# A NON-AUTONOMOUS VARIATIONAL PROBLEM DESCRIBING A NONLINEAR TIMOSHENKO BEAM 

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Abstract. We study the non-autonomous variational problem:

$$
\inf _{(\phi, \theta)}\left\{\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{(\phi-\theta)^{2}}{2}-V(x, \theta)\right) \mathrm{d} x\right\}
$$

where $k>0, V$ is a bounded continuous function, $(\phi, \theta) \in H^{1}([0,1]) \times L^{2}([0,1])$ and $\phi(0)=0$ in the sense of traces. The peculiarity of the problem is its setting in the product of spaces of different regularity order. Problems with this form arise in elastostatics, when studying the equilibria of a nonlinear Timoshenko beam under distributed load, and in classical dynamics of coupled particles in time-depending external fields. We prove the existence and qualitative properties of global minimizers and study, under additional assumptions on $V$, the existence and regularity of local minimizers.

## 1. Setting of the problem

Let us indicate by $L^{2}:=L^{2}([0,1], \mathbb{R})$ and $H^{1}:=H^{1}([0,1], \mathbb{R})$ the usual Lebesgue and Sobolev spaces, and by $H_{*}^{1} \subset H^{1}$ the subspace of functions $\phi$ verifying $\phi(0)=0$ in the sense of traces. For $k$ a strictly positive constant and $V:[0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ a bounded continuous function, we study the variational problem:

$$
\begin{equation*}
\inf _{(\phi, \theta) \in \mathfrak{S}}\left\{\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{(\phi-\theta)^{2}}{2}-V(x, \theta)\right) \mathrm{d} x\right\} \tag{1.1}
\end{equation*}
$$

where the pair $(\phi, \theta)$ is searched in

$$
\mathfrak{S}:=H_{*}^{1} \times L^{2}
$$

In the following, we endow $\mathfrak{S}$ with the natural product metric and topology. In particular, when we talk about a local minimizer of a functional $F$ defined over $\mathfrak{S}$, we mean a pair $(\widetilde{\phi}, \widetilde{\theta}) \in \mathfrak{S}$ such that $F(\widetilde{\phi}, \widetilde{\theta}) \leq F(\phi, \theta)$ for every $(\phi, \theta)$ belonging to a sufficiently small open ball, centered in $(\widetilde{\phi}, \widetilde{\theta})$, with respect to this product topology. Clearly, in our terminology, every global minimizer of a functional defined on $\mathfrak{S}$ is also a local minimizer. The problem admits, for instance, the following physical interpretations:
(1) $\phi, \theta$ represent the kinematical descriptors of an inextensible, geometrically nonlinear Timoshenko beam submitted to a distributed load depending on $V$;
(2) $\phi, \theta$ represent the Lagrangian coordinates of two bodies $B_{\phi}, B_{\theta}$ having quadratic attractive interaction potential and $x$ represents time; $B_{\phi}$ has mass $k$ while $B_{\theta}$ has negligible mass but is sensitive to an external time-dependent (electric or magnetic) field depending on $V$.

[^0]

Figure 1. A schematic representation of a Timoshenko beam. Notice that the inextensibility assumption $\sqrt{1.4}$ makes it possible to interpret the variable $x$ as a curvilinear abscissa along the middle line (the dashed curve in the picture).

In the following we mainly refer to the first interpretation. Let us therefore recall that a Timoshenko beam is a one-dimensional elastic body whose kinematics is described by a curvilinear parametrization $\chi:[0,1] \rightarrow \mathbb{R}^{2}$ and an extra kinematical variable $\phi$ interpreted as the orientation of the cross-section, hence it is the angle between the cross-section of the beam and a reference axis. A schematic representation of a Timoshenko beam is shown in Fig 1
When the material behavior is assumed to be linear and the model is also geometrically linearized, we get the original formulation of the Timoshenko beam elastic energy functional (see [20, 21]), namely

$$
\begin{equation*}
\int_{0}^{1}\left(\frac{k}{2}\left(\phi^{\prime}\right)^{2}+\frac{\left(\phi-\chi_{2}^{\prime}\right)^{2}}{2}\right) \mathrm{d} x \tag{1.2}
\end{equation*}
$$

where $\chi_{2}$ is the vertical component of $\chi$. If geometric nonlinearities are considered, the elastic energy of the beam reads as follows (for a detailed derivation, see [3]):

$$
\begin{equation*}
\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{(\phi-\theta)^{2}}{2}\right) \mathrm{d} x \tag{1.3}
\end{equation*}
$$

where it was assumed that the beam has length 1 and is inextensible, that is

$$
\begin{equation*}
\left\|\chi^{\prime}\right\|^{2} \equiv 1 \tag{1.4}
\end{equation*}
$$

The bending coefficient $k$ belongs to $\mathbb{R}^{+}$and the function $\theta$ verifies $\chi^{\prime}=(\cos \theta, \sin \theta)$. The potential due to a distributed load $\boldsymbol{b}(x)$ is $\int_{0}^{1}(\boldsymbol{b}(x) \cdot \boldsymbol{\chi}(x)) \mathrm{d} x$ which can be rewritten, using an integration by parts, as $\int_{0}^{1}\left(\boldsymbol{B}(x) \cdot \boldsymbol{\chi}^{\prime}(x)\right) \mathrm{d} x$, where $\boldsymbol{B}(x)=\int_{x}^{1} \boldsymbol{b}(\xi) \mathrm{d} \xi$ and the kinematical constraint

$$
\begin{equation*}
\chi(0)=0 \tag{1.5}
\end{equation*}
$$

was imposed. Notice that 1.3 reduces to 1.2 under the smallness assumption $\chi_{2}^{\prime}=$ $\sin \theta \approx \theta$. The minimization of the total energy, if the load is uniform and has zero horizontal component, is therefore of the form (1.1) with

$$
\begin{equation*}
V(x, \theta)=b(1-x) \sin \theta \tag{1.6}
\end{equation*}
$$

where $b>0$ corresponds to the load density per unit length. Adding to the constraint 1.5 the further requirement

$$
\begin{equation*}
\phi(0)=0, \tag{1.7}
\end{equation*}
$$

we obtain the conditions usually expressed saying that the beam is horizontally clamped at one of its extremes. Notice that, in the variational formulation (1.1), the constraint (1.7) is contained in the definition of $\mathfrak{S}$, while the constraint $\sqrt{1.5}$ has to be taken into account when reconstructing the vector field $\chi(x)$ from $\theta(x)$. The case in which $V$ does not depend on $x$ was studied in [3], while a numerical investigation of the case with distributed load was performed in [13].
The variational problem (1.1) is close to a model-case of non-autonomous, not strictly convex problem. The absence of a term with $\theta^{\prime 2}$ makes the integrand not (strictly) convex in the highest order derivative (so that convergence of minimizing sequences is not granted by standard arguments) and at the same time settles the problem in the "asymmetric" space $H^{1} \times L^{2}$. When $k=1$, an effective way to see the above mentioned asymmetry of the problem is writing it follows:

$$
\inf _{\phi, \theta}\left\{\frac{1}{2}\left(\|\phi\|_{H^{1}}^{2}+\|\theta\|_{L^{2}}^{2}\right)-(\phi, \theta)_{L^{2}}-\int_{0}^{1} V(x, \theta) \mathrm{d} x\right\} .
$$

When $k \neq 1$, an analogous representation can be obtained considering an equivalent metric on $H^{1}$.
The results developed herein all hold when $V$ is given by (1.6). However, we will not limit ourselves to this form of $V$ for the existence and the main properties of the global minimizer.
Specifically, in Section 2 the existence of a global minimizer of problem (1.1) will be proved assuming that $V$ is a bounded continuous function. Some general properties verified by global minimizers will be established under the further assumption that $\theta \rightarrow V(x, \theta)$ admits a global maximum at $a>0$ (independently of $x$ ) and that $V(x,-\theta)<V(x, \theta)$ for $\theta \in[0, a]$ and for almost every $x \in[0,1]$. In Section 3 the existence and regularity of local minimizers, different from the global ones, will be studied under the assumption that $V$ is given by (1.6), and that both $b$ and $k / b$ are sufficiently small. The main motivation of this study is the existence of minimizers with similar properties in the simpler cases of Euler beam under distributed load (see [11, 10]) and of Timoshenko beam under concentrated end load (see [3]).

## 2. GLOBAL MINIMIZERS

Let us assume that function $V$ in (1.1) is a bounded continuous function. We remark that, since we do not have a term depending on $\theta^{\prime}$ in the integral (1.1), the fact that a minimizing sequence $\left(\phi_{n}, \theta_{n}\right)$ has a bounded energy does not provide any information for the derivatives of $\theta_{n}$. Hence, the weak convergence in $L^{2}$ of $\theta_{n}$ to a function $\theta$ does not imply that $\theta$ minimizes the energy, so the usual direct method of the calculus of variations must be used with caution.
Let us reformulate problem (1.1) as

$$
\begin{equation*}
\inf _{(\phi, \theta) \in \mathfrak{S}} F(\phi, \theta) \tag{2.1}
\end{equation*}
$$

with

$$
\begin{equation*}
F(\phi, \theta)=\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{(\phi-\theta)^{2}}{2}-V(x, \theta)\right) \mathrm{d} x \tag{2.2}
\end{equation*}
$$

We establish the existence of a minimizer of $F(\phi, \theta)$ in the following proposition.
Proposition 2.1. Assume that $k>0$ and that $V$ is a bounded continuous function on $[0,1] \times \mathbb{R}$. Then Problem (2.1) admits a solution.

Proof. Fix $M>|V|$. Then $F(0,0) \leq M$ and the infimum can be searched among functions $(\phi, \theta)$ satisfying $F(\phi, \theta) \leq M$ and such functions satisfy

$$
\int_{0}^{1} \frac{k}{2} \phi^{\prime 2} \mathrm{~d} x \leq 2 M
$$

The infimum can therefore be searched assuming that $\|\phi\|_{H_{*}^{1}} \leq C$ for some constant $C$ depending only on $M$ and $k$, so that it follows that $\|\phi\|_{C^{0}([0,1])} \leq\|\phi\|_{H_{*}^{1}} \leq C$.
Let us define the function $H:[0,1] \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ as

$$
H(x, \phi, \theta):=-\phi \theta+\frac{\theta^{2}}{2}-V(x, \theta)
$$

so that we have

$$
F(\phi, \theta)=\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{\phi^{2}}{2}+H(x, \phi, \theta)\right) \mathrm{d} x
$$

Let us now set, for any $x \in[0,1]$ and $\phi \in[-C, C]$,

$$
\begin{equation*}
K(x, \phi):=\inf _{\theta \in \mathbb{R}}(H(x, \phi, \theta)) \tag{2.3}
\end{equation*}
$$

It is easy to check that any real number $\theta$ satisfying $H(x, \phi, \theta) \leq H(x, \phi, 0) \leq M$, satisfies $|\theta| \leq D:=C+\sqrt{2 M+C^{2}}$. Therefore the set of solutions of problem $\sqrt{2.3}$ is a non-empty closed subset of the compact $[-D, D]$. For every $x \in[0,1]$, let us indicate by $\theta_{\phi}(x)$ the smallest solution of (2.3). Being lower-semicontinuous, $\theta_{\phi}$ is a measurable function, and being bounded it is in $L^{2}([0,1])$.
Let $\left(x_{n}, \phi_{n}\right)$ be a sequence converging to $(x, \phi)$. Up to a subsequence, $\theta_{\phi_{n}}$ converges to some $\theta_{\phi} \in[-D, D]$ and we have

$$
H\left(x_{n}, \phi_{n}, \theta_{\phi_{n}}\right)=K\left(x_{n}, \phi_{n}\right) \leq H\left(x_{n}, \phi_{n}, \theta_{\phi}\right)
$$

As $H$ is continuous, passing to the limit we get

$$
K(x, \phi) \leq H\left(x, \phi, \theta_{\phi}\right)=\lim K\left(x_{n}, \phi_{n}\right) \leq H\left(x, \phi, \theta_{\phi}\right)=K(x, \phi)
$$

Hence $K$ is a bounded continuous function. By standard arguments of the calculus of variations (see for instance Tonelli's existence theorem in [7]), the problem

$$
\begin{equation*}
\inf _{\phi} \int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{\phi^{2}}{2}+K(\phi, x)\right) \mathrm{d} x \tag{2.4}
\end{equation*}
$$

admits a solution $\bar{\phi}$ in $H_{*}^{1}([0,1])$ and

$$
\inf _{(\phi, \theta)} F(\phi, \theta) \leq F\left(\bar{\phi}, \theta_{\bar{\phi}}\right)=\inf _{\phi} \int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{\phi^{2}}{2}+K(\phi, x)\right) \mathrm{d} x \leq \inf _{(\phi, \theta)} F(\phi, \theta)
$$

Hence, the pair $\left(\bar{\phi}, \theta_{\bar{\phi}}\right)$ is a solution to Problem 1.1).

Remark 2.2. In principle, the minimizer whose existence has been established in Proposition 2.1 may fail very badly to be unique, even whether the related Problem 2.4, which is a classical problem of calculus of variations, admits a unique solution $\bar{\phi}$.
Suppose, for instance, that $V(x, \theta)$ has the following form:

$$
V(x, \theta)=g(\theta-f(x))
$$

where $g$ is such that $\frac{\partial^{2} H}{\partial \theta^{2}}=1-g^{\prime \prime}(\theta-f(x))>0$ when $(\theta-f)$ belongs to some open set
$I$. Then every $\xi$ such that

$$
\xi-\phi-g^{\prime}(\xi-f(x))=0
$$

is a solution of problem (2.3) if $\xi-f \in I$.
Suppose now that $\left.\left.\bar{\phi}\right|_{A} \equiv f\right|_{A}$ on a set $A \subset[0,1]$ of positive measure, and that the equation $s=g^{\prime}(s)$ has two distinct solutions $s_{1}, s_{2} \in I$ such that, for $\theta_{1}(x)=s_{1}+\phi(x)$ and $\theta_{2}(x)=s_{2}+\phi(x)$, we have

$$
H\left(x, \phi, \theta_{1}\right)=H\left(x, \phi, \theta_{2}\right)
$$

Then the problem (2.3) is solved by both $\theta_{1}$ and $\theta_{2}$. In these hypotheses, $\left(\bar{\phi}, \theta^{*}\right)$ is a minimizer of $F$ for every $\theta^{*}$ defined as follows:

$$
\theta^{*}=\theta_{1} \text { for } x \in B, \quad \theta^{*}=\theta_{2} \text { for } x \in A \backslash B, \quad \theta^{*}=\theta_{\bar{\phi}} \text { for } x \in[0,1] \backslash A
$$

where $B \subset A$ is a (completely arbitrary) subset of positive measure.
Pathological phenomena of this type are well known (similar problems were already discussed, for instance, in the classical works [23, 24]), and are usually addressed by means of relaxation theory (see e.g. [12], Chapter III), which however has not been developed, to the best of our knowledge, for problems living in the product of Sobolev spaces of different regularity order. In the following, we will be mainly concerned with cases in which $V(x, \theta)$ does not produce such pathological multiplicity of minimizers.

We shall prove now some properties of the global minimizers of Problem 1.1. In addition to the information they provide on the problem, these results will ensure that the local minimizers studied in Section 3 are necessarily not global minimizers.

Lemma 2.3. Assume, in addition to the assumptions of Proposition 2.1 that there exists $a>0$ such that for almost every $x \in[0,1]$, for every $\theta \in \mathbb{R}, V(x, \theta) \leq V(x, a)$ and for every $\theta \in(0, a], V(x,-\theta)<V(x, \theta)$. Then any minimizer $(\widetilde{\phi}, \widetilde{\theta})$ of (1.1) takes values in $[0, a] \times[0, a]$.

Proof. Define $\widetilde{V}$ by $\widetilde{V}(x, \theta)=V(x, \theta)$ if $\theta<a, \widetilde{V}(x, \theta)=V(x, a)$ if $\theta \geq a$, so that $\widetilde{V}$ now satisfies, for almost every $x \in[0,1]$ and for every $\theta \in \mathbb{R}, \widetilde{V}(x, \theta) \leq \widetilde{V}(x, a)$ and $\widetilde{V}(x, \theta) \leq \widetilde{V}(x,|\theta|)$. We set

$$
\begin{equation*}
\widetilde{F}(\phi, \theta):=\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{(\phi-\theta)^{2}}{2}-\widetilde{V}(x, \theta)\right) \mathrm{d} x \tag{2.5}
\end{equation*}
$$

Clearly $\widetilde{F} \leq F$. Moreover, we have that

$$
\widetilde{F}(|\widetilde{\phi}|,|\widetilde{\theta}|) \leq \widetilde{F}(\widetilde{\phi}, \widetilde{\theta}) \quad \text { and } \quad \widetilde{F}(\min (|\widetilde{\phi}|, a), \min (|\widetilde{\theta}|, a)) \leq \widetilde{F}(|\widetilde{\phi}|,|\widetilde{\theta}|)
$$

as all integrands in (2.5) do not increase with these replacements. Set

$$
(\bar{\phi}, \bar{\theta}):=(\min (|\widetilde{\phi}|, a), \min (|\widetilde{\theta}|, a))
$$

As these functions take values in $[0, a]$, we have $\widetilde{F}(\bar{\phi}, \bar{\theta})=F(\bar{\phi}, \bar{\theta})$. Hence

$$
F(\bar{\phi}, \bar{\theta}) \leq \widetilde{F}(\widetilde{\phi}, \widetilde{\theta}) \leq F(\widetilde{\phi}, \widetilde{\theta}) \leq F(\bar{\phi}, \bar{\theta})
$$

This implies that the previous inequalities were in fact equalities. Since $\int_{0}^{1} \frac{k}{2} \phi^{\prime 2} \mathrm{~d} x$ does not decrease when replacing $|\widetilde{\phi}|$ by $\min \{|\widetilde{\phi}|, a\}$, it follows that $|\widetilde{\phi}(x)| \leq a$ almost everywhere (and thus everywhere as $\widetilde{\phi}$ is continuous). Moreover, since $\int_{0}^{1} \frac{(\phi-\theta)^{2}}{2} \mathrm{~d} x$ does not decrease when replacing $|\widetilde{\phi}|$ and $|\widetilde{\theta}|$ by $\min \{|\widetilde{\phi}|, a\}$ and $\min \{|\widetilde{\theta}|, a\}$, it follows that the sets $\{x,|\widetilde{\phi}(x)|>a\}$ and $\{x,|\widetilde{\theta}(x)|>a\}$ coincide up to a null set. Thus $|\widetilde{\theta}(x)| \leq a$ almost everywhere. Finally, noting that for every $\theta \in(0, a]$, we have $V(x,-\theta)<V(x, \theta)$ a.e. on $[0,1]$, and that $\int_{0}^{1} \widetilde{V}(x, \theta) \mathrm{d} x$ does not decrease when replacing $\widetilde{\theta}$ by $|\widetilde{\theta}|$, it follows that the set $\{x \in[0,1]: \widetilde{\theta}(x) \in[-a, 0)\}$ has null measure, hence $\widetilde{\theta}(x) \geq 0$ almost everywhere.

Proposition 2.4. In addition to the assumptions of Lemma 2.3. assume that, for every $x \in$ $[0,1]$, the function $\theta \rightarrow \underset{\sim}{V}(x, \theta)$ is of class $C^{k}(\mathbb{R})$, with $k \geq 2$, and satisfies $\frac{\partial^{2} V}{\partial \theta^{2}}(x, \theta) \neq 1$ in $[0,1] \times[0, a]$. If $(\widetilde{\phi}, \widetilde{\theta})$ is a minimizer of $(1.1)$, then $\widetilde{\theta} \in C^{k-1}(\mathbb{R})$ and $\widetilde{\phi} \in C^{k+1}(\mathbb{R})$. Moreover, if $V$ is a $C^{\infty}\left(C^{\omega}\right)$ function, then both $\widetilde{\phi}$ and $\widetilde{\theta}$ are $C^{\infty}\left(C^{\omega}\right)$ functions.
Proof. For every $x \in[0,1]$, we have

$$
\widetilde{\theta}=\min _{\theta}\left(-\widetilde{\phi}(x) \theta+\frac{\theta^{2}}{2}-V(x, \theta)\right)
$$

so $\tilde{\theta}(x)$ has to solve

$$
\begin{equation*}
-\widetilde{\phi}(x)+\widetilde{\theta}-\frac{\partial V}{\partial \theta}(x, \widetilde{\theta})=0 \tag{2.6}
\end{equation*}
$$

Let us set $f(x, \theta):=\theta-\frac{\partial V}{\partial \theta}(x, \theta)$. By the hypotheses of Lemma 2.3. for every $x \in[0,1]$ we have

$$
\frac{\partial V}{\partial \theta}(x, 0)>0 \quad \text { and } \quad \frac{\partial V}{\partial \theta}(x, a)=0
$$

thus

$$
f(x, 0)<0 \quad \text { and } \quad f(x, a)=a
$$

By Lemma 2.3. $\tilde{\theta}$ takes values in $[0, a]$, so, by hypothesis we have

$$
\begin{equation*}
\frac{\partial f}{\partial \theta}(x, \theta)=1-\frac{\partial^{2} V}{\partial \theta^{2}}(x, \theta) \neq 0 \tag{2.7}
\end{equation*}
$$

As a consequence, $f$ is strictly increasing with respect to $\theta$ in $[0,1] \times[0, a]$, and for every $x \in[0,1]$ there exists a unique value of $\widetilde{\theta} \in[0, a]$ such that (2.6) holds.
By the inverse function theorem, there exists a function $g:[0, a] \times[0,1] \rightarrow[0, a]$, with the same regularity of $f$, hence of class $C^{k-1}$, such that

$$
\begin{equation*}
\widetilde{\theta}(x)=g(x, \widetilde{\phi}(x)) \tag{2.8}
\end{equation*}
$$

As a consequence, $\widetilde{\theta} \in C^{0}([0,1])$. Since $(\widetilde{\phi}, \widetilde{\theta})$ is a minimizer, we have

$$
\mathrm{d} F(\widetilde{\phi}, \widetilde{\theta})[\xi, 0]=\int_{0}^{1}\left(k \widetilde{\phi}^{\prime} \xi^{\prime}+(\widetilde{\phi}-\widetilde{\theta}) \xi\right) \mathrm{d} x=0, \quad \forall \xi \in C_{c}^{\infty}([0,1])
$$

hence

$$
\begin{equation*}
k \tilde{\phi}^{\prime} \xi^{\prime}+(\tilde{\phi}-\tilde{\theta}) \xi=0 \tag{2.9}
\end{equation*}
$$

Since $\widetilde{\theta}$ is continuous, 2.9 implies that $\widetilde{\phi}$ is $C^{1}$. Then, using again 2.8), we obtain that $\widetilde{\theta}$ is of class $C^{1}$ and by 2.9 we get that $\widetilde{\phi}$ if of class $C^{2}$. Hence, by a standard argument, we obtain

$$
\begin{equation*}
k \widetilde{\phi}^{\prime \prime}=\widetilde{\phi}-\widetilde{\theta} \tag{2.10}
\end{equation*}
$$

and iterating (2.8) and 2.10 we obtain the desired regularity.
If $V$ is of class $C^{\infty}$, then by induction both $\widetilde{\phi}$ and $\widetilde{\theta}$ are of class $C^{\infty}$.
Assume finally that $V$ is real-analytic. Applying the real-analytic version of the inverse function theorem (see e.g. [16], p. 47) to 2.6, we can replace $\theta$ in (2.9) by an analytic function $G(\phi, x)$ to obtain the boundary value problem:

$$
\left\{\begin{array}{l}
-k \phi^{\prime \prime}+\phi-G(\phi, x)=0 \\
\phi(0)=0 \\
\phi^{\prime}(1)=0
\end{array}\right.
$$

which will be solved pointwise by $\widetilde{\phi}$. By Cauchy-Kovalevskaya theorem (for an ODE version, which is in fact a particular case, see for instance [19], theorem 4.1) we obtain $\widetilde{\phi} \in C^{\omega}([0,1])$ as well, whence $\widetilde{\theta} \in C^{\omega}([0,1])$ too.

From the previous proof, it is clear that the regularity of local minimizers of $F$ depends on the possibility to invert the function $f$, which is ensured if 2.7) holds. As a consequence, we have the following result.

Corollary 2.5. Let $V:[0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function of class $C^{k}$, with $k \geq 2$, such that

$$
\frac{\partial^{2} V}{\partial \theta^{2}}(x, \theta) \neq 1, \quad \forall(x, \theta) \in[0,1] \times \mathbb{R}
$$

If $(\widetilde{\phi}, \widetilde{\theta})$ is a local minimizer of $F$, then $\widetilde{\theta} \in C^{k-1}(\mathbb{R})$ and $\widetilde{\phi} \in C^{k+1}(\mathbb{R})$. Moreover, if $V$ is a $C^{\infty}\left(C^{\omega}\right)$ function, then both $\widetilde{\phi}$ and $\tilde{\theta}$ are $C^{\infty}\left(C^{\omega}\right)$ functions.
Lemma 2.3 and Proposition 2.4 apply to the problem 1.1) with

$$
V(x, \theta)=b(1-x) \sin \theta \quad \text { and } \quad a=\frac{\pi}{2}
$$

As a consequence, we can give the following result.
Lemma 2.6. If $(\widetilde{\phi}, \widetilde{\theta})$ solves problem (2.1)-(2.2) with $V(x, \theta)=b(1-x) \sin \theta$, then $\widetilde{\phi}$ is strictly increasing.

Proof. Since for every $(x, \theta) \in[0,1] \times[0, \pi / 2]$ we have

$$
\frac{\partial^{2} V}{\partial \theta^{2}}(x, \theta)=-b(1-x) \sin \theta \leq 0
$$

we can apply Proposition 2.4 and $\widetilde{\phi}$ is an analytic function. Therefore, it is piecewise monotonic and cannot be constant on an open interval without being constant (and thus equal to zero, as $\widetilde{\phi}(0)=0$ ) on $[0,1]$. We therefore just have to exclude that there exist $0 \leq \alpha<\beta \leq 1$ such that $\widetilde{\phi}$ is strictly decreasing on $[\alpha, \beta]$. Suppose now the contrary. Then define, on the interval $[\alpha, 1]$, the functions

$$
f(x):=\max \{\widetilde{\phi}(\alpha), \widetilde{\phi}(x)\}
$$

and

$$
g(x):= \begin{cases}f(x), & \text { if } f(x) \neq \widetilde{\phi}(x) \\ \widetilde{\theta}(x), & \text { otherwise }\end{cases}
$$

It is easily seen that $f \in H_{*}^{1}$ and $g \in L^{2}$. Moreover, $0 \leq \widetilde{\phi}(x) \leq f(x) \leq \frac{\pi}{2}$ and $0 \leq \widetilde{\theta}(x) \leq g(x) \leq \frac{\pi}{2}$. We have further that, a.e. on $[0,1], f^{\prime}(x) \leq \widetilde{\phi}^{\prime}(x)$ and that $|f-g| \leq|\widetilde{\phi}-\widetilde{\theta}|$. Since $\sin (\cdot)$ is strictly increasing in $\left[0, \frac{\pi}{2}\right]$, we also have $\sin \widetilde{\theta} \leq \sin g$. It follows that replacing $(\widetilde{\phi}, \tilde{\theta})$ by $(f, g)$ on $[\alpha, 1]$ the value of the functional 2.2 decreases, which is absurd.

Fig. 2 shows a numerical computation of the global minimizer of $F$ with $V=b(1-$ $x) \sin \theta$, with $b=1$ and $k=0.01$. It can be seen that $\widetilde{\phi}$ is strictly increasing, as it is ensured by Lemma 2.6


Figure 2. A numerically evaluated solution of (1.1) with $V(x, \theta)=$ $b(1-x) \sin \theta, b=1$ and $k=0.01$.

We end this section with the following result, which stems from the proof of Proposition 2.4 and it will be useful for the study of local minimizers different from the global one.

Corollary 2.7. Let $V(x, \theta)=b(1-x) \sin \theta$ and $b<1$. Then there exists a unique map $\Theta: H_{*}^{1} \rightarrow L^{2}$ such that for all $(\phi, \theta) \in H_{*}^{1} \times L^{2}$ we have

$$
F(\phi, \Theta(\phi)) \leq F(\phi, \theta)
$$

Proof. As in the proof of Proposition 2.4, let us set

$$
f(x, \theta)=\theta-\frac{\partial V}{\partial \theta}(x, \theta)=\theta+b(1-x) \cos \theta
$$

Under the hypothesis $b<1$, the function $f$ is strictly increasing. By the inverse function theorem, there exists a unique analytic function $g:[0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
f(x, g(x, \phi))-\phi=0
$$

for every $x \in[0,1]$ and $\phi \in \mathbb{R}$. Hence, we define the map $\Theta: H_{*}^{1} \rightarrow L^{2}$ as follows:

$$
(\Theta(\phi))(x):=g(x, \phi(x))
$$

where we notice that, being $\phi \in H_{*}^{1} \subset C^{0}, \Theta(\phi)$ is continuous, hence belongs to $L^{2}$.

## 3. LOCAL MINIMIZERS

The study of local minimizers in elastostatics is typically not easy, and fully general methods for establishing the existence of local minimizers which are not global ones, as famously asked by J.M. Ball in Problem 9 of [1], have not yet been found. In this section, we address the existence of local minimizers (different from the global one) of a particular case of the functional defined in (2.5), which we will indicate by $F_{b, k}: \mathfrak{S} \rightarrow \mathbb{R}$, defined as

$$
\begin{equation*}
F_{b, k}(\phi, \theta)=\int_{0}^{1}\left(\frac{k}{2} \phi^{\prime 2}+\frac{(\phi-\theta)^{2}}{2}-b(1-x) \sin \theta\right) \mathrm{d} x \tag{3.1}
\end{equation*}
$$

In particular, our main result is Theorem 3.1, which ensures the existence of a local minimizer $(\widetilde{\phi}, \widetilde{\theta})$ such that $\widetilde{\phi}(x)<0$ for all $x \in(0,1]$. We therefore extend the results of [3, 11, 10], where similar local minimizers where found for nonlinear Euler beams under distributed load and nonlinear Timoshenko beams under concentrated end-load (which leads to an autonomous variational problem).

Theorem 3.1. Let $F_{b, k}: \mathfrak{S} \rightarrow \mathbb{R}$ be as in 3.1. If both $b$ and $k / b$ are sufficiently small, then there exists a local minimizer $(\widetilde{\phi}, \widetilde{\theta})$ of $F_{b, k}$ such that

$$
\widetilde{\phi}(x)<0, \quad \forall x \in(0,1] .
$$

A numerically evaluated local minimizer when $b=1$ and $k=0.01$ is shown in Fig. 3.


Figure 3. A numerically evaluated local minimizer of 3.1 with $b=1$ and $k=0.01$.

From the statement of Theorem 3.1 it is immediately clear that the ratio $k / b$ plays a central role in the existence of local minimizers different from the global one. For the sake of presentation, for every fixed $b, k>0$ we will indicate by $\lambda$ the inverse of that ratio, hence

$$
\lambda=\frac{b}{k},
$$

so we will prove some of the following results provided that $\lambda$ is sufficiently large. A key ingredient for our proof is the function $\phi_{\lambda}^{*}$, which is given by the following definition. Indeed, as we are going to see during the different steps of the proof, it provides a "natural" upper bound for the component $\phi$ of the local minimizers, as it can be noted in Fig. 3 .

Definition 3.2. For every $b, k>0$ we denote by $\lambda$ the ratio $b / k$ and we define the function $\phi_{\lambda}:[0,1] \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
\phi_{\lambda}^{*}(x)=\max \left\{\frac{\lambda}{2} x^{2}\left(\frac{x}{3}-1\right)-\frac{1}{2} x^{2},-\pi\right\} \tag{3.2}
\end{equation*}
$$

We denote by $x_{\lambda}$ the least $x$ such that $\phi^{*}=-\pi$, that is:

$$
x_{\lambda}=\min \left\{x \in[0,1]: \phi_{\lambda}^{*}(x)=-\pi\right\}
$$

which is well defined if $\lambda$ is sufficiently large. Moreover, we set

$$
\begin{equation*}
\mathfrak{C}_{\lambda}^{*}:=\left\{\phi \in H_{*}^{1}: \phi(x) \leq \phi_{\lambda}^{*}(x), \forall x \in[0,1]\right\} \tag{3.3}
\end{equation*}
$$

and

$$
\mathfrak{S}_{\lambda}^{*}:=\mathfrak{C}_{\lambda, \epsilon}^{*} \times L^{2} \subset \mathfrak{S}
$$

The main idea of the proof of Theorem 3.1 is showing that the global minimizer of $F_{b, k}$ in $\mathfrak{S}^{*}$, denoted by $(\widetilde{\phi}, \widetilde{\theta})$, is strictly less than $\phi_{\lambda}^{*}$, except in 0 . The special form of $\phi_{\lambda}^{*}$ implies that $\widetilde{\phi}$ can "touch" it only at $x_{\lambda}$, and this is proved in Subsection 3.1. Subsection 3.2 is devoted to prove that $\widetilde{\phi}\left(x_{\lambda}\right)$ is actually also strictly less then $\phi_{\lambda}^{*}\left(x_{\lambda}\right)=-\pi$. As a first step we show that if $b$ is sufficiently small and $b / k=\lambda$ remains constant, then $\tilde{\phi}$ is arbitrarily close, with respect to the $C^{1}$ norm, to the minimizer of the Euler beam problem, hence to the global minimizer of

$$
\phi \mapsto \int_{0}^{1}\left(\frac{\phi^{\prime 2}}{2}-\lambda(1-x) \sin \phi\right) \mathrm{d} x
$$

subject to $\phi(0)=0$ and $\phi(x) \leq \phi_{\lambda}^{*}(x)$. As a second step, we prove that if $\lambda$ is sufficiently large, and thus if $k / b$ is sufficiently small, then such a minimizer is strictly less then $-\pi$ at $x_{\lambda}$. In Subsection 3.3 we formally give the proof of Theorem 3.1, recollecting all the previous results and using a $\Gamma$-convergence argument to show that $(\phi, \tilde{\theta})$ is indeed a local minimizer on the whole set $\mathfrak{S}$.
3.1. General results for minimizers in $\mathfrak{S}_{\lambda}^{*}$. In this section we provide some results that hold for the minimizers of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, independently of $b, k>0$. As a first step, we give the following existence result.
Proposition 3.3. For every $b, k>0$, there exists a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$.
Proof. The set $\mathfrak{C}_{\lambda}^{*} \subset H_{*}^{1}$ is convex and closed with respect to the $L^{\infty}$ norm and therefore it is closed with respect to the weak convergence in $H^{1}$. Since the minimizing sequences weakly converge in $H^{1}$, their weak limit belongs to $\mathfrak{C}_{\lambda}^{*}$ (see for instance Theorem 7.3.7 in [25]).

Since $\mathfrak{S}_{\lambda}^{*}$ is a closed set with boundary, a global minimizer does not satisfy the EulerLagrange equations in general. However, the form of $\phi_{\lambda}^{*}$ allows us to prove that this is actually the case, as stated by the following proposition, which is the main result of this subsection.

Proposition 3.4. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, then

$$
\begin{equation*}
k \widetilde{\phi}^{\prime \prime}=\widetilde{\phi}-\widetilde{\theta}, \quad \text { a.e. on }[0,1] \tag{3.4}
\end{equation*}
$$

Some preliminary definitions and results are required to prove Proposition 3.4 . In fact, an important step of the proof is showing that $\widetilde{\phi}$ is sufficiently smooth on the intervals $\left[0, x_{\lambda}\right]$ and $\left[x_{\lambda}, 1\right]$ (see Lemma 3.9; this can be achieved by exploiting the techniques given by [17], which have been used in different contexts to achieve the desired regularity of constrained minimizers (see e.g. [8, 14]). An important consequence of Proposition 3.4 and of the special definition of $\phi_{\lambda}^{*}$ is that the constrained minimizer $(\widetilde{\phi}, \widetilde{\theta}) \in \mathfrak{S}_{\lambda}^{*}$ is such that $\widetilde{\phi}$ equals $\phi_{\lambda}^{*}$ on 0 and, at most, on $x_{\lambda}$ : in other words $\widetilde{\phi}(x)<\phi_{\lambda}^{*}(x)$ for all $x \neq 0, x_{\lambda}$.

Definition 3.5. For every $\phi \in \mathfrak{C}_{\lambda}^{*}$, we define the set of infinitesimal admissible variations of $\phi$ in $\mathfrak{C}_{\lambda}^{*}$, denoted by $\mathcal{V}_{\lambda}^{*}$, the set

$$
\begin{equation*}
\mathcal{V}_{\lambda}^{*}(\phi):=\left\{\xi \in H_{*}^{1}([0,1]): \xi(x) \leq 0 \text { if } \phi(x)=\phi_{\lambda}^{*}(x)\right\} \tag{3.5}
\end{equation*}
$$

Lemma 3.6. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, then

$$
\begin{equation*}
\widetilde{\phi}-\widetilde{\theta}=-b(1-x) \cos \widetilde{\theta}, \quad \text { a.e. on }[0,1] . \tag{3.6}
\end{equation*}
$$

Proof. For every $\eta \in C_{0}^{\infty}([0,1]) \subset L^{2}([0,1])$ we have

$$
\mathrm{d} F_{b, k}(\widetilde{\phi}, \widetilde{\theta})[0, \eta]=-\int_{0}^{1}(\widetilde{\phi}-\tilde{\theta}+b(1-x) \cos \tilde{\theta}) \eta \mathrm{d} x=0
$$

from which 3.6 follows.
Lemma 3.7. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a global minimizer of $F_{b, k}$ on $\mathfrak{S}_{\lambda}^{*}$, then

$$
\widetilde{\phi}(x) \geq-\frac{3}{2} \pi, \quad \forall x \in[0,1] .
$$

Proof. Reasoning by contradiction, if there exists $\left.\left.x_{1} \in\right] 0,1\right]$ such that $\widetilde{\phi}\left(x_{1}\right)<-\frac{3}{2} \pi$, by continuity there exists $\left.x_{0} \in\right] 0, x_{1}$ [ such that $\widetilde{\phi}\left(x_{0}\right)=-\frac{3}{2} \pi$. Therefore, we can define the functions $\phi_{1}, \theta_{1}:[0,1] \rightarrow \mathbb{R}$ as

$$
\phi_{1}(x)=\left\{\begin{array}{ll}
\widetilde{\phi}(x), & \text { if } x \leq x_{0}, \\
-\frac{3}{2} \pi, & \text { if } x>x_{0},
\end{array} \quad \text { and } \quad \theta_{1}(x)= \begin{cases}\widetilde{\theta}(x), & \text { if } x \leq x_{0} \\
-\frac{3}{2} \pi, & \text { if } x>x_{0}\end{cases}\right.
$$

Since $\widetilde{\phi}\left(x_{1}\right)<-\frac{3}{2} \pi$, we have $\int_{x_{0}}^{1} \widetilde{\phi}^{\prime 2} \mathrm{~d} x \geq \int_{x_{0}}^{x_{1}} \widetilde{\phi}^{\prime 2} \mathrm{~d} x>0$, hence

$$
\begin{aligned}
F_{b, k}(\widetilde{\phi}, \widetilde{\theta})-F_{b, k}\left(\phi_{1}, \theta_{1}\right)=\int_{x_{0}}^{1}( & \left.k \frac{\widetilde{\phi}^{\prime 2}}{2}+\frac{(\widetilde{\phi}-\widetilde{\theta})^{2}}{2}\right) \mathrm{d} x \\
& +\int_{x_{0}}^{1} b(1-x)(1-\sin \widetilde{\theta}) \mathrm{d} x \geq \int_{x_{0}}^{1} k \frac{\widetilde{\phi}^{\prime 2}}{2} \mathrm{~d} x>0
\end{aligned}
$$

contradicting the minimality of $(\widetilde{\phi}, \widetilde{\theta})$.
Lemma 3.8. If $(\widetilde{\phi}, \widetilde{\theta})$ is a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, then

$$
\widetilde{\phi}(x)<-\pi, \quad \forall x \in\left(x_{\lambda}, 1\right] .
$$

Proof. Being a global minimizer of $F_{b, k}$, the restriction of $(\widetilde{\phi}, \widetilde{\theta})$ on the interval $\left[x_{\lambda}, 1\right]$, is a global minimizer for the functional

$$
(\phi, \theta) \mapsto \int_{x_{\lambda}}^{1}\left(k \frac{\phi^{\prime 2}}{2}+\frac{(\phi-\theta)^{2}}{2}-b(1-x) \sin \theta\right) \mathrm{d} x
$$

with the conditions $\phi\left(x_{\lambda}\right)=\widetilde{\phi}\left(x_{\lambda}\right) \leq-\pi$ and $\phi(x) \leq-\pi$. As a consequence, the pair $\left(\widetilde{\phi}_{1}, \widetilde{\theta}_{1}\right):=(\pi-\widetilde{\phi}, \pi-\widetilde{\theta})$ is the global minimizer of the same functional under the conditions $\phi\left(x_{\lambda}\right)=\pi-\widetilde{\phi}\left(x_{\lambda}\right) \geq 0$ and $\phi(x) \geq 0$. Thus, by Lemma 2.6, $\widetilde{\phi}_{1}$ is strictly increasing, so $\widetilde{\phi}$ is strictly decreasing. Since $\widetilde{\phi}(x) \leq \phi_{\lambda}^{*}(x)=-\pi$ for all $x \in\left[x_{\lambda}, 1\right]$, we obtain the thesis.

Lemma 3.9. If $(\widetilde{\phi}, \widetilde{\theta})$ is a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, then

$$
\left.\widetilde{\phi}\right|_{\left[0, x_{\lambda}\right]} \in W^{2, \infty}\left(\left[0, x_{\lambda}\right], \mathbb{R}\right)
$$

and

$$
\begin{equation*}
\left.\widetilde{\phi}\right|_{\left[x_{\lambda}, 1\right]} \in W^{2, \infty}\left(\left[x_{\lambda}, 1\right], \mathbb{R}\right) \tag{3.7}
\end{equation*}
$$

Proof. Thanks to Lemma 3.8, every $\xi \in C^{\infty}([0,1], \mathbb{R})$ with compact support in $\left(x_{\lambda}, 1\right)$ is an admissible variation. As a consequence, the regularity indicated by 3.7) can be obtained by standard arguments.
Therefore, from now on in this proof, we restrict our study on the interval $\left[0, x_{\lambda}\right]$. For the sake of presentation, we simply write $\widetilde{\phi}$ instead of $\left.\widetilde{\phi}\right|_{\left[0, x_{\lambda}\right]}$ and, similarly, the sets $\mathfrak{C}_{\lambda}^{*}$ and $\mathcal{V}_{\lambda}^{*}(\widetilde{\phi})$ have to be meant as defined on the interval $\left[0, x_{\lambda}\right]$. Since $(\widetilde{\phi}, \widetilde{\theta})$ is a global minimizer for $F_{b, k}, \widetilde{\phi}$ is a global minimizer for the functional $G: \mathfrak{C}_{\lambda} \rightarrow \mathbb{R}$ defined as

$$
G(\phi):=\int_{0}^{x_{\lambda}}\left(\frac{k}{2}\left(\phi^{\prime}\right)^{2}+\frac{(\phi-\tilde{\theta})^{2}}{2}\right) \mathrm{d} x
$$

so it satisfies

$$
\mathrm{d} G(\widetilde{\phi})[\xi]=\int_{0}^{x_{\lambda}}\left(k \phi^{\prime} \xi^{\prime}+(\phi-\theta) \xi\right) d x \geq 0, \quad \forall \xi \in \mathcal{V}_{\lambda}^{*}(\widetilde{\phi})
$$

Set $y=\widetilde{\phi}-\phi^{*}$. Since $\phi^{*}$ is of class $C^{2}$ on $\left[0, x_{\lambda}\right]$, our thesis can be obtained by proving that $y \in W^{2, \infty}\left(\left[0, x_{\lambda}\right]\right)$, thus by showing that $y^{\prime} \in W^{1, \infty}\left(\left[0, x_{\lambda}\right]\right)$. Defining the function $z:\left[0, x_{\lambda}\right] \rightarrow \mathbb{R}$ as

$$
z:=y+\phi^{*}-\tilde{\theta}-\left(\phi^{*}\right)^{\prime \prime}
$$

we can write the differential of $G$ as follows:

$$
\begin{aligned}
& \mathrm{d} G(\widetilde{\phi})[\xi]=\int_{0}^{x_{\lambda}}\left[k\left(y^{\prime}+\left(\phi^{*}\right)^{\prime}\right) \xi^{\prime}+\left(y+\phi^{*}-\widetilde{\theta}\right) \xi\right] \mathrm{d} x \\
& \quad=\int_{0}^{x_{\lambda}}\left[k y^{\prime} \xi^{\prime}+\left(y+\phi^{*}-\widetilde{\theta}-\left(\phi^{*}\right)^{\prime \prime}\right) \xi\right] \mathrm{d} x=\int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x
\end{aligned}
$$

For all $x \in\left[0, x_{\lambda}\right], \widetilde{\phi}(x) \leq \phi_{\lambda}^{*}(x)$ and, by Lemma 3.7, $\widetilde{\phi}(x) \geq-3 / 2 \pi$. Hence, $\widetilde{\phi}$ is bounded and, by (3.6), $\widetilde{\theta} \in L^{\infty}\left(\left[0, x_{\lambda}\right]\right)$. As a consequence, $z \in L^{\infty}\left(\left[0, x_{\lambda}\right]\right)$. Let us define

$$
J=\left\{x \in\left[0, x_{\lambda}\right]: \widetilde{\phi}(x)=\phi^{*}(x)\right\} \cup\left\{0, x_{\lambda}\right\} \quad \text { and } \quad I=\left[0, x_{\lambda}\right] \backslash J
$$

The set $I$ is an open set, hence it is a countable union of pairwise disjoint open intervals and we can write

$$
\left.I=\bigcup_{i \in A}\right] a_{i}, b_{i}[
$$

where $A$ is a countable set. Let us consider an arbitrary scalar field $\nu \in W_{0}^{1,2}\left(\left[0, x_{\lambda}\right]\right)$ such that $\nu(x)=0$ for all $x \in J$. As a consequence, both $\nu$ and $-\nu$ are infinitesimal admissible variations of $\widetilde{\phi}$ in $\mathfrak{C}_{\lambda}$ and we have

$$
\mathrm{d} G(\widetilde{\phi})[\nu]=\sum_{i \in A} \int_{a_{i}}^{b_{i}}\left(k y^{\prime} \nu^{\prime}+z \nu\right) \mathrm{d} x=0
$$

hence, for the arbitrariness of $V$,

$$
\int_{a_{i}}^{b_{i}}\left(k y^{\prime} \nu^{\prime}+z \nu\right) \mathrm{d} x=0, \quad \forall i \in A
$$

By a standard argument, we obtain that $y^{\prime}$ is absolutely continuous in $I$ and it satisfies

$$
\begin{equation*}
-k y^{\prime \prime}+z=0, \quad \text { a.e. in } I \tag{3.8}
\end{equation*}
$$

For an arbitrary $\xi \in W_{0}^{1,2}\left(\left[0, x_{\lambda}\right]\right)$, if we set $\eta(x)=\max \{\xi(x), 0\}$, then

$$
\zeta(x)=\xi(x)-\eta(x) \in \mathcal{V}_{\lambda}^{*}(\widetilde{\phi})
$$

hence

$$
\mathrm{d} G(\widetilde{\phi})[\zeta] \geq 0
$$

By (3.8), partial integration reduces to

$$
\int_{I}\left(k y^{\prime} \eta^{\prime}+z \eta\right) \mathrm{d} x=\sum_{i \in A}\left(y^{\prime}\left(b_{i}\right) \eta\left(b_{i}\right)-y^{\prime}\left(a_{i}\right) \eta\left(a_{i}\right)\right)
$$

Since $y=0$ in $J$ and $y<0$ in $I$ we have

$$
y^{\prime}\left(b_{i}\right) \geq 0, \quad y^{\prime}\left(a_{i}\right) \leq 0, \quad \forall i \in A
$$

except for $y^{\prime}(0)$ and $y^{\prime}(1)$, but in that cases $\eta(0)=\eta\left(x_{\lambda}\right)=0$. As a consequence,

$$
\int_{I}\left(k y^{\prime} \eta^{\prime}+z \eta\right) \mathrm{d} x \geq 0
$$

and we have

$$
\begin{aligned}
0 \leq \mathrm{d} G(\widetilde{\phi})[\zeta]=\int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x & -\int_{J}\left(k y^{\prime} \eta^{\prime}+z \eta\right) \mathrm{d} x-\int_{I}\left(k y^{\prime} \eta^{\prime}+z \eta\right) \mathrm{d} x \\
& \leq \int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x-\int_{J}\left(k y^{\prime} \eta^{\prime}+z \eta\right) \mathrm{d} x
\end{aligned}
$$

hence

$$
\int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x \geq \int_{J}\left(k y^{\prime} \eta^{\prime}+z \eta\right) \mathrm{d} x
$$

Since $y=0$ on $J$, then $y^{\prime}=0$ a.e. on $J$ (cf. [15] Lemma 7.7) and we obtain

$$
\begin{equation*}
\int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x \geq \int_{J} z \eta \mathrm{~d} x \tag{3.9}
\end{equation*}
$$

By (3.9), recalling that $|\eta| \leq|\xi|$, we obtain

$$
\left|\int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x\right| \leq\|z\|_{L^{\infty}}\|\xi\|_{L^{\infty}}
$$

whence

$$
\left|\int_{0}^{x_{\lambda}} k y^{\prime} \xi^{\prime} \mathrm{d} x\right| \leq\left|\int_{0}^{x_{\lambda}}\left(k y^{\prime} \xi^{\prime}+z \xi\right) \mathrm{d} x\right|+\left|\int_{0}^{x_{\lambda}} z \xi \mathrm{~d} x\right| \leq 2\|z\|_{L^{\infty}}\|\xi\|_{L^{\infty}}
$$

Since $\xi(0)=0$, then there exists a constant $c_{1}$ such that $\|\xi\|_{L^{\infty}} \leq c_{1}\left\|\xi^{\prime}\right\|_{L^{1}}$ and we obtain

$$
\left|\int_{0}^{x_{\lambda}} k y^{\prime} \xi^{\prime} \mathrm{d} x\right| \leq 2 c_{1}\|z\|_{L^{\infty}}\left\|\xi^{\prime}\right\|_{L^{1}}, \quad \forall \xi^{\prime} \in L^{1}\left(\left[0, x_{\lambda}\right]\right)
$$

Hence, $y^{\prime} \in L^{\infty}\left(\left[0, x_{\lambda}\right]\right)$ by the Riesz representation theorem. Using again 3.9), there exists a constant $c_{2}$ such that

$$
\left|\int_{0}^{x_{\lambda}} k y^{\prime} \xi^{\prime} \mathrm{d} x\right| \leq c_{2}\|z\|_{L^{\infty}}\|\xi\|_{L^{1}}
$$

By a standard argument (see, for instance, [6, Proposition 8.3]), this suffices to conclude that $y^{\prime} \in W^{1, \infty}\left(\left[0, x_{\lambda}\right]\right)$.

Now we are ready to prove Proposition 3.4
Proof of Proposition 3.4 Let $\mathcal{V}_{\lambda}^{*}(\widetilde{\phi})$ be the set of all admissible infinitesimal variations of $\widetilde{\phi}$ in $\mathfrak{C}_{\lambda}$, defined as in 3.5). Since $(\widetilde{\phi}, \widetilde{\theta})$ is a global minimizer,

$$
\begin{equation*}
\mathrm{d} F_{b, k}(\widetilde{\phi}, \widetilde{\theta})[\xi, 0]=\int_{0}^{1}\left(k \widetilde{\phi}^{\prime} \xi^{\prime}+(\widetilde{\phi}-\widetilde{\theta}) \xi\right) d x \geq 0, \quad \forall \xi \in \mathcal{V}_{\lambda}^{*}(\widetilde{\phi}) \tag{3.10}
\end{equation*}
$$

By Lemma 3.8, every function of class $C^{\infty}$ with compact support in $\left(x_{\lambda}, 1\right)$ belongs to $\mathcal{V}_{\lambda}^{*}(\widetilde{\phi})$. As a consequence, by a standard argument we obtain that

$$
k \widetilde{\phi}^{\prime \prime}=\widetilde{\phi}-\widetilde{\theta}, \quad \text { a.e. on }\left[x_{\lambda}, 1\right]
$$

and we can reduce our analysis on the interval $\left[0, x_{\lambda}\right]$. Let us now consider a variation in $\mathcal{V}_{\lambda}^{*}(\widetilde{\phi})$ with compact support in $\left(0, x_{\lambda}\right)$. By Lemma 3.9, $\widetilde{\phi} \in W^{2, \infty}\left(\left[0, x_{\lambda}\right]\right)$ so we can integrate by parts 3.10) and obtain

$$
\begin{equation*}
-k \widetilde{\phi}^{\prime \prime}+(\widetilde{\phi}-\widetilde{\theta})=\tau(x) \leq 0, \quad \text { a.e. on }\left[0, x_{\lambda}\right] \tag{3.11}
\end{equation*}
$$

where $\tau(x)=0$ if $\widetilde{\phi}(x)<\phi_{\lambda}^{*}(x)$. Set

$$
J=\left\{x \in\left[0, x_{\lambda}\right]: \widetilde{\phi}(x)=\phi_{\lambda}^{*}(x)\right\}
$$

Since $\widetilde{\phi} \in W^{2, \infty}\left(\left[0, x_{\tau}\right]\right)$, using [15] Lemma 7.7] we obtain that $\widetilde{\phi}^{\prime \prime}=\left(\phi_{\lambda}^{*}\right)^{\prime \prime}$ a.e. on $J$. Recalling also (3.6), we obtain

$$
\begin{aligned}
\tau=-k\left(\phi_{\lambda}^{*}\right)^{\prime \prime}-b(1-x) \cos \tilde{\theta} \geq & -k\left(\phi_{\lambda}^{*}\right)^{\prime \prime}-b(1-x) \\
& =b(1-x)+k-b(1-x)=k>0, \quad \text { a.e. on } J
\end{aligned}
$$

As a consequence, from (3.11) we deduce that $J$ is a set of measure zero and (3.4) follows.

Using Proposition 3.4 and exploiting again the properties of $\phi_{\lambda}^{*}$, we obtain that $\widetilde{\phi}$ can coincide with $\phi_{\lambda}^{*}$ only at 0 and $x_{\lambda}$. More formally, we have the following result.
Corollary 3.10. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$. Then

$$
\widetilde{\phi}(x)<\phi_{\lambda}^{*}(x), \quad \forall x \neq 0, x_{\lambda}
$$

Proof. Using again Lemma 3.8, it suffices to prove that $\widetilde{\phi}(x)<\phi_{\lambda}^{*}(x)$ for all $x \in\left(0, x_{\lambda}\right)$. Seeking a contradiction, let $\bar{x} \in\left(0, x_{\lambda}\right)$ be such that $\widetilde{\phi}(\bar{x})=\phi_{\lambda}^{*}(\bar{x})$. Since $\widetilde{\phi}(x) \leq \phi_{\lambda}^{*}(x)$
for all $x \in\left[0, x_{\lambda}\right]$ and, by Lemma 3.9, $\widetilde{\phi} \in W^{2, \infty}\left(\left[0, x_{\lambda}\right]\right) \subset C^{1}\left(\left[0, x_{\lambda}\right]\right)$ we have $\widetilde{\phi}^{\prime}(\bar{x})=$ $\phi_{\lambda}^{* \prime}(\bar{x})$. Hence, we obtain

$$
\begin{equation*}
0 \geq \widetilde{\phi}(x)-\phi_{\lambda}^{*}(x)=\int_{\bar{x}}^{x}\left(\int_{\bar{x}}^{s}\left(\widetilde{\phi}-\phi_{\lambda}^{*}\right)^{\prime \prime}(\tau) \mathrm{d} \tau\right) \mathrm{d} s, \quad \forall x \in\left[\bar{x}, x_{\lambda}\right] \tag{3.12}
\end{equation*}
$$

Since $\phi_{\lambda}^{*}$ is defined by (3.2), we have

$$
\phi_{\lambda}^{* \prime \prime}(x)=-(\lambda(1-x)+1), \quad \text { on }\left[0, x_{\lambda}\right]
$$

while (3.4) and (3.6) imply

$$
\widetilde{\phi}^{\prime \prime}(x)=-\frac{b}{k}(1-x) \cos \tilde{\theta} \geq-\lambda(1-x) \quad \text { a.e. on }\left[0, x_{\lambda}\right]
$$

As a consequence, from (3.12) we obtain that for every $\left.x \in] \bar{x}, x_{\lambda}\right]$ we have

$$
0 \geq \int_{\bar{x}}^{x}\left(\int_{\bar{x}}^{s}(-\lambda(1-\tau)+\lambda(1-\tau)+1) \mathrm{d} \tau\right) \mathrm{d} s=\frac{(x-\bar{x})^{2}}{2}>0
$$

which is absurd.

Remark 3.11. We notice that the proofs of Proposition 3.4 and of Corollary 3.10 rely only on the second order derivative of $\phi_{\lambda}^{*}$. As a consequence, if we substitute this constraint with another function with the same second order derivative we obtain analogous results. This observation will be useful in the final part of our work, when we will use a $\Gamma$-convergence argument to show the local minimality of $(\tilde{\phi}, \tilde{\theta})$.
3.2. Convergence to minimizers of the Euler beam. The following results are needed to show that, also for $x=x_{\lambda}$, the minimizer under $\phi_{\lambda}^{*}$ does not "touch" the constraint. This requires considerably more effort, and it is achieved through a comparison with the easier cases represented by the functionals describing the nonlinear Euler beam under uniformly distributed and concentrated load.
Recalling the definition of $\mathfrak{C}_{\lambda}^{*}$ given in (3.3), we define the functional $F_{\lambda}: \mathfrak{C}_{\lambda}^{*} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
F_{\lambda}(\phi):=\int_{0}^{1}\left(\frac{\left|\phi^{\prime}\right|^{2}}{2}-\lambda(1-x) \sin \phi\right) \mathrm{d} x \tag{3.13}
\end{equation*}
$$

which corresponds to the energy functional of a nonlinear Euler beam under distributed load (see e.g. [10]). Denoting by $\widetilde{\phi}_{\lambda}$ its minimizer, the main results of this subsection are the following:

- if $b$ is sufficiently small and $b / k=\lambda$, then the global minimizer $(\widetilde{\phi}, \widetilde{\theta})$ of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$ is such that $\left\|\widetilde{\phi}(x)-\widetilde{\phi}_{\lambda}(x)\right\|$ is arbitrarily small: in other words, the solutions of the problem of a nonlinear Timoshenko beam are similar to the ones of a nonlinear Euler beam;
- if $\lambda$ is sufficiently large, then $\widetilde{\phi}_{\lambda}\left(x_{\lambda}\right)$ is strictly less then $-\pi$ : this result will be achieved by a limit process that can get rid of the autonomous component in the functional 3.13.

Remark 3.12. The arguments used in the proofs of Proposition 3.4, of Lemma 3.9, and of Corollary 3.10 can be easily applied to $F_{\lambda}$. Therefore, if $\widetilde{\phi}_{\lambda}$ is a minimizer of $F$ in $\mathfrak{C}_{\lambda}^{*}$, we have

$$
\widetilde{\phi}_{\lambda}^{\prime \prime}+\lambda(1-x) \cos \widetilde{\phi}=0, \quad \text { a.e. on }[0,1]
$$

and

$$
\widetilde{\phi}_{\lambda}(x)<\phi_{\lambda}^{*}(x) \quad \forall x \neq 0, x_{\lambda}
$$

Proposition 3.13. Fix $\lambda_{0} \in \mathbb{R}$ and let $\left(b_{n}, k_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}^{+} \times \mathbb{R}^{+}$be such that

$$
\lim _{n \rightarrow \infty} b_{n}=0 \quad \text { and } \quad \frac{b_{n}}{k_{n}}=\lambda_{0}, \quad \forall n \in \mathbb{N}
$$

Let $\left(\widetilde{\phi}_{n}, \widetilde{\theta}_{n}\right) \in \mathfrak{S}_{\lambda_{0}}^{*}$ be the sequence of corresponding minimizers of $F_{b_{n}, k_{n}}$. Then, up to considering a subsequence, $\widetilde{\phi}_{n}$ and $\widetilde{\theta}_{n}$ converge in the $C^{1}$ norm and a.e., respectively, to a function $\widetilde{\phi}_{\lambda_{0}}$ which is a global minimizer of the functional $F_{\lambda_{0}}: \mathfrak{C}_{\lambda_{0}}^{*} \rightarrow \mathbb{R}$.

Proof. By Proposition 3.4 and Lemma 3.6, for every $n \in \mathbb{N}$ the pair $\left(\widetilde{\phi}_{n}, \widetilde{\theta}_{n}\right)$ satisfies almost everywhere the following system of equations:

$$
\left\{\begin{array}{l}
-k_{n} \widetilde{\phi}_{n}^{\prime \prime}+\widetilde{\phi}_{n}-\widetilde{\theta}_{n}=0 \\
\widetilde{\phi}_{n}-\widetilde{\theta}_{n}=-b_{n}(1-x) \cos \widetilde{\theta}_{n}
\end{array}\right.
$$

As a consequence,

$$
\widetilde{\phi}_{n}^{\prime \prime}=-\frac{b_{n}}{k_{n}}(1-x) \cos \widetilde{\theta}_{n} \leq \lambda_{0}
$$

so the sequence $\widetilde{\phi}_{n}^{\prime \prime}$ is equibounded with respect to the norm of $L^{\infty}([0,1])$. By the AscoliArzelà theorem, $\widetilde{\phi}_{n}$ converges, up to subsequences, in the $C^{1}([0,1])$ norm to a function $\widetilde{\phi}_{\lambda_{0}} \in \mathfrak{C}_{\lambda_{0}}$. By hypothesis, $b_{n} \rightarrow 0$, and using

$$
\widetilde{\theta}_{n}=\widetilde{\phi}_{n}-b_{n}(1-x) \cos \widetilde{\theta}_{n}, \quad \text { a.e. in }[0,1]
$$

we obtain that $\widetilde{\theta}$ converges to $\widetilde{\phi}_{\lambda_{0}}$ a.e.. Therefore, by the dominated convergence theorem we have

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \frac{1}{k_{n}} F_{b_{n}, k_{n}}\left(\widetilde{\phi}_{n}, \widetilde{\theta}_{n}\right)  \tag{3.14}\\
& =\lim _{n \rightarrow \infty} \int_{0}^{1}\left(\frac{\left|\widetilde{\phi}_{n}^{\prime}\right|^{2}}{2}+\frac{b_{n}^{2}(1-x)^{2} \cos ^{2} \widetilde{\theta}_{n}}{2 k_{n}}-\frac{b_{n}}{k_{n}}(1-x) \sin \widetilde{\theta}_{n}\right) \mathrm{d} x \\
& =\int_{0}^{1}\left(\frac{\left|\widetilde{\phi}_{\lambda_{0}}^{\prime}\right|^{2}}{2}-\lambda_{0}(1-x) \sin \widetilde{\phi}_{\lambda_{0}}\right) \mathrm{d} x=F_{\lambda_{0}}\left(\widetilde{\phi}_{\lambda_{0}}\right) .
\end{align*}
$$

It remains to show that $\widetilde{\phi}_{\lambda_{0}}$ is a minimizer for $F_{\lambda_{0}}$ in $\mathfrak{C}_{\lambda_{0}}^{*}$. By contradiction, let $\psi \in C_{\lambda_{0}}^{*}$ be such that $F_{\lambda_{0}}(\psi)<F_{\lambda_{0}}\left(\widetilde{\phi}_{\lambda_{0}}\right)$. Since

$$
\lim _{n \rightarrow \infty} \frac{1}{k_{n}} F_{b_{n}, k_{n}}(\psi, \psi)=F_{\lambda_{0}}(\psi)
$$

and 3.14 holds, there exist $\epsilon>0$ and $n$ such that

$$
\frac{1}{k_{n}} F_{b_{n}, k_{n}}\left(\widetilde{\phi}_{n}, \widetilde{\theta}_{n}\right)>F_{\lambda_{0}}\left(\widetilde{\phi}_{\lambda_{0}}\right)-\epsilon>F_{\lambda_{0}}(\psi)+\epsilon>\frac{1}{k_{n}} F_{b_{n}, k_{n}}(\psi, \psi),
$$

contradicting the minimality of $\left(\widetilde{\phi}_{n}, \widetilde{\theta}_{n}\right)$.
Proposition 3.13 entails that our aim is to study the behaviour of the minimizer $\widetilde{\phi}_{\lambda} \in \mathfrak{C}_{\lambda}^{*}$ of $F_{\lambda}$ as $\lambda$ goes to infinity. In particular, by Remark 3.12, we need to prove that if $\lambda$ is sufficiently large then $\widetilde{\phi}_{\lambda}\left(x_{\lambda}\right)<-\pi$. The next result provides a necessary condition for a function whose graph passes through $\left(x_{\lambda},-\pi\right)$ to be a minimizer. This condition involves the left and right derivatives at $x_{\lambda}$.

Lemma 3.14. Let $\widetilde{\phi}_{\lambda} \in \mathfrak{C}_{\lambda}^{*}$ be a minimizer of $F_{\lambda}$. If $\widetilde{\phi}_{\lambda}\left(x_{\lambda}\right)=-\pi$, then

$$
\begin{equation*}
\widetilde{\phi}_{\lambda}^{\prime}\left(x_{\lambda}^{-}\right) \leq \widetilde{\phi}_{\lambda}^{\prime}\left(x_{\lambda}^{+}\right) \tag{3.15}
\end{equation*}
$$

Proof. If $\widetilde{\phi}_{\lambda}$ is a minimizer, then for every $\xi \in C_{0}^{\infty}([0,1])$ such that

$$
\xi(x) \leq 0, \quad \forall x \in[0,1] \quad \text { and } \quad \xi\left(x_{\lambda}\right)<0
$$

we have $\mathrm{d} F_{\lambda}\left(\widetilde{\phi}_{\lambda}\right)[\xi] \geq 0$, hence

$$
\int_{0}^{1}\left(\widetilde{\phi}_{\lambda}^{\prime} \xi^{\prime}-\lambda(1-x) \cos \widetilde{\phi}_{\lambda} \xi\right) \mathrm{d} x \geq 0
$$

By Remark 3.12, $\widetilde{\phi}_{\lambda}$ does not coincide with $\phi_{\lambda}^{*}$ except in 0 and, by hypothesis, in $x_{\lambda}$. Therefore, it satisfies the Euler-Lagrange equation both in $\left(0, x_{\lambda}\right)$ and in $\left(x_{\lambda}, 1\right)$ and an integration by parts leads to

$$
\mathrm{d} F_{\lambda}\left(\widetilde{\phi}_{\lambda}\right)[\xi]=\left(\widetilde{\phi}_{\lambda}^{\prime}\left(x_{\lambda}^{-}\right)-\widetilde{\phi}_{\lambda}^{\prime}\left(x_{\lambda}^{+}\right)\right) \xi\left(x_{\lambda}\right) \geq 0
$$

By the arbitrariness of $\xi\left(x_{\lambda}\right)<0$, we obtain (3.15).
Due to Lemma 3.14, it becomes important to estimate the behaviour of the left and right derivatives at $x_{\lambda}$ of the minimizer of $F_{\lambda}$ among all the functions in $\mathfrak{C}_{\lambda}$ whose graph passes through $\left(x_{\lambda},-\pi\right)$. To this aim, we separately study the functional in the two intervals $\left[0, x_{\lambda}\right]$ and $\left[x_{\lambda}, 1\right]$ and we define the following sets

$$
\mathfrak{L}_{\lambda}:=\left\{\phi \in H^{1}\left(\left[0, x_{\lambda}\right], \mathbb{R}\right): \phi(x) \leq \phi_{\lambda}^{*}(x) \forall x \in\left[0, x_{\lambda}\right], \phi(0)=0 \text { and } \phi\left(x_{\lambda}\right)=-\pi\right\}
$$

and

$$
\mathfrak{R}_{\lambda}:=\left\{\phi \in H^{1}\left(\left[x_{\lambda}, 1\right], \mathbb{R}\right): \phi(x) \leq-\pi \forall x \in\left[x_{\lambda}, 1\right], \phi\left(x_{\lambda}\right)=-\pi\right\}
$$

On them, we define the functionals $L_{\lambda}: \mathfrak{L}_{\lambda} \rightarrow \mathbb{R}$ and $R_{\lambda}: \mathfrak{R}_{\lambda} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
L_{\lambda}(\phi):=\int_{0}^{x_{\lambda}}\left(\frac{\phi^{\prime 2}}{2}-\lambda(1-x) \sin \phi\right) \mathrm{d} x \tag{3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{\lambda}(\phi):=\int_{x_{\lambda}}^{1}\left(\frac{\phi^{\prime 2}}{2}-\lambda(1-x) \sin \phi\right) \mathrm{d} x . \tag{3.17}
\end{equation*}
$$

Let $\ell_{\lambda} \in \mathfrak{L}_{\lambda}$ and $r_{\lambda} \in \mathfrak{R}_{\lambda}$ be the minimizers of $L_{\lambda}$ and $R_{\lambda}$, respectively. Then a function $\psi_{\lambda} \in \mathfrak{C}_{\lambda}^{*}$ such that $\psi_{\lambda}\left(x_{\lambda}\right)=-\pi$ is a minimizer of $F_{\lambda}$ if and only if

$$
\psi_{\lambda}(x)= \begin{cases}\ell_{\lambda}(x), & \text { if } x \in\left[0, x_{\lambda}\right]  \tag{3.18}\\ r_{\lambda}(x), & \text { if } x \in\left[x_{\lambda}, 1\right]\end{cases}
$$

and, by Lemma 3.14, if

$$
\begin{equation*}
\ell_{\lambda}^{\prime}\left(x_{\lambda}\right) \leq r_{\lambda}^{\prime}\left(x_{\lambda}\right) \tag{3.19}
\end{equation*}
$$

As a consequence, proving that 3.19 does not hold for $\lambda$ sufficiently large implies that the minimizers of $F_{\lambda}$ in $\mathfrak{C}_{\lambda}^{*}$ does not pass through $\left(x_{\lambda},-\pi\right)$. Fig. 4 and Fig. 5 show $\ell_{\lambda}$ and $r_{\lambda}$ for $\lambda=15$ and $\lambda=100$, respectively. In Fig. 4, it can be noticed that 3.19 holds, and indeed $\psi_{\lambda}$ defined as in (3.18) is a global minimizer of $F_{\lambda}$ in $\mathfrak{C}_{\lambda}^{*}$. In Fig. 5 , it can be noticed that 3.19 does not holds, so the graph of the global minimizer $\widetilde{\phi}_{\lambda}$ does not pass through the point $\left(x_{\lambda},-\pi\right)$.


Figure 4. The functions $\ell_{\lambda}$ and $r_{\lambda}$ for $\lambda=15$; (numerically evaluated); it can be noticed that $\ell_{\lambda}^{\prime}\left(x_{\lambda}\right)<r_{\lambda}^{\prime}\left(x_{\lambda}\right)$ and the graph of $\widetilde{\phi}_{\lambda}$ passes through $\left(x_{\lambda},-\pi\right)$.


Figure 5. The functions $\ell_{\lambda}$ and $r_{\lambda}$ for $\lambda=100$ (numerically evaluated); it can be noticed that $\ell_{\lambda}^{\prime}\left(x_{\lambda}\right)>r_{\lambda}^{\prime}\left(x_{\lambda}\right)$, so by Lemma 3.14 a function whose graph passes through $\left(x_{\lambda},-\pi\right)$ cannot be a minimizer of $F_{\lambda}$ in $\mathfrak{C}_{\lambda}^{*}$.

Before giving the estimates for $\ell_{\lambda}^{\prime}\left(x_{\lambda}\right)$ and $r_{\lambda}^{\prime}\left(x_{\lambda}\right)$, we need to define a function $G: \mathbb{R}^{+} \rightarrow$ $\mathbb{R}$ as follows:

$$
G(\mu)=\int_{0}^{\pi} \frac{\mathrm{d} \sigma}{\sqrt{\mu+4 \pi \sin \sigma}}
$$

Remark 3.15. The function $G$ is strictly decreasing, $G(0)>1$ and $\lim _{\mu \rightarrow \infty} G(\mu)=0$. As a consequence, there exists an unique $E>0$ such that

$$
\begin{equation*}
\int_{0}^{\pi} \frac{\mathrm{d} \sigma}{\sqrt{E+4 \pi \sin \sigma}}=1 \tag{3.20}
\end{equation*}
$$

Remark 3.16. In both Lemma 3.17 and Lemma 3.18, we will exploit the following limit, obtained from the definition of $\phi_{\lambda}^{*}$ given in (3.2):

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty} \lambda x_{\lambda}^{2}=2 \pi \tag{3.21}
\end{equation*}
$$

Lemma 3.17. Let $\ell_{\lambda} \in \mathfrak{L}_{\lambda}$ be the minimizer of $L_{\lambda}$. Let $E>0$ be such that 3.20 holds. Then

$$
\lim _{\lambda \rightarrow \infty} x_{\lambda} \ell_{\lambda}^{\prime}\left(x_{\lambda}\right)=-E
$$

Proof. By the change of variable $x=x_{\lambda} t$, for every $\lambda$ we can reparametrize the integral functional $L_{\lambda}$ defined in (3.16) on the interval $[0,1]$. Hence, setting

$$
\varphi(t)=\phi\left(x_{\lambda} t\right)=\phi(x)
$$

we obtain

$$
\begin{equation*}
L_{\lambda}(\phi)=\frac{1}{x_{\lambda}} \int_{0}^{1}\left(\frac{1}{2}\left(\varphi^{\prime}\right)^{2}-\lambda x_{\lambda}^{2}\left(1-x_{\lambda} t\right) \sin \varphi\right) \mathrm{d} t \tag{3.22}
\end{equation*}
$$

As a consequence, the function $\widetilde{\ell}_{\lambda}:[0,1] \rightarrow \mathbb{R}$, defined as $\widetilde{\ell}_{\lambda}(t)=\ell_{\lambda}\left(x_{\lambda} t\right)$, minimizes the integral

$$
\int_{0}^{1}\left(\frac{1}{2}\left(\varphi^{\prime}\right)^{2}-\lambda x_{\lambda}^{2}\left(1-x_{\lambda} t\right) \sin \varphi\right) \mathrm{d} t
$$

and it never equals the function $\varphi_{\lambda}^{*}(t)=\phi_{\lambda}^{*}\left(x_{\lambda} t\right)$. Therefore, it is a solution of the following Dirichlet problem

$$
\left\{\begin{array}{l}
y^{\prime \prime}+\lambda x_{\lambda}^{2}\left(1-x_{\lambda} t\right) \cos y=0 \\
y(0)=0, \quad y(1)=-\pi
\end{array}\right.
$$

Arguing similarly as in the proof of Proposition 3.13 and recalling (3.21), $\tilde{\ell}_{\lambda}$ converges in the $C^{1}$ norm to the minimizer $\widetilde{z}$ of the functional

$$
L(z)=\int_{0}^{1}\left(\frac{1}{2}\left(z^{\prime}\right)^{2}-2 \pi \sin z\right) \mathrm{d} t
$$

subject to

$$
z(t) \leq \lim _{\lambda \rightarrow \infty} \varphi_{\lambda}^{*}(t)=\lim _{\lambda \rightarrow \infty}\left[\frac{\lambda x_{\lambda}^{2}}{2} t^{2}\left(\frac{x_{\lambda}}{3} t-1\right)-\frac{x_{\lambda}^{2}}{2} t^{2}\right]=-\pi t^{2}, \quad \forall t \in[0,1] .
$$

Therefore, $\tilde{z}$ is a solution of the following Dirichlet problem

$$
\left\{\begin{array}{l}
z^{\prime \prime}+2 \pi \cos z=0 \\
z(0)=0, \quad z(1)=-\pi
\end{array}\right.
$$

and

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty} x_{\lambda} \ell_{\lambda}^{\prime}\left(x_{\lambda}^{-}\right)=\lim _{\lambda \rightarrow \infty}\left(\widetilde{\ell}_{\lambda}\right)^{\prime}(1)=\widetilde{z}^{\prime}(1) \tag{3.23}
\end{equation*}
$$

Being the minimizer of a suitably regular autonomous problem, $\widetilde{z}$ is a monotone function (see, e.g., [9. Theorem 3.1]), so we have that $\widetilde{z}^{\prime}(t) \leq 0$ for all $t \in[0,1]$. Since $z^{\prime \prime}+$ $2 \pi \cos z=0$ is an autonomous differential equation, there exists $E \in \mathbb{R}$ such that

$$
\begin{equation*}
\frac{1}{2}\left(\widetilde{z}^{\prime}\right)^{2}+2 \pi \sin \widetilde{z}=\frac{1}{2} E, \quad \forall t \in[0,1] . \tag{3.24}
\end{equation*}
$$

Moreover, being $\widetilde{z}(1)=-\pi$, we obtain that $\left(\widetilde{z}^{\prime}(1)\right)^{2}=E$. Since $\widetilde{z}^{\prime}(t) \leq 0$, we have $\widetilde{z}^{\prime}(1)=-\sqrt{E}$ and from (3.24) we obtain

$$
\frac{\widetilde{z}^{\prime}}{\sqrt{E-4 \pi \sin \widetilde{z}}}=-1
$$

Recalling that $\widetilde{z}(0)=0$ and $\widetilde{z}(1)=-\pi$, with some easy computations we obtain

$$
\int_{0}^{\pi} \frac{\mathrm{d} \sigma}{\sqrt{E+4 \pi \sin \sigma}}=1
$$

hence $E$ satisfies 3.20 . By Remark 3.16, there exists a unique $E>0$ which satisfies (3.20) and using also (3.23) we have

$$
\lim _{\lambda \rightarrow \infty} x_{\lambda} \ell_{\lambda}^{\prime}\left(x_{\lambda}^{-}\right)=-\sqrt{E}
$$

Lemma 3.18. Let $r_{\lambda} \in \mathfrak{R}_{\lambda}$ be the minimizer of $R_{\lambda}$. Then

$$
\lim _{\lambda \rightarrow \infty} x_{\lambda} r^{\prime}\left(x_{\lambda}\right)=-2 \sqrt{\pi}
$$

Proof. The proof is similar to the one of Lemma 3.17 By the change of variable $x=x_{\lambda} t$, we re-parameterize the integral functional $R_{\lambda}$ defined in 3.17) on the interval $\left[1,1 / x_{\lambda}\right]$. As a consequence, the function $\widetilde{r}_{\lambda}:\left[1,1 / x_{\lambda}\right] \rightarrow \mathbb{R}$ given by $\widetilde{r}_{\lambda}(t)=r_{\lambda}\left(x_{\lambda} t\right)$ is a minimizer for the functional

$$
\varphi \mapsto \int_{0}^{1}\left(\frac{1}{2}\left(\varphi^{\prime}\right)^{2}-\lambda x_{\lambda}^{2}\left(1-x_{\lambda} t\right) \sin \varphi\right) \mathrm{d} t
$$

with the constraint $\widetilde{r}_{\lambda}(1)=-\pi$ and $\widetilde{r}_{\lambda}(t) \leq-\pi$. Let us notice that

$$
\begin{equation*}
x_{\lambda} r_{\lambda}^{\prime}\left(x_{\lambda}\right)=\widetilde{r}_{\lambda}(1) \tag{3.25}
\end{equation*}
$$

Using arguments similar to Lemma 3.7 and Lemma 3.8, we obtain that $\widetilde{r}_{\lambda}(t) \in\left(-\frac{3}{2} \pi,-\pi\right)$ for all $t \in\left(1,1 / x_{\lambda}\right]$. Therefore, it is the solution of the following Cauchy problem

$$
\left\{\begin{array}{l}
z^{\prime \prime}+\lambda x_{\lambda}^{2}\left(1-x_{\lambda} t\right) \cos z=0  \tag{3.26}\\
z(1)=-\pi \\
z^{\prime}(1)=-\nu_{\lambda}
\end{array}\right.
$$

for a suitable $\nu_{\lambda}>0$. Let us show that $\nu_{\lambda}>0$ is upper bounded. For every $\lambda$ we have

$$
z^{\prime}(t)+\nu_{\lambda}=\int_{1}^{t} z^{\prime \prime}(s) \mathrm{d} s
$$

If $\lambda$ is sufficiently large, $1 / x_{\lambda}>2$ and we can integrate the both sides of the previous equation on the interval $[1,2]$. Recalling that $z(1)=-\pi$ and that $z(2)>-\frac{3}{2} \pi$, we obtain

$$
\begin{aligned}
\nu_{\lambda} & =-\int_{1}^{2} z^{\prime}(t) \mathrm{d} t+\int_{1}^{2} \int_{1}^{t} z^{\prime \prime}(s) \mathrm{d} s \mathrm{~d} t \\
& =z(1)-z(2)-\int_{1}^{2} \int_{1}^{t} \lambda x_{\lambda}^{2}\left(1-x_{\lambda} s\right) \cos z \mathrm{~d} s \mathrm{~d} t \leq \frac{\pi+\lambda x_{\lambda}^{2}}{2}
\end{aligned}
$$

By (3.21), if $\lambda$ is sufficiently large we have

$$
\nu_{\lambda} \leq \frac{\pi+2 \pi}{2}+\frac{\pi}{2}=2 \pi
$$

Therefore, there exists $\nu>0$ such that, up to subsequences, $\nu_{\lambda} \rightarrow \nu$ as $\lambda \rightarrow \infty$ and, on every compact set $[1, M]$, the solutions of (3.26) uniformly converges to the solution $w:[1,+\infty) \rightarrow \mathbb{R}$ of the following Cauchy problem

$$
\left\{\begin{array}{l}
z^{\prime \prime}+2 \pi \cos z=0 \\
z(1)=0 \\
z^{\prime}(1)=-\nu
\end{array}\right.
$$

By the minimality conditions on $\widetilde{r}_{\lambda}$, in particular by $\widetilde{r}_{\lambda}^{\prime}\left(1 / x_{\lambda}\right)=0$, we obtain that $\nu>0$ is such that $\lim _{t \rightarrow \infty} w(t)=-\frac{3}{2} \pi$ and $\lim _{t \rightarrow \infty} w^{\prime}(t)=0$. Since $z^{\prime \prime}+2 \pi \cos z=0$ is an autonomous differential equation, there exists a constant $E>0$ such that

$$
\frac{1}{2}\left(w^{\prime}\right)^{2}+2 \pi \sin w=E
$$

and since $\sin w(1)=0$ we obtain

$$
E=\frac{1}{2} \nu^{2} .
$$

Moreover, since $\lim _{t \rightarrow \infty} w(t)=-\frac{3}{2} \pi$ and $\lim _{t \rightarrow \infty} w^{\prime}(t)=0$, we have that

$$
E=\frac{1}{2} \nu^{2}=\lim _{t \rightarrow \infty} \frac{1}{2}\left(w^{\prime}(t)\right)^{2}+2 \pi \sin w^{\prime}(t)=2 \pi
$$

hence $\nu^{2}=4 \pi$. Thus, using also 3.25, we obtain

$$
\lim _{\lambda \rightarrow \infty} x_{\lambda} r^{\prime}\left(x_{\lambda}\right)=\lim _{\lambda \rightarrow \infty}-\nu_{\lambda}=-\nu=-2 \sqrt{\pi}
$$

### 3.3. Existence of local minimizers distinct form the global ones.

Lemma 3.19. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a global minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, whose existence is ensured by Proposition 3.3. Then, for sufficiently small $b$ and sufficiently large $\lambda$, we have

$$
\widetilde{\phi}(x)<\phi_{\lambda}^{*}(x), \quad \forall x \neq 0
$$

Proof. By Corollary 3.10, we have that $\widetilde{\phi}$ can be equal to $\phi_{\lambda}^{*}$ only in 0 and $x_{\lambda}$. So our aim is to prove that if $b$ and $k / b$ are sufficiently small (or equivalently if $b$ is sufficiently small and $\lambda$ is sufficiently large), then

$$
\widetilde{\phi}\left(x_{\lambda}\right)<-\pi
$$

As a first step, let us show that if $\lambda=b / k$ is sufficiently large, then $\widetilde{\phi}_{\lambda}\left(x_{\lambda}\right)<-\pi$, where $\widetilde{\phi}_{\lambda}$ is the minimizer of the functional $F_{\lambda}$ defined in (3.13). By Lemma 3.14 and the definitions of $\ell_{\lambda}$ and $r_{\lambda}$, we need to prove that for $\lambda$ sufficiently large we have

$$
\ell_{\lambda}^{\prime}\left(x_{\lambda}\right)>r_{\lambda}^{\prime}\left(x_{\lambda}\right)
$$

namely we need to prove that 3.19 does not hold. Using Lemma 3.17 and Lemma 3.18 , it suffices to prove that

$$
-\sqrt{E}>-2 \sqrt{\pi}
$$

or, equivalently, that $4 \pi>E$. Since $E$ satisfies 3.20 and $G$ is a decreasing function, we need to show that

$$
1=G(E)>G(4 \pi)
$$

Since the $\sin (\sigma)>0$ for every $\sigma \in(0, \pi)$, we have

$$
G(4 \pi)=\int_{0}^{\pi} \frac{\mathrm{d} \sigma}{\sqrt{4 \pi+4 \pi \sin \sigma}}<\frac{\sqrt{\pi}}{2}<1
$$

hence we infer that if $\lambda$ is sufficiently large, then $\widetilde{\phi}_{\lambda}\left(x_{\lambda}\right)<-\pi$.
Therefore, let us fix such a $\lambda$. By contradiction, let $\left(b_{n}, k_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{R}^{+} \times \mathbb{R}^{+}$be a sequence such that $b_{n} \rightarrow 0, b_{n} / k_{n}=\lambda$ for all $n$ and the sequence of minimizers of $F_{b_{n}, k_{n}}$ in $\mathfrak{S}_{\lambda}^{*}$, denoted by $\left(\widetilde{\phi}_{n}, \widetilde{\theta}_{n}\right)$, is such that

$$
\widetilde{\phi}_{n}\left(x_{\lambda}\right)=-\pi, \quad \forall n \in \mathbb{N}
$$

By Proposition 3.13, there exists a subsequence of $\widetilde{\phi}_{n}$ that converges uniformly to $\widetilde{\phi}_{\lambda}$. Since $\widetilde{\phi}_{\lambda}\left(x_{\lambda}\right)<-\pi$, this is absurd, and we are done.

Remark 3.20. Lemma 3.19 is not enough to conclude that $(\widetilde{\phi}, \widetilde{\theta})$ is a local minimizer of the functional $F$ in $\mathfrak{S}$, since $\widetilde{\phi}$ belongs to $\partial \mathfrak{C}_{\lambda}^{*}$. Indeed, consider the sequence $\left(f_{k}\right)_{k} \subset \mathfrak{C}^{*}$ defined as follows:

$$
f_{k}= \begin{cases}k x, & \text { for } x \in\left[0, k^{-3}\right] \\ k^{-2}, & \text { for } x \in\left(k^{-3}, 1\right]\end{cases}
$$

Clearly we have $\left\|f_{k}\right\|_{H^{1}} \rightarrow 0$ when $k \rightarrow \infty$. For every $C$-Lipschitz $\phi \in \mathfrak{C}_{\lambda}^{*}, \phi(x)+$ $f_{k}(x)>0$ for $x \in\left[0, k^{-3}\right]$ if $k>C$. Therefore every Lipschitz-regular element of $\mathfrak{C}^{*}$ belongs to $\partial \mathfrak{C}^{*}$, hence $\tilde{\phi} \in \partial \mathfrak{C}^{*}$.
As a consequence of the previous remark, we need to show that there is a sufficiently small open ball $\widetilde{\mathcal{B}}$ centered in $\widetilde{\phi}$ such that $F(\widetilde{\phi}, \widetilde{\theta}) \leq F(\phi, \theta)$ for every $(\phi, \theta) \in \widetilde{\mathcal{B}} \times L^{2}$. We obtain this result by a $\Gamma$-convergence argument, for which we need the following definitions.

Definition 3.21. Fix $b, k>0$ and set $\lambda=b / k$. For each real number $\epsilon>0$ we define the function $\phi_{\lambda, \epsilon}^{*}:[0,1] \rightarrow \mathbb{R}$ as follows:

$$
\phi_{\lambda, \epsilon}^{*}(x)=\max \left\{\frac{\lambda}{2} x^{2}\left(\frac{x}{3}-1\right)-\frac{1}{2} x^{2}+\epsilon,-\pi\right\}
$$

We denote by $x_{\lambda, \epsilon}$ the least $x$ such that $\phi_{\lambda, \epsilon}^{*}=-\pi$, that is:

$$
x_{\lambda, \epsilon}=\min \left\{x \in[0,1]: \phi_{n}^{*}(x)=-\pi\right\}
$$

and we define

$$
\mathfrak{C}_{\epsilon}^{*}:=\left\{\phi \in H_{*}^{1}: \phi(x) \leq \phi_{\lambda, \epsilon}^{*}(x), \forall x \in[0,1]\right\}
$$

Recalling the definition of $\Theta: H_{*}^{1} \rightarrow L^{2}$ given in Corollary 2.7, we define the functional $\mathcal{F}_{\epsilon}: H_{*}^{1} \rightarrow \mathbb{R}$ as follows:

$$
\mathcal{F}_{\epsilon}(\phi)= \begin{cases}F_{b, k}(\phi, \Theta(\phi)), & \text { if } \phi \in \mathfrak{C}_{\epsilon}^{*} \\ +\infty, & \text { otherwise }\end{cases}
$$

and the functional $\mathcal{F}: H_{*}^{1}([0,1], \mathbb{R}) \rightarrow \mathbb{R}$ as follows:

$$
\mathcal{F}(\phi)= \begin{cases}F_{b, k}(\phi, \Theta(\phi)), & \text { if } \phi \in \mathfrak{C}^{*} \\ +\infty, & \text { otherwise }\end{cases}
$$

Remark 3.22. If $b<1$ and $(\widetilde{\phi}, \widetilde{\theta}) \in \mathfrak{S}^{*}$ is a global minimizer for $F_{b, k}$, then $\widetilde{\theta}=\Theta(\widetilde{\phi})$ a.e., thus

$$
F_{b, k}(\widetilde{\phi}, \widetilde{\theta})=\mathcal{F}(\widetilde{\phi})
$$

Lemma 3.23. For every $b, k, \epsilon>0, \mathcal{F}_{\epsilon}: H_{*}^{1} \rightarrow \mathbb{R}$ admits a global minimizer, that we denote by $\widetilde{\phi}_{\epsilon}$. Moreover,

$$
\begin{equation*}
\widetilde{\phi}_{\epsilon}(x)<\phi_{\lambda, \epsilon}^{*}(x), \quad \forall x \neq x_{\lambda, \epsilon} . \tag{3.27}
\end{equation*}
$$

Proof. Since $\mathfrak{C}_{\epsilon}^{*} \subset H_{*}^{1}$ is closed with respect to the $L^{\infty}$ norm, the existence of a global minimizer can be obtained following the same proof of Proposition 3.3. Moreover, as previously observed in Remark 3.11, the proof of Corollary 3.10 relies only on the second derivative of $\phi^{*}$, hence the analogous result given in 3.27) holds when we substitute $\phi^{*}$ with $\phi_{\epsilon}^{*}$.

Lemma 3.24. Let $\left(\epsilon_{n}\right)_{n} \subset \mathbb{R}$ be a strictly decreasing sequence such that $\epsilon_{n}>0$ for each $n$ and $\epsilon_{n} \rightarrow 0$. Then the sequence of functionals $\mathcal{F}_{\epsilon_{n}} \Gamma$-converges to the functional $\mathcal{F}$. Therefore, if $\left(\phi_{\epsilon_{n}}\right)_{n} \subset H_{*}^{1}$ is a sequence of absolute minimizers of $\mathcal{F}_{\epsilon_{n}}$, it converges in $L^{\infty}$-norm to a minimizer of $\mathcal{F}$.

Proof. Since $\epsilon_{n}$ is strictly decreasing, for every $n \in \mathbb{N}$ we have $\mathfrak{C}_{\lambda}^{*} \subset \mathfrak{C}_{\lambda, \epsilon_{n+1}}^{*} \subset \mathfrak{C}_{\epsilon_{n}}^{*}$. Therefore, the sequence $\mathcal{F}_{n}$ is pointwise non-decreasing, so that (see e.g. Remark 1.40 in [5])

$$
\Gamma-\lim _{n} \mathcal{F}_{n}=\sup _{n} \operatorname{sc}\left(\mathcal{F}_{n}\right)=\lim _{n} \operatorname{sc}\left(\mathcal{F}_{n}\right)=\lim _{n} \mathcal{F}_{n}=\mathcal{F},
$$

where $\operatorname{sc}(\cdot)$ indicates the lower-semicontinuous envelope and the penultimate equality holds because $\mathcal{F}_{n}$ is weakly lower-semicontinuous for every $n$. The second part of the statement follows from the basic properties of $\Gamma$-convergence (see Theorem 1.21 in [5]).

We are finally ready to prove our main result.
Proof of Theorem 3.1. By Lemma 3.19. if $b$ is sufficiently small and $\lambda$ is sufficiently large, any minimizer of $F_{b, k}$ in $\mathfrak{S}_{\lambda}^{*}$, denoted by $(\widetilde{\phi}, \widetilde{\theta})$, is such that $\widetilde{\phi}(x)<\phi_{\lambda}^{*}(x)<0$ for every $x \neq 0$. To prove Theorem 3.1, we need to prove that $(\widetilde{\phi}, \widetilde{\theta})$ is a local minimizer in the whole set $\mathfrak{S}=H_{*}^{1} \times L^{2}$. It is important to notice that we can choose $\lambda$ large enough such that $\widetilde{\phi}\left(x_{\lambda}\right)<\phi_{\lambda}^{*}\left(x_{\lambda}\right)=-\pi$ and that we can assume that $b<1$, so that Corollary 2.7 can be applied to define the function $\Theta$.
Seeking a contradiction, fix $\alpha>0$ and let $\widetilde{\mathcal{B}}_{\alpha} \subset H_{*}^{1}$ be the open ball with center $\widetilde{\phi}$ and radius $\alpha$ with respect to the $H^{1}$ norm. Then, by contradiction, there exists $\left(\phi_{\alpha}, \theta_{\alpha}\right) \in$ $\widetilde{\mathcal{B}}_{\alpha} \times L^{2}$ such that

$$
F_{b, k}\left(\phi_{\alpha}, \theta_{\alpha}\right)<F_{b, k}(\widetilde{\phi}, \widetilde{\theta})
$$

By definition of $\Theta$, this implies

$$
\mathcal{F}\left(\phi_{\alpha}\right) \leq F_{b, k}\left(\phi_{\alpha}, \theta_{\alpha}\right)<F_{b, k}(\widetilde{\phi}, \widetilde{\theta})=\mathcal{F}(\widetilde{\phi})
$$

Since $(\widetilde{\phi}, \widetilde{\theta})$ is a global minimizer of $F_{b, k}$ in $\mathfrak{S}^{*}=\mathfrak{C}^{*} \times L^{2}$, the previous chain of inequalities implies that $\phi_{\alpha} \notin \mathfrak{C}^{*}$. Let $\beta>0$ such that $\phi_{\alpha} \in \mathfrak{C}_{\beta}^{*}$. By Lemma 3.23, there exists a global minimizer of $\mathcal{F}_{\beta}$, that we denote by $\widetilde{\phi}_{\beta}$, and we have

$$
\mathcal{F}_{\beta}\left(\widetilde{\phi}_{\beta}\right) \leq \mathcal{F}_{\beta}\left(\phi_{\alpha}\right)<\mathcal{F}(\widetilde{\phi}) .
$$

Hence, $\widetilde{\phi}_{\beta} \in \mathfrak{C}_{\beta}^{*} \backslash \mathfrak{C}^{*}$ and there exists $\left.\left.\epsilon \in\right] 0, \beta\right]$ such that $\widetilde{\phi}_{\beta} \in \mathfrak{C}_{\epsilon}^{*}$ and

$$
\left\{x \in[0,1]: \widetilde{\phi}_{\beta}(x)=\phi_{\lambda, \epsilon}^{*}(x)\right\} \neq \emptyset .
$$

Since $\mathfrak{C}_{\epsilon}^{*} \subseteq \mathfrak{C}_{\beta}^{*}$, we have that we can take $\widetilde{\phi}_{\epsilon}=\widetilde{\phi}_{\beta}$, that is $\widetilde{\phi}_{\beta}$ is actually a global minimizer of $\mathcal{F}_{\epsilon}$. As a consequence, applying again Lemma 3.23, we have that $\widetilde{\phi}_{\epsilon}\left(x_{\lambda, \epsilon}\right)=$ $\phi_{\lambda, \epsilon}^{*}\left(x_{\lambda, \epsilon}\right)=-\pi$. For the sake of presentation, an illustration of the above construction is given in Fig. 6


Figure 6. Convergence of minimizers in the $\Gamma$-convergence argument for the proof of Theorem 3.1 .

Now, consider a strictly decreasing sequence $\left(\alpha_{n}\right)_{n} \subset \mathbb{R}^{+}$that converges to 0 . Applying the previous construction, we can construct a strictly decreasing sequence $\left(\epsilon_{n}\right)_{n} \subset \mathbb{R}^{+}$ that converges to 0 and a sequence $\left(\widetilde{\phi}_{\epsilon_{n}}\right)_{n} \subset H_{*}^{1}$ of global minimizers of $\mathcal{F}_{\epsilon_{n}}$ such that

$$
\widetilde{\phi}_{\epsilon_{n}}\left(x_{\lambda, \epsilon_{n}}\right)=-\pi .
$$

By Lemma 3.24, the sequence $\left(\widetilde{\phi}_{\epsilon_{n}}\right)_{n}$ uniformly converges to a minimizer $\bar{\phi}$ of $\mathcal{F}$. Since $\lim _{n \rightarrow \infty} x_{\lambda, \epsilon_{n}}=x_{\lambda}$ we have

$$
\bar{\phi}\left(x_{\lambda}\right)=\lim _{n \rightarrow \infty} \widetilde{\phi}_{\epsilon_{n}}\left(x_{\lambda, \epsilon_{n}}\right)=-\pi
$$

By Lemma 3.19, this is a contradiction and, finally, we are done.
3.4. Regularity of local minimizers. By Proposition 2.4, we obtain the following regularity result for the local minimizers of $F_{b, k}$ such that $\phi(x)<0$ in $(0,1]$, whose existence is ensured by Theorem 3.1 .
Corollary 3.25. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a local minimizer of $F_{b, k}(\phi, \theta)$ such that $\phi(x)<0$ in $(0,1]$. If $b \leq 1$, then both $\widetilde{\phi}$ and $\widetilde{\theta}$ are $C^{\infty}$.
A complementary result can be established assuming a lower bound on $b$ (depending on $x$ ). This is obtained in Proposition 3.27 , after we establish a lemma showing that $|\widetilde{\phi}-\widetilde{\theta}|$ cannot exceed $\pi$. Proposition 3.27 entails that, in general, there is no hope to be able to "neglect" the asymmetric nature of the variational problem, since the less regular component $(\theta)$ of the local minimizers can actually live in $L^{2} \backslash H^{1}$.

Lemma 3.26. Let $(\widetilde{\phi}, \widetilde{\theta})$ be a local minimizer of $F_{b, k}(\phi, \theta)$. Let $\left[(2 n-1) \frac{\pi}{2},(2 n+1) \frac{\pi}{2}\right]:=$ $I_{n}$ for $(n \in \mathbb{Z})$. If $\widetilde{\phi}(\bar{x}) \in\left(I_{n}\right)^{\circ}$ then $\widetilde{\theta}(x) \in I_{n}$ a.e. on a neighborhood of $\bar{x}$.

Proof. First of all notice that, for every measurable subset $M \subset[0,1]$, we have that $(\widetilde{\phi}, \widetilde{\theta})$ can be a (global or local) minimizer for $F_{b, k}$ only if

$$
\begin{equation*}
\widetilde{\theta}=\min _{\theta}\left(\frac{(\tilde{\phi}-\theta)^{2}}{2}-b(1-x) \sin \theta\right) \tag{3.28}
\end{equation*}
$$

a.e. on $M$. Since the function $\sin \left(x \pm(2 n \pm 1) \frac{\pi}{2}\right)$ is an even function, it follows that, if $\widetilde{\phi}(\bar{x}) \in\left(I_{n}\right)^{\circ}$, the minimization of the term $\frac{(\phi-\theta)^{2}}{2}$ implies $\widetilde{\theta}(x) \in I_{n}$ a.e. on a neighborhood of $\bar{x}$. Supposing indeed $\widetilde{\theta}(x) \in I_{n \pm 1}$, one can replace $\widetilde{\theta}(x)$ by the symmetric value with respect to $(2 n-1) \frac{\pi}{2}$ (if $\widetilde{\phi}$ belongs to the left half of $I_{n}$ ) or with respect to $(2 n+1) \frac{\pi}{2}$ (if $\widetilde{\phi}$ belongs to the right half of $I_{n}$ ), obtaining the same value for the term $b(1-x) \sin \theta$ and a strictly smaller value for the term $\frac{(\widetilde{\phi}-\theta)^{2}}{2}$.
Proposition 3.27. Suppose that there exists a local minimizer $(\widetilde{\phi}, \widetilde{\theta})$ of $F(\phi, \theta)$ in $\mathfrak{S}$ such that $\widetilde{\phi}(x)<-\frac{\pi}{2}$ for some $x \in(0,1)$ such that $b>\frac{1}{1-x}$. Then $\widetilde{\theta} \notin C^{0}$.
Proof. The function $\tilde{\theta}$ must solve the localized problem

$$
\inf _{\theta} \int_{S_{0}}\left[\frac{k}{2}\left(\widetilde{\phi}^{\prime}\right)^{2}+\frac{(\widetilde{\phi}-\theta)^{2}}{2}-b(1-x) \sin \theta\right] \mathrm{d} x
$$

where $S_{0}$ is any maximal sub-interval of $[0,1]$ such that $\widetilde{\phi}(x) \in\left(I_{n}\right)^{\circ}$ for $x \in S_{0}$. Therefore $\widetilde{\theta}(x) \in I_{0}$ a.e. in $S_{0}$.
On the other hand, $\widetilde{\theta}$ also solves the localized problem

$$
\inf _{\theta} \int_{S_{-1}}\left[\frac{k}{2}\left(\widetilde{\phi}^{\prime}\right)^{2}+\frac{(\widetilde{\phi}-\theta)^{2}}{2}-b(1-x) \sin \theta\right] d x
$$

where is any maximal sub-interval of $[0,1]$ such that $\widetilde{\phi}(x) \in\left(I_{-1}\right)^{\circ}$ for $x \in S_{-1}$. Therefore $\widetilde{\theta}(x) \in I_{-1}$ a.e. on $S_{-1}$.
Since $\widetilde{\phi}(0)=0$ and $\widetilde{\phi}(x)<-\frac{\pi}{2}$ for some $x \in(0,1)$, there exist two nonempty such intervals $S_{0}$ and $S_{-1}$. The continuity of $\widetilde{\phi}$ implies that $\widetilde{\theta}$ can be continuous only if $\widetilde{\theta}(x)=$ $-\frac{\pi}{2}$ at those $x$ such that $\widetilde{\phi}(x)=-\frac{\pi}{2}$. However we have:

$$
\begin{aligned}
& \left.\frac{\partial}{\partial \theta}\left(\frac{\theta^{2}}{2}-\phi \theta-b(1-x) \sin \theta\right)\right|_{\theta=\phi=-\frac{\pi}{2}}=0 \\
& \left.\frac{\partial^{2}}{\partial \theta^{2}}\left(\frac{\theta^{2}}{2}-\phi \theta-b(1-x) \sin \theta\right)\right|_{\theta=-\frac{\pi}{2}}=1-b(1-x)<0
\end{aligned}
$$

and therefore $\tilde{\theta}$ cannot verify 3.28 . This contradiction implies that $\widetilde{\theta}(x) \neq-\frac{\pi}{2}$, so $\widetilde{\theta} \notin$ $C^{0}$ 。

## 4. Further Questions

The results achieved in this paper open some new questions. The limit processes employed during the proof of Theorem 3.1 do not allow to estimate the lower bound for $\lambda$ and the upper bound for $b$ that ensure the existence of a local minimizer $(\widetilde{\phi}, \widetilde{\theta})$ of $F_{b, k}$ such that $\widetilde{\phi}(x) \leq 0$. By Corollary 3.10 and Proposition 3.13, the upper bound for $b$ depends on $\lambda$ : in particular, it depends on the distance between $\widetilde{\phi}_{\lambda}$, namely the minimizer of $F_{\lambda}$ in $\mathfrak{C}_{\lambda}^{*}$, and
$-\pi$ at $x_{\lambda}$. Even if giving such estimations is still an open problem, some numerical simulations conducted by the authors suggest that if $\lambda$ is greater than 42 , then the minimizer of $F_{\lambda}$ does not "touch" $\phi_{\lambda}^{*}$, as it can be seen in Fig. 7 .



Figure 7. Two minimizers of $F_{\lambda}$ in $\mathfrak{C}_{\lambda}^{*}$ : if $\lambda=42$, then the minimizer passes through $\left(x_{\lambda},-\pi\right)$ and it is not a local minimizer of $F_{\lambda}$ in $H_{*}^{1}([0,1])$; if $\lambda=43$, then the minimizer does not meet $\phi_{\lambda}^{*}$ and it is indeed a local minimizer of $F_{\lambda}$ in $H_{*}^{1}([0,1])$.

It is natural to try to generalize the results developed herein in various directions.
Firstly, from the point of view of calculus of variations, it would be interesting to investigate what happens to local minimizers when the potential $V$ has a more general form, rather than the form $b(1-x) \sin \theta$; this could be interesting for the application to onedimensional continua with more complicated microstructures than the one considered in the Timoshenko beam, as for instance the ones investigated in [2, 4, 18, 22].
Secondly, from the point of view of elasticity theory, the generalization of the inextensibility constraint 1.4 leads, in its simplest form, to a further additive term in the integrand of type $C\left(\left\|\chi^{\prime}\right\|-1\right)^{2} / 2$ and to a potential of the form

$$
V(x, \theta)=b\left\|\chi^{\prime}\right\|(1-x) \sin \theta
$$

and thus it also introduces new problems.
Finally, it is also natural to try to generalize the existence and regularity results concerning the global minimizer (developed in Section 2) to problems living in $W^{m, p} \times W^{n, p}$ of type:

$$
\inf _{u, v} \int_{\Omega}\left(f\left(\nabla^{m} u\right)+g\left(\nabla^{n} v\right)+h(u-v)-V(x, u, v)\right) \mathrm{d} x
$$

where $\Omega$ is a bounded domain of an Euclidean space and $m$ is strictly larger than $n$. One may expect that, assuming $f, g, h$ nice enough and suitable boundary conditions, the term in $u-v$ should allow to gain $W^{m, p}$ regularity for both elements of the minimizing pair $(\bar{u}, \bar{v})$.

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