Parabolic optimal control problems with combinatorial switching constraints Part I: Convex relaxations*

Christoph Buchheim, Alexandra Grütering and Christian Meyer[†] Department of Mathematics, TU Dortmund University, Germany

We consider optimal control problems for partial differential equations where the controls take binary values but vary over the time horizon, they can thus be seen as dynamic switches. The switching patterns may be subject to combinatorial constraints such as, e.g., an upper bound on the total number of switchings or a lower bound on the time between two switchings. While such combinatorial constraints are often seen as an additional complication that is treated in a heuristic postprocessing, the core of our approach is to investigate the convex hull of all feasible switching patterns in order to define a tight convex relaxation of the control problem. The convex relaxation is built by cutting planes derived from finite-dimensional projections, which can be studied by means of polyhedral combinatorics. A numerical example for the case of a bounded number of switchings shows that our approach can significantly improve the dual bounds given by the straightforward continuous relaxation, which is obtained by relaxing binarity constraints.

 $\mathbf{Keywords.}$ PDE-constrained optimization, switching time optimization, convex relaxations

1 Introduction

Mixed-integer optimal control of a system governed by partial or ordinary differential equations became a hot research topic in the last decade, as a variety of applications leads to such control problems. In particular, the control often comes in form of a finite set of switches which can be operated within a given continuous time horizon, e.g., by shifting of gear-switches in automotive engineering [22, 33, 45] or by switching of valves or compressors in gas and water networks [20, 27]. Consequently, various approaches are discussed in the literature to address optimal control problems with discrete control variables, often known as mixed-integer optimal control problems (MIOCPs). Direct methods, based on the firstdiscretize-then-optimize paradigm, are widely used to tackle MIOCPs; see for instance [22] and [52]. The control and, if desired, the state are discretized in time and space, in order to approximate the problem by a large, typically non-convex, finite-dimensional mixed-integer nonlinear programming problem (MINLP). The latter can be addressed by standard techniques; see [35] or [4] for surveys on algorithms for MINLPs. However, the size of the arising MINLPs easily becomes too large to solve them to proven optimality. In particular, direct methods are not promising for optimal control problems governed by partial differential equations (PDEs) [21, 46].

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^{†{}christoph.buchheim,alexandra.gruetering,christian.meyer}@math.tu-dortmund.de

In contrast, arbitrary close approximations of MIOCPs can be computed efficiently by first replacing the set of discrete control values by its convex hull and then appropriately rounding the result. The most common approximation methods for systems governed by ordinary differential equations (ODEs) are the Sum-Up Rounding strategy [41, 32] and the Next Force Rounding strategy [30]. PDE-constrained problems can also be addressed with the Sum-Up Ronding strategy [28]. However, in the presence of additional combinatorial constraints, the latter may be violated [36, Sect. 5.4], and the heuristics used to obtain feasible solutions often do not perform well [37, Example 3.2]. Therefore, when aiming at globally optimal solutions, such approaches may only serve for computing primal bounds. To minimize the integrality error, the Combinatorial Integral Approximation (CIA) [40] tracks the average of a relaxed solution over a given rounding grid by a piecewise constant integer control and the discretized problem is solved by a tailored branch-and-bound algorithm [31, 42]. The approach was again generalized to PDE-constrained problems [26]. To reduce the (undesired) chattering behavior of the rounded control the total variation is constrained [43] or switching cost aware rounding algorithms are considered [5, 1].

Other approaches optimize the switching times, e.g., by controlling the switching times through a continuous time control function which scales the length of minor time intervals [23, 38] or by including a fixed number of transition times as decision variables into the MIOCP and solving the corresponding finite-dimensional non-convex problems by gradient descent techniques [47, 18] or by second order methods [29, 48]. PDE-constrained optimal control problems can be addressed by the concept of switching time optimization as well [39]. Nevertheless, these methods have a limited applicability, since fairly restrictive assumptions on the objective and the state dynamics need to be made in order to guarantee differentiability in the discretized setting [19].

In the context of optimal control problems governed by PDEs, switching constraints are frequently imposed by penalty terms added to the objective functional [10, 15, 14]. The arising penalized problems are non-convex in general and are therefore convexified by means of the bi-conjugate functional associated with the penalty term. The desired switching structure of the optimal solutions of the convexified problems can however only be guaranteed under additional structural assumptions on the unknown solution. For the case of a switching between multiple constant control variables, a multi-bang approach might be favorable since optimal control problems subject to box constraints on the control may show a bang-bang behavior in the absence of a Tikhonov-type regularization term [49, 17, 9, 51]. However, the bang-bang structure of the optimal control cannot be guaranteed in general. In order to promote that the control attains the desired constant values, L^0 -penalty terms or suitable indicator functionals are added to the objective and convex relaxations of the penalty terms based on the bi-conjugate functional are employed to make the problem amenable for optimization algorithms [12, 16]. Again, as in case of the penalization of the switching constraints mentioned above, the multi-bang structure of the optimal solutions of the convexified problems can only be ensured under additional assumptions that cannot be verified a priori. In [13], the convexification of the L^0 -penalty by means of the bi-conjugate functional is employed in the context of topology optimization, in [11], the L^0 -penalty is enriched by the BV-seminorm. L^0 -penalization techniques that go without regularization or convexification are for instance addressed in [8] from a theoretical perspective and in [53] with regard to algorithms. However, to the best of our knowledge, additional combinatorial constraints on the switching structure have not yet been included in the penalization framework.

In summary, the design of global solvers for MIOCPs with dynamic switches and combinatorial switching constraints is an open field of research. The core of our new approach for addressing such problems is the computation of lower bounds by a tailored convexification of the set of feasible switching patterns in function space. A counterexample given in Section 3 shows that, even when the combinatorial constraint only consists in an upper bound on the total number of switchings, the naive approach of just relaxing the binarity

constraint does not lead to the convex hull of the set of feasible switching patterns. Our aim is to determine tighter approximations of this convex hull by considering finite-dimensional projections that allow for the efficient computation of cutting planes. Based on the resulting outer description of the convex hull, in the companion paper [6] we develop a tailored outer approximation algorithm which converges to a global minimizer of the convex relaxations. The resulting lower bounds could be used, e.g., in a branch-and-bound scheme to obtain globally optimal solutions of the control problems.

The remainder of this paper is organized as follows. In Section 2, we specify the prototypical optimal control problem as well as the class of combinatorial switching constraints considered in this work and show that the problem admits an optimal solution. In Section 3, we investigate the convex hull of feasible switching patterns and show that it can be fully described by cutting planes lifted from finite-dimensional projections. An example in Section 4 shows the strength of the lower bounds resulting from our tailored convexification.

2 Optimal control problem

For the sake of simplicity, throughout this paper, we restrict ourselves to a parabolic binary optimal control problem with switching constraints of the following form:

$$\begin{cases} & \min \quad J(y,u) = \frac{1}{2} \, \|y - y_{\rm d}\|_{L^2(Q)}^2 + \frac{\alpha}{2} \, \|u - \frac{1}{2}\|_{L^2(0,T;\mathbb{R}^n)}^2 \\ & \text{s.t.} \quad \partial_t y(t,x) - \Delta y(t,x) = \sum_{j=1}^n u_j(t) \, \psi_j(x) \quad \text{in } Q := \Omega \times (0,T), \\ & y(t,x) = 0 \qquad \qquad \text{on } \Gamma := \partial \Omega \times (0,T), \\ & y(0,x) = y_0(x) \qquad \qquad \text{in } \Omega, \end{cases}$$

Herein, T > 0 is a given final time and $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, denotes a bounded domain, where a domain is an open and connected subset of a finite-dimensional vector space, with Lipschitz boundary $\partial\Omega$ in the sense of [24, Def. 1.2.2.1]. The form functions $\psi_j \in H^{-1}(\Omega)$, $j = 1, \ldots, n$, as well as the initial state $y_0 \in L^2(\Omega)$ are given. Moreover,

$$D \subset \{u \in BV(0,T;\mathbb{R}^n) : u(t) \in \{0,1\}^n \text{ f.a.a. } t \in (0,T)\}$$

denotes the set of feasible switching controls. Finally, $y_d \in L^2(Q)$ is a given desired state and $\alpha \geq 0$ is a Tikhonov parameter weighting the mean deviation from $\frac{1}{2}$. Note that the choice of α does not have any impact on the set of optimal solutions of (P), as $u \in \{0,1\}^n$ a.e. in (0,T) and hence the Tikhonov term is constant. However, the convex relaxations of (P) considered in this paper as well as their optimal values are influenced by α .

The particular challenge of our problem are the combinatorial switching constraints modeled by the set D of feasible controls. It is supposed to satisfy the two following assumptions:

(D1)
$$D$$
 is a bounded set in $BV(0,T;\mathbb{R}^n)$,

(D2)
$$D$$
 is closed in $L^p(0,T;\mathbb{R}^n)$ for some fixed $p \in [1,\infty)$.

Here, $BV(0,T;\mathbb{R}^n)$ denotes the set of all vector-valued functions with bounded variation, i.e.,

$$BV(0,T;\mathbb{R}^n) := \{ u \in L^1(0,T;\mathbb{R}^n) : u_i \in BV(0,T) \text{ for } i = 1,\dots,n \}$$

equipped with the norm

$$||u||_{BV(0,T;\mathbb{R}^n)} := ||u||_{L^1(0,T;\mathbb{R}^n)} + \sum_{j=1}^n |u_j|_{BV(0,T)}.$$

For more details on the space of bounded variation functions, see, e.g., [3, Chap. 10]. Note that, in our case, the BV-seminorm $|u_j|_{BV(0,T)}$ agrees with the minimal number of switchings of any representative of u_j with values in $\{0,1\}$.

A possible example for such a set is

(2.1)
$$D_{\max} := \left\{ u \in BV(0, T; \mathbb{R}^n) : \ u(t) \in \{0, 1\}^n \text{ f.a.a. } t \in (0, T), \\ |u_j|_{BV(0, T)} \le \sigma_{\max} \ \forall j = 1, \dots, n \right\},$$

where $|\cdot|_{BV(0,T)}$ denotes the BV-seminorm and $\sigma_{\max} \in \mathbb{N}$ is a given number. The set D_{\max} meets the assumptions (D1) and (D2), as we will show in Example 3.1. This choice of D is motivated by the following application-driven scenario: suppose y is the temperature of a body covering the domain Ω and the aim of the optimization is to minimize the deviation of y from a given desired state y_d , by means of n given heat sources modeled by the form functions ψ_j , $j=1,\ldots,n$. These heat sources can be switched on and off at arbitrary points in time, but we are only allowed to shift each switch for at most σ_{\max} times. This leads to the set D_{\max} .

Various other practically relevant choices of D are conceivable. For instance, it could be required to bound the time interval between two shiftings of the same switch from below because of technical limitations; this kind of restriction is known as minimum dwell time constraints in the optimal control community and as min-up/min-down constraints in the unit commitment community. See Example 3.2 for a discussion and generalization of this class of constraints. Another condition may be that certain switches are not allowed to be used (or switched on) at the same time.

Our previous assumptions guarantee that the PDE contained in (P) admits a unique weak solution $y \in W(0,T) := H^1(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ for every $u \in D \subset L^2(0,T;\mathbb{R}^n)$; see [50, Chapter 3]. The associated solution operator $S \colon L^2(0,T;\mathbb{R}^n) \to W(0,T)$ is affine and continuous. Using this solution operator, the problem (P) can be written as

(P')
$$\begin{cases} & \min \ f(u) = J(Su, u) \\ & \text{s.t.} \ u \in D \ . \end{cases}$$

Note that the objective function $f \colon L^2(0,T;\mathbb{R}^n) \to \mathbb{R}$ is weakly lower semi-continuous because both $u \mapsto \|Su - y_{\mathrm{d}}\|_{L^2(Q)}^2$ and $u \mapsto \|u - \frac{1}{2}\|_{L^2(0,T;\mathbb{R}^n)}^2$ are convex and lower semi-continuous, thus weakly lower semi-continuous, and the solution operator S is affine and continuous, thus weakly continuous.

Theorem 2.1. Let $D \neq \emptyset$. Then Problem (P') admits a global minimizer.

Proof. Since $D \neq \emptyset$, we have $f^* := \inf_{u \in D} f(u) \in \mathbb{R} \cup \{-\infty\}$. Let $\{u^k\}_{k \in \mathbb{N}}$ in D be an infimal sequence with

$$\lim_{k \to \infty} f(u^k) = f^* .$$

We know that $\{u^k\}_{k\in\mathbb{N}}$ is a bounded sequence in $BV(0,T;\mathbb{R}^n)$, since D is a bounded set in $BV(0,T;\mathbb{R}^n)$ by assumption (D1), i.e.,

$$\sup_{k \in \mathbb{N}} \|u^k\|_{BV(0,T;\mathbb{R}^n)} = \sup_{k \in \mathbb{N}} \left(\|u^k\|_{L^1(0,T;\mathbb{R}^n)} + \sum_{j=1}^n |u_j^k|_{BV(0,T)} \right) < \infty.$$

By Theorem 10.1.3 and Theorem 10.1.4 in [3], $BV(0,T;\mathbb{R}^n)$ is compactly embedded in $L^p(0,T;\mathbb{R}^n)$, and hence there exists a strongly convergent subsequence, which we again denote by $\{u^k\}_{k\in\mathbb{N}}$, such that $u^k\to u^\star\in L^p(0,T;\mathbb{R}^n)$ for $k\to\infty$. Since D is closed in $L^p(0,T;\mathbb{R}^n)$ by condition (D2), we deduce that $u^\star\in D$. The weak lower semi-continuity of the objective function f leads to

$$f(u^*) \leq \liminf_{k \to \infty} f(u^k) = f^*$$
.

This implies $f^* > -\infty$ as well as the optimality of u^* for (P').

3 Convex hull description

The crucial ingredient of our approach is the outer description of the convex hull of the set D of feasible switching patterns by linear inequalities. In general, just replacing $\{0,1\}$ with [0,1] in the definition of D does not lead to the convex hull of D in any L^p -space. This is true even in the case of just one switch that can be changed at most once on the entire time horizon, i.e., if the feasible switching control is required to belong to

$$(3.1) D := \{ u \in BV(0,T) \colon \ u(t) \in \{0,1\} \text{ f.a.a. } t \in (0,T), |u|_{BV(0,T)} \le 1 \} \ .$$

Essentially, the naive approach does not consider the monotonicity of the switches in D, as we will see in the following counterexample.

Counterexample 3.1. Let D be defined as in (3.1) and consider the function

$$u(t) := \begin{cases} \frac{1}{2} & \text{if } t \in \left[\frac{1}{3}T, \frac{2}{3}T\right] \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, we have $u \in BV(0,T)$ with $u(t) \in [0,1]$ for $t \in (0,T)$ and $|u|_{BV(0,T)} = 1$. However, we claim that u does not belong to the closed convex hull of D in $L^p(0,T)$ for any $p \in [1,\infty)$.

Assume on contrary that $u \in \overline{\operatorname{conv}(D)}^{L^p(0,T)}$ for some $p \in [1,\infty)$. Then there exists a sequence $\{u^k\}_{k \in \mathbb{N}} \subset \operatorname{conv}(D)$ with $u^k \to u$ in $L^p(0,T)$ for $k \to \infty$. In particular, $\{u^k\}_{k \in \mathbb{N}}$ converges strongly to u in $L^1(0,T)$ due to $L^p(0,T) \hookrightarrow L^1(0,T)$, i.e.,

$$\int_0^T |u^k - u| \, \mathrm{d}t \to 0 \text{ for } k \to \infty.$$

Define $A^k := \{t \in [\frac{1}{3}T, \frac{2}{3}T] : u^k(t) \geq \frac{2}{5}\}$. We claim that there exists $k_0 \in \mathbb{N}$ such that the sets A^k , $k \geq k_0$, have a positive Lebesgue-measure. Indeed, if such a $k_0 \in \mathbb{N}$ did not exist, then we could find a subsequence, which we denote by the same symbol $\{A^k\}_{k \in \mathbb{N}}$ for simplicity, such that $\lambda(A^k) = 0$ for all $k \in \mathbb{N}$, where $\lambda(A^k)$ denotes the Lebesgue measure of A^k . With $\lambda(A^k) = 0$, it follows

$$\int_0^T |u^k - u| \, \mathrm{d}t \ge \int_{\frac{1}{3}T}^{\frac{2}{3}T} |u^k - \frac{1}{2}| \, \mathrm{d}t = \int_{\left[\frac{1}{3}T, \frac{2}{3}T\right] \setminus A^k} |u^k - \frac{1}{2}| \, \mathrm{d}t > \frac{1}{30}T,$$

where the last inequality holds due to $|u^k - \frac{1}{2}| > \frac{1}{10}$ for all $t \in [\frac{1}{3}T, \frac{2}{3}T] \setminus A^k$ by definition of A^k . This contradicts the strong convergence of u^k to u in $L^1(0,T)$. Thus, a number $k_0 \in \mathbb{N}$ exists with $\lambda(A^k) > 0$ for all $k \geq k_0$.

Now, let $k \geq k_0$ be arbitrary. We write $u^k \in \text{conv}(D)$ as a convex combination

$$u^k = \sum_{l=1}^{m_k} \mu_l^k y_l^k$$

of functions in D. Let $t_0 \in A^k$ be a Lebesgue point of all functions $y_l^k \in D$, $1 \le l \le m_k$, which exists since the set of all non-Lebesgue points of y_l^k is a set of Lebesgue measure zero. Then, we know

$$\frac{2}{5} \le u^k(t_0) = \sum_{l=1}^{m_k} \mu_l^k y_l^k(t_0) .$$

Set $I_k := \{l \in \{1, \dots, m_k\} : y_l^k(t_0) = 1\}$. The inequality then implies

(3.2)
$$\frac{2}{5} \le \sum_{l \in L} \mu_l^k .$$

Since $y_l^k(t_0)=1$ for $l\in I_k$ and y_l^k might shift at most once due to $|y_l^k|_{BV(0,T)}\leq 1$, we deduce that either y_l^k was first turned off and then turned on in $(0,t_0)$, such that $y_l^k(t)\equiv 1$ a.e. in (t_0,T) holds, or y_l^k was first turned on, i.e., $y_l^k(t)\equiv 1$ a.e. in $(0,t_0)$. Consequently, we get $y_l^k(t)\equiv 1$ a.e. in $(0,\frac{1}{3}T)$ or $(\frac{2}{3}T,T)$ for every $l\in I_k$. The latter, together with (3.2), yields

$$\int_0^T |u^k - u| \, \mathrm{d}t \ge \int_{(0, \frac{1}{3}T) \cup (\frac{2}{3}T, T)} |u^k| \, \mathrm{d}t \ge \sum_{l \in I_k} \mu_l^k \int_{(0, \frac{1}{3}T) \cup (\frac{2}{3}T, T)} y_l^k \, \mathrm{d}t \ge \frac{2}{15}T \,,$$

which contradicts the strong convergence of u^k to u in $L^1(0,T)$.

This counterexample shows that we cannot expect to obtain a tight description of $\operatorname{conv}(D)$ without a closer investigation of the specific switching constraint under consideration. Our basic idea is to reduce this investigation to a purely combinatorial task by projecting the set D to finite-dimensional spaces \mathbb{R}^M , by means of $M \in \mathbb{N}$ linear and continuous functionals $\Phi_i \in L^p(0,T;\mathbb{R}^n)^*$, $i=1,\ldots,M$. In the following, we restrict ourselves to local averaging operators of the form

(3.3)
$$\langle \Phi_{(j-1)N+i}, u \rangle := \frac{1}{\lambda(I_i)} \int_{I_i} u_j \, \mathrm{d}t$$

for j = 1, ..., n with suitably chosen subintervals $I_i \subset (0, T)$, i = 1, ..., N, and M := nN. The resulting projection then reads

(3.4)
$$\Pi \colon BV(0,T;\mathbb{R}^n) \ni u \mapsto \left(\langle \Phi_l, u \rangle\right)_{l=1}^M \in \mathbb{R}^M.$$

Note that Π is a linear mapping. The core result underlying our approach is that, for increasing N, projections Π_N can be designed such that

(3.5)
$$\overline{\operatorname{conv}(D)}^{L^p(0,T;\mathbb{R}^n)} = \bigcap_{N \in \mathbb{N}} \{ v \in L^p(0,T;\mathbb{R}^n) \colon \Pi_N(v) \in C_{D,\Pi_N} \}$$

where

$$C_{D,\Pi} := \operatorname{conv}\{\Pi(u) \colon u \in D\} \subset \mathbb{R}^M$$
.

In other words, an outer description of all finite-dimensional convex hulls $C_{D,\Pi}$ also leads to an outer description of the convex hull of D in function space.

We first observe that our general assumptions (D1) and (D2) guarantee the closedness of the finite-dimensional set $C_{D,\Pi}$ in \mathbb{R}^M .

Lemma 3.2. For any Π as in (3.4), the set $C_{D,\Pi}$ is closed in \mathbb{R}^M .

Proof. Let $\{\Pi(u^k)\}_{k\in\mathbb{N}}\subset\mathbb{R}^M$ be a convergent sequence in $\Pi(D)$, resulting from the projection of feasible switching controls $u^k\in D$ for $k\in\mathbb{N}$, with $\Pi(u^k)\to\omega$ in \mathbb{R}^M . The sequence $\{u^k\}_{k\in\mathbb{N}}\subset D$ is bounded in $BV(0,T;\mathbb{R}^n)$ by (D1). As in Theorem 2.1, the compactness of the embedding $BV(0,T;\mathbb{R}^n)\hookrightarrow L^p(0,T;\mathbb{R}^n)$ by Theorem 10.1.3 and Theorem 10.1.4 in [3] implies the existence of a strongly convergent subsequence, again denoted by $\{u^k\}_{k\in\mathbb{N}}$, such that $u^k\to u\in L^p(0,T;\mathbb{R}^n)$ for $k\to\infty$. Since D is closed in $L^p(0,T;\mathbb{R}^n)$ by (D2), we deduce $u\in D$. By continuity of Π in $L^p(0,T;\mathbb{R}^n)$, we then have

$$\omega = \lim_{k \to \infty} \Pi(u^k) = \Pi(u)$$

so that ω lies in $\Pi(D)$. Hence, the set $\Pi(D)$ is closed in \mathbb{R}^M . It is also bounded, thus compact, such that $C_{D,\Pi}$ is closed as the convex hull of a compact set in \mathbb{R}^M .

As a consequence, we obtain that the subset of $L^p(0,T;\mathbb{R}^n)$ corresponding to the finite-dimensional projection Π is convex and closed in $L^p(0,T;\mathbb{R}^n)$.

Lemma 3.3. For any Π as in (3.4), the set $\{v \in L^p(0,T;\mathbb{R}^n): \Pi(v) \in C_{D,\Pi}\}$ is convex and closed in $L^p(0,T;\mathbb{R}^n)$.

Proof. The convexity assertion follows from the convexity of $C_{D,\Pi}$ together with the linearity of Π . Closedness follows from Lemma 3.2 and the continuity of Π in $L^p(0,T;\mathbb{R}^n)$.

By the following observation, each projection Π gives rise to a relaxation of the closed convex hull of D in $L^p(0,T;\mathbb{R}^n)$. These relaxations can be used to derive outer approximations by linear inequalities.

Lemma 3.4. For any Π as in (3.4), we have

$$\overline{\operatorname{conv}(D)}^{L^p(0,T;\mathbb{R}^n)} \subseteq \{ v \in L^p(0,T;\mathbb{R}^n) \colon \Pi(v) \in C_{D,\Pi} \} =: V \ .$$

Proof. By construction of $C_{D,\Pi}$, every $u \in D$ satisfies $\Pi(u) \in C_{D,\Pi}$. The linearity of Π leads to $conv(D) \subset V$, using the convexity of V stated in Lemma 3.3. Again by Lemma 3.3, the set V is closed in $L^p(0,T;\mathbb{R}^n)$, which shows the desired result.

The following result shows that the convex hull of the set of feasible switching controls can be fully described with the help of appropriate finite-dimensional sets $C_{D,\Pi}$. With a little abuse of notation, we slightly change the notation of the local averaging operators in the sense that the number of subintervals now differs from the dimension M of the range of Π , see (3.6) below, in order to ease the proof of the following theorem.

Theorem 3.5. For each $k \in \mathbb{N}$, let $I_1^k, \ldots, I_{N_k}^k$, $N_k \in \mathbb{N}$, be disjoint open intervals in (0,T) such that

- (i) $\bigcup_{i=1}^{N_k} \overline{I_i^k} = [0,T]$ for all $k \in \mathbb{N}$ and
- (ii) $\max_{i=1,...,N_k} \lambda(I_i^k) \to 0$ for $k \to \infty$.

Set $M_k := n N_k$ and define projections $\Pi_k : BV(0,T;\mathbb{R}^n) \to \mathbb{R}^{M_k}$, for $k \in \mathbb{N}$, by

(3.6)
$$\langle \Phi_{(j-1)N_k+i}^k, u \rangle := \frac{1}{\lambda(I_i^k)} \int_{I_i^k} u_j(t) dt$$

for j = 1, ..., n and $i = 1, ..., N_k$. Moreover, set

$$V_k := \{ v \in L^p(0, T; \mathbb{R}^n) : \Pi_k(v) \in C_{D,\Pi_k} \}$$
.

Then

(3.7)
$$\overline{\operatorname{conv}(D)}^{L^p(0,T;\mathbb{R}^n)} = \bigcap_{k \in \mathbb{N}} V_k.$$

Proof. The inclusion " \subseteq " in (3.7) follows directly from Lemma 3.4, it thus remains to show " \supseteq ". For this, let

$$u \in \bigcap_{k \in \mathbb{N}} V_k$$
.

By definition of u, we have $\Pi_k(u) \in C_{D,\Pi_k}$. Hence, there exist $v_l^k \in D$ for $l = 1, \ldots, m$, where $m = m(k) \in \mathbb{N}$ may depend on k, as well coefficients $\mu_l^k \geq 0$ with $\sum_{l=1}^m \mu_l^k = 1$ and

$$\Pi_k(u) = \sum_{l=1}^m \mu_l^k \, \Pi_k(v_l^k) \; .$$

Set $u^k := \sum_{l=1}^m \mu_l^k v_l^k \in \text{conv}(D)$. By construction and the linearity of the projection, we have $\Pi_k(u^k) = \Pi_k(u)$, i.e.,

(3.8)
$$\int_{I_i^k} (u^k - u) \, dt = 0 \quad \forall i = 1, \dots, N_k, \ k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ be fixed. Thanks to assumption (i), we conclude that for every $\ell \in \mathbb{N}$ it holds

$$\lambda \left(I_i^{\ell} \setminus \bigcup_{I_r^k \subset I_i^{\ell}} I_r^k \right) \le 2 \max_{r=1,\dots,N_k} \lambda(I_r^k) .$$

Set $E_i^l := \bigcup_{I_r^k \subset I_r^\ell} I_r^k$ for all $\ell \in \mathbb{N}$ and $i = 1, \dots, N_\ell$. Then (3.8) implies

$$\int_{I_i^{\ell}} (u^k - u) dt = \int_{I_i^{\ell} \setminus E_i^{\ell}} (u^k - u) dt + \int_{E_i^{\ell}} (u^k - u) dt$$
$$= \int_{I_i^{\ell} \setminus E_i^{\ell}} (u^k - u) dt$$

and thus

(3.9)
$$\left| \int_{I_i^{\ell}} (u^k - u) \, \mathrm{d}t \right| \le \int_{I_i^{\ell} \setminus E_i^{\ell}} |u^k - u| \, \mathrm{d}t \le \lambda (I_i^{\ell} \setminus E_i^{\ell})$$

$$\le 2 \max_{r=1}^{k} \lambda (I_r^{k}) \quad \forall i = 1, \dots, N_{\ell}, \ \ell \in \mathbb{N}$$

Since $u^k(t) \in [0,1]^n$ holds almost everywhere in (0,T), there exists a weakly convergent subsequence, which we denote by the same symbol for simplicity, with $u^k \to \tilde{u}$ in $L^p(0,T;\mathbb{R}^n)$. Together with (3.9) and $\max_{r=1,\ldots,N_k} \lambda(I_r^k) \to 0$ for $k \to \infty$, the weak convergence of $\{u^k\}_{k\in\mathbb{N}}$ to \tilde{u} implies

(3.10)
$$\int_{I^{\ell}} (\tilde{u} - u) \, dt = 0 \quad \forall i = 1, \dots, N_{\ell}, \ \ell \in \mathbb{N} .$$

It is well known that the span of the characteristic functions $\chi_{I_i^\ell}$, $i=1,\ldots,N_\ell$ $\ell\in\mathbb{N}$, is dense in $L^p(0,T)$, so that (3.10) immediately yields $u=\tilde{u}$ in $L^p(0,T;\mathbb{R}^n)$. We thus obtain $u^k\rightharpoonup u$ in $L^p(0,T;\mathbb{R}^n)$. The set $\overline{\operatorname{conv} D}^{L^p(0,T;\mathbb{R}^n)}$ is convex and closed, thus weakly closed, so that we deduce $u\in\overline{\operatorname{conv} D}^{L^p(0,T;\mathbb{R}^n)}$.

Our aim is to exploit the result of Theorem 3.5 in order to obtain outer descriptions of the convex hull of D in function space from outer descriptions of finite-dimensional sets of the form $C_{D,\Pi}$. This approach is particularly appealing in case $C_{D,\Pi}$ is a polyhedron. Before discussing some relevant classes of constraints where this holds true, we first show that polyhedricity cannot be guaranteed in general. In fact, the following construction shows that every closed convex set $K \subseteq [0,1]^M$ can arise as $C_{D,\Pi}$ for some feasible set D.

Example 3.6. Let $M \in \mathbb{N}$ and $K \subseteq [0,1]^M$ be a closed convex set. Define T = M and

$$\begin{split} D_K := \left\{ u \in BV(0,T) \colon \ u(t) \in \{0,1\} \text{ f.a.a. } t \in (0,T), \\ |u|_{BV(0,T)} \leq M, \ (\int_{i-1}^i u \, \mathrm{d}t)_{i=1}^M \in K \right\}. \end{split}$$

By definition, the set D_K satisfies Assumption (D1). Also Assumption (D2) is easy to verify for arbitrary $p \in [1, \infty)$, using the closedness of K and Proposition 10.1.1(i) in [3], which guarantees, for any sequence $\{u^k\}_{k\in\mathbb{N}} \subset D_K$ converging to some u in $L^p(0,T) \hookrightarrow L^1(0,T)$, that

$$|u|_{BV(0,T)} \le \liminf_{k \to \infty} |u^k|_{BV(0,T)} \le M$$
.

Defining the projection Π by local averaging on the intervals $(i-1,i), i=1,\ldots,M$, we obtain $\Pi(D_K)=K$ and hence, due to convexity of K, we have $K=C_{D_K,\Pi}$.

In the following subsections, we discuss two of the practically most relevant classes of constraints D and investigate the associated sets $C_{D,\Pi}$. The first class includes D_{max} as defined in (2.1), whereas the second class includes the minimum dwell time constraints mentioned in the introduction. For the remainder of this section, we always assume that the intervals defining the projection Π are pairwise disjoint.

3.1 Pointwise combinatorial constraints

By Assumption (D1), the total number of shiftings of all switches is bounded by some $\sigma \in \mathbb{N}$. A relevant class of constraints arises when the switches must additionally satisfy certain combinatorial conditions at any point in time. As an example, it might be required that two specific switches are never used at the same time, or that some switch can only be used when another switch is also used, e.g., because they are connected in series. More formally, we assume that a set $U \subseteq \{0,1\}^n$ is given and consider the constraint

$$D_{\max}^{\Sigma}(U) := \Big\{ u \in BV(0,T;\mathbb{R}^n) \colon \ u(t) \in U \text{ f.a.a. } t \in (0,T), \sum_{j=1}^n |u_j|_{BV(0,T)} \le \sigma_{\max} \Big\}.$$

Lemma 3.7. The set $D_{\max}^{\Sigma}(U)$ satisfies Assumptions (D1) and (D2).

Proof. The set $D_{\max}^{\Sigma}(U)$ obviously satisfies (D1). Moreover, for any $p \in [1, \infty)$, Proposition 10.1.1(i) in [3] again guarantees for any sequence of controls $\{u^k\}_{k\in\mathbb{N}} \subset D_{\max}^{\Sigma}(U)$ in $L^p(0,T;\mathbb{R}^n) \hookrightarrow L^1(0,T;\mathbb{R}^n)$ that converges to some u that

$$|u_j|_{BV(0,T)} \le \liminf_{k \to \infty} |u_j^k|_{BV(0,T)} \le \sigma_{\max}$$

for $j=1,\ldots,n$, because of $\sup_{k\in\mathbb{N}}|u_j^k|_{BV(0,T)}\leq\sigma_{\max}$. Furthermore, since convergence in $L^p(0,T;\mathbb{R}^n)$ implies pointwise almost everywhere convergence for a subsequence, the limit also satisfies $u(t)\in U$ f.a.a. $t\in(0,T)$. It follows that $D_{\max}^\Sigma(U)$ is closed in $L^p(0,T;\mathbb{R}^n)$ and thus fulfills (D2).

We now show that the projections Π defined in (3.4) not only lead to polytopes when applied to $D_{\max}^{\Sigma}(U)$, but even yield integer polytopes, i.e., polytopes with integer vertices only.

Theorem 3.8. For any Π as in (3.4), the set $C_{D_{\max}^{\Sigma}(U),\Pi}$ is a 0/1-polytope in \mathbb{R}^{M} .

Proof. We claim that $C_{D_{\max}^{\Sigma}(U),\Pi} = \operatorname{conv}(K)$, where

$$K:=\{\Pi(u)\colon u\in D^\Sigma_{\max}(U) \text{ and for all } i=1,\ldots,M \text{ there exists } w_i\in U \text{ with } u(t)\equiv w_i \text{ f.a.a. } t\in I_i\}$$
 .

From this, the result follows directly, as $K \subseteq \{0,1\}^M$ holds by definition.

The direction " \supseteq " is trivial, since K is a subset of $\{\Pi(u) \colon u \in D^{\Sigma}_{\max}(U)\}$. It thus remains to show " \subseteq ". For this, let $u \in D^{\Sigma}_{\max}(U)$. We need to show that $\Pi(u)$ can be written as a convex combination of vectors in K. Let $m \in \{0, \ldots, M\}$ denote the number of intervals in which at least one of the switches is shifted in u. We prove the assertion by means of complete induction over the number m. For m = 0, we clearly have $\Pi(u) \in K \subset \text{conv}(K)$.

So let the number of intervals in which at least one of the switches is shifted be m+1. Additionally, let $\ell \in \{1, ..., M\}$ be an index so that at least one switch is shifted in the interval I_{ℓ} . Since we have the upper bound σ_{\max} on the total number of shiftings, only finitely many shiftings can be in the interval I_{ℓ} . Hence, I_{ℓ} can be divided into disjoint

subintervals $I_{\ell}^1, \ldots, I_{\ell}^s$ such that $\overline{I_{\ell}} = \bigcup_{k=1}^s \overline{I_{\ell}^k}$ and there exist $w_k \in U$ with $u(t) = w_k$ f.a.a. $t \in I_{\ell}^k$, $1 \le k \le s$. Define functions u^k for $k = 1, \ldots, s$ as follows:

$$u^k(t) := \begin{cases} w_k & \text{if } t \in I_\ell \\ u(t) & \text{otherwise} \end{cases}$$

Due to $u \in U$ a.e. in (0,T) and $w_k \in U$, $u^k(t)$ is a vector in U f.a.a. $t \in (0,T)$ and for $k = 1, \ldots, s$. Furthermore, u^k has at most as many shiftings as u in total and we thus obtain $u^k \in D_{\max}^{\Sigma}(U)$. By construction, we have

$$\frac{1}{\lambda(I_{\ell})} \int_{I_{\ell}} u(t) \, dt = \frac{1}{\lambda(I_{\ell})} \sum_{k=1}^{s} \int_{I_{\ell}^{k}} w_{k} \, dt = \sum_{k=1}^{s} \frac{\lambda(I_{\ell}^{k})}{\lambda(I_{\ell})} w_{k}$$

with $\lambda(I_\ell^k)/\lambda(I_\ell) \geq 0$ for every $k \in \{1,\ldots,s\}$ and $\sum_{k=1}^s \lambda(I_\ell^k)/\lambda(I_\ell) = 1$. Since the control is unchanged on the other intervals I_i , $i \neq \ell$, we obtain $\Pi(u) = \sum_{k=1}^s \lambda(I_\ell^k)/\lambda(I_\ell)\Pi(u^k)$. The functions u^k have no shifting in I_ℓ so that the number of intervals in which at least one of the switches is shifted is at most m. According to the induction hypothesis, the vectors $\Pi(u^k)$ can be written as a convex combinations of vectors in K and consequently, due to $\Pi(u) = \sum_{k=1}^s \lambda(I_\ell^k)/\lambda(I_\ell)\Pi(u^k)$, $\Pi(u)$ is also a convex combination of vectors in K. \square

It is easy to see that Theorem 3.8 also extends to the constraint D_{max} defined in (2.1). Indeed, whenever the constraint set D is defined by switch-wise constraints as in (2.1), polyhedricity and integrality can be verified for each switch individually, in which case D_{max} reduces to $D_{\text{max}}^{\Sigma}(\{0,1\})$.

The fact that $C_{D_{\max}^{\Sigma}(U),\Pi}$ is a polytope allows, in principle, to describe it by finitely many linear inequalities. However, the number of its facets may be exponential in n or M, so that a separation algorithm will be needed for the outer approximation algorithm presented in the companion paper [6]. It depends on the set U whether this separation problem can be performed efficiently. E.g., if U models arbitrary conflicts between switches that may not be used simultaneously, the separation problem turns out to be NP-hard, since U can model the independent set problem in this case.

Even for n=1 and $U=\{0,1\}$, the separation problem is non-trivial. In this case, the set K defined in Theorem 3.8 consists of all binary sequences $v_1,\ldots,v_M\in\{0,1\}$ such that $v_{i-1}\neq v_i$ for at most σ_{\max} indices $i\in\{2,\ldots,M\}$. For the slightly different setting where v_1 is fixed to zero, it is shown in [7] that the separation problem for $\operatorname{conv}(K)$ and hence for $C_{D_{\max},\Pi}$ can be solved in polynomial time. More precisely, a complete linear description of $C_{D_{\max},\Pi}$ is given by $v\in[0,1]^M$, $v_1=0$, and inequalities of the form

(3.11)
$$\sum_{i=1}^{m} (-1)^{j+1} v_{i_j} \le \left\lfloor \frac{\sigma_{\max}}{2} \right\rfloor,$$

where $i_1, \ldots, i_m \in \{2, \ldots, M\}$ is an increasing sequence of indices with $m - \sigma_{\max}$ odd and $m > \sigma_{\max}$. For given $\bar{v} \in [0, 1]^M$, a most violated inequality of the form (3.11) is obtained by choosing $\{i_1, i_3, \ldots\}$ as the local maximizers of \bar{v} and $\{i_2, i_4, \ldots\}$ as the local minimizers of \bar{v} (excluding 1); such an inequality can thus be computed in O(M) time. This separation algorithm is used in Section 4 to investigate the strength of our convex relaxation.

3.2 Switching point constraints

In this section, we focus on the case n = 1. It is well known that a function $u \in BV(0,T)$ admits a right-continuous representative given by $\hat{u}(t) = c + \mu([0,t]), t \in (0,T)$, where μ is the regular Borel measure on [0,T] associated with the distributional derivative of u

and $c \in \mathbb{R}$ a constant. Note that \hat{u} is unique on (0,T). Given $u \in BV(0,T)$ with its right-continuous representative \hat{u} , we denote the essential jump set of u by

$$J_u := \Big\{ t \in (0, T) \colon \lim_{\tau \to t} \hat{u}(\tau) \neq \lim_{\tau \to t} \hat{u}(\tau) \Big\}.$$

In the following, we assume that $u \in BV(0,T)$ always starts with zero. More formally, if $\lim_{\tau \searrow 0} \hat{u}(\tau) = 1$, we already count this as one switching from zero to one and add a switching point t = 0 to J_u . If J_u is a finite set, we denote its cardinality by $|J_u|$. For the rest of this section, let $\sigma \in \mathbb{N}$ be given.

Definition 3.9. Let $0 \le t_1 \le ... \le t_{\sigma} < \infty$ be given and set

$$\eta_{\leq} : \mathbb{R} \to \{0, \dots, \sigma\}, \quad \eta_{\leq}(t) := |\{i \in \{1, \dots, \sigma\} : t_i \leq t\}|
\eta_{=} : \mathbb{R} \to \{0, \dots, \sigma\}, \quad \eta_{=}(t) := |\{i \in \{1, \dots, \sigma\} : t_i = t\}|$$

with the usual convention $|\emptyset| = 0$. Then we define the function $u_{t_1,...,t_{\sigma}}$ by

(3.12)
$$u_{t_1,...,t_{\sigma}} : [0,T] \to \{0,1\},$$

$$u_{t_1,...,t_{\sigma}}(t) := \begin{cases} 0, & \text{if } \eta \leq (t) \text{ is even,} \\ 1, & \text{if } \eta \leq (t) \text{ is odd.} \end{cases}$$

It is easy to verify that $u_{t_1,...,t_{\sigma}}$ is a representative of u. Moreover, the function is right-continuous by construction, so that it agrees with the unique right-continuous representative $\hat{u} \in [u]$ on (0,T). Now, given any polytope $P \subseteq \mathbb{R}_+^{\sigma}$, we define the set of switching point constraints by

$$D_P := \{ u \in BV(0, T; \{0, 1\}) : \exists 0 \le t_1 \le \dots \le t_{\sigma} < \infty$$
s.t. $(t_1, \dots, t_{\sigma}) \in P, \ u_{t_1, \dots, t_{\sigma}} \in [u] \}$.

Lemma 3.10. The set D_P satisfies the assumptions in (D1) and (D2).

Proof. Since $u \in \{0,1\}$ a.e. in (0,T) and $|J_u| \leq \sigma$ holds for all $u \in D_P$ by construction, every $u \in D_P$ satisfies $|u|_{BV(0,T)} \leq \sigma$ such that (D1) is fulfilled.

To verify (D2), consider a sequence $\{u^k\} \subset D_P$ with $u^k \to u$ in $L^p(0,T)$. From (D1) and [3, 10.1.1(i)], we deduce $u \in BV(0,T)$. Moreover, there is a subsequence, denoted by the same symbol for convenience, such that the sequence of representatives $\{u_{t_1^k,\ldots,t_{\sigma}^k}\}$ converges pointwise almost everywhere in (0,T) to u. This yields $u \in \{0,1\}$ a.e. in (0,T). Furthermore, as a polytope, P is compact by definition, so that there is yet another subsequence such that $t^k := (t_1^k, \ldots, t_{\sigma}^k)$ converges to $\bar{t} \in \mathbb{R}^{\sigma}$ with $0 \le \bar{t}_1 \le \ldots \le \bar{t}_{\sigma} < \infty$ and $\bar{t} \in P$. The mapping

$$P \ni (t_1, \dots, t_{\sigma}) \mapsto u_{t_1, \dots, t_{\sigma}} \in L^p(0, T)$$

is continuous, which can be seen as follows. If $\{(t_1^k,\ldots,t_\sigma^k)\}_{k\in\mathbb{N}}\subseteq P$ converges to some $\bar{t}\in\mathbb{R}^\sigma$, then for every $t\in(0,T)\setminus\{\bar{t}_1,\ldots,\bar{t}_\sigma\}$ it is clear that

$$|\{i \in \{1, \dots, \sigma\} : t_i^k < t\}| = |\{i \in \{1, \dots, \sigma\} : \bar{t}_i < t\}|$$

holds for k sufficient large, so that $u_{t_1^k,...,t_{\sigma}^k}(t) \to u_{\bar{t}_1,...,\bar{t}_{\sigma}}(t)$ for $k \to \infty$ follows by the definition of the representatives in (3.12). Consequently, $\{u_{t_1^k,...,t_{\sigma}^k}\}_{k\in\mathbb{N}}$ converges pointwise almost everywhere to $u_{\bar{t}_1,...,\bar{t}_{\sigma}}$ in (0,T). By Lebesgue's dominated convergence theorem, see, e.g., [2, Lemma 3.25], $\{u_{t_1^k,...,t_{\sigma}^k}\}_{k\in\mathbb{N}}$ then also converges strongly to $u_{\bar{t}_1,...,\bar{t}_{\sigma}}$ in $L^p(0,T)$. Thus, we have

$$u = \lim_{k \to \infty} u^k = \lim_{k \to \infty} u_{t_1^k, \dots, t_{\sigma}^k} = u_{\bar{t}_1, \dots, \bar{t}_{\sigma}} \quad \text{in } L^p(0, T),$$

which gives $u \in D(P)$.

Theorem 3.11. For any Π as in (3.4), the set $C_{D_P,\Pi}$ is a polytope in \mathbb{R}^M .

Proof. Let $0 = s_0 < s_1 < \dots < s_{r-1} < s_r = \infty$ include all end points of the intervals I_1, \dots, I_M defining Π . Let Φ be the set of all maps $\varphi \colon \{1, \dots, \sigma\} \to \{1, \dots, r\}$. Then we have

(3.13)
$$\{(t_1, \dots, t_\sigma) \in P \colon t_1 \le \dots \le t_\sigma\} = \bigcup_{\varphi \in \Phi} P_{\varphi}$$

with

$$P_{\varphi} := \left\{ (t_1, \dots, t_{\sigma}) \in P : t_1 \le \dots \le t_{\sigma}, \ s_{\varphi(i)-1} \le t_i \le s_{\varphi(i)} \ \forall i = 1, \dots, \sigma \right\}.$$

Now each set P_{φ} is a (potentially empty) polytope. Moreover, by construction, the function $P_{\varphi} \ni (t_1, \dots, t_{\sigma}) \mapsto \Pi(u_{t_1, \dots, t_{\sigma}}) \in \mathbb{R}^M$ is linear, since

$$\Pi(u_{t_1,\dots,t_{\sigma}})_j = \frac{1}{\lambda(I_j)} \int_{I_j} u_{t_1,\dots,t_{\sigma}}(t) dt = \frac{1}{\lambda(I_j)} \sum_{i \in \{1,\dots,\sigma+1\}} \int_{I_j} \chi_{[t_i - t_{i-1}]} dt$$

for $j=1,\ldots,M$, where we set $t_0:=0,t_{\sigma+1}:=\infty$, and $\int_{I_j}\chi_{[t_i-t_{i-1}]}\,\mathrm{d}t$ is linear in t_i and t_{i-1} for a fixed assignment φ . It follows from (3.13) that $\Pi(D_P)$ is a finite union of polytopes and hence its convex hull $C_{D_P,\Pi}$ is a polytope again.

An important class of constraints of type D_P are the minimum dwell-time constraints. For a given minimum dwell time s > 0, it is required that the time elapsed between two switchings is at least s. This implies, in particular, that the number of such switchings is bounded by $\sigma := \lceil T/s \rceil$. We thus consider the constraint

$$D_s := \{ u \in BV(0,T) : \exists t_1, \dots, t_{\sigma} \ge 0 \\ \text{s.t. } t_i - t_{i-1} \ge s \ \forall \ j = 2, \dots, \sigma, \ u_{t_1, \dots, t_{\sigma}} \in [u] \}.$$

By Theorem 3.11, the set $C_{D_s,\Pi}$ is a polytope in \mathbb{R}^M . However, it is not a 0/1-polytope in general. As an example, consider the time horizon [0,3] with intervals $I_j := [j-1,j]$ for each j=1,2,3 and let $s=\frac{3}{2}$. Then it is easy to verify that $C_{D_s,\Pi}$ has several fractional vertices, e.g., the vector $(0,1,\frac{1}{2})^{\top}$, being the unique optimal solution when minimizing $(1,-1,\frac{1}{2})^{\top}x$ over $x \in C_{D_s,\Pi}$. Nevertheless, the separation problem for D_s can be solved efficiently, as we will show in the following. Our approach is thus well-suited to deal with minimum dwell time constraints as well.

In order to show tractability, we first argue that it is enough to consider as switching points the finitely many points in the set

$$S := [0, T] \cap \left(\mathbb{Z}s + (\{0, T\} \cup \{a_i, b_i : i = 1, \dots, M\}) \right)$$

where $I_i = [a_i, b_i]$ for i = 1, ..., M. The set S thus contains all end points of the intervals $I_1, ..., I_M$ and [0, T] shifted by arbitrary integer multiples of s, as long as they are included in [0, T]. Clearly, we can compute S in $O(M\sigma)$ time. Let $\tau_1, ..., \tau_{|S|}$ be the elements of S sorted in ascending order.

Lemma 3.12. Let v be a vertex of $C_{D_s,\Pi}$. Then there exists $u \in D_s$ with $\Pi(u) = v$ such that u switches only in S.

Proof. Choose $c \in \mathbb{R}^M$ such that v is the unique minimizer of $c^\top v$ with $v \in C_{D_s,\Pi}$. Moreover, choose any $u \in D_s$ with $\Pi(u) = v$ and let t_1, \ldots, t_σ be the switching points of u, i.e., let $0 \le t_1 \le \cdots \le t_\sigma < \infty$ such that $u_{t_1,\ldots,t_\sigma} \in [u]$. For the following, define

$$S'_i := \{t_\ell \mid \ell \in \{1, \dots, \sigma\}, \ t_\ell - t_i = s(\ell - j)\}$$

for $j = 1, \ldots, \sigma$.

Assume first that $t_j \in (a_i, b_i) \setminus S$ for some $i \in \{1, ..., M\}$ and some $j \in \{1, ..., \sigma\}$. By definition of S, all switching points having minimal distance to t_j do not belong to S as well, i.e., $S'_j \cap S = \emptyset$. Hence all points in S'_j can be shifted simultaneously by some small enough $\varepsilon > 0$, in both directions, maintaining feasibility with respect to D_s and without any of these points leaving or entering any of the intervals $I_1, ..., I_M$ and [0, T]. This shifting thus changes the value of $c^{\top}\Pi(u)$ linearly, as seen in the proof of Theorem 3.11, which is a contradiction to unique optimality of v.

We have thus shown that any $u \in D_s$ with $\Pi(u) = v$ must have all switching points either in S or outside of any interval I_i . So consider some $u \in D_s$ with $\Pi(u) = v$, defined by switching points t_1, \ldots, t_{σ} as above, and let $t_j \notin S$ be any switching point of u not belonging to any interval I_i . By shifting all switching points in S'_j simultaneously to the left until $S'_j \cap S \neq \emptyset$, taking into account that the set S'_j may increase when t_j decreases, we obtain another function $u' \in D_s$. By construction of S, no shifting point is moved beyond the next point in S to the left of its original position. In particular, none of the shifting points being moved enters any of the intervals I_i , so that we derive $\Pi(u') = \Pi(u) = v$, but u' has strictly less switching points outside of S than u. By repeatedly applying the same modification, we eventually obtain a function projecting to v with switching points only in S.

Theorem 3.13. One can optimize over $C_{D_s,\Pi}$ (and hence also separate from $C_{D_s,\Pi}$) in time polynomial in M and σ .

Proof. By Lemma 3.12, it suffices to optimize over the projections of all $u \in D_s$ with switchings only in S. This can be done by a simple dynamic programming approach: given $c \in \mathbb{R}^M$, we can compute the optimal value

$$c^*(t,b) := \min \ c^{\top} \Pi(u \cdot \chi_{[0,t]}) \ \text{s.t.} \ u \in D_s, \ \lim_{\tau \searrow t} \hat{u}(\tau) = b \text{ if } t < T$$

for $b \in \{0,1\}$ recursively for all $t \in S$. Starting with $c^*(\tau_1,b) = 0$, we obtain

$$c^*(\tau_j, b) = \min \begin{cases} c^*(\tau_{j-1}, b) + c^{\top} \Pi(b\chi_{[\tau_{j-1}, \tau_j]}) \\ c^*(\tau_j - s, 1 - b) + c^{\top} \Pi((1 - b)\chi_{[\tau_j - s, \tau_j]}), & \text{if } \tau_j \ge s \\ c^{\top} \Pi((1 - b)\chi_{[0, \tau_j]}), & \text{if } \tau_j < s, b = 1 \end{cases}$$

for $j=1,\ldots,|S|$. The desired optimal value is $\min\{c^*(T,0),c^*(T,1)\}$ then, and a corresponding optimal solution can be derived easily.

Note that σ is not polynomial in the input size in general, but only pseudopolynomial, if T and s are considered part of the input.

In practice, it is necessary to design an explicit separation algorithm for $C_{D_s,\Pi}$ instead of using the theoretical equivalence between separation and optimization. This might be possible by generalizing the results presented in [34]. In fact, in the special case that [0,T] is subdivided into intervals I_1,\ldots,I_M of the same size and this size is a divisor of s, it follows from Lemma 3.12 that $C_{D_s,\Pi}$ agrees with the min-up/min-down polytope investigated in [34]. In this case, $C_{D_s,\Pi}$ is a 0/1-polytope and a full linear description, together with an exact and efficient separation algorithm, is given in [34]. It might be possible to obtain similar polyhedral results for $C_{D_s,\Pi}$ also in the general case. We leave this as future work.

To conclude this section, we note that the latter results can easily be transferred to a situation where the minimum dwell time after switching up is different from the minimum dwell time after switching down, which is often considered in the literature. More generally, we may consider any $\bar{s} \in \mathbb{R}_+^{\sigma}$ and define

$$D_{\bar{s}} := \{ u \in BV(0,T) : \exists t_1, \dots, t_{\sigma} \ge 0 \\ \text{s.t. } t_1 \ge \bar{s}_1, \ t_j - t_{j-1} \ge \bar{s}_j \ \forall j = 2, \dots, \sigma, \ u_{t_1, \dots, t_{\sigma}} \in [u] \}.$$

In order to generalize the results obtained for D_s , it suffices to replace the set S used above by the set

$$\bar{S} := [0, T] \cap \left(\{0\} \cup \left\{ \pm \sum_{j=\ell_1}^{\ell_2} \bar{s}_j \mid 1 \le \ell_1 \le \ell_2 \le \sigma \right\} + \left(\{0, T\} \cup \{a_i, b_i \colon i = 1, \dots, M\} \right) \right),$$

which can be computed in $O(M\sigma^2)$ time. Using \bar{S} in place of S and following the same reasoning, both Lemma 3.12 and Theorem 3.13 also hold for $D_{\bar{s}}$.

4 Numerical evaluation of bounds

In this section, we test the quality of our outer description of the convex hull and, in particular, the strength of the resulting lower bounds. For this, we concentrate on the case of a single switch with an upper bound σ_{max} on the number of switchings, i.e., we consider

$$D := \big\{ u \in BV(0,T) \colon \ u(t) \in \{0,1\} \text{ f.a.a. } t \in (0,T), \ |u|_{BV(0,T)} \le \sigma_{\max} \big\}.$$

However, we assume that u is fixed to zero before the time horizon, so that we count it as a shift if u is 1 at the beginning. Moreover, we consider exemplarily a square domain $\Omega = [0,1]^2$, the end time T=2, the upper bound $\sigma_{\rm max}=2$ on the number of switchings and the form function ψ as well the desired state $y_{\rm d}$ given as

$$\psi(x) := 12\pi^2 \exp(x_1 + x_2) \sin(\pi x_1) \sin(\pi x_2)$$

$$y_d(t, x) := 2\pi^2 \max(\cos(2\pi t), 0) \sin(\pi x_1) \sin(\pi x_2).$$

We always choose $\alpha = 0$, so that the computed bounds are not deteriorated by the Tikhonov term.

For the discretization of the optimal control problem, we use the DUNE-library [44]. To obtain exact optimal solutions for comparison, we use the MINLP solver GUROBI 9.1.2 [25] for solving the discretized problem. The source code is part of the implementation at https://github.com/agruetering/dune-MIOCP. The spatial discretization uses a standard Galerkin method with continuous and piecewise linear functionals. For the state y and the desired temperature y_d we also use continuous and piecewise linear functionals in time, while the temporal discretization for the controls chooses piecewise constant functionals. The BV-seminorm condition then simplifies to

(4.1)
$$u_0 + \sum_{i=1}^{N_t - 1} |u_i - u_{i-1}| \le \sigma_{\max} ,$$

where the term u_0 is added in order to count a shift if $u_0 = 1$. We linearize (4.1) by introducing $N_t - 1$ additional real variables z_i expressing the absolute values $|u_i - u_{i-1}|$. More precisely, we require $z_i \geq u_i - u_{i-1}$ and $z_i \geq u_{i-1} - u_i$ and use the linear constraint $u_0 + \sum_{i=1}^{N_t} z_i \leq \sigma_{\max}$ instead of (4.1). The naive convex relaxation now replaces the binarity constraint $u_i \in \{0,1\}$ with $u_i \in [0,1]$ for $i=0,\ldots,N_t-1$. For the tailored convexification presented in this paper, we instead omit the constraint (4.1) and iteratively add a most violated cutting plane for $C_{D,\Pi}$, where the intervals I_1,\ldots,I_M for the projection are the ones given by the discretization in time, until the relative change of the bound is less than 0.1% in three successive iterations. To the best of our knowledge, there is no standard procedure for solving the convexified control problems with additional linear control constraints arising at this point, we thus also use Gurobi 9.1.2 [25] for this.

We investigate the bounds for a sequence of discretizations with various numbers N_t of time intervals and uniform spatial triangulations of Ω with $N_x \times N_x$ nodes. Gurobi is run with default settings except that the parallel mode is switched off for better comparison and the dual simplex method is used due to better performance. All computations have been

N_x	N_t	MINLP		naive rel.		tailored convexification				
		Obj	Time (s)	Obj	Gap	Obj	#Cuts	$\#\mathrm{Ex}$	Gap	Filled gap
10	20	13.69	4.38	8.41	38.60 %	9.67	21	5	29.39 %	23.85 %
	40	12.76	39.51	7.39	42.09~%	9.03	56	16	29.22~%	30.59~%
	60	12.51	152.44	7.29	41.71~%	8.86	108	30	29.19~%	30.01~%
	80	12.50	465.19	7.28	41.75~%	8.53	143	67	31.74~%	23.98~%
	100	12.54	663.20	7.25	42.18~%	7.95	136	88	36.62~%	13.18~%
15	20	13.69	32.07	8.38	38.79 %	9.67	21	5	29.40 %	24.20 %
	40	12.76	272.97	7.38	42.13~%	9.08	52	16	28.83~%	31.57~%
	60	12.51	1183.68	7.29	41.75~%	8.71	91	28	30.35~%	27.30~%
	80	12.50	2837.12	7.28	41.78~%	8.35	117	60	33.19~%	20.57~%
	100	12.54	4686.54	7.25	42.22~%	7.79	131	93	37.91~%	10.20~%
20	20	13.69	109.08	8.37	38.85 %	9.69	23	5	29.24 %	24.73 %
	40	12.76	1305.88	7.38	42.14~%	8.96	59	20	29.75~%	29.40~%
	60	12.51	5147.66	7.29	41.76~%	8.57	86	35	31.52~%	24.53~%
	80	12.50	15185.22	7.28	41.78~%	8.30	123	62	33.62~%	19.53~%
	100	12.54	19550.01	7.25	42.23~%	8.00	153	91	36.20~%	14.27~%

Table 1: Comparison of naive and tailored convexification.

performed on a 64bit Linux system with an Intel Xeon E5-2640 CPU @ $2.5~\mathrm{GHz}$ and $32~\mathrm{GB}$ RAM.

The results are presented in Table 1. For given choices of N_t and N_x , we report the objective values (Obj) obtained by the exact approach and the two relaxations. We emphasize that, for a given optimal solution of the respective problem, we recalculate the objective value with a much finer discretization, choosing $N_t = 200$ and $N_x = 100$. In particular, the bounds do not necessarily behave monotonously. It can be seen from the results that the new bounds are clearly stronger than the naive bounds. In the last column (Filled gap), we state how much of the gap left open by the naive relaxation is closed by the new relaxation. We also state how many cutting planes are computed altogether (#Cuts) and how many of them are needed to obtain at least the same bound as the naive relaxation (#Ex). The main message of Table 1 is that our new approach yields better bounds than the naive approach even after adding relatively few cutting planes. Additionally, the naive relaxation includes inequality constraints involving the BV-seminorm, such that its solution is very challenging in practice.

For the exact approach, we also state the time (in seconds) needed for the solution of the problem (Time). It is obvious from the results that only very coarse discretizations can be considered when using a straightforward MINLP-based approach. In the companion paper [6], we thus develop a tailored outer approximation algorithm based on the the convex hull description in (3.5) in order to compute the dual bounds of our convex relaxation of (P) more efficiently.

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