# Tree-degenerate graphs and nested dependent random choice

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January 27, 2022

#### Abstract

The celebrated dependent random choice lemma states that in a bipartite graph an average vertex (weighted by its degree) has the property that almost all small subsets S in its neighborhood has common neighborhood almost as large as in the random graph of the same edge-density. Two well-known applications of the lemma are as follows. The first is a theorem of Füredi [12] and of Alon, Krivelevich, and Sudakov [2] showing that the maximum number of edges in an n-vertex graph not containing a fixed bipartite graph with maximum degree at most r on one side is  $O(n^{2-1/r})$ . This was recently extended by Grzesik, Janzer and Nagy [14] to the family of so-called (r,t)-blowups of a tree. A second application is a theorem of Conlon, Fox, and Sudakov [5], confirming a special case of a conjecture of Erdős and Simonovits and of Sidorenko, showing that if H is a bipartite graph that contains a vertex complete to the other part and G is a graph then the probability that the uniform random mapping from V(H) to V(G) is a homomorphism is at least  $\left[\frac{2|E(G)|}{|V(G)|^2}\right]^{|E(H)|}$ .

In this note, we introduce a nested variant of the dependent random choice lemma, which might be of independent interest. We then apply it to obtain a common extension of the theorem of Conlon, Fox, and Sudakov and the theorem of Grzesik, Janzer, and Nagy, regarding Turán and Sidorenko properties of so-called tree-degenerate graphs.

## 1 Introduction

Given a graph G, let |G| denote its number of vertices. A homomorphism from a graph H to a graph G is a mapping  $f: V(H) \to V(G)$  such that for each edge uv in H, f(u)f(v) is an edge in G. Let  $h_H(G)$  denote the number of homomorphisms from H to G and  $t_H(G) = h_H(G)/|G|^{|H|}$ . Thus  $t_H(G)$  represents the fraction of mappings from V(H) to V(G) that are homomorphisms. Viewed probabilistically,  $t_H(G)$  is the probability that the uniform random mapping from V(H) to V(G)is a homomorphism. A beautiful conjecture of Sidorenko [22] is as follows.

**Conjecture 1.1** (Sidorenko[21]). For every bipartite graph H and every graph G,

$$t_H(G) \ge [t_{K_2}(G)]^{e(H)}$$

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Since  $[t_{K_2}(G)]^{e(H)} = \left(\frac{2e(G)}{n^2}\right)^{e(H)}$ , one may view Sidorenko's conjecture as saying that the number of homomorphic copies of H in an *n*-vertex graph G is asymptotically at least as large as in the *n*-vertex random graph with the same edge-density. The following lemma, based on tensor products, (see Remark 2 in the English version of [21] for instance), is commonly known and is used in many earlier papers (see [1],[5],[18] for instance). It reduces the conjecture to a slightly weaker statement.

**Lemma 1.2** ([21]). Let H be a bipartite graph. If there exists a positive constant c depending only on H such that for all graphs  $G t_H(G) \ge c[t_{K_2}(G)]^{e(H)}$  holds, then for all G,  $t_H(G) \ge [t_{K_2}(G)]^{e(H)}$ .

When the edge-density of G is sufficiently high, it is expected that many of the homomorphisms from H to G are injective. Erdős and Simonovits [10] made several conjectures regarding the number of injective homomorphisms. As usual, let ex(n, H) denote the Turán number of H, which is the maximum number of edges in an *n*-vertex graph not containing H. Let  $h_H^*(G)$  denote the number of injective homomorphisms from H to G, and let  $t_H^*(G) = h_H^*(G)/|G|^{H|}$ . The first conjecture of Erdős and Simonovits from [10] states that for every c > 0 there is a c' > 0 such that if  $e(G) > (1+c) \exp(n, H)$  then  $t_H^*(G) \ge c' t_{K_2}(G)^{e(H)}$ . The second, weaker, conjecture from [10] says if  $ex(n,H) = O(n^{2-\alpha})$  then there exist constants  $0 \le \tilde{\alpha} \le \alpha, c, c' > 0$  such that if  $e(G) > cn^{2-\tilde{\alpha}}$  then  $t_H^*(G) \geq c' t_{K_2}(G)^{e(H)}$ . It is known (see [21]) that this weaker version is equivalent to Sidorenko's conjecture. However, compared to the stronger conjecture of Erdős and Simonovits, Sidorenko's conjecture does not give an explicit sharp edge-density threshold on when to guarantee the stated number of injective homomorphisms. There is yet another version of the Erdős-Simonovits conjecture, given in [19], that is equivalent to saying that there exist two constants c, c' > 0 such that if G is an n-vertex graph G with  $e(G) > c \exp(n, H)$  then  $t_H^*(G) \ge c' [t_{K_2}(G)]^{e(H)}$ . Sidorenko [22] verified his own conjecture when H is a complete bipartite graph, an even cycle, a tree, or a bipartite graph with at most four vertices on one side. Hatami [15] proved that hypercubes satisfy Sidorenko's conjecture by developing a concept of norming graphs. The first major progress on Sidorenko's conjecture was made by Conlon, Fox and Sudakov [5], who used the celebrated dependent random choice method (see [11] for a survey) to show

**Theorem 1.3** (Conlon-Fox-Sudakov [5]). If H is a bipartite graph with m edges which has a vertex complete to the other part, then H satisfies Sidorenko's conjecture.

In fact, Conlon, Fox and Sudakov proved the stronger theorem that H contains a vertex complete to the other part and the minimum degree in the first part is at least d then  $t_H(G) \ge [t_{K_{r,d}}(G)]^{\frac{m}{rd}}$ . From Theorem 1.3, Conlon, Fox, and Sudakov [5] also deduced an approximate version of Sidorenko's conjecture. Since the work of Conlon, Fox, and Sudakov, there has been a lot of further progress on Sidorenko's conjecture. Li and Szegedy [20] used entropy method (presented in the form of logarithmic convexity inequalities) to the extend the result of Conlon, Fox and Sudakov to a more general family of graphs H, which they refer to as reflection trees. These ideas were further developed by Kim, Lee and Lee [18], who proved the conjecture for what they called tree-arrangeable graphs and showed that if T is a tree and H is a bipartite graph that satisfies Sidorenko's conjecture then the Cartesian product of T and H also satisfies Sidorenko's conjecture. Subsequently, Conlon, Kim, Lee and Lee [6, 7] and independently Szegedy [23] established more families of bipartite graphs H for which Sidorenko's conjecture holds. These include bipartite graphs that admit a certain type of tree decomposition, subdivisions of certain graphs including cliques, and certain cartesian products, and etc. More recently, Conlon and Lee [8] showed that Sidorenko's conjecture holds for any bipartite graph H with a bipartition (A, B) where the number of vertices in B of degree k satisfies a certain divisibility condition for each k. As a corollary, for every bipartite graph H with a bipartition (A, B)there is a positive integer p such that the blowup  $H_A^p$  formed by taking p vertex-disjoint copies of H and gluing all copies of A along corresponding vertices satisfies Sidorenko's conjecture.

Another line of work that motivates our result is related to a long-standing conjecture of Erdős regarding the Turán number of so-called r-degenerate graphs. Given a positive integer r, a graph H is r-degenerate if its vertices can be linearly ordered such that each vertex has back degree at most r.

**Conjecture 1.4** (Erdős [9]). Let r be a fixed positive integer. Let H be any r-degenerate bipartite graph. Then  $ex(n, H) = O(n^{2-1/r})$ .

The first major progress on Conjecture 1.4 was the following theorem, which was first obtained by Füredi [12] in an implicit form and then later reproved by Alon, Krivelevich, and Sudakov [2] using the dependent random choice method.

**Theorem 1.5** (Füredi [12], Alon-Krivelevich-Sudakov [2]). Let r be a positive integer. Let H be a bipartite graph with maximum degree at most r on one side. Then  $ex(n, H) = O(n^{2-1/r})$ .

The family of graphs satisfying the condition of Theorem 1.5 forms a very special family of rdegenerate bipartite graphs, which we will refer to as one side r-bounded bipartite graphs. Recently,
Grzesik, Janzer, and Nagy [14], among other things, extended Theorem 1.5 to a broader family of
graphs, called (r, t)-blowups of a tree.

**Definition 1.6** ((r,t)-blowups of a tree). Let  $r \leq t$  and m be positive integers. A bipartite graph H is an (r,t)-blowup of a tree (or (r,t)-blowup in short) with root block  $B_0$  and non-root blocks  $B_1, \ldots, B_m$  if  $B_0, B_1, \ldots, B_m$  partition V(H),  $|B_0| = r, |B_1| = \cdots = |B_m| = t$  and H can be constructed by joining  $B_1$  completely to  $B_0$  and for each  $2 \leq i \leq m$  joining  $B_i$  completely to a r-subset of  $B_{\gamma(i)}$  for some  $\gamma(i) \leq i - 1$ .

**Theorem 1.7** (Grzesik-Janzer-Nagy [14]). Let  $r \leq t$  be positive integers. If H is an (r, t)-blowup of a tree, then  $ex(n, H) = O(n^{2-1/r})$ .

Since every one-side r-bounded graph is a subgraph of an (r, t)-blowup with two blocks  $B_0, B_1$ , Theorem 1.7 substantially generalizes Theorem 1.5.

In this paper, we give a common strengthening of Theorem 1.3 and Theorem 1.7 by proving a general theorem on the Turán and Sidorenko properties of so-called tree-degenerate graphs.

**Definition 1.8** (Tree-degenerate graphs). A bipartite graph H is tree-degenerate with root blocks  $B_0$  and non-root blocks  $B_1, \ldots, B_m$  if  $B_0, B_1, \ldots, B_m$  partition V(H) and H can be constructed by letting  $P(B_1) = B_0$  and joining  $B_1$  completely to  $B_0$  and for each  $2 \le i \le m$  joining  $B_i$  completely to a subset  $P(B_i)$  of  $B_{\gamma(i)}$  for some  $1 \le \gamma(i) \le i-1$ , such that for all  $i \ge 2 |P(B_{\gamma(i)})| \le |P(B_i)|$ . We call  $P(B_i)$  the parent set of  $B_i$  and  $B_{\gamma(i)}$  the parent block of  $B_i$  and we call P the parent function. We call  $(B_0, \ldots, B_m, P)$  a block representation of H.

We present our main result in terms of so-called *r*-norm density. We will explain the advantage of doing so after the presenting the theorem. Let G be a graph with n vertices. For each positive integer r, we define the the *r*-norm density of G, denoted by  $p_r(G)$ , as

$$p_r(G) := t_{K_{1,r}}(G)^{1/r}$$

Note that  $p_1(G) = t_{K_2}(G) = \frac{2e(G)}{n^2}$ , is the usual edge-density of G. In general, one may view  $p_r(G)$  as a modified measure of edge-density of G that takes the degree distribution into account. Using convexity, one can show that  $p_r(G) \ge p_s(G)$  whenever  $r \ge s$  (see Lemma 2.3).

**Theorem 1.9** (Main theorem). For any graph G and positive integer  $\ell$ , let  $p_{\ell}(G) = t_{K_{1,\ell}}(G)^{1/\ell}$ . Let H be a tree-degenerate graph with a block representation  $(B_0, B_1, \ldots, B_m, P)$ . Let  $s = |B_0|$  and  $r = \max_i |P(B_i)|$ . There exist positive constants  $c_1 = c_1(H), c_2 = c_2(H), c_3 = c_3(H)$  depending only on H such that for any graph G

$$t_H(G) \ge c_1[p_s(G)]^{e(H)} \ge c_1[t_{K_2}(G)]^{e(H)}.$$

Furthermore, if  $h_{K_{1,r}}(G) > c_2 n^r$ , where n = |G|, then

$$t_H^*(G) \ge c_3[p_r(G)]^{e(H)} \ge c_3[t_{K_2}(G)]^{e(H)}.$$

The first part of the theorem and Lemma 1.2 imply the following.

**Corollary 1.10.** Let H be a tree-degenerate graph. Then H satisfies Sidorenko's conjecture.

Since a bipartite graph H containing a vertex complete to the other part is tree-degenerate with  $|B_0| = 1$  and  $\gamma(i) = 1$  for all  $i \ge 2$ , Corollary 1.10 generalizes Theorem 1.3. A special case of the second part of Theorem 1.9 yields the following.

**Corollary 1.11.** Let  $r \leq t$  be positive integers. Let H be an (r, t)-blowup of a tree with h vertices. Then  $ex(n, H) = O(n^{2-1/r})$ . Furthermore, there exist constants c, c' > 0 such that every n-vertex graph G with  $h_{K_{1,r}}(G) \geq cn^r$  contains at least  $c'n^h(\frac{2e(G)}{n^2})^{e(H)}$  copies of H.

Corollary 1.11 strengthens Theorem 1.7 in two ways. First, it relaxes the density requirement on G from  $e(G) = \Omega(n^{2-1/r})$  to  $h_{K_{1,r}}(G) = \Omega(n^r)$  (i.e. from  $p_1(G) = \Omega(n^{-1/r})$  to  $p_r(G) = \Omega(n^{-1/r})$ ). Second, it not only gives at least one copy of H, but an optimal number (up to a multiplicative constant) of copies of H. A closer examination of the proof of Theorem 1.7 given by Grzesik, Janzer, and Nagy in [14] shows that their proof can be strengthened to also give Corollary 1.11. However, Theorem 1.9 is more general than Corollary 1.11, as the counting statement applies to any tree-degenerate graph H, where parent set sizes can vary, instead of just to (r,t)-blowups. The relaxation of  $p_1(G) = \Omega(n^{-1/r})$  to  $p_r(G) = \Omega(n^{-1/r})$  is also a useful feature, as in bipartite Turán problems sometimes we need to handle cases where the host graph has very uneven degree distribution and hence high r-norm density, despite having relatively low 1-norm density (see [17] for an instance of this kind). To prove Theorem 1.9, we introduce a notion of goodness and prove a lemma that might be viewed as a nested variant of the dependent random choice lemma. Once we establish the lemma, the proof of Theorem 1.9 readily follows. Conceivably this variant could find more applications.

We organize our paper as follows. In Section 2, we introduce some preliminary lemmas. In Section 3, we establish the nested goodness lemma. In Section 4, we prove Theorem 1.9. In Section 5, we give some concluding remarks.

# 2 Preliminary lemmas

In this section, we first give some useful lemmas. They will be used in motivating some definitions and will also be used in the proofs in later sections. We start with a standard convexity-based inequality, which is sometimes referred to as the power means inequality. We include a proof for completeness.

**Lemma 2.1.** Let n be a positive integer. Let  $1 \le a \le b$  be reals. Let  $x_1, \ldots, x_n$  be reals. Then

$$\sum_{i=1}^{n} x_i^a \le n^{1-a/b} \cdot \left(\sum_{i=1}^{n} x_i^b\right)^{a/b}.$$

Equivalently,

$$\left(\frac{1}{n}\sum_{i=1}^{n}x_{i}^{a}\right)^{1/a} \leq \left(\frac{1}{n}\sum_{i=1}^{n}x_{i}^{b}\right)^{1/b}$$

**Proof.** Since the function  $x^{b/a}$  is either linear or is concave up, by Jensen's inequality, we have  $\sum_{i=1}^{n} x_i^b = \sum_{i=1}^{n} (x_i^a)^{b/a} \ge n [\frac{1}{n} \sum_{i=1}^{n} x_i^a]^{b/a}$ . Rearranging, we obtain the desired inequalities.

Let G be a graph with n vertices. Let r be a positive integer. Recall that  $p_r(G) := t_{K_{1,r}}(G)^{1/r}$ .

**Lemma 2.2.** For any graph G and positive integers r,  $p_r(G) = \frac{1}{n} \left( \frac{1}{n} \sum_{v \in V(G)} d(v)^r \right)^{1/r}$ .

**Proof.** Let n = |G|. Recall that  $t_{1,r}(G) = h_{K_{1,r}(G)}/n^{r+1}$ , where  $h_{K_{1,r}}(G)$  is the number of homomorphisms from  $K_{1,r}$  to G. It is easy to see that  $h_{K_{1,r}}(G) = \sum_{v \in V(G)} d(v)^r$ . Hence,  $p_r(G) = t_{K_{1,r}}(G)^{1/r} = \left(\frac{1}{n^{r+1}} \sum_{v \in V(G)} d(v)^r\right)^{1/r} = \frac{1}{n} \left(\frac{1}{n} \sum_{v \in V(G)} d(v)^r\right)^{1/r}$ .

Lemma 2.2 and Lemma 2.1 imply the following useful fact.

**Lemma 2.3.** For any graph G and positive integers  $r \ge s$ , we have  $p_r(G) \ge p_s(G)$ .

# 3 Nested goodness Lemma

Given a set W and a sequence S of elements of W, we call S a sequence in W for brevity. The length of S is defined to be number of elements in the sequence S (multiplicity counted) and is denoted by |S|. Given a positive integer k, we let  $W^k$  denote the set of sequences of length k in Wand we let  $W_k$  denote the set of sequences of length k in W in which the k elements are all different. Given a graph G and a sequence S in V(G), the common neighborhood N(S) is the set of vertices adjacent to every vertex in S.

We now introduce a goodness notion that is inspired by Lemma 2.1 of [5]. A more specialized version of it was introduced in [16].

**Definition 3.1** (*i*-good sequences). Let  $0 < \alpha, \beta < 1$  be reals. Let h, r be positive integers. Let G be an n-vertex graph. Let  $p = p_r(G) = t_{K_{1,r}}(G)^{1/r}$ . For each  $0 \le i \le h$ , we define an *i*-good sequence in V(G) relative to  $(\alpha, \beta, h, r)$  (or simply *i*-good in short) as follows. We say that a sequence T in V(G) is 0-good if  $|N(T)| \ge \alpha p^{|T|}n$ . For all  $1 \le i \le h$ , we say that a sequence S of length at most h in V(G) is *i*-good if S is 0-good and for each  $|S| \le k \le h$ , the number of (i-1)-good sequences of length k in N(S) is at least  $(1 - \beta)|N(S)|^k$ .

Below is our main theorem on the goodness notion.

**Theorem 3.2** (Nested goodness lemma). Let  $h \ge r$  be positive integers. Let  $0 < \beta < 1$  be a real. There is a positive real  $\alpha$  depending on h, r and  $\beta$  such that the following is true. Let G be any graph on n vertices. Let  $p = p_r(G) = t_{K_{1,r}}(G)^{1/r}$ . For any  $i, j \in [h]$ , let  $\mathcal{A}_{i,j}$  denote the set of *i*-good sequences of length j relative to  $(\alpha, \beta, h, r)$  in V(G). Then for each  $i \in [h]$  and  $r \le j \le h$ 

$$\sum_{S \in \mathcal{A}_{i,j}} |N(S)|^r \ge (1-\beta)n^{j+r}p^{jr}$$

In particular, there exists an *i*-good sequence S of size *j* such that  $|N(S)| \ge (1-\beta)^{1/r} p^j n$ .

Applying Theorem 3.2 with r = 1, we get the following theorem that is of independent interest.

**Theorem 3.3.** Let h be a positive integer. Let  $0 < \beta < 1$  be a real. There is a positive real  $\alpha$  depending on h and  $\beta$  such that the following is true. Let G be any graph on n vertices. Let  $p = \frac{2e(G)}{n^2}$ . For any  $i, j \in [h]$ , let  $\mathcal{A}_{i,j}$  denote the set of i-good sequences of length j relative to  $(\alpha, \beta, h, 1)$  in V(G). Then, for all  $i \in [h]$ 

$$\sum_{S \in \mathcal{A}_{i,j}} |N(S)| \ge (1-\beta)n^{j+1}p^j.$$

In particular, there exists an i-good sequence S of size j such that  $|N(S)| \ge (1-\beta)p^j n$ .

Loosely speaking, one may think of the usual dependent random choice lemma as saying that for any positive integers j, h and real  $0 < \beta < 1$ , there is a 1-good sequence S of size j relative to  $(\alpha, \beta, h, 1)$  for some appropriate  $\alpha > 0$  such that most of the subsets T in N(S) of size at most h have their common neighborhood fractionally as large as expected in the random graph of the same edge-density. In that regard, one may view Theorem 3.3 as a strengthening of the dependent random choice lemma to a stronger notion of goodness. Theorem 3.2 follows from the following more technical lemma.

**Lemma 3.4.** Let  $h \ge r$  be positive integers. Let  $0 < \beta < 1$  be a real. There exists a positive real  $\alpha$  depending on h, r and  $\beta$  such that the following is true. Let G be a graph on n vertices. Let G be any

graph on n vertices. Let  $p = p_r(G) = t_{K_{1,r}}(G)^{1/r}$ . For each  $0 \le i \le h$  and  $1 \le j \le h$ , let  $\mathcal{A}_{i,j}$  denote the set of *i*-good sequences of length *j* relative to  $(\alpha, \beta, h, r)$  in V(G) and let  $\mathcal{B}_{i,j} = [V(G)]^j \setminus \mathcal{A}_{i,j}$ . Then for each  $0 \le i \le h, 1 \le j \le h$  and  $1 \le \ell \le j$ ,

$$\sum_{S \in \mathcal{B}_{i,j}} |N(S)|^{\ell} \le \beta n^{j+\ell} p^{j\ell}$$

**Proof.** Suppose  $\alpha$  has been specified, we define a sequence  $\alpha_i$ ,  $0 \leq i \leq h$ , by letting  $\alpha_0 = \alpha$  and  $\alpha_i = \alpha + h(\alpha_{i-1}/\beta)^{1/h}$  for each  $i \in [h]$ . For fixed h and  $\beta$ , it is easy to see that by choosing  $\alpha$  to be small enough, we can ensure that  $\alpha_i$  in increasing in i and  $\alpha_h < \beta$ . Let us fix such an  $\alpha$ . Now, let  $\mathcal{A}_{i,j}$  and  $\mathcal{B}_{i,j}$  be defined as stated. We use induction on i to prove that for all  $0 \leq i \leq h, j \in [h]$ , and  $1 \leq \ell \leq j$ 

$$\sum_{S \in \mathcal{B}_{i,j}} |N(S)|^{\ell} \le \alpha_i n^{j+\ell} p^{j\ell}$$

For the basis step, let i = 0. Let  $j, \ell \in [h]$  where  $\ell \leq j$ . By definition,

$$\sum_{S \in \mathcal{B}_{0,j}} |N(S)|^{\ell} \le n^j (\alpha p^j n)^{\ell} \le \alpha n^{j+\ell} p^{j\ell} \le \alpha_0 n^{j+\ell} p^{j\ell}.$$
(1)

Hence the claim holds for i = 0. For the induction step, let  $i \ge 1$  and suppose the claims hold when i is replaced with i - 1. Let  $j \in [h]$ . For each  $j \le k \le h$ , let  $\mathcal{C}_{i,j}^k$  denote the set of sequences S in  $\mathcal{B}_{i,j}$  such that the number of sequences of length k in N(S) that are not (i - 1)-good is at least  $\beta |N(S)|^k$ . By definition,  $\mathcal{B}_{i,j} = \mathcal{B}_{0,j} \cup \bigcup_{k=j}^h \mathcal{C}_{i,j}^k$ . Let  $\mathcal{F}_k$  be the collection of pairs (S,T), where  $S \in \mathcal{C}_{i,j}^k$  and T is a sequence of length k in N(S) that is not (i - 1)-good. By our definition,

$$|\mathcal{F}_k| \ge \sum_{S \in \mathcal{C}_{i,j}^k} \beta |N(S)|^k = \beta \cdot \sum_{S \in \mathcal{C}_{i,j}^k} |N(S)|^k.$$

On the other hand, for each sequence T of length k in V(G) that is not (i-1)-good, the number of sequences S of length j that satisfy  $(S,T) \in \mathcal{F}_k$  is most  $|N(T)|^j$ . Hence,

$$|\mathcal{F}_k| \le \sum_{T \in \mathcal{B}_{i-1,k}} |N(T)|^j \le \alpha_{i-1} n^{j+k} p^{jk},$$

where the last inequality follows from the induction hypothesis. Combining the lower and upper bounds on  $|\mathcal{F}_k|$ , we get

$$\sum_{S \in \mathcal{C}_{i,j}^k} |N(S)|^k \le (\alpha_{i-1}/\beta) n^{j+k} p^{jk}.$$
(2)

Let  $\ell \in [h]$  such that  $\ell \leq j$ . Since  $j \leq k$ , we have  $\ell \leq k$ . Applying Lemma 2.1 with  $a = \ell, b = k$  and using  $|\mathcal{C}_{i,j}^k| \leq n^j$ , we get

$$\sum_{S \in \mathcal{C}_{i,j}^k} |N(S)|^{\ell} \le (n^j)^{1-\ell/k} (\alpha_{i-1}/\beta)^{\ell/k} (n^{j+k} p^{jk})^{\ell/k} \le (\alpha_{i-1}/\beta)^{1/h} n^{j+\ell} p^{j\ell}$$

where we used the fact that  $\alpha_{i-1}/\beta < 1$ . By (1) and (2), we have

$$\sum_{S \in \mathcal{B}_{i,j}} |N(S)|^{\ell} \le \sum_{S \in \mathcal{B}_{0,j}} |N(S)|^{\ell} + \sum_{k=j}^{h} \sum_{S \in \mathcal{C}_{i,j}^{k}} |N(S)|^{\ell} \le [\alpha + h(\alpha_{i-1}/\beta)^{1/h}] n^{j+\ell} p^{j\ell} \le \alpha_{i} n^{j+\ell} p^{j\ell}.$$

This completes the induction and the proof.

We need another quick lemma. Given two positive integers n, j, let  $n_j = n(n-1)\cdots(n-j+1)$ .

**Lemma 3.5.** Let G be a graph on n vertices and j, r positive integers. Let  $p = p_r(G) = t_{K_{1,r}}(G)^{1/r}$ . Then  $\sum_{S \in [V(G)]^j} |N(S)|^r \ge n^{j+r} p^{jr}$ . If  $p > 4jn^{-1/r}$  then  $\sum_{S \in [V(G)]_j} |N(S)|^r \ge \frac{1}{2^{j+1}} n^{j+r} p^{jr}$ .

**Proof.** First, note that  $\sum_{T \in [V(G)]^r} |N(T)| = h_{K_{1,r}}(G) = nt_{K_{1,r}}(G) = n^{r+1}p^r$ . Hence, by convexity

$$\sum_{\in [V(G)]^r} |N(T)|^j \ge n^r \left(\frac{\sum_{T \in [V(G)]^r} |N(T)|}{n^r}\right)^j = n^r (np^r)^j = n^{j+r} p^{jr}.$$

If  $p > 4jn^{-1/r}$ , then

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$$\sum_{T \in [V(G)]^r, |N(T)| \ge 2j} |N(T)|^j \ge n^{j+r} p^{jr} - n^r (2j)^j \ge \frac{1}{2} n^{j+r} p^{jr}.$$

Hence,

$$\sum_{T \in [V(G)]^r, |N(T)| \ge 2j} |N(T)|_j \ge \sum_{T \in [V(G)]^r, |N(T)| \ge 2j} (|N(T)|/2)^j \ge \frac{1}{2^{j+1}} n^{j+r} p^{jr}$$

To prove the first statement, note that  $\sum_{S \in [V(G)]^j} |N(S)|^r$  counts pairs (S,T), where S is a sequence of length j and T is a sequence of length r in N(S). By double counting, we have  $\sum_{S \in [V(G)]^j} |N(S)|^r = \sum_{T \in [V(G)]^r} |N(T)|^j \ge n^{j+r} p^{jr}$ .

For the second statement, note that  $\sum_{S \in [V(G)]_j} |N(S)|^r$  counts pairs (S,T), where S is a sequence of length j with no repetition and T is sequence of length r in N(S). By double counting and convexity, we have  $\sum_{S \in [V(G)]_j} |N(S)|^r = \sum_{T \in [V(G)]^r} |N(T)|_j \ge \frac{1}{2^{j+1}} n^{j+r} p^{jr}$ .

Now we are ready to prove Theorem 3.2.

**Proof of Theorem 3.2:** Let  $h \ge r$  be positive integers and  $0 < \beta < 1$  a real. Let  $\alpha$  be defined as in Lemma 3.4. Let  $i \in [h]$  and  $r \le j \le h$ . Let  $\mathcal{A}_{i,j}$  denote the set of *i*-good sequences of length *j* relative to  $(\alpha, \beta, h, r)$  in V(G) and let  $\mathcal{B}_{i,j} = [V(G)]^j \setminus \mathcal{A}_{i,j}$ . By Lemma 3.5,  $\sum_{S \in [V(G)]^j} |N(S)|^r \ge n^{j+r}p^{jr}$ . By Lemma 3.4,  $\sum_{S \in \mathcal{B}_{i,j}} |N(S)|^r \le \beta n^{j+r}p^{jr}$ . Hence,  $\sum_{S \in \mathcal{A}_{i,j}} |N(S)|^r \ge (1-\beta)n^{j+r}p^{jr}$ , as desired. This proves the first part of the theorem. Now, since  $|\mathcal{A}_{i,j}| \le n^j$ , by averaging, there exists an  $S \in \mathcal{A}_{i,j}$  such that  $|N(S)|^r \ge (1-\beta)p^{jr}$  and hence  $|N(S)| \ge (1-\beta)^{1/r}p^jn$ . This proves the second part of the theorem.

In order to prove the second part of Theorem 1.9, we need the following variant of Theorem 3.3. We omit the proof since it is almost identical to that of Theorem 3.3, except that we use the second statement of Lemma 3.5 instead of the first statement.

**Lemma 3.6.** Let  $h \ge r$  be positive integers. Let  $0 < \beta < 1$  be a real. There is a positive real  $\alpha$  depending on h, r and  $\beta$  such that the following is true. Let G be any graph on n vertices. Let  $p = p_r(G) = t_{K_{1,r}}(G)^{1/r}$ . For any  $i, j \in [h]$ , let  $\mathcal{A}_{i,j}^*$  denote the set of *i*-good sequences of length j relative to  $(\alpha, \beta, h, r)$  in V(G) that has no repetition. If  $p > 4jn^{-1/r}$ , then for each  $i \in [h]$  and  $r \le j \le h$ 

$$\sum_{S \in \mathcal{A}_{i,j}^*} |N(S)|^r \ge (\frac{1}{2^{j+1}} - \beta) n^{j+r} p^{jr}.$$

# 4 Proof of Theorem 1.9

**Proof of Theorem 1.9:** Let n = |G| and h = |H|. Let  $\beta = \frac{1}{2^{h+2}}$ . Let  $\alpha$  be the positive constant given by Theorem 3.2 for the given  $h, \beta$ , and r. Let

$$c_1 = c_1(H) = (1 - \beta)^{|B_1|/|B_0|} \alpha^{\sum_{i=2}^m |B_i|} (1 - h\beta)^{m-1}.$$

Let

$$c_2 = c_2(H) = 4h/\alpha$$
, and  $c_3 = c_3(H) = c_1/2^{h^2}$ .

Suppose H has root block  $B_0$  and non-root blocks  $B_1, \ldots, B_m$  such that  $B_1$  is completely joined to its parent set  $P(B_1) = B_0$  and for each  $i = 2, \ldots, m$ ,  $B_i$  is completely joined to its parent set  $P(B_i)$  where  $P(B_i) \subseteq B_{\gamma(i)}$  for some  $1 \leq \gamma(i) < i$  and  $|P(B_i)| \geq |P(B_{\gamma(i)})|$ . For each  $i \in [m]$ , let  $\mathcal{F}_i$  denote the collection of all the parent sets  $P(B_j)$  that are contained in  $B_i$ . Let T be a tree with  $V(T) := \{v_0, v_1, \ldots, v_m\}$  and edge set  $E(T) := v_0 v_1 \cup \{v_i v_{\gamma(i)} :\in [m]\}$ . We call T the *auxiliary tree* for H. For each  $i \in [m]$ , define the *depth* of  $B_i$ , denoted by  $d_i$ , to be the distance from  $v_0$  to  $v_i$  in T. Let q denote the maximum depth of a block. Then clearly  $q \leq m \leq h - 1$ .

Let G be any graph. For convenience, we say that a sequence in V(G) is *i*-good if it is *i*-good relative to  $(\alpha, \beta, h, r)$ . As in Theorem 3.2, for each  $0 \le i \le h$  and  $r \le j \le h$ , let  $\mathcal{A}_{i,j}$  be the set of *i*-good sequences of length *j* in V(G). Let  $\mathcal{B}_{i,j} = [V(G)]^j \setminus \mathcal{A}_{i,j}$ . Let  $\mathcal{A}_{i,j}^*$  be the set of *i*-good sequences of length *j* in V(G) that contains no repetition. Let *f* be the uniform random mapping from V(H) to V(G).

Let

- $E_1$  = the event that  $f(B_0) \in \mathcal{A}_{q,|B_0|}$  and  $f(B_1) \in [N(f(B_0))]^{|B_1|}$ ,
- $F_1$  = the event that each sequence in  $\mathcal{F}_1$  is mapped to a (q-1)-good sequence ,
- $E_1^*$  = the event that  $f(B_0) \in \mathcal{A}_{q,|B_0|}^*$  and  $f(B_1) \in [N(f(B_0))]^{|B_1|}$ .

For each  $i \in \{2, \ldots, m\}$ , let

 $E_i = \text{the event that } f(B_i) \in [N(f(P(B_i)))]^{|B_i|},$   $F_i = \text{the event that each sequence in } \mathcal{F}_i \text{ is mapped to an } (q - d_i)\text{-good sequence}$  $L_i = \text{the event that } f \text{ is injective on } B_0 \cup B_1 \cup \cdots \cup B_i.$  Recall that  $s = |B_0|$  and  $r = \max_i |P(B_i)|$ . By Theorem 3.2,

$$\sum_{S \in \mathcal{A}_{q,|B_0|}} |N(S)|^s \ge (1-\beta)n^{2s}p^{s^2}.$$
(3)

Furthermore, by Lemma 3.6 if  $p \ge 4jn^{-1/r}$  then

$$\sum_{S \in \mathcal{A}_{q,|B_0|}^*} |N(S)|^s \ge (\frac{1}{2^{j+1}} - \beta) n^{2s} p^{s^2}.$$
(4)

Hence, since  $|B_1| \ge |B_0| = s$ , using (3) and convexity we get

$$\sum_{S \in \mathcal{A}_{q,|B_0|}} |N(S)|^{|B_1|} = \sum_{S \in \mathcal{A}_{q,|B_0|}} (|N(S)|^s)^{\frac{|B_1|}{s}} \ge n^s (\frac{1}{n^s}(1-\beta)n^{2s}p^{s^2})^{\frac{|B_1|}{s}} = (1-\beta)^{\frac{|B_1|}{|B_0|}} n^{|B_0|+|B_1|}p^{|B_0||B_1|},$$

and if  $p \geq 4jn^{-1/r}$  then

$$\sum_{S \in \mathcal{A}_{q,|B_0|}^*} |N(S)|^{|B_1|} = \left(\frac{1}{2^{j+1}} - \beta\right)^{\frac{|B_1|}{|B_0|}} n^{|B_0| + |B_1|} p^{|B_0||B_1|}$$

Hence,

$$\mathbb{P}(E_1) = \sum_{S \in \mathcal{A}_{q,|B_0|}} \frac{1}{n^{|B_0|}} \cdot \frac{|N(S)|^{|B_1|}}{n^{|B_1|}} = \frac{1}{n^{|B_0| + |B_1|}} \sum_{S \in \mathcal{A}_{q,|B_0|}} |N(S)|^{|B_1|} \ge (1 - \beta)^{\frac{|B_1|}{|B_0|}} p^{|B_0||B_1|}, \quad (5)$$

and if  $p \ge 4jn^{-1/r}$  then

$$\mathbb{P}(E_1^*) = \sum_{S \in \mathcal{A}_{q,|B_0|}^*} \frac{1}{n^{|B_0|}} \cdot \frac{|N(S)|^{|B_1|}}{n^{B_1}} \ge (\frac{1}{2^{j+1}} - \beta)^{\frac{|B_1|}{|B_0|}} p^{|B_0||B_1|} \ge (\frac{1}{2^{h+2}})^{|B_1|} p^{|B_0||B_1|}.$$
(6)

We now bound  $\mathbb{P}(F_1|E_1)$ . Recall that  $\mathcal{F}_1$  consists of parent sets  $P(B_j)$  that are contained in  $B_1$ . By requirement, these sets have size at least  $|P(B_1)| = |B_0|$ . Let S be any fixed sequence in  $\mathcal{A}_{q,|B_0|}$ . By the definition of  $\mathcal{A}_{q,|B_0|}$ , for each  $|B_0| \leq k \leq h$ , the number of (q-1)-good sequences of length k in N(S) is at least  $(1 - \beta)|N(S)|^k$ . So, conditioning on f mapping  $B_0$  to S and  $B_1$  to N(S), the probability that f maps any particular sequence in  $\mathcal{F}_1$  to an (q-1)-good sequence is at least  $(1 - \beta)$ . Since there are clearly at most h sequences in  $\mathcal{F}_1$ , the probably that f maps every sequence in  $\mathcal{F}_1$  to a (q-1)-good sequence is at least  $1 - h\beta$ . Hence

$$\mathbb{P}(F_1|E_1) \ge 1 - h\beta. \tag{7}$$

For each i = 2, ..., h, we estimate  $\mathbb{P}(E_i | E_1 F_1 ... E_{i-1} F_{i-1})$ . Assume the event  $E_1 F_1 ... E_{i-1} F_{i-1}$ . Since  $P(B_i) \subseteq B_{\gamma(i)}$ , where  $\gamma(i) < i$ , by our assumption,  $P(B_i)$  is mapped to a  $(q - d_{\gamma(i)})$ -good sequence. Since a  $(q - d_{\gamma(i)})$ -sequence is 0-good by definition, we have  $|N(f(P(B_i)))| \ge \alpha p^{|P(B_i)|} n$ . Hence,

$$\mathbb{P}(E_i|E_1F_1\dots E_{i-1}F_{i-1}) = \frac{|N(f(P(B_i))|^{B_i|}}{n^{|B_i|}} \ge \frac{(\alpha p^{|P(B_i)|}n)^{|B_i|}}{n^{|B_i|}} = \alpha^{|B_i|}p^{|P(B_i)||B_i|}.$$
(8)

Now assume  $E_1F_1 \dots E_{i-1}F_{i-1}E_i$ . Since  $S := f(P(B_i))$  is a  $(q - d_{\gamma(i)})$ -good sequence, by definition, for each  $|S| \le k \le h$  the number of  $(q - 1 - d_{\gamma(i)})$ -good sequences is at least  $(1 - \beta)|N(S)|^k$ . Since there are at most h sequences in  $\mathcal{F}_i$ , as in deriving (7), we have

$$\mathbb{P}(F_i|E_1F_1\dots E_{i-1}F_{i-1}E_i) \ge 1 - h\beta.$$
(9)

By (5), (7), (8), and (9),

$$\mathbb{P}(f \text{ is a homomorphism}) \geq \mathbb{P}(E_1 F_1 \dots E_{m-1} F_{m-1} E_m)$$
  
$$\geq (1 - \beta)^{\frac{|B_1|}{|B_0|}} \alpha^{\sum_{i=2}^m |B_i|} (1 - h\beta)^{m-1} p^{|B_0||B_1| + \sum_{i=2}^m |P(B_i)||B_i|} \qquad (10)$$
  
$$= c_1 p^{e(H)}.$$

The proves the first and the main part of the theorem. For the second statement, suppose  $h_{K_{1,r}}(G) > c_2 n^r$ . Then

$$p = p_r = (h_{K_{1,r}}/n^{r+1})^{1/r} \ge c_2^{1/r} n^{-1/r} \ge (4h/\alpha)^{1/r} n^{-1/r}.$$

For each  $i \geq 2$ , we bound  $\mathbb{P}(L_i|E_1^*F_1E_2F_2L_2...L_{i-1}E_iF_i)$ . Assume  $E_1^*F_1E_2F_2L_2...L_{i-1}E_iF_i$ . By our assumption  $P(B_i)$  is mapped to a  $(q - d_{\gamma(i)})$ -good sequence and  $B_i$  is mapped into  $N(f(P(B_i)))$ . Since  $f(P(B_i))$  is 0-good,  $|N(f(P(B_i))| \geq \alpha p^{|P(B_i)|}n \geq \alpha [(4h/\alpha)^{1/r}n^{-1/r}]^r n = 4h$ , where we used the fact that  $|P(B_i)| \leq r$ . Given  $E_1^*F_1E_2F_2L_2...L_{i-1}F_{i-1}$ , the probability that f maps  $B_i$  injectively into  $N(F(P(B_i)))$  and avoids  $f(B_0 \cup B_1 \cup \cdots \cup B_{i-1})$  is at least  $(3h)_{|B_i|}/(4h)^{|B_i|} > (1/2)^{|B_i|}$ . Hence,

$$\mathbb{P}(L_i | E_1^* F_1 E_2 F_2 L_2 \dots L_{i-1} E_i F_i) > (1/2)^{|B_i|}.$$
(11)

By (6), (11), and a similar calculation as in (10), we have

$$\mathbb{P}(f \text{ is an injective homomorphism}) \geq \mathbb{P}(E_1^* F_1 E_2 F_2 L_2 \dots E_m F_m L_m) \\ \geq (\frac{1}{2^{h+2}})^{|B_1|} (\frac{1}{2})^{|B_2| + \dots + |B_m|} c_1 p^{e(H)} \\ \geq \frac{1}{2^{h^2}} c_1 p^{e(H)} = c_3 p^{e(H)}.$$

This proves the second part of the theorem.

# 5 Concluding remarks

In this note, we used a nested variant of the dependent random choice to not only embed an appropriate tree-degenerate bipartite graph H in a host graph G, but also give tight (up to a

multiplicative factor) counting bound on the number of copies of H in G. In this variant, we get extra goodness features almost for free. It will be interesting to find more applications of it.

Another interesting feature of Theorem 1.9 is that the condition of the host graph is relaxed from 1norm density to r-norm density, which makes the result more flexible for applications. In principle, one could study the so-called r-norm Turán problem for bipartite graphs, where one wants to determine the maximum r-norm density of an H-free graph on n-vertices for a given bipartite graph H. The problem seems particularly natural for the family of r-degenerate graphs. For hypergraph co-degree problems, such a study has recently been initiated by Balogh, Clemen, and Lidický [3, 4]. Last but not least, it will be highly desirable to make more progress on Conjecture 1.4 beyond the following general bound obtained by Alon, Krivelevich, and Sudakov [2], which has stood as the best known bound in the last two decades.

**Theorem 5.1** ([2]). If H is an r-degenerate bipartite graph, then  $ex(n, H) = O(n^{2-1/4r})$ .

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