$, if $d(v, u) \leq 2$, then $d\left(u, v^{l_{0}}\right) \leq 3$. Hence $d\left(v, x_{0}\right)=3$. Here, $x_{0}=v^{i, j, k}$ for some $i, j, k \in[n]$. If $i_{0} \notin\{i, j, k\}$, then $d\left(x_{0}, v^{i_{0}}\right)=4$, a contradiction as $v^{i_{0}} \in \sigma$. Hence $i_{0} \in\{i, j, k\}$. By similar arguments, we can show that $j_{0}, k_{0}, p_{0} \in\{i, j, k\}$. Hence $\left\{i_{0}, j_{0}, k_{0}, p_{0}\right\} \subseteq\{i, j, k\}$, which is not possible. Thus, $N(v) \subseteq \sigma$.

Suppose $v \notin \sigma$, then there exists a vertex $y_{0} \in \sigma$ such that $d\left(v, y_{0}\right) \geq 4$. Suppose $d\left(v, y_{0}\right)=4$. Let $y_{0}=v^{i, j, k, l}$. Since $n \geq 5$, there exists $t \in[n] \backslash\{i, j, k, l\}$. Then $d\left(y_{0}, v^{t}\right) \geq 4$, a contradiction as $v^{t} \in \sigma$. Hence $d\left(v, y_{0}\right) \geq 5$. But then $d\left(v^{i_{0}}, y_{0}\right) \geq 4$, again a contradiction. Hence $v \in \sigma$. Thus, $N[v] \subseteq \sigma$.

Lemma 4.10. Let $n \geq 5$ and let $\sigma \in \Delta_{n}$ be a maximal simplex. Suppose there exists a vertex $v$ such that $\left\{v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}}, v^{p_{0}}, v^{q_{0}}\right\} \subseteq \sigma$, where $p_{0}, q_{0} \notin\left\{i_{0}, j_{0}, k_{0}\right\}$. Then $\sigma=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$.

Proof. We first show that $v^{i_{0}} \in \sigma$. If $v^{i_{0}} \notin \sigma$, then there exists $y_{0} \in \sigma$ such that $d\left(v^{i_{0}}, y_{0}\right) \geq 4$. Observe that $d\left(v, y_{0}\right) \geq 3$. We have the following two cases.
(1) $d\left(v, y_{0}\right)=3$.

Here, $y_{0}=v^{i, j, k}$ for some $i, j, k \in[n]$. Since $d\left(y_{0}, v^{p_{0}}\right) \leq 3$ and $d\left(y_{0}, v^{q_{0}}\right) \leq 3$, we see that $p_{0}, q_{0} \in\{i, j, k\}$. Without loss of generality we assume that $i=p_{0}$ and $j=q_{0}$, i.e., $y_{0}=v^{p_{0}, q_{0}, k}$. Then either $d\left(y_{0}, v^{i_{0}, j_{0}}\right) \geq 4$, or $d\left(y_{0}, v^{i_{0}, k_{0}}\right) \geq 4$, or $d\left(y_{0}, v^{j_{0}, k_{0}}\right) \geq 4$, a contradiction as $v^{i_{0}, j_{0}}, v^{i_{0}, k_{0}}, v^{j_{0}, k_{0}} \in \sigma$.
(2) $d\left(v, y_{0}\right) \geq 4$.

Observe that if $d\left(v, y_{0}\right) \geq 5$, then $d\left(y_{0}, v^{p_{0}}\right) \geq 4$, which is not possible, since $y_{0}, v^{p_{0}} \in$ $\sigma$. Hence $d\left(v, y_{0}\right)=4$. There exist $i, j, k, l \in[n]$ such that $y_{0}=v^{i, j, k, l}$. Since $d\left(y_{0}, v^{p_{0}}\right) \leq$ 3 and $d\left(y_{0}, v^{q_{0}}\right) \leq 3$, we see that $p_{0}, q_{0} \in\{i, j, k, l\}$. Further, since $d\left(v^{i_{0}}, y_{0}\right) \geq 4, i_{0} \notin$ $\{i, j, k, l\}$. If $\left\{j_{0}, k_{0}\right\} \nsubseteq\{i, j, k, l\}$, then $d\left(y_{0}, v^{j_{0}, k_{0}}\right) \geq 4$, a contradiction as $v^{j_{0}, k_{0}} \in \sigma$. Hence $y_{0}=v^{p_{0}, q_{0}, j_{0}, k_{0}}$. But then $d\left(y_{0}, v^{i_{0}, j_{0}}\right) \geq 4$, a contradiction.
Hence $v^{i_{0}} \in \sigma$. By similar arguments $v^{j_{0}}, v^{k_{0}} \in \sigma$. Since $\left\{v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{p_{0}}, v^{q_{0}}\right\} \subseteq N(v) \cap \sigma$, from Lemma 4.9, $N[v] \subseteq \sigma$. Hence $N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}} \subseteq \sigma$. From Lemma 4.3, $N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$ is a maximal simplex and therefore $\sigma=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$.

Inspired by Lemma 4.7, we write the set of maximal simplices of $\Delta_{n}, M\left(\Delta_{n}\right)=\mathcal{A}_{n} \cup \mathcal{B}_{n} \cup \mathcal{C}_{n}$, where

$$
\begin{aligned}
\mathcal{A}_{n} & =\left\{\sigma \in M\left(\Delta_{n}\right): \sigma=N(v) \cup K_{v}^{i, j, k} \text { for some } v \in V\left(\mathbb{I}_{n}\right) \text { and } i, j, k \in[n]\right\}, \\
\mathcal{B}_{n} & =\left\{\sigma \in M\left(\Delta_{n}\right): \sigma=N(v) \cup N(w) \text { for some } v, w \in V\left(\mathbb{I}_{n}\right), v \sim w\right\} \text { and } \\
\mathcal{C}_{n} & =M\left(\Delta_{n}\right) \backslash\left(\mathcal{A}_{n} \cup \mathcal{B}_{n}\right) .
\end{aligned}
$$

Lemma 4.11. Let $n \geq 5$. Then by using a sequence of elementary 8 -collapses, $\Delta_{n}$ collapses to a subcomplex $\Delta_{n}^{\prime}$, where $M\left(\Delta_{n}^{\prime}\right)=\mathcal{B}_{n} \cup \mathcal{C}_{n} \cup\left\{K_{v}^{i, j, k} \cup\left\{v^{i}, v^{j}, v^{k}, v^{l}\right\}: v \in V\left(\mathbb{I}_{n}\right),\{i, j, k, l\} \subseteq[n]\right\}$.
Proof. Let $\sigma \in \mathcal{A}_{n}$. Then $\sigma=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$ for some $v \in V\left(\mathbb{I}_{n}\right)$ and $i_{0}, j_{0}, k_{0} \in[n]$. We will use the following claim.
Claim 2. $\Delta_{n}$ collapses to a subcomplex $X$, where the set of maximal simplices

$$
M(X)=M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{K_{v}^{i_{0}, j_{0}, k_{0}} \cup\left\{v^{i_{0}}, v^{j_{0}}, v^{k_{0}}, v^{i}\right\}: i \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}\right\}\right\} .
$$

Proof of Claim 2. Without loss of generality we assume that $\left\{i_{0}, j_{0}, k_{0}\right\}=\{1,2,3\}$. From Lemma 4.10, $\left(\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{4}, v^{5}\right\}, \sigma\right)$ is a collapsible pair. Thus, $\sigma \searrow \sigma \backslash\left\{v^{4}\right\}, \sigma \backslash\left\{v^{5}\right\}, \sigma \backslash$ $\left\{v^{1,2}\right\}, \sigma \backslash\left\{v^{1,3}\right\}, \sigma \backslash\left\{v^{2,3}\right\}$. Observe that $\sigma \backslash\left\{v^{1,2}\right\} \subseteq N(v) \cup N\left(v^{3}\right), \sigma \backslash\left\{v^{1,3}\right\} \subseteq N(v) \cup$ $N\left(v^{2}\right), \sigma \backslash\left\{v^{2,3}\right\} \subseteq N(v) \cup N\left(v^{1}\right)$. From Lemma 4.7, for any $u \sim w, N(u) \cup N(w)$ is a maximal simplex in $\Delta_{n}$ and therefore we see that $\Delta_{n}$ collapses to a subcomplex $X_{1}$, where $M\left(X_{1}\right)=M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{\sigma \backslash\left\{v^{4}\right\}, \sigma \backslash\left\{v^{5}\right\}\right\}$.
Hence claim is true if $n=5$. So assume that $n \geq 6$. From Lemma 4.10, $\left(\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{5}, v^{6}\right\}\right.$, $\sigma$ ) and ( $\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{4}, v^{6}\right\}, \sigma$ ) are collapsible pairs in $\Delta_{n}$. Hence $\left(\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{5}, v^{6}\right\}, \sigma \backslash\right.$ $\left\{v^{4}\right\}$ ) and ( $\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{4}, v^{6}\right\}, \sigma \backslash\left\{v^{5}\right\}$ ) are collapsible pairs in $X_{1}$. Thus, $\sigma \backslash\left\{v^{4}\right\} \searrow$ $\sigma \backslash\left\{v^{4}, v^{6}\right\}, \sigma \backslash\left\{v^{4}, v^{5}\right\}, \sigma \backslash\left\{v^{4}, v^{1,2}\right\}, \sigma \backslash\left\{v^{4}, v^{1,3}\right\}, \sigma \backslash\left\{v^{4}, v^{2,3}\right\}$ and $\sigma \backslash\left\{v^{5}\right\} \searrow \sigma \backslash\left\{v^{5}, v^{4}\right\}, \sigma \backslash$ $\left\{v^{5}, v^{6}\right\}, \sigma \backslash\left\{v^{5}, v^{1,2}\right\}, \sigma \backslash\left\{v^{5}, v^{1,3}\right\}, \sigma \backslash\left\{v^{5}, v^{2,3}\right\}$.

Observe that $\sigma \backslash\left\{v^{4}, v^{1,2}\right\}, \sigma \backslash\left\{v^{5}, v^{1,2}\right\} \subseteq N(v) \cup N\left(v^{3}\right), \sigma \backslash\left\{v^{4}, v^{1,3}\right\}, \sigma \backslash\left\{v^{5}, v^{1,3}\right\} \subseteq N(v) \cup$ $N\left(v^{2}\right), \sigma \backslash\left\{v^{4}, v^{2,3}\right\}, \sigma \backslash\left\{v^{5}, v^{2,3}\right\} \subseteq N(v) \cup N\left(v^{1}\right)$. Therefore, we conclude that $X_{1}$ collapses to the subcomplex $X_{2}$, where $M\left(X_{2}\right)=M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{\sigma \backslash\left\{v^{4}, v^{5}\right\}, \sigma \backslash\left\{v^{4}, v^{6}\right\}, \sigma \backslash\left\{v^{5}, v^{6}\right\}\right\}$.

Hence the claim is true if $n=6$. Let $n \geq 7$ and inductively assume that $\Delta_{n}$ collapses to the subcomplex $X_{n-5}$, where

$$
M\left(X_{n-5}\right)=M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{\sigma \backslash\left\{v^{l_{1}}, v^{l_{2}}, \ldots, v^{l_{n-5}}\right\}:\left\{l_{1}, l_{2}, \ldots, l_{n-5}\right\} \subseteq\{4,5, \ldots, n-1\}\right\} .
$$

Let $\left\{i_{1}, i_{2}, \ldots, i_{n-5}\right\} \subseteq\{4,5, \ldots, n-1\}$. Observe that $\left|\{4,5, \ldots, n-1, n\} \backslash\left\{i_{1}, i_{2}, \ldots, i_{n-5}\right\}\right|=$ 2 and $n \in\{4,5, \ldots, n-1, n\} \backslash\left\{i_{1}, i_{2}, \ldots, i_{n-5}\right\}$. Let $\{n, p\}=\{4,5, \ldots, n-1, n\} \backslash\left\{i_{1}, i_{2}, \ldots, i_{n-5}\right\}$. Using Lemma 4.10, we observe that $\sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}\right\}$ is the only maximal simplex in $X_{n-5}$,
which contains $\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{p}, v^{n}\right\}$. So, $\left(\left\{v^{1,2}, v^{1,3}, v^{2,3}, v^{p}, v^{n}\right\}, \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}\right\}\right)$ is a collapsible pair in $X_{n-5}$. Therefore $\sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}\right\} \searrow \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{p}\right\}, \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{n}\right\}$, $\sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{1,2}\right\}, \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{1,3}\right\}, \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{2,3}\right\}$. Clearly, $\sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}\right.$, $\left.v^{1,2}\right\} \subseteq N(v) \cup N\left(v^{3}\right), \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{1,3}\right\} \subseteq N(v) \cup N\left(v^{2}\right), \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{2,3}\right\} \subseteq$ $N(v) \cup N\left(v^{1}\right)$. Thus, we conclude that $X_{n-5}$ collapses to a subcomplex $X_{n-5}^{\prime}$, where

$$
\begin{aligned}
M\left(X_{n-5}^{\prime}\right)= & M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{\sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{p}\right\}, \sigma \backslash\left\{v^{i_{1}}, \ldots, v^{i_{n-5}}, v^{n}\right\}\right\} \cup \\
& \left\{\sigma \backslash\left\{v^{l_{1}}, v^{l_{2}}, \ldots, v^{l_{n-5}}\right\}:\left\{l_{1}, l_{2}, \ldots, l_{n-5}\right\} \subseteq\{4,5, \ldots, n-1\} \backslash\left\{i_{1}, \ldots, i_{n-5}\right\}\right\}
\end{aligned}
$$

By applying an argument similar as above for each $\left\{l_{1}, l_{2}, \ldots, l_{n-5}\right\} \subseteq\{4,5, \ldots, n-1\}$, we get that $X_{n-5}$ collapses to the subcomplex $X_{n-4}$, where

$$
M\left(X_{n-4}\right)=M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{\sigma \backslash\left\{v^{l_{1}}, v^{l_{2}}, \ldots, v^{l_{n-4}}\right\}:\left\{l_{1}, l_{2}, \ldots, l_{n-4}\right\} \subseteq\{4,5, \ldots, n\}\right\}
$$

Observe that $\left\{\sigma \backslash\left\{v^{l_{1}}, v^{l_{2}}, \ldots, v^{l_{n-4}}\right\}:\left\{l_{1}, l_{2}, \ldots, l_{n-4}\right\} \subseteq\{4,5, \ldots, n\}\right\}=\left\{K_{v}^{1,2,3} \cup\left\{v^{1}, v^{2}, v^{3}, v^{i}\right\}\right.$ : $i \in[n] \backslash\{1,2,3\}\}$. Thus, by induction we get that $\Delta_{n}$ collapses to a subcomplex $X_{n-4}$, where

$$
M\left(X_{n-4}\right)=M\left(\Delta_{n}\right) \backslash\{\sigma\} \cup\left\{K_{v}^{1,2,3} \cup\left\{v^{1}, v^{2}, v^{3}, v^{i}\right\}: 4 \leq i \leq n\right\}
$$

We take $X=X_{n-4}$ and this completes the proof of Claim 2.
By applying the Claim 2 for each $\tau \in \mathcal{A}_{n}$, we get that $\Delta_{n}$ collapses to a subcomplex $\Delta_{n}^{\prime}$, where $M\left(\Delta_{n}^{\prime}\right)=\mathcal{B}_{n} \cup \mathcal{C}_{n} \cup\left\{K_{v}^{i, j, k} \cup\left\{v^{i}, v^{j}, v^{k}, v^{l}\right\}: v \in V\left(\mathbb{I}_{n}\right),\{i, j, k, l\} \subseteq[n]\right\}$.

We are now ready to prove main theorem of this section.
Proof of Theorem $C$. We need to show that the collapsibility number of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$ is 8 . We first show that $\Delta_{n}$ is 8 -collapsible. It is easy to check that each maximal simplex of $\Delta_{4}$ is of dimension 7. Hence $\Delta_{4}$ is 8 -collapsible. So assume that $n \geq 5$. From Lemma 4.11, by using elementary 8-collapses, $\Delta_{n}$ collapses to a subcomplex $\Delta_{n}^{\prime}$, where $M\left(\Delta_{n}^{\prime}\right)=\mathcal{B}_{n} \cup \mathcal{C}_{n} \cup\left\{K_{v}^{i, j, k} \cup\left\{v^{i}, v^{j}, v^{k}, v^{l}\right\}\right.$ : $\left.v \in V\left(\mathbb{I}_{n}\right),\{i, j, k, l\} \subseteq[n]\right\}$. Let $\mathcal{D}_{n}=\left\{K_{v}^{i, j, k} \cup\left\{v^{i}, v^{j}, v^{k}, v^{l}\right\}: v \in V\left(\mathbb{I}_{n}\right),\{i, j, k, l\} \subseteq[n]\right\}$. Since $n \geq 5$, by using the cardinalities of the elements of $M\left(\Delta_{n}^{\prime}\right)$, we conclude that $M\left(\Delta_{n}^{\prime}\right)=$ $\mathcal{B}_{n} \sqcup \mathcal{C}_{n} \sqcup D_{n}$.

Choose a linear order $\prec_{1}$ on elements of $\mathcal{B}_{n}$. Extend $\prec_{1}$ to a linear order $\prec$ on maximal simplices of $\Delta_{n}^{\prime}$, where elements of $\mathcal{B}_{n}$ are ordered first, i.e., for any two $\sigma_{1}, \sigma_{2} \in M\left(\Delta_{n}^{\prime}\right)$, if $\sigma_{1} \in \mathcal{B}_{n}$ and $\sigma_{2} \in \mathcal{C}_{n} \cup \mathcal{D}_{n}$, then $\sigma_{1} \prec \sigma_{2}$. Let $\tau \in \Delta_{n}^{\prime}$. Let $\sigma$ be the smallest (with respect to $\prec)$ maximal simplex of $\Delta_{n}^{\prime}$ such that $\tau \subseteq \sigma$. If $\sigma \in \mathcal{C}_{n} \cup \mathcal{D}_{n}$, then $|\sigma|=8$ and thereby implying that $\left|M_{\prec}(\tau)\right| \leq 8$. So assume that $\sigma \in \mathcal{B}_{n}$. There exist $v, w \in V\left(\mathbb{I}_{n}\right)$ such that $v \sim w$ and $\sigma=N(v) \cup N(w)$. We first prove that $\left|M_{\prec}(\tau) \cap N(v)\right| \leq 4$.

Let $\operatorname{mes}_{\prec}(\tau)=\left(x_{1}, \ldots, x_{t}\right)$. Suppose $\left|M_{\prec}(\tau) \cap N(v)\right| \geq 5$. Let $k$ be the least integer such that $\left|\left\{x_{1}, \ldots, x_{k}\right\} \cap N(v)\right|=4$. Clearly, $k<t$. Let $\left\{x_{1}, \ldots, x_{k}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$. Observe that $x_{k} \in\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$. Let $\gamma$ be a maximal simplex such that $\gamma \prec \sigma$. If $\left\{x_{1}, \ldots, x_{k}\right\} \cap$ $(\sigma \backslash \gamma) \neq \emptyset$, then $x_{k+1} \in\left\{x_{1}, \ldots, x_{k}\right\}$. Hence $\left\{x_{1}, \ldots, x_{k+1}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$. If $\left\{x_{1}, \ldots, x_{k}\right\} \cap(\sigma \backslash \gamma)=\emptyset$, then $\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\} \subseteq \gamma$. From Lemma 4.9, $N(v) \subseteq \gamma$. Thus $x_{k+1} \notin N(v)$, thereby implying that $\left\{x_{1}, \ldots, x_{k+1}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$. If $k+1=t$, then we get a contradiction to the assumption that $\left|M_{\prec}(\tau) \cap N(v)\right| \geq 5$. Inductively assume that for all $k \leq l<t$, $\left\{x_{1}, \ldots, x_{l}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$. If $\left\{x_{1}, \ldots, x_{t-1}\right\} \cap(\sigma \backslash \gamma) \neq \emptyset$, then $x_{t} \in\left\{x_{1}, \ldots, x_{k}\right\}$. Hence $\left\{x_{1}, \ldots, x_{t}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$. If $\left\{x_{1}, \ldots, x_{t-1}\right\} \cap(\sigma \backslash \gamma)=\emptyset$, then $\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\} \subseteq \gamma$. From Lemma 4.9, $N(v) \subseteq \gamma$. Thus $x_{t} \notin N(v)$. Hence we get that $\left\{x_{1}, \ldots, x_{t}\right\} \cap N(v)=\left\{x_{i_{1}}, x_{i_{2}}, x_{i_{3}}, x_{i_{4}}\right\}$, which is a contradiction to the assumption that $\left|M_{\prec}(\tau) \cap N(v)\right| \geq 5$. Thus $\left|M_{\prec}(\tau) \cap N(v)\right| \leq 4$.

By using an argument similar as above, $\left|M_{\prec}(\tau) \cap N(w)\right| \leq 4$. Since $\tau \subseteq N(v) \cup N(w)$, we see that $M_{\prec}(\tau) \leq 8$. From Proposition 3.3, $\Delta_{n}$ is 8-collapsible.

Let $X$ be the Veitoris-Rips complex of a 4 -dimensional cube subgraph of $\mathbb{I}_{n}$. Then using Lemma 4.8, there exists a retraction $r: \Delta_{n} \rightarrow X$. Since $X \cong \Delta_{4}$ and $\Delta_{4} \cong S^{7}$, we see that $\tilde{H}_{7}(X ; \mathbb{Z}) \neq 0$. Further, since $r_{*}: \tilde{H}_{7}\left(\Delta_{n} ; \mathbb{Z}\right) \rightarrow \tilde{H}_{7}(X ; \mathbb{Z})$ is surjective, $\tilde{H}_{7}\left(\Delta_{n} ; \mathbb{Z}\right) \neq 0$. Using Proposition 1.2, we conclude that the collapsibility number of $\Delta_{n}$ is 8 .
4.3. Homology. The main aim of this section is to prove Theorem A. We first establish a series of lemmas, which we need to prove Theorem A. We always consider the reduced homology with integer coefficients.

For $1 \leq i \leq n$ and $\epsilon \in\{0,1\}$, let $\Delta_{n}^{i, \epsilon}=\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n}^{(i, \epsilon)} ; 3\right)$ and $\partial\left(\Delta_{n}\right)=\underset{i \in[n], \epsilon \in\{0,1\}}{ } \Delta_{n}^{i, \epsilon}$.
The following lemma plays a key role in the proof of Theorem A.
Lemma 4.12. Let $n \geq 5$ and let $p \leq n-2$. Then any $p$-cycle $c$ in $\Delta_{n}$ is homologous to $a$ p-cycle $\tilde{c}$ in $\partial\left(\Delta_{n}\right)$.

Proof. For any chain $z=\sum a_{i} \sigma_{i}$ in $\Delta_{n}$, if $a_{i} \neq 0$, then we say that $\sigma_{i} \in z$. For a cycle $z$ in $\Delta_{n}$, let $\mu(z)=\left\{\sigma \in z: \sigma \notin \partial\left(\Delta_{n}\right)\right\}$. Let $c$ be a $p$-cycle in $\Delta_{n}$. If $\mu(c)=\emptyset$, then $c$ is a $p$-cycle in $\partial\left(\Delta_{n}\right)$. Suppose $\mu(c) \neq \emptyset$. We show that $c$ is homologous to a $p$-cycle $c_{1}$ such that $\left|\mu\left(c_{1}\right)\right|<|\mu(c)|$. Let $\sigma \in c$ such that $\sigma \notin \partial\left(\Delta_{n}\right)$, i.e., $\sigma$ covers all places. Let $\tau$ be a maximal simplex such that $\sigma \subseteq \tau$. Using Lemmas 4.1, 4.3 and 4.6, we see that either $\tau=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$ for some $v$ and $i_{0}, j_{0}, k_{0} \in[n]$ or $\tau=N(v) \cup N(w)$ for some $v \sim w$.

Case 1. $\tau=N(v) \cup K_{v}^{i_{0}, j_{0}, k_{0}}$.
Let $i \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}\right\}$. Observe that for any $x \in \tau \backslash\left\{v^{i}\right\}, x(i)=v(i)$. Since $\sigma$ covers all places and $x(i)=v(i)$ for all $x \in \tau \backslash\left\{v^{i}\right\}$, we see that $v^{i} \in \sigma$. Thus, $\left\{v^{i}: i \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}\right\}\right\} \subseteq \sigma$. Clearly, $y(t)=v(t)$ for all $y \in\left\{v^{i}: i \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}\right\}\right\}$ and $t \in\left\{i_{0}, j_{0}, k_{0}\right\}$. Since $\sigma$ covers all places and $v^{i_{0}, j_{0}, k_{0}} \notin \tau$, we conclude that $|\sigma| \geq n-1$. Since $p \leq n-2$ and $\sigma$ is $p$-dimensional, we see that $|\sigma|=n-1$.

Suppose $v \in \sigma$. Let $\left\{x_{0}\right\}=\sigma \backslash\left\{\{v\} \cup\left\{v^{i}: i \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}\right\}\right\}\right\}$. For any $t \in\left\{i_{0}, j_{0}, k_{0}\right\}$ and $y \in\{v\} \cup\left\{v^{i}: i \in[n] \backslash\left\{i_{0}, j_{0}, k_{0}\right\}\right\}, y(t)=v(t)$. Hence the fact that $\sigma$ covers all places implies that $x_{0}=v^{i_{0}, j_{0}, k_{0}}$, which is not possible since $v^{i_{0}, j_{0}, k_{0}} \notin \tau$. So, $v \notin \sigma$. Clearly, $\sigma \cup\{v\} \in \Delta_{n}$.

Recall that for any simplex $\eta, B d(\eta)$ denotes the simplicial boundary of $\eta$. Let the coefficient of $\sigma$ in $c$ is $(-1)^{m} a_{\sigma}$ and the coefficient of $\sigma$ in $B d(\sigma \cup\{v\})$ is $(-1)^{s}$. Define a $p$-cycle $c_{1}$ as follows:

$$
c_{1}= \begin{cases}c-a_{\sigma} B d(\sigma \cup\{v\}) & \text { if } m \text { and } s \text { are of same parity, } \\ c+a_{\sigma} B d(\sigma \cup\{v\}) & \text { if } m \text { and } s \text { are of opposite parity. }\end{cases}
$$

Clearly, $c$ is homologous to $c_{1}$. Observe that $\sigma \notin c_{1}$. Let $\gamma \in c_{1}$ such that $\gamma \notin c$. Then observe that $v \in \gamma$ and $\gamma \subseteq \tau$. But we have seen above that if $v \in \gamma$, then $\gamma \in \partial\left(\Delta_{n}\right)$, i.e., $\gamma$ does not covers all places. Thus, we see that $\left|\mu\left(c_{1}\right)\right|<|\mu(c)|$. Since $|\mu(c)|$ is finite, by repeating the above argument finite number of times, we get a cycle $c_{k}$ such that $c$ is homologous to $c_{k}$ and $\left|\mu\left(c_{k}\right)\right|=0$, i.e., $c_{k}$ is a $p$-cycle in $\partial\left(\Delta_{n}\right)$. We take $\tilde{c}=c_{k}$.

Case 2. $\tau=N(v) \cup N(w)$.
For any $k \in[n]$ and $\gamma \in \Delta_{n}$, we say that $\gamma$ covers $k$-places, if there exist $i_{1}, \ldots, i_{k} \in[n]$ such for each $1 \leq l \leq k$, we get $x, y \in \gamma$ such that $x\left(i_{l}\right)=0$ and $y\left(i_{l}\right)=1$.

Observe that if $\sigma \subseteq N(v)$ or $\sigma \subseteq N(w)$, then $\sigma$ can covers at most $p+1$-places. Since $n>p+1, \sigma$ can not covers all places, a contradiction to the assumption that $\sigma$ covers all places. Hence $N(v) \cap \sigma \neq \emptyset$ and $N(w) \cap \sigma \neq \emptyset$.

Since $w \sim v, w=v^{q}$ for some $q \in[n]$. Suppose $v, w \in \sigma$. If $N(w) \cap \sigma=\{v\}$, then $\sigma=\left\{v, w, v^{i_{1}}, \ldots, v^{i_{p-1}}\right\}$ for some $i_{1}, i_{2}, \ldots, i_{p-1} \in[n] \backslash\{q\}$. Observe that $\sigma$ covers only $p$-places, namely $i_{1}, \ldots, i_{p-1}, q$. Hence $|N(w) \cap \sigma| \geq 2$. Then $\sigma=\left\{v, w, v^{i_{1}}, \ldots, v^{i_{s}}, v^{q, j_{1}}, \ldots, v^{q, j_{t}}\right\}$, for some $i_{1}, \ldots, i_{s}, j_{1} \ldots, j_{t} \in[n]$, where $s+t=p-1$. Here $\sigma$ can covers at most $p$ places, namely $i_{1}, \ldots, i_{s}, j_{1}, \ldots, j_{t}, q$ and $\sigma$ covers $p$ places only if $\left\{i_{1}, \ldots, i_{s}\right\} \cap\left\{j_{1}, \ldots, j_{t}\right\}=\emptyset$. Since $p<n, \sigma$ does not covers all places. Hence $\{v, w\} \nsubseteq \sigma$.

Suppose $v \in \sigma$. Then $w \notin \sigma$. If $N(w) \cap \sigma=\{v\}$, then $\sigma=\left\{v, v^{i_{1}}, \ldots, v^{i_{p}}\right\}$ for some $i_{1}, i_{2}, \ldots, i_{p} \in[n]$. Observe that $\sigma$ cover only $p$-places, namely $i_{1}, \ldots, i_{p}$. Hence $|N(w) \cap \sigma| \geq 2$. Let $\sigma=\left\{v, v^{i_{1}}, \ldots, v^{i_{s}}, v^{q, j_{1}}, \ldots, v^{q, j_{t}}\right\}$, where $i_{1}, \ldots, i_{s}, j_{1} \ldots, j_{t} \in[n]$ and $s+t=p$. Here $\sigma$ can covers at most $p+1$ places, namely $i_{1}, \ldots, i_{s}, j_{1}, \ldots, j_{t}, q$ and $\sigma$ covers $p+1$ places only if
$\left\{i_{1}, \ldots, i_{s}\right\} \cap\left\{j_{1}, \ldots, j_{t}\right\}=\emptyset, q \notin\left\{i_{1}, \ldots, i_{s}\right\}$. Thus, we conclude that $v \notin \sigma$. By an argument similar as above, $w \notin \sigma$.

Let the coefficient of $\sigma$ in $c$ is $(-1)^{m} a_{\sigma}$ and let the coefficient of $\sigma$ in $B d(\sigma \cup\{v\})$ is $(-1)^{r}$. Define a $p$-cycle $d_{1}$ as follows:

$$
d_{1}= \begin{cases}c-a_{\sigma} B d(\sigma \cup\{v\}) & \text { if } m \text { and } r \text { are of same parity } \\ c+a_{\sigma} B d(\sigma \cup\{v\}) & \text { if } m \text { and } r \text { are of opposite parity }\end{cases}
$$

Clearly, $c$ is homologous to $d_{1}$ and $\left|\mu\left(d_{1}\right)\right|<|\mu(c)|$. Since $|\mu(c)|$ is finite, by repeating the above argument finite number of times, we get a cycle $d_{k}$ such that $c$ is homologous to $d_{k}$ and $\left|\mu\left(d_{k}\right)\right|=0$. We take $\tilde{c}=d_{k}$.

This completes the proof.
For any $1 \leq t<n$ and two sequences $\left(\epsilon_{1}, \ldots, \epsilon_{t}\right)$ and $\left(j_{1}, \ldots, j_{t}\right)$, where for each $1 \leq l \leq t$, $\epsilon_{l} \in\{0,1\}$ and $j_{l} \in[n]$, let $\mathbb{I}_{n}^{\left(j_{1}, \epsilon_{1}\right), \ldots,\left(j_{t}, \epsilon_{t}\right)}$ denote the induced subgraph of $\mathbb{I}_{n}$ on the vertex set $\left\{v_{1} \ldots v_{n}: v_{j_{l}}=\epsilon_{l}, 1 \leq l \leq t\right\}$. Observe that for any $i<n$ and an $i$-dimensional cube subgraph $H$ of $\mathbb{I}_{n}$, there exist two sequences $\left(\epsilon_{1}, \ldots, \epsilon_{n-i}\right)$ and $\left(j_{1}, \ldots, j_{n-i}\right)$ such that $H=$ $\mathbb{I}_{n}^{\left(j_{1}, \epsilon_{1}\right), \ldots,\left(j_{n-i}, \epsilon_{n-i}\right)}$.

We now define a class, whose elements are the finite unions of the Vietoris-Rips complexes of cube graphs. For $n \geq 4$ and $3 \leq m \leq n$, let $\mathcal{W}_{n}^{m}$ denote the collection of all finite union $X=X_{1} \cup \ldots \cup X_{k}$ ( $k$ ranges over positive integers) such that $X$ satisfies the following conditions:

- for each $1 \leq j \leq k, X_{j}=\mathcal{V} \mathcal{R}\left(H_{j} ; 3\right)$ for some $m$-dimensional cube subgraph $H_{j}$ of $\mathbb{I}_{n}$.
- if $m \neq n$, then $X \subseteq \mathcal{V} \mathcal{R}(H ; 3)$ for some $(m+1)$-dimensional cube subgraph $H=$ $\mathbb{I}_{n}^{\left(j_{1}, \epsilon_{1}\right), \ldots,\left(j_{n-m-1}, \epsilon_{n-m-1}\right)}$ of $\mathbb{I}_{n}$. Further, if $X \neq \partial(\mathcal{V R}(H ; 3))$, then there exists $\lambda \in$ $[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}\right\}, \epsilon \in\{0,1\}$ such that $\mathcal{V} \mathcal{R}\left(H^{(\lambda, \epsilon)} ; 3\right) \subseteq X$ and $\mathcal{V} \mathcal{R}\left(H^{\left(\lambda, \epsilon^{\prime}\right)} ; 3\right) \nsubseteq X$, where $\epsilon^{\prime}=\{0,1\} \backslash\{\epsilon\}$.
Remark 3. Note that $\mathcal{W}_{n}^{n}=\left\{\Delta_{n}\right\}$ and $\partial\left(\Delta_{n}\right) \in \mathcal{W}_{n}^{n-1}$. Let $X=X_{1} \cup \ldots \cup X_{k} \in \mathcal{W}_{n}^{m}$ and suppose $X \subseteq \mathcal{V} \mathcal{R}(H ; 3)$, where $H=\mathbb{I}_{n}^{\left(j_{1}, \epsilon_{1}\right), \ldots,\left(j_{n-m-1}, \epsilon_{n-m-1}\right)}$. If $k=1$ or $X=\partial(\mathcal{V} \mathcal{R}(H ; 3))$, then clearly, $X$ is connected. If $X \neq \partial(\mathcal{V} \mathcal{R}(H ; 3))$, then there exists $\lambda \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}\right\}$ and $\epsilon \in\{0,1\}$ such that $\mathcal{V} \mathcal{R}\left(H^{(\lambda, \epsilon)} ; 3\right) \subseteq X$ and $\mathcal{V} \mathcal{R}\left(H^{\left(\lambda, \epsilon^{\prime}\right)} ; 3\right) \nsubseteq X$, where $\epsilon^{\prime}=\{0,1\} \backslash\{\epsilon\}$. Let $X_{p}=\mathcal{V} \mathcal{R}\left(H^{(\lambda, \epsilon)} ; 3\right)$. For each $p \neq j \in[k]$, since $X_{j} \cap X_{p} \cong \Delta_{m-1}$ is non empty and connected, we conclude that $X$ is connected.

For a $X=X_{1} \cup \ldots \cup X_{k} \in \mathcal{W}_{n}^{m}$, the following claim gives us a condition, when the intersection of $\bigcup_{l \neq i} X_{l}$ with $X_{i}(i \in[k])$ belongs to $\mathcal{W}_{n}^{m-1}$ or the intersection of $\bigcup_{l \neq i, j} X_{l}$ with $X_{i} \cap X_{j}(i, j \in[k])$ belongs to $\mathcal{W}_{n}^{m-2}$, which plays a key role in the proofs of Lemmas $4.15,4.18$ and 4.19 , while we use induction on $k$ and $m$.

Claim 3. Let $n \geq 4$ and $3 \leq m \leq n$ and let $X=X_{1} \cup \ldots \cup X_{k} \in \mathcal{W}_{n}^{m}$.
(i) If $k>1$, then there exists $q \in[k]$ such that $\bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m}$ and $X_{q} \cap \bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m-1}$.
(ii) If $k \geq 3$ and $m \geq 5$, then there exist $\lambda, q \in[k]$ such that for any subset $A \subseteq[k] \backslash\{q\}$ such that $\lambda \in A$ and $|A| \geq 2$, the following are true:

- there exists $p \in A \backslash\{\lambda\}$ such that if $\underset{i \in A \backslash\{p\}}{\bigcup}\left(X_{i} \cap X_{p} \cap X_{q}\right) \neq \emptyset$, then $\bigcup_{i \in A \backslash\{p\}}\left(X_{i} \cap\right.$

$$
\begin{aligned}
& \left.\quad X_{p} \cap X_{q}\right) \in \mathcal{W}_{n}^{m-2} \\
& \bullet \bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m} \text { and } X_{q} \cap \bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m-1} .
\end{aligned}
$$

Proof. Let $X \subseteq \mathcal{V} \mathcal{R}(H ; 3)$, where $H=\mathbb{I}_{n}^{\left(j_{1}, \epsilon_{1}\right), \ldots,\left(j_{n-m-1}, \epsilon_{n-m-1}\right)}$.
(i) If $X=\partial(\mathcal{V} \mathcal{R}(H ; 3))$, then choose $t \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}\right\}$ and $s \in\{0,1\}$. Clearly $\mathcal{V} \mathcal{R}\left(H^{t, s} ; 3\right) \subseteq X$. Without loss of generality we assume that $X_{1}=\mathcal{V} \mathcal{R}\left(H^{t, s} ; 3\right) \subseteq X$. Then $\bigcup_{j \neq 1} X_{j} \in \mathcal{W}_{n}^{m}$ and $X_{1} \cap \bigcup_{j \neq 1} X_{j}=\partial\left(X_{1}\right) \in \mathcal{W}_{n}^{m-1}$.

Let $X \neq \partial(\mathcal{V} \mathcal{R}(H ; 3))$. There exists $\mu \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}\right\}$ and $\epsilon \in\{0,1\}$ such that $\mathcal{V} \mathcal{R}\left(H^{(\mu, \epsilon)} ; 3\right) \subseteq X$ and $\mathcal{V} \mathcal{R}\left(H^{\left(\mu, \epsilon^{\prime}\right)} ; 3\right) \nsubseteq X$, where $\epsilon^{\prime}=\{0,1\} \backslash\{\epsilon\}$. Then $\mathcal{V R}\left(H^{(\mu, \epsilon)} ; 3\right)=X_{p}$ for some $1 \leq p \leq k$. Choose $l \in[n] \backslash\left\{j_{1}, \ldots j_{n-m-1}, \mu\right\}$ and $s \in\{0,1\}$ such that $\mathcal{V R}\left(H^{(l, s)} ; 3\right) \subseteq X$ (such $l$ exists because $m \geq 3$ ). There exists $q \in[k] \backslash\{p\}$ such that $X_{q}=\mathcal{V} \mathcal{R}\left(H^{(l, s)} ; 3\right)$. Let $Y=\bigcup_{j \neq q} X_{j}$. Since $X_{q} \cap X_{p} \neq \emptyset, X_{q} \cap Y \neq \emptyset$. Clearly, $Y \in \mathcal{W}_{n}^{m}$. Further, $X_{q} \cap Y \subseteq \mathcal{V} \mathcal{R}(T ; 3)$, where $T=\mathbb{I}_{n}^{\left(j_{1}, \epsilon_{1}\right), \ldots,\left(j_{n-m-1}, \epsilon_{n-m-1}\right),(l, s)}$. Observe that $\mathcal{V} \mathcal{R}\left(T^{(\mu, \epsilon)} ; 3\right) \subseteq X_{q} \cap Y$ and $\mathcal{V} \mathcal{R}\left(T^{\left(\mu, \epsilon^{\prime}\right)} ; 3\right) \nsubseteq X_{q} \cap Y$. Hence $X_{q} \cap Y \in \mathcal{W}_{n}^{m-1}$.
(ii) If $X=\partial(\mathcal{V} \mathcal{R}(H ; 3))$. Then choose $t \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}\right\}$. Take $X_{q}=\mathcal{V} \mathcal{R}\left(H^{(t, 0)} ; 3\right)$ and $X_{\lambda}=\mathcal{V} \mathcal{R}\left(H^{(t, 1)} ; 3\right)$. Then clearly, $\bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m}, X_{q} \cap \bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m-1}$. Further, since $X_{\lambda} \cap X_{q}=\emptyset$, for any choice of $p$ and $A$ containing $\lambda$, we get $\underset{i \in A \backslash\{p\}}{ }\left(X_{i} \cap X_{p} \cap X_{q}\right)=$ $\emptyset$, and therefore result is true.

Assume $X \neq \partial(\mathcal{V} \mathcal{R}(H ; 3))$. There exists $\mu \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}\right\}$ and $\epsilon \in\{0,1\}$ such that $\mathcal{V} \mathcal{R}\left(H^{(\mu, \epsilon)} ; 3\right) \subseteq X$ and $\mathcal{V} \mathcal{R}\left(H^{\left(\mu, \epsilon^{\prime}\right)} ; 3\right) \nsubseteq X$, where $\epsilon^{\prime}=\{0,1\} \backslash\{\epsilon\}$. Hence $X_{\lambda}=$ $\mathcal{V} \mathcal{R}\left(H^{(\mu, \epsilon)} ; 3\right)$ for some $\lambda \in[k]$. Choose $t_{1} \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}, \mu\right\}$ and $s_{1} \in\{0,1\}$ such that $\mathcal{V} \mathcal{R}\left(H^{\left(t_{1}, s_{1}\right)} ; 3\right) \subseteq X$. There exists $q \in[k] \backslash\{\lambda\}$ such that $X_{q}=\mathcal{V} \mathcal{R}\left(H^{\left(t_{1}, s_{1}\right)} ; 3\right)$. Let $A \subset[k] \backslash\{q\}$ such that $\lambda \in A$ and $|A| \geq 2$. Choose $p \in A \backslash\{\lambda\}$. There exists $t_{2} \in[n] \backslash\left\{j_{1}, \ldots, j_{n-m-1}, \mu\right\}$ and $s_{2} \in\{0,1\}$ such that $X_{p}=\mathcal{V} \mathcal{R}\left(H^{\left(t_{2}, s_{2}\right)} ; 3\right)$. Since $p \neq q,\left(t_{1}, s_{1}\right) \neq\left(t_{2}, s_{2}\right)$. Let $Z_{A}=\underset{i \in A \backslash\{p\}}{\bigcup}\left(X_{i} \cap X_{q}\right)$. If $Z_{A} \cap X_{p} \cap X_{q} \neq \emptyset$, then it is easy to check that $Z_{A} \cap X_{p} \cap X_{q} \subseteq \partial\left(\mathcal{V R}\left(H^{\left(t_{1}, s_{1}\right),\left(t_{2}, s_{2}\right)} ; 3\right)\right), \mathcal{V} \mathcal{R}\left(H^{(\mu, \epsilon),\left(t_{1}, s_{1}\right),\left(t_{2}, s_{2}\right)} ; 3\right) \subseteq$ $Z_{A} \cap X_{p} \cap X_{q}$ and $\mathcal{V} \mathcal{R}\left(H^{\left(\mu, \epsilon^{\prime}\right),\left(t_{1}, s_{1}\right),\left(t_{2}, s_{2}\right)} ; 3\right) \nsubseteq Z_{A} \cap X_{p} \cap X_{q}$. Hence $Z_{A} \cap X_{p} \cap X_{q} \in \mathcal{W}_{n}^{m-2}$. Clearly, $\bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m}$ and $X_{q} \cap \bigcup_{j \neq q} X_{j} \in \mathcal{W}_{n}^{m-1}$.

The nerve of a family of sets $\left(A_{i}\right)_{i \in I}$ is the simplicial complex $\mathbf{N}=\mathbf{N}\left(\left\{A_{i}\right\}\right)$ defined on the vertex set $I$ so that a finite subset $\sigma \subseteq I$ is in $\mathbf{N}$ precisely when $\bigcap_{i \in \sigma} A_{i} \neq \emptyset$.
Proposition 4.13. [13, Theorem 10.6] Let $\Delta$ be a simplicial complex and $\left(\Delta_{i}\right)_{i \in I}$ be a family of subcomplexes such that $\Delta=\bigcup_{i \in I} \Delta_{i}$. Suppose every nonempty finite intersection $\Delta_{i_{1}} \cap \ldots \cap \Delta_{i_{t}}$ for $i_{j} \in I, t \in \mathbb{N}$ is contractible, then $\Delta$ and $\mathbf{N}\left(\left\{\Delta_{i}\right\}\right)$ are homotopy equivalent.
Lemma 4.14. For any $X \in \mathcal{W}_{4}^{3}, \tilde{H}_{j}(X)=0$ for $0 \leq j \leq 2$.
Proof. Let $X=X_{1} \cup \ldots \cup X_{k}$. Observe that each non empty intersection $X_{i_{1}} \cap \ldots \cap X_{i_{t}}$ is homeomorphic to Vietoris-Rips complex of some cube subgraph of dimension less than 4 and therefore contractible. From Proposition 4.13, $X \simeq \mathbf{N}\left(\left\{X_{i}\right\}\right)$. For any $i, j \in[4]$ and $\epsilon, \delta \in\{0,1\}$, let $\overline{\{(i, \epsilon),(j, \delta)\}}$ be a simplicial complex on vertex set $\{(i, \epsilon),(j, \delta)\}$, which is isomorphic to $S^{0}$. If $X=\partial\left(\Delta_{4}\right)$, i.e., $X=\underset{i \in[4], \epsilon \in\{0,1\}}{ } \Delta_{4}^{i, \epsilon}$, then it is easy to check that

$$
\mathbf{N}\left(\left\{X_{i}\right\}\right) \cong \overline{\{(1,0),(1,1)\}} * \overline{\{(2,0),(2,1)\}} * \overline{\{(3,0),(3,1)\}} * \overline{\{(4,0),(4,1)\}},
$$

the join of 4-copies of $S^{0}$. Hence $X \simeq \mathbf{N}\left(\left\{X_{i}\right\}\right) \simeq S^{3}$ and therefore $\tilde{H}_{j}(X)=0$ for $0 \leq j \leq 2$.
If $X \neq \underset{i \in[4], \epsilon \in\{0,1\}}{ } \Delta_{4}^{i, \epsilon}$, then there exists $p \in[4], \epsilon \in\{0,1\}$ such that $\Delta_{4}^{p, \epsilon} \subseteq X$ but $\Delta_{4}^{p, \epsilon^{i}} \nsubseteq X$, where $\left\{\epsilon^{\prime}\right\}=\{0,1\} \backslash\{\epsilon\}$. It is easy to check that $\mathbf{N}\left(\left\{X_{i}\right\}\right)$ is a cone over the vertex $(p, \epsilon)$ and therefore it is contractible. Thus, we conclude that $\tilde{H}_{j}(X)=0$ for $0 \leq j \leq 2$.
Lemma 4.15. Let $n \geq 5$ and $4 \leq m \leq n$. For any $X \in \mathcal{W}_{n}^{m}$ and $0 \leq j \leq 3, \tilde{H}_{j}(X)=0$.
Proof. Let $X=X_{1} \cup \ldots \cup X_{p} \in \mathcal{W}_{n}^{m}$. Proof is by induction on $m$ and $p$. Let $m=4$. If $p=1$, then $X \cong \Delta_{4} \simeq S^{7}$. Hence $\tilde{H}_{j}(X)=0$ for $j \leq 3$. Let $p \geq 2$. Inductively assume that for any
$l<p$ and $i_{1}, \ldots, i_{l} \in[p]$, if $X_{i_{1}} \cup \ldots \cup X_{i_{l}} \in \mathcal{W}_{n}^{m}$, then $\tilde{H}_{j}\left(X_{i_{1}} \cup \ldots \cup X_{i_{l}}\right)=0$ for $j \leq 3$. From Claim 3 ( $i$, there exists $t \in[p]$ such that $\bigcup_{i \neq t} X_{i} \in \mathcal{W}_{n}^{m}$ and $X_{t} \cap \bigcup_{i \neq t} X_{i} \in \mathcal{W}_{n}^{m-1}$. Without loss of generality assume that $t=p$ and $Y=X_{1} \cup \ldots \cup X_{p-1}$. Then $\tilde{H}_{j}(Y)=0$ and $\tilde{H}_{j}\left(X_{p}\right)=0$ for $j \leq 3$. By Mayer-Vietoris sequence for homology, we have

$$
\cdots \longrightarrow \tilde{H}_{j}(Y) \oplus \tilde{H}_{j}\left(X_{p}\right) \longrightarrow \tilde{H}_{j}(X) \longrightarrow \tilde{H}_{j-1}\left(Y \cap X_{p}\right) \longrightarrow \tilde{H}_{j-1}(Y) \oplus \tilde{H}_{j-1}\left(X_{p}\right) \longrightarrow \cdots
$$

Since $Y \cap X_{p} \in \mathcal{W}_{n}^{3}, \tilde{H}_{j}\left(Y \cap X_{p}\right)=0$ for $j \leq 2$ by Lemma 4.14, . Thus, we conclude that $\tilde{H}_{j}(X)=0$ for $j \leq 3$. So for $m=4$, result is true. Let $m \geq 5$.

Induction hypothesis 1: For any $4 \leq r<m$ and $j \leq 3$, if $Y \in \mathcal{W}_{n}^{r}$, then $\tilde{H}_{j}(Y)=0$.
Let $4 \leq r<m$ and $Z \in \mathcal{W}_{n}^{r+1}$. Then $Z=Z_{1} \cup \ldots \cup Z_{q}$ for some $q$, where each $Z_{i}$ is the Vietoris-Rips complex of a $r+1$-dimensional cube subgraph of $\mathbb{I}_{n}$. We show that $\tilde{H}_{j}(Z)=0$ for $j \leq 3$. Proof is by induction on $q$. If $q=1$, then $Z \cong \Delta_{r+1}$. Since $r+1 \geq 5$, from Lemma 4.12, any $i$-cycle $c$ in $Z$ is homologous to an $i$-cycle $\tilde{c}$ in $\partial(Z)$ for $i \leq 3$. Hence it is enough to show that $\tilde{H}_{j}(\partial(Z))=0$ for $j \leq 3$. Clearly, $\partial(Z) \in \mathcal{W}_{n}^{r}$. From induction hypothesis 1 , we get that $\tilde{H}_{j}(\partial(Z))=0$ for $j \leq 3$. So assume that $q \geq 2$.

Induction hypothesis 2: For any $l<q, i_{1}, \ldots, i_{l} \in[q]$ and $j \leq 3$, if $Z_{i_{1}} \cup \ldots \cup Z_{i_{l}} \in \mathcal{W}_{n}^{r+1}$, then $\tilde{H}_{j}\left(Z_{i_{1}} \cup \ldots \cup Z_{i_{l}}\right)=0$.

From Claim $3(i)$, there exists $t \in[q]$ such that $\bigcup_{i \neq t} Z_{i} \in \mathcal{W}_{n}^{r+1}$ and $Z_{t} \cap \bigcup_{t \neq j} Z_{j} \in \mathcal{W}_{n}^{r}$. Without loss of generality assume that $t=q$. Let $U=Z_{1} \cup \ldots \cup Z_{q-1}$. Then $U \in \mathcal{W}_{n}^{r+1}$ and by induction hypothesis $2, \tilde{H}_{j}(U)=0$ for $0 \leq j \leq 3$. By Mayer-Vietoris sequence for homology, we have

$$
\cdots \longrightarrow \tilde{H}_{j}(U) \oplus \tilde{H}_{j}\left(Z_{q}\right) \longrightarrow \tilde{H}_{j}(Z) \longrightarrow \tilde{H}_{j-1}\left(U \cap Z_{q}\right) \longrightarrow \tilde{H}_{j-1}(U) \oplus \tilde{H}_{j-1}\left(Z_{q}\right) \longrightarrow \cdots
$$

From induction hypothesis $1, \tilde{H}_{j}\left(U \cap Z_{q}\right)=0$ for $0 \leq j \leq 3$. Therefore, we conclude that $\tilde{H}_{j}(Z)=0$ for $0 \leq j \leq 3$.

Thus, the proof is complete by induction.
Lemma 4.16. Let $n \geq m \geq 6$ and $k \geq 3$. For each $i \in[k]$, let $X_{i}$ be the Vietoris-Rips complex of some $m$-dimensional cube subgraph of $\mathbb{I}_{n}$. Let $Y=\bigcup_{l=1}^{k-2} X_{l} \cap X_{k}$ and $Y^{\prime}=X_{k-1} \cap X_{k}$ such that if $Y \cap Y^{\prime} \neq \emptyset$, then $Y \cap Y^{\prime} \in \mathcal{W}_{n}^{m-2}$. Then for each $x \in \tilde{H}_{4}\left(\bigcup_{l=1}^{k-1} X_{l} \cap X_{k}\right)$, there exist $x_{1} \in \tilde{H}_{4}(Y)$ and $x_{2} \in \tilde{H}_{4}\left(Y^{\prime}\right)$ such that $x=x_{1}+x_{2}$.
Proof. Let $X=\bigcup_{l=1}^{k-1} X_{l} \cap X_{k}$. Then $X=Y \cup Y^{\prime}$. Let $x \in \tilde{H}_{4}(X)$. If $Y \cap Y^{\prime}=\emptyset$, then $\tilde{H}_{4}(X)=\tilde{H}_{4}(Y) \oplus \tilde{H}_{4}\left(Y^{\prime}\right)$. Hence $x=x_{1}+x_{2}$ for some $x_{1} \in \tilde{H}_{4}(Y)$ and $x_{2} \in \tilde{H}_{4}\left(Y^{\prime}\right)$. Let $Y \cap Y^{\prime} \neq \emptyset$. By Mayer-Vietoris sequence for homology, we get

$$
\cdots \longrightarrow \tilde{H}_{4}(Y) \oplus \tilde{H}_{4}\left(Y^{\prime}\right) \xrightarrow{\psi} \tilde{H}_{4}(X) \xrightarrow{\phi} \tilde{H}_{3}\left(Y \cap Y^{\prime}\right) \longrightarrow \tilde{H}_{3}(Y) \oplus \tilde{H}_{3}\left(Y^{\prime}\right) \longrightarrow \cdots
$$

Since $m-2 \geq 4$, from Lemma $4.15 \tilde{H}_{3}\left(Y \cap Y^{\prime}\right)=0$. Hence $\psi: \tilde{H}_{4}(Y) \oplus \tilde{H}_{4}\left(Y^{\prime}\right) \longrightarrow \tilde{H}_{4}(X)$ given by $(\alpha, \beta) \mapsto \alpha+\beta$ is surjective. Thus $x=x_{1}+x_{2}$ for some $x_{1} \in \tilde{H}_{4}(Y)$ and $x_{2} \in$ $\tilde{H}_{4}\left(Y^{\prime}\right)$.
Lemma 4.17. Let $n \geq m \geq 6$ and $k \geq 2$. Let $X=X_{1} \cup \ldots \cup X_{k}$, where each $X_{i}$ is VietorisRips complex of some $m$-dimensional cube subgraph of $\mathbb{I}_{n}$. Further, assume that if $k \geq 3$, then for any set $A \subseteq[k-1]$ such that $1 \in A$ and $|A| \geq 2$, there exists $p \in A \backslash\{1\}$ such that $\underset{i \in A \backslash\{p\}}{\bigcup}\left(X_{i} \cap X_{p} \cap X_{k}\right) \neq \emptyset$ implies $\underset{i \in A \backslash\{p\}}{\bigcup}\left(X_{i} \cap X_{p} \cap X_{k}\right) \in \mathcal{W}_{n}^{m-2}$. Then the map $i_{*}$ : $\tilde{H}_{4}\left(\bigcup_{l=1}^{k-1} X_{l} \cap X_{k}\right) \rightarrow \tilde{H}_{4}\left(X_{k}\right)$ induced by the inclusion $\bigcup_{l=1}^{k-1} X_{l} \cap X_{k} \hookrightarrow X_{k}$, is injective.

Proof. Let $Y=\bigcup_{l=1}^{k-1} X_{l}$. If $Y \cap X_{k}=\emptyset$, then result is vacuously true. So assume that $Y \cap X_{k} \neq \emptyset$. If $k=2$, then $Y \cap X_{k} \cong \Delta_{m-1}$. From Lemma 4.8, there exists a retraction $X_{k} \rightarrow Y \cap X_{k}$ and therefore $i_{*}: \tilde{H}_{4}\left(Y \cap X_{k}\right) \rightarrow \tilde{H}_{4}\left(X_{k}\right)$ is injective.

Let $k \geq 3$ and inductively assume that for any $1 \leq t<k-1$ and $1 \in\left\{j_{1}, \ldots, j_{t}\right\} \subseteq[k-1]$, the map $i_{*}: \tilde{H}_{4}\left(\bigcup_{l=1}^{t} X_{j_{l}} \cap X_{k}\right) \rightarrow \tilde{H}_{4}\left(X_{k}\right)$ induced by the inclusion $\bigcup_{l=1}^{t} X_{j_{l}} \cap X_{k} \hookrightarrow X_{k}$, is injective.

Let $B=\left\{j_{1}, \ldots, j_{t+1}\right\} \subseteq[k-1]$ such that $1 \in B$. Let $Z=\bigcup_{l=1}^{t+1} X_{j_{l}}$. We show that the map $i_{*}: \tilde{H}_{4}\left(Z \cap X_{k}\right) \rightarrow \tilde{H}_{4}\left(X_{k}\right)$ is injective.

Let $0 \neq x \in \tilde{H}_{4}\left(Z \cap X_{k}\right)$. There exists $p \in B, p \neq 1$ such that $\underset{i \in B \backslash\{p\}}{\bigcup}\left(X_{i} \cap X_{p} \cap X_{k}\right) \neq \emptyset$ implies $\underset{i \in B \backslash\{p\}}{\bigcup}\left(X_{i} \cap X_{p} \cap X_{k}\right) \in \mathcal{W}_{n}^{m-2}$. From Lemma 4.16, there exist $x_{1} \in \tilde{H}_{4}\left(\underset{i \in B \backslash\{p\}}{\bigcup} X_{i} \cap X_{k}\right)$, $x_{2} \in \tilde{H}_{4}\left(X_{p} \cap X_{k}\right)$ such that $x=x_{1}+x_{2}$. Suppose $i_{*}(x)=0$ in $\tilde{H}_{4}\left(X_{k}\right)$. Since $x \neq 0$, at least one of $x_{1}$ or $x_{2}$ is a non zero element of $\tilde{H}_{4}\left(Z \cap X_{k}\right)$. Let $x_{1} \neq 0$ in $\tilde{H}_{4}\left(Z \cap X_{k}\right)$. Then $x_{1} \neq 0$ in $\tilde{H}_{4}\left(\bigcup_{i \in B \backslash\{p\}} X_{i} \cap X_{k}\right)$. From induction hypothesis the map $j_{*}: \tilde{H}_{4}\left(\bigcup_{i \in B \backslash\{p\}} X_{i} \cap X_{k}\right) \rightarrow \tilde{H}_{4}\left(X_{k}\right)$ induced by the inclusion $j: \bigcup_{i \in B \backslash\{p\}} X_{i} \cap X_{k} \rightarrow X_{k}$, is injective and therefore $j_{*}\left(x_{1}\right) \neq 0$. Since $i \in B \backslash\{p\}$ $i_{*}\left(x_{1}\right)=j_{*}\left(x_{1}\right)$, we see that $i_{*}\left(x_{1}\right) \neq 0$. Further, $i_{*}(x)=i_{*}\left(x_{1}+x_{2}\right)=x_{1}+x_{2}=0$ implies that $x_{1}=-x_{2}$. The injectivity of the map $j_{*}: \tilde{H}_{4}\left(\bigcup_{i \in B \backslash\{p\}} X_{i} \cap X_{k}\right) \rightarrow \tilde{H}_{4}\left(X_{k}\right)$ implies that $\tilde{H}_{4}\left(\underset{i \in B \backslash\{p\}}{\bigcup} X_{i} \cap X_{k}\right)$ is a subgroup of $\tilde{H}_{4}\left(X_{k}\right)$. Hence $x_{2} \in \tilde{H}_{4}\left(\underset{i \in B \backslash\{p\}}{\bigcup} X_{i} \cap X_{k}\right)$. Therefore $x_{1}+x_{2}=0$ in $\tilde{H}_{4}\left(\underset{i \in B \backslash\{p\}}{\bigcup} X_{i} \cap X_{k}\right)$. Hence $x=x_{1}+x_{2}=0$ in $\tilde{H}_{4}\left(Z \cap X_{k}\right)$, a contradiction. By an argument similar as above, we can show that, if $x_{2} \neq 0$, then $x_{1}+x_{2}=0$ in $\tilde{H}_{4}\left(Z \cap X_{k}\right)$, a contradiction. Thus $x \neq 0$ implies $i_{*}(x) \neq 0$. Therefore $i_{*}$ is injective. The proof is complete by induction.
Lemma 4.18. Let $n \geq 6$. For $X \in \mathcal{W}_{n}^{6}, \tilde{H}_{5}(X)=0$.
Proof. Let $X=X_{1} \cup \ldots \cup X_{k}$, where each $X_{i}$ is the Vietoris-Rips complex of a 6-dimensional cube subgraph of $\mathbb{I}_{n}$. If $k=1$, then $X \cong \Delta_{6}$ and hence result is true by Proposition 1.1. Let $k>1$ and assume that for any $l<k$ and $i_{1}, \ldots, i_{l} \in[k]$, if $X_{i_{1}} \cup \ldots \cup X_{i_{l}} \in \mathcal{W}_{n}^{6}$, then $\tilde{H}_{5}\left(X_{i_{1}} \cup \ldots \cup X_{i_{l}}\right)=0$.

If $k=2$, then from Claim $3(i)$, there exists $q_{1} \in[k]$ such that $\bigcup_{j \neq q_{1}} X_{j} \in \mathcal{W}_{n}^{6}$ and $X_{q_{1}} \cap$ $\bigcup_{j \neq q_{1}} X_{j} \in \mathcal{W}_{n}^{5}$.
Further, if $k \geq 3$, then from Claim 3 (ii) there exist $\lambda, q_{2} \in[k]$ such that $\underset{j \neq q_{2}}{\bigcup} X_{j} \in \mathcal{W}_{n}^{6}$, $X_{q_{2}} \cap \bigcup_{j \neq q_{2}} X_{j} \in \mathcal{W}_{n}^{5}$ and for any subset $A \subseteq[k] \backslash\left\{q_{2}\right\}$ containing $\lambda$, there exists $p \in A \backslash\{\lambda\}$ such that $\bigcup_{i \in A \backslash\{p\}}^{j \neq q_{2}}\left(X_{i} \cap X_{p} \cap X_{q_{2}}\right) \neq \emptyset$ implies $\bigcup_{i \in A \backslash\{p\}}\left(X_{i} \cap X_{p} \cap X_{q_{2}}\right) \in \mathcal{W}_{n}^{4}$.

Without loss of generality we assume that if $k=2$, then $q_{1}=k$ and if $k \geq 3$, then $q_{2}=k, \lambda=1$ and for $A=[k-1], p=k-1$. Let $Y=X_{1} \cup \ldots \cup X_{k-1}$. Then by induction hypothesis $\tilde{H}_{5}(Y)=0$ and $\tilde{H}_{5}\left(X_{k}\right)=0$. By Mayer-Vietoris sequence for homology, we have

$$
\cdots \longrightarrow \tilde{H}_{5}(Y) \oplus \tilde{H}_{5}\left(X_{p}\right) \longrightarrow \tilde{H}_{5}(X) \longrightarrow \tilde{H}_{4}\left(Y \cap X_{p}\right) \xrightarrow{h_{4}} \tilde{H}_{4}(Y) \oplus \tilde{H}_{4}\left(X_{p}\right) \longrightarrow \cdots
$$

Using Lemma 4.17, we conclude that the map $i_{*}: \tilde{H}_{4}\left(Y \cap X_{k}\right) \longrightarrow \tilde{H}_{4}\left(X_{k}\right)$ induced by the inclusion $Y \cap X_{k} \hookrightarrow X_{k}$, is injective and therefore the map $h_{4}: \tilde{H}_{4}\left(Y \cap X_{k}\right) \longrightarrow \tilde{H}_{4}(Y) \oplus \tilde{H}_{4}\left(X_{k}\right)$ is also injective. Since $\tilde{H}_{5}(Y)=0$ and $\tilde{H}_{5}\left(X_{k}\right)=0$, we conclude that $\tilde{H}_{5}(X)=0$.

Lemma 4.19. Let $m \geq 7$. For any $X \in \mathcal{W}_{n}^{m}$ and $j \in\{5,6\}, \tilde{H}_{j}(X)=0$.
Proof. Let $X=X_{1} \cup \ldots \cup X_{p}$, where each $X_{i}$ is the Vietoris-Rips complex of an $m$-dimensional cube subgraph of $\mathbb{I}_{n}$. Proof is by induction on $m$ and $p$. Let $m=7$. We show that $\tilde{H}_{j}(X)=0$ if $j \in\{5,6\}$.

Proof is by induction on $p$. If $p=1$, then $X \simeq \Delta_{7}$ and therefore result follows from Proposition 1.1. Let $p>1$. Inductively assume that for any $l<p$ and $i_{1}, \ldots, i_{l} \in[p]$, if $X_{i_{1}} \cup \ldots \cup X_{i_{l}} \in \mathcal{W}_{n}^{7}$, then $\tilde{H}_{j}\left(X_{i_{1}} \cup \ldots \cup X_{i_{l}}\right)=0$ for $j \in\{5,6\}$. From Claim 3 ( $i$ ), there exists $t \in[p]$ such that $\bigcup_{j \neq t} X_{j} \in \mathcal{W}_{n}^{7}$ and $X_{t} \cap \bigcup_{j \neq t} X_{j} \in \mathcal{W}_{n}^{6}$. Without loss of generality assume that $t=p$. Let $Y=X_{1} \cup \ldots \cup X_{p-1}$. Then by induction hypothesis $\tilde{H}_{j}(Y)=0$ and $\tilde{H}\left(X_{p}\right)=0$ for $j \in\{5,6\}$. By Mayer-Vietoris sequence for homology, we have

$$
\cdots \longrightarrow \tilde{H}_{j}(Y) \oplus \tilde{H}_{j}\left(X_{p}\right) \longrightarrow \tilde{H}_{j}(X) \longrightarrow \tilde{H}_{j-1}\left(Y \cap X_{p}\right) \xrightarrow{h_{j-1}} \tilde{H}_{j-1}(Y) \oplus \tilde{H}_{j-1}\left(X_{p}\right) \longrightarrow \cdots
$$

Since $Y \cap X_{p} \in \mathcal{W}_{n}^{6}, \tilde{H}_{5}\left(Y \cap X_{p}\right)=0$ by Lemma 4.18. If $j=6$, then since $\tilde{H}_{6}(Y)=0, \tilde{H}_{6}\left(X_{p}\right)=$ 0 and $\tilde{H}_{5}\left(Y \cap X_{p}\right)=0$, we see that $\tilde{H}_{6}(X)=0$. Using Claim 1 (ii) and Lemma 4.17, we conclude that the map $i_{*}: \tilde{H}_{4}\left(Y \cap X_{p}\right) \longrightarrow \tilde{H}_{4}\left(X_{p}\right)$ induced by the inclusion $Y \cap X_{p} \hookrightarrow X_{p}$, is injective and therefore the map $h_{4}: \tilde{H}_{4}\left(Y \cap X_{p}\right) \longrightarrow \tilde{H}_{4}(Y) \oplus \tilde{H}_{4}\left(X_{p}\right)$ is also injective. If $j=5$, then since $\tilde{H}_{5}(Y)=0, \tilde{H}_{5}\left(X_{p}\right)=0$, we conclude that $\tilde{H}_{5}(X)=0$. Hence result is true for $m=7$, i.e., for any $X \in \mathcal{W}_{n}^{7}, \tilde{H}_{j}(X)=0$ for $j \in\{5,6\}$. Now let $m \geq 8$.

Induction hypothesis 1: For any $7 \leq l<m$ and $j \in\{5,6\}$, if $X \in \mathcal{W}_{n}^{l}$, then $\tilde{H}_{j}(X)=0$.
Let $7 \leq l<m$ and suppose $Z \in \mathcal{W}_{n}^{l+1}$. Let $Z=Z_{1} \cup \ldots \cup Z_{q}$, where each $Z_{i}$ is the Vietoris-Rips complex of an $l+1$-dimensional cube subgraph of $\mathbb{I}_{n}$. We show that $\tilde{H}_{j}(Z)=0$ for $j \in\{5,6\}$.

Proof is by induction on $q$. If $q=1$, then $Z \simeq \Delta_{l+1}$. Since $l \geq 7$, from Lemma 4.12, any $j$-cycle $c$ in $Z$ is homologous to a $j$-cycle $\tilde{c}$ in $\partial(Z)$ for $j \in\{5,6\}$. Hence it is enough to show that $\tilde{H}_{j}(\partial(Z))=0$ for $j \in\{5,6\}$. Observe that $\partial(Z) \in \mathcal{W}_{n}^{l}$ and therefore by induction hypothesis 1 , $\tilde{H}_{j}(\partial(Z))=0$ for $j \in\{5,6\}$. Let $q>1$.

Induction hypothesis 2: For any $t<q, i_{1}, \ldots, i_{t} \in[q]$ and $j \in\{5,6\}$, if $X_{i_{1}} \cup \ldots \cup X_{i_{t}} \in \mathcal{W}_{n}^{l+1}$, then $\tilde{H}_{j}\left(X_{i_{1}} \cup \ldots \cup X_{i_{t}}\right)=0$.

From Claim $3(i)$, there exists $s \in[q]$ such that $\bigcup_{j \neq s} Z_{j} \in \mathcal{W}_{n}^{l+1}$ and $Z_{s} \cap \bigcup_{j \neq s} Z_{j} \in \mathcal{W}_{n}^{l}$. Without loss of generality assume that $s=q$. Let $U=Z_{1} \cup \ldots \cup Z_{q-1}$. By induction hypothesis 2, $\tilde{H}_{j}(U)=0$ and $\tilde{H}_{j}\left(Z_{q}\right)=0$ for $j \in\{5,6\}$. By Mayer-Vietoris sequence for homology, we have

$$
\cdots \longrightarrow \tilde{H}_{j}(U) \oplus \tilde{H}_{j}\left(Z_{q}\right) \longrightarrow \tilde{H}_{j}(Z) \longrightarrow \tilde{H}_{j-1}\left(U \cap Z_{q}\right) \xrightarrow{h_{j-1}} \tilde{H}_{j-1}(U) \oplus \tilde{H}_{j-1}\left(Z_{q}\right) \longrightarrow \cdots
$$

Since $U \cap Z_{q} \in \mathcal{W}_{n}^{l}$, from induction hypothesis $1, \tilde{H}_{j}\left(U \cap Z_{q}\right)=0$ for $j \in\{5,6\}$. If $j=6$, then since $\tilde{H}_{6}(U)=0, \tilde{H}_{6}\left(Z_{q}\right)=0$ and $\tilde{H}_{5}\left(U \cap Z_{q}\right)=0$, we see that $\tilde{H}_{6}(Z)=0$. If $j=5$, then since $\tilde{H}_{5}(U)=0, \tilde{H}_{5}\left(Z_{q}\right)=0$ and the map $h_{4}: \tilde{H}_{4}\left(U \cap Z_{q}\right) \longrightarrow \tilde{H}_{4}(U) \oplus \tilde{H}_{4}\left(Z_{q}\right)$ is injective by Lemma 4.17, we get that $\tilde{H}_{5}(Z)=0$.

This completes the proof.
We are now ready to prove main result of this section.
Proof of Theorem $A$. We must show that for $n \geq 5, \tilde{H}_{i}\left(\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right) ; \mathbb{Z}\right) \neq 0$ if and only if $i \in$ $\{4,7\}$. Using Theorem C and Proposition 1.2, we see that $\Delta_{n}$ is homotopy equivalent to a subcomplex of dimension less than 8 . Hence $\tilde{H}_{i}\left(\Delta_{n}\right)=$ for all $i \geq 8$. Let $X$ be the VietorisRips complex of a 4 -dimensional cube subgraph of $\mathbb{I}_{n}$. Then using Lemma 4.8, there exists a retraction $r: \Delta_{n} \rightarrow X$. Since $\Delta_{4} \cong S^{7}$ and $X \cong \Delta_{4}$, we see that $\tilde{H}_{7}(X) \neq 0$. Further, since $r_{*}: \tilde{H}_{7}\left(\Delta_{n}\right) \rightarrow \tilde{H}_{7}(X)$ is surjective, $\tilde{H}_{7}\left(\Delta_{n}\right) \neq 0$.

If $n \leq 6$, then result follows from Proposition 1.1. So assume that $n \geq 7$. Since $\Delta_{n} \in \mathcal{W}_{n}^{n}$, Lemma 4.19 implies that $\tilde{H}_{j}\left(\Delta_{n}\right)=0$ for $j \in\{5,6\}$. Let $Y$ be the Vietoris-Rips complex of a 5dimensional cube subgraph of $\mathbb{I}_{n}$. From Lemma 4.8, there exists a retraction $r_{1}: \Delta_{n} \rightarrow Y$. Since
$Y \cong \Delta_{5}$, using Proposition 1.1 we conclude that $\tilde{H}_{4}\left(\Delta_{n}\right) \neq 0$. From Lemma 4.15, $\tilde{H}_{i}\left(\Delta_{n}\right)=0$ for $i \leq 3$. This completes the proof.

## 5. Future Directions

In Theorem A, we have shown that $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 3\right)$ has non-trivial homology only in dimensions 4 and 7 . Further, the complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ is homotopy equivalent to a wedge sum of 3 -spheres. For $r \in\{2,3\}$, since $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 2\right)$ has non trivial homology only in dimension $i \in\left\{r+1,2^{r}-1\right\}$, we make the following conjecture.
Conjecture 1. For $n \geq r+2, \tilde{H}_{i}\left(\mathcal{V R}\left(\mathbb{I}_{n} ; r\right) ; \mathbb{Z}\right) \neq 0$ if and only if $i \in\left\{r+1,2^{r}-1\right\}$.
The following is a natural question to ask.
Question 5.1. Let $n \geq r+2$. Is $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; r\right)$ homotopy equivalent to a wedge sum of spheres of dimensions $r+1$ and $2^{r}-1$ ?

In Theorems B and C, we have proved that the collapsibility number of the complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; r\right)$ is $2^{r}$ for $r \in\{2,3\}$. The complex $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 1\right)$ is 1 -dimensional and it is isomorphic to the graph $\mathbb{I}_{n}$. Hence the collapsibility number of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; 1\right)$ is 2 . Further, it is easy to check that $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; n-1\right)$ is $\left(2^{n-1}-1\right)$-dimensional and it is isomorphic to the join of $2^{n-1}$-copies of $S^{0}$. Hence the collapsibility number of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; n-1\right)$ is $2^{n-1}$. This leads us to make the following conjecture.
Conjecture 2. For $n \geq r+1$, the collapsibility number of $\mathcal{V} \mathcal{R}\left(\mathbb{I}_{n} ; r\right)$ is $2^{r}$.

## Acknowledgements

I am very grateful to Basudeb Datta for many helpful discussions. A part of this work was completed when I was a postdoctoral fellow at IISc Bangalore, India. This work was partially supported by NBHM postdoctoral fellowship of the author.

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[^0]:    2020 Mathematics Subject Classification. 05E45, 55U10, 55N31.
    Key words and phrases. Vietoris-Rips complexes, d-collapsibility, Hypercube graphs, Homology.

