ON VIETORIS-RIPS COMPLEXES (WITH SCALE 3) OF HYPERCUBE GRAPHS

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ABSTRACT. For a metric space (X,d) and a scale parameter $r \geq 0$, the Vietoris-Rips complex $\mathcal{VR}(X;r)$ is a simplicial complex on vertex set X, where a finite set $\sigma \subseteq X$ is a simplex if and only if the diameter of σ is at most r. For $n \geq 1$, let \mathbb{I}_n denote the n-dimensional hypercube graph. In this paper, we show that $\mathcal{VR}(\mathbb{I}_n;r)$ has non trivial reduced homology only in dimensions 4 and 7. Therefore, we answer a question posed by Adamaszek and Adams recently.

A (finite) simplicial complex Δ is d-collapsible if it can be reduced to the void complex by repeatedly removing a face of size at most d that is contained in a unique maximal face of Δ . The collapsibility number of Δ is the minimum integer d such that Δ is d-collapsible. We show that the collapsibility number of $\mathcal{VR}(\mathbb{I}_n; r)$ is 2^r for $r \in \{2, 3\}$.

1. Introduction

For a metric space (X,d) and a scale parameter $r \geq 0$, the Vietoris-Rips complex $\mathcal{VR}(X;r)$ is a simplicial complex on vertex set X, where a finite set $\sigma \subseteq X$ is a simplex if and only if the diameter of σ is at most r, i.e., $\mathcal{VR}(X;r) = \{\sigma \subseteq X : |\sigma| < \infty \text{ and } d(x,y) \leq r \ \forall \ x,y \in \sigma\}$; here $|\cdot|$ denotes the cardinality of a set. The Vietoris-Rips complex was first introduced by Vietoris [38] to define a homology theory for metric spaces and independently re-introduced by E. Rips for studying hyperbolic groups, where it has been popularised as Rips-complex [26, 27]. The idea behind introducing these complexes was to create a finite simplicial model for metric spaces. The Vietoris-Rips complex and its homology have become an important tools in the applications of algebraic topology. In topological data analysis, it has been used to analyse data with persistent homology [10, 14, 42, 43]. These complexes have been used heavily in computational topology, as a simplicial model for point-cloud data [15, 16, 17, 20] and as simplicial completions of communication links in sensor networks [21, 22, 37]. For more on these complexes, the interested reader is referred to [1, 2, 4, 5, 6, 7, 18, 25, 34, 39, 40].

Consider any graph G as a metric space, where the distance between any two vertices is the length of a shortest path between them. The study of Vietoris-Rips complexes of hypercube graphs was initiated by Adamaszek and Adams in [3]. These questions on hypercubes arose from work by Kevin Emmett, Raúl Rabadán, and Daniel Rosenbloom related to the persistent homology formed from genetic trees, reticulate evolution, and medial recombination [23, 24].

For a positive integer n, let \mathbb{I}_n denote the n-dimensional hypercube graph (see Definition 2.2). In [3], Adamaszek and Adams proved that $\mathcal{VR}(\mathbb{I}_n; 2)$ is homotopy equivalent to a wedge sum of spheres of dimension 3. By using a computer calculation they proved the following.

Proposition 1.1. [3] Let $5 \le n \le 7$ and $0 \le i \le 7$. Then $\tilde{H}_i(\mathcal{VR}(\mathbb{I}_n;3);\mathbb{Z}) \ne 0$ if and only if $i \in \{4,7\}$.

Further, they asked, in what homological dimensions do the Vietoris-Rips complexes $\mathcal{VR}(\mathbb{I}_n;3)$ have nontrivial reduced homology? It is easy to check that the complexes $\mathcal{VR}(\mathbb{I}_n;3)$ are contractible for $1 \leq n \leq 3$ and $\mathcal{VR}(\mathbb{I}_4;3) \simeq S^7$. In this paper we prove the following.

Theorem A. Let $n \geq 5$. Then $\tilde{H}_i(\mathcal{VR}(\mathbb{I}_n;3);\mathbb{Z}) \neq 0$ if and only if $i \in \{4,7\}$.

Let Δ be a (finite) simplicial complex. Let $\gamma \in \Delta$ such that $|\gamma| \leq d$ and $\sigma \in \Delta$ is the only maximal simplex that contains γ . An elementary d-collapse of Δ is the simplicial complex Δ'

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obtained from Δ by removing all those simplices τ of Δ such that $\gamma \subseteq \tau \subseteq \sigma$, and we denote this elementary d-collapse by $\Delta \xrightarrow{\gamma} \Delta'$.

The complex Δ is called *d-collapsible* if there exists a sequence of elementary *d*-collapses

$$\Delta = \Delta_1 \xrightarrow{\gamma_1} \Delta_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{k-1}} \Delta_k = \emptyset$$

from Δ to the void complex \emptyset . Clearly, if Δ is d-collapsible and d < c, then Δ is c-collapsible. The collapsibility number of Δ is the minimal integer d such that Δ is d-collapsible.

The notion of d-collapsibility of simplicial complexes was introduced by Wegner [41]. In combinatorial topology it is an important problem to determine the collapsibility number or bounds for the collapsibility number of a simplicial complex and it has been widely studied (see [12, 19, 31, 35, 36]). A simple consequence of d-collapsibility is the following:

Proposition 1.2. [41] If X is d-collapsible then it is homotopy equivalent to a simplicial complex of dimension smaller than d.

Recently, Bigdeli and Faridi gave a connection between d-collapsibility and the chordal complexes; and proved that d-collapsibility is equivalent to the chordality of the Stanley-Reisner complexes of certain ideals [11]. For applications regarding Helly-type theorems, see [9, 29, 30]. One of the consequences of the topological colorful Helly theorem [30, Theorem 2.1] is the following.

Proposition 1.3. [31, Theorem 1.1] Let X be a d-collapsible simplicial complex on vertex set V, and let $X^c = \{ \sigma \subseteq V : \sigma \notin X \}$. Then, every collection of d+1 sets in X^c has a rainbow set belonging to X^c .

In this paper we prove the following.

Theorem B. For $n \geq 3$, the collapsibility number of $VR(\mathbb{I}_n; 2)$ is 4.

Theorem C. For $n \geq 4$, the collapsibility number of $VR(\mathbb{I}_n; 3)$ is 8.

Flow of the paper: In the following Section, we list out various definitions on graph theory and simplicial complexes that are used in this paper. We also fix a few notations, which we use throughout this paper. In Section 3, we consider the complex $\mathcal{VR}(\mathbb{I}_n; 2)$ and prove Theorem B. Section 4 is devoted to the complex $\mathcal{VR}(\mathbb{I}_n; 3)$ and divided into three subsections. In Section 4.1, we give a characterization of maximal simplices of $\mathcal{VR}(\mathbb{I}_n; 3)$. In Section 4.2, we prove Theorem C. Finally in Section 4.3, we prove Theorem A. In the last section, we posed a few conjectures and a question that arise naturally from the work done in this paper.

2. Preliminaries and Notations

A graph G is a pair (V(G), E(G)), where V(G) is the set of vertices of G and $E(G) \subseteq {V(G) \choose 2}$ denotes the set of edges. If $(x,y) \in E(G)$, it is also denoted by $x \sim y$ and we say that x is adjacent to y. A subgraph H of G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $U \subseteq V(G)$, the induced subgraph G[U] is the subgraph whose set of vertices is V(G[U]) = U and whose set of edges is $E(G[U]) = \{(a,b) \in E(G) \mid a,b \in U\}$.

A graph homomorphism from G to H is a function $\phi: V(G) \to V(H)$ such that, $(v, w) \in E(G) \implies (\phi(v), \phi(w)) \in E(H)$. A graph homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is also a graph homomorphism. Two graphs are called *isomorphic*, if there exists an isomorphism between them. If G and H are isomorphic, we write $G \cong H$.

Let G be a graph and v be a vertex of G. The neighbourhood of v is defined as $N_G(v) = \{w \in V(G) \mid (v, w) \in E(G)\}$ and the closed neighbourhood $N_G[v] = N_G(v) \cup \{v\}$.

Let x and y be two distinct vertices of G. A xy-path is a sequence $xv_0 \dots v_n y$ of vertices of G such that $x \sim v_0, v_n \sim y$ and $v_i \sim v_{i+1}$ for all $0 \leq i \leq n-1$. The length of a xy-path is the number of edges appearing in the path. The distance between x and y is the length of a shortest path (with respect to length) among all xy-paths and it is denoted by d(x,y). Clearly, if $(x,y) \in E(G)$, then d(x,y) = 1. By convention, d(v,v) = 0 for all $v \in V(G)$.

Definition 2.1. A (finite) abstract simplicial complex X is a collection of finite sets such that if $\tau \in X$ and $\sigma \subset \tau$, then $\sigma \in X$.

The elements of X are called *simplices* or *faces* of X. The *dimension* of a simplex σ is equal to $|\sigma|-1$. The dimension of an abstract simplicial complex is the maximum of the dimensions of its simplices. The 0-dimensional simplices are called *vertices* of X. If $\sigma \subset \tau$, we say that σ is a face of τ . If a simplex has dimension k, it is said to be k-dimensional or k-simplex. The boundary of a k-simplex σ is the simplicial complex, consisting of all faces of σ of dimension $\leq k-1$ and it is denoted by $Bd(\sigma)$. A simplex which is not a face of any other simplex is called a maximal simplex or facet. The set of maximal simplices of X is denoted by M(X).

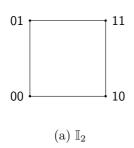
The *join* of two simplicial complexes \mathcal{K}_1 and \mathcal{K}_2 , denoted as $\mathcal{K}_1 * \mathcal{K}_2$, is a simplicial complex whose simplices are disjoint union of simplices of \mathcal{K}_1 and \mathcal{K}_2 . Let Δ^S denotes a (|S|-1)-dimensional simplex with vertex set S. The *cone* on \mathcal{K} with apex a, denoted as $C_a(\mathcal{K})$, is defined as

$$C_a(\mathcal{K}) := \mathcal{K} * \Delta^{\{a\}}.$$

In this article, we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer to book [33] by Kozlov. For terminologies of algebraic topology used in this article, we refer to [28].

Let X be a simplicial complex and $\tau, \sigma \in X$ such that $\sigma \subsetneq \tau$ and τ is the only maximal simplex in X that contains σ . A simplicial collapse of X is the simplicial complex Y obtained from X by removing all those simplices γ of X such that $\sigma \subseteq \gamma \subseteq \tau$. Here, σ is called a free face of τ and (σ, τ) is called a collapsible pair. We denote this collapse by $X \searrow Y$. In particular, if $X \searrow Y$, then $X \simeq Y$.

Definition 2.2. For a positive integer n, the n-dimensional Hypercube graph, denoted by \mathbb{I}_n , is a graph whose vertex set $V(\mathbb{I}_n) = \{x_1 \dots x_n : x_i \in \{0,1\} \ \forall \ 1 \leq i \leq n\}$ and any two vertices $x_1 \dots x_n$ and $y_1 \dots y_n$ are adjacent if and only if $\sum_{i=1}^n |x_i - y_i| = 1$, i.e., they are differ at exactly in one position (see Figure 2.1).



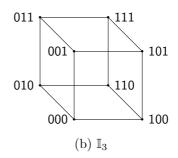


Figure 2.1

We now fix a few notations, which we use throughout this paper. For a positive integer n, we denote the set $\{1,\ldots,n\}$ by [n]. Let $v=v_1\ldots v_n\in V(\mathbb{I}_n)$. For any $i\in [n]$, we let $v(i)=v_i$. For $\{i_1,i_2,\ldots,i_k\}\subseteq [n]$, we let $v^{i_1,\ldots,i_k}\in V(\mathbb{I}_n)$ be defined by

$$v^{i_1,\dots,i_k}(j) = \begin{cases} v(j) & \text{if } j \notin \{i_1,\dots,i_k\}, \\ \{0,1\} \setminus \{v(j)\} & \text{if } j \in \{i_1,\dots,i_k\}. \end{cases}$$

Observe that for any two vertices $v, w \in V(\mathbb{I}_n)$, $d(v, w) = \sum_{i=1}^n |v(i) - w(i)|$ and d(v, w) = k if and only if $w = v^{i_1, \dots, i_k}$ for some $i_1, \dots, i_k \in [n]$. Clearly, $N_{\mathbb{I}_n}(v) = \{v^i : i \in [n]\}$. For $i, j, k \in [n]$, we let $K_v^{i,j,k} := \{v, v^{i,j}, v^{j,k}, v^{i,k}\}$. For the simplicity of notation, we write N(v) and N[v] for the sets $N_{\mathbb{I}_n}(v)$ and $N_{\mathbb{I}_n}[v]$ respectively.

Remark 1. The vertices of \mathbb{I}_n can be consider as subsets of [n], where the element $v_1 \dots v_n \in V(\mathbb{I}_n)$ correspond to the set $\{i : v_i = 1\}$. Hence the distance between any two vertices of \mathbb{I}_n is same as the cardinality of the symmetric difference of corresponding sets.

Using this notations, any $\sigma \in \mathcal{VR}(\mathbb{I}_n; r)$ is a set where the symmetric difference between any two elements in σ is at most r.

3. The complex $\mathcal{VR}(\mathbb{I}_n;2)$

In this section, we prove Theorem B. We first characterise the maximal simplices of $\mathcal{VR}(\mathbb{I}_n;2)$.

Lemma 3.1. Let $n \geq 3$ and σ be a maximal simplex of $VR(\mathbb{I}_n; 2)$. Then one of the following is true:

- (i) $\sigma = N[v]$ for some $v \in V(\mathbb{I}_n)$.
- (ii) $\sigma = \{v, v^{i_0}, v^{j_0}, v^{i_0, j_0}\}$ for some $v \in V(\mathbb{I}_n)$ and $i_0, j_0 \in [n]$.
- (ii) $\sigma = K_v^{i_0, j_0, k_0}$ for some $v \in V(\mathbb{I}_n)$ and $i_0, j_0, k_0 \in [n]$.

Proof. We consider the following cases.

Case 1. There exists a $w \in \sigma$ such that $N(w) \cap \sigma \neq \emptyset$.

Let us first assume that there exists a vertex $w \in \sigma$ such that $N(w) \cap \sigma = \{v\}$. Since $w \in N(v)$, $w = v^{p_0}$ for some $p_0 \in [n]$. We show that $\sigma = N[v]$. Suppose there exists $l_0 \in [n]$ such that $v^{l_0} \notin \sigma$. Since σ is maximal, there exists $x \in \sigma$ such that $d(x, v^{l_0}) \geq 3$. Further, since $v \in \sigma$, $d(v, x) \leq 2$. For any $t \in [n] \setminus \{l_0\}$, since $d(v^{l_0}, v^t) = 2$, we see that $x \neq v^t$ and therefore d(v, x) = 2. Hence $x = v^{i,j}$ for some $i, j \in [n]$. Since $N(w) \cap \sigma = \{v\}$, $p_0 \notin \{i, j\}$. But then d(x, w) = 3, a contradiction. Thus $N(v) \subseteq \sigma$. Since $v \in \sigma$, we see that $N[v] \subseteq \sigma$. Suppose there exists $v \in \sigma$ such that $v \notin N[v]$. Then d(v, v) = 0 and therefore $v = v^{s,t}$ for some $v \in \sigma$ such that $v \notin N[v]$. Then $v \in \sigma$ such that $v \notin N[v]$. Then $v \in \sigma$ such that $v \notin N[v]$. Then $v \in \sigma$ such that $v \in \sigma$ such that $v \notin N[v]$. Then $v \in \sigma$ such that $v \in \sigma$

Now assume that $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$. Let $u \in \sigma$. There exists $i_0, j_0 \in [n]$ such that $u^{i_0}, u^{j_0} \in \sigma$. Thus $\{u, u^{i_0}, u^{j_0}\} \subseteq \sigma$. Since $|N(u^{i_0}) \cap \sigma| \geq 2$ and $u^{i_0} \nsim u^{j_0}$, there exists $z \in \sigma \setminus \{u\}$ such that $z \sim u^{i_0}$. Then $z = u^{i_0,k}$ for some $k \in [n] \setminus \{i_0\}$. Since $u^{j_0} \in \sigma$, $d(u^{j_0}, y) \leq 2$, thereby implying that $k = j_0$. Thus $\{u, u^{i_0}, u^{j_0}, u^{i_0, j_0}\} \subseteq \sigma$. Suppose there exists $q \in \sigma \setminus \{u, u^{i_0}, u^{j_0}, u^{i_0, j_0}\}$. If $q \sim u$, then $q = u^i$ for some $i \in [n] \setminus \{i_0, j_0\}$. Here $d(q, u^{i_0, j_0}) = 3$, a contradiction. Hence $q \nsim u$, i.e., d(u, q) = 2. Then $q = v^{j,k}$ for some $j, k \in [n]$. If $\{i_0, j_0\} \cap \{j, k\} = \emptyset$, then $d(u^{i_0, j_0}, q) = 4$, a contradiction. Hence $\{i_0, j_0\} \cap \{j, k\} \neq \emptyset$. Without loss of generality we assume that $i_0 \in \{j, k\}$. In this case $d(q, u^{j_0}) = 3$, a contradiction. Thus $\sigma = \{u, u^{i_0}, u^{j_0}, u^{i_0, j_0}\}$. Hence σ is of the type (ii).

Case 2. $N(v) \cap \sigma = \emptyset$ for all $v \in \sigma$.

Let $v \in \sigma$. Clearly, $\{v\}$ is not a maximal simplex and therefore there exists $x \in \sigma, x \neq v$. Since $N(v) \cap \sigma = \emptyset$ and $d(v,x) \leq 2$, we see that d(v,x) = 2. There exist $i_0, j_0 \in [n]$ such that $x = v^{i_0,j_0}$. Hence $\{v, v^{i_0,j_0}\} \subseteq \sigma$. For any $t \in [n] \setminus \{i_0, j_0\}$, since $d(v^{i_0,t}, v) = 2 = d(v^{i_0,t}, v^{i_0,j_0})$, we see that $\{v, v^{i_0,j_0}, v^{i_0,t}\} \in \mathcal{VR}(\mathbb{I}_n; 2)$. Thus $\{v, v^{i_0,j_0}\}$ is not a maximal simplex and therefore there exists $y \in \sigma \setminus \{v, v^{i_0,j_0}\}$. Clearly, d(v,y) = 2. There exist $i,j \in [n]$ such that $y = v^{i,j}$. If $\{i,j\} \cap \{i_0,j_0\} = \emptyset$, then $d(y,v^{i_0,j_0}) \geq 3$, a contradiction. Hence $\{i,j\} \cap \{i_0,j_0\} \neq \emptyset$. Without loss of generality assume that $i=i_0$. Thus $\{v,v^{i_0,j_0},v^{i_0,j}\} \subseteq \sigma$. Since $N(v) \cap \sigma = \emptyset$, $v^{i_0} \notin \sigma$. Further, since σ is maximal, there exists $z \in \sigma$ such that $d(z,v^{i_0}) \geq 3$. Clearly d(v,z) = 2 and therefore $z = v^{k,l}$ for some $k,l \in [n]$. Since $d(z,v^{i_0}) \geq 3$, $i_0 \notin \{k,l\}$. Using the fact that $d(z,v^{i_0,j_0}) = 2 = d(z,v^{i_0,j_0})$, we conclude that $\{k,l\} = \{j_0,j\}$. Thus $\{v,v^{i_0,j_0},v^{i_0,j},v^{j_0,j}\} \subseteq \sigma$. Suppose there exists a $w \in \sigma$ such that $w \notin \{v,v^{i_0,j_0},v^{i_0,j},v^{j_0,j}\}$. Here, d(v,w) = 2 and therefore $w = v^{s,t}$ for some $s,t \in [n]$. Since $d(w,v^{i_0,j_0}) = 2$, $\{i_0,j_0\} \cap \{s,t\} \neq \emptyset$. Further, $d(w,v^{i_0,j_0}) = 2$ implies that $\{i_0,j\} \cap \{s,t\} \neq \emptyset$ and $d(w,v^{i_0,j_0}) = 2$ implies that $\{j_0,j\} \cap \{s,t\} \neq \emptyset$, which is not possible. Hence $\sigma = \{v,v^{i_0,j_0},v^{i_0,j},v^{j_0,j}\} = K_v^{i_0,j_0,j_0}$. Thus σ is of the type (iii).

Lemma 3.2. Let $n \geq 3$ and $\sigma \in \mathcal{VR}(\mathbb{I}_n; 2)$ be a maximal simplex. If for some $v, |N(v) \cap \sigma| \geq 3$, then either $\sigma = K_v^{i_0, j_0, k_0}$ for some $i_0, j_0, k_0 \in [n]$ or $N(v) \subseteq \sigma$.

Proof. Let $|N(v) \cap \sigma| \geq 3$. If n = 3, then |N(v)| = 3 and therefore $N(v) \subseteq \sigma$. So assume that $n \geq 4$. Suppose $N(v) \not\subseteq \sigma$. Then there exists $l_0 \in [n]$ such that $v^{l_0} \notin \sigma$. Since $|N(v) \cap \sigma| \geq 3$, there exist $i_0, j_0, k_0 \in [n]$ such that $\{v^{i_0}, v^{j_0}, v^{k_0}\} \subseteq \sigma$. Clearly, $l_0 \notin \{i_0, j_0, k_0\}$. Since $v^{l_0} \notin \sigma$ and σ is a maximal simplex, there exists $x \in \sigma$ such that $d(x, v^{l_0}) \geq 3$. Observe that for any vertex u, if d(v, u) = 1, then $d(u, v^{l_0}) \leq 2$. Hence $d(v, x) \geq 2$. If $d(v, x) \geq 4$, then $d(v^{i_0}, x) \geq 3$, a contradiction as $v^{i_0} \in \sigma$. Hence $d(v, x) \leq 3$. If d(v, x) = 3, then $x = v^{i,j,k}$ for some $i, j, k \in [n]$. Since $d(v^{i_0}, x) \leq 2$, $i_0 \in \{i, j, k\}$. Similarly $j_0, k_0 \in \{i, j, k\}$. Hence $\{i, j, k\} = \{i_0, j_0, k_0\}$. Thus $\sigma = \{v^{i_0}, v^{j_0}, v^{k_0}, v^{i_0, j_0, k_0}\} = K_v^{i_0, j_0, k_0}$.

Suppose d(v,x)=2. Here, $x=v^{i,j}$ for some $i,j\in[n]$. If $i_0\notin\{i,j\}$, then $d(x,v^{i_0})=3$, a contradiction as $v^{i_0}\in\sigma$. Hence $i_0\in\{i,j\}$. By similar argument, we can show that $j_0,k_0\in\{i,j\}$. Hence $\{i_0,j_0,k_0\}\subseteq\{i,j\}$, which is not possible. Thus, $N(v)\subseteq\sigma$.

We now review a result, which will play a key role in the proof of Theorem B.

Let X be a simplicial complex on vertex set [n] and let \prec : $\sigma_1, \ldots, \sigma_m$ be a linear ordering of the maximal simplices of X. Given a $\sigma \in X$, the minimal exclusion sequence $\operatorname{mes}_{\prec}(\sigma)$ is defined as follows. Let i denote the smallest index such that $\sigma \subseteq \sigma_i$. If i = 1, then $\operatorname{mes}_{\prec}(\sigma)$ is the null sequence. If $i \geq 2$, then $\operatorname{mes}_{\prec}(\sigma) = (v_1, \ldots, v_{i-1})$ is a finite sequence of length i - 1 such that $v_1 = \min(\sigma \setminus \sigma_1)$ and for each $k \in \{2, \ldots, i-1\}$,

$$v_k = \begin{cases} \min(\{v_1, \dots, v_{k-1}\} \cap (\sigma \setminus \sigma_k)) & \text{if } \{v_1, \dots, v_{k-1}\} \cap (\sigma \setminus \sigma_k) \neq \emptyset, \\ \min(\sigma \setminus \sigma_k) & \text{otherwise.} \end{cases}$$

Let $M_{\prec}(\sigma)$ denote the set of vertices appearing in mes $_{\prec}(\sigma)$. Define

$$d_{\prec}(X) := \max_{\sigma \in X} |M_{\prec}(\sigma)|.$$

The following result was stated and proved in [36, Proposition 1.3] as a special case where X is the nerve of a finite family of sets and then generalized by Lew for arbitrary simplical complex.

Proposition 3.3. [35, Theorem 6] If \prec is a linear ordering of the maximal simplices of X, then X is $d_{\prec}(X)$ -collapsible.

We are now ready to prove main result of this section.

Proof of Theorem B. We must show that for $n \geq 3$, the collapsibility number of $\mathcal{VR}(\mathbb{I}_n; 2)$ is 4. Since $\mathcal{VR}(\mathbb{I}_n; 2)$ is homotopy equivalent to a wedge sum of spheres of dimension 3 [3], $\tilde{H}_3(\mathcal{VR}(\mathbb{I}_n; 2)) \neq 0$ and therefore by using Proposition 1.2 we conclude that collapsibility of $\mathcal{VR}(\mathbb{I}_n; 2)$ is ≥ 4 . It is enough to show that $\mathcal{VR}(\mathbb{I}_n; 2)$ is 4-collapsible. From Lemma 3.1, each maximal simplex of $\mathcal{VR}(\mathbb{I}_n; 2)$ is of the form either (i) N[v] or (ii) $\{v, v^i, v^j, v^{i,j}\}$ or (iii) $K_v^{i_0,j_0,k_0}$. It is easy to check that these three sets of maximal simplices are pairwise disjoint sets. Choose a linear order \prec_1 on maximal simplices of the type (i). Extend \prec_1 to a linear order \prec on maximal simplices of $\mathcal{VR}(\mathbb{I}_n; 2)$, where maximal simplices of the type (i) are ordered first, i.e., for any two maximal simplices σ_1 and σ_2 , if $\sigma_1 = N[v]$ for some v and σ_2 is of the type (ii) or (iii), then $\sigma_1 \prec \sigma_2$. Let $\tau \in \mathcal{VR}(\mathbb{I}_n; 2)$. Let σ be the smallest (with respect to \prec) maximal simplex of $\mathcal{VR}(\mathbb{I}_n; 2)$ such that $\tau \subseteq \sigma$. If $\sigma \neq N[v]$ for all $v \in \mathcal{V}(\mathbb{I}_n)$, then $|\sigma| = 4$ and therefore by definition $|M_{\prec}(\tau)| \leq 4$. So, assume that $\sigma = N[v]$ for some $v \in \mathcal{V}(\mathbb{I}_n)$. We first prove that $|M_{\prec}(\tau) \cap N(v)| \leq 3$.

Let $\operatorname{mes}_{\prec}(\tau) = (x_1, \dots, x_t)$. Suppose $|M_{\prec}(\tau) \cap N(v)| \geq 4$. Let k be the least integer such that $|\{x_1, \dots, x_k\} \cap N(v)| = 3$. Clearly, k < t. Let $\{x_1, \dots, x_k\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. Observe that $x_k \in \{x_{i_1}, x_{i_2}, x_{i_3}\}$. We show that $\{x_1, \dots, x_{k+1}\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. Let γ be a maximal simplex such that $\gamma \prec \sigma$. If $\{x_1, \dots, x_k\} \cap (\sigma \setminus \gamma) \neq \emptyset$, then $x_{k+1} \in \{x_1, \dots, x_k\}$. Hence $\{x_1, \dots, x_{k+1}\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. If $\{x_1, \dots, x_k\} \cap (\sigma \setminus \gamma) = \emptyset$, then $\{x_{i_1}, x_{i_2}, x_{i_3}\} \subseteq \gamma$. From Lemma 3.2, either $N(v) \subseteq \gamma$ or $\gamma = K_v^{i_0, j_0, k_0}$. Since $\gamma \prec \sigma$, $\gamma \neq K_v^{i_0, j_0, k_0}$. Hence $N(v) \subseteq \gamma$. Thus $x_{k+1} \notin N(v)$, thereby implying that $\{x_1, \dots, x_{k+1}\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. If k+1=t, then we get a contradiction to the assumption that $|M_{\prec}(\tau) \cap N(v)| \geq 4$. Inductively assume that for all $k \leq l < t$, $\{x_1, \dots, x_l\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. By the argument similar as above we

can show that $\{x_1, \ldots, x_t\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}\}$, a contradiction. Thus $|M_{\prec}(\tau) \cap N(v)| \leq 3$. Since $\sigma = N[v]$, we conclude that $|M_{\prec}(\tau)| \leq 4$.

From Proposition 3.3, $VR(\mathbb{I}_n; 2)$ is 4-collapsible. This completes the proof.

4. The complex $VR(\mathbb{I}_n;3)$

In this section, we prove Theorem A and Theorem C. This section is divided into three subsections. In the next subsection, we characterise the maximal simplices of $\mathcal{VR}(\mathbb{I}_n;3)$. In subsection 4.2, using the minimal exclusion sequence, we prove Theorem C. Finally, in subsection 4.3, using the Mayer-Vietoris sequence for homology, we prove Theorem A.

We first fix some notations, which we use throughout this section. For any $n \geq 1$, let $\Delta_n = \mathcal{VR}(\mathbb{I}_n; 3)$. We say that a simplex $\sigma \in \Delta_n$ covers all places, if for each $i \in [n]$ there exist $v, w \in \sigma$ such that v(i) = 1 and w(i) = 0. For each $i \in \{n\}$ and $\epsilon \in \{0, 1\}$, let $\mathbb{I}_n^{(i,\epsilon)}$ be the induced subgraph of \mathbb{I}_n on the vertex set $\{v \in V(\mathbb{I}_n) : v(i) = \epsilon\}$. Observe that $\mathbb{I}_n^{(i,\epsilon)} \cong \mathbb{I}_{n-1}$.

4.1. **Maximal simplices.** We give a characterisation of the maximal simplices of Δ_n in Lemma 4.7. We first establish a few lemmas, which we need to prove Lemma 4.7.

Lemma 4.1. Let $n \geq 5$ and $\sigma \in \Delta_n$ be a maximal simplex such that σ covers all places. Then $N(v) \cap \sigma \neq \emptyset$ for all $v \in \sigma$.

Proof. Suppose there exists $v \in \sigma$ such that $N(v) \cap \sigma = \emptyset$. Since $v^1 \notin \sigma$ and σ is a maximal simplex, there exists $x \in \sigma$ such that $d(x, v^1) \geq 4$. It is easy to see that if $d(x, v) \leq 2$, then $d(x, v^1) \leq 3$. Hence d(v, x) = 3. Here, $x = v^{i_0, j_0, k_0}$ for some $i_0, j_0, k_0 \in [n]$. If $1 \in \{i_0, j_0, k_0\}$, then $d(v^1, x) = 2$. Hence $1 \notin \{i_0, j_0, k_0\}$. Since $v^{i_0} \notin \sigma$, there exists $y \in \sigma$ such that $d(y, v^{i_0}) \ge 4$. Here d(v,y) = 3. Hence $y = v^{i,j,k}$ for some $i,j,k \in [n]$. If $|\{i_0,j_0,k_0\} \cap \{i,j,k\}| \le 1$, then $d(y, v^{i_0, j_0, k_0}) \ge 4$, which contradict the fact that $v^{i_0, j_0, k_0} \in \sigma$. If $i_0 \in \{i, j, k\}$, then $d(v^{i_0}, y) = 2$, a contradiction. So $i_0 \notin \{i, j, k\}$ and therefore $\{i_0, j_0, k_0\} \cap \{i, j, k\} = \{j_0, k_0\}$. Hence $y = v^{j_0, k_0, l_0}$ for some $l_0 \in [n] \setminus \{i_0\}$. So, $\{v, v^{i_0, j_0, k_0}, v^{j_0, k_0, l_0}\} \subseteq \sigma$. Since $v^{j_0} \notin \sigma$, there exists $z \in \sigma$ such that $d(z, v^{j_0}) \ge 4$. Here, d(v, z) = 3. Since $d(z, v^{i_0, j_0, k_0}) \le 3$ and $d(z, v^{j_0, k_0, l_0}) \le 3$, we conclude that $z = v^{i_0,k_0,l_0}$. Further, since $v^{k_0} \notin \sigma$, there exists $w \in \sigma$ such that $d(w,v^{k_0}) \geq 4$. Here d(v, w) = 3. Since $d(w, v^{i_0, j_0, k_0}) \le 3$, $d(w, v^{j_0, k_0, l_0}) \le 3$ and $d(w, v^{i_0, k_0, l_0}) \le 3$, we conclude that $w = v^{i_0, j_0, l_0}$. So, $\{v, v^{i_0, j_0, k_0}, v^{j_0, k_0, l_0}, v^{i_0, k_0, l_0}, v^{i_0, j_0, l_0}\} \subseteq \sigma$. Since $n \ge 5$, there exists $p \in [n] \setminus \{i_0, j_0, k_0, l_0\}$. Observe that $v^{i_0, j_0, k_0}(p) = v^{j_0, k_0, l_0}(p) = v^{i_0, k_0, l_0}(p) = v^{i_0, j_0, l_0}(p) = v(p)$. Since σ covers all places, there exists $u \in \sigma$ such that $u(p) = \{0,1\} \setminus \{v(p)\}$. Since $N(v) \cap \sigma = \emptyset$, $u \neq v^p$. Thus, either d(v, u) = 2 or d(v, u) = 3. If d(v, u) = 2, then $u = v^{p,r}$ for some $r \in [n]$. If $r \notin \{i_0, j_0, k_0\}$, then $d(u, v^{i_0, j_0, k_0}) = 4$, a contradiction. Hence $r \in \{i_0, j_0, k_0\}$. Without loss of generality we assume that $r = i_0$. In this case $d(u, v^{j_0, k_0, l_0}) = 4$, a contradiction. Hence d(v, u) = 3. Here $u = v^{p,s,t}$ for some $s, t \in [n]$. If $|\{s, t\} \cap \{i_0, j_0, k_0\}| \le 1$, then $d(u, v^{i_0, j_0, k_0}) \ge 4$. Hence $\{s,t\}\subseteq\{i_0,j_0,k_0\}$. Without loss of generality we assume that $\{s,t\}=\{i_0,j_0\}$. Then $d(u, v^{i_0, k_0, l_0}) = 4$, a contradiction. Thus there exists no $u \in \sigma$ such that $u(p) = \{0, 1\} \setminus \{v(p)\}$, which is a contradiction to the hypothesis that σ covers all places. Hence $N(v) \cap \sigma \neq \emptyset$.

Lemma 4.2. Let $n \geq 5$ and let $\sigma \in \Delta_n$ be a maximal simplex such that σ covers all places. If there exists a $w \in \sigma$ such that $N(w) \cap \sigma = \{v\}$, then $N(v) \subseteq \sigma$.

Proof. Since $w \in N(v)$, $w = v^s$ for some $s \in [n]$. Without loss of generality we assume that $v = v_1 \dots v_n$, where $v_i = 0$ for each $i \in [n]$ and s = n, i.e., $w = v^n$. Suppose $N(v) \not\subseteq \sigma$. There exists $l_0 \in [n]$ such that $v^{l_0} \notin \sigma$. Clearly, $l_0 \neq n$. Since σ covers all places, there exists $x \in \sigma$ such that $x(l_0) = 1$. Further, since $x \neq v^{l_0}$, $d(x, v) \geq 2$. Thus, either d(x, v) = 3 or d(x, v) = 2. We consider the following two cases:

Case 1. d(x, v) = 3.

Here, $x = v^{l_0, i_0, j_0}$ for some $i_0, j_0 \in [n]$. If $n \notin \{i_0, j_0\}$, then $d(x, v^n) = 4$. Since $v^n \in \sigma$, $d(x, v^n) \leq 3$ and thereby implying that $n \in \{i_0, j_0\}$. Without loss of generality we assume that $n = j_0$, i.e., $x = v^{l_0, i_0, n}$.

From Lemma 4.1, there exists $y \in \sigma$ such that $y \sim x$. Clearly $y \neq v, v^n$. There exists $j \in [n]$ such that $y = x^j$. If $j \notin \{l_0, i_0, n\}$, then $d(y, v) \geq 4$. Hence $j \in \{l_0, i_0, n\}$ and thereby implying that d(y, v) = 2. If $j \neq n$, then $y = v^{l_0, n}$ or $y = v^{i_0, n}$. In both the cases $y \sim w = v^n$, which is not possible since $N(w) \cap \sigma = \{v\}$. Hence j = n and $y = v^{l_0, i_0}$. So, $\{v, v^n, v^{l_0, i_0, n}, v^{l_0, i_0}\} \subseteq \sigma$.

Since $v^{i_0,n} \sim v^n = w$ and $N(w) \cap \sigma = \{v\}$, we see that $v^{i_0,n} \notin \sigma$. Further, since σ is a maximal simplex, there exists $z \in \sigma$ such that $d(z,v^{i_0,n}) \geq 4$. Observe that for any vertex t, if $t \sim v$, then $d(t,v^{i_0,n}) \leq 3$ and therefore we see that $z \not\sim v$. Since $z,v \in \sigma, d(z,v) \leq 3$. Thus, either d(z,v)=3 or d(z,v)=2. If d(z,v)=3, then $z=v^{i,j,k}$ for some $i,j,k \in [n]$. Observe that if $n \notin \{i,j,k\}$, then $d(z,v^n)=4$, a contradiction as $v^n \in \sigma$. Hence $n \in \{i,j,k\}$. Without loss of generality we assume that i=n, i.e., $z=v^{n,j,k}$. But, then $d(z,v^{i_0,n}) \leq 3$, which is a contradiction as $d(z,v^{i_0,n}) \geq 4$. Thus d(z,v)=2. So, $z=v^{i,j}$ for some $i,j \in [n]$. If $\{i,j\} \cap \{i_0,n\} \neq \phi$, then $d(v^{i,j},v^{i_0,n}) \leq 3$. Hence $d(z,v^{i_0,n}) \geq 4$ implies that $\{i,j\} \cap \{i_0,n\} \neq \emptyset$. If $\{i,j\} \cap \{l_0,n,i_0\} = \emptyset$, then $d(v^{i,j},v^{l_0,i_0,n}) \geq 4$. Since $v^{l_0,i_0,n} \in \sigma$, $\{i,j\} \cap \{l_0,i_0,n\} \neq \emptyset$. Thus, we conclude that $\{i,j\} \cap \{l_0,i_0,n\} = \{l_0\}$. Hence $z=v^{l_0,k_0}$ for some $k_0 \neq i_0,n$. So, $\{v,v^n,v^{l_0,i_0,n},v^{l_0,i_0,n},v^{l_0,i_0,n},v^{l_0,i_0,n}\} \subseteq \sigma$.

Since $v^{l_0,n} \sim v^n$ and $N(v^n) \cap \sigma = \{v\}$, $v^{l_0,n} \notin \sigma$. Hence there exists $p \in \sigma$ such that $d(p,v^{l_0,n}) \geq 4$. Observe that for any $u \sim v$, $d(u,v^{l_0,n}) \leq 3$ and therefore $p \not\sim v$. Thus, $d(p,v) \geq 2$. Suppose d(p,v) = 3. Then $p = v^{i,j,k}$ for some $i,j,k \in [n]$. If $n \notin \{i,j,k\}$, then $d(p,v^n) = 4$, a contradiction as $v^n \in \sigma$. Hence $n \in \{i,j,k\}$. Without loss of generality we assume that i = n, i.e., $p = v^{n,j,k}$. But, then $d(p,v^{l_0,n}) \leq 3$, which is a contradiction. Thus d(p,v) = 2. So, $p = v^{i,j}$ for some $i,j \in [n]$. If $\{i,j\} \cap \{l_0,n\} \neq \phi$, then $d(v^{i,j},v^{l_0,n}) \leq 3$. Hence $\{i,j\} \cap \{l_0,n\} = \emptyset$. If $\{i,j\} \cap \{l_0,i_0,n\} = \emptyset$, then $d(v^{i,j},v^{l_0,i_0,n}) \geq 4$. Since $v^{l_0,i_0,n} \in \sigma$, $\{i,j\} \cap \{l_0,i_0,n\} \neq \emptyset$. Hence $\{i,j\} \cap \{l_0,i_0,n\} = \{i_0\}$. Thus $p = v^{i_0,j}$ for some $j \neq n,l_0$. If $j \neq k_0$, then $d(p,v^{l_0,k_0}) = 4$, which is a contradiction as $v^{l_0,k_0} \in \sigma$. Hence $p = v^{i_0,k_0}$. So, $\{v,v^n,v^{l_0,i_0,n},v^{l_0,i_0,n},v^{l_0,i_0,n},v^{l_0,i_0,k},v^{i_0,k_0}\} \subseteq \sigma$.

Since $n \geq 5$, there exists $j_0 \in [n] \setminus \{l_0, i_0, k_0, n\}$. Further, since σ covers all places, there exits $q \in \sigma$ such that $q(j_0) = 1$. Observe that $0 = v(j_0) = v^n(j_0) = v^{l_0, n, i_0}(j_0) = v^{l_0, i_0}(j_0) = v^{l_0, k_0}(j_0) = v^{l_0, k_0}(j_0) = v^{l_0, k_0}(j_0)$ and therefore $q \notin \{v, v^n, v^{l_0, i_0, n}, v^{l_0, i_0}, v^{l_0, k_0}, v^{i_0, k_0}\}$. Since $d(v^{j_0}, v^{l_0, i_0, n}) = 4$, $q \neq v^{j_0}$. Hence $d(q, v) \geq 2$.

(1.1) Suppose d(q, v) = 2.

Here, $q = v^{j_0,i}$ for some $i \in [n]$. If $i \notin \{l_0, i_0, n\}$, then $d(q, v^{l_0, i_0, n}) \ge 4$. Hence $v^{l_0, i_0, n} \in \sigma$ implies that $i \in \{l_0, i_0, n\}$. However, since $d(v^{j_0, l_0}, v^{i_0, k_0}) = 4 = d(v^{j_0, i_0}, v^{l_0, k_0}) = d(v^{j_0, n}, v^{i_0, k_0})$, we conclude that $i \notin \{l_0, i_0, n\}$. Thus, there exist no $q \in \sigma$ such that $q(j_0) = 1$, which is a contradiction to the assumption that σ covers all places.

(1.2) Suppose d(q, v) = 3.

Here, $q = v^{j_0,i,j}$ for some $i,j \in [n]$. Observe that if $|\{i,j\} \cap \{l_0,i_0,n\}| \leq 1$, then $d(q,v^{l_0,i_0,n}) \geq 4$, which is not possible since $v^{l_0,i_0,n} \in \sigma$. Hence $\{i,j\} \subset \{l_0,i_0,n\}$. If $n \notin \{i,j\}$, then $d(q,v^n) = 4$, a contradiction as $v^n \in \sigma$. Hence $n \in \{i,j\}$. Without loss of generality we assume that i = n, i.e., $q = v^{j_0,n,j}$, where $j \in \{l_0,i_0\}$. It is easy to see that $d(v^{j_0,n,l_0},v^{i_0,k_0}) \geq 4$ and $d(v^{j_0,n,i_0},v^{l_0,k_0}) \geq 4$. Since $v^{i_0,k_0},v^{l_0,k_0} \in \sigma$, we see that $q \notin \{v^{j_0,l_0,n},v^{j_0,i_0,n}\}$. Thus, there exists no $q \in \sigma$ such that $q(j_0) = 1$, which is a contradiction.

Case 2. d(x, v) = 2.

Here, $x=v^{l_0,s_0}$ for some $s_0\in[n]$. So, $\{v,v^n,v^{l_0,s_0}\}\subseteq\sigma$. Since $v^{l_0,n}\sim v^n=w$ and $N(w)\cap\sigma=\{v\},\,v^{l_0,n}\notin\sigma$. Hence there exists $y_0\in\sigma$ such that $d(y_0,v^{l_0,n})\geq 4$. Observe that for any $t\sim v,\,d(t,v^{l_0,n})\leq 3$ and therefore $y_0\not\sim v$. Thus, $d(y_0,v)\geq 2$. Suppose $d(y_0,v)=3$. Then $y_0=v^{i,j,k}$ for some $i,j,k\in[n]$. If $n\notin\{i,j,k\}$, then $d(y_0,v^n)\geq 4$, a contradiction as $y_0,v^n\in\sigma$. Hence $n\in\{i,j,k\}$. But, then $d(y_0,v^{l_0,n})\leq 3$, again a contradiction. Thus, $d(y_0,v)=2$. Here, $y_0=v^{i,j}$ for some $i,j\in[n]$. If $\{i,j\}\cap\{l_0,n\}\neq\emptyset$, then $d(y_0,v^{l_0,n})\leq 3$. Hence $\{i,j\}\cap\{l_0,n\}=\emptyset$. Further, if $s_0\notin\{i,j\}$, then $d(y_0,v^{l_0,s_0})\geq 4$. Hence $s_0\in\{i,j\}$. Therefore $y_0=v^{s_0,t_0}$ for some $t_0\in[n],t_0\neq l_0,n$. So, $\{v,v^n,v^{l_0,s_0},v^{s_0,t_0}\}\subseteq\sigma$.

Since $v^{s_0,n} \sim v^n$ and $N(v^n) \cap \sigma = \{v\}$, $v^{s_0,n} \notin \sigma$. Hence there exists $z_0 \in \sigma$ such that $d(z_0, v^{s_0,n}) \geq 4$. By the argument similar as above for the y_0 , we see that $d(z_0, v) = 2$. Therefore $z_0 = v^{i,j}$ for some $i, j \in [n]$. If $\{i, j\} \cap \{s_0, n\} \neq \emptyset$, then $d(z_0, v^{s_0,n}) \leq 3$. Hence $\{i, j\} \cap \{s_0, n\} = \emptyset$. If $l_0 \notin \{i, j\}$, then $d(z_0, v^{l_0, s_0}) \geq 4$ and if $t_0 \notin \{i, j\}$, then $d(z_0, v^{s_0, t_0}) \geq 4$. Since v^{l_0, s_0} , $v^{s_0, t_0} \in \sigma$, we conclude that $\{i, j\} = \{l_0, s_0\}$, i.e., $z_0 = v^{l_0, t_0}$. So, $\{v, v^n, v^{l_0, s_0}, v^{s_0, t_0}, v^{l_0, t_0}\} \subseteq \sigma$.

Since $n \geq 5$, there exists $m_0 \in [n] \setminus \{n, l_0, s_0, t_0\}$. Further, since σ covers all places, there exists $p_0 \in \sigma$ such that $p_0(m_0) = 1$. Clearly, $u(m_0) = 0$ for all $u \in \{v, v^n, v^{l_0, s_0}, v^{s_0, t_0}, v^{l_0, t_0}\}$. Hence $p_0 \notin \{v, v^n, v^{l_0, s_0}, v^{s_0, t_0}, v^{l_0, t_0}\}$.

Claim 1. $p_0 = v^{m_0}$.

Proof of Claim 1. Since $p_0, v \in \sigma, d(p_0, v) \leq 3$. Suppose $d(p_0, v) = 3$. Then $p_0 = v^{m_0, i, j}$ for some $i, j \in [n]$. If $n \notin \{i, j\}$, then $d(p_0, v^n) = 4$ and therefore $n \in \{i, j\}$. Further, if $\{s_0, l_0\} \cap \{i, j\} = \emptyset$, then $d(p_0, v^{l_0, s_0}) \geq 4$, which is not possible since $v^{l_0, s_0} \in \sigma$. Hence $\{i, j\} \cap \{l_0, s_0\} \neq \emptyset$ and therefore we see that p_0 is either v^{m_0, n, s_0} or v^{m_0, n, l_0} . But $d(v^{m_0, n, s_0}, v^{l_0, l_0}) = 5$ and $d(v^{m_0, n, l_0}, v^{s_0, l_0}) = 5$. Hence $p_0 \notin \{v^{m_0, n, s_0}, v^{m_0, n, l_0}\}$. Therefore $d(p_0, v) \leq 2$.

If $d(p_0, v) = 2$, then $p_0 = v^{m_0, i}$ for some $i \in [n]$. Since $v^{l_0, t_0}, v^{l_0, s_0} \in \sigma$ and $d(v^{m_0, s_0}, v^{l_0, t_0}) = 4 = d(v^{m_0, t_0}, v^{l_0, s_0})$, we conclude that $i \notin \{s_0, t_0\}$. But, then $d(p_0, v^{s_0, t_0}) = 4$, a contradiction as $v^{s_0, t_0} \in \sigma$. Hence $d(p_0, v) = 1$. Therefore $p_0 = v^{m_0}$. This completes the proof of Claim 1.

So, $\{v, v^n, v^{l_0, s_0}, v^{s_0, t_0}, v^{l_0, t_0}, v^{m_0}\} \subseteq \sigma$. Since σ is a maximal simplex and $v^{l_0} \notin \sigma$, there exists $q_0 \in \sigma$ such that $d(q_0, v^{l_0}) \geq 4$. Observe that for any t, if $d(v, t) \leq 2$, then $d(t, v^{l_0}) \leq 3$. Hence $d(v, q_0) = 3$. Here, $q_0 = v^{i,j,k}$ for some $i, j, k \in [n]$. If $n \notin \{i, j, k\}$, then $d(q_0, v^n) \geq 4$. Hence $n \in \{i, j, k\}$. If $l_0 \in \{i, j, k\}$, then $d(q_0, v^{l_0}) \leq 3$. Hence $l_0 \notin \{i, j, k\}$. Further, if $s_0 \notin \{i, j, k\}$, then $d(q_0, v^{l_0, s_0}) = 5$ and therefore we see that $s_0 \in \{i, j, k\}$. Without loss of generality we assume that i = n and $j = s_0$, i.e., $q_0 = v^{n, s_0, k}$. If $t_0 \neq k$, then $d(q_0, v^{l_0, t_0}) \geq 4$, a contradiction as $v^{l_0, t_0} \in \sigma$. Hence $k = t_0$, i.e., $q_0 = v^{n, s_0, t_0}$. But, then $d(q_0, v^{m_0}) = 4$, a contradiction. Thus, there exists no q_0 such that $d(q_0, v^{l_0}) \geq 4$, a contradiction.

Therefore we conclude that $N(v) \subseteq \sigma$. This completes the proof.

Recall that for a $v \in V(\mathbb{I}_n)$ and $i_0, j_0, k_0 \in [n]$, $K_v^{i_0, j_0, k_0} = \{v, v^{i_0, j_0}, v^{i_0, k_0}, v^{j_0, k_0}\}$.

Lemma 4.3. Let $n \ge 5$ and let $\sigma \in \Delta_n$ be a maximal simplex such that σ covers all places. If there exists a $w \in \sigma$ such that $N(w) \cap \sigma = \{v\}$, then $\sigma = N(v) \cup K_v^{i_0, j_0, k_0}$ for some $i_0, j_0, k_0 \in [n]$.

Proof. From Lemma 4.2, $N(v) \subseteq \sigma$. Since $w \sim v$, $w = v^{l_0}$ for some $l_0 \in [n]$. Suppose there exists $x \in \sigma$ such that d(x,v) = 3. Then $x = v^{i,j,k}$ for some $i,j,k \in [n]$. Choose $t \in [n] \setminus \{i,j,k\}$. Then $d(x,v^t) = 4$, a contradiction as $v^t \in N(v) \subseteq \sigma$. Hence $d(v,x) \le 2$ for all $x \in \sigma$. Since $N(w) \cap \sigma = \{v\}$, $v^{i,l_0} \notin \sigma$ for all $i \in [n], i \ne l_0$. Further, since σ is a maximal simplex and $v^{1,l_0} \notin \sigma$, there exists $x_0 \in \sigma$ such that $d(x_0,v^{1,l_0}) \ge 4$. For any $p \sim v$, $d(p,v^{1,l_0}) \le 3$ and therefore $d(x_0,v) = 2$. Hence $x_0 = v^{i_0,j_0}$ for some $i_0,j_0 \in [n]$. If $\{i_0,j_0\} \cap \{1,l_0\} \ne \emptyset$, then $d(x_0,v^{1,l_0}) \le 3$. Hence $\{i_0,j_0\} \cap \{1,l_0\} = \emptyset$. Thus $\{v,v^1,\ldots,v^n,v^{i_0,j_0}\} \subseteq \sigma$. Since $v^{i_0,l_0} \notin \sigma$, there exists $y_0 \in \sigma$ such that $d(y_0,v^{i_0,l_0}) \ge 4$. For any $q \in N(v)$, $d(q,v^{i_0,l_0}) \le 3$ and therefore $d(y_0,v) \ge 2$. Since $d(x,v) \le 2$ for all $x \in \sigma$, we see that $d(y_0,v) = 2$. Hence $y_0 = v^{i,j}$ for some i,j. If $\{i,j\} \cap \{i_0,j_0\} = \emptyset$, then $d(y_0,v^{i_0,j_0}) \ge 4$. Hence $\{i,j\} \cap \{i_0,j_0\} \ne \emptyset$. If $i_0 \in \{i,j\}$, then $d(y_0,v^{i_0,l_0}) \le 3$. Hence $i_0 \notin \{i,j\}$. Thus $i_0 = v^{i_0,k_0}$ for some $i_0 \ne i_0$. So, $i_0 \in [n]$ sugment similar as above for $i_0 \in [n]$ there exists $i_0 \in [n]$ such that $i_0 \in [n]$ by $i_0 \in [n]$ and argument similar as above for $i_0 \in [n]$ such that $i_0 \in [n]$ such that i

Suppose there exists $p \in \sigma$ such that $p \notin \{v, v^1, \dots, v^n, v^{i_0, j_0}, v^{i_0, k_0}, v^{j_0, k_0}\}$. Since $d(v, x) \leq 2$ for all $x \in \sigma$ and $p \notin N(v)$, we see that d(v, p) = 2. Here, $p = v^{i, j}$ for some $i, j \in [n]$. Since $d(p, v^{i_0, j_0}) \leq 3$, $d(p, v^{i_0, k_0}) \leq 3$ and $d(p, v^{j_0, k_0}) \leq 3$, we see that $\{i, j\} \cap \{i_0, j_0\} \neq \emptyset$, $\{i, j\} \cap \{i_0, k_0\} \neq \emptyset$ and $\{i, j\} \cap \{j_0, k_0\} \neq \emptyset$. But this is possible only if $\{i, j\} = \{i_0, j_0\}$, or $\{i, j\} = \{i_0, k_0\}$ or $\{i, j\} = \{j_0, k_0\}$. Thus $\sigma = \{v, v^1, \dots, v^n, v^{i_0, j_0}, v^{i_0, k_0}, v^{j_0, k_0}\} = N(v) \cup K_v^{i_0, j_0, k_0}$. \square

Lemma 4.4. Let $n \geq 5$ and $\sigma \in \Delta_n$ be a maximal simplex. If $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$, then there exists $\tilde{v} \in \sigma$ such that $|N(\tilde{v}) \cap \sigma| \geq 3$.

Proof. Let $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$. If $|N(w) \cap \sigma| \geq 3$ for all $w \in \sigma$, then we are done. So assume that there exists $v \in \sigma$ such that $|N(v) \cap \sigma| = 2$. There exist $i_0, j_0 \in [n]$ such that $\{v, v^{i_0}, v^{j_0}\} \subseteq \sigma$. Since $|N(v) \cap \sigma| = 2$, $v^i \notin \sigma$ for all $i \neq i_0, j_0$. Choose $p \in [n] \setminus \{i_0, j_0\}$. Since $v^p \notin \sigma$ and σ is maximal, there exists $x_0 \in \sigma$ such that $d(x_0, v^p) \geq 4$. Observe that for any $u \in V(\mathbb{I}_n)$, if $d(v, u) \leq 2$, then $d(v^p, u) \leq 3$. Hence $d(v, v^p) = 3$. Here, $v_0 = v^{i_0, j_0}$ for some $v_0 \in [n]$. If $v_0 \in \{i, j, k\}$, then $v_0 \in [n]$. If $v_0 \in \{i, j, k\}$. Thus $v_0 = v^{i_0, j_0, k_0}$ for some $v_0 \in [n]$. If $v_0 \in \{i, j, k\}$. Thus $v_0 = v^{i_0, j_0, k_0}$ for some $v_0 \in [n]$. If $v_0 \in [n]$ is $v_0 \in [n]$. If $v_0 \in [n]$ if $v_0 \in [n]$. If $v_0 \in [n]$ if

- $(1) \ z_0 = v^{i_0, j_0}.$
 - In this case, $\{v^{i_0}, v^{j_0}, v^{i_0, j_0, l_0}\} \subseteq N(v^{i_0, j_0}) \cap \sigma$. We take $\tilde{v} = v^{i_0, j_0}$.
- $(2) \ z_0 = v^{i_0, l_0}.$

In this case, $\{v, v^{i_0}, v^{j_0}, v^{i_0, j_0, k_0}, v^{i_0, j_0, l_0}, v^{i_0, l_0}, v^{i_0, l_0}\} \subseteq \sigma$. Since $|N(v^{i_0, j_0, k_0}) \cap \sigma| \ge 2$ and $N(v^{i_0, j_0, k_0}) \cap \{v, v^{i_0}, v^{i_0}, v^{i_0, j_0, l_0}, v^{i_0, j_0, k_0}, v^{i_0, l_0}\} = \emptyset$, there exists $u_0 \in \sigma$ such that $u_0 \sim v^{i_0, j_0, k_0}$. Now $d(v, u_0) \le 3$ implies that $u_0 \in \{v^{i_0, j_0}, v^{i_0, k_0}, v^{j_0, k_0}\}$. Since $d(v^{j_0, k_0}, v^{i_0, l_0}) = 4$ and $v^{i_0, l_0} \in \sigma$, we see that $u_0 \ne v^{j_0, k_0}$. If $u_0 = v^{i_0, j_0}$, then $\{v^{i_0}, v^{j_0}, v^{i_0, j_0, l_0}\} \subseteq N(u_0) \cap \sigma$ and we take $\tilde{v} = u_0$. If $u_0 = v^{i_0, k_0}$, then $\{v, v^{i_0, l_0}, v^{i_0, k_0}\} \subseteq N(v^{i_0}) \cap \sigma$ and we take $\tilde{v} = v^{i_0}$.

(3) $z_0 = v^{j_0, l_0}$.

In this case, $\{v, v^{i_0}, v^{j_0}, v^{i_0, j_0, k_0}, v^{i_0, j_0, l_0}, v^{j_0, l_0}\} \subseteq \sigma$. Since $|N(v^{i_0}) \cap \sigma| \ge 2$ and $N(v^{i_0}) \cap \{v, v^{i_0}, v^{j_0}, v^{i_0, j_0, k_0}, v^{i_0, j_0, l_0}, v^{j_0, l_0}\} = \{v\}$, there exists $w_0 \in \sigma, w \ne v$ such that $w_0 \sim v^{i_0}$. Since $w_0 \ne v$, we see that $w_0 = v^{i_0, i}$ for some $i \in [n]$. If $i \notin \{j_0, l_0\}$, then $d(w_0, v^{j_0, l_0}) = 4$, a contradiction as $v^{j_0, l_0} \in \sigma$. Hence $i \in \{j_0, l_0\}$. If $i = j_0$, then $w_0 = v^{i_0, j_0}$ and $\{v^{i_0}, v^{j_0}, v^{i_0, j_0, l_0}\} \subseteq N(w_0) \cap \sigma$. We take $\tilde{v} = w_0$. So, assume that $i = l_0$, $i.e., w_0 = v^{i_0, l_0}$.

Here $\{v, v^{i_0}, v^{j_0}, v^{i_0, j_0, k_0}, v^{i_0, j_0, l_0}, v^{j_0, l_0}, v^{i_0, l_0}\} \subseteq \sigma$. Since $|N(v^{i_0, j_0, k_0}) \cap \sigma| \ge 2$ and $N(v^{i_0, j_0, k_0}) \cap \{v, v^{i_0}, v^{i_0}, v^{i_0, j_0, k_0}, v^{i_0, j_0, l_0}, v^{i_0, l_0}, v^{i_0, l_0}\} = \emptyset$, there exists $q_0 \in \sigma$ such that $q_0 \sim v^{i_0, j_0, k_0}$. Since $d(v, q_0) \le 3$, we see that $q_0 \in \{v^{i_0, j_0}, v^{i_0, k_0}, v^{j_0, k_0}\}$. Further, since $d(v^{j_0, k_0}, v^{i_0, l_0}) = 4$, $q_0 \ne v^{j_0, k_0}$. If $q_0 = v^{i_0, j_0}$, then $\{v^{i_0}, v^{j_0}, v^{i_0, j_0, l_0}\} \subseteq N(q_0) \cap \sigma$ and we take $\tilde{v} = q_0$. If $q_0 = v^{i_0, k_0}$, then $\{v, v^{i_0, l_0}, v^{i_0, k_0}\} \subseteq N(v^{i_0}) \cap \sigma$ and we take $\tilde{v} = v^{i_0}$.

This completes the proof.

Lemma 4.5. Let $n \geq 5$ and $\sigma \in \Delta_n$ be a maximal simplex such that σ covers all places. Let $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$. If there exists a $v \in \sigma$ such that $|N(v) \cap \sigma| \geq 3$, then $N(v) \subseteq \sigma$.

Proof. Without loss of generality, we assume that $v=v_1\dots v_n$, where $v_i=0$ for all $i\in[n]$. Suppose $N(v)\not\subseteq\sigma$. Then there exists $l_0\in[n]$ such that $v^{l_0}\notin\sigma$. Since $|N(v)\cap\sigma|\geq 3$, there exist $i_0,j_0,k_0\in[n]\setminus\{l_0\}$ such that $\{v^{i_0},v^{j_0},v^{k_0}\}\subseteq\sigma$. Further, since σ is maximal and $v^{l_0}\notin\sigma$, there exists $x_0\in\sigma$ such that $d(x_0,v^{l_0})\geq 4$. Observe that $d(v,x_0)=3$ and therefore $x_0=v^{i,j,k}$ for some $i,j,k\in[n]$. Since $d(x_0,v^{l_0})\geq 4$, $l_0\notin\{i,j,k\}$. If $i_0\notin\{i,j,k\}$, then $d(x_0,v^{i_0})=4$, a contradiction as $v^{i_0}\in\sigma$. Hence $i_0\in\{i,j,k\}$. By similar arguments, $j_0,k_0\in\{i,j,k\}$ and therefore $x_0=v^{i_0,j_0,k_0}$. So, $\{v,v^{i_0},v^{j_0},v^{k_0},v^{i_0,j_0,k_0}\}\subseteq\sigma$.

Observe that for any $u \in \{v, v^{i_0}, v^{j_0}, v^{k_0}, v^{i_0,j_0,k_0}\}$, $u(l_0) = 0$. Since σ covers all places, there exists $y_0 \in \sigma$ such that $y_0(l_0) = 1$. Since $v^{l_0} \notin \sigma$, $y_0 \neq v^{l_0}$. Hence $d(v, y_0) \geq 2$. Suppose $d(v, y_0) = 3$. Then $y_0 = v^{l_0, i, j}$ for some i, j. If $k \in \{i_0, j_0, k_0\} \setminus \{i, j\}$, then $d(y_0, v^k) \geq 4$, a contradiction as $v^k \in \sigma$. Hence $d(v, y_0) = 2$. So, $y_0 = v^{l_0, i}$ for some $i \in [n]$. If $i \notin \{i_0, j_0, k_0\}$, then $d(y_0, v^{i_0, j_0, k_0}) \geq 4$. Since $v^{i_0, j_0, k_0} \in \sigma$, we see that $i \in \{i_0, j_0, k_0\}$. Without loss of generality we assume that $i = i_0$, i.e., $y_0 = v^{l_0, i_0}$. So, $\{v, v^{i_0}, v^{j_0}, v^{k_0}, v^{i_0, j_0, k_0}, v^{l_0, i_0}\} \subseteq \sigma$.

Observe that $N(v^{l_0,i_0}) \cap \{v,v^{i_0},v^{j_0},v^{k_0},v^{i_0,j_0,k_0},v^{l_0,i_0}\} = \{v^{i_0}\}$. Since $|N(v^{l_0,i_0}) \cap \sigma| \geq 2$, there exists $z_0 \in \sigma, z_0 \neq v^{i_0}$ such that $z_0 \sim v^{l_0,i_0}$. Further, since $z_0 \neq v^{i_0}$ and $v^{l_0} \notin \sigma$, $z_0 = v^{l_0,i_0,i}$ for some $i \in [n]$. If $i \neq j_0$, then $d(z_0,v^{j_0}) = 4$, a contradiction as $v^{j_0} \in \sigma$. Hence $z_0 = v^{l_0,i_0,j_0}$. But then $d(z_0,v^{k_0}) = 4$, a contradiction. Hence $N(v^{l_0,i_0}) \cap \sigma = \{v^{i_0}\}$, which is a contradiction.

Thus, we conclude that $N(v) \subseteq \sigma$.

Lemma 4.6. Let $n \geq 5$ and $\sigma \in \Delta_n$ be a maximal simplex such that σ covers all places. If $|N(w) \cap \sigma| \geq 2$ for all $w \in \sigma$, then there exist $v, w \in \sigma$ such that $v \sim w$ and $\sigma = N(v) \cup N(w)$.

Proof. Using Lemma 4.4 and Lemma 4.5, we conclude that there exists $v \in \sigma$ such that $N(v) \subseteq \sigma$. Hence $\{v, v^1, \dots, v^n\} \subseteq \sigma$. Observe that $N(v^1) \cap \{v, v^1, \dots, v^n\} = \{v\}$. Since $|N(v^1) \cap \sigma| \ge 2$, there exists $x_0 \in \sigma, x_0 \ne v$ such that $x_0 \sim v^1$. Then $x_0 = v^{1,i_1}$ for some $i_1 \in [n]$. So, $\{v, v^1, \dots, v^n, v^{1,i_1}\} \subseteq \sigma$. Choose $i_2 \in [n] \setminus \{1, i_1\}$.

 $\{v, v^1, \dots, v^n, v^{1,i_1}\} \subseteq \sigma. \text{ Choose } i_2 \in [n] \setminus \{1, i_1\}.$ Observe that $v^{i_2} \in \sigma$ and $N(v^{i_2}) \cap \{v, v^1, \dots, v^n, v^{1,i_1}\} = \{v\}. \text{ Since } |N(v^{i_2}) \cap \sigma| \geq 2, \text{ there exists } y_0 \in \sigma, y_0 \neq v \text{ such that } y_0 \sim v^{i_2}. \text{ Further, since } y_0 \neq v, \text{ we see that } y_0 = v^{i_2,i} \text{ for some } i \in [n]. \text{ If } i \notin \{1, i_1\}, \text{ then } d(y_0, v^{1,i_1}) = 4, \text{ a contradiction as } v^{1,i_1} \in \sigma. \text{ Hence either } y_0 = v^{i_2,1} \text{ or } y_0 = v^{i_2,i_1}. \text{ If } y_0 = v^{i_2,1}, \text{ then } \{v, v^{1,i_1}, v^{i_2,1}\} \subseteq N(v^1) \cap \sigma. \text{ Hence from Lemma 4.5, } N(v_1) \subseteq \sigma. \text{ Thus } N(v) \cup N(v^1) \subseteq \sigma. \text{ If } y_0 = v^{i_2,i_1}, \text{ then } \{v, v^{1,i_1}, v^{i_2,i_1}\} \subseteq N(v^{i_1}) \cap \sigma \text{ and therefore Lemma 4.5 implies that } N(v^{i_1}) \subseteq \sigma. \text{ Hence } N(v) \cup N(v^{i_1}) \subseteq \sigma.$

Thus, we have shown that there exist vertices $v, w \in \sigma$ such that $v \sim w$ and $N(v) \cup N(w) \subseteq \sigma$. We now show that $\sigma \subseteq N(v) \cup N(w)$. Suppose there exists $z_0 \in \sigma$ such that $z_0 \notin N(v) \cup N(w)$. Since $w \sim v$, $w = v^{l_0}$ for some $l_0 \in [n]$. Clearly, $d(z_0, v) \geq 2$. Suppose $d(z_0, v) = 2$. Then $z_0 = v^{i,j}$ for some $i, j \in [n]$. If $l_0 \in \{i, j\}$, then $z_0 \sim v^{l_0}$, which is a contradiction as $z_0 \notin N(w)$. Hence $l_0 \notin \{i, j\}$. Choose $k_0 \in [n] \setminus \{l_0, i, j\}$. Since $N(v^{l_0}) \subseteq \sigma$ and $v^{l_0} \sim v^{l_0, k_0}$, we see that $v^{l_0, k_0} \in \sigma$. But then $d(z_0, v^{l_0, k_0}) = 4$, a contradiction. Now let $d(z_0, v) = 3$. Then $z_0 = v^{i, j, k}$ for some i, j, k. Choose $p \in [n] \setminus \{i, j, k\}$. Then $N(v) \subseteq \sigma$ implies that $v^p \in \sigma$. But $d(z_0, v^p) = 4$, a contradiction. Thus, we conclude that $N(v) \cup N(w) = \sigma$.

We are now ready to give a characterization of maximal simplices of Δ_n . Recall that for $i \in [n]$ and $\epsilon \in \{0,1\}$, $\mathbb{I}_n^{(i,\epsilon)}$ is the induced subgraph of \mathbb{I}_n on the vertex set $\{v \in V(\mathbb{I}_n) : v(i) = \epsilon\}$.

Lemma 4.7. Let $n \ge 4$ and let $\sigma \in \Delta_n$ be a maximal simplex. Then $dim(\sigma) \in \{7, n+3, 2n-1\}$. Moreover, if $dim(\sigma) \ne 7$, then either $\sigma = N(v) \cup N(w)$ for some $v \sim w$, or $\sigma = N(u) \cup K_u^{i,j,k}$ for some u and $i, j, k \in [n]$.

Proof. Proof is by induction on n. Let n=4. For any two vertices $v,w\in V(\mathbb{I}_4)$, let $\overline{\{v,w\}}$ denote a simplicial complex on two vertices, i.e., $\overline{\{v,w\}}\cong S^0$. Let v=0000. It is easy to check that

$$\Delta_n = \overline{\{v, v^{1,2,3,4}\}} * \overline{\{v^1, v^{2,3,4}\}} * \overline{\{v^2, v^{1,3,4}\}} * \overline{\{v^3, v^{1,2,4}\}} * \overline{\{v^4, v^{1,2,3}\}} * \overline{\{v^{1,2}, v^{3,4}\}} * \overline{\{v^{1,3}, v^{2,4}\}} * \overline{\{v^{1,4}, v^{2,3}\}},$$

the join of 8-copies of S^0 . Therefore each maximal simplex of Δ_4 is of dimension 7. So assume that $n \geq 5$. Inductively assume that result is true for each $\mathcal{VR}(\mathbb{I}_r;3)$, where $4 \leq r < n$.

Let $\sigma \in \Delta_n$ be a maximal simplex. Suppose σ covers all places. Then from Lemma 4.1, $|N(v) \cap \sigma| \ge 1$ for all $v \in \sigma$. If there exists a vertex $w \in \sigma$ such that $N(w) \cap \sigma = \{v\}$, then from Lemma 4.3, $\sigma = N(v) \cup K_v^{i_0, j_0, k_0}$ for some $i_0, j_0, k_0 \in [n]$. Clearly $\dim(\sigma) = n + 3$. If $|N(v) \cap \sigma| \ge 2$ for all $v \in \sigma$, then from Lemma 4.6, there exist $v, w \in \sigma$ such that $v \sim w$ and $\sigma = N(v) \cup N(w)$. It is easy to check that $\dim(\sigma) = 2n - 1$.

So, assume that σ does not covers all places. There exists $l \in [n]$ such that v(l) = w(l) for all $v, w \in \sigma$. Without loss of generality we assume that v(l) = 0 for all $v \in \sigma$. Observe that $\sigma \in \mathcal{VR}(\mathbb{I}_n^{(l,0)};3)$. Since σ is a maximal simplex in Δ_n , σ is maximal in $\mathcal{VR}(\mathbb{I}_n^{(l,0)};3)$. Since $\mathbb{I}_n^{(l,0)} \cong \mathbb{I}_{n-1}$, by induction hypothesis, either $\dim(\sigma) = 7$ or; $\sigma = N_{\mathbb{I}_n^{l,0}}(v) \cup N_{\mathbb{I}_n^{(l,0)}}(w)$ for some $v, w \in V(\mathbb{I}_n^{(l,0)}), v \in N_{\mathbb{I}_n^{(l,0)}}(w)$ or $\sigma = N_{\mathbb{I}_n^{(l,0)}}(u) \cup K_u^{(i,j,k)}$ for some $i, j, k \in [n] \setminus \{l\}$. Suppose $\dim(\sigma) \neq 7$. Then either $\sigma = N_{\mathbb{I}_n^{(l,0)}}(v) \cup N_{\mathbb{I}_n^{(l,0)}}(w)$ for some $v, w \in V(\mathbb{I}_n^{(l,0)}), v \in N_{\mathbb{I}_n^{(l,0)}}(w)$ or

 $\sigma = N_{\mathbb{I}_n^{(l,0)}}(v) \cup K_v^{i,j,k}$ for some $i,j,k \in [n] \setminus \{l\}$. In either case $v^l \notin \sigma$ and $\sigma \cup \{v^l\}$ is a simplex in Δ_n , which contradicts the maximality of σ . Hence $\dim(\sigma) = 7$. This completes the proof. \square

Remark 2. In [32], Kleitmann proved that for $n \geq 2l+1$, the largest set family of subsets of [n] with pairwise symmetric difference at most 2l contains no more than $\sum_{t=0}^{l} {n \choose t}$ elements. Hence it gives the maximum possible size of a maximal simplex of $\mathcal{VR}(\mathbb{I}_n; 2)$ (l=1) in Lemma 3.1 and the maximum possible size of a maximal simplex of $\mathcal{VR}(\mathbb{I}_n; 3)$ (l=2) in Lemma 4.7

4.2. Collapsibility. In this section, we prove Theorem C. We first establish a few lemmas, which we need to prove Theorem C.

Let X be a topological space and A be a subspace of X. Recall that a retraction of X onto A is a map $r: X \to A$ such that r(a) = a for all $a \in A$.

Lemma 4.8. Let n > m and let H be an m-dimensional cube subgraph of \mathbb{I}_n . Then there exists a retraction $r : \Delta_n \to \mathcal{VR}(H; 3)$.

Proof. Observe that, there exist sequences (i_1, \ldots, i_{n-m}) and $(\epsilon_1, \ldots, \epsilon_{n-m})$, where $i_1, \ldots, i_{n-m} \in [n], \epsilon_1, \ldots, \epsilon_{n-m} \in \{0, 1\}$ such that H is the induced subgraph of \mathbb{I}_n on the vertex set $\{v \in V(\mathbb{I}_n) : v(i_j) = \epsilon_j \ \forall \ 1 \leq j \leq n-m\}$. Define $r_1 : V(I_n) \to V(I_n^{i_1, \epsilon_1})$ as follows: for $v \in V(\mathbb{I}_n)$ and $t \in [n]$,

$$r_1(v)(t) = \begin{cases} v(t) & \text{if } t \neq i_1, \\ \epsilon_1 & \text{if } t = i_1. \end{cases}$$

We extend the map r_1 to $\tilde{r}_1:\Delta_n\to \mathcal{VR}(\mathbb{I}_n^{i_1,\epsilon_1};3)$ by $\tilde{r}_1(\sigma):=\{r_1(v):v\in\sigma\}$ for all $\sigma\in\Delta_n$. Let $\sigma\in\Delta_n$ and let $v,w\in\sigma$. Then $d(v,w)\leq 3$. If $v(i_1)=w(i_1)$, then $r_1(v)=r_1(w)$ and therefore $d(r_1(v),r_1(w))=d(v,w)$. If $v(i_1)\neq w(i_1)$, then $d(r_1(v),r_1(w))=d(v,w)-1$. So, $d(r_1(v),r_1(w))\leq d(v,w)\leq 3$. Thus, $\tilde{r}_1(\sigma)\in\mathcal{VR}(\mathbb{I}_n^{i_1,\epsilon_1};3)$. Hence \tilde{r}_1 is well defined. Clearly \tilde{r}_1 is onto and for any $\sigma\in\mathcal{VR}(\mathbb{I}_n^{i_1,\epsilon_1};3)$, $\tilde{r}_1(\sigma)=\sigma$. Hence \tilde{r}_1 is a retraction. If m=n-1, then we take $r=\tilde{r}_1$. Suppose m< n-1. Let n-m=k. Assume that we have a retraction $\tilde{r}_{k-1}:\Delta_n\to\mathcal{VR}(H_{k-1};3)$, where H_{k-1} is the induced subgraph of \mathbb{I}_n on the vertex set $\{v\in V(\mathbb{I}_n): v(i_j)=\epsilon_j\;\forall\;1\leq j\leq k-1\}$. Define $r_k:V(H_{k-1})\to V(H)$ as follows: for $v\in V(H_{k-1})$ and $t\in[n]$,

$$r_k(v)(t) = \begin{cases} v(t) & \text{if } t \neq i_k, \\ \epsilon_k & \text{if } t = i_k. \end{cases}$$

Extend the map r_k to $\tilde{r}_k : \mathcal{VR}(H_{k-1};3) \to \mathcal{VR}(H;3)$ by $\tilde{r}_k(\sigma) := \{r_k(v) : v \in \sigma\}$ for all $\sigma \in \Delta_n$. Clearly, \tilde{r}_k is a retraction. We take r as the composition of the maps \tilde{r}_k and \tilde{r}_{k-1} . This completes the proof.

Lemma 4.9. Let $n \geq 5$ and $\sigma \in \Delta_n$ be a maximal simplex. If for some v, $|N(v) \cap \sigma| \geq 4$, then $N[v] \subseteq \sigma$.

Proof. Let $|N(v) \cap \sigma| \geq 4$. Suppose $N(v) \not\subseteq \sigma$. Then there exists a $l_0 \in [n]$ such that $v^{l_0} \notin \sigma$. Since $|N(v) \cap \sigma| \geq 4$, there exist $i_0, j_0, k_0, p_0 \in [n]$ such that $\{v^{i_0}, v^{j_0}, v^{k_0}, v^{p_0}\} \subseteq \sigma$. Clearly $l_0 \notin \{i_0, j_0, k_0, p_0\}$. Since $v^{l_0} \notin \sigma$ and σ is a maximal simplex, there exists $x_0 \in \sigma$ such that $d(x_0, v^{l_0}) \geq 4$. Observe that for any vertex u, if $d(v, u) \leq 2$, then $d(u, v^{l_0}) \leq 3$. Hence $d(v, x_0) = 3$. Here, $x_0 = v^{i,j,k}$ for some $i, j, k \in [n]$. If $i_0 \notin \{i, j, k\}$, then $d(x_0, v^{i_0}) = 4$, a contradiction as $v^{i_0} \in \sigma$. Hence $i_0 \in \{i, j, k\}$. By similar arguments, we can show that $j_0, k_0, p_0 \in \{i, j, k\}$. Hence $\{i_0, j_0, k_0, p_0\} \subseteq \{i, j, k\}$, which is not possible. Thus, $N(v) \subseteq \sigma$.

Suppose $v \notin \sigma$, then there exists a vertex $y_0 \in \sigma$ such that $d(v, y_0) \geq 4$. Suppose $d(v, y_0) = 4$. Let $y_0 = v^{i,j,k,l}$. Since $n \geq 5$, there exists $t \in [n] \setminus \{i,j,k,l\}$. Then $d(y_0, v^t) \geq 4$, a contradiction as $v^t \in \sigma$. Hence $d(v, y_0) \geq 5$. But then $d(v^{i_0}, y_0) \geq 4$, again a contradiction. Hence $v \in \sigma$. Thus, $N[v] \subseteq \sigma$.

Lemma 4.10. Let $n \geq 5$ and let $\sigma \in \Delta_n$ be a maximal simplex. Suppose there exists a vertex v such that $\{v^{i_0,j_0}, v^{i_0,k_0}, v^{j_0,k_0}, v^{p_0}, v^{q_0}\} \subseteq \sigma$, where $p_0, q_0 \notin \{i_0, j_0, k_0\}$. Then $\sigma = N(v) \cup K_v^{i_0,j_0,k_0}$.

Proof. We first show that $v^{i_0} \in \sigma$. If $v^{i_0} \notin \sigma$, then there exists $y_0 \in \sigma$ such that $d(v^{i_0}, y_0) \ge 4$. Observe that $d(v, y_0) \geq 3$. We have the following two cases.

(1) $d(v, y_0) = 3$.

Here, $y_0 = v^{i,j,k}$ for some $i, j, k \in [n]$. Since $d(y_0, v^{p_0}) \leq 3$ and $d(y_0, v^{q_0}) \leq 3$, we see that $p_0, q_0 \in \{i, j, k\}$. Without loss of generality we assume that $i = p_0$ and $j = q_0$, i.e., $y_0 = v^{p_0,q_0,k}$. Then either $d(y_0,v^{i_0,j_0}) \geq 4$, or $d(y_0,v^{i_0,k_0}) \geq 4$, or $d(y_0,v^{j_0,k_0}) \geq 4$, a contradiction as $v^{i_0,j_0}, v^{i_0,k_0}, v^{j_0,k_0} \in \sigma$.

(2) $d(v, y_0) \ge 4$.

Observe that if $d(v, y_0) \ge 5$, then $d(y_0, v^{p_0}) \ge 4$, which is not possible, since $y_0, v^{p_0} \in$ σ . Hence $d(v, y_0) = 4$. There exist $i, j, k, l \in [n]$ such that $y_0 = v^{i, j, k, l}$. Since $d(y_0, v^{p_0}) \leq 1$ 3 and $d(y_0, v^{q_0}) \leq 3$, we see that $p_0, q_0 \in \{i, j, k, l\}$. Further, since $d(v^{i_0}, y_0) \geq 4$, $i_0 \notin \{i, j, k, l\}$. If $\{j_0, k_0\} \not\subseteq \{i, j, k, l\}$, then $d(y_0, v^{j_0, k_0}) \geq 4$, a contradiction as $v^{j_0, k_0} \in \sigma$. Hence $y_0 = v^{p_0,q_0,j_0,k_0}$. But then $d(y_0,v^{i_0,j_0}) \geq 4$, a contradiction.

Hence $v^{i_0} \in \sigma$. By similar arguments $v^{j_0}, v^{k_0} \in \sigma$. Since $\{v^{i_0}, v^{j_0}, v^{k_0}, v^{p_0}, v^{q_0}\} \subseteq N(v) \cap \sigma$, from Lemma 4.9, $N[v] \subseteq \sigma$. Hence $N(v) \cup K_v^{i_0,j_0,k_0} \subseteq \sigma$. From Lemma 4.3, $N(v) \cup K_v^{i_0,j_0,k_0}$ is a maximal simplex and therefore $\sigma = N(v) \cup K_v^{i_0, j_0, k_0}$

Inspired by Lemma 4.7, we write the set of maximal simplices of Δ_n , $M(\Delta_n) = \mathcal{A}_n \cup \mathcal{B}_n \cup \mathcal{C}_n$, where

$$\mathcal{A}_n = \{ \sigma \in M(\Delta_n) : \sigma = N(v) \cup K_v^{i,j,k} \text{ for some } v \in V(\mathbb{I}_n) \text{ and } i, j, k \in [n] \},$$

$$\mathcal{B}_n = \{ \sigma \in M(\Delta_n) : \sigma = N(v) \cup N(w) \text{ for some } v, w \in V(\mathbb{I}_n), v \sim w \} \text{ and }$$

$$\mathcal{C}_n = M(\Delta_n) \setminus (\mathcal{A}_n \cup \mathcal{B}_n).$$

Lemma 4.11. Let $n \geq 5$. Then by using a sequence of elementary 8-collapses, Δ_n collapses to a subcomplex Δ'_n , where $M(\Delta'_n) = \mathcal{B}_n \cup \mathcal{C}_n \cup \{K_v^{i,j,k} \cup \{v^i,v^j,v^k,v^l\} : v \in V(\mathbb{I}_n), \{i,j,k,l\} \subseteq [n]\}.$

Proof. Let $\sigma \in \mathcal{A}_n$. Then $\sigma = N(v) \cup K_v^{i_0, j_0, k_0}$ for some $v \in V(\mathbb{I}_n)$ and $i_0, j_0, k_0 \in [n]$. We will use the following claim.

Claim 2. Δ_n collapses to a subcomplex X, where the set of maximal simplices

$$M(X) = M(\Delta_n) \setminus \{\sigma\} \cup \{K_v^{i_0, j_0, k_0} \cup \{v^{i_0}, v^{j_0}, v^{k_0}, v^i\} : i \in [n] \setminus \{i_0, j_0, k_0\}\}.$$

Proof of Claim 2. Without loss of generality we assume that $\{i_0, j_0, k_0\} = \{1, 2, 3\}$. From Lemma 4.10, $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^4, v^5\}, \sigma)$ is a collapsible pair. Thus, $\sigma \searrow \sigma \setminus \{v^4\}, \sigma \setminus \{v^5\}, \sigma \setminus \{v^{1,2}\}, \sigma \setminus \{v^{1,3}\}, \sigma \setminus \{v^{2,3}\}$. Observe that $\sigma \setminus \{v^{1,2}\} \subseteq N(v) \cup N(v^3), \sigma \setminus \{v^{1,3}\} \subseteq N(v) \cup N(v^3)$ $N(v^2), \sigma \setminus \{v^{2,3}\} \subseteq N(v) \cup N(v^1)$. From Lemma 4.7, for any $u \sim w$, $N(u) \cup N(w)$ is a maximal simplex in Δ_n and therefore we see that Δ_n collapses to a subcomplex X_1 , where $M(X_1) = M(\Delta_n) \setminus \{\sigma\} \cup \{\sigma \setminus \{v^4\}, \sigma \setminus \{v^5\}\}.$

Hence claim is true if n=5. So assume that $n\geq 6$. From Lemma 4.10, $(\{v^{1,2},v^{1,3},v^{2,3},v^5,v^6\},$ σ) and $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^4, v^6\}, \sigma)$ are collapsible pairs in Δ_n . Hence $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^5, v^6\}, \sigma)$ of and $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^{4}, v^{5}\}, \sigma)$ are conapsible pairs in Δ_n . Hence $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^{4}, v^{5}\}, \sigma \setminus \{v^{4}\})$ and $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^{4}, v^{6}\}, \sigma \setminus \{v^{5}\}, \sigma \setminus \{v^{4}, v^{1,2}\}, \sigma \setminus \{v^{4}, v^{1,3}\}, \sigma \setminus \{v^{4}, v^{2,3}\} \text{ and } \sigma \setminus \{v^{5}\} \setminus \sigma \setminus \{v^{5}, v^{4}\}, \sigma \setminus \{v^{5}, v^{6}\}, \sigma \setminus \{v^{5}, v^{1,2}\}, \sigma \setminus \{v^{5}, v^{1,3}\}, \sigma \setminus \{v^{5}, v^{2,3}\}.$ Observe that $\sigma \setminus \{v^{4}, v^{1,2}\}, \sigma \setminus \{v^{5}, v^{1,2}\} \subseteq N(v) \cup N(v^{3}), \sigma \setminus \{v^{4}, v^{1,3}\}, \sigma \setminus \{v^{5}, v^{1,3}\} \subseteq N(v) \cup N(v^{2}), \sigma \setminus \{v^{4}, v^{2,3}\}, \sigma \setminus \{v^{5}, v^{2,3}\} \subseteq N(v) \cup N(v^{1}).$ Therefore, we conclude that X_1 collapses

to the subcomplex X_2 , where $M(X_2) = M(\Delta_n) \setminus \{\sigma\} \cup \{\sigma \setminus \{v^4, v^5\}, \sigma \setminus \{v^4, v^6\}, \sigma \setminus \{v^5, v^6\}\}$.

Hence the claim is true if n=6. Let $n\geq 7$ and inductively assume that Δ_n collapses to the subcomplex X_{n-5} , where

$$M(X_{n-5}) = M(\Delta_n) \setminus \{\sigma\} \cup \{\sigma \setminus \{v^{l_1}, v^{l_2}, \dots, v^{l_{n-5}}\} : \{l_1, l_2, \dots, l_{n-5}\} \subseteq \{4, 5, \dots, n-1\}\}.$$

Let $\{i_1, i_2, \dots, i_{n-5}\} \subseteq \{4, 5, \dots, n-1\}$. Observe that $|\{4, 5, \dots, n-1, n\} \setminus \{i_1, i_2, \dots, i_{n-5}\}| =$ 2 and $n \in \{4, 5, \dots, n-1, n\} \setminus \{i_1, i_2, \dots, i_{n-5}\}$. Let $\{n, p\} = \{4, 5, \dots, n-1, n\} \setminus \{i_1, i_2, \dots, i_{n-5}\}$. Using Lemma 4.10, we observe that $\sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}\}$ is the only maximal simplex in X_{n-5} , which contains $\{v^{1,2}, v^{1,3}, v^{2,3}, v^p, v^n\}$. So, $(\{v^{1,2}, v^{1,3}, v^{2,3}, v^p, v^n\}, \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}\})$ is a collapsible pair in X_{n-5} . Therefore $\sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}\} \setminus \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^p\}, \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^n\}, \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^{1,2}\}, \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^{1,3}\}, \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^{2,3}\}$. Clearly, $\sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v$

$$M(X'_{n-5}) = M(\Delta_n) \setminus \{\sigma\} \cup \{\sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^p\}, \sigma \setminus \{v^{i_1}, \dots, v^{i_{n-5}}, v^n\}\} \cup \{\sigma \setminus \{v^{l_1}, v^{l_2}, \dots, v^{l_{n-5}}\} : \{l_1, l_2, \dots, l_{n-5}\} \subseteq \{4, 5, \dots, n-1\} \setminus \{i_1, \dots, i_{n-5}\}\}.$$

By applying an argument similar as above for each $\{l_1, l_2, \dots, l_{n-5}\} \subseteq \{4, 5, \dots, n-1\}$, we get that X_{n-5} collapses to the subcomplex X_{n-4} , where

$$M(X_{n-4}) = M(\Delta_n) \setminus \{\sigma\} \cup \{\sigma \setminus \{v^{l_1}, v^{l_2}, \dots, v^{l_{n-4}}\} : \{l_1, l_2, \dots, l_{n-4}\} \subseteq \{4, 5, \dots, n\}\}.$$

Observe that $\{\sigma \setminus \{v^{l_1}, v^{l_2}, \dots, v^{l_{n-4}}\} : \{l_1, l_2, \dots, l_{n-4}\} \subseteq \{4, 5, \dots, n\}\} = \{K_v^{1,2,3} \cup \{v^1, v^2, v^3, v^i\} : i \in [n] \setminus \{1, 2, 3\}\}$. Thus, by induction we get that Δ_n collapses to a subcomplex X_{n-4} , where

$$M(X_{n-4}) = M(\Delta_n) \setminus \{\sigma\} \cup \{K_n^{1,2,3} \cup \{v^1, v^2, v^3, v^i\} : 4 \le i \le n\}.$$

We take $X = X_{n-4}$ and this completes the proof of Claim 2.

By applying the Claim 2 for each $\tau \in \mathcal{A}_n$, we get that Δ_n collapses to a subcomplex Δ'_n , where $M(\Delta'_n) = \mathcal{B}_n \cup \mathcal{C}_n \cup \{K_v^{i,j,k} \cup \{v^i, v^j, v^k, v^l\} : v \in V(\mathbb{I}_n), \{i, j, k, l\} \subseteq [n]\}.$

We are now ready to prove main theorem of this section.

Proof of Theorem C. We need to show that the collapsibility number of $\mathcal{VR}(\mathbb{I}_n;3)$ is 8. We first show that Δ_n is 8-collapsible. It is easy to check that each maximal simplex of Δ_4 is of dimension 7. Hence Δ_4 is 8-collapsible. So assume that $n \geq 5$. From Lemma 4.11, by using elementary 8-collapses, Δ_n collapses to a subcomplex Δ'_n , where $M(\Delta'_n) = \mathcal{B}_n \cup \mathcal{C}_n \cup \{K_v^{i,j,k} \cup \{v^i,v^j,v^k,v^l\}: v \in V(\mathbb{I}_n), \{i,j,k,l\} \subseteq [n]\}$. Let $\mathcal{D}_n = \{K_v^{i,j,k} \cup \{v^i,v^j,v^k,v^l\}: v \in V(\mathbb{I}_n), \{i,j,k,l\} \subseteq [n]\}$. Since $n \geq 5$, by using the cardinalities of the elements of $M(\Delta'_n)$, we conclude that $M(\Delta'_n) = \mathcal{B}_n \cup \mathcal{C}_n \cup D_n$.

Choose a linear order \prec_1 on elements of \mathcal{B}_n . Extend \prec_1 to a linear order \prec on maximal simplices of Δ'_n , where elements of \mathcal{B}_n are ordered first, *i.e.*, for any two $\sigma_1, \sigma_2 \in M(\Delta'_n)$, if $\sigma_1 \in \mathcal{B}_n$ and $\sigma_2 \in \mathcal{C}_n \cup \mathcal{D}_n$, then $\sigma_1 \prec \sigma_2$. Let $\tau \in \Delta'_n$. Let σ be the smallest (with respect to \prec) maximal simplex of Δ'_n such that $\tau \subseteq \sigma$. If $\sigma \in \mathcal{C}_n \cup \mathcal{D}_n$, then $|\sigma| = 8$ and thereby implying that $|M_{\prec}(\tau)| \leq 8$. So assume that $\sigma \in \mathcal{B}_n$. There exist $v, w \in V(\mathbb{I}_n)$ such that $v \sim w$ and $\sigma = N(v) \cup N(w)$. We first prove that $|M_{\prec}(\tau) \cap N(v)| \leq 4$.

Let $\operatorname{mes}_{\prec}(\tau) = (x_1, \dots, x_t)$. Suppose $|M_{\prec}(\tau) \cap N(v)| \geq 5$. Let k be the least integer such that $|\{x_1, \dots, x_k\} \cap N(v)| = 4$. Clearly, k < t. Let $\{x_1, \dots, x_k\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. Observe that $x_k \in \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. Let γ be a maximal simplex such that $\gamma \prec \sigma$. If $\{x_1, \dots, x_k\} \cap (\sigma \setminus \gamma) \neq \emptyset$, then $x_{k+1} \in \{x_1, \dots, x_k\}$. Hence $\{x_1, \dots, x_{k+1}\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. If $\{x_1, \dots, x_k\} \cap (\sigma \setminus \gamma) = \emptyset$, then $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\} \subseteq \gamma$. From Lemma 4.9, $N(v) \subseteq \gamma$. Thus $x_{k+1} \notin N(v)$, thereby implying that $\{x_1, \dots, x_{k+1}\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. If k+1=t, then we get a contradiction to the assumption that $|M_{\prec}(\tau) \cap N(v)| \geq 5$. Inductively assume that for all $k \leq l < t$, $\{x_1, \dots, x_l\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. If $\{x_1, \dots, x_{t-1}\} \cap (\sigma \setminus \gamma) \neq \emptyset$, then $x_t \in \{x_1, \dots, x_k\}$. Hence $\{x_1, \dots, x_t\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$. If $\{x_1, \dots, x_{t-1}\} \cap (\sigma \setminus \gamma) = \emptyset$, then $\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\} \subseteq \gamma$. From Lemma 4.9, $N(v) \subseteq \gamma$. Thus $x_t \notin N(v)$. Hence we get that $\{x_1, \dots, x_t\} \cap N(v) = \{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}$, which is a contradiction to the assumption that $|M_{\prec}(\tau) \cap N(v)| \geq 5$. Thus $|M_{\prec}(\tau) \cap N(v)| \leq 4$.

By using an argument similar as above, $|M_{\prec}(\tau) \cap N(w)| \leq 4$. Since $\tau \subseteq N(v) \cup N(w)$, we see that $M_{\prec}(\tau) \leq 8$. From Proposition 3.3, Δ_n is 8-collapsible.

Let X be the Veitoris-Rips complex of a 4-dimensional cube subgraph of \mathbb{I}_n . Then using Lemma 4.8, there exists a retraction $r: \Delta_n \to X$. Since $X \cong \Delta_4$ and $\Delta_4 \cong S^7$, we see that $\tilde{H}_7(X;\mathbb{Z}) \neq 0$. Further, since $r_*: \tilde{H}_7(\Delta_n;\mathbb{Z}) \to \tilde{H}_7(X;\mathbb{Z})$ is surjective, $\tilde{H}_7(\Delta_n;\mathbb{Z}) \neq 0$. Using Proposition 1.2, we conclude that the collapsibility number of Δ_n is 8.

4.3. **Homology.** The main aim of this section is to prove Theorem A. We first establish a series of lemmas, which we need to prove Theorem A. We always consider the reduced homology with integer coefficients.

teger coefficients. For
$$1 \le i \le n$$
 and $\epsilon \in \{0,1\}$, let $\Delta_n^{i,\epsilon} = \mathcal{VR}(\mathbb{I}_n^{(i,\epsilon)};3)$ and $\partial(\Delta_n) = \bigcup_{i \in [n], \epsilon \in \{0,1\}} \Delta_n^{i,\epsilon}$.

The following lemma plays a key role in the proof of Theorem A.

Lemma 4.12. Let $n \geq 5$ and let $p \leq n-2$. Then any p-cycle c in Δ_n is homologous to a p-cycle \tilde{c} in $\partial(\Delta_n)$.

Proof. For any chain $z = \sum a_i \sigma_i$ in Δ_n , if $a_i \neq 0$, then we say that $\sigma_i \in z$. For a cycle z in Δ_n , let $\mu(z) = \{\sigma \in z : \sigma \notin \partial(\Delta_n)\}$. Let c be a p-cycle in Δ_n . If $\mu(c) = \emptyset$, then c is a p-cycle in $\partial(\Delta_n)$. Suppose $\mu(c) \neq \emptyset$. We show that c is homologous to a p-cycle c_1 such that $|\mu(c_1)| < |\mu(c)|$. Let $\sigma \in c$ such that $\sigma \notin \partial(\Delta_n)$, i.e., σ covers all places. Let τ be a maximal simplex such that $\sigma \subseteq \tau$. Using Lemmas 4.1, 4.3 and 4.6, we see that either $\tau = N(v) \cup K_v^{i_0, j_0, k_0}$ for some v and v and v and v and v and v are v are v and v are v are v and v are v are v and v are v and v are v and v are v are v and v are v and v are v are v and v are v are v are v are v and v are v and v are v are v are v and v are v and v are v are v and v are v are v and v are v and v are v are v are v and v and v are v and v are v are v and v are v are v are v and v are v and v are v are v are v are v and v are v are v and v are v are v and v are v are v are v and v are v are v are v are v are v and v are v and v are v and v are v are v are v are v are v and v are v are v are v and v are v are v are v are v are v an

Case 1. $\tau = N(v) \cup K_v^{i_0, j_0, k_0}$.

Let $i \in [n] \setminus \{i_0, j_0, k_0\}$. Observe that for any $x \in \tau \setminus \{v^i\}$, x(i) = v(i). Since σ covers all places and x(i) = v(i) for all $x \in \tau \setminus \{v^i\}$, we see that $v^i \in \sigma$. Thus, $\{v^i : i \in [n] \setminus \{i_0, j_0, k_0\}\} \subseteq \sigma$. Clearly, y(t) = v(t) for all $y \in \{v^i : i \in [n] \setminus \{i_0, j_0, k_0\}\}$ and $t \in \{i_0, j_0, k_0\}$. Since σ covers all places and $v^{i_0, j_0, k_0} \notin \tau$, we conclude that $|\sigma| \ge n - 1$. Since $p \le n - 2$ and σ is p-dimensional, we see that $|\sigma| = n - 1$.

Suppose $v \in \sigma$. Let $\{x_0\} = \sigma \setminus \{\{v\} \cup \{v^i : i \in [n] \setminus \{i_0, j_0, k_0\}\}\}$. For any $t \in \{i_0, j_0, k_0\}$ and $y \in \{v\} \cup \{v^i : i \in [n] \setminus \{i_0, j_0, k_0\}\}$, y(t) = v(t). Hence the fact that σ covers all places implies that $x_0 = v^{i_0, j_0, k_0}$, which is not possible since $v^{i_0, j_0, k_0} \notin \tau$. So, $v \notin \sigma$. Clearly, $\sigma \cup \{v\} \in \Delta_n$.

Recall that for any simplex η , $Bd(\eta)$ denotes the simplicial boundary of η . Let the coefficient of σ in c is $(-1)^m a_{\sigma}$ and the coefficient of σ in $Bd(\sigma \cup \{v\})$ is $(-1)^s$. Define a p-cycle c_1 as follows:

$$c_1 = \begin{cases} c - a_{\sigma}Bd(\sigma \cup \{v\}) & \text{if } m \text{ and } s \text{ are of same parity,} \\ c + a_{\sigma}Bd(\sigma \cup \{v\}) & \text{if } m \text{ and } s \text{ are of opposite parity.} \end{cases}$$

Clearly, c is homologous to c_1 . Observe that $\sigma \notin c_1$. Let $\gamma \in c_1$ such that $\gamma \notin c$. Then observe that $v \in \gamma$ and $\gamma \subseteq \tau$. But we have seen above that if $v \in \gamma$, then $\gamma \in \partial(\Delta_n)$, i.e., γ does not covers all places. Thus, we see that $|\mu(c_1)| < |\mu(c)|$. Since $|\mu(c)|$ is finite, by repeating the above argument finite number of times, we get a cycle c_k such that c is homologous to c_k and $|\mu(c_k)| = 0$, i.e., c_k is a p-cycle in $\partial(\Delta_n)$. We take $\tilde{c} = c_k$.

Case 2. $\tau = N(v) \cup N(w)$.

For any $k \in [n]$ and $\gamma \in \Delta_n$, we say that γ covers k-places, if there exist $i_1, \ldots, i_k \in [n]$ such for each $1 \le l \le k$, we get $x, y \in \gamma$ such that $x(i_l) = 0$ and $y(i_l) = 1$.

Observe that if $\sigma \subseteq N(v)$ or $\sigma \subseteq N(w)$, then σ can covers at most p+1-places. Since n > p+1, σ can not covers all places, a contradiction to the assumption that σ covers all places. Hence $N(v) \cap \sigma \neq \emptyset$ and $N(w) \cap \sigma \neq \emptyset$.

Since $w \sim v$, $w = v^q$ for some $q \in [n]$. Suppose $v, w \in \sigma$. If $N(w) \cap \sigma = \{v\}$, then $\sigma = \{v, w, v^{i_1}, \dots, v^{i_{p-1}}\}$ for some $i_1, i_2, \dots, i_{p-1} \in [n] \setminus \{q\}$. Observe that σ covers only p-places, namely i_1, \dots, i_{p-1}, q . Hence $|N(w) \cap \sigma| \geq 2$. Then $\sigma = \{v, w, v^{i_1}, \dots, v^{i_s}, v^{q,j_1}, \dots, v^{q,j_t}\}$, for some $i_1, \dots, i_s, j_1, \dots, j_t \in [n]$, where s + t = p - 1. Here σ can covers at most p places, namely $i_1, \dots, i_s, j_1, \dots, j_t, q$ and σ covers p places only if $\{i_1, \dots, i_s\} \cap \{j_1, \dots, j_t\} = \emptyset$. Since $p < n, \sigma$ does not covers all places. Hence $\{v, w\} \not\subseteq \sigma$.

Suppose $v \in \sigma$. Then $w \notin \sigma$. If $N(w) \cap \sigma = \{v\}$, then $\sigma = \{v, v^{i_1}, \dots, v^{i_p}\}$ for some $i_1, i_2, \dots, i_p \in [n]$. Observe that σ cover only p-places, namely i_1, \dots, i_p . Hence $|N(w) \cap \sigma| \geq 2$. Let $\sigma = \{v, v^{i_1}, \dots, v^{i_s}, v^{q,j_1}, \dots, v^{q,j_t}\}$, where $i_1, \dots, i_s, j_1, \dots, j_t \in [n]$ and s + t = p. Here σ can covers at most p + 1 places, namely $i_1, \dots, i_s, j_1, \dots, j_t, q$ and σ covers p + 1 places only if

 $\{i_1,\ldots,i_s\}\cap\{j_1,\ldots,j_t\}=\emptyset, q\notin\{i_1,\ldots,i_s\}$. Thus, we conclude that $v\notin\sigma$. By an argument similar as above, $w \notin \sigma$.

Let the coefficient of σ in c is $(-1)^m a_{\sigma}$ and let the coefficient of σ in $Bd(\sigma \cup \{v\})$ is $(-1)^r$. Define a p-cycle d_1 as follows:

$$d_1 = \begin{cases} c - a_{\sigma} B d(\sigma \cup \{v\}) & \text{if } m \text{ and } r \text{ are of same parity,} \\ c + a_{\sigma} B d(\sigma \cup \{v\}) & \text{if } m \text{ and } r \text{ are of opposite parity.} \end{cases}$$

Clearly, c is homologous to d_1 and $|\mu(d_1)| < |\mu(c)|$. Since $|\mu(c)|$ is finite, by repeating the above argument finite number of times, we get a cycle d_k such that c is homologous to d_k and $|\mu(d_k)| = 0$. We take $\tilde{c} = d_k$.

This completes the proof.

For any $1 \le t < n$ and two sequences $(\epsilon_1, \ldots, \epsilon_t)$ and (j_1, \ldots, j_t) , where for each $1 \le l \le t$, $\epsilon_l \in \{0,1\}$ and $j_l \in [n]$, let $\mathbb{I}_n^{(j_1,\epsilon_1),\dots,(j_t,\epsilon_t)}$ denote the induced subgraph of \mathbb{I}_n on the vertex set $\{v_1 \dots v_n : v_{i_l} = \epsilon_l, 1 \leq l \leq t\}$. Observe that for any i < n and an i-dimensional cube subgraph H of \mathbb{I}_n , there exist two sequences $(\epsilon_1, \ldots, \epsilon_{n-i})$ and (j_1, \ldots, j_{n-i}) such that H = $\mathbb{I}_n^{(j_1,\epsilon_1),\dots,(j_{n-i},\epsilon_{n-i})}.$

We now define a class, whose elements are the finite unions of the Vietoris-Rips complexes of cube graphs. For $n \geq 4$ and $3 \leq m \leq n$, let \mathcal{W}_n^m denote the collection of all finite union $X = X_1 \cup \ldots \cup X_k$ (k ranges over positive integers) such that X satisfies the following conditions:

- for each $1 \leq j \leq k$, $X_j = \mathcal{VR}(H_j; 3)$ for some m-dimensional cube subgraph H_j of \mathbb{I}_n . if $m \neq n$, then $X \subseteq \mathcal{VR}(H; 3)$ for some (m+1)-dimensional cube subgraph $H = \mathbb{I}_n^{(j_1,\epsilon_1),\dots,(j_{n-m-1},\epsilon_{n-m-1})}$ of \mathbb{I}_n . Further, if $X \neq \partial(\mathcal{VR}(H; 3))$, then there exists $\lambda \in \mathbb{I}_n^{(j_1,\epsilon_1),\dots,(j_{n-m-1},\epsilon_{n-m-1})}$ $[n] \setminus \{j_1, \ldots, j_{n-m-1}\}, \epsilon \in \{0, 1\} \text{ such that } \mathcal{VR}(H^{(\lambda, \epsilon)}; 3) \subseteq X \text{ and } \mathcal{VR}(H^{(\lambda, \epsilon')}; 3) \not\subseteq X,$ where $\epsilon' = \{0, 1\} \setminus \{\epsilon\}.$

Remark 3. Note that $W_n^n = \{\Delta_n\}$ and $\partial(\Delta_n) \in W_n^{n-1}$. Let $X = X_1 \cup \ldots \cup X_k \in W_n^m$ and suppose $X \subseteq \mathcal{VR}(H;3)$, where $H = \mathbb{I}_n^{(j_1,\epsilon_1),\ldots,(j_{n-m-1},\epsilon_{n-m-1})}$. If k = 1 or $X = \partial(\mathcal{VR}(H;3))$, then clearly, X is connected. If $X \neq \partial(\mathcal{VR}(H;3))$, then there exists $\lambda \in [n] \setminus \{j_1, \ldots, j_{n-m-1}\}$ and $\epsilon \in \{0,1\}$ such that $\mathcal{VR}(H^{(\lambda,\epsilon)};3) \subseteq X$ and $\mathcal{VR}(H^{(\lambda,\epsilon')};3) \not\subseteq X$, where $\epsilon' = \{0,1\} \setminus \{\epsilon\}$. Let $X_p = \mathcal{VR}(H^{(\lambda,\epsilon)};3)$. For each $p \neq j \in [k]$, since $X_j \cap X_p \cong \Delta_{m-1}$ is non empty and connected, we conclude that X is connected.

For a $X = X_1 \cup ... \cup X_k \in \mathcal{W}_n^m$, the following claim gives us a condition, when the intersection of $\bigcup X_l$ with X_i ($i \in [k]$) belongs to \mathcal{W}_n^{m-1} or the intersection of $\bigcup X_l$ with $X_i \cap X_j$ ($i, j \in [k]$) belongs to W_n^{m-2} , which plays a key role in the proofs of Lemmas 4.15, 4.18 and 4.19, while we use induction on k and m.

- Claim 3. Let $n \geq 4$ and $3 \leq m \leq n$ and let $X = X_1 \cup ... \cup X_k \in \mathcal{W}_n^m$. (i) If k > 1, then there exists $q \in [k]$ such that $\bigcup_{j \neq q} X_j \in \mathcal{W}_n^m$ and $X_q \cap \bigcup_{j \neq q} X_j \in \mathcal{W}_n^{m-1}$.
 - (ii) If $k \geq 3$ and $m \geq 5$, then there exist $\lambda, q \in [k]$ such that for any subset $A \subseteq [k] \setminus \{q\}$ such that $\lambda \in A$ and $|A| \geq 2$, the following are true:
 - there exists $p \in A \setminus \{\lambda\}$ such that if $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_q) \neq \emptyset$, then $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_q) \neq \emptyset$
 - $X_p \cap X_q) \in \mathcal{W}_n^{m-2}.$ $\bigcup_{i \neq q} X_j \in \mathcal{W}_n^m \text{ and } X_q \cap \bigcup_{j \neq q} X_j \in \mathcal{W}_n^{m-1}.$

Proof. Let $X \subseteq \mathcal{VR}(H;3)$, where $H = \mathbb{I}_n^{(j_1,\epsilon_1),\dots,(j_{n-m-1},\epsilon_{n-m-1})}$.

(i) If $X = \partial(\mathcal{VR}(H;3))$, then choose $t \in [n] \setminus \{j_1, \dots, j_{n-m-1}\}$ and $s \in \{0,1\}$. Clearly $\mathcal{VR}(H^{t,s};3)\subseteq X$. Without loss of generality we assume that $X_1=\mathcal{VR}(H^{t,s};3)\subseteq X$. Then $\bigcup_{j\neq 1} X_j \in \mathcal{W}_n^m$ and $X_1 \cap \bigcup_{j\neq 1} X_j = \partial(X_1) \in \mathcal{W}_n^{m-1}$.

Let $X \neq \partial(\mathcal{VR}(H;3))$. There exists $\mu \in [n] \setminus \{j_1, \dots, j_{n-m-1}\}$ and $\epsilon \in \{0,1\}$ such that $\mathcal{VR}(H^{(\mu,\epsilon)};3) \subseteq X$ and $\mathcal{VR}(H^{(\mu,\epsilon')};3) \not\subseteq X$, where $\epsilon' = \{0,1\} \setminus \{\epsilon\}$. Then $\mathcal{VR}(H^{(\mu,\epsilon)};3) = X_p$ for some $1 \leq p \leq k$. Choose $l \in [n] \setminus \{j_1, \dots, j_{n-m-1}, \mu\}$ and $s \in \{0,1\}$ such that $\mathcal{VR}(H^{(l,s)};3) \subseteq X$ (such l exists because $m \geq 3$). There exists $q \in [k] \setminus \{p\}$ such that $X_q = \mathcal{VR}(H^{(l,s)};3)$. Let $Y = \bigcup_{j \neq q} X_j$. Since $X_q \cap X_p \neq \emptyset$, $X_q \cap Y \neq \emptyset$. Clearly,

 $Y \in \mathcal{W}_n^m$. Further, $X_q \cap Y \subseteq \mathcal{VR}(T;3)$, where $T = \mathbb{I}_n^{(j_1,\epsilon_1),\dots,(j_{n-m-1},\epsilon_{n-m-1}),(l,s)}$. Observe that $\mathcal{VR}(T^{(\mu,\epsilon)};3) \subseteq X_q \cap Y$ and $\mathcal{VR}(T^{(\mu,\epsilon')};3) \not\subseteq X_q \cap Y$. Hence $X_q \cap Y \in \mathcal{W}_n^{m-1}$.

(ii) If $X = \partial(\mathcal{VR}(H;3))$. Then choose $t \in [n] \setminus \{j_1, \dots, j_{n-m-1}\}$. Take $X_q = \mathcal{VR}(H^{(t,0)};3)$ and $X_\lambda = \mathcal{VR}(H^{(t,1)};3)$. Then clearly, $\bigcup_{j \neq q} X_j \in \mathcal{W}_n^m$, $X_q \cap \bigcup_{j \neq q} X_j \in \mathcal{W}_n^{m-1}$. Further, since $X_\lambda \cap X_q = \emptyset$, for any choice of p and A containing λ , we get $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_q) = \emptyset$, and therefore result is true.

Assume $X \neq \partial(\mathcal{VR}(H;3))$. There exists $\mu \in [n] \setminus \{j_1, \dots, j_{n-m-1}\}$ and $\epsilon \in \{0, 1\}$ such that $\mathcal{VR}(H^{(\mu,\epsilon)};3) \subseteq X$ and $\mathcal{VR}(H^{(\mu,\epsilon')};3) \not\subseteq X$, where $\epsilon' = \{0, 1\} \setminus \{\epsilon\}$. Hence $X_{\lambda} = \mathcal{VR}(H^{(\mu,\epsilon)};3)$ for some $\lambda \in [k]$. Choose $t_1 \in [n] \setminus \{j_1, \dots, j_{n-m-1}, \mu\}$ and $s_1 \in \{0, 1\}$ such that $\mathcal{VR}(H^{(t_1,s_1)};3) \subseteq X$. There exists $q \in [k] \setminus \{\lambda\}$ such that $X_q = \mathcal{VR}(H^{(t_1,s_1)};3)$. Let $A \subset [k] \setminus \{q\}$ such that $\lambda \in A$ and $|A| \geq 2$. Choose $p \in A \setminus \{\lambda\}$. There exists $t_2 \in [n] \setminus \{j_1, \dots, j_{n-m-1}, \mu\}$ and $s_2 \in \{0, 1\}$ such that $X_p = \mathcal{VR}(H^{(t_2,s_2)};3)$. Since $p \neq q, (t_1, s_1) \neq (t_2, s_2)$. Let $Z_A = \bigcup_{i \in A \setminus \{p\}} (X_i \cap X_q)$. If $Z_A \cap X_p \cap X_q \neq \emptyset$, then it is

easy to check that $Z_A \cap X_p \cap X_q \subseteq \partial(\mathcal{VR}(H^{(t_1,s_1),(t_2,s_2)};3)), \mathcal{VR}(H^{(\mu,\epsilon),(t_1,s_1),(t_2,s_2)};3) \subseteq Z_A \cap X_p \cap X_q \text{ and } \mathcal{VR}(H^{(\mu,\epsilon'),(t_1,s_1),(t_2,s_2)};3) \not\subseteq Z_A \cap X_p \cap X_q. \text{ Hence } Z_A \cap X_p \cap X_q \in \mathcal{W}_n^{m-2}.$ Clearly, $\bigcup_{j \neq q} X_j \in \mathcal{W}_n^m \text{ and } X_q \cap \bigcup_{j \neq q} X_j \in \mathcal{W}_n^{m-1}.$

The nerve of a family of sets $(A_i)_{i\in I}$ is the simplicial complex $\mathbf{N} = \mathbf{N}(\{A_i\})$ defined on the vertex set I so that a finite subset $\sigma \subseteq I$ is in \mathbf{N} precisely when $\bigcap_{i\in\sigma} A_i \neq \emptyset$.

Proposition 4.13. [13, Theorem 10.6] Let Δ be a simplicial complex and $(\Delta_i)_{i \in I}$ be a family of subcomplexes such that $\Delta = \bigcup_{i \in I} \Delta_i$. Suppose every nonempty finite intersection $\Delta_{i_1} \cap \ldots \cap \Delta_{i_t}$ for $i_j \in I, t \in \mathbb{N}$ is contractible, then Δ and $\mathbf{N}(\{\Delta_i\})$ are homotopy equivalent.

Lemma 4.14. For any $X \in W_4^3$, $\tilde{H}_j(X) = 0$ for $0 \le j \le 2$.

Proof. Let $X = X_1 \cup \ldots \cup X_k$. Observe that each non empty intersection $X_{i_1} \cap \ldots \cap X_{i_t}$ is homeomorphic to Vietoris-Rips complex of some cube subgraph of dimension less than 4 and therefore contractible. From Proposition 4.13, $X \simeq \mathbf{N}(\{X_i\})$. For any $i, j \in [4]$ and $\epsilon, \delta \in \{0, 1\}$, let $\overline{\{(i, \epsilon), (j, \delta)\}}$ be a simplicial complex on vertex set $\{(i, \epsilon), (j, \delta)\}$, which is isomorphic to S^0 . If $X = \partial(\Delta_4)$, i.e., $X = \bigcup_{i \in [4], \epsilon \in \{0, 1\}} \Delta_4^{i, \epsilon}$, then it is easy to check that

$$\mathbf{N}(\{X_i\}) \cong \overline{\{(1,0),(1,1)\}} * \overline{\{(2,0),(2,1)\}} * \overline{\{(3,0),(3,1)\}} * \overline{\{(4,0),(4,1)\}}$$

the join of 4-copies of S^0 . Hence $X \simeq \mathbf{N}(\{X_i\}) \simeq S^3$ and therefore $\tilde{H}_j(X) = 0$ for $0 \le j \le 2$. If $X \ne \bigcup_{i \in [4], \epsilon \in \{0,1\}} \Delta_4^{i,\epsilon}$, then there exists $p \in [4], \epsilon \in \{0,1\}$ such that $\Delta_4^{p,\epsilon} \subseteq X$ but $\Delta_4^{p,\epsilon'} \not\subseteq X$,

where $\{\epsilon'\} = \{0,1\} \setminus \{\epsilon\}$. It is easy to check that $\mathbf{N}(\{X_i\})$ is a cone over the vertex (p,ϵ) and therefore it is contractible. Thus, we conclude that $\tilde{H}_j(X) = 0$ for $0 \le j \le 2$.

Lemma 4.15. Let $n \geq 5$ and $4 \leq m \leq n$. For any $X \in \mathcal{W}_n^m$ and $0 \leq j \leq 3$, $\tilde{H}_j(X) = 0$.

Proof. Let $X = X_1 \cup \ldots \cup X_p \in \mathcal{W}_n^m$. Proof is by induction on m and p. Let m = 4. If p = 1, then $X \cong \Delta_4 \simeq S^7$. Hence $\tilde{H}_j(X) = 0$ for $j \leq 3$. Let $p \geq 2$. Inductively assume that for any

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of generality assume that t = p and $Y = X_1 \cup ... \cup X_{p-1}$. Then $\tilde{H}_j(Y) = 0$ and $\tilde{H}_j(X_p) = 0$ for $j \leq 3$. By Mayer-Vietoris sequence for homology, we have

$$\cdots \longrightarrow \tilde{H}_{j}(Y) \oplus \tilde{H}_{j}(X_{p}) \longrightarrow \tilde{H}_{j}(X) \longrightarrow \tilde{H}_{j-1}(Y \cap X_{p}) \longrightarrow \tilde{H}_{j-1}(Y) \oplus \tilde{H}_{j-1}(X_{p}) \longrightarrow \cdots$$

Since $Y \cap X_p \in \mathcal{W}_n^3$, $\tilde{H}_j(Y \cap X_p) = 0$ for $j \leq 2$ by Lemma 4.14, . Thus, we conclude that $\tilde{H}_j(X) = 0$ for $j \leq 3$. So for m = 4, result is true. Let $m \geq 5$.

Induction hypothesis 1: For any $4 \le r < m$ and $j \le 3$, if $Y \in \mathcal{W}_n^r$, then $\tilde{H}_j(Y) = 0$.

Let $4 \leq r < m$ and $Z \in \mathcal{W}_n^{r+1}$. Then $Z = Z_1 \cup \ldots \cup Z_q$ for some q, where each Z_i is the Vietoris-Rips complex of a r+1-dimensional cube subgraph of \mathbb{I}_n . We show that $\tilde{H}_j(Z) = 0$ for $j \leq 3$. Proof is by induction on q. If q = 1, then $Z \cong \Delta_{r+1}$. Since $r+1 \geq 5$, from Lemma 4.12, any i-cycle c in Z is homologous to an i-cycle \tilde{c} in $\partial(Z)$ for $i \leq 3$. Hence it is enough to show that $\tilde{H}_j(\partial(Z)) = 0$ for $j \leq 3$. Clearly, $\partial(Z) \in \mathcal{W}_n^r$. From induction hypothesis 1, we get that $\tilde{H}_j(\partial(Z)) = 0$ for $j \leq 3$. So assume that $q \geq 2$.

Induction hypothesis 2: For any $l < q, i_1, \ldots, i_l \in [q]$ and $j \leq 3$, if $Z_{i_1} \cup \ldots \cup Z_{i_l} \in \mathcal{W}_n^{r+1}$, then $\tilde{H}_j(Z_{i_1} \cup \ldots \cup Z_{i_l}) = 0$.

From Claim 3 (i), there exists $t \in [q]$ such that $\bigcup_{i \neq t} Z_i \in \mathcal{W}_n^{r+1}$ and $Z_t \cap \bigcup_{t \neq j} Z_j \in \mathcal{W}_n^r$. Without loss of generality assume that t = q. Let $U = Z_1 \cup \ldots \cup Z_{q-1}$. Then $U \in \mathcal{W}_n^{r+1}$ and by induction hypothesis 2, $\tilde{H}_j(U) = 0$ for $0 \leq j \leq 3$. By Mayer-Vietoris sequence for homology, we have

$$\cdots \longrightarrow \tilde{H}_{j}(U) \oplus \tilde{H}_{j}(Z_{q}) \longrightarrow \tilde{H}_{j}(Z) \longrightarrow \tilde{H}_{j-1}(U \cap Z_{q}) \longrightarrow \tilde{H}_{j-1}(U) \oplus \tilde{H}_{j-1}(Z_{q}) \longrightarrow \cdots$$

From induction hypothesis 1, $\tilde{H}_j(U \cap Z_q) = 0$ for $0 \le j \le 3$. Therefore, we conclude that $\tilde{H}_j(Z) = 0$ for $0 \le j \le 3$.

Thus, the proof is complete by induction.

Lemma 4.16. Let $n \geq m \geq 6$ and $k \geq 3$. For each $i \in [k]$, let X_i be the Vietoris-Rips complex of some m-dimensional cube subgraph of \mathbb{I}_n . Let $Y = \bigcup_{l=1}^{k-2} X_l \cap X_k$ and $Y' = X_{k-1} \cap X_k$ such that if $Y \cap Y' \neq \emptyset$, then $Y \cap Y' \in \mathcal{W}_n^{m-2}$. Then for each $x \in \tilde{H}_4(\bigcup_{l=1}^{k-1} X_l \cap X_k)$, there exist $x_1 \in \tilde{H}_4(Y)$ and $x_2 \in \tilde{H}_4(Y')$ such that $x = x_1 + x_2$.

Proof. Let $X = \bigcup_{l=1}^{k-1} X_l \cap X_k$. Then $X = Y \cup Y'$. Let $x \in \tilde{H}_4(X)$. If $Y \cap Y' = \emptyset$, then $\tilde{H}_4(X) = \tilde{H}_4(Y) \oplus \tilde{H}_4(Y')$. Hence $x = x_1 + x_2$ for some $x_1 \in \tilde{H}_4(Y)$ and $x_2 \in \tilde{H}_4(Y')$. Let $Y \cap Y' \neq \emptyset$. By Mayer-Vietoris sequence for homology, we get

$$\cdots \longrightarrow \tilde{H}_4(Y) \oplus \tilde{H}_4(Y') \xrightarrow{\psi} \tilde{H}_4(X) \xrightarrow{\phi} \tilde{H}_3(Y \cap Y') \longrightarrow \tilde{H}_3(Y) \oplus \tilde{H}_3(Y') \longrightarrow \cdots$$

Since $m-2 \geq 4$, from Lemma 4.15 $\tilde{H}_3(Y \cap Y') = 0$. Hence $\psi : \tilde{H}_4(Y) \oplus \tilde{H}_4(Y') \longrightarrow \tilde{H}_4(X)$ given by $(\alpha, \beta) \mapsto \alpha + \beta$ is surjective. Thus $x = x_1 + x_2$ for some $x_1 \in \tilde{H}_4(Y)$ and $x_2 \in \tilde{H}_4(Y')$.

Lemma 4.17. Let $n \geq m \geq 6$ and $k \geq 2$. Let $X = X_1 \cup ... \cup X_k$, where each X_i is Vietoris-Rips complex of some m-dimensional cube subgraph of \mathbb{I}_n . Further, assume that if $k \geq 3$, then for any set $A \subseteq [k-1]$ such that $1 \in A$ and $|A| \geq 2$, there exists $p \in A \setminus \{1\}$ such that $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_k) \neq \emptyset$ implies $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_k) \in \mathcal{W}_n^{m-2}$. Then the map i_* :

$$\tilde{H}_4(\bigcup_{l=1}^{k-1} X_l \cap X_k) \to \tilde{H}_4(X_k)$$
 induced by the inclusion $\bigcup_{l=1}^{k-1} X_l \cap X_k \hookrightarrow X_k$, is injective.

Proof. Let $Y = \bigcup_{l=1}^{k-1} X_l$. If $Y \cap X_k = \emptyset$, then result is vacuously true. So assume that $Y \cap X_k \neq \emptyset$. If k=2, then $Y \cap X_k \cong \Delta_{m-1}$. From Lemma 4.8, there exists a retraction $X_k \to Y \cap X_k$ and therefore $i_*: \tilde{H}_4(Y \cap X_k) \to \tilde{H}_4(X_k)$ is injective.

Let $k \geq 3$ and inductively assume that for any $1 \leq t < k-1$ and $1 \in \{j_1, \ldots, j_t\} \subseteq [k-1]$, the map $i_*: \tilde{H}_4(\bigcup_{l=1}^t X_{j_l} \cap X_k) \to \tilde{H}_4(X_k)$ induced by the inclusion $\bigcup_{l=1}^t X_{j_l} \cap X_k \hookrightarrow X_k$, is injective.

Let $B = \{j_1, \ldots, j_{t+1}\} \subseteq [k-1]$ such that $1 \in B$. Let $Z = \bigcup_{l=1}^{t+1} X_{j_l}$. We show that the map $i_*: \tilde{H}_4(Z \cap X_k) \to \tilde{H}_4(X_k)$ is injective.

Let $0 \neq x \in \tilde{H}_4(Z \cap X_k)$. There exists $p \in B$, $p \neq 1$ such that $\bigcup_{i \in B \setminus \{p\}} (X_i \cap X_p \cap X_k) \neq \emptyset$ implies $\bigcup_{i \in B \setminus \{p\}} (X_i \cap X_p \cap X_k) \in \mathcal{W}_n^{m-2}$. From Lemma 4.16, there exist $x_1 \in \tilde{H}_4(\bigcup_{i \in B \setminus \{p\}} X_i \cap X_k)$,

 $x_2 \in \tilde{H}_4(X_p \cap X_k)$ such that $x = x_1 + x_2$. Suppose $i_*(x) = 0$ in $\tilde{H}_4(X_k)$. Since $x \neq 0$, at least one of x_1 or x_2 is a non zero element of $\tilde{H}_4(Z \cap X_k)$. Let $x_1 \neq 0$ in $\tilde{H}_4(Z \cap X_k)$. Then $x_1 \neq 0$ in $\tilde{H}_4(\bigcup_{i \in B \setminus \{p\}} X_i \cap X_k)$. From induction hypothesis the map $j_* : \tilde{H}_4(\bigcup_{i \in B \setminus \{p\}} X_i \cap X_k) \to \tilde{H}_4(X_k)$

induced by the inclusion $j: \bigcup_{i \in B \setminus \{p\}} X_i \cap X_k \to X_k$, is injective and therefore $j_*(x_1) \neq 0$. Since

 $i_*(x_1) = j_*(x_1)$, we see that $i_*(x_1) \neq 0$. Further, $i_*(x) = i_*(x_1 + x_2) = x_1 + x_2 = 0$ implies that $x_1 = -x_2$. The injectivity of the map $j_* : \tilde{H}_4(\bigcup_{i \in B \setminus \{p\}} X_i \cap X_k) \to \tilde{H}_4(X_k)$ implies that

 $\tilde{H}_4(\bigcup_{i\in B\setminus\{p\}}X_i\cap X_k)$ is a subgroup of $\tilde{H}_4(X_k)$. Hence $x_2\in \tilde{H}_4(\bigcup_{i\in B\setminus\{p\}}X_i\cap X_k)$. Therefore

 $x_1 + x_2 = 0$ in $\tilde{H}_4(\bigcup_{i \in R \setminus \{p\}} X_i \cap X_k)$. Hence $x = x_1 + x_2 = 0$ in $\tilde{H}_4(Z \cap X_k)$, a contradiction. By

an argument similar as above, we can show that, if $x_2 \neq 0$, then $x_1 + x_2 = 0$ in $H_4(Z \cap X_k)$, a contradiction. Thus $x \neq 0$ implies $i_*(x) \neq 0$. Therefore i_* is injective. The proof is complete by induction.

Lemma 4.18. Let $n \geq 6$. For $X \in \mathcal{W}_n^6$, $\tilde{H}_5(X) = 0$.

Proof. Let $X = X_1 \cup \ldots \cup X_k$, where each X_i is the Vietoris-Rips complex of a 6-dimensional cube subgraph of \mathbb{I}_n . If k=1, then $X\cong \Delta_6$ and hence result is true by Proposition 1.1. Let k > 1 and assume that for any l < k and $i_1, \ldots, i_l \in [k]$, if $X_{i_1} \cup \ldots \cup X_{i_l} \in \mathcal{W}_n^6$, then $\tilde{H}_5(X_{i_1} \cup \ldots \cup X_{i_l}) = 0.$

If k=2, then from Claim 3 (i), there exists $q_1 \in [k]$ such that $\bigcup_{j \neq q_1} X_j \in \mathcal{W}_n^6$ and $X_{q_1} \cap$

 $\bigcup X_j \in \mathcal{W}_n^5$.

Further, if $k \geq 3$, then from Claim 3 (ii) there exist $\lambda, q_2 \in [k]$ such that $\bigcup X_j \in \mathcal{W}_n^6$,

 $X_{q_2} \cap \bigcup_{i \neq q_2} X_j \in \mathcal{W}_n^5$ and for any subset $A \subseteq [k] \setminus \{q_2\}$ containing λ , there exists $p \in A \setminus \{\lambda\}$ such

that $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_{q_2}) \neq \emptyset$ implies $\bigcup_{i \in A \setminus \{p\}} (X_i \cap X_p \cap X_{q_2}) \in \mathcal{W}_n^4$.

Without loss of generality we assume that if k=2, then $q_1=k$ and if $k\geq 3$, then $q_2=k, \lambda=1$ and for A = [k-1], p = k-1. Let $Y = X_1 \cup ... \cup X_{k-1}$. Then by induction hypothesis $\tilde{H}_5(Y) = 0$ and $H_5(X_k) = 0$. By Mayer-Vietoris sequence for homology, we have

$$\cdots \longrightarrow \tilde{H}_5(Y) \oplus \tilde{H}_5(X_p) \longrightarrow \tilde{H}_5(X) \longrightarrow \tilde{H}_4(Y \cap X_p) \xrightarrow{h_4} \tilde{H}_4(Y) \oplus \tilde{H}_4(X_p) \longrightarrow \cdots$$

Using Lemma 4.17, we conclude that the map $i_*: \tilde{H}_4(Y \cap X_k) \longrightarrow \tilde{H}_4(X_k)$ induced by the inclusion $Y \cap X_k \hookrightarrow X_k$, is injective and therefore the map $h_4 : \tilde{H}_4(Y \cap X_k) \longrightarrow \tilde{H}_4(Y) \oplus \tilde{H}_4(X_k)$ is also injective. Since $\tilde{H}_5(Y) = 0$ and $\tilde{H}_5(X_k) = 0$, we conclude that $\tilde{H}_5(X) = 0$.

Lemma 4.19. Let $m \geq 7$. For any $X \in \mathcal{W}_n^m$ and $j \in \{5,6\}$, $\tilde{H}_j(X) = 0$.

Proof. Let $X = X_1 \cup \ldots \cup X_p$, where each X_i is the Vietoris-Rips complex of an m-dimensional cube subgraph of \mathbb{I}_n . Proof is by induction on m and p. Let m = 7. We show that $\tilde{H}_j(X) = 0$ if $j \in \{5, 6\}$.

Proof is by induction on p. If p=1, then $X\simeq \Delta_7$ and therefore result follows from Proposition 1.1. Let p>1. Inductively assume that for any l< p and $i_1,\ldots,i_l\in [p]$, if $X_{i_1}\cup\ldots\cup X_{i_l}\in \mathcal{W}_n^7$, then $\tilde{H}_j(X_{i_1}\cup\ldots\cup X_{i_l})=0$ for $j\in\{5,6\}$. From Claim 3 (i), there exists $t\in [p]$ such that $\bigcup_{j\neq t}X_j\in \mathcal{W}_n^7$ and $X_t\cap\bigcup_{j\neq t}X_j\in \mathcal{W}_n^6$. Without loss of generality assume that

t = p. Let $Y = X_1 \cup ... \cup X_{p-1}$. Then by induction hypothesis $\tilde{H}_j(Y) = 0$ and $\tilde{H}(X_p) = 0$ for $j \in \{5, 6\}$. By Mayer-Vietoris sequence for homology, we have

$$\cdots \longrightarrow \tilde{H}_{j}(Y) \oplus \tilde{H}_{j}(X_{p}) \longrightarrow \tilde{H}_{j}(X) \longrightarrow \tilde{H}_{j-1}(Y \cap X_{p}) \xrightarrow{h_{j-1}} \tilde{H}_{j-1}(Y) \oplus \tilde{H}_{j-1}(X_{p}) \longrightarrow \cdots$$

Since $Y \cap X_p \in \mathcal{W}_n^6$, $\tilde{H}_5(Y \cap X_p) = 0$ by Lemma 4.18. If j = 6, then since $\tilde{H}_6(Y) = 0$, $\tilde{H}_6(X_p) = 0$ and $\tilde{H}_5(Y \cap X_p) = 0$, we see that $\tilde{H}_6(X) = 0$. Using Claim 1 (ii) and Lemma 4.17, we conclude that the map $i_* : \tilde{H}_4(Y \cap X_p) \longrightarrow \tilde{H}_4(X_p)$ induced by the inclusion $Y \cap X_p \hookrightarrow X_p$, is injective and therefore the map $h_4 : \tilde{H}_4(Y \cap X_p) \longrightarrow \tilde{H}_4(Y) \oplus \tilde{H}_4(X_p)$ is also injective. If j = 5, then since $\tilde{H}_5(Y) = 0$, $\tilde{H}_5(X_p) = 0$, we conclude that $\tilde{H}_5(X) = 0$. Hence result is true for m = 7, i.e., for any $X \in \mathcal{W}_n^7$, $\tilde{H}_j(X) = 0$ for $j \in \{5,6\}$. Now let $m \geq 8$.

Induction hypothesis 1: For any $7 \le l < m$ and $j \in \{5,6\}$, if $X \in \mathcal{W}_n^l$, then $\tilde{H}_j(X) = 0$.

Let $7 \leq l < m$ and suppose $Z \in W_n^{l+1}$. Let $Z = Z_1 \cup \ldots \cup Z_q$, where each Z_i is the Vietoris-Rips complex of an l+1-dimensional cube subgraph of \mathbb{I}_n . We show that $\tilde{H}_j(Z) = 0$ for $j \in \{5,6\}$.

Proof is by induction on q. If q = 1, then $Z \simeq \Delta_{l+1}$. Since $l \geq 7$, from Lemma 4.12, any j-cycle c in Z is homologous to a j-cycle \tilde{c} in $\partial(Z)$ for $j \in \{5,6\}$. Hence it is enough to show that $\tilde{H}_j(\partial(Z)) = 0$ for $j \in \{5,6\}$. Observe that $\partial(Z) \in \mathcal{W}_n^l$ and therefore by induction hypothesis 1, $\tilde{H}_j(\partial(Z)) = 0$ for $j \in \{5,6\}$. Let q > 1.

Induction hypothesis 2: For any $t < q, i_1, \ldots, i_t \in [q]$ and $j \in \{5, 6\}$, if $X_{i_1} \cup \ldots \cup X_{i_t} \in \mathcal{W}_n^{l+1}$, then $\tilde{H}_j(X_{i_1} \cup \ldots \cup X_{i_t}) = 0$.

From Claim 3 (i), there exists $s \in [q]$ such that $\bigcup_{j \neq s} Z_j \in \mathcal{W}_n^{l+1}$ and $Z_s \cap \bigcup_{j \neq s} Z_j \in \mathcal{W}_n^l$. Without loss of generality assume that s = q. Let $U = Z_1 \cup \ldots \cup Z_{q-1}$. By induction hypothesis $2, \tilde{H}_j(U) = 0$ and $\tilde{H}_j(Z_q) = 0$ for $j \in \{5, 6\}$. By Mayer-Vietoris sequence for homology, we have

$$\cdots \longrightarrow \tilde{H}_{j}(U) \oplus \tilde{H}_{j}(Z_{q}) \longrightarrow \tilde{H}_{j}(Z) \longrightarrow \tilde{H}_{j-1}(U \cap Z_{q}) \xrightarrow{h_{j-1}} \tilde{H}_{j-1}(U) \oplus \tilde{H}_{j-1}(Z_{q}) \longrightarrow \cdots$$

Since $U \cap Z_q \in \mathcal{W}_n^l$, from induction hypothesis 1, $\tilde{H}_j(U \cap Z_q) = 0$ for $j \in \{5,6\}$. If j = 6, then since $\tilde{H}_6(U) = 0$, $\tilde{H}_6(Z_q) = 0$ and $\tilde{H}_5(U \cap Z_q) = 0$, we see that $\tilde{H}_6(Z) = 0$. If j = 5, then since $\tilde{H}_5(U) = 0$, $\tilde{H}_5(Z_q) = 0$ and the map $h_4 : \tilde{H}_4(U \cap Z_q) \longrightarrow \tilde{H}_4(U) \oplus \tilde{H}_4(Z_q)$ is injective by Lemma 4.17, we get that $\tilde{H}_5(Z) = 0$.

This completes the proof.

We are now ready to prove main result of this section.

Proof of Theorem A. We must show that for $n \geq 5$, $\tilde{H}_i(\mathcal{VR}(\mathbb{I}_n; 3); \mathbb{Z}) \neq 0$ if and only if $i \in \{4,7\}$. Using Theorem C and Proposition 1.2, we see that Δ_n is homotopy equivalent to a subcomplex of dimension less than 8. Hence $\tilde{H}_i(\Delta_n) = \text{for all } i \geq 8$. Let X be the Vietoris-Rips complex of a 4-dimensional cube subgraph of \mathbb{I}_n . Then using Lemma 4.8, there exists a retraction $r: \Delta_n \to X$. Since $\Delta_4 \cong S^7$ and $X \cong \Delta_4$, we see that $\tilde{H}_7(X) \neq 0$. Further, since $r_*: \tilde{H}_7(\Delta_n) \to \tilde{H}_7(X)$ is surjective, $\tilde{H}_7(\Delta_n) \neq 0$.

If $n \leq 6$, then result follows from Proposition 1.1. So assume that $n \geq 7$. Since $\Delta_n \in \mathcal{W}_n^n$, Lemma 4.19 implies that $\tilde{H}_j(\Delta_n) = 0$ for $j \in \{5,6\}$. Let Y be the Vietoris-Rips complex of a 5-dimensional cube subgraph of \mathbb{I}_n . From Lemma 4.8, there exists a retraction $r_1 : \Delta_n \to Y$. Since

 $Y \cong \Delta_5$, using Proposition 1.1 we conclude that $\tilde{H}_4(\Delta_n) \neq 0$. From Lemma 4.15, $\tilde{H}_i(\Delta_n) = 0$ for $i \leq 3$. This completes the proof.

5. Future Directions

In Theorem A, we have shown that $\mathcal{VR}(\mathbb{I}_n;3)$ has non-trivial homology only in dimensions 4 and 7. Further, the complex $\mathcal{VR}(\mathbb{I}_n;2)$ is homotopy equivalent to a wedge sum of 3-spheres. For $r \in \{2,3\}$, since $\mathcal{VR}(\mathbb{I}_n;2)$ has non trivial homology only in dimension $i \in \{r+1,2^r-1\}$, we make the following conjecture.

Conjecture 1. For $n \geq r+2$, $\tilde{H}_i(\mathcal{VR}(\mathbb{I}_n;r);\mathbb{Z}) \neq 0$ if and only if $i \in \{r+1,2^r-1\}$.

The following is a natural question to ask.

Question 5.1. Let $n \ge r + 2$. Is $VR(\mathbb{I}_n; r)$ homotopy equivalent to a wedge sum of spheres of dimensions r + 1 and $2^r - 1$?

In Theorems B and C, we have proved that the collapsibility number of the complex $\mathcal{VR}(\mathbb{I}_n; r)$ is 2^r for $r \in \{2, 3\}$. The complex $\mathcal{VR}(\mathbb{I}_n; 1)$ is 1-dimensional and it is isomorphic to the graph \mathbb{I}_n . Hence the collapsibility number of $\mathcal{VR}(\mathbb{I}_n; 1)$ is 2. Further, it is easy to check that $\mathcal{VR}(\mathbb{I}_n; n-1)$ is $(2^{n-1}-1)$ -dimensional and it is isomorphic to the join of 2^{n-1} -copies of S^0 . Hence the collapsibility number of $\mathcal{VR}(\mathbb{I}_n; n-1)$ is 2^{n-1} . This leads us to make the following conjecture.

Conjecture 2. For $n \ge r+1$, the collapsibility number of $VR(\mathbb{I}_n;r)$ is 2^r .

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References

- M. Adamaszek, Clique complexes and graph powers, Israel Journal of Mathematics, 196(1)(2013), pp. 295–319.
- [2] M. Adamaszek and H. Adams, *The Vietoris–Rips complexes of a circle*, Pacific Journal of Mathematics, 290(217), pp. 1–40.
- [3] M. Adamaszek and H. Adams, On Vietoris-Rips complexes of hypercube graphs. Journal of Applied and Computational Topology, 2(2022), pp. 177–192.
- [4] M. ADAMASZEK, H. ADAMS AND F. FRICK, Metric reconstruction via optimal transport, SIAM Journal on Applied Algebra and Geometry, 2(4)(2018), pp. 597–619.
- [5] M. Adamaszek, H. Adams, F. Frick, C. Peterson and C. Previte-Johnson, *Nerve complexes of circular arcs*, Discrete & Computational Geometry, 56(2016), pp. 251–273.
- [6] M. ADAMASZEK, H. ADAMS, E. GASPAROVIC, M. GOMMEL, E. PURVINE, R. SAZDANOVIC, B. WANG, Y.WANG AND L. ZIEGELMEIER, On homotopy types of Vietoris-Rips complexes of metric gluings, Journal of Applied and Computational Topology, 4(2020), pp. 424–454.
- [7] M. Adamaszek, F. Frick and A. Vakili, On homotopy types of Euclidean Rips complexes, Discrete & Computational Geometry, 58(3)(2017), pp. 526–542.
- [8] H. Adams, M. Heim and C Peterson, *Metric thickenings and group actions*, Journal of Topology and Analysis, 2020, pp. 1–27.
- [9] R. Aharoni, R. Holzman and Z. Jiang, *Rainbow fractional matchings*, Combinatorica, 39(6)(2019), pp. 1191–1202.
- [10] U. Bauer, Ripser: efficient computation of Vietoris-Rips persistence barcodes, Journal of Applied and Computational Topology, 2021, 1–33.
- [11] M. BIGDELI AND S. FARIDI, Chordality, d-collapsibility, and componentwise linear ideals, Journal of Combinatorial Theory, Series A, 172(2020), pp. 105204.
- [12] T. BIYIKOĞLU AND Y. CIVAN, Collapsibility of simplicial complexes and graphs, arXiv preprint arxiv:2201.13046.
- [13] A. BJÖRNER, Topological methods, Handbook of combinatorics, 2(1995), pp. 819–1872.
- [14] G. CARLSSON, Topology and data, Bulletin of the American Mathematical Society, 46(2)(2009), pp. 255–308.
- [15] E. CARLSSON, G. CARLSSON AND V. DE SILVA, An algebraic topological method for feature identification, International Journal of Computational Geometry & Applications, 16(4)(2006), pp. 291–314.

- [16] G. CARLSSON, T. ISHKHANOV, V. DE SILVA AND A. ZOMORODIAN, On the local behavior of spaces of natural images, International Journal of Computer Vision, 76(2008), pp. 1–12.
- [17] G. CARLSSON, A. ZOMORODIAN, A. COLLINS AND L. GUIBAS, Persistence barcodes for shapes, International Journal of Shape Modeling, 11(2005), pp. 149–187.
- [18] E. W. CHAMBERS, V. DE SILVA, J. ERICKSON AND R. GHRIST, Vietoris-Rips complexes of planar point sets, Discrete & Computational Geometry, 44(1)(2010), 75–90.
- [19] I. Choi, J. Kim and B. Park, *Collapsibility of non-cover complexes of graphs*, Electronic Journal of Combinatorics, 27(1)(2020), pp. P1.20.
- [20] V. DE SILVA AND G. CARLSSON, Topological estimation using witness complexes, in SPBG04 Symposium on Point-Based Graphics, 2004, pp. 157–166.
- [21] V. DE SILVA AND R. GHRIST, Coordinate-free coverage in sensor networks with controlled boundaries via homology, The International Journal of Robotics Research, 25(12)(2006), pp. 1205–1222.
- [22] V. DE SILVA AND R. GHRIST, Coverage in sensor networks via persistent homology, Algebraic & Geometric Topology, 7.1(2007), 339–358.
- [23] K. J. EMMETT, Topology of Reticulate Evolution, PhD thesis, Columbia University, 2016.
- [24] K. EMMETT AND R. RABADÁN, Quantifying reticulation in phylogenetic complexes using homology, arXiv preprint arXiv:1511.01429, 2015.
- [25] E. GASPAROVIC, M. GOMMEL, E. PURVINE, R. SAZDANOVIC, B. WANG, Y. WANG AND L. ZIEGELMEIER, A complete characterization of the one-dimensional intrinsic Cech persistence diagrams for metric graphs, In Research in Computational Topology, Springer, 2018, pp. 33–56.
- [26] É. Ghys and P. de la Harpe, Espaces métriques hyperboliques, Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988), 1990, pp. 27–45.
- [27] M. Gromov, Hyperbolic groups, Essays in group theory, Springer, New York, NY, 1987, pp. 75–263.
- [28] A. HATCHER, Algebraic Topology, Cambridge University Press, 2002.
- [29] G. KALAI, Intersection patterns of convex sets, Israel Journal of Mathematics, 48(2-3)(1984), pp. 161-174.
- [30] G. KALAI AND R. MESHULAM, A topological colorful Helly theorem, Advances in Mathematics, 191(2)(2005), pp. 305–311.
- [31] M. Kim and A. Lew, Complexes of graphs with bounded independence number, Israel Journal of Mathematics, 249(2002), pp. 83–120 (2022).
- [32] D. J. KLEITMAN, On a combinatorial conjecture of Erdős, Journal of Combinatorial Theory 1 (1966), 209– 214.
- [33] D. KOZLOV, Combinatorial algebraic topology, volume 21 of Algorithms and Computation in Mathematics, Springer, Berlin, 2008.
- [34] M. LESNICK, R. RABADÁN AND D. IS ROSENBLOOM, Quantifying genetic innovation: Mathematical foundations for the topological study of reticulate evolution, SIAM Journal on Applied Algebra and Geometry, 4(1)(2020), pp. 141–184.
- [35] A. Lew, Collapsibility of simplicial complexes of hypergraphs, Electronic Journal of Combinatorics, 26(4)(2019), pp. P4.10.
- [36] J. MATOUŠEK AND M. TANCER Dimension gaps between representability and collapsibility, Discrete & Computational Geometry, 42(4)(2009), pp. 631–639.
- [37] A. Muhammad and A. Jadbabaie, Dynamic coverage verification in mobile sensor networks via switched higher order Laplacians, in Robotics: Science & Systems, 2007, pp. p.72.
- [38] L. VIETORIS, Uber den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen, Mathematische Annalen, 97(1)(1927), pp. 454–472.
- [39] Ž. Virk, 1-dimensional intrinsic persistence of geodesic spaces, Journal of Topology and Analysis, 2018, pp. 1–39.
- [40] Ž. VIRK, Approximations of 1-dimensional intrinsic persistence of geodesic spaces and their stability, Revista Matemática Complutense, 32(2019), pp. 195–213.
- [41] G. Wegner, d-Collapsing and nerves of families of convex sets, Archiv der Mathematik, 26(1)(1975), pp. 317–321.
- [42] A. ZOMORODIAN, Fast construction of the Vietoris-Rips complex, Computers & Graphics, 34(3)(2010), pp. 263–271.
- [43] A. ZOMORODIAN AND G. CARLSSON, Computing persistent homology, Discrete & Computational Geometry, 33(2)(2005), pp. 249–274.

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