Intertwining connectivities for vertex-minors and pivot-minors

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Abstract

We show that for pairs (Q, R) and (S, T) of disjoint subsets of vertices of a graph G, if G is sufficiently large, then there exists a vertex v in $V(G) - (Q \cup R \cup S \cup T)$ such that there are two ways to reduce G by a vertex-minor operation that removes v while preserving the connectivity between Q and R and the connectivity between S and T. Our theorem implies an analogous theorem of Chen and Whittle (2014) for matroids restricted to binary matroids.

1 Introduction

Oum [7] proved a vertex-minor analog of Tutte's Linking Theorem on matroids [11]. Roughly speaking, the theorem of Oum says that for every pair of disjoint sets Q, R of vertices of a graph G, there are at least two ways to reduce G by a vertex-minor operation while keeping the 'connectivity' between Qand R, where this connectivity will be defined using the rank function of matrices. We prove that if the graph is large, for any two pairs (Q, R) and (S, T) of disjoint sets of vertices, there exist two ways to reduce the graph by a vertex-minor operation while preserving the connectivity between Q and R, and the connectivity between S and T.

To state the main theorem precisely, we introduce a few concepts. A graph is simple if it has neither loops nor parallel edges. In this paper, all graphs are finite and simple. For a vertex v of a graph G, the local complementation at v is an operation that, for each pair x, y of distinct neighbors of v, adds an edge xy if x and y are non-adjacent in G and removes an edge xy otherwise. Let G * v be the graph obtained from G by applying the local complementation at v. A graph H is a vertex-minor of G if it can be obtained from G by applying a sequence of local complementations and deletions of vertices. For an edge uv of a graph G, let $G \wedge uv = G * u * v * u$. We remark that the pivoting operation is well defined since G * u * v * u = G * v * u * v. The operation obtaining $G \wedge uv$ from Gis called pivoting uv. A graph H is a pivot-minor of G if it can be obtained from G by applying a sequence of pivoting edges and deleting vertices.

For a graph G, the *cut-rank* function ρ_G is a function that maps a set X of vertices of G to the rank of an $X \times (V(G) - X)$ matrix¹ over GF(2) whose (i, j)-entry is 1 if i and j are adjacent and 0 otherwise. For disjoint sets S, T of vertices of G, the *connectivity between* S and T, denoted by $\kappa_G(S,T)$, is defined by

$$\min_{S \subseteq X \subseteq V(G) - T} \rho_G(X).$$

Now we are ready to state the analog of Tutte's Linking Theorem for vertex-minors as reformulated by Geelen, Kwon, McCarty, and Wollan [4, Theorem 4.1].

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¹For two sets A and B, an $A \times B$ -matrix denotes an $|A| \times |B|$ matrix whose rows and columns are indexed by the elements of A and B respectively.

Theorem 1.1 (Oum [7]). Let G be a graph and Q, R be disjoint subsets of V(G). Let $\kappa_G(Q, R) = k$ and $F = V(G) - (Q \cup R)$. For each vertex v of F, at least two of the following hold:

- (i) $\kappa_{G\setminus v}(Q,R) = k.$
- (ii) $\kappa_{G*v \setminus v}(Q, R) = k$.
- (iii) $\kappa_{G \wedge uv \setminus v}(Q, R) = k$ for each neighbor u of v.

Theorem 1.1 is about preserving the rank-connectivity of one pair of vertex sets while taking vertex-minors. Here is our main theorem which considers two pairs of vertex sets.

Theorem 1.2. Let G be a graph and Q, R, S, and T be subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$. Let $\kappa_G(Q, R) = k$, $\kappa_G(S, T) = \ell$, and $F = V(G) - (Q \cup R \cup S \cup T)$. If $|F| \ge (2\ell + 1)2^{2k}$, then there exists a vertex v in F such that at least two of the following hold:

- (i) $\kappa_{G\setminus v}(Q, R) = k$ and $\kappa_{G\setminus v}(S, T) = \ell$.
- (ii) $\kappa_{G*v \setminus v}(Q, R) = k \text{ and } \kappa_{G*v \setminus v}(S, T) = \ell.$
- (iii) $\kappa_{G \wedge uv \setminus v}(Q, R) = k$ and $\kappa_{G \wedge uv \setminus v}(S, T) = \ell$ for each neighbor u of v.

Since at least two of (i), (ii), and (iii) hold, we deduce that (i) or (iii) holds. Thus, we have the following corollary for pivot-minors.

Corollary 1.3. Let G be a graph and Q, R, S, and T be subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$. Let $\kappa_G(Q, R) = k$, $\kappa_G(S, T) = \ell$, and $F = V(G) - (Q \cup R \cup S \cup T)$. If $|F| \ge (2\ell + 1)2^{2k}$, then there exists a vertex v in F such that at least one of the following holds:

- (i) $\kappa_{G \setminus v}(Q, R) = k$ and $\kappa_{G \setminus v}(S, T) = \ell$.
- (ii) $\kappa_{G \land uv \land v}(Q, R) = k$ and $\kappa_{G \land uv \land v}(S, T) = \ell$ for each neighbor u of v.

Our proof is inspired by the proof of the following theorem of Chen and Whittle [2] who proved the analog for matroids, which was conjectured by Geelen, and proved for representable matroids by Huynh and van Zwam [6].

Theorem 1.4 (Chen and Whittle [2]). Let M be a matroid and Q, R, S, and T be subsets of E(M) such that $Q \cap R = S \cap T = \emptyset$. Let $\kappa_G(Q, R) = k$, $\kappa_G(S, T) = \ell$, and $F = E(M) - (Q \cup R \cup S \cup T)$. If $|F| \ge (2\ell + 1)2^{2k+1}$, then there exists an element e of E(M) such that at least one of the following holds:

- (i) $\kappa_{M\setminus e}(Q, R) = k$ and $\kappa_{M\setminus e}(S, T) = \ell$.
- (ii) $\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = \ell$.

In fact, Corollary 1.3 implies Theorem 1.4 restricted to binary matroids by using a relation between pivot-minors of bipartite graphs and minors of matroids [7]. One of the key differences between our proof and the proof of Chen and Whittle is that we use a new way of measuring the local connectivity, $\tilde{\sqcap}(S,T) = \frac{1}{2}(\rho_G(S) + \rho_G(T) - \rho_G(S \cup T))$. The purpose of having $\frac{1}{2}$ in the previous definition is to ensure that $\tilde{\sqcap}_G[S, V(G) - S] = \rho_G(S)$.

Our theorem is motivated by the following conjecture for pivot-minors. A pivot-minor H of a graph G is proper if |V(H)| < |V(G)|. A graph G is an *intertwine* of graphs H_1 and H_2 for pivot-minors if it contains both H_1 and H_2 as pivot-minors and no proper pivot-minor of G contains both H_1 and H_2 as pivot-minors.

Conjecture 1.5 (Intertwining conjecture for pivot-minors). For graphs G_1 and G_2 , there are only finitely many intertwines of G_1 and G_2 for pivot-minors.



Figure 1: G and $G \wedge uv$.

Together with Theorem 1.1, Conjecture 1.5 implies Corollary 1.3 without an explicit function. Suppose that G is a graph and Q, R, S, and T are subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$, $\kappa_G(Q, R) = k$, and $\kappa_G(S, T) = \ell$. By Theorem 1.1, G has pivot-minors G_1 and G_2 such that $V(G_1) = Q \cup R$, $V(G_2) = S \cup T$, $\rho_{G_1}(Q) = k$, and $\rho_{G_2}(S) = \ell$. If Conjecture 1.5 holds, then there exists an integer n such that every intertwine of G_1 and G_2 for pivot-minors has at most n vertices. If |V(G)| > n, then G is not an intertwine of G_1 and G_2 for pivot-minors. Hence, there exists a proper pivot-minor H of G having both G_1 and G_2 as pivot-minors. Let v be a vertex in V(G) - V(H). Then it is easy to see that (i) or (ii) of Corollary 1.3 holds.

The following conjecture of Oum [8] implies the intertwining conjecture for pivot-minors.

Conjecture 1.6 (Well-quasi-ordering conjecture for pivot-minors). For every infinite sequence G_1 , G_2 , ... of graphs, there exist i < j such that G_i is isomorphic to a pivot-minor of G_j .

Although the analog of Conjecture 1.6 for vertex-minors is still open, Geelen and Oum [5] proved the analog of Conjecture 1.5 for vertex-minors.

This paper is organized as follows. In Section 2, we introduce concepts of vertex-minors and pivotminors, and review several inequalities for cut-rank functions. In Section 3, we present simple lemmas on the cut-rank function. In Section 4, we prove Theorem 1.2.

2 Preliminaries

For a graph G and a vertex v of G, let $N_G(v)$ be the set of vertices adjacent to v in G. For a graph G and a subset X of V(G), let G[X] be the induced subgraph of G on X. For two sets A and B, let $A \triangle B = (A - B) \cup (B - A)$.

Vertex-minors and pivot-minors Note that for a graph G and a vertex v of G, the local complementation at v replaces $G[N_G(v)]$ with its complement. A graph H is *locally equivalent* to a graph G if H can be obtained from G by applying a sequence of local complementations. Recall that a graph H is a *vertex-minor* of a graph G if H can be obtained from G by applying a sequence of local complementations. Recall that a graph H is a *vertex-minor* of a graph G if H can be obtained from G by applying a sequence of local complementations.

For an edge uv of a graph G, let $G \wedge uv = G * u * v * u$. Then $G \wedge uv$ is obtained from G by pivoting uv. Alternatively, pivoting uv can be understood as an operation that removes an edge xy if x, y are non-adjacent and adds an edge xy otherwise for every pair $(x, y) \in (X_1 \times X_2) \cup (X_2 \times X_3) \cup (X_3 \times X_1)$ where X_1 is the set of common neighbors of u and v, X_2 is the set of neighbors of u that are non-neighbors of v, and X_3 is the set of neighbors of v that are non-neighbors of u and then swaps the labels of u and v, see Oum [7] and Figure 1. The graph $G \wedge uv$ is well defined since G * u * v * u = G * v * u * v [7, Corollary 2.2]. A graph H is a pivot-minor of a graph G if H can be obtained from G by a sequence of pivoting and deleting vertices.

Lemma 2.1 (Oum [7]). Let G be a graph and v be a vertex of G. If x and y are neighbors of v in G, then $(G \wedge vx) \setminus v$ is locally equivalent to $(G \wedge vy) \setminus v$.

For a vertex v of G with a neighbor u, we write G/v to denote $G \wedge uv \setminus v$. If v has no neighbor in G, then we let G/v denote $G \setminus v$. Then the graph G/v is well-defined up to local equivalence by Lemma 2.1. The following lemma can be easily deduced from isotropic systems [1], and Geelen and Oum provide an elementary graph-theoretic proof. **Lemma 2.2** (Geelen and Oum [5, Lemma 3.1]). Let G be a graph and v and w be vertices of G. Then the following hold:

- (1) If $v \neq w$ and $vw \notin E(G)$, then $(G * w) \setminus v$, $(G * w * v) \setminus v$, and (G * w)/v are locally equivalent to $G \setminus v$, $G * v \setminus v$, and G/v respectively.
- (2) If $v \neq w$ and $vw \in E(G)$, then $(G * w) \setminus v$, $(G * w * v) \setminus v$, and (G * w)/v are locally equivalent to $G \setminus v$, G/v, and $(G * v) \setminus v$ respectively.
- (3) If v = w, then $(G * w) \setminus v$, $(G * w * v) \setminus v$, and (G * w)/v are locally equivalent to $G * v \setminus v$, $G \setminus v$, and G/v respectively.

From Lemma 2.2, we can deduce the following lemma easily.

Lemma 2.3. Let H be a vertex-minor of a graph G and v be a vertex of H. Let $H_1 = H \setminus v$, $H_2 = H * v \setminus v$, and $H_3 = H/v$ and let $G_1 = G \setminus v$, $G_2 = G * v \setminus v$, and $G_3 = G/v$. Then there exists a permutation $\sigma : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ such that H_i is a vertex-minor of $G_{\sigma(i)}$ for each $i \in \{1, 2, 3\}$.

Proof. Since H is a vertex-minor of G, there exist a sequence u_1, \ldots, u_m of vertices of G and a subset X of V(G) such that $H = G * u_1 * \cdots * u_m \setminus X$. We proceed by induction on m. If m = 0, then $H = G \setminus X$. Obviously, $H_i = G_i \setminus X$ for each $i \in \{1, 2\}$. We claim that $H_3 = G_3 \setminus X$. If there is a neighbor w of v in G which is not in X, then $H_3 = H \wedge vw \setminus v = (G \wedge vw \setminus v) \setminus X = G_3 \setminus X$. If $N_G(v) \subseteq X$, then $H_3 = H \setminus v = G \setminus X \setminus v$. Since X contains all the neighbors of v, it is easy to check that $G_3 \setminus X = ((G \wedge uv) \setminus v) \setminus X = G \setminus X \setminus v = H_3$.

Therefore we may assume that $m \neq 0$. Let $H' = G * u_1$. Then $H = H' * u_2 * \cdots * u_m \setminus X$, $H'_1 = H' \setminus v$, $H'_2 = H' * v \setminus v$, and $H'_3 = H'/v$. By the induction hypothesis, there is a permutation $\sigma_1 : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ such that H_i is a vertex-minor of $H'_{\sigma_1(i)}$ for each $i \in \{1, 2, 3\}$. By Lemma 2.2, there is a permutation $\sigma_2 : \{1, 2, 3\} \rightarrow \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ such that H'_j is locally equivalent to $G_{\sigma_2(j)}$ for each $j \in \{1, 2, 3\}$. Let $\sigma = \sigma_2 \circ \sigma_1$. Then H_i is a vertex-minor of $G_{\sigma(i)}$ for each $i \in \{1, 2, 3\}$.

Cut-rank function and connectivity For a finite set V, a $V \times V$ -matrix A, and subsets X and Y of V, let A[X, Y] be the $X \times Y$ -submatrix of A. For a graph G, let A_G be the adjacency matrix of G over the binary field GF(2). The *cut-rank* $\rho_G(X)$ of $X \subseteq V(G)$ is defined by

$$\rho_G(X) = \operatorname{rank}(A_G[X, V(G) - X]).$$

It is obvious to check that $\rho_G(X) = \rho_G(V(G) - X)$.

The following lemmas give some properties of the cut-rank function.

Lemma 2.4 (see Oum [7, Proposition 2.6]). If a graph G' is locally equivalent to a graph G, then $\rho_G(X) = \rho_{G'}(X)$ for each $X \subseteq V(G)$.

Lemma 2.5 (see Oum [7, Corollary 4.2]). Let G be a graph and let X, Y be subsets of V(G). Then

$$\rho_G(X) + \rho_G(Y) \ge \rho_G(X \cap Y) + \rho_G(X \cup Y).$$

Lemma 2.6 (Oum [9, Lemma 2.3]). Let G be a graph and v be a vertex of G. Let X and Y be subsets of $V(G) - \{v\}$. Then the following hold:

- (S1) $\rho_{G\setminus v}(X) + \rho_G(Y \cup \{v\}) \ge \rho_{G\setminus v}(X \cap Y) + \rho_G(X \cup Y \cup \{v\}).$
- (S2) $\rho_{G\setminus v}(X) + \rho_G(Y) \ge \rho_G(X \cap Y) + \rho_{G\setminus v}(X \cup Y).$

Lemma 2.7. Let G be a graph and v be a vertex of G. For a subset X of $V(G) - \{v\}$, we have

- (i) $\rho_{G\setminus v}(X) + 1 \ge \rho_G(X) \ge \rho_{G\setminus v}(X).$
- (ii) $\rho_{G\setminus v}(X) + 1 \ge \rho_G(X \cup \{v\}) \ge \rho_{G\setminus v}(X).$

Proof. Observe that removing a row or a column of a matrix decreases the rank by at most 1 and never increases the rank. \Box

Let G be a graph and S, T be disjoint subsets of V(G). The connectivity between S and T in G, denoted by $\kappa_G(S,T)$, is defined by $\min_{S \subset X \subset V(G)-T} \rho_G(X)$.

Lemma 2.8. Let H be a vertex-minor of a graph G and S and T be disjoint subsets of V(H). Then $\kappa_H(S,T) \leq \kappa_G(S,T)$.

Proof. The conclusion follows from Lemma 2.4 and (i) of Lemma 2.7.

Lemma 2.9 (Ourn and Seymour [10, Lemma 1]). Let G be a graph and X_1 , X_2 , Y_1 , and Y_2 be subsets of V(G) such that $X_1 \cap X_2 = Y_1 \cap Y_2 = \emptyset$. Then, we have

$$\kappa_G(X_1, X_2) + \kappa_G(Y_1, Y_2) \ge \kappa_G(X_1 \cap Y_1, X_2 \cup Y_2) + \kappa_G(X_1 \cup Y_1, X_2 \cap Y_2).$$

The following corollaries are easy consequences of Theorem 1.1.

Corollary 2.10. Let G be a graph and Q, R, S, and T be subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$. Let $F = V(G) - (Q \cup R \cup S \cup T)$, $k = \kappa_G(Q, R)$, and $\ell = \kappa_G(S, T)$. For every vertex v of F, at least one of the following holds:

- (i) $\kappa_{G\setminus v}(Q, R) = k$ and $\kappa_{G\setminus v}(S, T) = \ell$.
- (ii) $\kappa_{G*v\setminus v}(Q, R) = k$ and $\kappa_{G*v\setminus v}(S, T) = \ell$.
- (iii) $\kappa_{G \wedge uv \setminus v}(Q, R) = k$ and $\kappa_{G \wedge uv \setminus v}(S, T) = \ell$ for each neighbor u of v.

Proof. By Theorem 1.1, at least two graphs H_1 , H_2 among $G \setminus v$, $G * v \setminus v$, and G/v have the property that $\kappa_{H_1}(Q, R) = \kappa_{H_2}(Q, R) = k$. Again by Theorem 1.1, at least one graph H of H_1 or H_2 satisfies the property that $\kappa_H(S, T) = \ell$.

Corollary 2.11. Let G be a graph and Q, R, S, and T be subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$. Let F be a subset of $V(G) - (Q \cup R \cup S \cup T)$, $k = \kappa_G(Q, R)$, and $\ell = \kappa_G(S, T)$. Then there exists a vertex-minor H of G such that V(H) = V(G) - F, $\kappa_H(Q, R) = k$, and $\kappa_H(S, T) = \ell$.

Proof. We proceed by induction on |F|. We may assume that $|F| \ge 1$. Let v be a vertex of F. By Corollary 2.10, there is a graph $G_1 \in \{G \setminus v, G * v \setminus v, G/v\}$ such that $\kappa_{G_1}(Q, R) = k$ and $\kappa_{G_1}(S, T) = \ell$. By the induction hypothesis, there is a vertex-minor H of G_1 such that $V(H) = V(G_1) - (F - \{v\}) = V(G) - F$, $\kappa_H(Q, R) = \kappa_{G_1}(Q, R) = k$, and $\kappa_H(S, T) = \kappa_{G_1}(S, T) = \ell$. Therefore, the conclusion follows since H is a vertex-minor of G.

The following lemma is the analog of [3, Lemma 4.7].

Lemma 2.12. Let G be a graph and S and T be disjoint subsets of V(G). Then there exist $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_G(S_1, T_1) = \kappa_G(S, T)$.

Proof. By Lemma 2.9, there exists a matroid M_1 on V(G) - T whose rank function is $\kappa_G(X,T)$ for each subset X of V(G) - T. Let S_1 be a maximal independent set of M_1 contained in S. Then we have $|S_1| = \kappa_G(S_1,T) = \kappa_G(S,T)$. By Lemma 2.9, there is a matroid M_2 on $V(G) - S_1$ whose rank function is $\kappa_G(X,S_1)$ for every subset X of $V(G) - S_1$. Let T_1 be a maximal independent set of M_2 contained in T. Then $|T_1| = \kappa_G(T_1,S_1) = \kappa_G(T,S_1)$ and so we finish the proof.

3 Lemmas on the cut-rank function.

In this section, we present simple lemmas on the cut-rank function. A subset X of V(G) is an (S,T)-separating set of order k in G if $S \subseteq X \subseteq V(G) - T$ and $\rho_G(X) = k$.

For a graph G and disjoint subsets S, T of V(G), let $\tilde{\sqcap}_G[S,T] = \frac{1}{2}(\rho_G(S) + \rho_G(T) - \rho_G(S \cup T)).$

Lemma 3.1. Let G be a graph and S and T be disjoint subsets of V(G). If A and B are (S,T)-separating sets of order $k := \kappa_G(S,T)$ in G, then both $A \cap B$ and $A \cup B$ are (S,T)-separating sets of order k in G.

Proof. Since both $A \cap B$ and $A \cup B$ are (S,T)-separating sets, $\rho_G(A \cap B) \ge k$ and $\rho_G(A \cup B) \ge k$. By Lemma 2.5,

$$2k = \rho_G(A) + \rho_G(B) \ge \rho_G(A \cup B) + \rho_G(A \cap B) \ge 2k$$

and therefore $\rho_G(A \cup B) = \rho_G(A \cap B) = k$.

Lemma 3.2. Let G be a graph and S and T be disjoint subsets of V(G) such that $\rho_G(S) = \kappa_G(S,T)$. Let U be a subset of S. Let v be a vertex in $V(G) - (S \cup T)$. If $\kappa_{G\setminus v}(U,T) < \kappa_G(U,T)$, then $\kappa_{G\setminus v}(S,T) < \kappa_G(S,T)$.

Proof. Let $k = \rho_G(S) = \kappa_G(S, T)$. Suppose that $\kappa_{G\setminus v}(S, T) = k$. Let X be a (U, T)-separating set in $G \setminus v$. By (S2) of Lemma 2.6,

$$\rho_{G\setminus v}(X) + \rho_G(S) \ge \rho_G(X \cap S) + \rho_{G\setminus v}(X \cup S)$$

and since $X \cup S$ is (S, T)-separating in $G \setminus v$, we have $\rho_{G \setminus v}(X \cup S) \ge k = \rho_G(S)$. Hence, we deduce that $\rho_{G \setminus v}(X) \ge \rho_G(X \cap S) \ge \kappa_G(U, T)$. So $\kappa_{G \setminus v}(U, T) \ge \kappa_G(U, T)$, contradicting the assumption. \Box

Lemma 3.3. Let G be a graph and X_2 and Y be disjoint subsets of V(G). Let X_1 be a subset of X_2 . Then $\tilde{\sqcap}_G[X_1, Y] \leq \tilde{\sqcap}_G[X_2, Y]$.

Proof. Since $X_1 \subseteq X_2$, by Lemma 2.5, we have

$$\rho_G(X_2) + \rho_G(X_1 \cup Y) \ge \rho_G(X_2 \cup (X_1 \cup Y)) + \rho_G(X_2 \cap (X_1 \cup Y))$$

= $\rho_G(X_2 \cup Y) + \rho_G(X_1).$

Hence, $2\tilde{\sqcap}_G(X_1, Y) = \rho_G(X_1) + \rho_G(Y) - \rho_G(X_1 \cup Y) \le \rho_G(X_2) + \rho_G(Y) - \rho_G(X_2 \cup Y) = 2\tilde{\sqcap}_G(X_2, Y).$

Lemma 3.4. Let G be a graph and Q and R be disjoint subsets of V(G) such that $\rho_G(Q) = \kappa_G(Q, R)$. Let v be a vertex of $V(G) - (Q \cup R)$ such that $\kappa_{G \setminus v}(Q, R) < \kappa_G(Q, R)$. Then the following hold:

(Q1) $\rho_G(Q \cup \{v\}) \ge \rho_G(Q).$

(Q2) If $\rho_{G\setminus v}(Q) = \rho_G(Q)$, then $\rho_G(Q \cup \{v\}) = \rho_G(Q) + 1$.

Proof. (Q1) holds clearly since $\rho_G(Q) = \kappa_G(Q, R)$.

To prove (Q2), let $k = \kappa_G(Q, R)$. Since $\kappa_{G\setminus v}(Q, R) < k$, there is a subset X of V(G) such that $Q \subseteq X \subseteq V(G) - (R \cup \{v\})$ and $\rho_{G\setminus v}(X) \leq k - 1$. Then $\rho_{G\setminus v}(X) < k \leq \rho_G(X \cup \{v\})$ because $Q \subseteq X \cup \{v\} \subseteq V(G) - R$ and by (S1) of Lemma 2.6, we have that

$$\rho_{G\setminus v}(X) + \rho_G(Q \cup \{v\}) \ge \rho_{G\setminus v}(Q) + \rho_G(X \cup \{v\}) > \rho_{G\setminus v}(Q) + \rho_{G\setminus v}(X).$$

Hence, by Lemma 2.7, $\rho_G(Q \cup \{v\}) = \rho_{G \setminus v}(Q) + 1 = \rho_G(Q) + 1$.

 \Box

4 Proof of Theorem 1.2

For disjoint subsets S and T of vertices of a graph G, a vertex $v \in V(G) - (S \cup T)$ is (S, T)-flexible if $\kappa_{G\setminus v}(S,T) = \kappa_{G*v\setminus v}(S,T) = \kappa_{G\wedge uv\setminus v}(S,T) = \kappa_G(S,T)$ for each $u \in N_G(v)$. Note that every isolated vertex is (S,T)-flexible.

Lemma 4.1. Let S, T be disjoint sets of vertices of a graph G. If a vertex v is (S,T)-flexible in G, then it is (S,T)-flexible in every graph locally equivalent to G.

Proof. Let G' be a graph locally equivalent to G. Let $k = \kappa_G(S,T)$, $G_1 = G \setminus v$, $G_2 = G * v \setminus v$, and $G_3 = G/v$. Since v is (S,T)-flexible in G, we have $\kappa_{G_1}(S,T) = \kappa_{G_2}(S,T) = \kappa_{G_3}(S,T) = k$. Let $H_1 = G' \setminus v$, $H_2 = G' * v \setminus v$, and $H_3 = G'/v$. Then by Lemma 2.3, there is a permutation $\sigma : \{1,2,3\} \rightarrow \{1,2,3\}$ such that H_i is locally equivalent to $G_{\sigma(i)}$ for each $i \in \{1,2,3\}$. Hence, by Lemma 2.4, we have $\kappa_{H_i}(S,T) = \kappa_{G_{\sigma(i)}}(S,T) = k$ for each $i \in \{1,2,3\}$. Therefore, v is (S,T)-flexible in G'.

The following lemma finds a nested set of (S, T)-separating sets of order $\kappa_G(S, T)$ for disjoint sets S and T of vertices of a graph G.

Lemma 4.2. Let G be a graph and S and T be disjoint subsets of V(G). Let $k = \kappa_G(S,T)$ and $F \subseteq V(G) - (S \cup T)$ be a set of n vertices which are not (S,T)-flexible. Then there exist an ordering f_1, \ldots, f_n of vertices in F and a sequence A_1, \ldots, A_n of (S,T)-separating sets of order k in G such that the following hold:

- (i) $A_i \subseteq A_{i+1}$ for each $1 \le i \le n-1$.
- (ii) $A_i \cap F = \{f_1, \dots, f_i\}$ for each $1 \le i \le n$.

Proof. We prove by induction on n = |F|. We may assume that $n \ge 1$. We first claim that for every $v \in F$, there exists an (S,T)-separating set of order k in G containing v. Since v is not (S,T)-flexible in G, there exists a graph $G' \in \{G \setminus v, G * v \setminus v, G/v\}$ such that $\kappa_{G'}(S,T) < \kappa_G(S,T)$. So there is a subset A of $V(G) - \{v\}$ such that $S \subseteq A \subseteq V(G) - (T \cup \{v\})$ and $\rho_{G'}(A) \le k - 1$. There exists a graph H locally equivalent to G such that $H \setminus v = G'$. Therefore, since $S \subseteq A \cup \{v\} \subseteq V(G) - T$, by Lemmas 2.4 and 2.7, we have $k \le \rho_G(A \cup \{v\}) = \rho_H(A \cup \{v\}) \le \rho_{H\setminus v}(A) + 1 = \rho_{G'}(A) + 1 \le k$ and so $\rho_G(A \cup \{v\}) = k$. Now it follows that $A \cup \{v\}$ is an (S,T)-separating set of order k in G containing v.

For each $u \in F$, let A_u be an (S, T)-separating set of order k in G containing u such that $|A_u|$ is minimum. Let x be a vertex of F such that $|A_x| \leq |A_u|$ for each $u \in F$.

Now we claim that $A_x \cap F = \{x\}$. Suppose that there exists an element $y \in (A_x - \{x\}) \cap F$. Then, by Lemma 3.1, both $A_x \cap A_y$ and $A_x \cup A_y$ are (S,T)-separating sets of order k in G. Hence, $A_y \subseteq A_x$ by the choice of A_y . Then we have $A_x = A_y$ because $|A_x| \leq |A_u|$ for every $u \in F$. Since y is not (S,T)-flexible, there exists a graph $G'' \in \{G \setminus y, G * y \setminus y, G/y\}$ such that $\kappa_{G''}(S,T) < \kappa_G(S,T)$. By Lemma 2.4, we may assume that $G'' = G \setminus y$. Then there exists $S \subseteq X \subseteq V(G) - (T \cup \{y\})$ such that $\rho_{G \setminus y}(X) = k - 1$. By Lemma 2.7, $\rho_G(X) = k$ and $\rho_G(X \cup \{y\}) = k$. So $X \cup \{y\}$ is an (S,T)-separating set of order k in G containing y. By Lemma 3.1, $A_y \cap (X \cup \{y\})$ is an (S,T)-separating set of order kin G. Therefore, by the choice of A_y , we have $A_y \subseteq X \cup \{y\}$ and so $A_y - \{y\} \subseteq X$. By applying (S1) of Lemma 2.6,

$$2k - 1 = \rho_{G \setminus y}(X) + \rho_G(A_y) = \rho_{G \setminus y}(X) + \rho_G((A_y - \{y\}) \cup \{y\})$$

$$\geq \rho_{G \setminus y}(X \cap (A_y - \{y\})) + \rho_G(X \cup (A_y - \{y\}) \cup \{y\})$$

$$= \rho_{G \setminus y}(A_y - \{y\}) + \rho_G(X \cup \{y\}).$$

Since $\rho_G(X \cup \{y\}) = k$, we know that $\rho_{G\setminus y}(A_y - \{y\}) \leq k - 1$ and so $\rho_G(A_y - \{y\}) \leq k$ by Lemma 2.7. Recall that $S \subseteq A_y - \{y\} \subseteq V(G) - T$ and $k = \kappa_G(S, T)$. Therefore, $\rho_G(A_y - \{y\}) = k$. Since $A_x = A_y$, this is a contradiction to the minimality of A_x . Thus $A_x \cap F = \{x\}$.

Let $f_1 = x$ and $A_1 = A_x$. Then $k = \kappa_G(S,T) \le \kappa_G(A_1,T) \le \rho_G(A_1) = k$ and therefore we have that $\kappa_G(A_1,T) = k$. By Lemmas 2.4 and 3.2, no vertex of $F - \{f_1\}$ is (A_1,T) -flexible. Hence, by the induction hypothesis, there exist an ordering f_2, \ldots, f_n of elements of $F - \{f_1\}$ and a sequence A_2, \ldots, A_n of (A_1, T) -separating sets of order k in G such that (i) and (ii) hold.

So we finish the proof with the fact that A_2, \ldots, A_n are also (S, T)-separating sets of order k in G.

Our proof of Theorem 1.2 consists of two parts. In the first part, we will assume that S and T are small and prove the theorem. In the second part, we will show how to reduce the size of S and T. The following lemma will be used at the key step in the first part.

Lemma 4.3. Let G be a graph and Q, R, S, and T be subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$ and $S \cup T \subseteq Q \cup R$. Let $F = V(G) - (Q \cup R) \neq \emptyset$ and $k = \kappa_G(Q, R)$ and $\ell = \kappa_G(S, T)$. If $\rho_G(Q) = \rho_G(R) = k$ and no vertex of F is (Q, R)-flexible or (S, T)-flexible, then (1) or (2) holds:

- (1) There exists a vertex v of F such that at least two of the following hold:
 - (i) $\kappa_{G\setminus v}(Q, R) = k$ and $\kappa_{G\setminus v}(S, T) = \ell$.
 - (ii) $\kappa_{G*v\setminus v}(Q,R) = k \text{ and } \kappa_{G*v\setminus v}(S,T) = \ell.$
 - (iii) $\kappa_{G \wedge uv \setminus v}(Q, R) = k$ and $\kappa_{G \wedge uv \setminus v}(S, T) = \ell$ for each $u \in N_G(v)$.
- (2) There exist disjoint subsets Q' and R' of V(G) such that the following hold:
 - (i) $Q \subseteq Q', R \subseteq R'$ and $\rho_G(Q') = \rho_G(R') = k$.
 - (ii) $\tilde{\sqcap}_G[Q', R'] \ge \tilde{\sqcap}_G[Q, R] + \frac{1}{2}$.
 - (iii) $|V(G) (Q' \cup R')| \ge \lfloor \frac{1}{2} |F| \rfloor$.

Proof. Assume that (1) does not hold. Let n = |F|. Since no vertex of F is (Q, R)-flexible, by Lemma 4.2, there exists an ordering f_1, \ldots, f_n of vertices of F such that $Q \cup \{f_1, \ldots, f_i\}$ is a (Q, R)-separating set of order k in G for each $i \in \{1, \ldots, n\}$. Let $A_i = Q \cup \{f_1, \ldots, f_i\}$ for each $1 \le i \le n$.

No vertex of F is (S,T)-flexible and so, by Lemma 4.2, there exist a vertex g in F and an (S,T)seperating set C of order ℓ in G such that $C - (Q \cup R) = \{g\}$.

By Theorem 1.1, there are graphs $G'_1, G'_2 \in \{G \setminus g, G * g \setminus g, G/g\}$ such that $\kappa_{G'_i}(S,T) = \kappa_G(S,T)$ for $i \in \{1,2\}$. Since (1) does not hold, there exists $G' \in \{G'_1, G'_2\}$ such that $\kappa_{G'}(Q,R) < \kappa_G(Q,R)$. Then by Lemma 2.4, we may assume that $G' = G \setminus g$.

Since $\kappa_{G\setminus g}(S,T) = \ell$ and $S \subseteq C - \{g\} \subseteq V(G\setminus g) - T$, we have $\ell \leq \rho_{G\setminus g}(C - \{g\}) \leq \rho_G(C) = \ell$ and therefore $\rho_{G\setminus g}(C - \{g\}) = \rho_G(C)$. Since $C - \{g\} \subseteq Q \cup R$, by (S1) of Lemma 2.6,

$$\rho_{G\backslash g}(Q\cup R) + \rho_G(C) \ge \rho_{G\backslash g}((Q\cup R)\cap C) + \rho_G((Q\cup R)\cup C)$$
$$= \rho_{G\backslash g}(C - \{g\}) + \rho_G(Q\cup R\cup \{g\}).$$

Hence $\rho_G(Q \cup R \cup \{g\}) \leq \rho_{G \setminus g}(Q \cup R)$ because $\rho_{G \setminus g}(C - \{g\}) = \rho_G(C)$. By Lemma 2.7, $\rho_{G \setminus g}(Q \cup R) \leq \rho_G(Q \cup R \cup \{g\})$ and therefore $\rho_{G \setminus g}(Q \cup R) = \rho_G(Q \cup R \cup \{g\})$.

Now we claim that $\tilde{\sqcap}_G(Q \cup \{g\}, R) \geq \tilde{\sqcap}_G(Q, R) + \frac{1}{2}$. Observe that it is equivalent to show that

$$\rho_G(Q \cup \{g\}) + \rho_G(R) - \rho_G(Q \cup R \cup \{g\}) \ge \rho_G(Q) + \rho_G(R) - \rho_G(Q \cup R) + 1.$$

We have $\rho_G(Q \cup R) \ge \rho_{G\backslash g}(Q \cup R) = \rho_G(Q \cup R \cup \{g\})$ and, by (Q1) of Lemma 3.4, $\rho_G(Q \cup \{g\}) \ge \rho_G(Q)$. Therefore, it is enough to prove that $\rho_G(Q \cup R) \ge \rho_G(Q \cup R \cup \{g\}) + 1$ or $\rho_G(Q \cup \{g\}) \ge \rho_G(Q) + 1$. Suppose that $\rho_G(Q \cup R) = \rho_G(Q \cup R \cup \{g\}) = \rho_{G\backslash g}(Q \cup R)$. Then, by (S2) of Lemma 2.6, we have

$$\rho_{G\setminus g}(Q) + \rho_G(Q \cup R) \ge \rho_{G\setminus g}(Q \cup R) + \rho_G(Q).$$

So $\rho_{G\setminus g}(Q) \ge \rho_G(Q)$ and we have $\rho_{G\setminus g}(Q) = \rho_G(Q)$ by Lemma 2.7. Then by (Q2) of Lemma 3.4, $\rho_G(Q \cup \{g\}) = \rho_G(Q) + 1$, proving the claim.

Similarly, we have $\tilde{\sqcap}_G(Q, R \cup \{g\}) \geq \tilde{\sqcap}_G(Q, R) + \frac{1}{2}$. Let *i* be an integer such that $f_i = g$ and let

$$(Q', R') = \begin{cases} (A_i, R) & \text{if } i \le \lfloor \frac{n}{2} \rfloor, \\ (Q, V(G) - A_{i-1}) & \text{otherwise.} \end{cases}$$

Then by Lemma 3.3,

$$\tilde{\sqcap}_G(Q',R') \ge \min\left(\tilde{\sqcap}_G(Q \cup \{g\},R), \tilde{\sqcap}_G(Q,R \cup \{g\})\right) \ge \tilde{\sqcap}_G(Q,R) + \frac{1}{2}.$$

So (ii) holds and (i) and (iii) hold by the construction.

Now we are ready to prove Theorem 1.2 when S and T are small.

Proposition 4.4. Let G be a graph and Q, R, S, and T be subsets of V(G) such that $Q \cap R = S \cap T = \emptyset$ and $F = V(G) - (Q \cup R \cup S \cup T)$. Let $k = \kappa_G(Q, R)$ and $\ell = \kappa_G(S, T)$. If $|S| = |T| = \ell$ and $|F| \ge (2\ell + 1)2^{2k}$, then there is a vertex $v \in F$ such that at least two of the following hold:

- (1) $\kappa_{G\setminus v}(Q, R) = k \text{ and } \kappa_{G\setminus v}(S, T) = \ell.$
- (2) $\kappa_{G*v \setminus v}(Q, R) = k \text{ and } \kappa_{G*v \setminus v}(S, T) = \ell.$
- (3) $\kappa_{G \wedge uv \setminus v}(Q, R) = k$ and $\kappa_{G \wedge uv \setminus v}(S, T) = \ell$ for every neighbor u of v.

Proof. If F has a vertex which is (S,T)-flexible or (Q,R)-flexible, then our conclusion follows by Theorem 1.1. So we can assume that no vertex of F is (S,T)-flexible or (Q,R)-flexible. Let n = |F|.

By Lemma 4.2, there exist an ordering f_1, \ldots, f_n of vertices of F and a sequence A_1, \ldots, A_n of (Q, R)-separating sets of order k in G satisfying the following:

- $A_i \subseteq A_{i+1}$ for each $1 \le i \le n-1$.
- $A_i \cap F = \{f_1, \ldots, f_i\}$ for each $1 \le i \le n$.

For each $1 \leq i \leq n$, let $B_i = V(G) - A_i$. Let $q = 2^{2k}$ and $A_0 = Q$. For $1 \leq i \leq 2\ell + 1$, let $X_i = A_{iq} - A_{(i-1)q}$. Since $|S| = |T| = \ell$, there exists $1 \leq m \leq 2\ell + 1$ such that $X_m \cap (S \cup T) = \emptyset$. Let j = (m-1)q. Then we have $Q \cup R \cup S \cup T \subseteq A_j \cup B_{j+q}$.

Assume that our conclusion fails and so every vertex of F satisfies at most one of (1), (2), and (3). We claim that, for each $1 \le i \le 2k + 2$, there exist disjoint subsets Q_i and R_i of V(G) satisfying the following.

(i) $Q \subseteq Q_i, R \subseteq R_i$, and $\rho_G(Q_i) = \rho_G(R_i) = k$.

(ii)
$$\tilde{\sqcap}_G[Q_i, R_i] \geq \frac{i-1}{2}$$

(iii) $|V(G) - (Q_i \cup R_i)| \ge \lfloor 2^{2k+1-i} \rfloor.$

We proceed by the induction on *i*. Let $Q_1 = A_j$, $R_1 = B_{j+q}$, and $F_1 = V(G) - (Q_1 \cup R_1)$. Then $|F_1| = 2^{2k}$ and so (Q_1, R_1) satisfies the claim. Therefore we may assume that $i \ge 2$. By the induction hypothesis, there exist disjoint subsets Q_{i-1} and R_{i-1} of V(G) satisfying (i), (ii), and (iii) for i-1. By Lemmas 2.4 and 3.2, no vertex of $V(G) - (Q_{i-1} \cup R_{i-1})$ is (Q_{i-1}, R_{i-1}) -flexible. If there is a vertex v of $V(G) - (Q_{i-1} \cup R_{i-1})$ satisfying (1) of Lemma 4.3 for two pairs (Q_{i-1}, R_{i-1}) and (S, T), then by Lemmas 2.4 and 3.2, v satisfies at least two of (1), (2), and (3), contradicting our assumption. So we may assume that $V(G) - (Q_{i-1} \cup R_{i-1})$ has no such vertex. Hence, by Lemma 4.3, there exist disjoint subsets Q_i and R_i of V(G) such that the following hold:

(a) $Q_{i-1} \subseteq Q_i$, $R_{i-1} \subseteq R_i$ and $\rho_G(Q_i) = \rho_G(R_i) = k$.

(b)
$$\tilde{\sqcap}_G[Q_i, R_i] \ge \tilde{\sqcap}_G[Q_{i-1}, R_{i-1}] + \frac{1}{2} \ge \frac{i-2}{2} + \frac{1}{2} = \frac{i-1}{2}$$
.

(c) $|V(G) - (Q_i \cup R_i)| \ge \lfloor \frac{1}{2} |V(G) - (Q_{i-1} \cup R_{i-1})| \rfloor \ge \lfloor \frac{1}{2} \cdot 2^{2k+2-i} \rfloor = \lfloor 2^{2k+1-i} \rfloor.$

This proves our claim. Then by (ii) and Lemma 3.3, $k + \frac{1}{2} \leq \tilde{\sqcap}_G(Q_{2k+2}, R_{2k+2}) \leq \tilde{\sqcap}_G(Q_{2k+2}, V(G) - Q_{2k+2}) = \rho_G(Q_{2k+2}) = k$, which is a contradiction. Therefore our conclusion holds.

Now we are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.12, there exist $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_G(S_1, T_1) = \kappa_G(S, T)$. Let $X = (S \cup T) - (Q \cup R \cup S_1 \cup T_1)$. By Corollary 2.11, there is a vertexminor H of G such that V(H) = V(G) - X, $\kappa_H(Q, R) = k$, and $\kappa_H(S_1, T_1) = \ell$.

For a vertex v of $V(H) - (Q \cup R \cup S_1 \cup T_1)$, let $H_1^v = H \setminus v$, $H_2^v = H * v \setminus v$, and $H_3^v = H/v$ and let $G_1^v = G \setminus v$, $G_2^v = G * v \setminus v$, and $G_3^v = G/v$. Then by Lemma 2.3, there exists a permutation $\sigma_v : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ such that H_i^v is a vertex-minor of $G_{\sigma(i)}^v$ for each $i \in \{1, 2, 3\}$. By Lemma 2.8, $\kappa_{H_i^v}(S_1, T_1) \leq \kappa_{G_{\sigma(i)}^v}(S_1, T_1) \leq \kappa_{G_{\sigma(i)}^v}(S, T) \leq \kappa_G(S, T) = \ell$ and $\kappa_{H_i^v}(Q, R) \leq \kappa_{G_{\sigma(i)}^v}(Q, R) \leq \kappa_G(Q, R) = k$ for each $i \in \{1, 2, 3\}$.

Since $|V(H) - (Q \cup R \cup S_1 \cup T_1)| = |F| \ge (2\ell + 1)2^{2k}$, by Proposition 4.4, there exist a vertex v of $V(H) - (Q \cup R \cup S_1 \cup T_1) = F$ and $i, j \in \{1, 2, 3\}$ such that $i \ne j$ and $\kappa_{H_i^v}(Q, R) = \kappa_{H_j^v}(Q, R) = k$ and $\kappa_{H_i^v}(S_1, T_1) = \kappa_{H_j^v}(S_1, T_1) = \ell$. Therefore, $\kappa_{G_{\sigma(i)}^v}(S, T) = \kappa_{G_{\sigma(j)}^v}(S, T) = \ell$ and $\kappa_{G_{\sigma(i)}^v}(Q, R) = k$.

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