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NEW ASSUMPTIONS FOR STABILITY ANALYSIS IN ELLIPTIC OPTIMAL CONTROL PROBLEMS*

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Abstract. This paper is dedicated to the stability analysis of the optimal solutions of a control problem associated with a semilinear elliptic equation. The linear differential operator of the equation is neither monotone nor coercive due to the presence of a convection term. The control appears only linearly, or may not even appear explicitly in the objective functional. Under new assumptions, we prove Lipschitz stability of the optimal controls and associated states with respect to not only perturbations in the equation and the objective functional but also the Tikhonov regularization parameter.

Key words. semilinear elliptic equations, optimality conditions, stability analysis, Tikhonov regularization

MSC codes. 35J61, 49J20, 49K20, 49K40

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1. Introduction.

In this paper, we study the optimal control problem

$$\text{(P)} \qquad \min_{u\in\mathcal{U}_{ad}} J(u) := \int_\Omega L(x, y_u(x), u(x))\,\mathrm{d}x,$$

where $\mathcal{U}_{ad} = \{u \in L^2(\Omega) : u_a \le u(x) \le u_b \text{ for a.a. } x \in \Omega\}$, $-\infty < u_a < u_b < +\infty$. Here, y_u denotes the solution of the following semilinear elliptic equation:

$$\text{(1.1)} \qquad \begin{cases} -\operatorname{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y + f(x,y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases}$$

Assumptions on the data of the control problem (P) will be given below. The aim of this paper is to prove stability results for the local minimizers of (P) with respect to perturbations in the data of the control problem. There are quite a few previous papers devoted to this issue; see, for instance, [14], [15], [16], [17]. In all of these cases, the second derivative of L with respect to u is bounded from below by a positive constant. This is the case where the Tikhonov term is involved in the objective functional. Under this condition and assuming sufficient second order optimality conditions (SSOC), the Lipschitz stability of the optimal controls is proved. Here, we assume that u appears linearly in $L(x, y, u)$ or does not even appear at all. Therefore, the previous results do not apply. In this case, under (SSOC) for optimality, Lipschitz stability of the optimal states can be proved; see [7]. In section 4, we obtain

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analogous estimates for the optimal states replacing (SSOC) by a weaker condition; see (3.13). It is weaker because (SSOC) implies our assumption, but they are not equivalent. In addition, our assumption implies strict local optimality of the control; see Theorem 3.5.

In order to prove stability of the optimal controls when they are not explicitly involved in the objective functional, besides (SSOC) an additional structural hypothesis is usually assumed. This situation was studied in [21], where the authors proved Lipschitz stability of the control with respect to linear perturbations simultaneously appearing in the state equation and the objective functional. The drawback is that the additional hypothesis is satisfied only by bang-bang controls. Here, we obtain analogous estimates changing the mentioned assumption by a weaker one, see (5.2). Though this second assumption (5.2) is stronger than (3.13), it can be satisfied by optimal controls independently if they are bang-bang or not. Moreover our assumption (5.2) is satisfied if the (SSOC) and the additional hypothesis are assumed.

Finally, under the assumption (5.2), Lipschitz stability is established for the optimal states with respect to simultaneous perturbations in the equations and in the objective functional with respect to the state and the control, and with respect to the Tikhonov regularization parameter. The stability with respect to the Tikhonov regularization has been studied in [7] and [20]. In [7], Hölder stability of the states is proved. In [20], stability of the control is proved under (SSOC) and the structural assumption. The reader is also referred to [23], [24], [25] for the case of linear partial differential equations.

In this paper, besides providing some new sufficient conditions for Lipschitz stability for the optimal control and associated states, we deal with a semilinear elliptic state equation that is neither monotone nor coercive. Though some crucial results for this state equation are taken from [6], some estimates have been proved that are not available in the literature.

The plan of this paper is as follows. In section 2, we analyze the state equation. First, we establish some properties of the linear differential operator of the state equation, and the full semilinear equation is analyzed in the second part of the section. The control problem (P) is studied in section 3. We prove that our assumption (3.13) is a sufficient condition for strong local optimality. Section 4 is dedicated to the proof of Lipschitz stability of the optimal states. In section 5 we introduce the stronger condition (5.2) replacing (3.13) that allows us to establish the Lipschitz stability of the optimal controls. Finally, in section 6, the Tikhonov regularization is considered.

2. Analysis of the partial differential equation.

In this section we analyze the equation (1.1). We split the section in two parts. In the first part, we establish the results concerning the linear operator of the elliptic equation. In the second subsection, the nonlinear equation will be studied.

2.1. Analysis of the linear differential operator.

We define the differential operator $\mathcal{A} : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ by

$$\mathcal{A}y = -\mathrm{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y.$$

The following assumptions are supposed to hold throughout the paper. They ensure that the mathematical objects under consideration are well defined.

Assumption 2.1. The following statements are fulfilled.

(i) The set $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with a Lipschitz boundary Γ. The mapping $A : \Omega \longrightarrow \mathbb{R}^{n\times n}$ is measurable and bounded in Ω, and there exists $\Lambda_A > 0$ such that $\xi^\top A(x)\xi \geq \Lambda_A|\xi|^2$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

(ii) We assume that $b \in L^p(\Omega;\mathbb{R}^n)$ with $p \geq 3$ if $n=3$ and $p>2$ arbitrary if $n=2$.

Under these assumptions it is known that $\mathcal{A}: H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism despite the fact that the operator is neither coercive nor monotone; see [6], [13, Theorem 8.3], [22]. The following identity is satisfied

$$\langle \mathcal{A}y, z\rangle = \int_\Omega A\nabla y \cdot \nabla z \,\mathrm{d}x + \int_\Omega b\cdot \nabla y z \,\mathrm{d}x \quad \forall y, z \in H_0^1(\Omega),$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Along this paper we will set

$$\|y\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla y(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The next lemma states some properties of $\mathcal{A}$ that will be used later.

Lemma 2.2. *The following statements are fulfilled:*

(i) *There exists a constant $C_{\Lambda_A,b}$ such that Gårding's inequality holds*

$$\langle \mathcal{A}y, y\rangle \geq \frac{\Lambda_A}{4}\|y\|^2_{H_0^1(\Omega)} - C_{\Lambda_A,b}\|y\|^2_{L^2(\Omega)} \quad \forall y \in H_0^1(\Omega). \tag{2.1}$$

(ii) *Let $a \in L^\infty(\Omega)$ be a nonnegative function and $h \in H^{-1}(\Omega)$. If $y \in H_0^1(\Omega)$ satisfies $\mathcal{A}y + ay = h$ and h is a nonnegative linear form, then y is a nonnegative function as well.*

(iii) *Let a be as above and $h \in L^r(\Omega)$ with $r > \frac{n}{2}$. Then, the solution y of the above equation belongs to $H_0^1(\Omega) \cap C(\bar\Omega)$. Moreover, there exists a constant C_r independent of a and h such that*

$$\|y\|_{H_0^1(\Omega)} + \|y\|_{C(\bar\Omega)} \leq C_r \|h\|_{L^r(\Omega)}. \tag{2.2}$$

Proof. The proof of (2.1) can be found in [6]; see also [13, Lemma 8.4]. For the proof of (ii) the reader is referred again to [6] and [13, Theorem 8.1]. The $H_0^1(\Omega)\cap C(\bar\Omega)$ regularity of y for functions $h \in L^r(\Omega)$ is well known; see [13, Lemma 8.31]. It remains to prove the estimates (2.2) for a constant C_r independent of h and a. Let us denote by $y_{a,h} \in H_0^1(\Omega)\cap C(\bar\Omega)$ the solution of $\mathcal{A}y + ay = h$. With $y_{0,h}$ we denote the solution corresponding to $a \equiv 0$. Then, the estimate $\|y_{0,h}\|_{C(\bar\Omega)} \leq C\|h\|_{L^r(\Omega)}$ is well known for a constant C depending on r, but independent of h. Let us write $h = h^+ - h^-$. From (ii) we know that $y_{a,h^+} \geq 0$ and $y_{a,h^-} \geq 0$. Now, since $\mathcal{A}(y_{a,h^+} - y_{0,h^+}) + a(y_{a,h^+} - y_{0,h^+}) = -ay_{0,h^+}$, again by item (ii), we obtain $0 \leq y_{a,h^+} \leq y_{0,h^+}$ and consequently $\|y_{a,h^+}\|_{C(\bar\Omega)} \leq \|y_{0,h^+}\|_{C(\bar\Omega)}$. Analogously, by the same argument $0 \leq y_{a,h^-} \leq y_{0,h^-}$ and consequently $\|y_{a,h^-}\|_{C(\bar\Omega)} \leq \|y_{0,h^-}\|_{C(\bar\Omega)}$. Therefore,

$$\begin{aligned}\|y_{a,h}\|_{C(\bar\Omega)} &\leq \|y_{a,h^+}\|_{C(\bar\Omega)} + \|y_{a,h^-}\|_{C(\bar\Omega)} \leq \|y_{0,h^+}\|_{C(\bar\Omega)} + \|y_{0,h^-}\|_{C(\bar\Omega)} \\ &\leq C\Big(\|h^+\|_{L^r(\Omega)} + \|h^-\|_{L^r(\Omega)}\Big) \leq 2C\|h\|_{L^r(\Omega)},\end{aligned}$$

where C is independent of a and h. To prove the corresponding estimate in $H_0^1(\Omega)$ we use Gårding's inequality (2.1) and the above estimate:

$$\begin{aligned}&\frac{\Lambda_A}{4}\|y_{a,h}\|^2_{H_0^1(\Omega)} \leq \langle \mathcal{A}y_{a,h}, y_{a,h}\rangle + C_{\Lambda_A,b}\|y_{a,h}\|^2_{L^2(\Omega)} \\ &\leq \langle \mathcal{A}y_{a,h}, y_{a,h}\rangle + \int_\Omega a y_{a,h}^2 \,\mathrm{d}x + C_{\Lambda_A,b}\|y_{a,h}\|^2_{L^2(\Omega)}\end{aligned}$$

$$= \int_\Omega h y_{a,h}\,\mathrm{d}x + C_{\Lambda_A,b}\|y_{a,h}\|^2_{L^2(\Omega)} \le |\Omega|^{\frac{r-1}{r}}\|h\|_{L^r(\Omega)}\|y_{a,h}\|_{C(\bar\Omega)} + C_{\Lambda_A,b}|\Omega|\|y_{a,h}\|^2_{C(\bar\Omega)}$$
$$\le 2C\Big(|\Omega|^{\frac{r-1}{r}} + 2CC_{\Lambda_A,b}|\Omega|\Big)\|h\|^2_{L^r(\Omega)},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Since the above constants are independent of a and h, the inequality completes the proof of (2.2). ∎

Now, we consider the adjoint operator $\mathcal{A}^*: H^1_0(\Omega) \to H^{-1}(\Omega)$ of $\mathcal{A}$. Since $\mathcal{A}$ is an isomorphism, $\mathcal{A}^*$ is also an isomorphism. It is obvious that $\mathcal{A}^*\varphi = -\mathrm{div}(A^\top\nabla\varphi) - \mathrm{div}(\varphi b)$. The operator $\mathcal{A}^*$ satisfies the same properties established in Lemma 2.2. Indeed, the Gårding's inequality is a consequence of (2.1) and the identity $\langle\mathcal{A}^*\varphi,\varphi\rangle = \langle\mathcal{A}\varphi,\varphi\rangle$. The proof of the estimate (2.2) is the same for the operator $\mathcal{A}^*$. We only prove the statement (ii). Let $h \in H^{-1}(\Omega)$ be a nonnegative linear form. This means that $\langle h, y\rangle \ge 0$ for every nonnegative function $y \in H^1_0(\Omega)$. Let $\varphi \in H^1_0(\Omega)$ satisfy $\mathcal{A}^*\varphi + a\varphi = h$. Now, given a nonnegative function $w \in L^2(\Omega)$ we take $y \in H^1_0(\Omega)$ satisfying $\mathcal{A}y + ay = w$. By Lemma 2.2(ii) we have that $y \ge 0$. Then, we obtain

$$\int_\Omega w\varphi\,dx = \langle\mathcal{A}y + ay, \varphi\rangle = \langle\mathcal{A}^*\varphi + a\varphi, y\rangle = \langle h, y\rangle \ge 0.$$

Since w is an arbitrary nonnegative function of $L^2(\Omega)$, this inequality yields $\varphi \ge 0$.

We finish this subsection by proving an $L^s(\Omega)$ estimate.

LEMMA 2.3. *Assume that $s \in [1, \frac{n}{n-2})$, s' is its conjugate, and let $a \in L^\infty(\Omega)$ be a nonnegative function. Then, there exists a constant $C_{s'}$ independent of a such that*

$$\left\{\begin{array}{l} \|y_h\|_{L^s(\Omega)} \le C_{s'}\|h\|_{L^1(\Omega)} \\ \|\varphi_h\|_{L^s(\Omega)} \le C_{s'}\|h\|_{L^1(\Omega)} \end{array}\right. \quad \forall h \in H^{-1}(\Omega)\cap L^1(\Omega), \tag{2.3}$$

where y_h and φ_h satisfy the equations $\mathcal{A}y_h + ay_h = h$ and $\mathcal{A}^*\varphi_h + a\varphi_h = h$, respectively, and $C_{s'}$ is given by (2.2) *with $r = s'$.*

Proof. We prove the estimate (2.3) for φ_h and $n = 3$, the proof being identical for y_h and analogous for $n = 2$ with minor modifications. First we observe that $H^1_0(\Omega) \subset L^6(\Omega) \subset L^3(\Omega)$, hence $\varphi_h \in L^s(\Omega)$. As a consequence we obtain that $|\varphi_h|^{s-1}\mathrm{sign}(\varphi_h) \in L^{s'}(\Omega)$. Moreover, $s < 3$ implies that $s' > \frac{3}{2}$. According to Lemma 2.2(iii), the solution of $\mathcal{A}y + ay = |\varphi_h|^{s-1}\mathrm{sign}(\varphi_h)$ belongs to $H^1_0(\Omega)\cap C(\bar\Omega)$ and satisfies $\|y\|_{C(\bar\Omega)} \le C_{s'}\||\varphi_h|^{s-1}\mathrm{sign}(\varphi_h)\|_{L^{s'}(\Omega)} = C_{s'}\|\varphi_h\|^{s-1}_{L^s(\Omega)}$, where $C_{s'}$ is independent of a and h. Using these facts we infer

$$\|\varphi_h\|^s_{L^s(\Omega)} = \int_\Omega |\varphi_h|^s\,\mathrm{d}x = \langle\mathcal{A}y + ay, \varphi_h\rangle = \langle\mathcal{A}^*\varphi_h + a\varphi_h, y\rangle$$
$$= \int_\Omega hy\,\mathrm{d}x \le \|h\|_{L^1(\Omega)}\|y\|_{C(\bar\Omega)} \le C_{s'}\|h\|_{L^1(\Omega)}\|\varphi_h\|^{s-1}_{L^s(\Omega)}.$$

This proves (2.3) for φ_h. ∎

2.2. Analysis of the semilinear equation. In this subsection, we formulate some results concerning the semilinear equation (1.1). For this purpose we make the following assumptions on the nonlinear term of the equation.

Assumption 2.4. We assume that $f: \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable satisfying

$$f(\cdot, 0) \in L^r(\Omega) \text{ with } r > \frac{n}{2} \text{ and } \frac{\partial f}{\partial y}(x,y) \ge 0\ \forall y \in \mathbb{R}, \tag{2.4}$$

$$\forall M>0\ \exists C_{f,M}>0 \text{ such that } \left|\frac{\partial f}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| \le C_{f,M}\ \forall |y|\le M, \tag{2.5}$$

$$\begin{cases} \forall M>0 \text{ and } \forall \varepsilon>0\ \exists \delta>0 \text{ such that} \\ \left|\dfrac{\partial^2 f}{\partial y^2}(x,y_2) - \dfrac{\partial^2 f}{\partial y^2}(x,y_1)\right| < \varepsilon \ if\ |y_1|,|y_2|\le M \text{ and } |y_2-y_1|\le\delta, \end{cases} \tag{2.6}$$

for almost every $x\in\Omega$.

THEOREM 2.5. *Let Assumptions* 2.1 *and* 2.4 *hold. If u belongs to $L^r(\Omega)$ for some $r>n/2$, then there exists a unique solution $y_u\in H_0^1(\Omega)\cap C(\bar\Omega)$ of* (1.1)*. Moreover, there exists a constant $K_{f,r}$ independent of u such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar\Omega)} \le K_{f,r}\big(\|u\|_{L^r(\Omega)} + \|f(\cdot,0)\|_{L^r(\Omega)} + 1\big). \tag{2.7}$$

Further, if $\{u_k\}_{k=1}^\infty$ is a sequence converging weakly to u in $L^r(\Omega)$, then $y_{u_k}\to y_u$ strongly in $H_0^1(\Omega)\cap C(\bar\Omega)$.

The reader is referred to [6] for the proof of this result. As a consequence of (2.7) we get

$$\exists K_U>0 \text{ such that } \|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar\Omega)} \le K_U \quad \forall u\in\mathcal{U}_{ad}. \tag{2.8}$$

For each $r>n/2$, we define the map $G_r: L^r(\Omega)\to H_0^1(\Omega)\cap C(\bar\Omega)$ by $G_r(u)=y_u$.

THEOREM 2.6. *Let Assumptions* 2.1 *and* 2.4 *hold. For every $r>\frac{n}{2}$ the map G_r is of class C^2, and the first and second derivatives at $u\in L^r(\Omega)$ in the directions $v,v_1,v_2\in L^r(\Omega)$, denoted by $z_{u,v}=G'_r(u)v$ and $z_{u,v_1,v_2}=G''_r(u)(v_1,v_2)$, are the solutions of the equations*

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x,y_u)z = v, \tag{2.9}$$

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x,y_u)z = -\frac{\partial^2 f}{\partial y^2}(x,y_u)z_{u,v_1}z_{u,v_2}, \tag{2.10}$$

respectively.

The proof of this theorem is an easy application of the implicit function theorem; see [6].

LEMMA 2.7. *The following statements are fulfilled.*

(i) *Suppose that $r>\frac{n}{2}$ and $s\in[1,\frac{n}{n-2})$. Then, there exist constants K_r depending on r and M_s depending on s such that for every $u,\bar u\in\mathcal{U}_{ad}$*

$$\|y_u - y_{\bar u} - z_{\bar u,u-\bar u}\|_{C(\bar\Omega)} \le K_r\|y_u-y_{\bar u}\|^2_{L^{2r}(\Omega)}, \tag{2.11}$$

$$\|y_u - y_{\bar u} - z_{\bar u,u-\bar u}\|_{L^s(\Omega)} \le M_s\|y_u-y_{\bar u}\|^2_{L^2(\Omega)}. \tag{2.12}$$

(ii) *Taking $C_X=K_2\sqrt{|\Omega|}$ if $X=C(\bar\Omega)$ and $C_X=M_2$ if $X=L^2(\Omega)$, the following inequality holds*

$$\|z_{u,v}-z_{\bar u,v}\|_X \le C_X\|y_u-y_{\bar u}\|_X\|z_{\bar u,v}\|_X \ \forall u,\bar u\in\mathcal{U}_{ad} \text{ and } \forall v\in L^2(\Omega). \tag{2.13}$$

(iii) *Let X be as in* (ii). *There exists $\varepsilon > 0$ such that for all $\bar{u}, u \in \mathcal{U}_{ad}$ with $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} \le \varepsilon$ the following inequalities are satisfied:*

$$\frac{1}{2}\|y_u - y_{\bar{u}}\|_X \le \|z_{\bar{u},u-\bar{u}}\|_X \le \frac{3}{2}\|y_u - y_{\bar{u}}\|_X, \tag{2.14}$$

$$\frac{1}{2}\|z_{\bar{u},v}\|_X \le \|z_{u,v}\|_X \le \frac{3}{2}\|z_{\bar{u},v}\|_X \quad \forall v \in L^2(\Omega). \tag{2.15}$$

Proof. Let us set $\phi = y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}} \in H_0^1(\Omega) \cap C(\bar{\Omega})$. From the equations satisfied by the three functions and using the mean value theorem we get

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x, y_{\bar{u}})\phi = \left[\frac{\partial f}{\partial y}(x, y_{\bar{u}}) - \frac{\partial f}{\partial y}(x, y_\theta)\right](y_u - y_{\bar{u}}),$$

where $y_\theta(x) = y_{\bar{u}}(x) + \theta(x)(y_u(x) - y_{\bar{u}}(x))$ with $\theta : \Omega \longrightarrow [0,1]$ measurable. Using again the mean value theorem we deduce

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x, y_{\bar{u}})\phi = -\theta\frac{\partial^2 f}{\partial y^2}(x, y_\vartheta)(y_u - y_{\bar{u}})^2$$

with $y_\vartheta(x) = y_{\bar{u}}(x) + \vartheta(x)(y_\theta(x) - y_{\bar{u}}(x))$ and $\vartheta : \Omega \longrightarrow [0,1]$ measurable. By Lemma 2.2(iii) and taking into account (2.5) and (2.8) we infer the existence of C_r independent of $u, \bar{u} \in \mathcal{U}_{ad}$ such that

$$\|\phi\|_{C(\bar{\Omega})} \le C_r C_{f,K_U}\|(y_u - y_{\bar{u}})^2\|_{L^r(\Omega)} = C_r C_{f,K_U}\|y_u - y_{\bar{u}}\|^2_{L^{2r}(\Omega)},$$

which proves (2.11) with $K_r = C_r C_{f,K_U}$. To prove (2.12) we use Lemma 2.3 to obtain

$$\|\phi\|_{L^s(\Omega)} \le C_{s'} C_{f,K_U}\|(y_u - y_{\bar{u}})^2\|_{L^1(\Omega)} = C_{s'} C_{f,K_U}\|y_u - y_{\bar{u}}\|^2_{L^2(\Omega)}.$$

Taking $M_s = C_{s'} C_{f,K_U}$, (2.12) follows.

Now we prove (2.13) for $X = C(\bar{\Omega})$. Setting $\psi = z_{u,v} - z_{\bar{u},v}$ and subtracting the corresponding equations, we infer with the mean value theorem

$$\mathcal{A}\psi + \frac{\partial f}{\partial y}(x, y_u)\psi = \left[\frac{\partial f}{\partial y}(x, y_{\bar{u}}) - \frac{\partial f}{\partial y}(x, y_u)\right] z_{\bar{u},v} = \frac{\partial^2 f}{\partial y^2}(x, y_\theta)(y_{\bar{u}} - y_u) z_{\bar{u},v}.$$

Taking $r = 2$ in (2.2) and using (2.5) and (2.8), it follows from the above equation that

$$\|\psi\|_{C(\bar{\Omega})} \le C_2 C_{f,K_U}\|(y_{\bar{u}} - y_u) z_{\bar{u},v}\|_{L^2(\Omega)} \le K_2\sqrt{|\Omega|}\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})}\|z_{\bar{u},v}\|_{C(\bar{\Omega})},$$

which proves (2.13) for $X = C(\bar{\Omega})$. The proof for $X = L^2(\Omega)$ is analogous; we use the estimate (2.3) for $s = 2$ instead of (2.2).

To prove (2.14) for $X = C(\bar{\Omega})$ we use (2.11) with $r = 2$ to get

$$\begin{aligned}\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} &\le \|\phi\|_{C(\bar{\Omega})} + \|z_{\bar{u},u-\bar{u}}\|_{C(\bar{\Omega})} \le K_2\|y_u - y_{\bar{u}}\|^2_{L^4(\Omega)} + \|z_{\bar{u},u-\bar{u}}\|_{C(\bar{\Omega})} \\ &\le K_2\sqrt{|\Omega|}\|y_u - y_{\bar{u}}\|^2_{C(\bar{\Omega})} + \|z_{\bar{u},u-\bar{u}}\|_{C(\bar{\Omega})}.\end{aligned}$$

Choosing $\varepsilon_1 = [2K_2\sqrt{|\Omega|}]^{-1}$, we see the first inequality of (2.14) follows if $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} < \varepsilon_1$. To deal with the case $X = L^2(\Omega)$ we use (2.12) with $s = 2$ and obtain

$$\begin{aligned}\|y_u - y_{\bar u}\|_{L^2(\Omega)} &\le \|\phi\|_{L^2(\Omega)} + \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)} \le M_2\|y_u - y_{\bar u}\|^2_{L^2(\Omega)} + \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\ &\le M_2\sqrt{|\Omega|}\|y_u - y_{\bar u}\|_{C(\bar\Omega)}\|y_u - y_{\bar u}\|_{L^2(\Omega)} + \|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}.\end{aligned}$$

Hence, taking $\varepsilon_2 = [2M_2\sqrt{|\Omega|}]^{-1}$ we obtain the first inequality of (2.14) with $X = L^2(\Omega)$ if $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon_2$.

To prove the second inequality of (2.14) for $X = C(\bar\Omega)$, we proceed as follows:

$$\begin{aligned}\|z_{\bar u,u-\bar u}\|_{C(\bar\Omega)} &\le \|\phi\|_{C(\bar\Omega)} + \|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le K_2\sqrt{|\Omega|}\|y_u - y_{\bar u}\|^2_{C(\bar\Omega)} + \|y_u - y_{\bar u}\|_{C(\bar\Omega)}\\ &\le \frac{3}{2}\|y_u - y_{\bar u}\|_{C(\bar\Omega)} \quad \text{if } \|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon_1.\end{aligned}$$

Similarly the second inequality of (2.14) follows if $X = L^2(\Omega)$ with ε_2 replacing ε_1.

Finally, we prove (2.15). Using (2.13) we obtain

$$\begin{aligned}\|z_{u,v}\|_X &\le \|z_{u,v} - z_{\bar u,v}\|_X + \|z_{\bar u,v}\|_X \le C_X\|y_u - y_{\bar u}\|_X\|z_{\bar u,v}\|_X + \|z_{\bar u,v}\|_X,\\ \|z_{\bar u,v}\|_X &\le \|z_{u,v} - z_{\bar u,v}\|_X + \|z_{u,v}\|_X \le C_X\|y_u - y_{\bar u}\|_X\|z_{\bar u,v}\|_X + \|z_{u,v}\|_X. \qquad \square\end{aligned}$$

Therefore, selecting $\varepsilon = [2C_2]^{-1}$ for $X = C(\bar\Omega)$ and $\varepsilon = [2C_2\sqrt{|\Omega|}]^{-1}$ for $X = L^2(\Omega)$, we see (2.15) follows if $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le \varepsilon$.

3. The control problem.

In this section, we make assumptions on the objective functional J so that (P) has at least one solution and the first and second order conditions for local optimality can be established. Since the problem is not convex, we will consider not only global minimizers but also local minimizers. Throughout this paper, we will say that $\bar u$ is a local minimizer of (P) if $\bar u \in \mathcal{U}_{ad}$ and there exists a ball $B_\rho(\bar u) \subset L^2(\Omega)$ such that $J(\bar u) \le J(u)$ for every $u \in \mathcal{U}_{ad} \cap B_\rho(\bar u)$. We will also say that $\bar u$ is a strong local minimizer of (P) if $\bar u \in \mathcal{U}_{ad}$ and there exists $\varepsilon > 0$ such that $J(\bar u) \le J(u)$ for every $u \in \mathcal{U}_{ad}$ with $\|y_u - y_{\bar u}\|_{C(\bar\Omega)} < \varepsilon$. If the previous inequalities are strict whenever $u \ne \bar u$, then we say that $\bar u$ is a strict (strong) local minimizer. As far as we know, the notion of strong local minimizers in the framework of control of partial differential equations was introduced for the first time in [1]; see also [2].

We make the following assumptions on L.

Assumption 3.1. The function $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is Carathéodory and of class C^2 with respect to the second variable. In addition, we assume that

$$L(x,y,u) = L_0(x,y) + g(x)u \quad \text{with} \quad L_0(\cdot,0) \in L^1(\Omega) \quad \text{and} \quad g \in L^\infty(\Omega), \tag{3.1}$$

$$\left\{\begin{array}{l}\forall M > 0\ \exists \psi_M \in L^2(\Omega) \text{ and } C_{L,M} > 0 \text{ such that}\\ \left|\dfrac{\partial L}{\partial y}(x,y,u)\right| \le \psi_M(x) \quad \text{and} \quad \left|\dfrac{\partial^2 L}{\partial y^2}(x,y,u)\right| \le C_{L,M}\ \forall |y| \le M,\end{array}\right. \tag{3.2}$$

$$\left\{\begin{array}{l}\forall M > 0 \text{ and } \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that}\\ \left|\dfrac{\partial^2 L}{\partial y^2}(x,y_2,u) - \dfrac{\partial^2 L}{\partial y^2}(x,y_1,u)\right| < \varepsilon \text{ if } |y_1|, |y_2| \le M,\ |y_2 - y_1| \le \delta,\end{array}\right. \tag{3.3}$$

for almost every $x \in \Omega$.

Using Theorem 2.5, the assumptions on L, and the boundedness of $\mathcal{U}_{ad}$ in $L^\infty(\Omega)$, the existence of at least one solution of (P) follows. Indeed, if we take a minimizing sequence $\{u_k\}_{k=1}^\infty$, we can assume that $u_k \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$. Then Theorem 2.5 implies that $y_{u_k} \to y_{\bar u}$ strongly in $H_0^1(\Omega)\cap C(\bar\Omega)$. Further, using (2.8) and (3.2) with $M = K_U$ we infer with the mean value theorem that

$$|L_0(x, y_{u_k}(x))| \le |L_0(x,0)| + \psi_{K_U}(x)K_U.$$

Then we can apply Lebesgue's dominated convergence theorem to pass to the limit in the objective functional and to obtain $J(u_k) \to J(\bar u)$.

In order to derive the first order optimality conditions satisfied by a local minimizer we address the issue of the differentiability of the objective functional J.

Theorem 3.2. *Suppose that $r > \frac{n}{2}$. Then, the functional $J : L^r(\Omega) \longrightarrow \mathbb{R}$ is of class C^2. Moreover, given $u, v, v_1, v_2 \in L^r(\Omega)$ we have*

$$J'(u)v = \int_\Omega (\varphi_u + g)v\,\mathrm{d}x, \tag{3.4}$$

$$J''(u)(v_1,v_2) = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,y_u,u) - \varphi_u \frac{\partial^2 f}{\partial y^2}(x,y_u)\right] z_{u,v_1} z_{u,v_2}\,\mathrm{d}x, \tag{3.5}$$

where $\varphi_u \in H_0^1(\Omega)\cap C(\bar\Omega)$ is the unique solution of the adjoint equation

$$\begin{cases} \mathcal{A}^*\varphi + \dfrac{\partial f}{\partial y}(x,y_u)\varphi = \dfrac{\partial L}{\partial y}(x,y_u,u) \text{ in } \Omega,\\ \varphi = 0 \text{ on } \Gamma. \end{cases} \tag{3.6}$$

This is a straightforward consequence of Theorem 2.6, Assumption 3.1, and the chain rule. The only critical issue is the existence, uniqueness, and regularity of φ_u. But this is an immediate consequence of Lemma 2.2(iii) that, as already mentioned, applies to the operator $\mathcal{A}^*$ as well. From this theorem, the optimality conditions follow in the classical way.

Theorem 3.3. *Let $\bar u$ be a (strong or not strong) local minimizer of (P); then there exist two unique elements $\bar y, \bar\varphi \in H_0^1(\Omega)\cap C(\bar\Omega)$ such that*

$$\begin{cases} \mathcal{A}\bar y + f(x,\bar y) = \bar u \text{ in } \Omega,\\ \bar y = 0 \text{ on } \Gamma, \end{cases} \tag{3.7}$$

$$\begin{cases} \mathcal{A}^*\bar\varphi + \dfrac{\partial f}{\partial y}(x,\bar y)\bar\varphi = \dfrac{\partial L}{\partial y}(x,\bar y,\bar u) \text{ in } \Omega,\\ \bar\varphi = 0 \text{ on } \Gamma, \end{cases} \tag{3.8}$$

$$\int_\Omega (\bar\varphi + g)(u - \bar u)\,dx \ge 0 \quad \forall u \in \mathcal{U}_{ad}. \tag{3.9}$$

The derivation of sufficient second order conditions for local optimality is more delicate. First, we introduce the cone of critical directions on which we formulate the necessary second order conditions for optimality: if $\bar u \in \mathcal{U}_{ad}$ is a local minimizer of (P), we define

$$C_{\bar u} = \{v \in L^2(\Omega) : J'(\bar u)v = 0 \text{ and } v \text{ satisfies the sign conditions (3.10)}\},$$

$$v(x)\begin{cases} \ge 0 & \text{if } \bar u(x) = u_a,\\ \le 0 & \text{if } \bar u(x) = u_b. \end{cases} \tag{3.10}$$

As usual, from (3.9) we deduce that $(\bar\varphi + g)(x)v(x) \geq 0$ for almost all $x \in \Omega$ if $v \in L^2(\Omega)$ satisfies (3.10). Therefore, the condition $J'(\bar u)v = 0$ for v satisfying (3.10) is possible only if $v(x) = 0$ for almost every $x \in \Omega$ such that $(\bar\varphi + g)(x) \neq 0$. Therefore, $C_{\bar u}$ can be written

$$C_{\bar u} = \{v \in L^2(\Omega) : \text{satisfying (3.10) and } v(x) = 0 \text{ if } |(\bar\varphi + g)(x)| > 0\}.$$

It is well known that every local minimizer $\bar u$ satisfies the second order necessary optimality condition $J''(\bar u)v^2 \geq 0$ for all $v \in C_{\bar u}$; see, for instance, [8]. However, based on $C_{\bar u}$ it is not possible to get sufficient second order conditions for local optimality. The reader is referred to [12] for a counterexample. A procedure suggested by several authors consists of extending the cone of critical directions $C_{\bar u}$; see [10], [11], [18], [19]. Two possible extensions of $C_{\bar u}$ seem natural after the above comments: for $\tau > 0$ we define the extended cones

$$D^\tau_{\bar u} = \{v \in L^2(\Omega) : \text{satisfying (3.10) and } v(x) = 0 \text{ if } |(\bar\varphi + g)(x)| > \tau\},$$
$$G^\tau_{\bar u} = \{v \in L^2(\Omega) : \text{satisfying (3.10) and } J'(\bar u)v \leq \tau \|z_v\|_{L^1(\Omega)}\}.$$

On any of these cones we can formulate sufficient second order conditions for local optimality. Obviously, both are extensions of $C_{\bar u}$. In [3], the authors introduced the cone $C^\tau_{\bar u} = D^\tau_{\bar u} \cap G^\tau_{\bar u}$, which is also an extension of $C_{\bar u}$. They proved that the first order optimality conditions (3.7)–(3.9), along with the condition

$$\exists \delta > 0 \text{ such that } J''(\bar u)v^2 \geq \delta \|z_v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar u}, \tag{3.11}$$

imply the existence of $\kappa > 0$ and $\varepsilon > 0$ such that

$$J(\bar u) + \frac{\kappa}{2}\|y_u - \bar y\|^2_{L^2(\Omega)} \leq J(u) \quad \forall u \in \mathcal{U}_{ad} \text{ such that } \|y_u - \bar y\|_{C(\bar\Omega)} < \varepsilon. \tag{3.12}$$

Actually, the proof in [3] was carried out for a parabolic control problem with $g = 0$. However, the same proof works for the elliptic case and $g \neq 0$. Here, we replace (3.11) with a new assumption that also implies (3.12)

Assumption 3.4. There exist numbers $\alpha > 0$ and $\gamma > 0$ such that

$$J'(\bar u)(u - \bar u) + J''(\bar u)(u - \bar u)^2 \geq \gamma \|z_{\bar u, u-\bar u}\|^2_{L^2(\Omega)} \ \forall u \in \mathcal{U}_{ad} \text{ with } \|y_u - \bar y\|_{C(\bar\Omega)} < \alpha. \tag{3.13}$$

It was proved in [4] that (3.11) implies (3.13). Therefore, (3.13) appears as a weaker assumption. However, the next theorem proves that it is sufficient to imply (3.12).

THEOREM 3.5. *Let $\bar u \in \mathcal{U}_{ad}$ satisfy the optimality conditions* (3.7)–(3.9) *and Assumption* 3.4. *Then, there exist $\varepsilon > 0$ and $\kappa > 0$ such that* (3.12) *holds.*

Before proving this theorem we establish some lemmas.

LEMMA 3.6. *Let $\bar u \in \mathcal{U}_{ad}$ be fixed with associated state $\bar y$. Then, the following inequality holds for all $\theta \in [0, 1]$ and $u \in \mathcal{U}_{ad}$:*

$$\|y_{\bar u + \theta(u - \bar u)} - \bar y\|_{C(\bar\Omega)} \leq (C_2 C_{f,K_U}\sqrt{|\Omega|}\|y_u - \bar y\|_{C(\bar\Omega)} + 1)\|y_u - \bar y\|_{C(\bar\Omega)}, \tag{3.14}$$

where C_2 is the constant of (2.2) *with $r = 2$, and C_{f,K_U} is the one deduced from* (2.5) *and* (2.8).

Proof. The proof of this lemma is based on the analogous result for parabolic control problems established in [5]. We take $\theta \in [0,1]$ and $u \in \mathcal{U}_{ad}$. We set $\phi = y_{\bar u+\theta(u-\bar u)} - [\bar y + \theta(y_u - \bar y)]$. Then, we have

$$\mathcal{A}\phi + f(x, y_{\bar u+\theta(u-\bar u)}) - [f(x,\bar y) + \theta(f(x,y_u) - f(x,\bar y))] = 0.$$

Applying the mean value theorem, we obtain measurable functions $\theta_i : \Omega \longrightarrow [0,1]$, $i = 1,2$, such that

$$f(x, y_{\bar u+\theta(u-\bar u)}) - f(x,\bar y) = \frac{\partial f}{\partial y}(x,y_1)(y_{\bar u+\theta(u-\bar u)} - \bar y) \text{ and } y_1 = \bar y + \theta_1(y_{\bar u+\theta(u-\bar u)} - \bar y),$$

$$f(x,y_u) - f(x,\bar y) = \frac{\partial f}{\partial y}(x,y_2)(y_u - \bar y) \text{ with } y_2 = \bar y + \theta_2(y_u - \bar y).$$

Inserting these identities into the above partial differential equation, we infer

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x,y_1)(y_{\bar u+\theta(u-\bar u)} - \bar y) - \theta\frac{\partial f}{\partial y}(x,y_2)(y_u - \bar y) = 0.$$

Noting that $y_{\bar u+\theta(u-\bar u)} - \bar y = \phi + \theta(y_u - \bar y)$, we see the above equality and a new application of the mean value theorem lead to

$$\mathcal{A}\phi + \frac{\partial f}{\partial y}(x,y_1)\phi = \theta\left[\frac{\partial f}{\partial y}(x,y_2) - \frac{\partial f}{\partial y}(x,y_1)\right](y_u - \bar y) = \theta\frac{\partial^2 f}{\partial y^2}(x,y_3)(y_u - \bar y)^2,$$

where $y_3 = y_1 + \theta_3(y_2 - y_1)$. Using (2.2) with $r = 2$, (2.5), and (2.8) we infer

$$\|\phi\|_{C(\bar\Omega)} \le C_2 C_{f,K_U}\|(y_u - \bar y)^2\|_{L^2(\Omega)} \le C_2 C_{f,K_U}\sqrt{|\Omega|}\|y_u - \bar y\|^2_{C(\bar\Omega)}.$$

This implies

$$\begin{aligned}\|y_{\bar u+\theta(u-\bar u)} - \bar y\|_{C(\bar\Omega)} &= \|\phi + \theta(y_u - \bar y)\|_{C(\bar\Omega)}\\ &\le (C_2 C_{f,K_U}\sqrt{|\Omega|}\|y_u - \bar y\|_{C(\bar\Omega)} + 1)\|y_u - \bar y\|_{C(\bar\Omega)}.\end{aligned}$$ ∎

Lemma 3.7. *There exists a constant $M_U > 0$ such that*

$$\|\varphi_u\|_{C(\bar\Omega)} \le M_U \quad \forall u \in \mathcal{U}_{ad}. \tag{3.15}$$

Moreover, given $\bar u \in \mathcal{U}_{ad}$ with associated state $\bar y$ and adjoint state $\bar\varphi$, we have

$$\|\varphi_{\bar u+\theta(u-\bar u)} - \bar\varphi\|_{C(\bar\Omega)} \le C\|y_u - \bar y\|_{C(\bar\Omega)} \quad \forall\theta \in [0,1] \text{ and } \forall u \in \mathcal{U}_{ad}, \tag{3.16}$$

where C depends only on f, L, $\mathcal{U}_{ad}$, and Ω.

Proof. For the proof of (3.15) we use (2.2) with $r = 2$, (2.8), and (3.2) as follows:

$$\|\varphi_u\|_{C(\bar\Omega)} \le C_2\left\|\frac{\partial L}{\partial y}(x,y_u,u)\right\|_{L^2(\Omega)} \le M_U = C_2\|\psi_{K_U}\|_{L^2(\Omega)}.$$

Let us prove (3.16). Given $u \in \mathcal{U}_{ad}$ and $\theta \in [0,1]$ let us denote $u_\theta = \bar u + \theta(u - \bar u)$, $y_\theta = y_{u_\theta}$, and $\varphi_\theta = \varphi_{u_\theta}$. Subtracting the equations satisfied by φ_θ and $\bar\varphi$, we get with the mean value theorem

$$\begin{aligned}\mathcal{A}^*(\varphi_\theta - \bar\varphi) + \frac{\partial f}{\partial y}(x,\bar y)(\varphi_\theta - \bar\varphi) &= \frac{\partial L}{\partial y}(x,y_\theta,u_\theta) - \frac{\partial L}{\partial y}(x,\bar y,\bar u)\\ + \left[\frac{\partial f}{\partial y}(x,\bar y) - \frac{\partial f}{\partial y}(x,y_\theta)\right]\varphi_\theta &= \left[\frac{\partial^2 L}{\partial y^2}(x,y_\vartheta,u_\vartheta) - \varphi_\theta\frac{\partial^2 f}{\partial y^2}(x,y_\vartheta)\right](y_\theta - \bar y),\end{aligned}$$

where $y_\vartheta = \bar y + \vartheta(y_\theta - \bar y)$ for some measurable function $\vartheta : \Omega \longrightarrow [0,1]$. Now, we apply (2.2) with $r = 2$, (2.8), (3.15), (2.5), and (3.2) to get, from the above equation,

$$\|\varphi_\theta - \bar\varphi\|_{C(\bar\Omega)} \le C_2(C_{L,K_U} + M_U C_{f,K_U})\sqrt{|\Omega|}\|y_\theta - \bar y\|_{C(\bar\Omega)}.$$

Then, (3.16) follows from Lemma 3.6. ∎

LEMMA 3.8. *For every $\rho > 0$ there exists $\varepsilon > 0$ such that if $u \in \mathcal{U}_{ad}$ and $\|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon$, then*

$$|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})]v^2| < \rho\|z_{\bar{u},v}\|^2_{L^2(\Omega)} \quad \forall v \in L^2(\Omega) \text{ and } \forall\theta \in [0, 1]. \tag{3.17}$$

Proof. First, let us denote u_θ, y_θ, and φ_θ as in the proof of Lemma 3.7. From (3.5) we get

$$\begin{aligned}
&|[J''(\bar{u} + \theta(u - \bar{u})) - J''(\bar{u})]v^2| \\
&\leq \int_\Omega \left| \left[\frac{\partial^2 L}{\partial y^2}(x, y_\theta, u_\theta) - \frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) \right] z^2_{u_\theta,v} \right| \mathrm{d}x \\
&\quad + \int_\Omega \left| (\varphi_\theta - \bar{\varphi}) \frac{\partial^2 f}{\partial y^2}(x, y_\theta) z^2_{u_\theta,v} \right| \mathrm{d}x + \int_\Omega \left| \bar{\varphi} \left[\frac{\partial^2 f}{\partial y^2}(x, y_\theta) - \frac{\partial^2 f}{\partial y^2}(x, \bar{y}) \right] z^2_{u_\theta,v} \right| \mathrm{d}x \\
&\quad + \int_\Omega \left| \left[\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) - \bar{\varphi} \frac{\partial^2 f}{\partial y^2}(x, \bar{y}) \right] (z^2_{u_\theta,v} - z^2_{\bar{u},v}) \right| \mathrm{d}x \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us estimate the terms I_i. For I_1 we deduce from (3.3), (2.15), and (3.14) that for every $\rho > 0$ there exists $\varepsilon > 0$ such that $I_1 \leq \rho\|z_{\bar{u},v}\|^2_{L^2(\Omega)}$ if $\|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon$. The same estimate can be deduced for I_2 using (2.5), (2.8), (2.15), and (3.16). The estimate for I_3 follows from (2.6), (2.8), (2.15), (3.14), and (3.15). Finally, we estimate I_4 by using (2.5), (2.8), (2.13), (2.15), (3.2), (3.14), and (3.15) to infer that

$$\begin{aligned}
I_4 &\leq (C_{L,K_U} + M_U C_{f,K_U})\|z_{u_\theta,v} + z_{\bar{u},v}\|_{L^2(\Omega)}\|z_{u_\theta,v} - z_{\bar{u},v}\|_{L^2(\Omega)} \\
&\leq \frac{5}{2}(C_{L,K_U} + M_U C_{f,K_U})C_{L^2(\Omega)}|\Omega|^{\frac{1}{2}}\|z_{\bar{u},v}\|_{L^2(\Omega)}\|y_\theta - \bar{y}\|_{C(\bar{\Omega})}\|z_{\bar{u},v}\|_{L^2(\Omega)} \\
&\leq \rho\|z_{\bar{u},v}\|^2_{L^2(\Omega)} \quad \text{if } \|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon.
\end{aligned}$$

Hence, (3.17) is a straightforward consequence of the above estimates. ∎

Proof of Theorem 3.5. Let us take $u \in \mathcal{U}_{ad}$ with $\|y_u - \bar{y}\|_{C(\bar{\Omega})} < \alpha$. By performing a Taylor expansion and using the fact that $J'(\bar{u})(u - \bar{u}) \geq 0$, we obtain

$$\begin{aligned}
J(u) &= J(\bar{u}) + J'(\bar{u})(u - \bar{u}) + \frac{1}{2}J''(u_\theta)(u - \bar{u})^2 \\
&\geq J(\bar{u}) + \frac{1}{2}[J'(\bar{u})(u - \bar{u}) + J''(\bar{u})(u - \bar{u})^2] + \frac{1}{2}[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2 \\
&\geq J(\bar{u}) + \frac{\gamma}{2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)} - \frac{1}{2}|[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2|.
\end{aligned}$$

Lemma 3.8 implies the existence of $\varepsilon \in (0, \alpha]$ such that $|[J''(u_\theta) - J''(\bar{u})](u - \bar{u})^2| < \frac{\gamma}{2}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)}$ for every $u \in \mathcal{U}_{ad}$ with $\|y_u - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon$. Inserting this estimate into the above expression and taking ε still smaller if necessary, we can apply (2.14) to deduce

$$J(u) \geq J(\bar{u}) + \frac{\gamma}{4}\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)} \geq J(\bar{u}) + \frac{\gamma}{16}\|y_u - \bar{y}\|^2_{L^2(\Omega)}.$$

This inequality yields (3.12) with $\kappa = \frac{\gamma}{8}$. ∎

4. Stability of the states. In this section, we consider the following perturbations of the control problem (P):

$$(\mathrm{P}_\varepsilon) \quad \min_{u \in \mathcal{U}_{ad}} J_\varepsilon(u) := \int_\Omega [L(x, y^\varepsilon_u(x), u(x)) + \eta_\varepsilon(x)y^\varepsilon_u(x)]\,\mathrm{d}x,$$

where y_u^ε is the solution of the equation

$$\left\{\begin{array}{l} -\mathrm{div}\big(A(x)\nabla y\big) + b(x)\cdot\nabla y + f(x,y) = u + \xi_\varepsilon \ \text{ in } \ \Omega,\\ y = 0 \ \text{ on } \ \Gamma. \end{array}\right. \tag{4.1}$$

Here we assume that $\{\xi_\varepsilon\}_{\varepsilon>0}$ and $\{\eta_\varepsilon\}_{\varepsilon>0}$ are bounded families in $L^2(\Omega)$ satisfying $(\xi_\varepsilon,\eta_\varepsilon)\to(0,0)$ in $L^2(\Omega)^2$ as $\varepsilon\to 0$. As a consequence of Theorem 2.5 we get the existence and uniqueness of a solution $y_u^\varepsilon\in H_0^1(\Omega)\cap C(\bar\Omega)$ of (4.1). Moreover, using (2.7) with $r=2$ and the boundedness of $\{\xi_\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ we infer that the set $\{y_u^\varepsilon : u\in\mathcal{U}_{ad}$ and $\varepsilon>0\}$ is bounded in $H_0^1(\Omega)\cap C(\bar\Omega)$. Therefore, increasing the value of K_U, if necessary, we can assume that (2.8) and the inequality

$$\|y_u^\varepsilon\|_{H_0^1(\Omega)} + \|y_u^\varepsilon\|_{C(\bar\Omega)} \le K_U \quad \forall u\in\mathcal{U}_{ad} \ \text{ and } \ \forall\varepsilon>0 \tag{4.2}$$

hold. We will prove that the solutions of problems (P$_\varepsilon$) converge to the solutions of (P) in some sense to be made precise below. Conversely, we will also prove that any strict strong local minimizer of (P) can be approximated by strong local minimizers of problems (P$_\varepsilon$). Finally, the Lipschitz stability of the optimal states with respect to the perturbations is established. We begin by analyzing the difference between the solutions of (1.1) and (4.1).

THEOREM 4.1. *The following inequalities hold for every $\varepsilon>0$:*

$$\|y_u^\varepsilon - y_u\|_{H_0^1(\Omega)} + \|y_u^\varepsilon - y_u\|_{C(\bar\Omega)} \le C_2\|\xi_\varepsilon\|_{L^2(\Omega)} \ \forall u\in L^2(\Omega), \tag{4.3}$$

$$\|z_{u,v}^\varepsilon - z_{u,v}\|_{L^2(\Omega)} \le C_2^2 C_{f,K_U}\|\xi_\varepsilon\|_{L^2(\Omega)}\|z_{u,v}\|_{L^2(\Omega)} \ \forall (u,v)\in\mathcal{U}_{ad}\times L^2(\Omega), \tag{4.4}$$

where C_2 is the constant given in (2.2) *for $r=2$, C_{f,K_U} is the constant $C_{f,M}$ of* (2.5) *with $M=K_U$ given as in* (2.8) *or* (4.2), *and $z_{u,v}^\varepsilon$ denotes the solution of* (2.9) *with y_u^ε replacing y_u.*

Proof. Subtracting the equations (4.1) and (1.1) and using the mean value theorem, we obtain

$$\mathcal{A}(y_u^\varepsilon - y_u) + \frac{\partial f}{\partial y}(x,y_\theta)(y_u^\varepsilon - y_u) = \xi_\varepsilon.$$

Then, (2.2) implies (4.3). To prove (4.4) we subtract the equations satisfied by $z_{u,v}^\varepsilon$ and $z_{u,v}$ to obtain

$$\mathcal{A}(z_{u,v}^\varepsilon - z_{u,v}) + \frac{\partial f}{\partial y}(x,y_u^\varepsilon)(z_{u,v}^\varepsilon - z_{u,v}) = \left[\frac{\partial f}{\partial y}(x,y_u) - \frac{\partial f}{\partial y}(x,y_u^\varepsilon)\right] z_{u,v}.$$

Now, using (2.3) with $s=2$, (2.5), (2.8), and (4.3), from the previous equation with the mean value theorem we obtain

$$\begin{aligned}
&\|z_{u,v}^\varepsilon - z_{u,v}\|_{L^2(\Omega)} \le C_2\left\|\left[\frac{\partial f}{\partial y}(x,y_u) - \frac{\partial f}{\partial y}(x,y_u^\varepsilon)\right] z_{u,v}\right\|_{L^1(\Omega)}\\
&\le C_2 C_{f,K_U}\|(y_u^\varepsilon - y_u)z_{u,v}\|_{L^1(\Omega)}\\
&\le C_2 C_{f,K_U}\|y_u^\varepsilon - y_u\|_{L^2(\Omega)}\|z_{u,v}\|_{L^2(\Omega)} \le C_2^2 C_{f,K_U}\|\xi_\varepsilon\|_{L^2(\Omega)}\|z_{u,v}\|_{L^2(\Omega)}. \qquad \square
\end{aligned}$$

Now we analyze the convergence of problems (P$_\varepsilon$) to (P).

THEOREM 4.2. *Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of problems* (P_ε). *Any control $\bar u$ that is a weak* limit in $L^\infty(\Omega)$ of a sequence $\{u_{\varepsilon_k}\}_{k=1}^\infty$ with $\varepsilon_k \to 0$ as $k\to\infty$ is a solution of (P). Moreover, the strong convergence $y^{\varepsilon_k}_{u_{\varepsilon_k}} \to y_{\bar u}$ in $H_0^1(\Omega)\cap C(\bar\Omega)$ holds.*

Proof. The existence of the sequences $\{u_{\varepsilon_k}\}_{k=1}^\infty$ converging to $\bar u$ weakly* in $L^\infty(\Omega)$ is a consequence of the boundedness of $\mathcal{U}_{ad}$ in $L^\infty(\Omega)$. From Theorem 2.5 and (4.3) we infer

$$\begin{aligned}
&\|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{\bar u}\|_{H_0^1(\Omega)} + \|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{\bar u}\|_{C(\bar\Omega)}\\
&\quad \le \|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{u_{\varepsilon_k}}\|_{H_0^1(\Omega)} + \|y^{\varepsilon_k}_{u_{\varepsilon_k}} - y_{u_{\varepsilon_k}}\|_{C(\bar\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar u}\|_{H_0^1(\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar u}\|_{C(\bar\Omega)}\\
&\quad \le C_2\|\xi_\varepsilon\|_{L^2(\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar u}\|_{H_0^1(\Omega)} + \|y_{u_{\varepsilon_k}} - y_{\bar u}\|_{C(\bar\Omega)} \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Using this fact, the convergence $\eta_\varepsilon \to 0$ as $\varepsilon\to 0$, (3.2), the optimality of u_{ε_k} for $(\mathrm{P}_{\varepsilon_k})$, and again (4.3), we get

$$J(\bar u) = \lim_{k\to\infty} J_{\varepsilon_k}(u_{\varepsilon_k}) \le \lim_{k\to\infty} J_{\varepsilon_k}(u) = J(u) \quad \forall u \in \mathcal{U}_{ad},$$

which proves that $\bar u$ is a solution of (P). ∎

Now, we establish a kind of converse result.

THEOREM 4.3. *Let $\bar u$ be a strict strong local minimizer of (P). Then, there exist $\varepsilon_0 > 0$ and a family of strong local minimizers $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of problems (P_ε) such that $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$ and $y^\varepsilon_{u_\varepsilon} \to y_{\bar u}$ strongly in $H_0^1(\Omega)\cap C(\bar\Omega)$ as $\varepsilon\to 0$.*

Proof. Since $\bar u$ is a strict strong local minimizer of (P), there exists $\rho > 0$ such that $\bar u$ is the unique solution of the problem

$$(\mathrm{P}_\rho) \quad \min_{u\in\mathcal{U}_\rho} J(u),$$

where $\mathcal{U}_\rho = \{u \in \mathcal{U}_{ad} : \|y_u - y_{\bar u}\|_{C(\bar\Omega)} \le \rho\}$. Now, for every $\varepsilon > 0$ we define the problems

$$(\mathrm{P}_{\rho,\varepsilon}) \quad \min_{u\in\mathcal{U}_\rho} J_\varepsilon(u).$$

Using Theorem 2.5 we deduce that $\mathcal{U}_\rho$ is weakly* closed in $L^\infty(\Omega)$, and hence the existence of a solution u_ε of $(\mathrm{P}_{\rho,\varepsilon})$ can be proved as we indicated for (P). Moreover, arguing as in the proof of Theorem 4.2, we deduce the existence of sequences $\{u_{\varepsilon_k}\}_{k=1}^\infty$ converging weakly* to a solution u of (P_ρ) in $L^\infty(\Omega)$ and such that $y^{\varepsilon_k}_{u_{\varepsilon_k}} \to y_u$ strongly in $H_0^1(\Omega)\cap C(\bar\Omega)$. Since $\bar u$ is the unique solution of (P_ρ), we conclude the convergence $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$ and $y^\varepsilon_{u_\varepsilon} \to y_{\bar u}$ in $H_0^1(\Omega)\cap C(\bar\Omega)$ as $\varepsilon\to 0$. Therefore, there exists $\varepsilon_0 > 0$ such that $\|y^\varepsilon_{u_\varepsilon} - y_{\bar u}\|_{C(\bar\Omega)} < \rho$ for every $\varepsilon < \varepsilon_0$. This implies that u_ε is a strong local minimizer of (P_ε) for every $\varepsilon < \varepsilon_0$, which completes the proof. ∎

Now we establish our main theorem of this section.

THEOREM 4.4. *Let $\bar u$ be a local minimizer of (P) satisfying Assumption* 3.4 *and $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ be a family of local solutions of problems (P_ε) such that $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$ as $\varepsilon\to 0$. Then, there exist $\hat\varepsilon \in (0,\varepsilon_0)$ and a constant $C > 0$ such that*

$$\|y^\varepsilon_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} \le C\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big) \quad \forall \varepsilon < \hat\varepsilon, \tag{4.5}$$

where $\bar y = y_{\bar u}$.

Let us observe that Assumption 3.4 implies that $\bar{u}$ satisfies (3.12). Hence, $\bar{u}$ is a strict strong local minimizer of (P), and, consequently, Theorem 4.3 ensures the existence of a family $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ of strong local minimizers of problems (P_ε) satisfying the conditions of the above theorem. Before proving this theorem we establish the following lemma.

LEMMA 4.5. *Let $\bar{u}$ satisfy the assumptions of Theorem 4.4. Then, there exists $\varepsilon>0$ such that*

$$J'(u)(u-\bar{u}) \geq \frac{\gamma}{2}\|z_{u,u-\bar{u}}\|^2_{L^2(\Omega)} \quad \forall u \in \mathcal{U}_{ad} \text{ with } \|y_u-\bar{y}\|_{C(\bar{\Omega})}<\varepsilon, \tag{4.6}$$

where γ is given as in Assumption 3.4.

Proof. We denote by $H:\Omega\times\mathbb{R}^3\longrightarrow\mathbb{R}$ as follows the Hamiltonian associated with the control problem (P):

$$H(x,y,\varphi,u)=L(x,y,u)+\varphi[u-f(x,y)].$$

For every $u\in\mathcal{U}_{ad}$ and $v\in L^2(\Omega)$, we define $\psi_{u,v}\in H^1_0(\Omega)\cap C(\bar{\Omega})$ as the function satisfying

$$\mathcal{A}^*\psi_{u,v}+\frac{\partial f}{\partial y}(x,y_u)\psi_{u,v}=\frac{\partial^2 H}{\partial y^2}(x,y_u,\varphi_u,u)z_{u,v}.$$

We split the proof into two steps.

Step I. Here we prove that for every $\rho>0$ there exists $\varepsilon>0$ such that for every $u\in\mathcal{U}_{ad}$ with $\|y_u-\bar{y}\|_{C(\bar{\Omega})}<\varepsilon$ we have

$$\left|\int_\Omega(\varphi_u-\bar{\varphi}-\psi_{\bar{u},u-\bar{u}})(u-\bar{u})\,\mathrm{d}x\right|\leq\rho\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)}. \tag{4.7}$$

Setting $\pi=\varphi_u-\bar{\varphi}-\psi_{\bar{u},u-\bar{u}}$ and subtracting their respective equations, it follows with the mean value theorem that

$$\begin{aligned}
\mathcal{A}^*\pi+\frac{\partial f}{\partial y}(x,\bar{y})\pi &= \frac{\partial H}{\partial y}(x,y_u,\varphi_u,u)-\frac{\partial H}{\partial y}(x,\bar{y},\bar{\varphi},\bar{u})\\
&\quad-\frac{\partial^2 H}{\partial y^2}(x,\bar{y},\bar{\varphi},\bar{u})z_{\bar{u},u-\bar{u}}-\frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar{y},\bar{\varphi},\bar{u})(\varphi_u-\bar{\varphi})\\
&=\frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)(y_u-\bar{y})-\frac{\partial^2 H}{\partial y^2}(x,\bar{y},\bar{\varphi},\bar{u})z_{\bar{u},u-\bar{u}}\\
&\quad+\left[\frac{\partial^2 H}{\partial y\partial\varphi}(x,y_\theta,\varphi_\theta,u_\theta)-\frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar{y},\bar{\varphi},\bar{u})\right](\varphi_u-\bar{\varphi})\\
&=\frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)(y_u-\bar{y}-z_{\bar{u},u-\bar{u}})\\
&\quad+\left[\frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)-\frac{\partial^2 H}{\partial y^2}(x,\bar{y},\bar{\varphi},\bar{u})\right]z_{\bar{u},u-\bar{u}}\\
&\quad+\left[\frac{\partial^2 H}{\partial y\partial\varphi}(x,y_\theta,\varphi_\theta,u_\theta)-\frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar{y},\bar{\varphi},\bar{u})\right](\varphi_u-\bar{\varphi}).
\end{aligned}$$

This implies

$$
\begin{aligned}
\int_\Omega \pi(u-\bar u)\,\mathrm{d}x &= \int_\Omega \pi\left(\mathcal{A} z_{\bar u,u-\bar u} + \frac{\partial f}{\partial y}(x,\bar y) z_{\bar u,u-\bar u}\right)\mathrm{d}x\\
&= \int_\Omega \left(\mathcal{A}^*\pi + \frac{\partial f}{\partial y}(x,\bar y)\pi\right) z_{\bar u,u-\bar u}\,\mathrm{d}x\\
&= \int_\Omega \frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta)(y_u-\bar y - z_{\bar u,u-\bar u}) z_{\bar u,u-\bar u}\,\mathrm{d}x\\
&\quad + \int_\Omega \left[\frac{\partial^2 H}{\partial y^2}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u)\right] z^2_{\bar u,u-\bar u}\,\mathrm{d}x\\
&\quad + \int_\Omega \left[\frac{\partial^2 H}{\partial y\partial\varphi}(x,y_\theta,\varphi_\theta,u_\theta) - \frac{\partial^2 H}{\partial y\partial\varphi}(x,\bar y,\bar\varphi,\bar u)\right](\varphi_u-\bar\varphi) z_{\bar u,u-\bar u}\,\mathrm{d}x\\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

We estimate every term I_i. For the first term we use (2.5), (2.8), (2.12) with $s=2$, (2.14) with $X=L^2(\Omega)$, (3.2), and (3.15) as follows:

$$
\begin{aligned}
|I_1| &\le (C_{L,K_U} + M_U C_{f,K_U})\|y_u - \bar y - z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le (C_{L,K_U} + M_U C_{f,K_U}) M_2 \|y_u-\bar y\|^2_{L^2(\Omega)}\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le 2(C_{L,K_U} + M_U C_{f,K_U}) M_2 \sqrt{|\Omega|}\varepsilon \|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

The second term is estimated with (2.6), (2.8), (3.3), (3.14), (3.15), (3.16), leading to $|I_2| \le \rho\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}$ for ρ arbitrarily small if ε is taken according to ρ. Finally, for the last term we use the same inequalities as for I_2 the fact that (3.16) holds true with $L^2(\Omega)$ instead of $C(\bar\Omega)$ and additionally (2.15) with $X=L^2(\Omega)$ to get

$$
\begin{aligned}
|I_3| &\le \rho\|\varphi_u-\bar\varphi\|_{L^2(\Omega)}\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le \rho C_2 (C_{L,K_U} + M_U C_{f,K_U})\|y_u-\bar y\|_{L^2(\Omega)}\|z_{\bar u,u-\bar u}\|_{L^2(\Omega)}\\
&\le 2\rho C_2(C_{L,K_U} + M_U C_{f,K_U})\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where again ρ is arbitrarily small if ε is chosen according to it. Thus, (4.7) follows from the proved estimates.

Step II. Now, we prove (4.6). First, we observe that for every $v\in L^2(\Omega)$,

$$
\begin{aligned}
\int_\Omega \psi_{\bar u,v} v\,\mathrm{d}x &= \int_\Omega \psi_{\bar u,v}\left(\mathcal{A} z_{\bar u,v} + \frac{\partial f}{\partial y}(x,\bar y) z_{\bar u,v}\right)\mathrm{d}x\\
&= \int_\Omega \left(\mathcal{A}^*\psi_{\bar u,v} + \frac{\partial f}{\partial y}(x,\bar y)\psi_{\bar u,v}\right) z_{\bar u,v}\,\mathrm{d}x = \int_\Omega \frac{\partial^2 H}{\partial y^2}(x,\bar y,\bar\varphi,\bar u) z^2_{\bar u,v}\,\mathrm{d}x\\
&= J''(\bar u)v^2,
\end{aligned}
$$

where the last inequality follows from (3.5) and the definition of the Hamiltonian. Let $\varepsilon>0$ be such that (4.7) holds with $\rho=\frac{\gamma}{2}$. Then, using Assumption 3.4 and (4.7), for $u\in\mathcal{U}_{ad}$ with $\|y_u-\bar y\|_{C(\bar\Omega)}<\varepsilon$ we get

$$
\begin{aligned}
J'(u)(u-\bar u) &= \int_\Omega (\varphi_u + g)(u-\bar u)\,\mathrm{d}x\\
&= \int_\Omega (\varphi_u - \bar\varphi - \psi_{\bar u,u-\bar u})(u-\bar u)\,\mathrm{d}x + \int_\Omega (\bar\varphi + g + \psi_{\bar u,u-\bar u})(u-\bar u)\,\mathrm{d}x\\
&\ge -\frac{\gamma}{2}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)} + [J'(\bar u)(u-\bar u) + J''(\bar u)(u-\bar u)^2] \ge \frac{\gamma}{2}\|z_{\bar u,u-\bar u}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

□

Remark 4.6. Let us note that if $\bar{u}$ is a local minimizer of (P) satisfying Assumption 3.4, then there exists $\varepsilon > 0$ such that there is no stationary point $\hat{u}$ of (P) different from $\bar{u}$ such that $\|y_{\hat{u}} - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon$. We say that $\hat{u}$ is a stationary point of (P) if it satisfies the first order optimality condition. In particular, if $\hat{u}$ is a stationary point, then $J'(\hat{u})(\bar{u} - \hat{u}) \geq 0$. This contradicts (4.6) if $\|y_{\hat{u}} - \bar{y}\|_{C(\bar{\Omega})} < \varepsilon$.

Proof of Theorem 4.4. Using the local optimality of u_ε, we get

$$\begin{aligned} 0 &\geq J'_\varepsilon(u_\varepsilon)(u_\varepsilon - \bar{u}) \\ &= J'(u_\varepsilon)(u_\varepsilon - \bar{u}) + \int_\Omega \left[\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon) - \frac{\partial L}{\partial y}(x, y_{u_\varepsilon}, u_\varepsilon)\right] z_{u_\varepsilon, u_\varepsilon - \bar{u}} \,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon)(z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar{u}} - z_{u_\varepsilon, u_\varepsilon - \bar{u}}) \,\mathrm{d}x + \int_\Omega \eta_\varepsilon z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar{u}} \,\mathrm{d}x. \end{aligned} \tag{4.8}$$

We estimate each one of these four terms. First, we observe that the convergence $u_\varepsilon \rightharpoonup \bar{u}$ in $L^2(\Omega)$ implies that $\|y_{u_\varepsilon} - \bar{y}\|_{C(\bar{\Omega})} \to 0$; see Theorem 2.5. Hence, from Lemma 4.5 we deduce the existence of $\varepsilon_1 > 0$ such that

$$J'(u_\varepsilon)(u_\varepsilon - \bar{u}) \geq \frac{\gamma}{2} \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|^2_{L^2(\Omega)} \quad \forall \varepsilon < \varepsilon_1. \tag{4.9}$$

For the second term we use Schwarz's inequality, the mean value theorem, (2.8), (4.2), (3.2), and (4.3) to get

$$\begin{aligned} &\int_\Omega \left|\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon) - \frac{\partial L}{\partial y}(x, y_{u_\varepsilon}, u_\varepsilon)\right| |z_{u_\varepsilon, u_\varepsilon - \bar{u}}| \,\mathrm{d}x \\ &\quad \leq C_{L,K_U} \|y^\varepsilon_{u_\varepsilon} - y_{u_\varepsilon}\|_{L^2(\Omega)} \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)} \\ &\quad \leq C_{L,K_U} \sqrt{|\Omega|} C_2 \|\xi_\varepsilon\|_{L^2(\Omega)} \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)}. \end{aligned} \tag{4.10}$$

Now we estimate the third term with (3.2), (4.2), Schwarz's inequality, and (4.4) to get

$$\begin{aligned} &\int_\Omega \left|\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon)\right| |z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar{u}} - z_{u_\varepsilon, u_\varepsilon - \bar{u}}| \,\mathrm{d}x \\ &\quad \leq \int_\Omega \psi_{K_U} |z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar{u}} - z_{u_\varepsilon, u_\varepsilon - \bar{u}}| \,\mathrm{d}x \\ &\quad \leq \|\psi_{K_U}\|_{L^2(\Omega)} C_2^2 C_{f,K_U} \|\xi_\varepsilon\|_{L^2(\Omega)} \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)}. \end{aligned} \tag{4.11}$$

For the last term, we use again (4.4) and the fact that $\{\xi_\varepsilon\}_{\varepsilon > 0}$ is bounded in $L^2(\Omega)$, obtaining

$$\begin{aligned} \int_\Omega |\eta_\varepsilon z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar{u}}| \,\mathrm{d}x &\leq \|\eta_\varepsilon\|_{L^2(\Omega)} \Big(\|z^\varepsilon_{u_\varepsilon, u_\varepsilon - \bar{u}} - z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)} + \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)}\Big) \\ &\leq \Big(C_2^2 C_{f,K_U} \|\xi_\varepsilon\|_{L^2(\Omega)} + 1\Big) \|\eta_\varepsilon\|_{L^2(\Omega)} \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)} \\ &\leq C \|\eta_\varepsilon\|_{L^2(\Omega)} \|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)}. \end{aligned} \tag{4.12}$$

Inserting the estimates (4.9)–(4.12) into (4.8), for some constant $C' > 0$ and every $\varepsilon < \varepsilon_1$, we obtain

$$\|z_{u_\varepsilon, u_\varepsilon - \bar{u}}\|_{L^2(\Omega)} \leq C' \Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big).$$

Finally, using (2.14) and (4.3) we deduce the existence of $\varepsilon_2 \in (0, \varepsilon_1]$ such that for every $\varepsilon < \varepsilon_2$, we have

$$\begin{aligned}\|y^\varepsilon_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} &\le \|y^\varepsilon_{u_\varepsilon} - y_{u_\varepsilon}\|_{L^2(\Omega)} + \|y_{u_\varepsilon} - \bar y\|_{L^2(\Omega)}\\ &\le C_2\sqrt{|\Omega|}\|\xi_\varepsilon\|_{L^2(\Omega)} + 2\|z_{u_\varepsilon, u_\varepsilon - \bar u}\|_{L^2(\Omega)}\\ &\le C_2\sqrt{|\Omega|}\|\xi_\varepsilon\|_{L^2(\Omega)} + 2C'\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big),\end{aligned}$$

which proves (4.5). ∎

5. Stability of the controls. In the previous section, we established Lipschitz stability for the optimal states with respect to state perturbations in the objective functional and to the force in the state equation. In order to obtain stability of the optimal controls, an additional assumption is usually required. The reader is referred to Qui and Wachsmuth [21] for the following assumption:

$$\exists C > 0 \text{ such that } |\{x \in \Omega : |(\varphi + g)(x)| \le \varepsilon\}| \le C\varepsilon \quad \forall \varepsilon > 0. \tag{5.1}$$

Using this assumption and sufficient second order optimality conditions, they proved Lipschitz stability of the controls in the $L^1(\Omega)$ norm. However, the assumption (5.1) implies that $\bar u$ is bang-bang. As far as we know, there is no proof of stability for the optimal controls when they are not bang-bang. Assumption 3.4 considered in the previous sections is applicable for the case of optimal controls that are not bang-bang. Nevertheless, it leads only to Lipschitz stability of the optimal states. Here, we modify Assumption 3.4 as follows.

Assumption 5.1. There exist numbers $\alpha > 0$ and $\gamma > 0$ such that for all $u \in \mathcal{U}_{ad}$ with $\|y_u - \bar y\|_{C(\bar\Omega)} < \alpha$ the following inequality is fulfilled:

$$J'(\bar u)(u - \bar u) + J''(\bar u)(u - \bar u)^2 \ge \gamma \|z_{\bar u, u - \bar u}\|_{L^2(\Omega)} \|u - \bar u\|_{L^1(\Omega)}. \tag{5.2}$$

Under this assumption we will prove Lipschitz stability of the optimal controls. It has been proved in [9] that the sufficient second order conditions plus the structural assumption (5.1) imply the existence of positive numbers γ and α such that

$$J'(\bar u)(u - \bar u) + J''(\bar u)(u - \bar u)^2 \ge \gamma \|u - \bar u\|^2_{L^1(\Omega)} \ \forall u \in \mathcal{U}_{ad} \text{ with } \|u - \bar u\|_{L^1(\Omega)} < \alpha. \tag{5.3}$$

We have the next equivalence.

PROPOSITION 5.2. *The statement* (5.3) *is equivalent to the existence of positive numbers γ' and α' such that*

$$J'(\bar u)(u - \bar u) + J''(\bar u)(u - \bar u)^2 \ge \gamma' \|u - \bar u\|^2_{L^1(\Omega)} \ \forall u \in \mathcal{U}_{ad} \text{ with } \|y_u - \bar y\|_{C(\bar\Omega)} < \alpha'. \tag{5.4}$$

Proof. Let us assume that (5.3) holds, but (5.4) is false. Then, for every integer $k \ge 1$ there exists an element $u_k \in \mathcal{U}_{ad}$ such that

$$J'(\bar u)(u_k - \bar u) + J''(\bar u)(u_k - \bar u)^2 < \frac{1}{k}\|u_k - \bar u\|^2_{L^1(\Omega)} \quad \text{and} \quad \|y_{u_k} - \bar y\|_{C(\bar\Omega)} < \frac{1}{k}. \tag{5.5}$$

Since $\{u_k\}_{k=1}^\infty \subset \mathcal{U}_{ad}$ is bounded in $L^\infty(\Omega)$, we can extract a subsequence, denoted in the same way, such that $u_k \stackrel{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$. On the one hand, (5.5) implies that $y_{u_k} \to \bar y$ in $C(\bar\Omega)$. On the other hand, from Theorem 2.5 the convergence $y_{u_k} \to y_u$ in

$C(\bar\Omega)$ follows. Then, $y_u = \bar y$ and, consequently, $u = \bar u$ holds. But (5.3) implies that $\bar u$ is bang-bang, and, hence, the weak convergence $u_k \stackrel{*}{\rightharpoonup} \bar u$ yields the strong convergence $u_k \to \bar u$ in $L^1(\Omega)$; see [9, Proposition 12 and Lemma 6]. Then, (5.5) contradicts (5.3).

Let us prove the converse implication. First, we observe that given $u \in \mathcal{U}_{ad}$, with the mean value theorem, we get

$$\mathcal{A}(y_u - \bar y) + \frac{\partial f}{\partial y}(x, \bar y + \theta(y_u - \bar y))(y_u - \bar y) = u - \bar u.$$

Now, using (2.2) with $r = 2$ we get

$$\|y_u - \bar y\|_{C(\bar\Omega)} \le C_2 \|u - \bar u\|_{L^2(\Omega)} \le C_2 \sqrt{u_b - u_a}\, \|u - \bar u\|^{\frac{1}{2}}_{L^1(\Omega)}.$$

Then, taking $\alpha = \frac{{\alpha'}^2}{C_2^2(u_b - u_a)}$, we obtain that (5.4) implies (5.3) with $\gamma = \gamma'$. ∎

From (2.3) we infer that (5.4) implies (5.2). Hence, the combination of sufficient second order conditions plus (5.1) is a stronger assumption than (5.2).

THEOREM 5.3. *Let $\bar u$ be a local minimizer of (P) satisfying Assumption* 5.1 *and $\{u_\varepsilon\}_{\varepsilon < \varepsilon_0}$ be a family of local solutions of problems (P_ε) such that $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$. Then, there exist $\hat\varepsilon \in (0, \varepsilon_0)$ and a constant $C > 0$ such that*

$$\|u_\varepsilon - \bar u\|_{L^1(\Omega)} \le C\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big) \quad \forall \varepsilon < \hat\varepsilon, \tag{5.6}$$

where $\bar y = y_{\bar u}$.

The proof of this theorem follows the steps of that of Theorem 4.4 with Lemma 4.5 replaced by the following.

LEMMA 5.4. *Let $\bar u$ satisfy the assumptions of Theorem* 5.3. *Then, there exists $\varepsilon > 0$ such that*

$$J'(u)(u - \bar u) \ge \frac{\gamma}{2}\|z_{u,u-\bar u}\|_{L^2(\Omega)}\|u - \bar u\|_{L^1(\Omega)} \ \forall u \in \mathcal{U}_{ad} \text{ with } \|y_u - \bar y\|_{C(\bar\Omega)} < \varepsilon, \tag{5.7}$$

where γ is given as in Assumption 5.1.

Proof. We use (4.7) with $\rho = \frac{\gamma}{2C_2}$, Assumption 5.1, and (2.3) to deduce for $\varepsilon > 0$ small enough that

$$\begin{aligned} J'(u)(u - \bar u) &= \int_\Omega (\varphi_u + g)(u - \bar u)\,\mathrm{d}x \\ &= \int_\Omega (\varphi_u - \bar\varphi - \psi_{\bar u, u-\bar u})(u - \bar u)\,\mathrm{d}x + \int_\Omega (\bar\varphi + g + \psi_{\bar u, u-\bar u})(u - \bar u)\,\mathrm{d}x \\ &\ge -\frac{\gamma}{2C_2}\|z_{\bar u, u-\bar u}\|^2_{L^2(\Omega)} + [J'(\bar u)(u - \bar u) + J''(\bar u)(u - \bar u)^2] \\ &\ge -\frac{\gamma}{2}\|z_{\bar u, u-\bar u}\|_{L^2(\Omega)}\|u - \bar u\|_{L^1(\Omega)} + \gamma\|z_{\bar u, u-\bar u}\|_{L^2(\Omega)}\|u - \bar u\|_{L^1(\Omega)}, \end{aligned}$$

which proves (5.7). ∎

Proof of Theorem 5.3. We follow the proof of Theorem 4.4, replacing the estimate (4.9) by (5.7) to deduce with (4.8) and (4.10)–(4.12) the inequality

$$\begin{aligned} 0 \ge {J'}_\varepsilon(u_\varepsilon)(u_\varepsilon - \bar u) &\ge \frac{\gamma}{2}\|z_{u_\varepsilon, u_\varepsilon - \bar u}\|_{L^2(\Omega)}\|u_\varepsilon - \bar u\|_{L^1(\Omega)} \\ &\quad - C_1\|z_{u_\varepsilon, u_\varepsilon - \bar u}\|_{L^2(\Omega)}\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big). \end{aligned}$$

Then, dividing this inequality by $\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}$ we get

$$\|u_\varepsilon - \bar u\|_{L^1(\Omega)} \le \frac{2C_1}{\gamma}\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big),$$

which proves (5.6) with $C = \frac{2C_1}{\gamma}$. ∎

6. Some final state stability results. In this section we see how Assumption 5.1 allows us to prove Lipschitz stability for the optimal states for more general perturbations of (P). Here, we consider the following simultaneous perturbations on the control and state variables of (P):

$$(\mathrm{P}_\varepsilon) \quad \min_{u\in\mathcal{U}_{ad}} J_\epsilon(u) := \int_\Omega L_\varepsilon(x, y_u^\varepsilon(x), u(x))\,\mathrm{d}x,$$

where y_u^ε is the solution of (4.1), and for every $\epsilon > 0$,

$$L_\varepsilon(x,y,u) = L_0(x,y) + \eta_\varepsilon y + g_\varepsilon u + \frac{\varepsilon}{2}u^2.$$

As in section 4, we assume that $\{\xi_\varepsilon\}_{\varepsilon>0}$ and $\{\eta_\varepsilon\}_{\varepsilon>0}$ are bounded families in $L^2(\Omega)$ satisfying $(\xi_\varepsilon,\eta_\varepsilon) \to (0,0)$ in $L^2(\Omega)^2$ as $\varepsilon \to 0$. Moreover, we suppose that $\|g_\varepsilon - g\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$. Under these assumptions, it is immediate to check that (P_ε) is an approximation of (P) in the sense of Theorems 4.2 and 4.3. Moreover, we have the following Lipschitz stability property for the optimal states.

THEOREM 6.1. *Let $\bar u$ be a local minimizer of (P) satisfying Assumption* 5.1 *and $\{u_\varepsilon\}_{\varepsilon<\varepsilon_0}$ be a family of local solutions of problems (P_ε) such that $u_\varepsilon \stackrel{*}{\rightharpoonup} \bar u$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$. Then, there exist $\hat\varepsilon \in (0,\varepsilon_0)$ and a constant $C > 0$ such that*

$$\|y^\varepsilon_{u_\varepsilon} - \bar y\|_{L^2(\Omega)} \le C\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)} + \|g_\varepsilon - g\|_{L^\infty(\Omega)} + \varepsilon\Big) \quad \forall \varepsilon < \hat\varepsilon, \tag{6.1}$$

where $\bar y = y_{\bar u}$.

Proof. Similarly to (4.8) we have

$$\begin{aligned}0 \ge J'_\varepsilon(u_\varepsilon)(u_\varepsilon - \bar u) &= J'(u_\varepsilon)(u_\varepsilon - \bar u) + \int_\Omega (\varepsilon u_\varepsilon + g_\varepsilon - g)(u_\varepsilon - \bar u)\,\mathrm{d}x\\ &+ \int_\Omega \Big[\frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon) - \frac{\partial L}{\partial y}(x, y_{u_\varepsilon}, u_\varepsilon)\Big] z_{u_\varepsilon,u_\varepsilon-\bar u}\,\mathrm{d}x\\ &+ \int_\Omega \frac{\partial L}{\partial y}(x, y^\varepsilon_{u_\varepsilon}, u_\varepsilon)(z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u} - z_{u_\varepsilon,u_\varepsilon-\bar u})\,\mathrm{d}x + \int_\Omega \eta_\varepsilon z^\varepsilon_{u_\varepsilon,u_\varepsilon-\bar u}\,\mathrm{d}x.\end{aligned}$$

Then, using (5.7) and (4.10)–(4.12) we obtain with (2.3) that

$$\begin{aligned}0 &\ge \frac{\gamma}{2}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}\|u_\varepsilon - \bar u\|_{L^1(\Omega)} - \Big(\varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)} + \|g_\varepsilon - g\|_{L^\infty(\Omega)}\Big)\|u_\varepsilon - \bar u\|_{L^1(\Omega)}\\ &- C_1\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}\Big(\|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big) \ge \frac{\gamma}{2}\|z_{u_\varepsilon,u_\varepsilon-\bar u}\|_{L^2(\Omega)}\|u_\varepsilon - \bar u\|_{L^1(\Omega)}\\ &- C'\Big(\varepsilon + \|g_\varepsilon - g\|_{L^\infty(\Omega)} + \|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big)\|u_\varepsilon - \bar u\|_{L^1(\Omega)},\end{aligned}$$

where $C' = \max\{1, |u_a|, |u_b|, C_1C_2\}$. Dividing the above expression by $\|u_\varepsilon - \bar u\|_{L^1(\Omega)}$ and using (2.14), we infer

$$\|y_{u_\varepsilon} - \bar{y}\|_{L^2(\Omega)} \leq \frac{4C'}{\gamma}\Big(\varepsilon + \|g_\varepsilon - g\|_{L^\infty(\Omega)} + \|\xi_\varepsilon\|_{L^2(\Omega)} + \|\eta_\varepsilon\|_{L^2(\Omega)}\Big).$$

Now, the rest follows as in the proof of Theorem 4.4. ∎

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