# Intertwining connectivities for vertex-minors and pivot-minors 

Duksang Lee ${ }^{* 2,1}$ and Sang-il Oum ${ }^{* 1,2}$<br>${ }^{1}$ Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea<br>${ }^{2}$ Department of Mathematical Sciences, KAIST, Daejeon, South Korea<br>Email: duksang@kaist.ac.kr, sangil@ibs.re.kr

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#### Abstract

We show that for pairs $(Q, R)$ and $(S, T)$ of disjoint subsets of vertices of a graph $G$, if $G$ is sufficiently large, then there exists a vertex $v$ in $V(G)-(Q \cup R \cup S \cup T)$ such that there are two ways to reduce $G$ by a vertex-minor operation that removes $v$ while preserving the connectivity between $Q$ and $R$ and the connectivity between $S$ and $T$. Our theorem implies an analogous theorem of Chen and Whittle (2014) for matroids restricted to binary matroids.


## 1 Introduction

Oum [7] proved a vertex-minor analog of Tutte's Linking Theorem on matroids [11]. Roughly speaking, the theorem of Oum says that for every pair of disjoint sets $Q, R$ of vertices of a graph $G$, there are at least two ways to reduce $G$ by a vertex-minor operation while keeping the 'connectivity' between $Q$ and $R$, where this connectivity will be defined using the rank function of matrices. We prove that if the graph is large, for any two pairs $(Q, R)$ and $(S, T)$ of disjoint sets of vertices, there exist two ways to reduce the graph by a vertex-minor operation while preserving the connectivity between $Q$ and $R$, and the connectivity between $S$ and $T$.

To state the main theorem precisely, we introduce a few concepts. A graph is simple if it has neither loops nor parallel edges. In this paper, all graphs are finite and simple. For a vertex $v$ of a graph $G$, the local complementation at $v$ is an operation that, for each pair $x, y$ of distinct neighbors of $v$, adds an edge $x y$ if $x$ and $y$ are non-adjacent in $G$ and removes an edge $x y$ otherwise. Let $G * v$ be the graph obtained from $G$ by applying the local complementation at $v$. A graph $H$ is a vertex-minor of $G$ if it can be obtained from $G$ by applying a sequence of local complementations and deletions of vertices. For an edge $u v$ of a graph $G$, let $G \wedge u v=G * u * v * u$. We remark that the pivoting operation is well defined since $G * u * v * u=G * v * u * v$. The operation obtaining $G \wedge u v$ from $G$ is called pivoting uv. A graph $H$ is a pivot-minor of $G$ if it can be obtained from $G$ by applying a sequence of pivoting edges and deleting vertices.

For a graph $G$, the cut-rank function $\rho_{G}$ is a function that maps a set $X$ of vertices of $G$ to the rank of an $X \times(V(G)-X)$ matrix $^{1}$ over $\mathrm{GF}(2)$ whose $(i, j)$-entry is 1 if $i$ and $j$ are adjacent and 0 otherwise. For disjoint sets $S, T$ of vertices of $G$, the connectivity between $S$ and $T$, denoted by $\kappa_{G}(S, T)$, is defined by

$$
\min _{S \subseteq X \subseteq V(G)-T} \rho_{G}(X) .
$$

Now we are ready to state the analog of Tutte's Linking Theorem for vertex-minors as reformulated by Geelen, Kwon, McCarty, and Wollan [4, Theorem 4.1].

[^0]Theorem 1.1 (Oum [7]). Let $G$ be a graph and $Q, R$ be disjoint subsets of $V(G)$. Let $\kappa_{G}(Q, R)=k$ and $F=V(G)-(Q \cup R)$. For each vertex $v$ of $F$, at least two of the following hold:
(i) $\kappa_{G \backslash v}(Q, R)=k$.
(ii) $\kappa_{G * v \backslash v}(Q, R)=k$.
(iii) $\kappa_{G \wedge u v \backslash v}(Q, R)=k$ for each neighbor $u$ of $v$.

Theorem 1.1 is about preserving the rank-connectivity of one pair of vertex sets while taking vertex-minors. Here is our main theorem which considers two pairs of vertex sets.

Theorem 1.2. Let $G$ be a graph and $Q, R, S$, and $T$ be subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$. Let $\kappa_{G}(Q, R)=k, \kappa_{G}(S, T)=\ell$, and $F=V(G)-(Q \cup R \cup S \cup T)$. If $|F| \geq(2 \ell+1) 2^{2 k}$, then there exists a vertex $v$ in $F$ such that at least two of the following hold:
(i) $\kappa_{G \backslash v}(Q, R)=k$ and $\kappa_{G \backslash v}(S, T)=\ell$.
(ii) $\kappa_{G * v \backslash v}(Q, R)=k$ and $\kappa_{G * v \backslash v}(S, T)=\ell$.
(iii) $\kappa_{G \wedge u v \backslash v}(Q, R)=k$ and $\kappa_{G \wedge u v \backslash v}(S, T)=\ell$ for each neighbor $u$ of $v$.

Since at least two of (i), (ii), and (iii) hold, we deduce that (i) or (iii) holds. Thus, we have the following corollary for pivot-minors.

Corollary 1.3. Let $G$ be a graph and $Q, R, S$, and $T$ be subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$. Let $\kappa_{G}(Q, R)=k, \kappa_{G}(S, T)=\ell$, and $F=V(G)-(Q \cup R \cup S \cup T)$. If $|F| \geq(2 \ell+1) 2^{2 k}$, then there exists a vertex $v$ in $F$ such that at least one of the following holds:
(i) $\kappa_{G \backslash v}(Q, R)=k$ and $\kappa_{G \backslash v}(S, T)=\ell$.
(ii) $\kappa_{G \wedge u v \backslash v}(Q, R)=k$ and $\kappa_{G \wedge u v \backslash v}(S, T)=\ell$ for each neighbor $u$ of $v$.

Our proof is inspired by the proof of the following theorem of Chen and Whittle [2] who proved the analog for matroids, which was conjectured by Geelen, and proved for representable matroids by Huynh and van Zwam [6].

Theorem 1.4 (Chen and Whittle [2]). Let $M$ be a matroid and $Q, R$, $S$, and $T$ be subsets of $E(M)$ such that $Q \cap R=S \cap T=\emptyset$. Let $\kappa_{G}(Q, R)=k, \kappa_{G}(S, T)=\ell$, and $F=E(M)-(Q \cup R \cup S \cup T)$. If $|F| \geq(2 \ell+1) 2^{2 k+1}$, then there exists an element $e$ of $E(M)$ such that at least one of the following holds:
(i) $\kappa_{M \backslash e}(Q, R)=k$ and $\kappa_{M \backslash e}(S, T)=\ell$.
(ii) $\kappa_{M / e}(Q, R)=k$ and $\kappa_{M / e}(S, T)=\ell$.

In fact, Corollary 1.3 implies Theorem 1.4 restricted to binary matroids by using a relation between pivot-minors of bipartite graphs and minors of matroids [7]. One of the key differences between our proof and the proof of Chen and Whittle is that we use a new way of measuring the local connectivity, $\tilde{\Pi}(S, T)=\frac{1}{2}\left(\rho_{G}(S)+\rho_{G}(T)-\rho_{G}(S \cup T)\right)$. The purpose of having $\frac{1}{2}$ in the previous definition is to ensure that $\tilde{\Pi}_{G}[S, V(G)-S]=\rho_{G}(S)$.

Our theorem is motivated by the following conjecture for pivot-minors. A pivot-minor $H$ of a graph $G$ is proper if $|V(H)|<|V(G)|$. A graph $G$ is an intertwine of graphs $H_{1}$ and $H_{2}$ for pivot-minors if it contains both $H_{1}$ and $H_{2}$ as pivot-minors and no proper pivot-minor of $G$ contains both $H_{1}$ and $H_{2}$ as pivot-minors.

Conjecture 1.5 (Intertwining conjecture for pivot-minors). For graphs $G_{1}$ and $G_{2}$, there are only finitely many intertwines of $G_{1}$ and $G_{2}$ for pivot-minors.


Figure 1: $G$ and $G \wedge u v$.
Together with Theorem 1.1, Conjecture 1.5 implies Corollary 1.3 without an explicit function. Suppose that $G$ is a graph and $Q, R, S$, and $T$ are subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$, $\kappa_{G}(Q, R)=k$, and $\kappa_{G}(S, T)=\ell$. By Theorem 1.1, $G$ has pivot-minors $G_{1}$ and $G_{2}$ such that $V\left(G_{1}\right)=$ $Q \cup R, V\left(G_{2}\right)=S \cup T, \rho_{G_{1}}(Q)=k$, and $\rho_{G_{2}}(S)=\ell$. If Conjecture 1.5 holds, then there exists an integer $n$ such that every intertwine of $G_{1}$ and $G_{2}$ for pivot-minors has at most $n$ vertices. If $|V(G)|>n$, then $G$ is not an intertwine of $G_{1}$ and $G_{2}$ for pivot-minors. Hence, there exists a proper pivot-minor $H$ of $G$ having both $G_{1}$ and $G_{2}$ as pivot-minors. Let $v$ be a vertex in $V(G)-V(H)$. Then it is easy to see that (i) or (ii) of Corollary 1.3 holds.

The following conjecture of Oum [8] implies the intertwining conjecture for pivot-minors.
Conjecture 1.6 (Well-quasi-ordering conjecture for pivot-minors). For every infinite sequence $G_{1}$, $G_{2}, \ldots$ of graphs, there exist $i<j$ such that $G_{i}$ is isomorphic to a pivot-minor of $G_{j}$.

Although the analog of Conjecture 1.6 for vertex-minors is still open, Geelen and Oum [5] proved the analog of Conjecture 1.5 for vertex-minors.

This paper is organized as follows. In Section 2, we introduce concepts of vertex-minors and pivotminors, and review several inequalities for cut-rank functions. In Section 3, we present simple lemmas on the cut-rank function. In Section 4, we prove Theorem 1.2.

## 2 Preliminaries

For a graph $G$ and a vertex $v$ of $G$, let $N_{G}(v)$ be the set of vertices adjacent to $v$ in $G$. For a graph $G$ and a subset $X$ of $V(G)$, let $G[X]$ be the induced subgraph of $G$ on $X$. For two sets $A$ and $B$, let $A \triangle B=(A-B) \cup(B-A)$.

Vertex-minors and pivot-minors Note that for a graph $G$ and a vertex $v$ of $G$, the local complementation at $v$ replaces $G\left[N_{G}(v)\right]$ with its complement. A graph $H$ is locally equivalent to a graph $G$ if $H$ can be obtained from $G$ by applying a sequence of local complementations. Recall that a graph $H$ is a vertex-minor of a graph $G$ if $H$ can be obtained from $G$ by applying a sequence of local complementations and deletions of vertices.

For an edge $u v$ of a graph $G$, let $G \wedge u v=G * u * v * u$. Then $G \wedge u v$ is obtained from $G$ by pivoting $u v$. Alternatively, pivoting $u v$ can be understood as an operation that removes an edge $x y$ if $x, y$ are non-adjacent and adds an edge $x y$ otherwise for every pair $(x, y) \in\left(X_{1} \times X_{2}\right) \cup\left(X_{2} \times X_{3}\right) \cup\left(X_{3} \times X_{1}\right)$ where $X_{1}$ is the set of common neighbors of $u$ and $v, X_{2}$ is the set of neighbors of $u$ that are nonneighbors of $v$, and $X_{3}$ is the set of neighbors of $v$ that are non-neighbors of $u$ and then swaps the labels of $u$ and $v$, see Oum [7] and Figure 1. The graph $G \wedge u v$ is well defined since $G * u * v * u=G * v * u * v[7$, Corollary 2.2]. A graph $H$ is a pivot-minor of a graph $G$ if $H$ can be obtained from $G$ by a sequence of pivoting and deleting vertices.

Lemma 2.1 (Oum [7]). Let $G$ be a graph and $v$ be a vertex of $G$. If $x$ and $y$ are neighbors of $v$ in $G$, then $(G \wedge v x) \backslash v$ is locally equivalent to $(G \wedge v y) \backslash v$.

For a vertex $v$ of $G$ with a neighbor $u$, we write $G / v$ to denote $G \wedge u v \backslash v$. If $v$ has no neighbor in $G$, then we let $G / v$ denote $G \backslash v$. Then the graph $G / v$ is well-defined up to local equivalence by Lemma 2.1. The following lemma can be easily deduced from isotropic systems [1], and Geelen and Oum provide an elementary graph-theoretic proof.

Lemma 2.2 (Geelen and Oum [5, Lemma 3.1]). Let $G$ be a graph and $v$ and $w$ be vertices of $G$. Then the following hold:
(1) If $v \neq w$ and $v w \notin E(G)$, then $(G * w) \backslash v,(G * w * v) \backslash v$, and $(G * w) / v$ are locally equivalent to $G \backslash v, G * v \backslash v$, and $G / v$ respectively.
(2) If $v \neq w$ and $v w \in E(G)$, then $(G * w) \backslash v,(G * w * v) \backslash v$, and $(G * w) / v$ are locally equivalent to $G \backslash v, G / v$, and $(G * v) \backslash v$ respectively.
(3) If $v=w$, then $(G * w) \backslash v,(G * w * v) \backslash v$, and $(G * w) / v$ are locally equivalent to $G * v \backslash v, G \backslash v$, and $G / v$ respectively.

From Lemma 2.2, we can deduce the following lemma easily.
Lemma 2.3. Let $H$ be a vertex-minor of a graph $G$ and $v$ be a vertex of $H$. Let $H_{1}=H \backslash v$, $H_{2}=H * v \backslash v$, and $H_{3}=H / v$ and let $G_{1}=G \backslash v, G_{2}=G * v \backslash v$, and $G_{3}=G / v$. Then there exists a permutation $\sigma:\{1,2,3\} \rightarrow\{1,2,3\}$ such that $H_{i}$ is a vertex-minor of $G_{\sigma(i)}$ for each $i \in\{1,2,3\}$.

Proof. Since $H$ is a vertex-minor of $G$, there exist a sequence $u_{1}, \ldots, u_{m}$ of vertices of $G$ and a subset $X$ of $V(G)$ such that $H=G * u_{1} * \cdots * u_{m} \backslash X$. We proceed by induction on $m$. If $m=0$, then $H=G \backslash X$. Obviously, $H_{i}=G_{i} \backslash X$ for each $i \in\{1,2\}$. We claim that $H_{3}=G_{3} \backslash X$. If there is a neighbor $w$ of $v$ in $G$ which is not in $X$, then $H_{3}=H \wedge v w \backslash v=(G \wedge v w \backslash v) \backslash X=G_{3} \backslash X$. If $N_{G}(v) \subseteq X$, then $H_{3}=H \backslash v=G \backslash X \backslash v$. Since $X$ contains all the neighbors of $v$, it is easy to check that $G_{3} \backslash X=((G \wedge u v) \backslash v) \backslash X=G \backslash X \backslash v=H_{3}$.

Therefore we may assume that $m \neq 0$. Let $H^{\prime}=G * u_{1}$. Then $H=H^{\prime} * u_{2} * \cdots * u_{m} \backslash X$, $H_{1}^{\prime}=H^{\prime} \backslash v, H_{2}^{\prime}=H^{\prime} * v \backslash v$, and $H_{3}^{\prime}=H^{\prime} / v$. By the induction hypothesis, there is a permutation $\sigma_{1}:\{1,2,3\} \rightarrow\{1,2,3\}$ such that $H_{i}$ is a vertex-minor of $H_{\sigma_{1}(i)}^{\prime}$ for each $i \in\{1,2,3\}$. By Lemma 2.2, there is a permutation $\sigma_{2}:\{1,2,3\} \rightarrow\{1,2,3\}$ such that $H_{j}^{\prime}$ is locally equivalent to $G_{\sigma_{2}(j)}$ for each $j \in\{1,2,3\}$. Let $\sigma=\sigma_{2} \circ \sigma_{1}$. Then $H_{i}$ is a vertex-minor of $G_{\sigma(i)}$ for each $i \in\{1,2,3\}$.

Cut-rank function and connectivity For a finite set $V$, a $V \times V$-matrix $A$, and subsets $X$ and $Y$ of $V$, let $A[X, Y]$ be the $X \times Y$-submatrix of $A$. For a graph $G$, let $A_{G}$ be the adjacency matrix of $G$ over the binary field $\mathrm{GF}(2)$. The cut-rank $\rho_{G}(X)$ of $X \subseteq V(G)$ is defined by

$$
\rho_{G}(X)=\operatorname{rank}\left(A_{G}[X, V(G)-X]\right) .
$$

It is obvious to check that $\rho_{G}(X)=\rho_{G}(V(G)-X)$.
The following lemmas give some properties of the cut-rank function.
Lemma 2.4 (see Oum [7, Proposition 2.6]). If a graph $G^{\prime}$ is locally equivalent to a graph $G$, then $\rho_{G}(X)=\rho_{G^{\prime}}(X)$ for each $X \subseteq V(G)$.
Lemma 2.5 (see Oum [7, Corollary 4.2]). Let $G$ be a graph and let $X, Y$ be subsets of $V(G)$. Then

$$
\rho_{G}(X)+\rho_{G}(Y) \geq \rho_{G}(X \cap Y)+\rho_{G}(X \cup Y)
$$

Lemma 2.6 (Oum [9, Lemma 2.3]). Let $G$ be a graph and $v$ be a vertex of $G$. Let $X$ and $Y$ be subsets of $V(G)-\{v\}$. Then the following hold:

$$
\begin{align*}
& \rho_{G \backslash v}(X)+\rho_{G}(Y \cup\{v\}) \geq \rho_{G \backslash v}(X \cap Y)+\rho_{G}(X \cup Y \cup\{v\}) .  \tag{S1}\\
& \rho_{G \backslash v}(X)+\rho_{G}(Y) \geq \rho_{G}(X \cap Y)+\rho_{G \backslash v}(X \cup Y) .
\end{align*}
$$

Lemma 2.7. Let $G$ be a graph and $v$ be a vertex of $G$. For a subset $X$ of $V(G)-\{v\}$, we have
(i) $\rho_{G \backslash v}(X)+1 \geq \rho_{G}(X) \geq \rho_{G \backslash v}(X)$.
(ii) $\rho_{G \backslash v}(X)+1 \geq \rho_{G}(X \cup\{v\}) \geq \rho_{G \backslash v}(X)$.

Proof. Observe that removing a row or a column of a matrix decreases the rank by at most 1 and never increases the rank.

Let $G$ be a graph and $S, T$ be disjoint subsets of $V(G)$. The connectivity between $S$ and $T$ in $G$, denoted by $\kappa_{G}(S, T)$, is defined by $\min _{S \subseteq X \subseteq V(G)-T} \rho_{G}(X)$.

Lemma 2.8. Let $H$ be a vertex-minor of a graph $G$ and $S$ and $T$ be disjoint subsets of $V(H)$. Then $\kappa_{H}(S, T) \leq \kappa_{G}(S, T)$.

Proof. The conclusion follows from Lemma 2.4 and (i) of Lemma 2.7.
Lemma 2.9 (Oum and Seymour [10, Lemma 1]). Let $G$ be a graph and $X_{1}, X_{2}, Y_{1}$, and $Y_{2}$ be subsets of $V(G)$ such that $X_{1} \cap X_{2}=Y_{1} \cap Y_{2}=\emptyset$. Then, we have

$$
\kappa_{G}\left(X_{1}, X_{2}\right)+\kappa_{G}\left(Y_{1}, Y_{2}\right) \geq \kappa_{G}\left(X_{1} \cap Y_{1}, X_{2} \cup Y_{2}\right)+\kappa_{G}\left(X_{1} \cup Y_{1}, X_{2} \cap Y_{2}\right) .
$$

The following corollaries are easy consequences of Theorem 1.1.
Corollary 2.10. Let $G$ be a graph and $Q, R, S$, and $T$ be subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$. Let $F=V(G)-(Q \cup R \cup S \cup T), k=\kappa_{G}(Q, R)$, and $\ell=\kappa_{G}(S, T)$. For every vertex $v$ of $F$, at least one of the following holds:
(i) $\kappa_{G \backslash v}(Q, R)=k$ and $\kappa_{G \backslash v}(S, T)=\ell$.
(ii) $\kappa_{G * v \backslash v}(Q, R)=k$ and $\kappa_{G * v \backslash v}(S, T)=\ell$.
(iii) $\kappa_{G \wedge u v \backslash v}(Q, R)=k$ and $\kappa_{G \wedge u v \backslash v}(S, T)=\ell$ for each neighbor $u$ of $v$.

Proof. By Theorem 1.1, at least two graphs $H_{1}, H_{2}$ among $G \backslash v, G * v \backslash v$, and $G / v$ have the property that $\kappa_{H_{1}}(Q, R)=\kappa_{H_{2}}(Q, R)=k$. Again by Theorem 1.1, at least one graph $H$ of $H_{1}$ or $H_{2}$ satisfies the property that $\kappa_{H}(S, T)=\ell$.

Corollary 2.11. Let $G$ be a graph and $Q, R, S$, and $T$ be subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$. Let $F$ be a subset of $V(G)-(Q \cup R \cup S \cup T), k=\kappa_{G}(Q, R)$, and $\ell=\kappa_{G}(S, T)$. Then there exists a vertex-minor $H$ of $G$ such that $V(H)=V(G)-F, \kappa_{H}(Q, R)=k$, and $\kappa_{H}(S, T)=\ell$.

Proof. We proceed by induction on $|F|$. We may assume that $|F| \geq 1$. Let $v$ be a vertex of $F$. By Corollary 2.10, there is a graph $G_{1} \in\{G \backslash v, G * v \backslash v, G / v\}$ such that $\kappa_{G_{1}}(Q, R)=k$ and $\kappa_{G_{1}}(S, T)=\ell$. By the induction hypothesis, there is a vertex-minor $H$ of $G_{1}$ such that $V(H)=V\left(G_{1}\right)-(F-\{v\})=$ $V(G)-F, \kappa_{H}(Q, R)=\kappa_{G_{1}}(Q, R)=k$, and $\kappa_{H}(S, T)=\kappa_{G_{1}}(S, T)=\ell$. Therefore, the conclusion follows since $H$ is a vertex-minor of $G$.

The following lemma is the analog of [3, Lemma 4.7].
Lemma 2.12. Let $G$ be a graph and $S$ and $T$ be disjoint subsets of $V(G)$. Then there exist $S_{1} \subseteq S$ and $T_{1} \subseteq T$ such that $\left|S_{1}\right|=\left|T_{1}\right|=\kappa_{G}\left(S_{1}, T_{1}\right)=\kappa_{G}(S, T)$.

Proof. By Lemma 2.9, there exists a matroid $M_{1}$ on $V(G)-T$ whose rank function is $\kappa_{G}(X, T)$ for each subset $X$ of $V(G)-T$. Let $S_{1}$ be a maximal independent set of $M_{1}$ contained in $S$. Then we have $\left|S_{1}\right|=\kappa_{G}\left(S_{1}, T\right)=\kappa_{G}(S, T)$. By Lemma 2.9, there is a matroid $M_{2}$ on $V(G)-S_{1}$ whose rank function is $\kappa_{G}\left(X, S_{1}\right)$ for every subset $X$ of $V(G)-S_{1}$. Let $T_{1}$ be a maximal independent set of $M_{2}$ contained in $T$. Then $\left|T_{1}\right|=\kappa_{G}\left(T_{1}, S_{1}\right)=\kappa_{G}\left(T, S_{1}\right)$ and so we finish the proof.

## 3 Lemmas on the cut-rank function.

In this section, we present simple lemmas on the cut-rank function. A subset $X$ of $V(G)$ is an (S,T)-separating set of order $k$ in $G$ if $S \subseteq X \subseteq V(G)-T$ and $\rho_{G}(X)=k$.

For a graph $G$ and disjoint subsets $S, T$ of $V(G)$, let $\tilde{\Pi}_{G}[S, T]=\frac{1}{2}\left(\rho_{G}(S)+\rho_{G}(T)-\rho_{G}(S \cup T)\right)$.
Lemma 3.1. Let $G$ be a graph and $S$ and $T$ be disjoint subsets of $V(G)$. If $A$ and $B$ are $(S, T)$ separating sets of order $k:=\kappa_{G}(S, T)$ in $G$, then both $A \cap B$ and $A \cup B$ are $(S, T)$-separating sets of order $k$ in $G$.

Proof. Since both $A \cap B$ and $A \cup B$ are $(S, T)$-separating sets, $\rho_{G}(A \cap B) \geq k$ and $\rho_{G}(A \cup B) \geq k$. By Lemma 2.5,

$$
2 k=\rho_{G}(A)+\rho_{G}(B) \geq \rho_{G}(A \cup B)+\rho_{G}(A \cap B) \geq 2 k
$$

and therefore $\rho_{G}(A \cup B)=\rho_{G}(A \cap B)=k$.
Lemma 3.2. Let $G$ be a graph and $S$ and $T$ be disjoint subsets of $V(G)$ such that $\rho_{G}(S)=\kappa_{G}(S, T)$. Let $U$ be a subset of $S$. Let $v$ be a vertex in $V(G)-(S \cup T)$. If $\kappa_{G \backslash v}(U, T)<\kappa_{G}(U, T)$, then $\kappa_{G \backslash v}(S, T)<\kappa_{G}(S, T)$.
Proof. Let $k=\rho_{G}(S)=\kappa_{G}(S, T)$. Suppose that $\kappa_{G \backslash v}(S, T)=k$. Let $X$ be a $(U, T)$-separating set in $G \backslash v$. By (S2) of Lemma 2.6,

$$
\rho_{G \backslash v}(X)+\rho_{G}(S) \geq \rho_{G}(X \cap S)+\rho_{G \backslash v}(X \cup S)
$$

and since $X \cup S$ is $(S, T)$-separating in $G \backslash v$, we have $\rho_{G \backslash v}(X \cup S) \geq k=\rho_{G}(S)$. Hence, we deduce that $\rho_{G \backslash v}(X) \geq \rho_{G}(X \cap S) \geq \kappa_{G}(U, T)$. So $\kappa_{G \backslash v}(U, T) \geq \kappa_{G}(U, T)$, contradicting the assumption.

Lemma 3.3. Let $G$ be a graph and $X_{2}$ and $Y$ be disjoint subsets of $V(G)$. Let $X_{1}$ be a subset of $X_{2}$. Then $\tilde{\Pi}_{G}\left[X_{1}, Y\right] \leq \tilde{\Pi}_{G}\left[X_{2}, Y\right]$.

Proof. Since $X_{1} \subseteq X_{2}$, by Lemma 2.5, we have

$$
\begin{aligned}
\rho_{G}\left(X_{2}\right)+\rho_{G}\left(X_{1} \cup Y\right) & \geq \rho_{G}\left(X_{2} \cup\left(X_{1} \cup Y\right)\right)+\rho_{G}\left(X_{2} \cap\left(X_{1} \cup Y\right)\right) \\
& =\rho_{G}\left(X_{2} \cup Y\right)+\rho_{G}\left(X_{1}\right) .
\end{aligned}
$$

Hence, $2 \tilde{\Pi}_{G}\left(X_{1}, Y\right)=\rho_{G}\left(X_{1}\right)+\rho_{G}(Y)-\rho_{G}\left(X_{1} \cup Y\right) \leq \rho_{G}\left(X_{2}\right)+\rho_{G}(Y)-\rho_{G}\left(X_{2} \cup Y\right)=2 \tilde{\Pi}_{G}\left(X_{2}, Y\right)$.
Lemma 3.4. Let $G$ be a graph and $Q$ and $R$ be disjoint subsets of $V(G)$ such that $\rho_{G}(Q)=\kappa_{G}(Q, R)$. Let $v$ be a vertex of $V(G)-(Q \cup R)$ such that $\kappa_{G \backslash v}(Q, R)<\kappa_{G}(Q, R)$. Then the following hold:
(Q1) $\rho_{G}(Q \cup\{v\}) \geq \rho_{G}(Q)$.
(Q2) If $\rho_{G \backslash v}(Q)=\rho_{G}(Q)$, then $\rho_{G}(Q \cup\{v\})=\rho_{G}(Q)+1$.
Proof. (Q1) holds clearly since $\rho_{G}(Q)=\kappa_{G}(Q, R)$.
To prove (Q2), let $k=\kappa_{G}(Q, R)$. Since $\kappa_{G \backslash v}(Q, R)<k$, there is a subset $X$ of $V(G)$ such that $Q \subseteq X \subseteq V(G)-(R \cup\{v\})$ and $\rho_{G \backslash v}(X) \leq k-1$. Then $\rho_{G \backslash v}(X)<k \leq \rho_{G}(X \cup\{v\})$ because $Q \subseteq X \cup\{v\} \subseteq V(G)-R$ and by (S1) of Lemma 2.6, we have that

$$
\rho_{G \backslash v}(X)+\rho_{G}(Q \cup\{v\}) \geq \rho_{G \backslash v}(Q)+\rho_{G}(X \cup\{v\})>\rho_{G \backslash v}(Q)+\rho_{G \backslash v}(X) .
$$

Hence, by Lemma 2.7, $\rho_{G}(Q \cup\{v\})=\rho_{G \backslash v}(Q)+1=\rho_{G}(Q)+1$.

## 4 Proof of Theorem 1.2

For disjoint subsets $S$ and $T$ of vertices of a graph $G$, a vertex $v \in V(G)-(S \cup T)$ is $(S, T)$-flexible if $\kappa_{G \backslash v}(S, T)=\kappa_{G * v \backslash v}(S, T)=\kappa_{G \wedge u v \backslash v}(S, T)=\kappa_{G}(S, T)$ for each $u \in N_{G}(v)$. Note that every isolated vertex is $(S, T)$-flexible.

Lemma 4.1. Let $S, T$ be disjoint sets of vertices of a graph $G$. If a vertex $v$ is $(S, T)$-flexible in $G$, then it is $(S, T)$-flexible in every graph locally equivalent to $G$.

Proof. Let $G^{\prime}$ be a graph locally equivalent to $G$. Let $k=\kappa_{G}(S, T), G_{1}=G \backslash v, G_{2}=G * v \backslash v$, and $G_{3}=G / v$. Since $v$ is $(S, T)$-flexible in $G$, we have $\kappa_{G_{1}}(S, T)=\kappa_{G_{2}}(S, T)=\kappa_{G_{3}}(S, T)=k$. Let $H_{1}=G^{\prime} \backslash v, H_{2}=G^{\prime} * v \backslash v$, and $H_{3}=G^{\prime} / v$. Then by Lemma 2.3, there is a permutation $\sigma:\{1,2,3\} \rightarrow\{1,2,3\}$ such that $H_{i}$ is locally equivalent to $G_{\sigma(i)}$ for each $i \in\{1,2,3\}$. Hence, by Lemma 2.4, we have $\kappa_{H_{i}}(S, T)=\kappa_{G_{\sigma(i)}}(S, T)=k$ for each $i \in\{1,2,3\}$. Therefore, $v$ is $(S, T)$-flexible in $G^{\prime}$.

The following lemma finds a nested set of $(S, T)$-separating sets of order $\kappa_{G}(S, T)$ for disjoint sets $S$ and $T$ of vertices of a graph $G$.

Lemma 4.2. Let $G$ be a graph and $S$ and $T$ be disjoint subsets of $V(G)$. Let $k=\kappa_{G}(S, T)$ and $F \subseteq V(G)-(S \cup T)$ be a set of $n$ vertices which are not $(S, T)$-flexible. Then there exist an ordering $f_{1}, \ldots, f_{n}$ of vertices in $F$ and a sequence $A_{1}, \ldots, A_{n}$ of $(S, T)$-separating sets of order $k$ in $G$ such that the following hold:
(i) $A_{i} \subseteq A_{i+1}$ for each $1 \leq i \leq n-1$.
(ii) $A_{i} \cap F=\left\{f_{1}, \ldots, f_{i}\right\}$ for each $1 \leq i \leq n$.

Proof. We prove by induction on $n=|F|$. We may assume that $n \geq 1$. We first claim that for every $v \in F$, there exists an $(S, T)$-separating set of order $k$ in $G$ containing $v$. Since $v$ is not $(S, T)$-flexible in $G$, there exists a graph $G^{\prime} \in\{G \backslash v, G * v \backslash v, G / v\}$ such that $\kappa_{G^{\prime}}(S, T)<\kappa_{G}(S, T)$. So there is a subset $A$ of $V(G)-\{v\}$ such that $S \subseteq A \subseteq V(G)-(T \cup\{v\})$ and $\rho_{G^{\prime}}(A) \leq k-1$. There exists a graph $H$ locally equivalent to $G$ such that $H \backslash v=G^{\prime}$. Therefore, since $S \subseteq A \cup\{v\} \subseteq V(G)-T$, by Lemmas 2.4 and 2.7, we have $k \leq \rho_{G}(A \cup\{v\})=\rho_{H}(A \cup\{v\}) \leq \rho_{H \backslash v}(A)+1=\rho_{G^{\prime}}(A)+1 \leq k$ and so $\rho_{G}(A \cup\{v\})=k$. Now it follows that $A \cup\{v\}$ is an (S,T)-separating set of order $k$ in $G$ containing $v$.

For each $u \in F$, let $A_{u}$ be an $(S, T)$-separating set of order $k$ in $G$ containing $u$ such that $\left|A_{u}\right|$ is minimum. Let $x$ be a vertex of $F$ such that $\left|A_{x}\right| \leq\left|A_{u}\right|$ for each $u \in F$.

Now we claim that $A_{x} \cap F=\{x\}$. Suppose that there exists an element $y \in\left(A_{x}-\{x\}\right) \cap F$. Then, by Lemma 3.1, both $A_{x} \cap A_{y}$ and $A_{x} \cup A_{y}$ are ( $S, T$ )-separating sets of order $k$ in $G$. Hence, $A_{y} \subseteq A_{x}$ by the choice of $A_{y}$. Then we have $A_{x}=A_{y}$ because $\left|A_{x}\right| \leq\left|A_{u}\right|$ for every $u \in F$. Since $y$ is not $(S, T)$-flexible, there exists a graph $G^{\prime \prime} \in\{G \backslash y, G * y \backslash y, G / y\}$ such that $\kappa_{G^{\prime \prime}}(S, T)<\kappa_{G}(S, T)$. By Lemma 2.4, we may assume that $G^{\prime \prime}=G \backslash y$. Then there exists $S \subseteq X \subseteq V(G)-(T \cup\{y\})$ such that $\rho_{G \backslash y}(X)=k-1$. By Lemma 2.7, $\rho_{G}(X)=k$ and $\rho_{G}(X \cup\{y\})=k$. So $X \cup\{y\}$ is an (S,T)-separating set of order $k$ in $G$ containing $y$. By Lemma 3.1, $A_{y} \cap(X \cup\{y\})$ is an $(S, T)$-separating set of order $k$ in $G$. Therefore, by the choice of $A_{y}$, we have $A_{y} \subseteq X \cup\{y\}$ and so $A_{y}-\{y\} \subseteq X$. By applying (S1) of Lemma 2.6,

$$
\begin{aligned}
2 k-1 & =\rho_{G \backslash y}(X)+\rho_{G}\left(A_{y}\right)=\rho_{G \backslash y}(X)+\rho_{G}\left(\left(A_{y}-\{y\}\right) \cup\{y\}\right) \\
& \geq \rho_{G \backslash y}\left(X \cap\left(A_{y}-\{y\}\right)\right)+\rho_{G}\left(X \cup\left(A_{y}-\{y\}\right) \cup\{y\}\right) \\
& =\rho_{G \backslash y}\left(A_{y}-\{y\}\right)+\rho_{G}(X \cup\{y\}) .
\end{aligned}
$$

Since $\rho_{G}(X \cup\{y\})=k$, we know that $\rho_{G \backslash y}\left(A_{y}-\{y\}\right) \leq k-1$ and so $\rho_{G}\left(A_{y}-\{y\}\right) \leq k$ by Lemma 2.7. Recall that $S \subseteq A_{y}-\{y\} \subseteq V(G)-T$ and $k=\kappa_{G}(S, T)$. Therefore, $\rho_{G}\left(A_{y}-\{y\}\right)=k$. Since $A_{x}=A_{y}$, this is a contradiction to the minimality of $A_{x}$. Thus $A_{x} \cap F=\{x\}$.

Let $f_{1}=x$ and $A_{1}=A_{x}$. Then $k=\kappa_{G}(S, T) \leq \kappa_{G}\left(A_{1}, T\right) \leq \rho_{G}\left(A_{1}\right)=k$ and therefore we have that $\kappa_{G}\left(A_{1}, T\right)=k$. By Lemmas 2.4 and 3.2, no vertex of $F-\left\{f_{1}\right\}$ is $\left(A_{1}, T\right)$-flexible. Hence, by
the induction hypothesis, there exist an ordering $f_{2}, \ldots, f_{n}$ of elements of $F-\left\{f_{1}\right\}$ and a sequence $A_{2}, \ldots, A_{n}$ of $\left(A_{1}, T\right)$-separating sets of order $k$ in $G$ such that (i) and (ii) hold.

So we finish the proof with the fact that $A_{2}, \ldots, A_{n}$ are also $(S, T)$-separating sets of order $k$ in $G$.

Our proof of Theorem 1.2 consists of two parts. In the first part, we will assume that $S$ and $T$ are small and prove the theorem. In the second part, we will show how to reduce the size of $S$ and $T$. The following lemma will be used at the key step in the first part.

Lemma 4.3. Let $G$ be a graph and $Q, R, S$, and $T$ be subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$ and $S \cup T \subseteq Q \cup R$. Let $F=V(G)-(Q \cup R) \neq \emptyset$ and $k=\kappa_{G}(Q, R)$ and $\ell=\kappa_{G}(S, T)$. If $\rho_{G}(Q)=\rho_{G}(R)=k$ and no vertex of $F$ is $(Q, R)$-flexible or $(S, T)$-flexible, then (1) or (2) holds:
(1) There exists a vertex $v$ of $F$ such that at least two of the following hold:
(i) $\kappa_{G \backslash v}(Q, R)=k$ and $\kappa_{G \backslash v}(S, T)=\ell$.
(ii) $\kappa_{G * v \backslash v}(Q, R)=k$ and $\kappa_{G * v \backslash v}(S, T)=\ell$.
(iii) $\kappa_{G \wedge u v \backslash v}(Q, R)=k$ and $\kappa_{G \wedge u v \backslash v}(S, T)=\ell$ for each $u \in N_{G}(v)$.
(2) There exist disjoint subsets $Q^{\prime}$ and $R^{\prime}$ of $V(G)$ such that the following hold:
(i) $Q \subseteq Q^{\prime}, R \subseteq R^{\prime}$ and $\rho_{G}\left(Q^{\prime}\right)=\rho_{G}\left(R^{\prime}\right)=k$.
(ii) $\tilde{\Pi}_{G}\left[Q^{\prime}, R^{\prime}\right] \geq \tilde{\Pi}_{G}[Q, R]+\frac{1}{2}$.
(iii) $\left|V(G)-\left(Q^{\prime} \cup R^{\prime}\right)\right| \geq\left\lfloor\frac{1}{2}|F|\right\rfloor$.

Proof. Assume that (1) does not hold. Let $n=|F|$. Since no vertex of $F$ is $(Q, R)$-flexible, by Lemma 4.2, there exists an ordering $f_{1}, \ldots, f_{n}$ of vertices of $F$ such that $Q \cup\left\{f_{1}, \ldots, f_{i}\right\}$ is a $(Q, R)$ seperating set of order $k$ in $G$ for each $i \in\{1, \ldots, n\}$. Let $A_{i}=Q \cup\left\{f_{1}, \ldots, f_{i}\right\}$ for each $1 \leq i \leq n$.

No vertex of $F$ is $(S, T)$-flexible and so, by Lemma 4.2, there exist a vertex $g$ in $F$ and an $(S, T)$ seperating set $C$ of order $\ell$ in $G$ such that $C-(Q \cup R)=\{g\}$.

By Theorem 1.1, there are graphs $G_{1}^{\prime}, G_{2}^{\prime} \in\{G \backslash g, G * g \backslash g, G / g\}$ such that $\kappa_{G_{i}^{\prime}}(S, T)=\kappa_{G}(S, T)$ for $i \in\{1,2\}$. Since (1) does not hold, there exists $G^{\prime} \in\left\{G_{1}^{\prime}, G_{2}^{\prime}\right\}$ such that $\kappa_{G^{\prime}}(Q, R)<\kappa_{G}(Q, R)$. Then by Lemma 2.4, we may assume that $G^{\prime}=G \backslash g$.

Since $\kappa_{G \backslash g}(S, T)=\ell$ and $S \subseteq C-\{g\} \subseteq V(G \backslash g)-T$, we have $\ell \leq \rho_{G \backslash g}(C-\{g\}) \leq \rho_{G}(C)=\ell$ and therefore $\rho_{G \backslash g}(C-\{g\})=\rho_{G}(C)$. Since $C-\{g\} \subseteq Q \cup R$, by (S1) of Lemma 2.6,

$$
\begin{aligned}
\rho_{G \backslash g}(Q \cup R)+\rho_{G}(C) & \geq \rho_{G \backslash g}((Q \cup R) \cap C)+\rho_{G}((Q \cup R) \cup C) \\
& =\rho_{G \backslash g}(C-\{g\})+\rho_{G}(Q \cup R \cup\{g\}) .
\end{aligned}
$$

Hence $\rho_{G}(Q \cup R \cup\{g\}) \leq \rho_{G \backslash g}(Q \cup R)$ because $\rho_{G \backslash g}(C-\{g\})=\rho_{G}(C)$. By Lemma 2.7, $\rho_{G \backslash g}(Q \cup R) \leq$ $\rho_{G}(Q \cup R \cup\{g\})$ and therefore $\rho_{G \backslash g}(Q \cup R)=\rho_{G}(Q \cup R \cup\{g\})$.

Now we claim that $\tilde{\Pi}_{G}(Q \cup\{g\}, R) \geq \tilde{\Pi}_{G}(Q, R)+\frac{1}{2}$. Observe that it is equivalent to show that

$$
\rho_{G}(Q \cup\{g\})+\rho_{G}(R)-\rho_{G}(Q \cup R \cup\{g\}) \geq \rho_{G}(Q)+\rho_{G}(R)-\rho_{G}(Q \cup R)+1 .
$$

We have $\rho_{G}(Q \cup R) \geq \rho_{G \backslash g}(Q \cup R)=\rho_{G}(Q \cup R \cup\{g\})$ and, by $(\mathrm{Q} 1)$ of Lemma 3.4, $\rho_{G}(Q \cup\{g\}) \geq \rho_{G}(Q)$. Therefore, it is enough to prove that $\rho_{G}(Q \cup R) \geq \rho_{G}(Q \cup R \cup\{g\})+1$ or $\rho_{G}(Q \cup\{g\}) \geq \rho_{G}(Q)+1$. Suppose that $\rho_{G}(Q \cup R)=\rho_{G}(Q \cup R \cup\{g\})=\rho_{G \backslash g}(Q \cup R)$. Then, by (S2) of Lemma 2.6, we have

$$
\rho_{G \backslash g}(Q)+\rho_{G}(Q \cup R) \geq \rho_{G \backslash g}(Q \cup R)+\rho_{G}(Q) .
$$

So $\rho_{G \backslash g}(Q) \geq \rho_{G}(Q)$ and we have $\rho_{G \backslash g}(Q)=\rho_{G}(Q)$ by Lemma 2.7. Then by (Q2) of Lemma 3.4, $\rho_{G}(Q \cup\{g\})=\rho_{G}(Q)+1$, proving the claim.

Similarly, we have $\tilde{\Pi}_{G}(Q, R \cup\{g\}) \geq \tilde{\Pi}_{G}(Q, R)+\frac{1}{2}$. Let $i$ be an integer such that $f_{i}=g$ and let

$$
\left(Q^{\prime}, R^{\prime}\right)= \begin{cases}\left(A_{i}, R\right) & \text { if } i \leq\left\lfloor\frac{n}{2}\right\rfloor \\ \left(Q, V(G)-A_{i-1}\right) & \text { otherwise } .\end{cases}
$$

Then by Lemma 3.3,

$$
\tilde{\Pi}_{G}\left(Q^{\prime}, R^{\prime}\right) \geq \min \left(\tilde{\Pi}_{G}(Q \cup\{g\}, R), \tilde{\Pi}_{G}(Q, R \cup\{g\})\right) \geq \tilde{\Pi}_{G}(Q, R)+\frac{1}{2} .
$$

So (ii) holds and (i) and (iii) hold by the construction.
Now we are ready to prove Theorem 1.2 when $S$ and $T$ are small.
Proposition 4.4. Let $G$ be a graph and $Q, R, S$, and $T$ be subsets of $V(G)$ such that $Q \cap R=S \cap T=\emptyset$ and $F=V(G)-(Q \cup R \cup S \cup T)$. Let $k=\kappa_{G}(Q, R)$ and $\ell=\kappa_{G}(S, T)$. If $|S|=|T|=\ell$ and $|F| \geq(2 \ell+1) 2^{2 k}$, then there is a vertex $v \in F$ such that at least two of the following hold:
(1) $\kappa_{G \backslash v}(Q, R)=k$ and $\kappa_{G \backslash v}(S, T)=\ell$.
(2) $\kappa_{G * v \backslash v}(Q, R)=k$ and $\kappa_{G * v \backslash v}(S, T)=\ell$.
(3) $\kappa_{G \wedge u v \backslash v}(Q, R)=k$ and $\kappa_{G \wedge u v \backslash v}(S, T)=\ell$ for every neighbor $u$ of $v$.

Proof. If $F$ has a vertex which is $(S, T)$-flexible or $(Q, R)$-flexible, then our conclusion follows by Theorem 1.1. So we can assume that no vertex of $F$ is $(S, T)$-flexible or $(Q, R)$-flexible. Let $n=|F|$.

By Lemma 4.2, there exist an ordering $f_{1}, \ldots, f_{n}$ of vertices of $F$ and a sequence $A_{1}, \ldots, A_{n}$ of $(Q, R)$-seperating sets of order $k$ in $G$ satisfying the following:

- $A_{i} \subseteq A_{i+1}$ for each $1 \leq i \leq n-1$.
- $A_{i} \cap F=\left\{f_{1}, \ldots, f_{i}\right\}$ for each $1 \leq i \leq n$.

For each $1 \leq i \leq n$, let $B_{i}=V(G)-A_{i}$. Let $q=2^{2 k}$ and $A_{0}=Q$. For $1 \leq i \leq 2 \ell+1$, let $X_{i}=A_{i q}-A_{(i-1) q}$. Since $|S|=|T|=\ell$, there exists $1 \leq m \leq 2 \ell+1$ such that $X_{m} \cap(S \cup T)=\emptyset$. Let $j=(m-1) q$. Then we have $Q \cup R \cup S \cup T \subseteq A_{j} \cup B_{j+q}$.

Assume that our conclusion fails and so every vertex of $F$ satisfies at most one of (1), (2), and (3). We claim that, for each $1 \leq i \leq 2 k+2$, there exist disjoint subsets $Q_{i}$ and $R_{i}$ of $V(G)$ satisfying the following.
(i) $Q \subseteq Q_{i}, R \subseteq R_{i}$, and $\rho_{G}\left(Q_{i}\right)=\rho_{G}\left(R_{i}\right)=k$.
(ii) $\tilde{\Pi}_{G}\left[Q_{i}, R_{i}\right] \geq \frac{i-1}{2}$.
(iii) $\left|V(G)-\left(Q_{i} \cup R_{i}\right)\right| \geq\left\lfloor 2^{2 k+1-i}\right\rfloor$.

We proceed by the induction on $i$. Let $Q_{1}=A_{j}, R_{1}=B_{j+q}$, and $F_{1}=V(G)-\left(Q_{1} \cup R_{1}\right)$. Then $\left|F_{1}\right|=2^{2 k}$ and so ( $Q_{1}, R_{1}$ ) satisfies the claim. Therefore we may assume that $i \geq 2$. By the induction hypothesis, there exist disjoint subsets $Q_{i-1}$ and $R_{i-1}$ of $V(G)$ satisfying (i), (ii), and (iii) for $i-1$. By Lemmas 2.4 and 3.2, no vertex of $V(G)-\left(Q_{i-1} \cup R_{i-1}\right)$ is ( $\left.Q_{i-1}, R_{i-1}\right)$-flexible. If there is a vertex $v$ of $V(G)-\left(Q_{i-1} \cup R_{i-1}\right)$ satisfying (1) of Lemma 4.3 for two pairs ( $Q_{i-1}, R_{i-1}$ ) and ( $S, T$ ), then by Lemmas 2.4 and 3.2, $v$ satisfies at least two of (1), (2), and (3), contradicting our assumption. So we may assume that $V(G)-\left(Q_{i-1} \cup R_{i-1}\right)$ has no such vertex. Hence, by Lemma 4.3, there exist disjoint subsets $Q_{i}$ and $R_{i}$ of $V(G)$ such that the following hold:
(a) $Q_{i-1} \subseteq Q_{i}, R_{i-1} \subseteq R_{i}$ and $\rho_{G}\left(Q_{i}\right)=\rho_{G}\left(R_{i}\right)=k$.
(b) $\tilde{\Pi}_{G}\left[Q_{i}, R_{i}\right] \geq \tilde{\Pi}_{G}\left[Q_{i-1}, R_{i-1}\right]+\frac{1}{2} \geq \frac{i-2}{2}+\frac{1}{2}=\frac{i-1}{2}$.
(c) $\left|V(G)-\left(Q_{i} \cup R_{i}\right)\right| \geq\left\lfloor\frac{1}{2}\left|V(G)-\left(Q_{i-1} \cup R_{i-1}\right)\right|\right\rfloor \geq\left\lfloor\frac{1}{2} \cdot 2^{2 k+2-i}\right\rfloor=\left\lfloor 2^{2 k+1-i}\right\rfloor$.

This proves our claim. Then by (ii) and Lemma 3.3, $k+\frac{1}{2} \leq \tilde{\Pi}_{G}\left(Q_{2 k+2}, R_{2 k+2}\right) \leq \tilde{\Pi}_{G}\left(Q_{2 k+2}, V(G)-\right.$ $\left.Q_{2 k+2}\right)=\rho_{G}\left(Q_{2 k+2}\right)=k$, which is a contradiction. Therefore our conclusion holds.

Now we are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.12, there exist $S_{1} \subseteq S$ and $T_{1} \subseteq T$ such that $\left|S_{1}\right|=\left|T_{1}\right|=$ $\kappa_{G}\left(S_{1}, T_{1}\right)=\kappa_{G}(S, T)$. Let $X=(S \cup T)-\left(Q \cup R \cup S_{1} \cup T_{1}\right)$. By Corollary 2.11, there is a vertexminor $H$ of $G$ such that $V(H)=V(G)-X, \kappa_{H}(Q, R)=k$, and $\kappa_{H}\left(S_{1}, T_{1}\right)=\ell$.

For a vertex $v$ of $V(H)-\left(Q \cup R \cup S_{1} \cup T_{1}\right)$, let $H_{1}^{v}=H \backslash v, H_{2}^{v}=H * v \backslash v$, and $H_{3}^{v}=H / v$ and let $G_{1}^{v}=$ $G \backslash v, G_{2}^{v}=G * v \backslash v$, and $G_{3}^{v}=G / v$. Then by Lemma 2.3, there exists a permutation $\sigma_{v}:\{1,2,3\} \rightarrow$ $\{1,2,3\}$ such that $H_{i}^{v}$ is a vertex-minor of $G_{\sigma(i)}^{v}$ for each $i \in\{1,2,3\}$. By Lemma 2.8, $\kappa_{H_{i}^{v}}\left(S_{1}, T_{1}\right) \leq$ $\kappa_{G_{\sigma(i)}^{v}}\left(S_{1}, T_{1}\right) \leq \kappa_{G_{\sigma(i)}^{v}}^{v}(S, T) \leq \kappa_{G}(S, T)=\ell$ and $\kappa_{H_{i}^{v}}(Q, R) \leq \kappa_{G_{\sigma(i)}^{v}}(Q, R) \leq \kappa_{G}(Q, R)=k$ for each $i \in\{1,2,3\}$.

Since $\left|V(H)-\left(Q \cup R \cup S_{1} \cup T_{1}\right)\right|=|F| \geq(2 \ell+1) 2^{2 k}$, by Proposition 4.4, there exist a vertex $v$ of $V(H)-\left(Q \cup R \cup S_{1} \cup T_{1}\right)=F$ and $i, j \in\{1,2,3\}$ such that $i \neq j$ and $\kappa_{H_{i}^{v}}(Q, R)=\kappa_{H_{j}^{v}}(Q, R)=k$ and $\kappa_{H_{i}^{v}}\left(S_{1}, T_{1}\right)=\kappa_{H_{j}^{v}}\left(S_{1}, T_{1}\right)=\ell$. Therefore, $\kappa_{G_{\sigma(i)}^{v}}(S, T)=\kappa_{G_{\sigma(j)}^{v}}(S, T)=\ell$ and $\kappa_{G_{\sigma(i)}^{v}}(Q, R)=$ $\kappa_{G_{\sigma(j)}^{v}}(Q, R)=k$.

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    ${ }^{1}$ For two sets $A$ and $B$, an $A \times B$-matrix denotes an $|A| \times|B|$ matrix whose rows and columns are indexed by the elements of $A$ and $B$ respectively.

