Nonconforming virtual elements for the biharmonic equation with Morley degrees of freedom on polygonal meshes

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Abstract

The lowest-order nonconforming virtual element extends the Morley triangular element to polygons for the approximation of the weak solution $u \in V := H_0^2(\Omega)$ to the biharmonic equation. The abstract framework allows (even a mixture of) two examples of the local discrete spaces $V_h(P)$ and a smoother allows rough source terms $F \in V^* = H^{-2}(\Omega)$. The a priori and a posteriori error analysis in this paper circumvents any trace of second derivatives by some computable conforming companion operator $J: V_h \to V$ from the nonconforming virtual element space V_h . The operator J is a right-inverse of the interpolation operator and leads to optimal error estimates in piecewise Sobolev norms without any additional regularity assumptions on $u \in V$. As a smoother the companion operator modifies the discrete right-hand side and then allows a quasi-best approximation. An explicit residual-based a posteriori error estimator is reliable and efficient up to data oscillations. Numerical examples display the predicted empirical convergence rates for uniform and optimal convergence rates for adaptive mesh-refinement.

Keywords: biharmonic equation, virtual elements, nonconforming, polytopes, enrichment, a priori, a posteriori, adaptive mesh-refinement, companion operator, smoother

AMS subject classifications: 65N12, 65N15, 65N30, 65N50.

1 Introduction

The popular nonconforming Morley finite element method (FEM) for fourth-order problems allows a generalization from triangular domains to polygons in the class of nonconforming virtual element methods (ncVEM). Two ncVEM have been introduced in [2, 21, 35] for H^3 regular solutions, while a medius analysis in [30] allows minimal regularity. In comparison to the existing literature on ncVEM for biharmonic problems, this paper presents an abstract framework and identifies two hypotheses (H1)-(H2) for a unified stability and a priori error analysis of at least two different ncVEM each with an individual parameter r = -1, 0, 1, 2 (r = -1 for original VE spaces and r = 0, 1, 2 for enhanced VE spaces [1]) and even a mixture of those. This paper adds a new analysis with a computable conforming companion that allows a quasi-best approximation

$$|u - Gu_h|_{2,pw} \lesssim \min_{v_h \in V_h} |u - Gv_h|_{2,pw}$$
 (1.1)

with the local Galerkin projection G onto piecewise quadratics and a general source function with a smoother for the first time in ncVEM and completes the a priori error convergence analysis in piecewise energy and weaker Sobolev norms. The lower-order estimates are available in the literature for enhanced VE spaces, e.g., Zhao et al. discuss piecewise H^1 error estimate in [35] for an enhanced VE space (r=0) and this paper proves it also for original VE spaces (r=-1). The design of companion operators started in [13] for second-order and in [11, 20, 32] for fourth-order problems. It is related to enrichments in multigrid methods [6] and to reliable a posteriori error control [17]. Its role as a smoother in ncVEM generalizes [18, 19, 32] for the Morley FEM. The first paper [16] on an a posteriori error analysis for ncVEM is restricted to second-order problems and includes many references on an a posteriori error analysis for the conforming VEM. This is

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the first paper on an *a posteriori* virtual element error control for fourth-order problems with reliable and efficient error estimators and a suggested adaptive mesh-refining algorithm. The presented *a posteriori* error analysis also covers conforming VEM [10].

Main results. This paper contributes to the understanding of the ncVEM for a class of examples that includes the two known examples of discrete VE spaces for fourth-order problems

- a computable conforming companion operator,
- a priori error estimates in piecewise H^1 and H^2 norms,
- quasi-best approximation for a smoother for any source term $F \in H^{-2}(\Omega)$,
- reliable and (up to data oscillations) efficient a posteriori error control,
- adaptive mesh-refinement algorithm with improved empirical convergence rates.

The results are displayed for 2D and the lowest-order case only corresponding to the Morley degrees of freedom, but the arguments allow a higher dimension and higher degrees.

Outline and organisation of the paper. Section 2 describes the admissible partitions of the domain Ω into polygonal domains and defines the local and global Morley degrees of freedom. Subsection 2.2 establishes a Poincaré-Friedrichs inequality on polygons and Subsection 2.4 recall the local Galerkin projection G and establish associated error estimates. Section 3 explains an abstract framework with hypotheses (H1)-(H2) and presents two affirmative examples of virtual elements $V_h(P)$. The interpolation operator is defined in Subsection 3.3 and its error estimates follow in Subsection 3.4. Section 4 designs a computable conforming companion operator, which is a right-inverse of the interpolation operator, and provides the fundamental approximation error estimates. Subsection 5.1 introduces the natural stabilization and Subsection 5.2 the discrete problem with two choices for the right-hand side. Subsection 5.3 provides the a priori error estimates in piecewise H^1 and H^2 norms in the best-approximation form up to data oscillations; a smoother in the right-hand side eliminates the oscillations. Section 6 developes an explicit residual-based reliable and (up to data oscillations) efficient error analysis for ncVEM that also applies for the conforming VEM. The stabilization term is efficient with respect to the sum of the error $u - Gu_h$ and $u - u_h$ in their piecewise H^2 seminorms. Subsection 7.1 suggests an adaptive mesh-refinement algorithm. Numerical results support the theoretical predictions in Subsections 7.2-7.3 and provide striking numerical evidence of optimal empirical convergence rates for adaptive mesh-refining. Supplement material accompanies this paper with details on the local virtual element spaces and is referred to as Appendix A, B, and C throughout this paper. This contains partly established or routine results that are somehow standard but seemingly not available in the literature in this form.

Notation. Standard notation on Lebesgue and Sobolev spaces and norms applies throughout this paper, e.g., $\|\cdot\|_{s,\mathcal{D}}$ (resp. seminorm $|\cdot|_{s,\mathcal{D}}$) for $s\geq 0$ denotes norm on the Sobolev space $H^s(\mathcal{D}):=H^s(\operatorname{int}(\mathcal{D}))$ of order $s\in\mathbb{R}$ defined in the interior $\operatorname{int}(\mathcal{D})$ of a domain \mathcal{D} , while $(\cdot,\cdot)_{L^2(\mathcal{D})}$ and $\|\cdot\|_{L^2(\mathcal{D})}$ denote the L^2 scalar product and L^2 norm in \mathcal{D} . Let $|\mathcal{D}|$ denote the area of a domain \mathcal{D} , and $f_{\mathcal{D}} \bullet dx := |\mathcal{D}|^{-1} \int_{\mathcal{D}} \bullet dx$ denote the integral mean on \mathcal{D} . Define the Sobolev space $V=H_0^2(\Omega):=\{v\in H^2(\Omega):v=v_{\mathbf{n}}=0\}$ for the derivative $v_{\mathbf{n}}=\nabla v\cdot\mathbf{n}$ in the direction of outward unit normal \mathbf{n} along the boundary $\partial\mathcal{D}$. The vector space $C^r(\mathcal{D})$ is the set of C^r -continuous functions defined on a domain \mathcal{D} for $r\in\mathbb{N}_0$. Let $\mathcal{P}_k(\mathcal{D})$ denote the set of polynomials of degree at most $k\in\mathbb{N}_0$ defined on a domain \mathcal{D} and $\mathcal{P}_k(\mathcal{M})$ denote the set of piecewise polynomials on an admissible partition $\mathcal{M}\in\mathbb{M}$ (defined in Subsection 2.1). The piecewise seminorm and norm in $H^s(\mathcal{M})$ for $s\in\mathbb{R}$ (see the definition of $|\cdot|_{s,P}=|\cdot|_{H^s(P)}$ in, e.g., [8, Chapter 14]) read $|\cdot|_{s,\mathrm{pw}}:=\left(\sum_{P\in\mathcal{M}}|\cdot|_{s,P}^2\right)^{1/2}$ and $\|\cdot\|_{s,\mathrm{pw}}:=\left(\sum_{P\in\mathcal{M}}\|\cdot\|_{s,P}^2\right)^{1/2}$. Let Π_k denote the L^2 projection on $\mathcal{P}_k(\mathcal{M})$ for $k\in\mathbb{N}_0$. The oscillation of $f\in L^2(\Omega)$ reads

$$\operatorname{osc}_2(f, \mathcal{M}) := \left(\sum_{P \in \mathcal{M}} \operatorname{osc}_2^2(f, P)\right)^{1/2} \quad \text{for} \quad \operatorname{osc}_2^2(f, P) := \|h_P^2(1 - \Pi_2)f\|_{L^2(P)}^2.$$

Let $\mathbb S$ be the set of 2×2 symmetric matrices in $\mathbb R^{2 \times 2}$ and let δ_{jk} denote the Kronecker delta $(\delta_{jk} = 0 \text{ if } j \neq k \text{ and } \delta_{jj} = 1)$. Let $\alpha := (\alpha_1, \alpha_2)$ denote a multi-index with $\alpha_j \in \mathbb N_0$ for j = 1, 2

and $|\alpha| := \alpha_1 + \alpha_2$. The outward normal and tangential derivatives of first and higher orders are written as subscripts $\mathbf{n}, \boldsymbol{\tau}, \mathbf{nn}, \boldsymbol{\tau\tau}, \mathbf{n\tau\tau}$ etc. for the exterior unit normal vector \mathbf{n} and the tangential vector $\boldsymbol{\tau}$ along the boundary ∂P of the (polygonal Lipschitz) domain $P \in \mathcal{M} \in \mathbb{M}$ (from Subsection 2.1). An inequality $A \lesssim B$ abbreviates $A \leq CB$ for a generic constant C, that exclusively depends on the domain Ω and on the mesh-parameter ρ (from (M2) below).

2 Virtual element method

2.1 Admissible partitions

Let M be a family of decompositions of $\overline{\Omega}$ into polygonal domains satisfying the two mesh conditions (M1)-(M2) with a universal positive constant ρ .

- (M1) Admissibility. Any two distinct polygonal domains P and P' in $\mathcal{M} \in \mathbb{M}$ are disjoint or share a finite number of edges and vertices.
- (M2) Mesh regularity. Every polygonal domain P of diameter h_P is star-shaped with respect to every point of a ball of radius greater than equal to ρh_P and every edge E of P has a length h_E greater than equal to ρh_P .

Here and throughout this paper, $h_{\mathcal{M}}|_P := h_P := \operatorname{diam}(P)$ denotes the piecewise constant mesh-size $h_{\mathcal{M}} \in \mathcal{P}_0(\mathcal{M})$ and $h_{\max} := \max_{P \in \mathcal{M}} h_P$ denotes the maximum diameter over all $P \in \mathcal{M} \in \mathbb{M}$. Let $\mathcal{V}(P)$ (resp. \mathcal{V}) denote the set of vertices of P (resp. of \mathcal{M}) and let $\mathcal{E}(P)$ (resp. \mathcal{E}) denote the set of edges of P (resp. of \mathcal{M}). Denote the interior and boundary edges of \mathcal{M} by $\mathcal{E}(\Omega)$ and $\mathcal{E}(\partial\Omega)$. Let $|\mathcal{V}|$ (resp. $|\mathcal{E}|$) denote the number of vertices (resp. edges) and $N := |\mathcal{V}| + |\mathcal{E}|$.

The standard notation of the polygonal domain P with N_P edges and N_P vertices is depicted in Figure 2.1.a. Note that $3 \leq N_P \leq M(\rho)$ for a global number $M(\rho)$ that exclusively depends on ρ [4]. We enumerate the vertices $\mathcal{V}(P) := \{z_1, \ldots, z_{N_P}\}$ and edges $\mathcal{E}(P) := \{E(1), \ldots, E(N_P)\}$ consecutively, i.e., $E(j) = \text{conv}\{z_j, z_{j+1}\}$ for $j = 1, \ldots, N_P$ with $z_{N_P+1} := z_1$ and enumerate z_1, \ldots, z_{N_P} counterclockwise along the boundary ∂P . (M2) implies that each polygonal domain $P \in \mathcal{M}$ can be divided into triangles $T(j) := \text{conv}\{z_0, z_j, z_{j+1}\}$ for all $j = 1, \ldots, N_P$ and for the midpoint z_0 of the ball from (M2) in Figure 2.1.b. It is known [7] that the resulting subtriangulation $\mathcal{T}|_P := \mathcal{T}(P) := \bigcup_{j=1}^{N_P} T(j)$ of $P \in \mathcal{M}$ is uniformly shape-regular; i.e, the minimum angle in each triangle $T \in \mathcal{T}(P), P \in \mathcal{M} \in \mathbb{M}$, is bounded below by some positive constant $w_0 > 0$ that exclusively depends on ρ . Let $\widehat{\mathcal{V}}$ (resp. $\widehat{\mathcal{V}}(P)$) denote the set of vertices and $\widehat{\mathcal{E}}$ (resp. $\widehat{\mathcal{E}}(P)$) denote the set of edges in \mathcal{T} (resp. $\mathcal{T}(P)$).

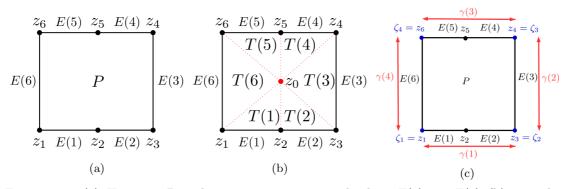


Figure 2.1: (a) Hexagon P with vertices z_1, \ldots, z_6 and edges $E(1), \ldots, E(6)$ (b) its subtriangulation $\mathcal{T}(P)$ and (c) P with corner points ζ_1, \ldots, ζ_4 and sides $\gamma(1), \ldots, \gamma(4)$.

With a counterclockwise orientation along the polygonal boundary ∂P , assign the piecewise constant tangential unit vector $\boldsymbol{\tau}_P$ and the outer normal unit vector \mathbf{n}_P . Define the local degrees of freedom (dofs) $\mathrm{dof}_1(v),\ldots,\mathrm{dof}_{2N_P}(v)$, for $v\in H^2(P)$, by

$$dof_{j}(v) = \begin{cases} v(z_{j}) & \text{for } j = 1, \dots, N_{P}, \\ \int_{E(k)} v_{\mathbf{n}} ds & \text{for } j = N_{P} + 1, \dots, 2N_{P} \text{ and } k = j - N_{P} \end{cases}$$
 (2.1)

with the vertices z_j and the edges $E(k) \in \mathcal{E}(P)$ of the polygonal domain $P \in \mathcal{M}$ in Figure 2.1.a. Given a polygonal domain $P \in \mathcal{M}$, the dofs in (2.1) are collected in the linear map Dof := $(\text{dof}_1, \ldots, \text{dof}_{2N_P}) : H^2(P) \to \mathbb{R}^{2N_P}$.

Fix the orientation of a unit normal vector \mathbf{n}_E to each edge $E \in \mathcal{E}$. The sign of the jump $[\bullet]_E$ across an edge $E \in \mathcal{E}$ follows from the subsequent convention of the orientation of a unit normal vector \mathbf{n}_E along an edge E: Label the two neighbouring polygons P_{\pm} sharing the interior edge $E = \partial P_+ \cap \partial P_-$ such that $\mathbf{n}_{P_+}|_E = \mathbf{n}_E$ and $\mathbf{n}_{P_-}|_E = -\mathbf{n}_E$. This defines the sign in the jump $[\bullet]_E := \bullet|_{P_+} - \bullet|_{P_-}$ across $E \in \mathcal{E}(\Omega)$. For a boundary edge $E \in \mathcal{E}(\partial\Omega)$, set $\mathbf{n}_{\Omega}|_E = \mathbf{n}_E$ and $[\bullet]_E := \bullet|_E$. For $v_{\rm nc} \in V_{\rm nc} := \{v \in H^2(\mathcal{M}) : v \text{ is continuous at interior vertices and zero at boundary vertices of <math>\mathcal{M}$, and $\int_E [v_{\mathbf{n}}]_E ds = 0$ for all $E \in \mathcal{E}$ }, the global Morley dofs read

$$\operatorname{dof}_{j}(v_{\operatorname{nc}}) = \begin{cases} v_{\operatorname{nc}}(z_{j}) & \text{for } j = 1, \dots, |\mathcal{V}|, \\ \int_{E(k)} (v_{\operatorname{nc}})_{\mathbf{n}} ds & \text{for } j = |\mathcal{V}| + 1, \dots, N \text{ and } k = j - |\mathcal{V}|. \end{cases}$$

$$(2.2)$$

The global degrees of freedom $\operatorname{dof}_{j}(v_{\operatorname{nc}})$ from (2.2) coincide with the local degrees of freedom $\operatorname{Dof}(v_{\operatorname{nc}}|_{P})$ from (2.1) for each polygonal domain $P \in \mathcal{M}$ up to a (known) change of signs of \mathbf{n}_{E} and $\mathbf{n}_{P}|_{E}$ for an interior edge $E \in \mathcal{E}(\Omega)$.

Lemma 2.1. $|\cdot|_{2,pw}$ defines a norm on V_{nc} equivalent to $||\cdot||_{2,pw}$.

Proof. This is shown in [2, Lemma 3.1] and in [35, Lemma 5.1] based on the Poincaré-Friedrichs inequality for piecewise H^2 functions in [9].

2.2 Poincaré-Friedrichs inequality

This subsection provides a Poincaré-Friedrichs inequality for a polygonal domain $P \in \mathcal{M} \in \mathbb{M}$ with explicit constants that exclusively depend on ρ from Subsection 2.1.

Theorem 2.2 (Poincaré-Friedrichs inequality). There exists a positive constant C_{PF} (that exclusively depends on ρ) such that

(a)
$$||f||_{L^{2}(P)} \leq C_{PF}h_{P}|f|_{1,P}$$
 holds for any $f \in H^{1}(P)$ with $0 \in \text{conv} \Big\{ \int_{E(1)} f \, ds, \dots, \int_{E(N_{P})} f \, ds \Big\}$,
(b) $\sum_{m=0}^{1} h_{P}^{m-2}|f|_{m,P} \leq C_{PF}|f|_{2,P}$ holds for any $f \in H^{2}(P)$ with $0 \in \text{conv} \{f(z_{1}), \dots, f(z_{N_{P}})\}$ and $0 \in \text{conv} \Big\{ \int_{E(1)} \frac{\partial f}{\partial x_{j}} \, ds, \dots, \int_{E(N_{P})} \frac{\partial f}{\partial x_{j}} \, ds \Big\}$ for $j = 1$ and $j = 2$.

Proof. The proof of (a) is included in [16, Lemma 2.1]. The proof of (b) considers the subtriangulation $\mathcal{T}(P)$ of the polygonal domain P from Figure 2.1.b. Define the linear interpolation $I_1 f \in S_1(\mathcal{T}(P)) := \mathcal{P}_1(\mathcal{T}(P)) \cap C^0(P)$ with $(f - I_1 f)(z_k) = 0$ for all $k = 0, \ldots, N_P$. The triangle inequality shows

$$||f||_{L^{2}(P)} \le ||f - I_{1}f||_{L^{2}(P)} + ||I_{1}f||_{L^{2}(P)}. \tag{2.3}$$

The Bramble Hilbert lemma [7, 22] leads to a positive constant $C_{\rm BH}$ (that exclusively depends on the shape of the triangles $T(1), \ldots, T(N_P)$ and so merely on ρ) in the error estimate

$$||f - I_1 f||_{L^2(P)} \le C_{\text{BH}} h_P^2 |f|_{2,P}.$$
 (2.4)

(Explicit formulas for $C_{\rm BH}$ in terms of the maximal angle in a triangle can be found in [14].) Let $\varphi_0, \ldots, \varphi_{N_P}$ be the nodal basis functions of $S_1(\mathcal{T}(P))$ with $\varphi_k(z_\ell) = \delta_{k\ell}$ for $k, \ell = 0, \ldots, N_P$; whence $I_1 f = \sum_{k=0}^{N_P} f(z_k) \varphi_k$. The local mass matrix with entries $\int_T \lambda_j \lambda_k dx = (1 + \delta_{jk}) |T|/12$ for j, k = 1, 2, 3 and the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ in a triangle $T \in \mathcal{T}(P)$ has the eigenvalues |T|/12 (twice) and |T|/3. Rayleigh quotients with the local mass matrix reveal

$$||I_{1}f||_{L^{2}(P)}^{2} = \sum_{T \in \mathcal{T}(P)} ||\sum_{z \in \mathcal{V}(T)} f(z)\varphi_{z}||_{L^{2}(T)}^{2} \leq \sum_{T \in \mathcal{T}(P)} \frac{|T|}{3} \sum_{z \in \mathcal{V}(T)} f(z)^{2}$$

$$\leq \frac{1}{3} \sum_{j=0}^{N_{P}} f(z_{j})^{2} \sum_{\substack{T \in \mathcal{T}(P) \\ z_{j} \in \mathcal{V}(T)}} |T| \leq \frac{|P|}{3} \sum_{k=0}^{N_{P}} f(z_{k})^{2}.$$

$$(2.5)$$

Since $0 \in \text{conv}\{f(z_1), \dots, f(z_{N_P})\}$, there exists convex coefficients $0 \leq \mu_1, \dots, \mu_{N_P} \leq 1$ with $\sum_{\ell=1}^{N_P} \mu_\ell = 1$ and $\sum_{\ell=1}^{N_P} \mu_\ell f(z_\ell) = 0$. This implies $\min_{j=1}^{N_P} f(z_j) \leq 0 \leq \max_{j=1}^{N_P} f(z_j)$ and so $x = \{f(z_1), \dots, f(z_{N_P})\}$ in [15, Lemma 4.2] guarantees

$$\sum_{k=0}^{N_P} f(z_k)^2 \le f(z_0)^2 + M \sum_{j=1}^{N_P} (f(z_j)) - f(z_{j+1})^2$$
(2.6)

for the constant $M:=(2(1-\cos(\pi/N_P)))^{-1}$ (that exclusively depends on $M(\rho)\geq N_P$ and so on ρ). The underlying inequality $\sum_{j=1}^{N_P}x_j^2\leq M\sum_{j=1}^{N_P}(x_{j+1}-x_j)^2$ with $x_{N_P+1}=x_1$ follows for $x_\ell:=\min\{x_1,\ldots,x_{N_P}\}\leq 0\leq \max\{x_1,\ldots,x_{N_P}\}=:x_m$ for some indices $\ell,m\in\{1,\ldots,N_P\}$ immediately from $\max\{|x_1|,\ldots,|x_{N_P}|\}\leq |x_\ell|+|x_m|=|x_\ell-x_m|\leq \sum_{j=1}^{N_P}|x_{j+1}-x_j|\leq N_P^{1/2}(\sum_{j=1}^{N_P}|x_{j+1}-x_j|^2)^{1/2}$ with triangle and Cauchy-Schwarz inequalities. The optimal constants in [15] require little matrix analysis. The above coefficients μ_ℓ and a Cauchy-Schwarz inequality show that

$$f(z_0)^2 = \left(\sum_{\ell=1}^{N_P} \mu_\ell(f(z_0) - f(z_\ell))\right)^2 \le N_P \sum_{\ell=1}^{N_P} (f(z_0) - f(z_\ell))^2.$$
 (2.7)

The combination (2.6)-(2.7) results in

$$\sum_{k=0}^{N_P} f(z_k)^2 \le N_P \sum_{\ell=1}^{N_P} (f(z_0)) - f(z_\ell)^2 + M \sum_{j=1}^{N_P} (f(z_j)) - f(z_{j+1})^2.$$
 (2.8)

For any edge $E = \text{conv}\{a,b\} \in \mathcal{E}(\mathcal{T}(P))$ with vertices $a,b \in \mathcal{V}(E) \subset \mathcal{V}(\mathcal{T}(P))$ and an aligned triangle $T(E) = \text{conv}\{z_0,E\} \supset E$, the tangential derivative shows $|f(a) - f(b)| = \left| \int_E f_{\tau} \, ds \right|$. The trace identity $\int_E f \, ds = \int_{T(E)} f \, dx + \frac{1}{2} \int_{T(E)} (x - z_0) \cdot \nabla f(x) \, dx$ [15, Lemma 2.6] implies that

$$2h_E^{-1}|T(E)| |f(a) - f(b)| \le 2 \left| \int_{T(E)} \nabla f(x) \cdot \tau_E \, dx \right| + \left| \int_{T(E)} (x - z_0) \cdot D^2 f(x) \tau_E \, dx \right|.$$

A Cauchy-Schwarz inequality provides

$$|f(a) - f(b)| \le h_E |T(E)|^{-1/2} (|f|_{1,T(E)} + h_{T(E)}|f|_{2,T(E)}). \tag{2.9}$$

Recall ω_0 from Subsection 2.1 and note that $\sin(\omega_0)h_{T(E)}^2 \leq 4|T(E)|$. This and $h_E \leq h_{T(E)} \leq h_P$ for all $E \in \mathcal{E}(P)$ lead in (2.9) to

$$|f(a) - f(b)|^2 \le \frac{8}{\sin(\omega_0)} (|f|_{1,T(E)}^2 + h_P^2 |f|_{2,T(E)}^2).$$

This applies to the triangles $T(E) = T(j) \in \mathcal{T}(P)$ with the edges $\operatorname{conv}\{a, b\} = \operatorname{conv}\{z_j, z_{j+1}\}$ and $\operatorname{conv}\{z_0, z_j\}$ for $j = 1, \ldots, N_P$ from Figure 2.1.b. Hence (2.8) shows that

$$(N_P + M)^{-1} \frac{\sin \omega_0}{8} \sum_{k=0}^{N_P} f(z_k)^2 \le \sum_{E \in \mathcal{E}(P)} (|f|_{1,T(E)}^2 + h_P^2 |f|_{2,T(E)}^2) = |f|_{1,P}^2 + h_P^2 |f|_{2,P}^2.$$

This and (2.5) result in $||I_1f||_{L^2(P)} \leq C(\rho)(h_P|f|_{1,P} + h_P^2|f|_{2,P})$ with $|P| \leq \pi h_P^2$ from [?, Lemma 1.12], $N_P \leq M(\rho)$ from Subsection 2.1, and $C(\rho)^2 := \frac{8\pi(M(\rho)+M)}{3\sin(\omega_0)}$. The part (a) of the lemma applies to $\partial f/\partial x_j$ for j=1 and for j=2, and so controls the term $|f|_{1,P} \leq C_{\mathrm{PF}}h_P|f|_{2,P}$. Consequently, $||I_1f||_{L^2(P)} \leq C(\rho)(1+C_{\mathrm{PF}})h_P^2|f|_{2,P}$. This and (2.3)-(2.4) show $||f||_{L^2(P)} \leq (C_{\mathrm{BH}} + C(\rho)(1+C_{\mathrm{PF}}))h_P^2|f|_{2,P}$ and conclude the proof of (b) with a re-labelled constant C_{PF} .

Recall Dof: $H^2(P) \to \mathbb{R}^{2N_P}$ for a polygonal domain $P \in \mathcal{M} \in \mathbb{M}$ from (2.1).

Lemma 2.3. Any $v \in H^2(P)$ with Dof(v) = 0 satisfies $h_P^{-2} ||v||_{L^2(P)} + h_P^{-1} |v|_{1,P} \le C_{PF} |v|_{2,P}$.

Proof. If $\mathbf{n} := (\mathbf{n}_x, \mathbf{n}_y)$ is an outward unit normal to an edge E, then the unit tangential vector along E is $\boldsymbol{\tau} := (\boldsymbol{\tau}_x, \boldsymbol{\tau}_y) = (-\mathbf{n}_y, \mathbf{n}_x)$. This leads to the split $\nabla v = v_{\mathbf{n}} \mathbf{n} + v_{\boldsymbol{\tau}} \boldsymbol{\tau}$. An integration along E(k) of the tangential derivative implies for all $k = 1, \ldots, N_P$ that

$$\int_{E(k)} \nabla v \, ds = \left(\int_{E(k)} v_{\mathbf{n}} \, ds \right) \mathbf{n}_{E(k)} + \left(v(z_{k+1}) - v(z_k) \right) \tau_{E(k)}. \tag{2.10}$$

This and the re-summation $\sum_{k=1}^{N_P} (v(z_{k+1}) - v(z_k)) \boldsymbol{\tau}_{E(k)} = \sum_{k=1}^{N_P} (\boldsymbol{\tau}_{E(k-1)} - \boldsymbol{\tau}_{E(k)}) v(z_k)$ lead to

$$\int_{E(k)} \nabla v \, ds = \operatorname{dof}_{N_P + k}(v) \mathbf{n}_{E(k)} + (\tau_{E(k-1)} - \tau_{E(k)}) \operatorname{dof}_k(v). \tag{2.11}$$

If $\operatorname{Dof}(v) = 0$, then $\int_{E(k)} \nabla v \, ds = 0$ for all $k = 1, \dots, N_P$ from (2.11). This and $v(z_j) = \operatorname{dof}_j(v) = 0$ for all $j = 1, \dots, N_P$ allow the Poincaré-Friedrichs inequality in Theorem 2.2.b.

2.3 Biharmonic model problem

Define the scalar product $a(\cdot,\cdot):V\times V\to\mathbb{R}$ for $u,v\in V=H_0^2(\Omega)$ by

$$a(u,v) := \int_{\Omega} D^2 u : D^2 v \, dx \quad \text{with } D^2 u := \begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix}$$
 (2.12)

with $D^2u: D^2v:=u_{11}v_{11}+2u_{12}v_{12}+u_{22}v_{22}$. The subscripts $\alpha, \beta=1,2$ abbreviate the secondorder partial derivatives $u_{\alpha\beta}:=\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta}$. The bilinear form a_{pw} and the differential operator D^2_{pw} denote the corresponding piecewise versions (with respect to a partition \mathcal{M} or \mathcal{T} suppressed in the notation). The local contribution $a^P(\cdot,\cdot)$ is the semi-scalar product

$$a^{P}(u,v) := \int_{P} D^{2}u : D^{2}v \, dx \quad \text{for } u,v \in H^{2}(P).$$

The scalar product $a(\cdot,\cdot)$ induces the energy norm $|v|_{2,\Omega}:=a(v,v)^{1/2}$ equivalent to the Sobolev norm $\|\cdot\|_{2,\Omega}$ owing to the Friedrichs inequality [8, Sec. 10.6] and $(V,a(\cdot,\cdot))$ is a Hilbert space. Given any $F\in V^*=H^{-2}(\Omega)$, the Riesz representation is the weak solution $u\in V$ to

$$a(u, v) = F(v)$$
 for all $v \in V$. (2.13)

Elliptic regularity. For the fixed polygonal bounded Lipschitz domain Ω , there exist positive constants $\sigma_{\text{reg}} > 1/2$ and C_{reg} [3, 5, 19] such that $F \in H^{-s}(\Omega)$ and $2 - \sigma \leq s \leq 2$ for $\sigma := \min\{\sigma_{\text{reg}}, 1\}$ imply $u \in V \cap H^{4-s}$ and

$$||u||_{4-s,\Omega} < C_{\text{reg}}||F||_{-s,\Omega}.$$
 (2.14)

2.4 Galerkin projection

The H^2 elliptic projection operator $G: H^2(P) \to \mathcal{P}_2(P)$ is defined, for any $v \in H^2(P)$, by $Gv \in \mathcal{P}_2(P)$ and

$$a^{P}(Gv, \chi) = a^{P}(v, \chi) \quad \text{for all } \chi \in \mathcal{P}_{2}(P)$$
 (2.15)

with the additional conditions (i.e., three equations to fix the affine contribution)

$$\frac{1}{N_P} \sum_{j=1}^{N_P} Gv(z_j) = \frac{1}{N_P} \sum_{j=1}^{N_P} v(z_j) \quad \text{and} \quad \int_{\partial P} \nabla Gv \, ds = \int_{\partial P} \nabla v \, ds. \tag{2.16}$$

Equation (2.15) determines $Gv \in \mathcal{P}_2(P)$ up to affine functions and the additional three equations in (2.16) define $Gv \in \mathcal{P}_2(P)$ uniquely for $v \in H^2(P)$. The linear operator $G: H^2(P) \to \mathcal{P}_2(P)$ is a projection onto $\mathcal{P}_2(P)$. An integration by parts and (2.16) imply, for all $v \in H^2(P)$, that

$$\Pi_0 D^2 v = \frac{1}{|P|} \int_{\partial P} \nabla v \, ds = \frac{1}{|P|} \int_{\partial P} \nabla G v \, ds = \Pi_0 D^2 G v = D^2 G v. \tag{2.17}$$

Lemma 2.4 (approximation error of G). Any $v \in H^2(P)$ with $Gv \in \mathcal{P}_2(P)$ from (2.15)-(2.16) satisfies $C_{\mathrm{PF}}^{-1} \sum_{m=0}^{1} h_P^{m-2} |v - Gv|_{m,P} \leq |v - Gv|_{2,P} \leq |v|_{2,P}$ and there exists a positive constant C_{apx} (that exclusively depends on ρ) such that $|v - Gv|_{2,P} \leq C_{\mathrm{apx}} h_P^s |v|_{2+s,P}$ for $v \in H^{2+s}(P)$ and $0 < s \leq 1$.

Proof. The condition (2.16) implies that $\sum_{j=1}^{N_P} (v - Gv)(z_j) = 0$ and $\int_{\partial P} \nabla (v - Gv) ds = 0$. Hence the Poincaré-Friedrichs inequality in Theorem 2.2.b proves that $C_{\mathrm{PF}}^{-1} \sum_{m=0}^{1} h_P^{m-2} |v - Gv|_{m,P} \leq |v - Gv|_{2,P}$. The Pythagoras identity $|v - Gv|_{2,P}^2 + |Gv|_{2,P}^2 = |v|_{2,P}^2$ from (2.15) provides $|v - Gv|_{2,P} \leq |v|_{2,P}$. The definition of G in (2.15) shows that $|v - Gv|_{2,P} \leq |v - \chi|_{2,P}$ for any $\chi \in \mathcal{P}_2(P)$. The Bramble-Hilbert lemma [25, Thm. 6.1] concludes the proof.

Recall Dof: $H^2(P) \to \mathbb{R}^{2N_P}$ for a polygonal domain $P \in \mathcal{M} \in \mathbb{M}$ from (2.1).

Lemma 2.5 (boundedness of Dof). There exists a positive constant C_d (that exclusively depends on ρ) such that any $v \in H^2(P)$ satisfies $|\text{Dof}(v - Gv)|_{\ell^2} \leq C_d h_P |v - Gv|_{2,P}$.

Proof. The scaled Sobolev inequality from [7, Sec. 2.1.3] leads for w := v - Gv to

$$|w(z_0)| \le ||w||_{L^{\infty}(P)} \le C_S \sum_{m=0}^{2} h_P^{m-1} |w|_{m,P}$$
 (2.18)

with a positive constant C_S (that exclusively depends on ρ). A Cauchy-Schwarz inequality and the trace inequality $\|w\|_{L^2(E)}^2 \leq C_T(h_E^{-1}\|w\|_{L^2(P)}^2 + h_E\|\nabla w\|_{L^2(P)}^2)$ (e.g., from [7, p. 554]) for any edge $E \in \mathcal{E}(P)$ result in

$$\left| \int_{E} w_{\mathbf{n}} \, ds \right| \le h_{E}^{1/2} \|w_{\mathbf{n}}\|_{L^{2}(\partial P)} \le C_{T}(|w|_{1,P} + h_{E}|w|_{2,P}). \tag{2.19}$$

The combination of (2.18)-(2.19) and Lemma 2.4 imply that

$$|w(z_0)| \le C_S(1 + C_{PF})h_P|w|_{2,P}$$
 and $\left| \int_F w_{\mathbf{n}} ds \right| \le C_T(1 + C_{PF})h_P|w|_{2,P}.$ (2.20)

This concludes the proof of the lemma with $C_d := (C_S + C_T)(1 + C_{PF})$.

The following lemma estimates $|Gv|_{m,P}$ for m=0,1,2 and the Galerkin projection G in terms of dofs of v for any $v \in H^2(P)$.

Lemma 2.6 (G as a function of Dof). The projection operator Gv is computable in terms of the degrees of freedom $Dof(v) \in \mathbb{R}^{2N_P}$ for any $v \in H^2(P)$ and $\sum_{m=0}^2 h_P^{m-1} |Gv|_{m,P} \leq C_g |Dof(v)|_{\ell^2}$ holds with a positive constant C_g (that exclusively depends on ρ).

Proof. Antonietti et al. discuss the proof [2, Lemma 3.3] of the computability of the projection operator G in terms of the dofs from (2.1). Appendix A provides details of this first part and the proof of the estimates of $|Gv|_{m,P}$ for m = 0, 1, 2.

3 Abstract framework and fundamental estimates

3.1 Hypotheses

Given any polygonal domain $P \in \mathcal{M} \in \mathbb{M}$, recall the geometry and the local degrees of freedom from Subsection 2.1 and merely suppose the hypotheses (H1)-(H2) throughout this paper.

(H1) The vector space $V_h(P)$ is of dimension $2N_P$, satisfies $\mathcal{P}_2(P) \subseteq V_h(P) \subset H^2(P)$, and the triplet $(P, V_h(P), (\text{dof}_1, \dots, \text{dof}_{2N_P}))$ is a finite element in the sense of Ciarlet.

The unique existence of a nodal basis $\psi_1, \ldots, \psi_{2N_P}$ of $V_h(P)$ with $\operatorname{dof}_k(\psi_j) = \delta_{jk}$ for all $j, k = 1, \ldots, 2N_P$ is a consequence for any finite element in the sense of Ciarlet [8, Chapter 3].

(**H2**) The aforementioned nodal basis functions $\psi_1, \ldots, \psi_{2N_P}$ satisfy $h_P \left(\sum_{j=1}^{2N_P} |\psi_j|_{2,P}^2 \right)^{1/2} \le C_{\text{stab}}$ for a positive constant C_{stab} (that exclusively depends on ρ).

Notice that (H1)-(H2) also imply the uniform stability of the discrete problem for the natural stabilization term s_h in (5.1) below. The road map of the proofs in the two examples below is outlined in the seminal contribution [21] for a related virtual element space $V_h(P)$. Appendix B and C independently provide details for the two examples below.

3.2 Examples of the discrete space $V_h(P)$

This subsection presents two examples [2, 35] of the local discrete space $V_h(P)$ with (H1)-(H2). Recall that $\mathcal{P}_r(P)$ is the vector space of polynomials of degree $\leq r$ regarded as functions in P, and fix the parameter r = -1, 0, 1, 2 with the convention $\mathcal{P}_{-1}(P) = \{0\}$.

3.2.1 First example of $V_h(P)$ from [2, 21]

The discrete space in [2, Sec. 3] and in [21, Sec. 2.3] solves the biharmonic problem with boundary conditions for $\mathcal{P}_0(\mathcal{E}(P)) := \{ q \in L^{\infty}(\partial P) : \forall E \in \mathcal{E}(P) \quad q|_E \in \mathcal{P}_0(E) \},$

$$\widehat{V}_{h}(P) := \begin{cases} v \in H^{2}(P) : \exists f \in \mathcal{P}_{r}(P) \exists g \in \mathcal{P}_{0}(\mathcal{E}(P)) \exists a_{1}, \dots, a_{N_{P}} \in \mathbb{R} \\ \forall w \in H^{2}(P) \quad a^{P}(v, w) = (f, w)_{L^{2}(P)} + (g, w_{\mathbf{n}})_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} a_{j}w(z_{j}) \end{cases}, \quad (3.1)$$

$$V_h(P) := \{ v \in \widehat{V}_h(P) : v - Gv \perp \mathcal{P}_r(P) \text{ in } L^2(P) \}.$$
(3.2)

Proposition 3.1. The discrete space $V_h(P)$ from (3.1)-(3.2) satisfies (H1)-(H2).

Proof. The arguments in [21, Lemma 3.4-3.5] and in [21, Appendix A] can be adopted for the proof of (H1) and of (H2) for r = -1. Appendix B presents a simpler proof that also covers r = 0, 1, 2.

Second example of $V_h(P)$ generalizes [35]

Recall the set $\mathcal{E}(P) = \{E(1), \dots, E(N_P)\}\$ of edges and the set $\mathcal{V}(P) = \{z_1, \dots, z_{N_P}\}\$ of vertices along the polygon $\partial P = E(1) \cup \cdots \cup E(N_P)$ as in Subsection 2.1. The following generalization of $V_h(P)$ in [35, Sec. 4] requires further notation with corners as depicted in Figure 2.1.c. The boundary of the polygon $\partial P := \bigcup_{j=1}^{J} \operatorname{conv}\{\zeta_j, \zeta_{j+1}\}$ is also a polygon of the corner points $\zeta_1, \ldots, \zeta_J \subset \{z_1, \ldots, z_{N_P}\}$ with indices $\zeta_j = z_{k(j)}, \zeta_{j+1} = z_{k(j)+m(j)}, k(j+1) = k(j)+m(j)$, and k(J+1) := k(1). By definition, the interior angle at a corner ζ_j is different from $0, \pi, 2\pi$, while it is equal to π at all other vertices $z_j \in \mathcal{V}(P) \setminus \{\zeta_1, \ldots, \zeta_J\}$. Given the one-dimensional side $\overline{\gamma(j)} := \operatorname{conv}\{\zeta_j, \zeta_{j+1}\} = E(k(j)) \cup \cdots \cup E(k(j) + m(j)), \text{ consider the } (m(j) + 2)\text{-dimensional}$ quadratic C^1 spline space

$$S(j) := \mathcal{P}_2(\mathcal{E}(\gamma(j))) \cap C^1(\gamma(j))$$
 for $j = 1, \dots, J$.

The vertices $z_{k(j)}, \ldots, z_{k(j)+m(j)}$ on $\gamma(j)$ lead to a partition of $\gamma(j)$, written as $\mathcal{E}(\gamma(j))$, and the subset of functions in S(j) that vanish at all those vertices form a one-dimensional subspace $\operatorname{span}\{\psi_j\}$ of S(j). This is elementary to verify and Appendix C exploits pictures and norms of ψ_j . It turns out that two conditions on the sign $\psi_j|_{E(k(j))} \geq 0$ and on the scaling $\|\psi_j\|_{L^{\infty}(\gamma(j))} = 1$ determine ψ_i uniquely. So $\psi_i \in S(j)$ is fixed by the geometry of P in Figure 2.1.c. The second class of VEs is generalized through a linear functional $\Lambda_j:S(j)\to\mathbb{R}$ with the normalization $\Lambda_j(\psi_j) = 1$ and the boundedness $\|\Lambda_j\| \leq C_{\Lambda}$ of the operator norm $\|\Lambda_j\| := \sup\{\Lambda_j(f) : f \in \Lambda_j(f) : f \in \Lambda_j(f) = 1\}$ $S(j), ||f||_{L^{\infty}(\gamma(j))} = 1$ of Λ_j , provided S(j) is endowed with the maximum norm. We suppose that the upper bound C_{Λ} exclusively depends on ρ . Abbreviate $W:=H_0^1(P)\cap H^2(P)$ and define

$$\widehat{W}_{h}(P) := \left\{ \begin{array}{ll} w \in H^{2}(P) & : w|_{\partial P} \in C^{0}(\partial P), \quad \forall j = 1, \dots, J \quad w|_{\gamma(j)} \in S(j), \text{ and} \\ \exists f \in \mathcal{P}_{r}(P) \quad \exists g \in \mathcal{P}_{0}(\mathcal{E}(P)) \quad \forall \phi \in W \\ a^{P}(w, \phi) = (f, \phi)_{L^{2}(P)} + (g, \phi_{\mathbf{n}})_{L^{2}(\partial P)} \end{array} \right\}, \qquad (3.3)$$

$$W_{h}(P) := \left\{ \begin{array}{ll} w \in \widehat{W}_{h}(P) : \quad \forall j = 1, \dots, J \quad \Lambda_{j}((w - Gw)|_{\gamma(j)}) = 0 \text{ and} \\ w - Gw \perp \mathcal{P}_{r}(P) \quad \text{in } L^{2}(P) \end{array} \right\}. \qquad (3.4)$$

$$W_h(P) := \left\{ \begin{array}{ll} w \in \widehat{W}_h(P) : & \forall j = 1, \dots, J \quad \Lambda_j((w - Gw)|_{\gamma(j)}) = 0 \text{ and} \\ w - Gw \perp \mathcal{P}_r(P) & \text{in } L^2(P) \end{array} \right\}. \tag{3.4}$$

Proposition 3.2. The discrete space $W_h(P)$ from (3.3)-(3.4) satisfies (H1)-(H2).

Proof. The proof of (H1) for Λ_i from Example 3.4 below and for r=-1 is given in [35, Lemma 4.1]. A proof of (**H2**) with clear dependence of the constant C_{stab} on the mesh regularity parameter ρ seems missing in the literature. Appendix C contains further details for the general case and the proof of (H2) with explicit constant.

Example 3.3 (Example of Λ_j). Given the first edge E(k(j)) in $\gamma(j)$, define $\Lambda_j(v) := \frac{3}{2} \oint_{E(k(j))} v \, ds$ for $v \in L^1(\gamma(j))$. Then $\Lambda_j(\psi_j) = 1$ and $\|\Lambda_j\| \leq C_{\Lambda} = 3/2$.

Example 3.4 (Comparison with [35]). Zhao et al. consider merely convex polygons in [35] and we interpret that this implies that the sides are the edges $(\zeta_j = z_j \text{ for all } j = 1, ..., J \text{ and } J = N_P$; all interior angles in P are different from π). Then their choice $\Lambda_j(\bullet) = \int_{E(j)} \bullet ds$ coincides with Example 3.3 and recovers $V_b(P)$ in [35] for certain geometries.

We continue the more general discussion and point out that, for each $P \in \mathcal{M}$, $V_h(P)$ can even be selected from either (3.1)-(3.2) or (3.3)-(3.4) and a mixture is allowed in the abstract framework at hand. In particular, for different polygons $P \in \mathcal{M}$, the space $V_h(P)$ and the parameter r could be different.

3.3 Interpolation

Given $P \in \mathcal{M} \in \mathbb{M}$, recall from **(H1)**-(**H2)** the nodal basis $(\psi_1, \ldots, \psi_{2N_P})$ of $V_h(P)$ with $dof_j(\psi_k) = \delta_{jk}$ for $j, k = 1, \ldots, 2N_P$.

Definition 3.5 (local interpolation operator). Define $I_h^P: H^2(P) \to V_h(P)$ by

$$I_h^P v = \sum_{j=1}^{2N_P} \operatorname{dof}_j(v) \psi_j \quad \text{for all } v \in H^2(P).$$
(3.5)

Recall the notation $V_{\rm nc}$ from Subsection 2.1 for $\mathcal{M} \in \mathbb{M}$.

Definition 3.6 (global discrete space). The (nonconforming) global virtual element space is the collection of all local spaces $V_h(P)$ for $P \in \mathcal{M}$ with well-defined global dofs from (2.2), namely

$$V_h := \{ v_h \in V_{\text{nc}} : \forall P \in \mathcal{M} \quad v_h|_P \in V_h(P) \}.$$

Definition 3.7 (global interpolation operator). Define the global interpolation operator $I_h: V_{\rm nc} \to V_h$ by $(I_h v_{\rm nc})|_P := I_h^P(v_{\rm nc}|_P)$ for all $P \in \mathcal{M}$. (The global interpolation I_h is well-defined because the dofs from (2.1) are uniquely defined for any $v_{\rm nc} \in V_{\rm nc}$.)

3.4 Interpolation error estimates

The main results about \mathcal{I}_h are summarized as follows.

Theorem 3.8 (interpolation). There exists a positive constant C_I (that exclusively depends on ρ) such that any $v \in H^2(P)$ and its interpolation $I_h^P v \in V_h(P)$ from Definition 3.5 satisfy

(a)
$$D_{\mathrm{pw}}^2(v - I_h^P v) \perp \mathcal{P}_0(P; \mathbb{S})$$
 in $L^2(P; \mathbb{S})$,

(b)
$$GI_h^P v = Gv \text{ and } \sum_{m=0}^2 h_P^{m-2} |v - GI_h^P v|_{m,P} \le (1 + C_{PF}) |v - Gv|_{2,P},$$

(c) $|I_h^P v|_{2,P} \le C_{\text{Ib}} |v|_{2,P}$,

(d)
$$\sum_{m=0}^{2} h_{P}^{m-2} |v - I_{h}^{P} v|_{m,P} \le C_{\mathrm{I}} |v - G v|_{2,P} \le C_{\mathrm{I}} C_{\mathrm{apx}} h_{P}^{s} |v|_{2+s,P}$$

provided $v \in H^{2+s}(P)$ for $0 < s \le 1$ in the last estimate in (d) with C_{apx} from Lemma 2.4.

Proof of (a). Since v and $I_h^P v$ coincide at the vertices from Definition 3.5 of I_h^P , their tangential derivatives satisfy

$$\int_{E} (I_h^P v)_{\tau} ds = I_h^P v(z_2) - I_h^P v(z_1) = v(z_2) - v(z_1) = \int_{E} v_{\tau} ds$$
(3.6)

for the vertices z_1, z_2 of an edge E directed from z_1 to $z_2 = z_1 + h_E \tau_E$. An integration by parts proves the first step and the split $\nabla v = v_{\mathbf{n}} \mathbf{n} + v_{\tau} \tau$ proves the second step in

$$\int_{P} D^{2}v \, dx = \sum_{E \in \mathcal{E}(P)} \left(\int_{E} \nabla v \, ds \right) \otimes \mathbf{n}_{P}|_{E} = \sum_{E \in \mathcal{E}(P)} \left(\int_{E} (v_{\mathbf{n}} \mathbf{n} + v_{\tau} \boldsymbol{\tau}) \, ds \right) \otimes \mathbf{n}_{P}|_{E}. \tag{3.7}$$

The combination of (3.6)-(3.7) and $\int_E (I_h^P v)_{\mathbf{n}} ds = \int_E (v)_{\mathbf{n}} ds$ from Definition 3.5 shows

$$\int_{P} D^{2}v \, dx = \sum_{E \in \mathcal{E}(P)} \left(\int_{E} ((I_{h}^{P}v)_{\mathbf{n}}\mathbf{n} + (I_{h}^{P}v)_{\boldsymbol{\tau}}\boldsymbol{\tau}) \, ds \right) \otimes \mathbf{n}_{P}|_{E} = \int_{P} D^{2}I_{h}^{P}v \, dx \tag{3.8}$$

with the split for $\nabla I_h^P v = (I_h^P v)_{\mathbf{n}} \mathbf{n} + (I_h^P v)_{\boldsymbol{\tau}} \boldsymbol{\tau}$ and an integration by parts in the last step. \square

Proof of (b). Since the dofs of $I_h^P v$ and v coincide by Definition 3.5 and G is uniquely determined by Lemma 2.6, $GI_h^P v = Gv$ for $v \in H^2(P)$. Hence Lemma 2.4 concludes the proof.

Proof of (c). Since $I_h^P v - Gv \in V_h(P)$ for any $v \in H^2(P)$, the nodal basis functions $\psi_j \in V_h(P)$ for $j = 1, ..., 2N_P$ and the dofs from (2.1) with $Dof(I_h^P v) = Dof(v)$ lead to

$$I_h^P v - Gv = \sum_{j=1}^{2N_P} \operatorname{dof}_j (I_h^P v - Gv) \psi_j = \sum_{j=1}^{2N_P} \operatorname{dof}_j (v - Gv) \psi_j.$$

The Cauchy-Schwarz inequality $|I_h^P v - G v|_{2,P} \le h_P^{-1} |\text{Dof}(v - G v)|_{\ell^2} h_P \left(\sum_{j=1}^{2N_P} |\psi_j|_{2,P}^2 \right)^{1/2}$, Lemma 2.4-2.5, and **(H2)** result in

$$|I_h^P v - Gv|_{2,P} \le C_d C_{\text{stab}} |v|_{2,P}.$$
 (3.9)

Since $|Gv|_{2,P} \leq |v|_{2,P}$ (from (2.15)), (3.9) and the triangle inequality $|I_h^Pv|_{2,P} \leq |I_h^Pv - Gv|_{2,P} + |Gv|_{2,P}$ conclude the proof with $C_{\text{Ib}} := 1 + C_d C_{\text{stab}}$.

Proof of (d). Substitute $w := v - I_h^P v \in H^2(P)$ for v in (b) and observe $I_h^P w = 0$, so $GI_h^P w = 0$, to derive

$$(1 + C_{PF})^{-1} \sum_{m=0}^{2} h_P^{m-2} |w|_{m,P} \le |w - Gw|_{2,P} \le |v - Gv|_{2,P} + |I_h^P v - GI_h^P v|_{2,P}$$
(3.10)

with a triangle inequality in the last step. Let $g:=v-GI_h^Pv\in H^2(P)$ with $I_h^Pg=I_h^Pv-GI_h^Pv$ from $I_h^P(GI_h^Pv)=GI_h^Pv$. Then (b)-(c) result in $|I_h^Pg|_{2,P}\leq C_{\mathrm{Ib}}|g|_{2,P}\leq C_{\mathrm{Ib}}|v-Gv|_{2,P}$. This and (3.10) conclude the proof of (d) with $C_{\mathrm{I}}:=(1+C_{\mathrm{PF}})(1+C_{\mathrm{Ib}})$.

4 Conforming companion

Recall that \mathcal{T} is a shape-regular sub-triangulation of the polygonal mesh $\mathcal{M} \in \mathbb{M}$ of the domain Ω from Subsection 2.1 with the set of edges $\widehat{\mathcal{E}} \supset \mathcal{E}$.

4.1 Morley interpolation

The Morley finite element space $M(\mathcal{T})$ is defined as

$$M(\mathcal{T}) := \left\{ \begin{aligned} v_M \in \mathcal{P}_2(\mathcal{T}) : & v_M \text{ is continuous at interior vertices and zero at boundary} \\ & \text{vertices of } \mathcal{T}, \text{ and } (v_M)_{\mathbf{n}} \text{ is continuous at midpoints of interior} \\ & \text{edges and zero at midpoints of boundary edges of } \widehat{\mathcal{E}} \end{aligned} \right\}.$$

The Morley interpolation operator $I_M: V_h \to M(\mathcal{T})$ is (uniquely) defined by

$$I_{\mathbf{M}}v_h(z) = v_h(z)$$
 at $z \in \widehat{\mathcal{V}}$ and $\int_E (I_{\mathbf{M}}v_h)_{\mathbf{n}} ds = \int_E (v_h)_{\mathbf{n}} ds$ on $E \in \widehat{\mathcal{E}}$. (4.1)

A well-known consequence of (4.1) [19, Lemma 3.1] (follows as the proof of Theorem 3.8.a) is

$$D_{\mathrm{pw}}^{2}(v_{h} - I_{\mathrm{M}}v_{h}) \perp \mathcal{P}_{0}(\mathcal{T}; \mathbb{S}) \quad \text{in } L^{2}(\Omega; \mathbb{S}). \tag{4.2}$$

The interpolation error estimate from [11, Theorem 3] (for functions in $H^2(T)$) implies

$$\frac{1}{2} \sum_{m=0}^{2} |h_{\mathcal{T}}^{m-2}(v_h - I_{\mathcal{M}}v_h)|_{m,pw} \le |v_h - Gv_h|_{2,pw}.$$
(4.3)

Lemma 4.1 (conforming companion for Morley [27]). There exists a linear map $J': M(\mathcal{T}) \to \mathcal{T}$ $H_0^2(\Omega)$ and a constant $C_{J'}$ (that exclusively depends on ρ) such that any $v_M \in M(\mathcal{T})$ satisfies (a) $J'v_{\mathrm{M}}(z) = v_{\mathrm{M}}(z)$ at all vertices $z \in \widehat{\mathcal{V}}$, (b) $f_{E}(J'v_{\mathrm{M}})_{n} ds = f_{E}(v_{\mathrm{M}})_{n} ds$ on all edges $E \in \widehat{\mathcal{E}}$, (c) $D_{\mathrm{pw}}^{2}(v_{\mathrm{M}} - J'v_{\mathrm{M}}) \perp \mathcal{P}_{0}(\mathcal{T}; \mathbb{S})$ in $L^{2}(\Omega; \mathbb{S})$,

(d)
$$\sum_{m=0}^{2} |h_{\mathcal{T}}^{m-2}(v_{\mathcal{M}} - J'v_{\mathcal{M}})|_{m, pw} \le C_{J'} \min_{v \in V} |v_{\mathcal{M}} - v|_{2, pw}.$$

Computable Morley interpolation

The virtual element function $v_h \in V_h$ is given implicitly such that the computation of $I_M v_h$ is possible in principle, but too costly. The aim in this section is the analysis of a function $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$ that is computable in terms of dofs of $v_h \in V_h$. Given any $v_h \in V_h$ set

$$v_{\mathcal{M}}(z) := \begin{cases} v_{h}(z) \text{ at } z \in \mathcal{V}, \\ Gv_{h}(z) \text{ at } z \in \widehat{\mathcal{V}} \setminus \mathcal{V} \end{cases} \quad \text{and} \quad \int_{E} (v_{\mathcal{M}})_{\mathbf{n}} \, ds := \begin{cases} \int_{E} (v_{h})_{\mathbf{n}} \, ds \text{ on } E \in \mathcal{E}, \\ \int_{E} (Gv_{h})_{\mathbf{n}} \, ds \text{ on } E \in \widehat{\mathcal{E}} \setminus \mathcal{E}. \end{cases}$$
(4.4)

Notice that $v_{\mathrm{M}} \in \mathrm{M}(\mathcal{T})$ is well-defined and computable in terms of the dofs of $v_h \in V_h$ (for $z \in \mathcal{V}, E \in \mathcal{E}$) and from $Gv_h \in \mathcal{P}_2(\mathcal{T})$ with Lemma 2.6 (for $z \in \mathcal{V} \setminus \mathcal{V}, E \in \mathcal{E} \setminus \mathcal{E}$). The following estimates control the difference between $I_{\mathcal{M}}v_h$ and $v_{\mathcal{M}}$ in $\mathcal{M}(\mathcal{T})$ for $v_h \in V_h$.

Lemma 4.2 (key). There exists a positive constant $C_{\rm M}$ (that exclusively depends on ρ) with

$$\sum_{m=0}^{2} |h_{\mathcal{T}}^{m-2}(I_{\mathcal{M}}v_h - v_{\mathcal{M}})|_{m, pw} \le C_{\mathcal{M}}|v_h - Gv_h|_{2, pw} \quad \text{for any } v_h \in V_h \text{ and } v_{\mathcal{M}} \in \mathcal{M}(\mathcal{T}) \text{ with } (4.4).$$

Proof. Recall the sub-triangulation $\mathcal{T}(P)$ of P with mid-point z_0 and the set of edges $\widehat{\mathcal{E}}(P)$ (resp. $\mathcal{E}(P)$ in $\mathcal{T}(P)$ (resp. P) from Subsection 2.1 for a polygonal domain $P \in \mathcal{M}$. Let ψ_z and ψ_E be the nodal basis functions of $M(\mathcal{T})$ for $z \in \widehat{\mathcal{V}}$ and $E \in \widehat{\mathcal{E}}$, and write $I_M v_h - v_M \in M(\mathcal{T})$ as

$$(I_{\mathcal{M}}v_h - v_{\mathcal{M}})|_P = (v_h - Gv_h)(z_0)\psi_{z_0} + \sum_{E \in \widehat{\mathcal{E}}(P) \setminus \mathcal{E}(P)} \left(\oint_E (v_h - Gv_h)_{\mathbf{n}} \, ds \right) \psi_E. \tag{4.5}$$

The basis functions ψ_z and ψ_E are written explicitly, e.g., in [12, Sec. 6] and scale like

$$|\psi_{z_0}|_{m,T} \approx h_T^{1-m}$$
 and $|\psi_E|_{m,T} \approx h_T^{2-m}$ for $m = 0, 1, 2$ (4.6)

in a triangle $T \in \mathcal{T}(P)$ with $E \subset \partial T$. Note that the integral means of normal derivatives of ψ_E over edges enter (4.5) and not the dofs for the ncVEM from (2.1). Lemma 2.5 with integral means along edges $E \in \widehat{\mathcal{E}}(P) \setminus \mathcal{E}(P)$ in (2.20) and $h_P \leq \rho^{-1} h_E$ from (M2) imply that

$$h_P^{-1}|(v_h - Gv_h)(z_0)| + \left| \oint_E (v_h - Gv_h)_{\mathbf{n}} \, ds \right| \lesssim |v_h - Gv_h|_{2,P}.$$
 (4.7)

The combination (4.5)-(4.7) concludes the proof.

Computable companion for VEM 4.3

The new tool in this paper is the conforming companion operator J established in Theorem 4.3. The estimate in part (c) of which is sharp in the sense that the reverse inequality also holds (with a multiplicative constant 2 from a triangle inequality).

Theorem 4.3 (companion operator for VEM). There exist a linear map $J: V_h \to H_0^2(\Omega)$, which is a right-inverse to the interpolation operator I_h , and a universal constant C_J (that exclusively depends on ρ) such that any $v_h \in V_h$ satisfies (a)-(c).

depends on
$$\rho$$
) such that any $v_h \in V_h$ satisfies (a)-(c).
(a) $D^2_{pw}(v_h - Jv_h) \perp \mathcal{P}_0(\mathcal{M}; \mathbb{S})$ in $L^2(\Omega; \mathbb{S})$, (b) $Gv_h - Jv_h \perp \mathcal{P}_2(\mathcal{M})$ in $L^2(\Omega)$,

$$(c) \sum_{m=0}^{2} (|h_{\mathcal{M}}^{m-2}(Gv_h - Jv_h)|_{m,pw} + |h_{\mathcal{M}}^{m-2}(v_h - Jv_h)|_{m,pw}) \le C_J (|v_h - Gv_h|_{2,pw} + \min_{v \in V} |v_h - v|_{2,pw}).$$

Proof. Construction of J. For a given $v_h \in V_h$, a composite function $J'v_M$ belongs to $V = H_0^2(\Omega)$ and we need a modification to achieve (b). For each $T \in \mathcal{T}(P)$ from Subsection 2.1, the cubic bubble-function $b_T := 27\lambda_1\lambda_2\lambda_3 \in H_0^1(T)$ for the barycentric co-ordinates $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{P}_1(T)$ of T, leads to $b_T^2 \in H_0^2(T)$ with $0 \le b_T^2 \le 1$ and $\int_T b_T^2 dx = 81/280$. Hence $\int_P b_P dx = 1$ follows for

$$b_P := \frac{280}{81} \sum_{T \in \mathcal{T}(P)} b_T^2 \in H_0^2(P) \subset H_0^2(\Omega) \quad \text{for } P \in \mathcal{M}.$$
 (4.8)

Let $v_P \in \mathcal{P}_2(P)$ be the Riesz representation of the linear functional $\mathcal{P}_2(P) \to \mathbb{R}$, $w_2 \mapsto (Gv_h - J'v_M, w_2)_{L^2(P)}$ in the Hilbert space $\mathcal{P}_2(P)$ endowed with the weighted scalar product $(b_P \bullet, \bullet)_{L^2(P)}$. In other words, given $v_h \in V_h$, $v_M \in \mathcal{P}_2(\mathcal{M})$ with $v_M|_P := v_P$ satisfies

$$(b_P v_{\mathcal{M}}, w_2)_{L^2(\Omega)} = (G v_h - J' v_{\mathcal{M}}, w_2)_{L^2(\Omega)} \text{ for all } w_2 \in \mathcal{P}_2(\mathcal{M});$$
 (4.9)

whence $\Pi_2(Gv_h - J'v_M) = \Pi_2(b_Pv_P)$. Given $v_h \in V_h$ with $v_M \in \mathcal{P}_2(\mathcal{M})$ define

$$Jv_h := J'v_M + \sum_{P \in \mathcal{M}} b_P v_{\mathcal{M}} \in V. \tag{4.10}$$

Proof of (a). The definitions of J and $b_P \in H_0^2(P)$ imply for any vertex $z \in \mathcal{V}$ and for any edge $E \in \mathcal{E}$ that $Jv_h(z) = J'v_M(z)$ and $f_E(Jv_h)_{\mathbf{n}} ds = f_E(J'v_M)_{\mathbf{n}} ds$. This, Lemma 4.1.a-b as $\mathcal{V} \subset \widehat{\mathcal{V}}$ and $\mathcal{E} \subset \widehat{\mathcal{E}}$, and (4.4) lead to

$$Jv_h(z) = J'v_M(z) = v_M(z) = v_h(z),$$
 (4.11)

$$\oint_{E} (Jv_{h})_{\mathbf{n}} ds = \oint_{E} (J'v_{M})_{\mathbf{n}} ds = \oint_{E} (v_{M})_{\mathbf{n}} ds = \oint_{E} (v_{h})_{\mathbf{n}} ds.$$
 (4.12)

The proof of (a) follows from (4.11)-(4.12) as in the proof of Theorem 3.8.a.

Proof of (b). The definition of
$$J$$
 in (4.10) directly shows $\Pi_2(Gv_h) = \Pi_2(Jv_h)$.

Proof of (c). Abbreviate $\overline{v} := N_P^{-1} \sum_{j=1}^{N_P} v(z_j)$ for any $v \in H^2(P)$. The definition of G in (2.16) and (4.11) imply $\overline{Gv_h - Jv_h} = \overline{Gv_h - v_h} + \overline{v_h - Jv_h} = 0$. The split $\nabla Jv_h := (Jv_h)_{\mathbf{n}} \mathbf{n} + (Jv_h)_{\boldsymbol{\tau}} \boldsymbol{\tau}$ and an integration of the tangential component show for $k = 1, \ldots, N_P$ that

$$\int_{E(k)} \nabla J v_h \, ds = \int_{E(k)} (J v_h)_{\mathbf{n}} \mathbf{n}_{E(k)} \, ds + (J v_h(z_{k+1}) - J v_h(z_k))) \boldsymbol{\tau}_{E(k)}
= \int_{E(k)} (v_h)_{\mathbf{n}} \mathbf{n}_{E(k)} \, ds + (v_h(z_{k+1}) - v_h(z_k))) \boldsymbol{\tau}_{E(k)} = \int_{E(k)} \nabla v_h \, ds \qquad (4.13)$$

with (4.11)-(4.12) in the second step and $\nabla v_h := (v_h)_{\mathbf{n}} \mathbf{n} + (v_h)_{\boldsymbol{\tau}} \boldsymbol{\tau}$ in the last step. This and (2.16) lead to $\int_{\partial P} \nabla (Gv_h - Jv_h) \, ds = \int_{\partial P} \nabla (Gv_h - v_h) \, ds + \int_{\partial P} \nabla (v_h - Jv_h) \, ds = 0$. Hence the Poincaré-Friedrich inequality in Theorem 2.2.b implies

$$\sum_{m=0}^{2} |h_{\mathcal{M}}^{m-2} (Gv_h - Jv_h)|_{m, pw} \le (1 + C_{PF}) |Gv_h - Jv_h|_{2, pw}.$$
(4.14)

It remains to control $|Gv_h - Jv_h|_{2,pw}$. There exists a positive constant C_b in the inverse estimates

$$C_b^{-1} \|\chi\|_{L^2(P)}^2 \le (b_P, \chi^2)_{L^2(P)} \le C_b \|\chi\|_{L^2(P)}^2, \tag{4.15}$$

$$C_b^{-1} \|\chi\|_{L^2(P)} \le \sum_{m=0}^2 h_P^m |b_P \chi|_{m,P} \le C_b \|\chi\|_{L^2(P)} \quad \text{for any } \chi \in \mathcal{P}_2(P).$$
 (4.16)

The inequalities (4.15)-(4.16) are standard inverse estimates for a (shape-regular) triangle (replace P by $T \in \mathcal{T}(P)$ therein) and (4.8) reveals (4.15)-(4.16) as a sum of those. Hence C_b depends exclusively on ρ . The definition of J in (4.10) shows $|J'v_{\rm M} - Jv_h|_{2,P} = |b_P v_P|_{2,P}$ and the inverse inequality (4.16) proves $|b_P v_P|_{2,P} \leq C_b h_P^{-2} ||v_P||_{L^2(P)}$. The first inequality in (4.15) and the definition of $v_P \in \mathcal{P}_2(P)$ in (4.9) lead to

$$C_b^{-1} \|v_P\|_{L^2(P)}^2 \le (b_P v_P, v_P)_{L^2(P)} = (Gv_h - J'v_M, v_P)_{L^2(P)} \le \|Gv_h - J'v_M\|_{L^2(P)} \|v_P\|_{L^2(P)}$$

with a Cauchy-Schwarz inequality in the last step. Hence $||v_P||_{L^2(P)} \le C_b ||Gv_h - J'v_M||_{L^2(P)}$ and the combination with the above estimates verifies

$$|J'v_{\mathcal{M}} - Jv_{h}|_{2,P} = |b_{P}v_{P}|_{2,P} \le C_{h}^{2}h_{P}^{-2}||(Gv_{h} - J'v_{\mathcal{M}})||_{L^{2}(P)}.$$

Triangle inequalities and Lemma 4.1.d lead, for any $v \in V$, to

$$C_b^{-2}|J'v_{\mathcal{M}} - Jv_h|_{2,\mathrm{pw}} \le \|h_{\mathcal{M}}^{-2}(Gv_h - I_{\mathcal{M}}v_h)\|_{L^2(\Omega)} + \|h_{\mathcal{M}}^{-2}(I_{\mathcal{M}}v_h - v_{\mathcal{M}})\|_{L^2(\Omega)} + C_{J'}(|v_{\mathcal{M}} - I_{\mathcal{M}}v_h|_{2,\mathrm{pw}} + |I_{\mathcal{M}}v_h - v_h|_{2,\mathrm{pw}} + |v_h - v|_{2,\mathrm{pw}}).$$
(4.17)

The estimation of the upper bound in (4.17) involves a few arguments like the triangle inequality $\|h_{\mathcal{M}}^{-2}(Gv_h - I_{\mathcal{M}}v_h)\|_{L^2(\Omega)} \leq \|h_{\mathcal{M}}^{-2}(Gv_h - v_h)\|_{L^2(\Omega)} + \|h_{\mathcal{M}}^{-2}(v_h - I_{\mathcal{M}}v_h)\|_{L^2(\Omega)}, \ \|h_{\mathcal{M}}^{-2}(Gv_h - v_h)\|_{L^2(\Omega)} \leq C_{\mathrm{PF}}|v_h - Gv_h|_{2,\mathrm{pw}} \text{ from Lemma 2.4, } \|h_{\mathcal{M}}^{-2}(v_h - I_{\mathcal{M}}v_h)\|_{L^2(\Omega)} + |v_h - I_{\mathcal{M}}v_h|_{2,\mathrm{pw}} \leq 2|v_h - Gv_h|_{2,\mathrm{pw}} \text{ from (4.3), and } |v_{\mathcal{M}} - I_{\mathcal{M}}v_h| \leq C_{\mathcal{M}}|v_h - Gv_h|_{2,\mathrm{pw}} \text{ from Lemma 4.2. This and the fact that } v \in V \text{ is arbitrary eventually result in}$

$$|J'v_{\mathcal{M}} - Jv_{h}|_{2, pw} \le C_{b}^{2} (2 + C_{PF} + C_{\mathcal{M}} + C_{J'}(2 + C_{\mathcal{M}})) \Big(|v_{h} - Gv_{h}|_{2, pw} + \min_{v \in V} |v_{h} - v|_{2, pw} \Big)$$

$$(4.18)$$

as well as $|v_{\rm M}-v_h|_{\rm 2,pw} \leq |v_{\rm M}-I_{\rm M}v_h|_{\rm 2,pw} + |I_{\rm M}v_h-v_h|_{\rm 2,pw} \leq (2+C_{\rm M})|v_h-Gv_h|_{\rm 2,pw}$. The latter estimate, Lemma 4.1.d, and a triangle inequality lead, for any $v \in V$, to

$$C_{J'}^{-1}|v_{\mathcal{M}} - J'v_{\mathcal{M}}|_{2,pw} \le |v_{\mathcal{M}} - v|_{2,pw} \le (2 + C_{\mathcal{M}})|v_{h} - Gv_{h}|_{2,pw} + |v_{h} - v|_{2,pw}. \tag{4.19}$$

Recall $|I_{\rm M}v_h - v_{\rm M}|_{2,\rm pw} \le C_{\rm M}|v_h - Gv_h|_{2,\rm pw}$ from Lemma 4.2 and deduce $|Gv_h - I_{\rm M}v_h|_{2,\rm pw} \le |v_h - Gv_h|_{2,\rm pw}$ from (4.2). The aforementioned estimates and (4.18)-(4.19) allow the estimation of all terms in the upper bound of the triangle inequality

$$|Gv_h - Jv_h|_{2,pw} \le |Gv_h - I_M v_h|_{2,pw} + |I_M v_h - v_M|_{2,pw} + |v_M - J'v_M|_{2,pw} + |J'v_M - Jv_h|_{2,\Omega}$$

$$\le C_{JP} (|v_h - Gv_h|_{2,pw} + \min_{v \in V} |v_h - v|_{2,pw})$$

with $C_{JP} := 1 + C_{\rm M} + C_{J'}(2 + C_{\rm M}) + C_b^2(2 + C_{\rm PF} + C_{\rm M} + C_{J'}(2 + C_{\rm M}))$. This and (4.14) show

$$\sum_{m=0}^{2} |h_{\mathcal{M}}^{m-2} (Gv_h - Jv_h)|_{m,pw} \le (1 + C_{PF}) C_{JP} (|v_h - Gv_h|_{2,pw} + \min_{v \in V} |v_h - v|_{2,pw}). \tag{4.20}$$

Lemma 2.4, (4.20), and triangle inequalities

$$\sum_{m=0}^{2} |h_{\mathcal{M}}^{m-2}(v_h - Jv_h)|_{m,pw} \le \sum_{m=0}^{2} |h_{\mathcal{M}}^{m-2}(v_h - Gv_h)|_{m,pw} + \sum_{m=0}^{2} |h_{\mathcal{M}}^{m-2}(Gv_h - Jv_h)|_{m,pw}$$

conclude the proof of (c) with $C_J := (1 + C_{PF})(1 + C_{JP})$.

Proof of $I_hJ=\operatorname{id}$ in V_h . Recall $N=|\mathcal{V}|+|\mathcal{E}|$ from Subsection 2.1. Definition 3.6 and (H1)-(H2) imply the existence of a (global) nodal basis (ψ_1,\ldots,ψ_N) of V_h with $\operatorname{dof}_j(\psi_k)=\delta_{jk}$ for $j,k=1,\ldots,N$. Definition 3.7 of I_h and $\operatorname{dof}_j(Jv_h)=\operatorname{dof}_j(v_h)$ from (4.11)-(4.12) show that

$$I_h(Jv_h) = \sum_{j=1}^{N} \text{dof}_j(Jv_h)\psi_j = \sum_{j=1}^{N} \text{dof}_j(v_h)\psi_j = v_h.$$

Remark 1 (another companion). The properties (a)-(c) in Theorem 4.3 can also be satisfied by other smoothers, e.g., by $J'I_{\rm M}$. The latter is not immediately computable; whence the new definition of J is adopted in this paper to be used as a smoother Q in Section 5.

Remark 2 (generalizations). Any linear operator $J: V_h \to V$ with (a)-(c) allow the *a priori* and *a posteriori* error estimates in Section 5-6. The analog design of such a conforming companion is possible in 3D with the Morley companion operator J' from [20] in 3D.

5 Discrete problem and a priori error analysis

5.1 Stabilization

The discrete VE functions $v_h \in V_h$ will not be computed explicitly, but Gv_h will. The resulting discrete counterpart $a_h(\cdot,\cdot)$ in V_h of the scalar product $a(\cdot,\cdot)$ in (2.12) requires a stabilization. Recall the semi-scalar product a^P from Subsection 2.3, the number N_P of vertices for $P \in \mathcal{M}$, $dof_1, \ldots, dof_{2N_P}$ from (2.1), and define the semi-scalar product $S^P : V_h(P) \times V_h(P) \to \mathbb{R}$,

$$S^{P}(v_h, w_h) = h_P^{-2} \sum_{j=1}^{2N_P} \operatorname{dof}_{j}(v_h) \operatorname{dof}_{j}(w_h) \quad \text{for } v_h, w_h \in V_h(P).$$
 (5.1)

Lemma 5.1. There exists a positive constant C_s (that exclusively depends on ρ) such that

$$C_s^{-1}a^P(w_h, w_h) \le S^P(w_h, w_h) \le C_s a^P(w_h, w_h)$$
 for all $w_h \in (1 - G)V_h(P)$. (5.2)

Proof. For any $w_h = \sum_{j=1}^{2N_P} \operatorname{dof}_j(w_h) \psi_j \in V_h(P)$ with the nodal basis functions $\psi_1, \dots, \psi_{2N_P}$ of $V_h(P)$ from (H1)-(H2), a Cauchy-Schwarz inequality and (H2) show that

$$|w_h|_{2,P} \le |\mathrm{Dof}(w_h)|_{\ell^2} \Big(\sum_{j=1}^{2N_P} |\psi_j|_{2,P}^2\Big)^{1/2} \le C_{\mathrm{stab}} h_P^{-1} |\mathrm{Dof}(w_h)|_{\ell^2}.$$

The sum over all squared estimates for $P \in \mathcal{M}$ proves the first inequality in (5.2) for $C_s = C_{\text{stab}}^2$. Note that $Gw_h = 0$ for each $w_h \in (1 - G)V_h(P)$ and Lemma 2.4-2.5 result in

$$h_P^{-1}|\mathrm{Dof}(w_h)|_{\ell^2} = h_P^{-1}|\mathrm{Dof}(w_h - Gw_h)|_{\ell^2} \le C_d|w_h - Gw_h|_{2,P} \le C_d|w_h|_{2,P}.$$

The sum over all squared estimates for $P \in \mathcal{M}$ proves the second inequality in (5.2). This concludes the proof with $C_s := \max\{C_{\text{stab}}^2, C_d^2\}$.

Remark 3 (generalization). The *a priori* and *a posteriori* error analysis in Section 5-6 hold for any semi-scalar product $S^P: V_h(P) \times V_h(P) \to \mathbb{R}$ with (5.2).

5.2 Discrete problem

Recall a_{pw} from Subsection 2.3 and $(S^P: P \in \mathcal{M})$ from (5.1). Define the discrete semi-scalar products, for $v_h, w_h \in V_h$, by

$$a_h(v_h, w_h) := \sum_{P \in \mathcal{M}} a_h^P(v_h, w_h) := a_{pw}(Gv_h, Gw_h) + s_h(v_h, w_h), \tag{5.3}$$

$$s_h(v_h, w_h) := \sum_{P \in \mathcal{M}} S^P((1 - G)v_h, (1 - G)w_h).$$
(5.4)

Recall that $|\cdot|_{2,pw}$ defines a norm in V_h (cf. Lemma 2.1), so $a_h(\cdot,\cdot)$ is a scalar product in V_h .

Lemma 5.2 (boundedness and ellipticity of a_h). Any $v_h, w_h \in V_h$ satisfy

$$a_h(v_h, w_h) \le (1 + C_s)|v_h|_{2, pw} |w_h|_{2, pw} \quad and \quad C_s^{-1}|v_h|_{2, pw}^2 \le a_h(v_h, v_h).$$

Proof. The boundedness follows from the definition of a_h and (5.2). The ellipticity follows from (5.2),(2.15), and the Pythagoras identity $|v_h|_{2,P}^2 = |(1-G)v_h|_{2,P}^2 + |Gv_h|_{2,P}^2$.

The Riesz representation theorem guarantees the unique existence of a solution $u_h \in V_h$ to

$$a_h(u_h, v_h) = F_h(v_h) \quad \text{for all } v_h \in V_h \tag{5.5}$$

for the right-hand side $F_h(v_h) := F(Qv_h)$ with Q = G (standard VEM) or Q = J (smoother).

5.3 A priori error estimates

This subsection establishes an error estimate with respect to the norms $|\cdot|_{2,\text{pw}}$ and $|\cdot|_{1,\text{pw}}$. Recall elliptic regularity and $0 < \sigma \le 1$ from (2.14) for the weak solution $u \in H_0^2(\Omega) \cap H^{2+\sigma}(\Omega)$ to (B.15) provided $f \in L^2(\Omega)$ with $F(\cdot) := (f, \cdot)_{L^2(\Omega)}$ in V. Let $u_h \in V_h$ solve (5.5) and recall the maximal mesh-size h_{max} . The second part of the assertion implies (1.1).

Theorem 5.3 (error estimates). There exist positive constants C_1 and C_2 (that exclusively depend on ρ) such that $f \in L^2(\Omega)$ and Q = G or Q = J imply

$$h_{max}^{-\sigma}(|u - u_h|_{1,pw} + |u - Gu_h|_{1,pw}) + |u - u_h|_{2,pw} + |u - Gu_h|_{2,pw}$$

$$\leq C_1(|u - Gu|_{2,pw} + \operatorname{osc}_2(f, \mathcal{M})) \leq C_2 h_{max}^{\sigma} ||f||_{L^2(\Omega)}.$$

For $F \in V^* = H^{-2}(\Omega)$ and solely for Q = J, it holds (even without extra regularity of $u \in V$)

$$h_{max}^{-\sigma}(|u-u_h|_{1,pw}+|u-Gu_h|_{1,pw})+|u-u_h|_{2,pw}+|u-Gu_h|_{2,pw}\leq C_1|u-Gu|_{2,pw}.$$

Proof. Key identity. Let $e_h := I_h u - u_h \in V_h$ and $v_h \in V_h$ with the interpolation I_h from Definition 3.7 and $GI_h u = Gu$ from Theorem 3.8.b. The discrete problem (5.5) and (5.3) imply

$$a_h(e_h, v_h) = a_{DW}(Gu, Gv_h) + s_h(I_hu, v_h) - F(Qv_h) = a_{DW}(Gu, Jv_h) + s_h(I_hu, v_h) - F(Qv_h)$$

with $a_{pw}(Gu, Gv_h) = a_{pw}(Gu, v_h) = a_{pw}(Gu, Jv_h)$ from (2.15) and Theorem 4.3.a in the last step. The continuous problem (B.15) with a test function $v = Jv_h$ reveals

$$a_h(e_h, v_h) = a_{\text{DW}}(Gu - u, Jv_h) + s_h(I_h u, v_h) + F(Jv_h) - F(Qv_h).$$
(5.6)

Estimate of $a_{pw}(Gu-u, Jv_h)$. A triangle inequality, Theorem 4.3.c, and $|v_h-Gv_h|_{2,pw} \le |v_h|_{2,pw}$ (cf. Lemma 2.4) show

$$|Jv_h|_{2,\Omega} \le |v_h - Jv_h|_{2,\mathrm{DW}} + |v_h|_{2,\mathrm{DW}} \le C_J(|v_h - Gv_h|_{2,\mathrm{DW}} + |v_h|_{2,\mathrm{DW}}) + |v_h|_{2,\mathrm{DW}} \le (1 + 2C_J)|v_h|_{2,\mathrm{DW}}.$$

This and a Cauchy-Schwarz inequality provide

$$a_{\text{DW}}(Gu - u, Jv_h) \le (1 + 2C_J)|u - Gu|_{2,\text{DW}}|v_h|_{2,\text{DW}}.$$
 (5.7)

Estimates of $s_h(I_hu, v_h)$. A Cauchy-Schwarz inequality for the semi-scalar product $s_h(\cdot, \cdot)$ and (5.2) provide $C_s^{-1}s_h(I_hu, v_h) \leq |(1-G)I_hu|_{2,pw}|(1-G)v_h|_{2,pw}$. A triangle equality, Lemma 2.4, and Theorem 3.8.b-d in the end result in

$$C_s^{-1} s_h(I_h u, v_h) \le (|u - I_h u|_{2, \text{DW}} + |u - G u|_{2, \text{DW}})|v_h|_{2, \text{DW}} \le (1 + C_I)|u - G u|_{2, \text{DW}}|v_h|_{2, \text{DW}}.$$
 (5.8)

Estimate of $F(Jv_h) - F(Qv_h)$. The term $F(Jv_h) - F(Qv_h)$ vanishes for Q = J, so let Q = G. The orthogonality $Jv_h - Gv_h \perp \mathcal{P}_2(\mathcal{M})$ in $L^2(\Omega)$ from Theorem 4.3.b result in

$$F(Jv_h) - F(Qv_h) = (h_{\mathcal{M}}^2(f - \Pi_2 f), h_{\mathcal{M}}^{-2}(Jv_h - Gv_h))_{L^2(\Omega)} \le 2C_J \operatorname{osc}_2(f, \mathcal{M})|v_h|_{2, pw}.$$
(5.9)

The last step follows from a Cauchy-Schwarz inequality, Theorem 4.3.c, and Lemma 2.4. Estimate of $|u - u_h|_{2,pw}$. The key identity (5.6) with $v_h = e_h$, the coercivity of a_h from Lemma 5.2, and (5.7)-(5.8) lead to $C_3 := 1 + 2C_J + C_s(1 + C_I)$ in

$$|e_h|_{2,\text{DW}} \le C_3(|u - Gu|_{2,\text{DW}} + \text{osc}_2(f, \mathcal{M})).$$
 (5.10)

A triangle inequality for $u - u_h = (u - I_h u) + e_h$, Theorem 3.8.d, and (5.10) show that

$$|u - u_h|_{2,\text{DW}} \le (C_I + C_3)(|u - Gu|_{2,\text{DW}} + \text{osc}_2(f, \mathcal{M})).$$
 (5.11)

Recall that $\operatorname{osc}_2(f, \mathcal{M})$ can be omitted in (5.9)-(5.11) (and below in (5.18)-(5.19)) if Q = J. Duality solution and its regularity. Let $z \in H_0^2(\Omega) \cap H^{2+\sigma}(\Omega)$ be the weak solution to $\Delta^2 z = -\Delta J e_h \in L^2(\Omega)$ and recall $||z||_{2+\sigma,\Omega} \leq C_{\text{reg}}|\Delta J e_h|_{-1,\Omega} \leq C_{\text{reg}}|J e_h|_{1,\Omega}$ from (2.14). The weak formulation of $\Delta^2 z = -\Delta J e_h \in L^2(\Omega)$ leads to $|J e_h|_{1,\Omega}^2 = a(J e_h, z)$.

Reduction to the key term $a_h(e_h, I_h z)$. Elementary algebra reveals (with $GI_h z = Gz$)

$$|Je_h|_{1,\Omega}^2 = a_{\text{pw}}(Je_h - e_h, z) + a_{\text{pw}}(e_h, z - Gz) + a_{\text{pw}}(e_h, Gz).$$
(5.12)

Theorem 4.3.a implies $a_{pw}(Je_h - e_h, z) = a_{pw}(Je_h - e_h, z - Gz)$. Hence a Cauchy-Schwarz inequality and Theorem 4.3.c imply that

$$a_{pw}(Je_h - e_h, z) \le (|e_h - Ge_h|_{2,pw}| + |e_h|_{2,pw})|z - Gz|_{2,pw}$$

$$\le 2C_{apx}C_J h_{max}^{\sigma}|e_h|_{2,pw}|z|_{2+\sigma,\Omega}$$
(5.13)

with Lemma 2.4 in the end. A Cauchy-Schwarz inequality and Lemma 2.4 show

$$a_{\text{pw}}(e_h, z - Gz) \le |e_h|_{2,\text{pw}} |z - Gz|_{2,\text{pw}} \le C_{\text{apx}} h_{\text{max}}^{\sigma} |e_h|_{2,\text{pw}} |z|_{2+\sigma,\Omega}.$$
 (5.14)

Key identity revisited. The definition (5.3) and the identity (5.6) lead, for $v_h = I_h z$, to

$$a_{pw}(e_h, Gz) = a_h(e_h, I_h z) - s_h(e_h, I_h z) = a_{pw}(Gu - u, JI_h z) + F(JI_h z) - F(QI_h z).$$

Estimate of $a_{pw}(Gu-u,JI_hz)$. The definition of G in (2.15) shows $a_{pw}(Gu-u,JI_hz)=a_{pw}(Gu-u,JI_hz-GI_hz)$. Theorem 4.3.c (with $v_h=I_hz$ and v=z) and Theorem 3.8.b imply $|JI_hz-GI_hz|_{2,pw} \leq C_J(|I_hz-Gz|_{2,pw}+|I_hz-z|_{2,pw})$. A triangle inequality reveals

$$C_J^{-1}|JI_hz - GI_hz|_{2,\text{pw}} \le (2|z - I_hz| + |z - Gz|_{2,\text{pw}}) \le (2C_I + C_{\text{apx}})h_{\text{max}}^{\sigma}|z|_{2+\sigma,\Omega}$$
 (5.15)

with Theorem 3.8.d and Lemma 2.4 in the last step. This and a Cauchy-Schwarz inequality prove

$$a_{\text{DW}}(Gu - u, JI_h z) \le C_J(2C_I + C_{\text{adx}})h_{\text{max}}^{\sigma}|u - Gu|_{2,\text{DW}}|z|_{2+\sigma,\Omega}.$$
 (5.16)

Estimate of $F(JI_hz) - F(QI_hz)$. The term $F(JI_hz) - F(QI_hz)$ vanishes for Q = J, so let Q = G. The estimate (5.9) for $v_h = I_hz$ provides $F(JI_hz) - F(QI_hz) \le \operatorname{osc}_2(f, \mathcal{M}) \|h_{\mathcal{M}}^{-2}(JI_hz - GI_hz)\|_{L^2(\Omega)}$. Theorem 4.3.c (with v = z), Theorem 3.8.b, and (5.15) show

$$F(JI_{h}z) - F(QI_{h}z) \le C_{J} \operatorname{osc}_{2}(f, \mathcal{M})(|I_{h}z - Gz|_{2, pw} + |I_{h}z - z|_{2, pw})$$

$$\le C_{J}(2C_{I} + C_{apx})h_{\max}^{\sigma} \operatorname{osc}_{2}(f, \mathcal{M})|z|_{2+\sigma,\Omega}.$$
(5.17)

Estimate of $|u-u_h|_{1,pw}$. The combination of (5.12)-(5.14) and (5.16)-(5.17) imply

$$|Je_h|_{1,\Omega}^2 \le C_4 h_{\max}^{\sigma}(|u - Gu|_{2,\text{DW}} + \text{osc}_2(f,\mathcal{M}))|z|_{2+\sigma,\Omega}$$

with $C_4 := C_{\text{apx}}(1 + 2C_J) + C_J(2C_I + C_{\text{apx}})$. This and the aforementioned regularity show

$$|Je_h|_{1,\Omega} < C_{\text{reg}}C_4 h_{\text{max}}^{\sigma}(|u - Gu|_{2,\text{nw}} + \text{osc}_2(f, \mathcal{M})).$$
 (5.18)

A triangle inequality for $u - u_h = (u - I_h u) + (e_h - Je_h) + Je_h$, Theorem 3.8.d for the first term, Theorem 4.3.c and (5.10) for the second term, and (5.18) for the last term result in

$$|u - u_h|_{1,pw} \le C_{\rm I} h_{\rm max} |u - Gu|_{2,pw} + 2C_J h_{\rm max} |e_h|_{2,pw} + |Je_h|_{1,\Omega}$$

$$\le (C_{\rm I} + 2C_J C_3 + C_4 C_{\rm reg}) h_{\rm max}^{\sigma} (|u - Gu|_{2,pw} + osc_2(f, \mathcal{M})).$$
(5.19)

Estimate of $|u - Gu_h|_{2,pw}$. Recall that $|\cdot|_s := s_h(\cdot,\cdot)^{1/2}$ for $s_h(\cdot,\cdot)$ from Subsection 4.1 defines a seminorm in V_h that is equivalent to $|(1-G)\cdot|_{2,pw}$ owing to (5.2). It follows that

$$|u_h - Gu_h|_{2,P} \le C_s^{1/2} S^P ((1 - G)u_h, (1 - G)u_h)^{1/2}.$$
(5.20)

This, triangle inequalities, and $|(1-G)(u_h-I_hu)|_{2,pw} \le |u_h-I_hu|_{2,pw}$ from Lemma 2.4 with Theorem 3.8.b show

$$|u_h|_s \le |u_h - I_h u|_s + |I_h u|_s \le C_s^{1/2} (|u_h - I_h u|_{2,pw} + |(1 - G)I_h u|_{2,pw})$$

$$\le C_s^{1/2} (|u_h - I_h u|_{2,pw} + |u - I_h u|_{2,pw} + |u - Gu|_{2,pw}).$$
(5.21)

The combination of (5.10)-(5.11) and (5.20)-(5.21) with the triangle inequality $|u - Gu_h|_{2,pw} \le |u - u_h|_{2,pw} + |u_h - Gu_h|_{2,pw}$ results in

$$|u - Gu_h|_{2,pw} \le (C_s + (1 + C_s)(C_I + C_3))(|u - Gu|_{2,pw} + osc_2(f, \mathcal{M})).$$

Estimate of $|u - Gu_h|_{1,pw}$. A triangle inequality and Lemma 2.4 provide

$$|u - Gu_h|_{1,pw} \le |u - u_h|_{1,pw} + C_{PF}h_{max}|u_h - Gu_h|_{2,pw}.$$

This and the combination of (5.19)-(5.21) conclude the proof.

6 A posteriori error analysis

This section establishes a reliable and (up to data oscillations) efficient explicit residual-based a posteriori error estimator for a source term $f \in L^2(\Omega)$ for both cases Q = G and Q = J. Given any polygon $P \in \mathcal{M}$ and the discrete solution $u_h \in V_h$ to (5.5), define the computable terms

$$\begin{split} \eta_P^2 &:= h_P^4 \|f\|_{L^2(P)}^2 & \text{(volume residual)}, \\ \zeta_P^2 &:= S^P((1-G)u_h, (1-G)u_h) & \text{(stabilization)}, \\ \Xi_P^2 &:= \sum_{E \in \mathcal{E}(P)} \left(h_E^{-3} \|[Gu_h]_E\|_{L^2(E)}^2 + h_E^{-1} \|[(Gu_h)_{\mathbf{n}}]_E\|_{L^2(E)}^2 \right) & \text{(nonconformity)}, \\ \mu_P^2 &:= \eta_P^2 + \zeta_P^2 + \Xi_P^2 & \text{(error estimator)}. \end{split}$$

Those local quantities $\bullet|_P$ form a family $(\bullet|_P : P \in \mathcal{M})$ over the index set \mathcal{M} and their Euclid vector norms $\bullet|_{\mathcal{M}}$ enter the upper error bounds

$$\eta_{\mathcal{M}} := (\sum_{P \in \mathcal{M}} \eta_P^2)^{1/2}, \quad \zeta_{\mathcal{M}} := (\sum_{P \in \mathcal{M}} \zeta_P^2)^{1/2}, \quad \Xi_{\mathcal{M}} := (\sum_{P \in \mathcal{M}} \Xi_P^2)^{1/2}, \quad \mu_{\mathcal{M}} := (\sum_{P \in \mathcal{M}} \mu_P^2)^{1/2}.$$

Theorem 6.1 (reliability). There exist positive constants C_{r1} and C_{r2} (that exclusively depend on ρ), such that, for m = 1, 2,

$$C_{\rm rm}^{-2}(|u-u_h|_{m,pw}^2 + |u-Gu_h|_{m,pw}^2) \le \sum_{P \in \mathcal{M}} h_P^{2\sigma(2-m)} \mu_P^2.$$
 (6.1)

The proof of Theorem 6.1 uses an enrichment operator $E_h: \mathcal{P}_2(\mathcal{T}) \to H_0^2(\Omega)$ from [28].

Lemma 6.2. There exists a positive constant C_a (that exclusively depends on ρ) such that any $v_2 \in \mathcal{P}_2(\mathcal{T})$ satisfies

$$|v_2 - E_h v_2|_{2,\text{pw}}^2 \le C_a^2 \sum_{E \in \mathcal{E}} \left(h_E^{-3} \| [v_2]_E \|_{L^2(E)}^2 + h_E^{-1} \| [(v_2)_n]_E \|_{L^2(E)}^2 \right).$$
 (6.2)

Proof. There exists a positive constant C_a (that exclusively depends on ρ) such that any $v_2 \in \mathcal{P}_2(\mathcal{T})$ and its enrichment $E_h v_2 \in H_0^1(\Omega)$ from [28, Lemma 3.1] satisfy

$$|v_2 - E_h v_2|_{2,\mathcal{T}}^2 \le C_a^2 \sum_{E \in \widehat{\mathcal{E}}} \left(h_E^{-3} \| [v_2]_E \|_{L^2(E)}^2 + h_E^{-1} \| [(v_2)_{\mathbf{n}}]_E \|_{L^2(E)}^2 \right).$$

The constant C_a depends on the shape-regularity of the sub-triangulation \mathcal{T} from Subsection 2.1 and so depends on ρ . Since any edge $E \in \mathcal{E}$ is unrefined in the sub-triangulation \mathcal{T} , the above inequality reduces to (6.2) for any $v_{2|P} \in H^2(P)$ and $P \in \mathcal{M}$. This concludes the proof.

The composition $E_h \circ G : V_h \to V$ connects a given function $v_h \in V_h$ to a conforming function,

$$V_h \xrightarrow{G} \mathcal{P}_2(\mathcal{M}) \hookrightarrow \mathcal{P}_2(\mathcal{T}) \xrightarrow{E_h} V$$

Proof of Theorem 6.1 for m=2. Let $e:=u-E_hGu_h\in V=H_0^2(\Omega)$. The scalar product $a(\cdot,\cdot)$ and the continuous problem (B.15) lead to $|u-E_hGu_h|_{2,\Omega}^2=a(u-E_hGu_h,e)=F(e)-a(E_hGu_h,e)$. Recall $a_{\mathrm{pw}}(Gu_h,GI_he)=a_{\mathrm{pw}}(Gu_h,I_he)=a_{\mathrm{pw}}(Gu_h,e)$ from (2.15) and Theorem 3.8.a. This and the discrete problem (5.5) result in

$$|u - E_h G u_h|_{2,\Omega}^2 = F(e) - F(Q I_h e) + a_{pw}(G u_h - E_h G u_h, e) + s_h(u_h, I_h e).$$
(6.3)

Estimate of $F(e) - F(QI_h e)$. Case 1 (Q = G). Theorem 3.8.b and Lemma 2.4 show

$$||e - GI_h e||_{L^2(P)} \le C_{PF} h_P^2 |e - Ge|_{2,P} \le C_{PF} h_P^2 |e|_{2,P}.$$

Case 2 (Q = J). A triangle inequality and Theorem 4.3.c lead to

$$||e - JI_h e||_{L^2(P)} \le ||e - I_h e||_{L^2(P)} + C_J h_P^2 (|I_h e - GI_h e|_{2,P} + |I_h e|_{2,P}).$$

Theorem 3.8.c and Lemma 2.4 show $|I_h e - GI_h e|_{2,P} + |I_h e|_{2,P} \le 2C_{\text{Ib}}|e|_{2,P}$. The previous estimates and Theorem 3.8.d result in $||e - JI_h e||_{L^2(P)} \le (C_{\text{I}} + 2C_J C_{\text{Ib}})h_P^2|e|_{2,P}$. A Cauchy-Schwarz inequality in the right-hand side of $F(e) - F(QI_h e) = (f, e - QI_h e)_{L^2(\Omega)}$ and the above estimates in case Q = G or Q = J provide an estimate for each $P \in \mathcal{M}$. Their sum reads

$$F(e) - F(QI_h e) \le (C_{PF} + C_I + 2C_J C_{Ib}) \eta_{\mathcal{M}} |e|_{2,\Omega}.$$
 (6.4)

Estimate of $a_{pw}(Gu_h - E_hGu_h, e)$. A Cauchy-Schwarz inequality and Lemma 6.2 lead to

$$a_{\text{DW}}(Gu_h - E_h Gu_h, e) \le C_a \Xi_{\mathcal{M}} |e|_{2,\Omega}. \tag{6.5}$$

Estimate of $s_h(u_h, I_h e)$. A Cauchy-Schwarz inequality and (5.2) result in

$$s_h(u_h, I_h e) \le C_s^{1/2} s_h^{1/2}(u_h, u_h) |(1 - G)I_h e|_{2, pw} \le C_s^{1/2} C_{Ib} s_h^{1/2}(u_h, u_h) |e|_{2, \Omega}$$

$$(6.6)$$

with $|(1-G)I_he|_{2,pw} \leq |I_he|_{2,pw} \leq C_{\text{Ib}}|e|_{2,\Omega}$ from Lemma 2.4 and Theorem 3.8.c in the last step. Estimate of $|u-Gu_h|_{2,pw}$. The combination of (6.3)-(6.6) shows $C_5 := C_{\text{Ip}} + C_{\text{I}} + 2C_JC_{\text{Ib}} + C_a + C_s^{1/2}C_{\text{Ib}}$ in $|u-E_hGu_h|_{2,\Omega} \leq C_5\mu_{\mathcal{M}}$. This, Lemma 6.2, and a triangle inequality for the error $u-Gu_h = (u-E_hGu_h) + (E_hGu_h-Gu_h)$ reveal

$$|u - Gu_h|_{2,pw} \le (C_5 + C_a)\mu_{\mathcal{M}}.$$
 (6.7)

Estimate of $|u - u_h|_{2,pw}$. Recall (5.20) in the form $C_s^{1/2}|u_h - Gu_h|_{2,pw} \leq \zeta_{\mathcal{M}}$. This, (6.7), and a triangle inequality lead to

$$|u - u_h|_{2,pw} \le |u - Gu_h|_{2,pw} + |u_h - Gu_h|_{2,pw} \le (C_5 + C_a + C_s^{1/2})\mu_{\mathcal{M}}.$$

This and (6.7) verify (6.1) for
$$m = 2$$
 with $C_{r2} := \sqrt{3}(2(C_5 + C_a) + C_s^{1/2})$.

Proof of Theorem 6.1 for m=1. Let $\Psi \in V=H_0^2(\Omega)$ solve the dual problem $a(v,\Psi)=(\Delta(u-J'I_{\mathbf{M}}u_h),v)_{L^2(\Omega)}$ for all $v\in V$. Elliptic regularity (2.14) provides $\Psi\in H^{2+\sigma}(\Omega)$ and the estimate

$$\|\Psi\|_{2+\sigma,\Omega} < C_{\text{reg}}|u - J'I_{\text{M}}u_h|_{1,\Omega}. \tag{6.8}$$

The test function $v = u - J'I_{\mathbf{M}}u_h \in V$ in the dual problem shows

$$|u - J'I_{\mathcal{M}}u_h|_{1,\Omega}^2 = a(u - J'I_{\mathcal{M}}u_h, \Psi) = F(\Psi) - F(QI_h\Psi) + a_h(u_h, I_h\Psi) - a(J'I_{\mathcal{M}}u_h, \Psi)$$
 (6.9)

with the continuous problem (B.15) and the discrete problem (5.5) in the last step. Theorem 3.8.b provides $GI_h\Psi=G\Psi$ and (2.15) shows $a_{\mathrm{pw}}(Gu_h,GI_h\Psi)=a_{\mathrm{pw}}(Gu_h,\Psi)$. Notice that $a_{\mathrm{pw}}(Gu_h-J'I_{\mathrm{M}}u_h,G\Psi)=a_{\mathrm{pw}}(Gu_h-u_h,G\Psi)+a_{\mathrm{pw}}(u_h-I_{\mathrm{M}}u_h,G\Psi)+a_{\mathrm{pw}}(I_{\mathrm{M}}u_h-J'I_{\mathrm{M}}u_h,G\Psi)=0$ follows from (2.15), (4.2), and Lemma 4.1.c. This and (6.9) result in

$$|u - J'I_{\mathcal{M}}u_h|_{1,\Omega}^2 = F(\Psi) - F(QI_h\Psi) + a_{pw}(Gu_h - J'I_{\mathcal{M}}u_h, \Psi - G\Psi) + s_h(u_h, I_h\Psi).$$
 (6.10)

Estimate of $F(\Psi) - F(QI_h\Psi)$. Case1 (Q = G). Theorem 3.8.b and Lemma 2.4 show that

$$\|\Psi - GI_h\Psi\|_{L^2(P)} \le C_{PF}C_{apx}h_P^{2+\sigma}|\Psi|_{2+\sigma,P}$$

Case 2 (Q = J). A triangle inequality and Theorem 4.3.c (with $v = \Psi$) reveal that

$$\|\Psi - JI_h\Psi\|_{L^2(P)} \le \|\Psi - I_h\Psi\|_{L^2(P)} + C_J h_P^2 (|I_h\Psi - GI_h\Psi|_{2,P} + |I_h\Psi - \Psi|_{2,P})$$

$$\le (C_{\rm I} + C_J (2C_{\rm I} + C_{\rm apx})) h_P^{2+\sigma} |\Psi|_{2+\sigma,P}$$

with Theorem 3.8.d and (5.15) in the last step. The two estimates for $\|\Psi - QI_h\Psi\|_{L^2(P)}$ (for Q = G and Q = J) and the Cauchy-Schwarz inequality $\int_P f(\Psi - QI_h\Psi) dx \leq \|f\|_{L^2(P)} \|\Psi - QI_h\Psi\|_{L^2(P)}$ prove an estimate for each $P \in \mathcal{M}$. The sum of all those provides

$$F(\Psi) - F(QI_h\Psi) \le (C_{PF}C_{apx} + C_I + C_J(2C_I + C_{apx}))|\Psi|_{2+\sigma,\Omega} \sum_{P \in \mathcal{M}} h_P^{\sigma} \eta_P.$$
 (6.11)

Estimate of $a_{pw}(Gu_h - J'I_Mu_h, \Psi - G\Psi)$. A triangle inequality and Lemma 4.1.d (with $v = E_hGu_h$) imply $|Gu_h - J'I_Mu_h|_{2,pw} \le |Gu_h - I_Mu_h|_{2,pw} + C_{J'}|I_Mu_h - E_hGu_h|_{2,pw}$. This and $|Gu_h - I_Mu_h|_{2,pw} \le |u_h - Gu_h|_{2,pw}$ from (4.2) lead to

$$|Gu_h - J'I_M u_h|_{2,pw} \le (1 + C_{J'})|u_h - Gu_h|_{2,pw} + C_{J'}|Gu_h - E_h Gu_h|_{2,pw}$$

$$\le (1 + C_{J'})C_s^{1/2}\zeta_{\mathcal{M}} + C_{J'}C_a\Xi_{\mathcal{M}}$$
(6.12)

with (5.20) and Lemma 6.2 in the last step. A Cauchy-Schwarz inequality, (6.12), and Lemma 2.4 prove with $C_6 := C_{\text{apx}}((1 + C_{J'})C_s^{1/2} + C_{J'}C_a)$ that

$$a_{\mathrm{pw}}(Gu_h - J'I_{\mathrm{M}}u_h, \Psi - G\Psi) \le C_6|\Psi|_{2+\sigma,\Omega} \sum_{P \in \mathcal{M}} h_P^{\sigma}(\zeta_P + \Xi_P). \tag{6.13}$$

Estimate of $s_h(u_h, I_h\Psi)$. Argue as in (6.6) for the stability term and proceed with $|(1 - G)I_h\Psi|_{2,pw} \leq |I_h\Psi - G\Psi|_{2,pw}$ from Theorem 3.8.b to deduce that

$$s_h(u_h, I_h \Psi) \le C_s^{1/2} \zeta_{\mathcal{M}} |I_h \Psi - G\Psi|_{2, \text{pw}} \le C_s^{1/2} (C_I + C_{\text{apx}}) |\Psi|_{2+\sigma, \Omega} \sum_{P \in \mathcal{M}} h_P^{\sigma} \zeta_P$$
 (6.14)

with a triangle inequality, Theorem 3.8.d, and Lemma 2.4 in the last step. Estimate of $|u - Gu_h|_{1,pw}$. The combination of (6.10)-(6.11) and (6.13)-(6.14) provides

$$|u - J' I_{\mathcal{M}} u_h|_{1,\Omega}^2 \le C_7 |\Psi|_{2+\sigma,\Omega} \sum_{P \in \mathcal{M}} h_P^{\sigma} \mu_P$$
 (6.15)

with $C_7 := C_{\rm PF}C_{\rm apx} + C_{\rm I} + C_J(2C_{\rm I} + C_{\rm apx}) + C_6 + C_s^{1/2}(C_{\rm I} + C_{\rm apx})$. Argue as in (4.13) to prove $\int_{\partial P} \nabla (J'I_{\rm M}u_h - u_h) \, ds = 0$ so that the Poincaré-Friedrichs inequality from Theorem 2.2.a applies. This and the triangle inequality $|u - Gu_h|_{1,pw} \leq |u - J'I_{\rm M}u_h|_{1,\Omega} + |J'I_{\rm M}u_h - Gu_h|_{1,pw}$ result in

$$|u - Gu_h|_{1,pw} \le |u - J'I_Mu_h|_{1,\Omega} + C_{PF}|h_{\mathcal{M}}(J'I_Mu_h - Gu_h)|_{2,pw}.$$

The regularity estimate (6.8), (6.15), and (6.12) lead in the previous displayed estimate to

$$|u - Gu_h|_{1,pw} \le C_{reg}(C_7 + C_{PF}((1 + C_{J'})C_s^{1/2} + C_{J'}C_a)) \sum_{P \in \mathcal{M}} h_P^{\sigma} \mu_P.$$

Estimate of $|u - u_h|_{1,pw}$. The triangle inequality $|u - u_h|_{1,pw} \le |u - Gu_h|_{1,pw} + |Gu_h - u_h|_{1,pw}$, Lemma 2.4, and (5.20) conclude the proof of (6.1) for m = 1 with $C_{r1} := \sqrt{3(2C_{reg}(C_7 + C_{PF}((1 + C_{J'})C_s^{1/2} + C_{J'}C_a) + C_s^{1/2})}$.

Let $z \in \mathcal{V}$ be a vertex in \mathcal{M} with the neighbouring polygons $\mathcal{M}(z) := \{P' \in \mathcal{M} : z \in P'\}$ and define the vertex patch $\omega_z := \operatorname{int}(\cup \mathcal{M}(z))$ and the larger neighbourhood $\Omega(P) := \cup_{z \in \mathcal{V}(P)} \omega_z$. The edge patch $\mathcal{M}(E) := \{P' \in \mathcal{M} : E \subset \partial P'\}$ consists of one or two neighbouring polygons that share an edge $E \in \mathcal{E}$ and this defines $\omega(E) := \operatorname{int}(\cup \mathcal{M}(E))$.

Theorem 6.3 (local efficiency up to data oscillations). For any $P \in \mathcal{M}$ it holds

$$\zeta_P^2 \lesssim |u - u_h|_{2,P}^2 + |u - Gu_h|_{2,P}^2,$$
(6.16)

$$\eta_P^2 \lesssim |u - Gu_h|_{2,P}^2 + \text{osc}_2^2(f, P),$$
(6.17)

$$\Xi_P^2 \lesssim \sum_{E \in \mathcal{E}(P)} \sum_{P' \in \Omega(\omega(E))} (|u - u_h|_{2,P'}^2 + |u - Gu_h|_{2,P'}^2). \tag{6.18}$$

Proof of (6.16). The upper bound in (5.2) and a triangle inequality lead to

$$\zeta_P^2 \le C_s |(1-G)u_h|_{2P}^2 \le 2C_s (|u-u_h|_{2P}^2 + |u-Gu_h|_{2P}^2).$$

Proof of (6.17). Abbreviate $\theta_P := (f - \Pi_2 f)|_P$, recall the bubble-function b_P from (4.8), and substitute $v = b_P \Pi_2 f \in V$ in (B.15) to obtain

$$(\Pi_2 f, v)_{L^2(P)} = a^P(u, v) - (\theta_P, v)_{L^2(P)} = a^P(u - Gu_h, v) - (\theta_P, v)_{L^2(P)}$$

with $a^P(Gu_h, v) = a^P(Gu_h, Gv) = 0$ from (2.15) and Gv = 0 (from Lemma 2.6 for Dof(v) = 0) in the last step. This and Cauchy-Schwarz inequalities show

$$(\Pi_2 f, v)_{L^2(P)} \le |u - Gu_h|_{2,P} |v|_{2,P} + \|\theta_P\|_{L^2(P)} \|v\|_{L^2(P)}. \tag{6.19}$$

The first inequality in (4.15) shows $C_b^{-1} \|\Pi_2 f\|_{L^2(P)}^2 \leq (\Pi_2 f, v)_{L^2(P)}$ and the second inequality in (4.16) verifies $\sum_{m=0}^2 h_P^m |v|_{2,P} \leq C_b \|\Pi_2 f\|_{L^2(P)}$. Those estimates prove in (6.19) that

$$C_b^{-2} h_P^2 \|\Pi_2 f\|_{L^2(P)} \le |u - Gu_h|_{2,P} + \operatorname{osc}_2(f, P).$$

This and the triangle inequality $\|\eta_P^2\|_{L^2(P)} \le \|h_P^2(f - \Pi_2 f)\|_{L^2(P)} + \|h_P^2 \Pi_2 f\|_{L^2(P)}$ conclude the proof of (6.17).

Proof of (6.18). Since $[Gu_h]_E = [Gu_h - J'I_Mu_h]_E$, the trace inequality leads to

$$C_T^{-1} \| [Gu_h]_E \|_{L^2(E)} \le h_E^{-1/2} \| Gu_h - J'I_M u_h \|_{L^2(\omega(E))} + h_E^{1/2} \| \nabla_{\text{pw}} (Gu_h - J'I_M u_h) \|_{L^2(\omega(E))}.$$

Rewrite $Gu_h - J'I_Mu_h = (Gu_h - u_h) + (u_h - I_Mu_h) + (I_Mu_h - J'I_Mu_h)$. Abbreviate $|\cdot|_{2,\mathcal{T}(P')} := \sum_{T \in \mathcal{T}(P')} |\cdot|_{2,T}$ for $P' \in \mathcal{M}$. Lemma 2.4, (4.3), and Lemma 4.1.d for $P' \in \mathcal{M}(E)$ lead to

$$C_T^{-1} h_E^{-3/2} || [Gu_h]_E ||_{L^2(E)} \le (2 + C_{PF}) \sum_{P' \in \mathcal{M}(E)} (|u_h - Gu_h|_{2,P'} + |(1 - J')I_M u_h|_{2,\mathcal{T}(P')}). \quad (6.20)$$

There exists a local version of Lemma 4.1.d established in [20, Lemma 5.1] with a positive constant C_8 (that exclusively depends on the shape regularity of \mathcal{T}) such that

$$|(1-J')I_{\mathcal{M}}u_h|_{2,\mathcal{T}(P')} \leq C_8 \min_{v \in V} \|D_{\mathrm{pw}}^2(I_{\mathcal{M}}u_h - v)\|_{L^2(\Omega(P'))} \leq C_8 \|D_{\mathrm{pw}}^2(I_{\mathcal{M}}u_h - u)\|_{L^2(\Omega(P'))}.$$

A triangle inequality shows

$$|(1-J')I_{\mathcal{M}}u_h|_{2,\mathcal{T}(P')} \le C_8(\|D^2_{\mathrm{pw}}(I_{\mathcal{M}}u_h - u_h)\|_{L^2(\Omega(P'))} + \|D^2_{\mathrm{pw}}(u_h - u)\|_{L^2(\Omega(P'))}).$$

This estimate for each $P' \in \mathcal{M}(E)$, $|u_h - I_M u_h|_{2,pw} \le |u_h - G u_h|_{2,pw}$ from (4.2), (6.20), and a triangle inequality imply

$$h_E^{-3/2} \| [Gu_h]_E \|_{L^2(E)} \le C_T (2 + C_{PF}) (1 + C_8) \sum_{P' \in \Omega(\mathcal{M}(E))} (|u - u_h|_{2,P'} + |u - Gu_h|_{2,P'}).$$
(6.21)

It remains to control the term $h_E^{-1/2} \| [(Gu_h)_{\mathbf{n}}]_E \|_{L^2(E)}$ for each $E \in \mathcal{E}(P)$. Since $u - u_h \in V_{\mathrm{nc}}$, $\alpha_E := \int_E (u - u_h)_{\mathbf{n}} ds \in \mathbb{R}$ is uniquely defined. Rewrite $[(Gu_h)_{\mathbf{n}}]_E = [(Gu_h - u)_{\mathbf{n}} + \alpha_E]_E$ for $E \in \mathcal{E}(P)$. The triangle inequality $\| [(Gu_h - u)_{\mathbf{n}} + \alpha_E]_E \|_{L^2(E)} \le \| [(Gu_h - u_h)_{\mathbf{n}}]_E \|_{L^2(E)} + \| [(u_h - u)_{\mathbf{n}} + \alpha_E]_E \|_{L^2(E)}$ and the trace inequality lead to

$$h_E^{-1/2} \| [(Gu_h)_{\mathbf{n}}]_E \|_{L^2(E)} \lesssim (h_E^{-1} \| \nabla_{pw} (Gu_h - u_h) \|_{L^2(\omega(E))} + \| D_{pw}^2 (Gu_h - u_h) \|_{L^2(\omega(E))})$$

$$+ (h_E^{-1} \| (u_h - u)_{\mathbf{n}} + \alpha_E \|_{L^2(\omega(E))} + \| D_{pw}^2 (u - u_h) \|_{L^2(\omega(E))}). \quad (6.22)$$

Since $\int_E ((u_h - u)_{\mathbf{n}} + \alpha_E) ds = 0$, the Poincaré-Friedrichs inequality in Theorem 2.2.a applies to $\mathbf{n}_E \cdot \nabla (u_h - u) + \alpha_E = f$ in each $P' \in \mathcal{M}(E)$ and asserts $\|(u_h - u)_{\mathbf{n}} + \alpha_E\|_{L^2(\omega(E))} \le C_{\mathrm{PF}} h_P \|D^2_{\mathrm{pw}}(u_h - u)\|_{L^2(\omega(E))}$. This, $h_E \ge \rho h_{P'}$ from (M2), and Lemma 2.4 show

$$h_E^{-1/2} \| [(Gu_h)_{\mathbf{n}}]_E \|_{L^2(E)} \lesssim \| D_{\mathrm{pw}}^2 (Gu_h - u_h) \|_{L^2(\omega(E))} + \| D_{\mathrm{pw}}^2 (u - u_h) \|_{L^2(\omega(E))}.$$

This and a triangle inequality result in

$$h_E^{-1/2} \| [(Gu_h)_{\mathbf{n}}]_E \|_{L^2(E)} \lesssim \sum_{P' \in \mathcal{M}(E)} (|u - u_h|_{2,P'} + |u - Gu_h|_{2,P'}).$$

Remark 4 (efficiency of H^1 error control). The upper bounds in (6.16)-(6.18) with a multiplication factor $h_P^{2\sigma}$ for $0 < \sigma \le 1$ in front of μ_P^2 show that the error estimators in (6.1) for m = 1 converge (at least) with the expected convergence rate of the piecewise H^1 error.

Remark 5 (higher-order ncVEM and 3D). The upper bound (6.1) for the error $|u-u_h|_{2,pw}$ can be generalized to ncVEM of higher order $r \geq 3$ (see [2] for the discrete setting). The enrichment operator E_h can be defined from the piecewise polynomial space $\mathcal{P}_r(\mathcal{T})$ to $H_0^2(\Omega)$ [28] and the arguments in this section hold in the three-dimensional case as well.

Remark 6 (conforming VEM). There are papers on the *a priori* error estimates for the conforming VEM, but there is no work on the *a posteriori* VE analysis for the biharmonic problem in the current literature. The analysis in Section 6 applies to the conforming case with $E_h \circ G = 1$ and $J' \circ I_M = 1$ in the proof of (6.1) for m = 2 and m = 1. This establishes the reliable and efficient *a posteriori* error estimator $\eta_M + \zeta_M$ for the conforming VEM.

Remark 7 (extensions). The source term F is assumed to be an L^2 function in the main parts of this paper for simplicity and brevity. A class of more general sources $F \in H^{-2}(\Omega)$ is discussed in Theorem 5.3 only for the smoother Q = J in the discrete problem and then avoids the data oscillations. More examples on a class of right-hand sides F are discussed in [18] and, in particular, the *a posteriori* error estimates can be generalized for this class of source terms as well. The arguments of [18] apply here as well and further details are omitted for brevity.

7 Numerical results

This section discusses two numerical experiments with uniform and adaptive mesh-refinement.

7.1 Adaptive algorithm

A standard adaptive algorithm with the loop Solve \rightarrow Estimate \rightarrow Mark \rightarrow Refine from [16, Sec. 6.1] is performed in two computational benchmarks.

Step 1 (SOLVE). Find the solution u_h to (5.5) for Q = G in the right-hand side and compute the errors H1e and H2e, $Hme := |u - Gu_h|_{m,pw}$ for m = 1, 2, using polygauss quadrature rule [31] for the input parameter n = 10.

Step 2 (ESTIMATE). Compute the local residuals in Theorem 6.1 and collect all these contributions for $P \in \mathcal{M}$ to obtain the upper bound $H1\mu$ and $H2\mu$, $Hm\mu^2 := \sum_{P \in \mathcal{M}} h_P^{2\sigma(2-m)} \mu_P^2$ for the piecewise H^m error for m = 1, 2. Abbreviate the number of degrees of freedom by ndof.

Step 3 (MARK). The Dörfler marking strategy [33] determines $\mathcal{D}_m \subset \mathcal{M}$ for m = 1, 2 with

$$Hm\mu^2 \leq 0.5 \sum_{P \in \mathcal{D}_m} h_P^{2\sigma(2-m)} \mu_P^2.$$

Step 4 (REFINE). A refinement strategy in VEM divides the marked polygonal domains by connecting the mid-points of the edges to the centroid and allow at most one hanging node per edge; cf. [34] for further details on a MATLAB implementation.

7.2 Numerical example in L-shaped domain

This subsection considers an L-shaped domain of Figure 7.1 with the exact solution of the model problem in polar co-ordinates (r, θ)

$$u(r,\theta) = r^{5/3} \sin\left(\frac{5\theta}{3}\right)$$
 in $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$.

In this example, both u and u_n are not zero along the boundary $\partial\Omega$ and f=0. The upper bound for inhomogeneous boundary data can be established with minor modifications: The term Ξ_P in the error estimator for $P \in \mathcal{M}$, which share a boundary edge, changes to

$$\Xi_{P}^{2} = \sum_{E \in \mathcal{E}(P) \cap \mathcal{E}(\Omega)} \left(\frac{1}{h_{E}^{3}} \| [Gu_{h}]_{E} \|_{L^{2}(E)}^{2} + \frac{1}{h_{E}} \| [(Gu_{h})_{\mathbf{n}}]_{E} \|_{L^{2}(E)}^{2} \right)$$

$$+ \sum_{E \in \mathcal{E}(P) \cap \mathcal{E}(\partial \Omega)} \left(\frac{1}{h_{E}^{3}} \| [Gu_{h} - u]_{E} \|_{L^{2}(E)}^{2} + \frac{1}{h_{E}} \| [(Gu_{h} - u)_{\mathbf{n}}]_{E} \|_{L^{2}(E)}^{2} \right).$$

Figure 7.1 displays strong local mesh-refinement at the re-entry corner in the adaptive mesh-refining. Figure 7.2 shows that uniform refinement yields the sub-optimal convergence rate, whereas adaptive refinements recover the optimal convergence rate.

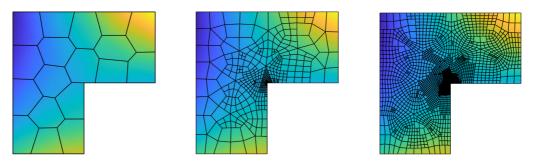


Figure 7.1: Output $\mathcal{M}_1, \mathcal{M}_{10}, \mathcal{M}_{15}$ of the adaptive algorithm in Subsection 7.2.

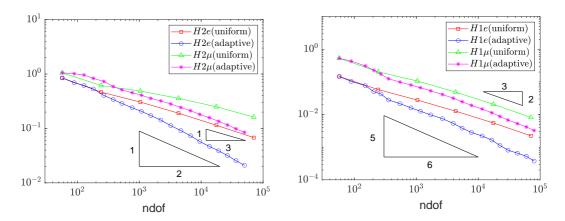


Figure 7.2: Convergence history plot of the errors resp. error estimators H2e resp. $H2\mu$ (left) and H1e resp. $H1\mu$ (right) vs ndof for the L-shaped domain in Subsection 7.2.

7.3 Numerical example in Z-shaped domain

The subsection considers the polygonal domain Ω with the vertices (0,0),(1,0),(1,1),(-1,1),(-1,-1),(1,-1) of Figure 7.3. Define the right-hand side function f in the polar co-ordinates (r,θ) with the exact solution

$$u(r,\theta) = (1 - r^2 \cos^2(\theta))^2 (1 - r^2 \sin^2(\theta))^2 r^{(1+z)} g(\theta).$$

Here z = 0.505009698896589 is a noncharacteristic root of $\sin^2(z\omega) = z^2 \sin^2(\omega)$, $\omega = 7\pi/4$ and $g(\theta)$ is as given in [29, p. 107]. Figure 7.3 and 7.4 display the numerical results.

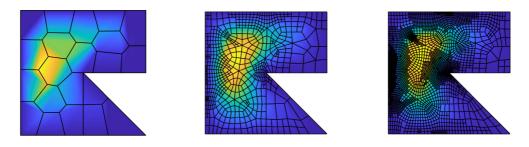


Figure 7.3: Output $\mathcal{M}_1, \mathcal{M}_{10}, \mathcal{M}_{15}$ of the adaptive algorithm in Subsection 7.3.

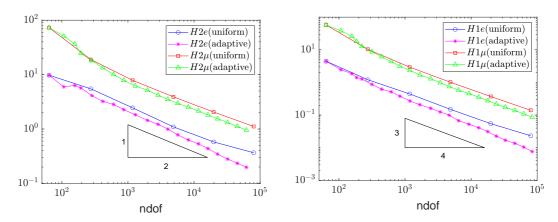


Figure 7.4: Convergence history plot of the errors resp. error estimators H2e resp. $H2\mu$ (left) and H1e resp. $H1\mu$ (right) vs ndof for the Z-shaped domain in Subsection 7.3.

7.4 Evaluation

Empirical convergence rates. The two domains from Subsection 7.2 resp. 7.3 have weak solutions $u \in H^{2+\sigma-\epsilon}(\Omega)$ for any $\epsilon > 0$ with the typical corner singularity for $\sigma = 2/3$ resp. $\sigma = z = 0.505$ and hence we expect and observe the empirical convergence rates in the H^2 norm (resp. H^1 norm) $\sigma/2$ (resp. σ) in terms of ndof^{-1/2} for uniform mesh-refinements. The adaptive mesh-refining improves the empirical convergence rates to the optimal values 1/2 (resp. $(1+\sigma)/2$).

Efficiency indices. The ratio of the error estimator and the total error (effectivity index) in the piecewise H^2 and H^1 norm remains bounded: $3 \le H2\mu/H2e \le 5$ and $5.5 \le H1\mu/H1e \le 7$ in both examples. This confirms empirically that the error estimator mimics the behaviour of the total error and also validates Theorem 6.1.

Dominant error contributions. Figure 7.5 displays the individual components $\eta_{\ell}, \zeta_{\ell}, \Xi_{\ell}^{1}, \Xi_{\ell}^{2}$ in the error estimator, which abbreviate $\eta_{\mathcal{M}_{\ell}}, \zeta_{\mathcal{M}_{\ell}}, \Xi_{\mathcal{M}_{\ell}}^{1}, \Xi_{\mathcal{M}_{\ell}}^{2}$ and shows the dominance of Ξ_{ℓ}^{1} . The remaining part Ξ_{ℓ}^{2} of $\Xi_{\ell} := \Xi_{\ell}^{1} + \Xi_{\ell}^{2}$ and the volume residual η_{ℓ} converge more rapidly. The error estimator components for the H^{1} error in the adaptive refinement behave similar.

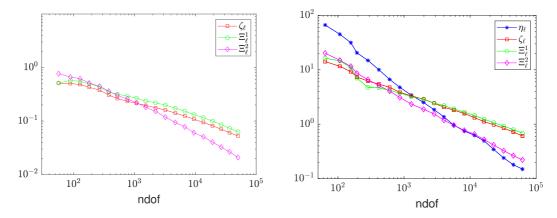


Figure 7.5: Convergence history plot of the error estimator components corresponding to H2e for adaptive mesh-refinement vs ndof in Subsection 7.2 (left) and 7.3 (right).

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A Proof of Lemma 2.6

The integral $a^P(v,w) = \int_P D^2v : D^2w \, dx$ for $v \in H^4(P)$ and $w \in H^2(P)$ allows an integration by parts formula [2, Sec. 2.1] with the boundary terms $\mathcal{M}_{\mathbf{nn}}(v) := v_{\mathbf{nn}}, \ T(v) := (\Delta v)_{\mathbf{n}} + v_{\mathbf{n}\tau\tau}, \ \mathcal{M}_{\mathbf{n}\tau}(v) := v_{\mathbf{n}\tau}$ from [2] and the abbreviations $[M_{\mathbf{n}\tau}(v)]_{z_j} := M_{\mathbf{n}\tau}(v)|_{E(j-1)}(z_j) - M_{\mathbf{n}\tau}(v)|_{E(j)}(z_j)$ with $E(0) := E(N_P)$ for a cyclic notation along ∂P . Those boundary terms are well-defined as traces of a smooth function $v \in H^4(P)$ and then the formula reads

$$a^{P}(v,w) = (\Delta^{2}v, w)_{L^{2}(P)} + (M_{\mathbf{nn}}(v), w_{\mathbf{n}})_{L^{2}(\partial P)} - (T(v), w)_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} [M_{\mathbf{n}\tau}(v)]_{z_{j}} w(z_{j}).$$
(A.1)

The boundary terms require a smooth function v like the quadratic polynomial $\chi \in \mathcal{P}_2(P)$ with $\Delta^2 \chi = 0 = T(\chi)$ and with piecewise constants $M_{\mathbf{nn}}(\chi)|_{E(k)} = \chi_{\mathbf{nn}}|_{E(k)}$ and $M_{\mathbf{n\tau}}(\chi)|_{E(k)} = \chi_{\mathbf{n\tau}}|_{E(k)}$ for $k = 1, \ldots, N_P$, which are computable in terms of $D^2 \chi \in \mathbb{S}$ and of the geometry of P from Figure 2.1.a. This and the definition of G from (2.15) lead, for $\chi \in \mathcal{P}_2(P)$ and $w \in H^2(P)$ in (A.1), to

$$a^{P}(Gw,\chi) = a^{P}(\chi,w) = (M_{\mathbf{nn}}(\chi), w_{\mathbf{n}})_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} [M_{\mathbf{n\tau}}(\chi)]_{z_{j}} w(z_{j}). \tag{A.2}$$

The dofs of w from (2.1) allow for a re-writing of the right-hand side of (A.2), namely

$$a^{P}(Gw,\chi) = \sum_{k=1}^{N_{P}} \chi_{\mathbf{n}\mathbf{n}}|_{E(k)} \operatorname{dof}_{N_{P}+k}(w) + \sum_{j=1}^{N_{P}} (\chi_{\mathbf{n}\boldsymbol{\tau}}|_{E(j-1)} - \chi_{\mathbf{n}\boldsymbol{\tau}}|_{E(j)}) \operatorname{dof}_{j}(w). \tag{A.3}$$

This defines $Gw \in \mathcal{P}_2(P)$ up to an affine contribution fixed in (2.16). The first condition in (2.16) reads

$$N_P^{-1} \sum_{j=1}^{N_P} Gw(z_j) = N_P^{-1} \sum_{j=1}^{N_P} w(z_j) = N_P^{-1} \sum_{j=1}^{N_P} \operatorname{dof}_j(w).$$
 (A.4)

For the second condition in (2.16), the identity $\int_{E(k)} \nabla w \, ds = \operatorname{dof}_{N_P+k}(w) \mathbf{n}_{E(k)} + (\boldsymbol{\tau}_{E(k-1)} - \boldsymbol{\tau}_{E(k)}) \operatorname{dof}_k(w)$ for $k = 1, \ldots, N_P$ from (2.11) shows

$$\int_{\partial P} \nabla w \, ds = \sum_{k=1}^{N_P} \left(\int_{E(k)} \nabla w \, ds \right) = \sum_{k=1}^{N_P} (\operatorname{dof}_{N_P + k}(w) \mathbf{n}_{E(k)} + (\boldsymbol{\tau}_{E(k-1)} - \boldsymbol{\tau}_{E(k)}) \operatorname{dof}_k(w)). \quad (A.5)$$

The equations (A.3)-(A.5) form a linear system of 6 equations for $Gw \in \mathcal{P}_2(P)$ and the right-hand sides in (A.3)-(A.5) are computable in terms of $\operatorname{dof}_1(w), \ldots, \operatorname{dof}_{2N_P}(w)$ for any $w \in H^2(P)$. Hence its solution Gw is computable in terms of the dofs of w. It is elementary to see that Gw is uniquely determined and the 6×6 coefficient matrix in the resulting linear system of equations is regular. (Another proof for this follows from the estimates in the second part of the proof below.) This proves the first part of the lemma.

The second part part of the proof estimates $|Gw|_{m,P}$ for m=0,1,2 in terms of $Dof(w)=(dof_1(w),\ldots,dof_{2N_P}(w))$. The definition of G with $\chi=Gw\in\mathcal{P}_2(P)$ in (2.15) and (A.3) imply

$$|Gw|_{2,P}^2 = a^P(w,Gw) \le \sum_{k=1}^{N_P} |\chi_{\mathbf{nn}}|_{E(k)} ||\operatorname{dof}_{N_P+k}(w)| + \sum_{j=1}^{N_P} (|\chi_{\mathbf{n\tau}}|_{E(j-1)} - \chi_{\mathbf{n\tau}}|_{E(j)} |) |\operatorname{dof}_j(w)|.$$

Cauchy-Schwarz inequalities and $|\chi_{\mathbf{nn}}|_{E(k)}|, |\chi_{\mathbf{n\tau}}|_{E(k)}| \leq |D^2\chi|$ reveal

$$|Gw|_{2,P}^2 \le 2|D^2\chi|\sum_{i=1}^{2N_P}|\mathrm{dof}_j(w)| \le \sqrt{8N_P}|P|^{-1/2}|Gw|_{2,P}|\mathrm{Dof}(w)|_{\ell^2}.$$

The bounds $|P|^{-1/2} \le \pi^{-1/2} C_{\rm sr}^{-2} h_P^{-1}$ (cf. [4, Chap. 1]) with a shape regularity constant $C_{\rm sr}$ of $\mathcal{T}(P)$ (that exclusively depends on ρ) and $N_P \le M(\rho)$ from Subsection 2.1 show

$$|Gw|_{2,P} \le \sqrt{8\pi^{-1}C_{\rm sr}^{-4}M(\rho)}h_P^{-1}|{\rm Dof}(w)|_{\ell^2}.$$
 (A.6)

Define $a:=N_P^{-1}\sum_{j=1}^{N_P}w(z_j)\in\mathbb{R},\,B:=|\partial P|^{-1}\int_{\partial P}\nabla w\,ds\in\mathbb{R}^2$, and the affine function $g(x):=a+B(x-N_P^{-1}\sum_{j=1}^{N_P}z_j)$ to control the lower-order terms $\sum_{m=0}^1h_P^{m-1}|Gw|_{m,P}$. The definition (2.16) of G provides

$$\sum_{j=1}^{N_P} (Gw - g)(z_j) = \left(\sum_{j=1}^{N_P} w(z_j)\right) - aN_P = 0, \quad \int_{\partial P} \nabla (Gw - g) \, ds = \int_{\partial P} \nabla w \, ds - |\partial P| B = 0.$$

The Poincaré-Friedrichs inequality from Theorem 2.2.b, therefore, shows for Gw - g that

$$h_P^{-2} \| Gw - g \|_{L^2(P)} + h_P^{-1} | Gw - g |_{1,P} \le C_{PF} | Gw |_{2,P}.$$
 (A.7)

The Cauchy-Schwarz inequality implies $|a| \leq N_P^{-1/2}|\mathrm{Dof}(w)|_{\ell^2}$. Since $|\mathbf{n}_{E(k)}| = 1 = |\boldsymbol{\tau}_{E(k)}|$, (A.5) shows that $|\int_{\partial P} \nabla w \, ds| \leq 2 \sum_{j=1}^{2N_P} |\mathrm{dof}_j(w)|$. A Cauchy-Schwarz inequality and $|\partial P|^{-1} \leq N_P^{-1} \rho^{-1} h_P^{-1}$ from (M2) imply that $|B| \leq 2\sqrt{2}N_P^{-1/2} \rho^{-1} h_P^{-1}|\mathrm{Dof}(w)|_{\ell^2}$. The definition of g and the previous two estimates for |a| and |B| result in

$$||g||_{L^2(P)} \le |P|^{1/2} (|a| + h_P|B|) \le (\frac{\pi}{N_P})^{1/2} (1 + 2\sqrt{2}\rho^{-1}) h_P |\operatorname{Dof}(w)|_{\ell^2}$$
 (A.8)

with the coarse bound $|P| \leq \pi h_P^2$ in the last step. This and the inverse estimate for $|g|_{1,P}$ (collected from an inverse estimate in the triangle $T \in \mathcal{T}(P)$ and so with a well-established bound for C_{inv}) show

$$|g|_{1,P} \le C_{\text{inv}} h_P^{-1} ||g||_{L^2(P)} \le \left(\frac{\pi}{N_P}\right)^{1/2} (1 + 2\sqrt{2}\rho^{-1}) C_{\text{inv}} |\text{Dof}(w)|_{\ell^2}.$$
 (A.9)

Triangle inequalities $\sum_{m=0}^{2} h_P^{m-2} |Gw|_{m,P} \le \sum_{m=0}^{2} h_P^{m-2} (|Gw-g|_{m,P} + |g|_{m,P})$, (A.7)-(A.9), and the abbreviation $C_9 := 1 + C_{\text{PF}} + (1 + C_{\text{inv}}) \pi^{1/2} (1 + 2\sqrt{2}\rho^{-1})$ prove

$$\sum_{m=0}^{2} h_{P}^{m-1} |Gw|_{m,P} \le C_9(h_P |Gw|_{2,P} + |\text{Dof}(w)|_{\ell^2}) \le C_g |\text{Dof}(w)|_{\ell^2}$$

with (A.6) and $C_g := C_9(1 + \sqrt{8\pi^{-1}C_{\rm sr}^{-4}M(\rho)})$ in the last step. This concludes the proof. (Notice that, in particular, Dof(w) = 0 implies Gw = 0 and this proves that the linear system of equations (A.3)-(A.5) involves a regular coefficient matrix as announced.)

B Proof of Proposition 3.1

Step 1 defines an HCT finite element space. Recall the sub-triangulation $\mathcal{T}(P)$ from Subsection 2.1 and decompose any triangle $T \in \mathcal{T}(P)$ further into three sub-triangles $\mathcal{K}(T)$ depicted in Figure B.1.c. Then the Hsieh-Clough-Tocher (HCT) finite element space [22] reads

$$HCT(\mathcal{T}(P)) := \{ \hat{v} \in H^2(P) : \forall T \in \mathcal{T} \quad \hat{v}|_T \in \mathcal{P}_3(\mathcal{K}(T)) \}.$$

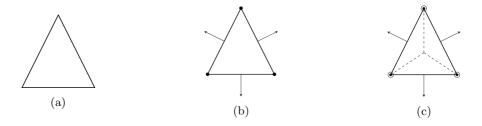


Figure B.1: (a) Triangle T, (b) Morley, (c) HCT.

The standard degrees of freedom (dofs) in the HCT finite element (cf. [22, Chap. 6] or [27, Sec. 2.3]) are the nodal values of the function and its first-order derivatives at each vertex and the mid-point values of the normal derivatives along each edge of a triangle as depicted in Figure B.1.c. This paper utilizes the integral means instead of the mid-point values of normal derivatives along edges and all other dofs (i.e., nodal values) are unchanged. Let $\psi_1^{\text{HCT}}, \ldots, \psi_{2N_P}^{\text{HCT}}$ be the $2N_P$ nodal basis functions in $\text{HCT}(\mathcal{T}(P))$ with $\nabla \psi_j^{\text{HCT}}(z_\ell) = 0$ for all $j = 1, \ldots, 2N_P$ and, for all $k, \ell = 1, \ldots, N_P$,

$$\psi_k^{\text{HCT}}(z_\ell) = \delta_{k\ell}, \ f_{E(\ell)}(\psi_k^{\text{HCT}})_{\mathbf{n}} \ ds = 0, \ \text{and} \ \psi_{k+N_P}^{\text{HCT}}(z_\ell) = 0, \ f_{E(\ell)}(\psi_{k+N_P}^{\text{HCT}})_{\mathbf{n}} \ ds = \delta_{k\ell}.$$
 (B.1)

In contrast to this, let $\widetilde{\psi}_{\ell+N_P}^{\rm HCT}\in {\rm HCT}(\mathcal{T}(P))$ be a nodal basis function in the standard HCT finite element with $\widetilde{\psi}_{\ell+N_P}^{\rm HCT}({\rm mid}(E(\ell)))=1$ for $\ell=1,\ldots,N_P$, while all other dofs vanish. Then $4\int_0^1 s(1-s)\,ds=2/3$ leads to $\psi_{\ell+N_P}^{\rm HCT}=\frac{3}{2}\widetilde{\psi}_{\ell+N_P}^{\rm HCT}$ for $\int_{E(\ell)}(\psi_{\ell+N_P}^{\rm HCT})_{\bf n}\,ds=1$ and $\ell=1,\ldots,N_P$. This observation, the scaling of the standard HCT basis functions from [27, Prop. 2.5], and the bound $h_T^{-1}\leq \rho^{-1}h_P^{-1}$ from (M2) for all $T\in\mathcal{T}(P)$ provide a positive constant $C_{\rm HCT}$ (that exclusively depends on ρ) in

$$\max_{k=1}^{N_P} h_P |\psi_k^{\text{HCT}}|_{2,P} + \max_{\ell=1}^{N_P} |\psi_{\ell+N_P}^{\text{HCT}}|_{2,P} \le C_{\text{HCT}}.$$
 (B.2)

Step 2 constructs an HCT interpolation. Recall the nodal basis functions $\psi_1^{\text{HCT}}, \dots, \psi_{2N_P}^{\text{HCT}}$ of HCT($\mathcal{T}(P)$) selected in Step 1. The HCT interpolation $w_{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ of a given $w \in H^2(P)$ reads

$$w_{\text{HCT}} := \sum_{k=1}^{N_P} w(z_k) \psi_k^{\text{HCT}} + \sum_{\ell=1}^{N_P} \left(\oint_{E(\ell)} w_{\mathbf{n}} \, ds \right) \psi_{\ell+N_P}^{\text{HCT}}. \tag{B.3}$$

The duality relations (B.1) imply for $m = 1, ..., N_P$ in (B.3) that $w_{\text{HCT}}(z_m) = w(z_m)$ and $\int_{E(m)} (w_{\text{HCT}})_{\mathbf{n}} ds = \int_{E(m)} w_{\mathbf{n}} ds$. In other words, $\text{Dof}(w_{\text{HCT}}) = \text{Dof}(w)$ for the vector Dof with components from (2.1).

Step 3 defines the Hilbert space (V_0, a^P) . The kernel of the linear map $\text{Dof}: H^2(P) \to \mathbb{R}^{2N_P}$ is the closed subspace

$$V_0 := \left\{ v \in H^2(P) : v(z_j) = 0 = \int_{E(j)} v_{\mathbf{n}} \, ds \quad \text{for } j = 1, \dots, N_P \right\}$$

of the Hilbert space $H^2(P)$ and so complete. Hence (V_0, a^P) is a Hilbert space. Notice that $w - w_{\text{HCT}} \in V_0$ for any $w \in H^2(P)$ from Step 2 and Gw = 0 for $w \in V_0$ (from Lemma 2.6 with Dof(w) = 0 as explained in Appendix A).

Step 4 proves that $V_0^{\perp} := \{v \in H^2(P) : a^P(v, w) = 0 \text{ for all } w \in V_0\} \subset \widehat{V}_h(P)$. Given any $v \in V_0^{\perp}$ and $w \in H^2(P)$, define the HCT interpolation w_{HCT} of w from Step 2. Then $w - w_{\text{HCT}} \in V_0$ from Step 3 implies that

$$a^{P}(v,w) = a^{P}(v,w_{\text{HCT}}) = \sum_{k=1}^{N_{P}} a^{P}(v,\psi_{k}^{\text{HCT}})w(z_{k}) + \sum_{\ell=1}^{N_{P}} a^{P}(v,\psi_{\ell+N_{P}}^{\text{HCT}})h_{E(\ell)}^{-1} \int_{E(\ell)} w_{\mathbf{n}} ds. \quad (B.4)$$

Define $f := 0 \in \mathcal{P}_{-1}(P)$, $a_{\ell} := a^{P}(v, \psi_{\ell}^{\text{HCT}})$, and $g \in \mathcal{P}_{0}(\mathcal{E}(P))$ by $g|_{E(\ell)} := a^{P}(v, \psi_{\ell+N_{P}}^{\text{HCT}})h_{E(\ell)}^{-1}$ for $\ell = 1, \ldots, N_{P}$ to rewrite the right-hand side in (B.4) as

$$a^{P}(v, w) = (g, w_{\mathbf{n}})_{L^{2}(\partial P)} + \sum_{\ell=1}^{N_{P}} a_{\ell} w(z_{\ell})$$
 for all $w \in H^{2}(P)$.

This implies $v \in \widehat{V}_h(P)$ for r = -1 and concludes the proof of Step 4.

Step 5 proves that Dof: $\widehat{V}_h(P) \to \mathbb{R}^{2N_P}$ is surjective. Given any $x = (x_1, \dots, x_{2N_P}) \in \mathbb{R}^{2N_P}$ and the functions $\psi_1^{\text{HCT}}, \dots, \psi_{2N_P}^{\text{HCT}}$ selected in Step 1, define $u_{BC} := \sum_{k=1}^{2N_P} x_k \psi_k^{\text{HCT}} \in H^2(P)$ with $\text{Dof}(u_{BC}) = x$ from Step 2. Let $u_0 \in V_0$ denote the Riesz representation of $a^P(u_{BC}, \cdot)$ in (V_0, a^P) , i.e., $a^P(u_0, \cdot) = a^P(u_{BC}, \cdot)$ in V_0 . Since $a^P(u_0 - u_{BC}, \cdot) = 0$ in V_0 , Step 4 shows that $\widehat{u}_h := u_{BC} - u_0 \in \widehat{V}_h(P)$. Recall $\text{Dof}(u_{BC}) = x$ and $\text{Dof}(u_0) = 0$ to deduce $\text{Dof}(\widehat{u}_h) = x$.

Step 6 establishes an inclusion in $\widehat{V}_h(P)$ with a non-zero f. Given $f \in \mathcal{P}_r(P)$, the Riesz-representation theorem guarantees the unique existence of the weak solution $u(f) \in V_0$ to

$$a^{P}(u(f), v) = (f, v)_{L^{2}(P)}$$
 for all $v \in V_{0}$. (B.5)

Recall that r is a fixed parameter in $\{-1,0,1,2\}$ and r=-1 is trivial in this step. Since (B.5) implies $\Delta^2 u(f) = f$ in P, it remains to prove that $u(f) \in \widehat{V}_h(P)$. Given any $w \in H^2(P)$ with w_{HCT} from (B.3), $w - w_{\text{HCT}} \in V_0$ leads in (B.5) to $a^P(u(f), w - w_{\text{HCT}}) = (f, w - w_{\text{HCT}})_{L^2(P)}$. Hence

$$a^{P}(u(f), w) = (f, w - w_{HCT})_{L^{2}(P)} + a^{P}(u(f), w_{HCT}) = (f, w)_{L^{2}(P)} + \Lambda(w_{HCT})$$
(B.6)

with the linear functional $\Lambda(v) := a^P(u(f), v) - (f, v)_{L^2(P)}$ for any $v \in H^2(P)$. The representation (B.3) of w_{HCT} shows

$$a^{P}(u(f), w) = (f, w)_{L^{2}(P)} + \sum_{k=1}^{N_{P}} w(z_{k}) \Lambda(\psi_{k}^{\text{HCT}}) + \sum_{\ell=1}^{N_{P}} \left(\oint_{E(\ell)} w_{\mathbf{n}} \, ds \right) \Lambda(\psi_{\ell+N_{P}}^{\text{HCT}})$$

$$= (f, w)_{L^{2}(P)} + (g, w_{\mathbf{n}})_{L^{2}(\partial P)} + \sum_{k=1}^{N_{P}} a_{k} w(z_{k}) \quad \text{for all } w \in H^{2}(P)$$
(B.7)

with the definition of $a_{\ell} := \Lambda(\psi_{\ell}^{\text{HCT}})$ and $g \in \mathcal{P}_0(\mathcal{E}(P))$ by $g|_{E(\ell)} := |E(\ell)|^{-1}\Lambda(\psi_{\ell+N_P}^{\text{HCT}})$ for $\ell = 1, \ldots, N_P$. This implies that $u(f) \in \widehat{V}_h(P)$. Notice that u(f) depends linearly on $f \in \mathcal{P}_r(P)$ and so $\mathcal{P}_r(P) \to \widehat{V}_h(P)$, $f \mapsto u(f)$ defines a linear map.

Step 7 proves that $\mathcal{L}: \mathcal{P}_r(P) \to \mathcal{P}_r(P), f \mapsto \Pi_r u(f)$ is an isomorphism. For any $f \in \mathcal{P}_r(P)$ with $\mathcal{L}f = 0$, the orthogonality $(1 - \Pi_r)u(f) \perp \mathcal{P}_r(P)$ in $L^2(P)$ shows $0 = \int_P (\mathcal{L}f)f \, dx = \int_P u(f)f \, dx$. This and v = u(f) in (B.5) result in $0 = a^P(u(f), u(f)) = |u(f)|_{2,P}$. Consequently $u(f) \in \mathcal{P}_1(P) \cap V_0$ and so u(f) = 0. Thus $f = \Delta^2 u(f) = 0$ and \mathcal{L} is injective; whence bijective. \square

Step 8 proves that Dof: $V_h(P) \to \mathbb{R}^{2N_P}$ is an isomorphism.

Proof of surjectivity. Given any $x \in \mathbb{R}^{2N_P}$, there exists $\widehat{u}_h \in \widehat{V}_h(P)$ with $\operatorname{Dof}(\widehat{u}_h) = x$ from Step 4. This leads to $g := \Pi_r G \widehat{u}_h - \Pi_r \widehat{u}_h \in \mathcal{P}_r(P)$. Since \mathcal{L} is bijective in $\mathcal{P}_r(P)$ (from Step 7), there exists $f \in \mathcal{P}_r(P)$ with $\Pi_r u(f) = g$. Recall that $u(f) \in V_0$ implies $\operatorname{Dof}(u(f)) = 0$ and Gu(f) = 0. Altogether, $u_h^P := u(f) + \widehat{u}_h \in \widehat{V}_h(P)$ satisfies $\operatorname{Dof}(u_h^P) = x$ and $\Pi_r G u_h^P = \Pi_r G \widehat{u}_h = g + \Pi_r \widehat{u}_h = \Pi_r u(f) + \Pi_r \widehat{u}_h = \Pi_r u_h^P$. Hence $u_h^P \in V_h(P)$.

Proof of injectivity. Suppose $v_h \in V_h(P)$ satisfies $Dof(v_h) = 0$. Recall $v_h \in V_h(P) \cap V_0$ and $Gv_h = 0$ from Step 3. The definition (3.1) (for $v = v_h$ with Dof(v) = 0) leads for some $f \in \mathcal{P}_r(P)$ such that $a^P(v_h, v_h) = (f, v_h)_{L^2(P)}$. This and (3.2) reveal

$$|v_h|_{2,P}^2 = a^P(v_h, v_h) = (f, v_h)_{L^2(P)} = (f, \Pi_r v_h)_{L^2(P)} = (f, \Pi_r G v_h)_{L^2(P)} = 0.$$

Hence $v_h \in V_0 \cap \mathcal{P}_1(P)$ and so $v_h = 0$.

Proof of (H1). The key observation from Step 8 is that $V_h(P)$ has the dimension $2N_P$ and $dof_1, \ldots, dof_{2N_P}$ from (2.1) are linear independent. Consequently $(P, V_h(P), (dof_1, \ldots, dof_{2N_P}))$ is a finite element in the sense of Ciarlet.

Step 9 provides the scaling of an HCT interpolation. Let $\psi_h \equiv \psi_p$ be a nodal basis function of the finite element $(P, V_h(P), \text{Dof})$ for some $p \in \{1, \dots, 2N_P\}$ and let ψ_{HCT} be its HCT interpolation from Step 2, namely

$$\psi_{\text{HCT}} = \sum_{k=1}^{N_P} \psi_h(z_k) \psi_k^{\text{HCT}} + \sum_{\ell=1}^{N_P} \left(\oint_{E(\ell)} (\psi_h)_{\mathbf{n}} \, ds \right) \psi_{\ell+N_P}^{\text{HCT}}. \tag{B.8}$$

The definition of a nodal basis function ψ_h shows that $\psi_h(z_k)$ and $\int_{E(\ell)} (\psi_h)_{\mathbf{n}} ds$ are zero or one for $k, \ell = 1, \ldots, N_P$. The scaling of $\psi_1^{\text{HCT}}, \ldots, \psi_{2N_P}^{\text{HCT}}$ from (B.2), and the bound $h_E^{-1} \leq \rho^{-1} h_P^{-1}$ for all $E \in \mathcal{E}(P)$ from (M2) lead to

$$|\psi_{\text{HCT}}|_{2,P} \le C_{\text{HCT}} (1 + \rho^{-1}) h_P^{-1}.$$
 (B.9)

Step 10 controls a nodal basis function in $V_h(P)$ by its HCT interpolation. For a given nodal basis function $\psi_h \in V_h(P)$, its HCT interpolation ψ_{HCT} from (B.8), $\text{Dof}(\psi_h - \psi_{\text{HCT}}) = 0$ from Step 3, and the test function $w = \psi_h - \psi_{\text{HCT}}$ lead in (3.1) to

$$a^{P}(\psi_{h}, \psi_{h} - \psi_{HCT}) = (f, \psi_{h} - \psi_{HCT})_{L^{2}(P)}$$
 (B.10)

for some $f \in \mathcal{P}_r(P)$. The definition of the L^2 projection Π_r and the relation $\Pi_r G \psi_h = \Pi_r \psi_h$ from (3.2) for r = -1, 0, 1, 2 show that

$$(f, \psi_h)_{L^2(P)} = (f, \Pi_r G \psi_h)_{L^2(P)} = (f, \Pi_r G \psi_{HCT})_{L^2(P)}$$

with $G\psi_h = G\psi_{HCT}$ from $Dof(\psi_h) = Dof(\psi_{HCT})$ and Lemma 2.6 in the last step. This, (B.10), and a Cauchy-Schwarz inequality imply that

$$|\psi_h|_{2,P}^2 \le |\psi_h|_{2,P} |\psi_{\text{HCT}}|_{2,P} + ||f||_{L^2(P)} ||\Pi_r G \psi_{\text{HCT}} - \psi_{\text{HCT}}||_{L^2(P)}.$$
(B.11)

Recall f=0 for r=-1 so suppose r=0,1,2 for the time being. Since $\int_P (1-\Pi_r)G\psi_{\mathrm{HCT}}\,dx=0$ from the definition of the L^2 projection Π_r for r=0,1,2, the Poincaré inequality from [7, Subsec. 2.1.5] shows that $\|(1-\Pi_r)G\psi_{\mathrm{HCT}}\|_{L^2(P)} \leq C_{\mathrm{P}}h_P|G\psi_{\mathrm{HCT}}|_{1,P}$ with a positive constant C_{P} (that exclusively depends on ρ). Since ψ_h is a nodal basis function, $\int_E (\psi_{\mathrm{HCT}})_{\mathbf{n}}\,ds=\int_E (\psi_h)_{\mathbf{n}}\,ds$ for all $E\in\mathcal{E}(P)$ imply that $\int_E (\psi_{\mathrm{HCT}})_{\mathbf{n}}\,ds=0$ for all but at most one $E\in\mathcal{E}(P)$ and the Poincaré-Friedrichs inequality from Theorem 2.2.a shows $|\psi_{\mathrm{HCT}}|_{1,P}\leq C_{\mathrm{PF}}h_P|\psi_{\mathrm{HCT}}|_{2,P}$. This, Lemma 2.4, and a triangle inequality result in $|G\psi_{\mathrm{HCT}}|_{1,P}\leq |G\psi_{\mathrm{HCT}}-\psi_{\mathrm{HCT}}|_{1,P}+|\psi_{\mathrm{HCT}}|_{1,P}\leq 2C_{\mathrm{PF}}h_P|\psi_{\mathrm{HCT}}|_{2,P}$. Hence the previous estimates lead to

$$\|(1-\Pi_r)G\psi_{HCT}\|_{L^2(P)} \le 2C_PC_{PF}h_P^2|\psi_{HCT}|_{2.P}.$$

This, Lemma 2.4, and $\|(1 - \Pi_r G)\psi_{\text{HCT}}\|_{L^2(P)} \le \|(1 - \Pi_r)G\psi_{\text{HCT}}\|_{L^2(P)} + \|(1 - G)\psi_{\text{HCT}}\|_{L^2(P)}$ result in

$$\|(1 - \Pi_r G)\psi_{HCT}\|_{L^2(P)} \le C_{PF}(1 + 2C_P)h_P^2|\psi_{HCT}|_{2,P}.$$
 (B.12)

Step 11 bounds the term $||f||_{L^2(P)}$. Recall the bubble-function b_P from the proof of Theorem 4.3. The substitution of $\chi = f \in \mathcal{P}_r(P)$ in the first estimate of (4.15) proves that

$$C_b^{-1} ||f||_{L^2(P)}^2 \le (f, b_P f)_{L^2(P)} = a^P(\psi_h, b_P f)$$

with $w = b_P f \in H_0^2(P)$ in (3.1) in the last step. A Cauchy-Schwarz inequality and the second inverse estimate of (4.16) imply

$$C_b^{-1} \|f\|_{L^2(P)}^2 \le |\psi_h|_{2,P} |b_P f|_{2,P} \le C_b h_P^{-2} |\psi_h|_{2,P} \|f\|_{L^2(P)}.$$

Consequently, $||f||_{L^2(P)} \le C_b^2 h_P^{-2} |\psi_h|_{2,P}$.

Proof of (H2). The last estimate and the combination of (B.11)-(B.12) result in

$$|\psi_h|_{2,P} \le (1 + C_b^2(C_{PF}(1 + 2C_P)))|\psi_{HCT}|_{2,P}$$

for r=0,1,2. For r=-1, f=0 and (B.10) show $|\psi_h|_{2,P} \leq |\psi_{\text{HCT}}|_{2,P}$. The combination with (B.9) verifies **(H2)** with $C_{\text{stab}} := C_{\text{HCT}}(1+\rho^{-1})(1+C_b^2(C_{\text{PF}}(1+2C_{\text{P}})))$ for a nodal basis function ψ_h in $V_h(P)$ from (3.1)-(3.2) and concludes the proof.

Remark 8 (trace of H^2 functions). The trace operator $\operatorname{tr} := (\gamma_0, \gamma_1) : H^2(P) \to H^{3/2}(\partial P) \times H^{1/2}(\partial P)$ for a polygon P is not surjective, i.e., $v|_{\partial P}$ and $v_{\mathbf{n}}|_{\partial P}$ are not independent of each other [23, 26]. One consequence for weak solutions is that we cannot immediately split the boundary conditions in the weak form (A.1) into the two strong formulations $M_{\mathbf{nn}}(v_h)|_{\partial P} \in \mathcal{P}_0(\mathcal{E}(P))$ and $T(v_h)|_{\partial P} = 0$ for $v_h \in V_h(P)$.

Remark 9 (weak and strong formulations of VE functions). Compared to the current VE literature on fourth-order problems [2, 21, 24, 30, 35], the definition of $V_h(P)$ in (3.1)-(3.2) or (3.3)-(3.4) looks different. In Example 1, for instance, the analog to $V_h(P)$ in [2, 21] reads

$$V_h^s(P) := \begin{cases} v_h \in H^2(P) : & \Delta^2 v_h \in \mathcal{P}_r(P), \ \mathcal{M}_{\mathbf{nn}}(v_h)|_{\partial P} \in \mathcal{P}_0(\mathcal{E}(P)) \text{ and} \\ & T(v_h)|_{\partial P} \in \mathcal{P}_{-1}(\partial P), \ v_h - Gv_h \perp \mathcal{P}_r(P) \text{ in } L^2(P) \end{cases}$$
(B.13)

We refer to this as the strong formulation, but utilize the weak form of $V_h(P)$ in (3.1)-(3.2) throughout this paper. The point is the regularity of the weak solution $v \in H^2(P)/\mathcal{P}_1(P)$ to

$$a^{P}(v,w) = (f,w)_{L^{2}(P)} + (g,w_{\mathbf{n}})_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} a_{j}w(z_{j}) \quad \text{for all } w \in H^{2}(P).$$
 (B.14)

The weak solution $v \in H^2(P)$ is unique up to affine functions $\mathcal{P}_1(P)$ and the right-hand side displays a given $f \in \mathcal{P}_r(P), g \in \mathcal{P}_0(\mathcal{E}(P))$ and $a_1, \ldots, a_{N_P} \in \mathbb{R}$. The integration by parts formula from (A.1) exploits the formula (B.14) as

$$a^{P}(v,w) = (\Delta^{2}v,w)_{L^{2}(P)} + (M_{\mathbf{nn}}(v),w_{\mathbf{n}})_{L^{2}(\partial P)} - (T(v),w)_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} [M_{\mathbf{n}\tau}(v)]_{z_{j}} w(z_{j})$$

that holds for smooth $v \in H^4(P)$ and for all $w \in H^2(P)$. The technical issue is that only for smooth v, it holds $M_{\mathbf{nn}}(v)|_{\partial P} \in \mathcal{P}_0(\mathcal{E}(P))$ and $T(v)|_{\partial P} \in \mathcal{P}_{-1}(\partial P)$ and, to the best knowledge of the authors, it is unclear whether the weak solution $v \in H^2(P) \cap C^{\infty}(\operatorname{int}(P))$ to (B.14) allows a weak definition of the individual terms $M_{\mathbf{nn}}(v)|_{\partial P}$ and $T(v)|_{\partial P}$ on the boundary ∂P despite the fact that elliptic regularity guarantees that $v \in C^{\infty}(\operatorname{int}(P))$ is smooth inside the polygonal domain P. A routine argument with a test function $w \in \mathcal{D}(\operatorname{int}(P))$ leads to

$$(g, w_{\mathbf{n}})_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} a_{j}w(z_{j}) = (M_{\mathbf{n}\mathbf{n}}(v), w_{\mathbf{n}})_{L^{2}(\partial P)} - (T(v), w)_{L^{2}(\partial P)} + \sum_{j=1}^{N_{P}} [M_{\mathbf{n}\boldsymbol{\tau}}(v)]_{z_{j}}w(z_{j})$$
(B.15)

provided $M_{\mathbf{nn}}(v)|_{\partial P}$ and $T(v)|_{\partial P}$ can be defined well. Remark 8 already gives a warning for this and the consequence in the literature on fourth-order problems is, cf., e.g. [26], that only the sum of the right-hand side in (B.15) is a well-defined linear functional and it is never split into $M_{\mathbf{nn}}(v)|_{\partial P}$ and $T(v)|_{\partial P}$. Clearly, once we knew that both $M_{\mathbf{nn}}(v)|_{\partial P}$ and $T(v)|_{\partial P}$ are distributions, the identity (B.15) might reveal that $g = M_{\mathbf{nn}}(v)|_{\partial P}$ and $T(v)|_{\partial P} = 0$, but we do not know that. On the formal level, if we have the information that $v \in V_h(P)$ is smooth up to the boundary, then $v \in V_h^s(P)$ would follow. In this sense, the notation $V_h^s(P)$ is interpreted as a strong formulation of $V_h(P)$ from (3.1)-(3.2). We understand that, with the substitution of $V_h^s(P)$ from (B.13) as $V_h(P)$, the results of [2, 21] are well-defined and remain valid. At least the paper [21] already adopts this point of view in [21, Lemma 3.4].

Analog remarks apply to [35] and Example 2 in this paper.

C Proof of Proposition 3.2

C.1 One-dimensional finite element

Recall the vertices z_1, \ldots, z_{N_P} and the edges $E(1), \ldots, E(N_P)$, the corners ζ_1, \ldots, ζ_J and the sides $\gamma(1), \ldots, \gamma(J)$ of a polygonal domain $P \in \mathcal{M}$, and recall the vector space S(j) and a linear functional Λ_j for $j = 1, \ldots, J$ from Subsection 3.2.2. Change the coordinate system to let $\gamma(j) = (0, L)$ belong to the real axis $\mathbb{R} \times \{0\}$ and identify $(z_{k(j)}, \ldots, z_{k(j)+m(j)}) \equiv (0, s_1, \ldots, s_m)$ with a quasi-uniform partition $0 = s_0 < s_1 < \cdots < s_m = L$. Then $S(j) = \mathcal{P}_2(\mathcal{E}(\gamma(j))) \cap C^1[0, L]$ has a known B_2^2 spline basis, while we define $\varphi_\ell \in S(j)$ for $\ell = 0, \ldots, m$ by

$$\varphi_{\ell}(s_{\ell}) = 1, \varphi'_{\ell}(s_{\ell}) = 0 = \varphi_{\ell}(s_k)$$
 for all $k = 0, \dots, m$ and $k \neq \ell$.

Lemma C.1 (construction of φ_{ℓ}). The functions $\varphi_0, \ldots, \varphi_m$ belong to S(j) and there exists a positive constant C_{10} (that exclusively depends on ρ) such that

$$\|\varphi_{\ell}\|_{L^{\infty}(\gamma(j))} \leq C_{10} \quad \text{for } \ell = 0, \dots, m.$$

Proof. The construction of φ_{ℓ} first determines the piecewise quadratic φ_{ℓ} uniquely in the intervals $(s_{\ell-1},s_{\ell})$ and $(s_{\ell},s_{\ell+1})$. This determines the derivative φ'_{ℓ} at the end points $s_{\ell-1}$ and $s_{\ell+1}$. The values $\varphi'_{\ell}(s_{\ell-1})$ and $\varphi_{\ell}(s_{\ell-2}) = 0 = \varphi_{\ell}(s_{\ell-1})$ lead to a unique quadratic polynomial $\varphi_{\ell}|_{(s_{\ell-2},s_{\ell-1})}$. A successive application of this argument to the remaining intervals leads to a $\varphi_{\ell} \in S(j)$. Let $h_{\ell} := s_{\ell} - s_{\ell-1}$ for $\ell = 1, \ldots, m$. A direct computation of $\varphi_0, \ldots, \varphi_m$ displayed in Figure C.1 and C.2 controls the extrema $\|\varphi_{\ell}\|_{L^{\infty}(\gamma(j))} \leq \max\{1, \max_{p,q=0,\ldots,m} h_p/(2h_q)\}$. It follows from (M2) that $h_{\ell} \approx L$ and so $\|\varphi_{\ell}\|_{L^{\infty}(\gamma(j))} \leq C_{10}$ for $\ell = 0,\ldots,m$ with a positive constant C_{10} that exclusively depends on ρ .

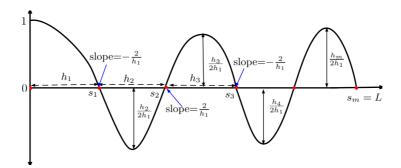


Figure C.1: The nodal basis function φ_0 .

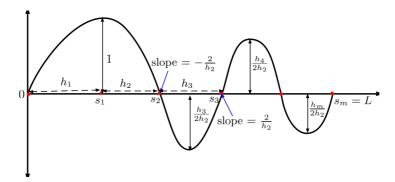


Figure C.2: The nodal basis function φ_1 .

Define $\widetilde{\psi}$ with $\widetilde{\psi}(0) = 0 = \widetilde{\psi}(s_1)$ and $\widetilde{\psi}(h_1/2) = 1$ in the first interval $(0, s_1)$. Then compute the slope $-4/h_1$ at s_1 and define $\widetilde{\psi}$ uniquely in (s_1, s_2) with $\widetilde{\psi}'(s_1) = -4/h_1$ and $\widetilde{\psi}(s_1) = 0 = \widetilde{\psi}(s_2)$. Continue the procedure by preassigning the derivative values $(-1)^{\ell-1}4/h_1$ at the left vertex $s(\ell-1)$ and function values $\widetilde{\psi}(s_{\ell-1}) = 0 = \widetilde{\psi}(s_{\ell})$ for all $\ell=2,\ldots,m$ so that $\widetilde{\psi} \in S(j)$, and verify that extrema in each interval $(s_{\ell-1},s_{\ell})$ is $(-1)^{\ell-1}h_{\ell}/h_1$. Since $\|\widetilde{\psi}\|_{L^{\infty}(\gamma(j))} \geq 1$, rescale $\psi := \frac{\widetilde{\psi}}{\|\widetilde{\psi}\|_{L^{\infty}(\gamma(j))}}$ to derive $\|\psi\|_{L^{\infty}(\gamma(j))} = 1$.

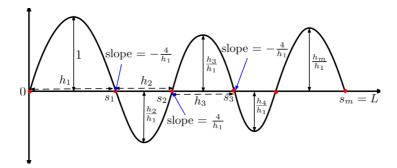


Figure C.3: The nodal basis function ψ with $\|\psi\|_{L^{\infty}(\gamma(j))} = 1$ (here $h_1 = \max_{\ell=1}^m h_{\ell}$).

Lemma C.2 (basis of S(j)). The functions $\varphi_0, \ldots, \varphi_m, \psi$ form a basis of S(j).

Proof. Let $\alpha, \alpha_0, \ldots, \alpha_m \in \mathbb{R}$ satisfy $\sum_{\ell=0}^m \alpha_\ell \varphi_\ell + \alpha \psi = 0$. This implies, for any $k = 0, \ldots, m$, that

$$0 = \sum_{\ell=0}^{m} \alpha_{\ell} \varphi_{\ell}(s_k) + \alpha \psi(s_k) = \sum_{\ell=0}^{m} \alpha_{\ell} \delta_{k\ell} = \alpha_k.$$

The function ψ attains a positive value at the midpoint $h_1/2$. So $\alpha\psi(h_1/2) = 0$ shows that $\alpha = 0$. Consequently, $\varphi_0, \ldots, \varphi_m, \psi$ are linearly independent. Since $\dim(S(j)) = m + 2$, they form a basis of S(j).

Given the basis functions $\varphi_0, \ldots, \varphi_m, \psi$ as in Lemma C.2, define

$$\psi_{\ell} := \varphi_{\ell} - \Lambda_j(\varphi_{\ell})\psi$$
 for all $\ell = 0, ..., m$ and $\psi_{m+1} := \psi$.

Lemma C.3 (finite element and 1D stability). The triple $(\gamma(j), S(j), (dof_{k(j)}, \ldots, dof_{k(j)+m(j)}, \Lambda_j))$ forms a finite element in the sense of Ciarlet. The functions $\psi_0, \ldots, \psi_{m+1}$ form a nodal basis of S(j) and

$$1 \leq \|\psi_{\ell}\|_{L^{\infty}(\gamma(i))} \leq C_{11}$$
 for all $\ell = 0, ..., m+1$

holds with a positive constant C_{11} (that exclusively depends on ρ).

Proof. The functions $\psi_0, \ldots, \psi_{m+1}$ satisfy, for all $k, \ell = 0, \ldots, m$, the duality relations

$$dof_k(\psi_\ell) = \psi_\ell(s_k) = \varphi_\ell(s_k) = \delta_{k\ell}, \quad dof_k(\psi_{m+1}) = \psi_{m+1}(s_k) = \psi(s_k) = 0,$$

$$\Lambda_j(\psi_\ell) = \Lambda_j(\phi_\ell) - \Lambda_j(\phi_\ell)\Lambda_j(\psi) = 0, \quad \Lambda_j(\psi_{m+1}) = \Lambda_j(\psi) = 1.$$

Hence the functions $\psi_0, \ldots, \psi_{m+1}$ form a nodal basis of S(j). Consequently the triple $(\gamma(j), S(j), (\text{dof}_{k(j)}, \ldots, \text{dof}_{k(j)+m(j)}, \Lambda_j)$ forms a finite element in the sense of Ciarlet. Recall $\|\psi\|_{L^{\infty}(\gamma(j))} = 1$ and notice $\|\psi_{m+1}\|_{L^{\infty}(\gamma(j))} = \|\psi\|_{L^{\infty}(\gamma(j))} = 1$. The definitions of φ_{ℓ} show, for $\ell = 0, \ldots, m$, that

$$1 \le \|\psi_{\ell}\|_{L^{\infty}(\gamma(j))} \le \|\varphi_{\ell}\|_{L^{\infty}(\gamma(j))} + |\Lambda_{j}(\varphi_{\ell})| \le (1 + C_{\Lambda})\|\varphi_{\ell}\|_{L^{\infty}(\gamma(j))} \le (1 + C_{\Lambda})C_{10}$$

with the assumption $\|\Lambda_j\| \leq C_{\Lambda}$ from Subsection 3.2.2 in the second last step and $\|\varphi_{\ell}\|_{L^{\infty}(\gamma(j))} \leq C_{10}$ from Lemma C.1 in the last step.

C.2 Proof of (H1)

Recall $\mathrm{Dof}(\bullet) = (\mathrm{dof}_1(\bullet), \ldots, \mathrm{dof}_{2N_P}(\bullet))$ from Subsection 2.1 for the polygon P with N_P edges and J sides and recall $\Lambda_1, \ldots, \Lambda_J$ from Subsection 3.2.2. Let $\Lambda(\bullet) := (\Lambda_1(\bullet), \ldots, \Lambda_J(\bullet))$ and abbreviate $\mathrm{Dof} \oplus \Lambda := (\mathrm{dof}_1, \ldots, \mathrm{dof}_{2N_P}, \Lambda_1, \ldots, \Lambda_J) : H^2(P) \to \mathbb{R}^{2N_P + J}$.

Step 1 designs an HCT interpolation. Given $w_h \in W_h(P)$, its trace $w_h|_{\gamma(j)} \in S(j) \subset C^1(\gamma(j))$ allows for well-defined tangential derivatives at all the vertices, that are utilized to define the HCT interpolation as follows. Define the normal derivative of $w_{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ to be zero at any vertex z, which is not a corner. This and the tangential derivative uniquely define $\nabla w_h(z) = \nabla w_{\text{HCT}}(z)$. There are two linearly independent tangential derivatives at a corner ζ_j

along $\gamma(j)$ and $\gamma(j+1)$ and they uniquely define a vector $\nabla w_h(\zeta_j) = \nabla w_{\text{HCT}}(\zeta_j)$ for $j = 1, \ldots, J$. Those values and the point evaluations $w_h(z) = w_{\text{HCT}}(z)$ at $z \in \mathcal{V}(P)$ allow for the HCT interpolation $w_{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ of $w_h \in W_h(P)$ with

$$w_{\text{HCT}}(z_k) = w_h(z_k), \quad \nabla w_{\text{HCT}}(z_k) = \nabla w_h(z_k), \quad \text{and} \quad \int_{E(\ell)} (w_{\text{HCT}})_{\mathbf{n}} \, ds = \int_{E(\ell)} (w_h)_{\mathbf{n}} \, ds$$

for all $k, \ell = 1, \ldots, N_P$, while all other dofs of w_{HCT} in the HCT finite element space vanish. Let $\psi_1^{\text{HCT}}, \ldots, \psi_{4N_P}^{\text{HCT}}$ denote the $4N_P$ nodal basis functions in $\text{HCT}(\mathcal{T}(P))$ with $\psi_1^{\text{HCT}}, \ldots, \psi_{2N_P}^{\text{HCT}}$ from Step 1 in Appendix B. For any $k = 2N_P + 1, \ldots, 4N_P$, define some remaining nodal basis functions $\psi_k^{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ uniquely by $\psi_k^{\text{HCT}}(z_\ell) = 0 = \int_{E(\ell)} (\psi_k^{\text{HCT}})_{\mathbf{n}} \, ds$ for $\ell = 1, \ldots, N_P$, and

$$\partial_x(\psi_k^{\text{HCT}})(z_\ell) = \delta_{(k-2N_P)\ell}, \quad \partial_y(\psi_k^{\text{HCT}})(z_\ell) = 0 \quad \text{if } k = 2N_P + 1, \dots, 3N_P,$$

$$\partial_x(\psi_k^{\text{HCT}})(z_\ell) = 0, \quad \partial_y(\psi_k^{\text{HCT}})(z_\ell) = \delta_{(k-3N_P)\ell} \quad \text{if } k = 3N_P + 1, \dots, 4N_P.$$

The scaling of the standard HCT basis functions from [27, Prop. 2.5], the observation from Step 1 in Appendix B for the scaling of $\psi_1^{\text{HCT}}, \dots, \psi_{4N_P}^{\text{HCT}}$, and the bound $h_T^{-1} \leq \rho^{-1} h_P^{-1}$ for all $T \in \mathcal{T}(P)$ from (M2) provide a positive constant C_{HCT} (that exclusively depends on ρ) in

$$\max_{k=1}^{N_P} h_P |\psi_k^{\text{HCT}}|_{2,P} + \max_{\ell=1}^{3N_P} |\psi_{\ell+N_P}^{\text{HCT}}|_{2,P} \le C_{\text{HCT}}.$$
 (C.1)

(This extends (B.2) with a possibly different constant C_{HCT} for additional $2N_P$ nodal basis functions $\psi_{2N_P+1}^{\text{HCT}}, \dots, \psi_{4N_P}^{\text{HCT}}$.) Then, given $w_h \in W_h(P)$, the HCT interpolation w_{HCT} reads

$$w_{\text{HCT}} = \sum_{k=1}^{N_P} \text{dof}_k(w_h) \psi_k^{\text{HCT}} + \sum_{k=N_P+1}^{2N_P} \text{dof}_k(w_h) h_{E(k-N_P)}^{-1} \psi_k^{\text{HCT}} + \sum_{k=2N_P+1}^{3N_P} \partial_x(w_h) (z_{k-2N_P}) \psi_k^{\text{HCT}} + \sum_{k=3N_P+1}^{4N_P} \partial_y(w_h) (z_{k-3N_P}) \psi_k^{\text{HCT}}.$$

Step 2 defines the Hilbert space W_0 . The kernel $\{v \in H^2(P) : \forall \ell = 1, \dots, N_P \quad \int_{E(\ell)} v_{\mathbf{n}} ds = 0\}$ of the linear map $(\mathrm{dof}_{N_P+1}, \dots, \mathrm{dof}_{2N_P}) : H^2(P) \cap H^1_0(P) \to \mathbb{R}^{N_P}$ is a Hilbert space $H^2(P)$ and its intersection

$$W_0 := \left\{ v \in W : \int_{E(\ell)} v_{\mathbf{n}} \, ds = 0 \quad \text{for } \ell = 1, \dots, N_P \right\} \subset W := H_0^1(P) \cap H^2(P)$$

is complete. Therefore (W_0, a^P) is a Hilbert space.

Step 3 proves $w_h - w_{\text{HCT}} \in W_0$ for any $w_h \in W_h(P)$ and its HCT interpolation $w_{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ from Step 1. The design leads to $(w_h - w_{\text{HCT}})|_{\gamma(j)} \in \mathcal{P}_3(\mathcal{E}(\gamma(j)))$. Step 1 shows that $(w_h - w_{\text{HCT}})(z_k)$ and $\nabla(w_h - w_{\text{HCT}})(z_k)$ vanish for all $k = 0, \ldots, m$. These values and $(w_h - w_{\text{HCT}})|_{\gamma(j)} \in \mathcal{P}_3(\mathcal{E}(\gamma(j))) \cap C^1(\gamma(j))$ uniquely determine $(w_h - w_{\text{HCT}})|_{\gamma(j)} = 0$ for $j = 1, \ldots, J$ and so $(w_h - w_{\text{HCT}})|_{\partial P} = 0$. Consequently, $w_h - w_{\text{HCT}} \in W$. Since $\int_{\mathcal{E}(\ell)} (w_{\text{HCT}})_{\mathbf{n}} ds = \int_{\mathcal{E}(\ell)} (w_h)_{\mathbf{n}} ds$ for all $\ell = 1, \ldots, N_P$ (by design of w_{HCT}), $w_h - w_{\text{HCT}} \in W_0$ follows from the definition of W_0 in Step 2.

Step 4 establishes a sufficient criterion for the inclusion in $\widehat{W}_h(P)$: If $w \in H^2(P)$ and $f \in \mathcal{P}_r(P)$ satisfy $a^P(w, \phi_0) = (f, \phi_0)_{L^2(P)}$ for all $\phi_0 \in W_0$, then there exists $g \in \mathcal{P}_0(\mathcal{E}(P))$ such that

$$a^P(w,\phi) = (f,\phi)_{L^2(P)} + (g,\phi_{\mathbf{n}})_{L^2(\partial P)}$$
 for all $\phi \in W$.

Proof: Let $\varphi_{\ell} \in S^1(\mathcal{T}(P)) := \{ v \in C^0(\mathcal{T}(P)) : \forall T \in \mathcal{T}(P) \ v|_T \in \mathcal{P}_1(T) \}$ be the nodal basis functions in the Courant FEM $(\mathcal{P}_1 \text{ conforming})$ associated with the vertices z_{ℓ} and define the

quadratic edge-bubble function $b_{E(\ell)} := 4\varphi_{\ell}\varphi_{\ell+1}$ with support $T(E(\ell))$ depicted in Figure 2.1.b. Define $\psi_1, \ldots, \psi_{N_P} \in W$ by

$$\psi_{\ell}(x) := \frac{8}{15} (x - \operatorname{mid}(E(\ell))) \cdot \mathbf{n}_{E(\ell)} b_{E(\ell)}^{2}(x) \quad \text{for all } x \in P$$

so that $f_{E(k)}(\psi_{\ell})_{\mathbf{n}} ds = \delta_{k\ell}$ for all $k, \ell = 1, \dots, N_P$. Given any $\phi \in W$, define

$$\phi_0 := \phi - \sum_{\ell=1}^{N_P} \Big(\int_{E(\ell)} \phi_{\mathbf{n}} \, ds \Big) \psi_{\ell} \in W_0.$$

Then $a^P(w,\phi) = a^P(w,\phi_0) + a^P(w,\phi-\phi_0)$ and the assumption $a^P(w,\phi_0) = (f,\phi_0)_{L^2(P)}$ imply

$$a^{P}(w,\phi) = (f,\phi_{0})_{L^{2}(P)} + \sum_{\ell=1}^{N_{P}} \Big(\int_{E(\ell)} \phi_{\mathbf{n}} \, ds \Big) a^{P}(w,\psi_{\ell}) = (f,\phi)_{L^{2}(P)} + (g,\phi_{\mathbf{n}})_{L^{2}(\partial P)}$$

for $g \in \mathcal{P}_0(\mathcal{E}(P))$ with $g|_{E(\ell)} := a^P(w, \psi_\ell) - (f, \psi_\ell)_{L^2(P)}$ for $\ell = 1, \dots, N_P$ in the last step. This concludes the proof of the claim.

Step 5 proves that $Dof \oplus \Lambda : \widehat{W}_h(P) \to \mathbb{R}^{2N_P+J}$ is surjective. Given any $x \in \mathbb{R}^{2N_P+J}$, Lemma C.3 leads to some $w_j \in S(j)$ such that $\operatorname{dof}_{k(j)+\ell}(w_j) = x_{k(j)+\ell}$ for $\ell = 0, \ldots, m(j)$ and $\Lambda_j(w_j) = x_{2N_P+j}$. This holds for all $j = 1, \ldots, J$ and defines a continuous $w \in \mathcal{P}^2(\mathcal{E}(P)) \cap C^0(\partial P)$ with $w|_{\gamma(j)} = w_j \in S(j)$ on the boundary ∂P . Since $w \in C^0(\partial P)$ satisfies $w|_{\gamma(j)} \in C^1(\gamma(j))$ for all $j = 1, \ldots, J$, the tangential derivatives of w define $\nabla w(\zeta_j)$ at each corner point ζ_j , while at each other vertex $z \in \mathcal{V}(P)$ (that is not a corner) the tangential derivative $w'(z) = w_{\tau}(z)$ and the vanishing normal derivative determine a unique vector $\nabla w(z) \in \mathbb{R}^2$. Given those values of w and ∇w at all vertices in $\mathcal{V}(P)$, define an HCT interpolation $u_{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ as in Step 1 with

$$u_{ ext{HCT}}(z_k) = w(z_k), \quad \nabla u_{ ext{HCT}}(z_k) = \nabla w(z_k), \quad ext{and} \quad \int_{E(\ell)} (u_{ ext{HCT}})_{\mathbf{n}} \, ds = x_{\ell+N_P}$$

for $k, \ell = 1, \ldots, N_P$, while all other dofs vanish. This and Step 3 reveal that $u_{\text{HCT}}|_{\partial P} = w|_{\partial P}$. Let $u_0 \in W_0$ denote the Riesz representation of the linear functional $a^P(u_{\text{HCT}}, \cdot)$ in the Hilbert space (W_0, a^P) , i.e., $a^P(u_0, \cdot) = a^P(u_{\text{HCT}}, \cdot)$ in W_0 . Let $\widehat{u}_h := u_{\text{HCT}} - u_0$ and deduce $\widehat{u}_h|_{\partial P} = u_{\text{HCT}}|_{\partial P} \in S^2(\mathcal{E}(P))$ from $u_0|_{\partial P} = 0$. Since $a^P(\widehat{u}_h, \varphi_0) = 0$ for all $\varphi_0 \in \mathcal{D}(\text{int}(P))$, it follows $\Delta^2\widehat{u}_h = 0$ in P. Step 4 with $f = 0 \in \mathcal{P}_{-1}(P)$ implies the existence of $g \in \mathcal{P}_0(\mathcal{E}(P))$ with

$$a^{P}(\widehat{u}_{h}, \phi) = (g, \phi_{\mathbf{n}})_{L^{2}(\partial P)}$$
 for all $\phi \in W$

and proves that $\widehat{u}_h \in \widehat{W}_h(P)$. Recall (Dof $\oplus \Lambda$)(\widehat{u}_h) = x from the design of w and w_{HCT} in the very beginning of the proof.

Step 6 establishes an inclusion in $\widehat{W}_h(P)$ with a non-zero f. Given $f \in \mathcal{P}_r(P)$, the Riesz-representation theorem guarantees the unique existence of the weak solution $u(f) \in W_0$ to

$$a^{P}(u(f), v) = (f, v)_{L^{2}(P)}$$
 for all $v \in W_{0}$. (C.2)

Consequently $\Delta^2 u(f) = f$ in P. This and Step 4 show that $u(f) \in \widehat{W}_h(P)$.

Step 7 proves that $\mathcal{L}: \mathcal{P}_r(P) \to \mathcal{P}_r(P), f \mapsto \Pi_r u(f)$ is an isomorphism. For any $f \in \mathcal{P}_r(P)$ with $\mathcal{L}f = 0$, the orthogonality $(1 - \Pi_r)u(f) \perp \mathcal{P}_r(P)$ in $L^2(P)$ shows $0 = \int_P (\mathcal{L}f)f \, dx = \int_P u(f)f \, dx$. This and v = u(f) in (C.2) result in $0 = a^P(u(f), u(f)) = |u(f)|_{2,P}^2$. Consequently $u(f) \in \mathcal{P}_1(P) \cap W_0$ and so u(f) = 0. Thus $f = \Delta^2 u(f) = 0$ and \mathcal{L} is injective; whence bijective.

Step 8 proves that Dof: $W_h(P) \to \mathbb{R}^{2N_P}$ is an isomorphism.

Proof of surjectivity: Given $x \in \mathbb{R}^{2N_P}$, there exists some $v \in H^2(P)$ with $\mathrm{Dof}(v) = x$ (for a proof we may utilize Step 5 for $(x,0,\dots,0) \in \mathbb{R}^{2N_P+J}$ and obtain at least one $v \in \widehat{W}_h(P) \subset H^2(P)$).

Given $v \in H^2(P)$, let $\chi := Gv \in \mathcal{P}_2(P)$ and notice χ is computable with Lemma 2.6 from x in a unique way. Set $y_j := \Lambda_j(\chi|_{\gamma(j)}) \in \mathbb{R}$ for all $j = 1, \ldots, J$. Step 5 proves that given $y = (x_1, \ldots, x_{2N_P}, y_1, \ldots, y_J) \in \mathbb{R}^{2N_P + J}$, there exists some $\widehat{u}_h \in \widehat{W}_h(P)$ with $(\text{Dof} \oplus \Lambda)(\widehat{u}_h) = y$, i.e., $\text{Dof}(\widehat{u}_h) = x$ and $\Lambda_j(\widehat{u}_h|_{\gamma(j)}) = y_j$ for $j = 1, \ldots, J$. Since $G\widehat{u}_h = Gv = \chi$ for any $v \in H^2(P)$ with Dof(v) = x from Lemma 2.6, $\Lambda_j(\widehat{u}_h|_{\gamma(j)}) = y_j = \Lambda_j(\chi|_{\gamma(j)}) = \Lambda_j(G\widehat{u}_h|_{\gamma(j)})$. This leads to $g := \prod_r G\widehat{u}_h - \prod_r \widehat{u}_h \in \mathcal{P}_r(P)$. Since \mathcal{L} is bijective in $\mathcal{P}_r(P)$ (from Step 7), there exists $f \in \mathcal{P}_r(P)$ with $\prod_r u(f) = g$. Recall $u(f) \in \widehat{W}_h(P) \cap W_0$ from Step 6 and Dof(u(f)) = 0, G(u(f)) = 0, and $\Lambda(u(f)) = 0$. Altogether, $u_h^P := u(f) + \widehat{u}_h \in \widehat{W}_h(P)$ satisfies $\text{Dof}(u_h^P) = x$, $\Lambda(u_h^P - Gu_h^P) = 0$, and $\prod_r Gu_h^P = \prod_r G\widehat{u}_h = g + \prod_r \widehat{u}_h = \prod_r u_h^P$. This concludes the proof of $u_h^P \in W_h(P)$ with $\text{Dof}(u_h^P) = x$.

Proof of injectivity. Suppose $w_h \in W_h(P)$ satisfies $\operatorname{Dof}(w_h) = 0$. This and Lemma 2.6 imply $Gw_h = 0$ and hence the condition $\Lambda_j((w_h - Gw_h)|_{\gamma(j)}) = 0$ in (3.4) shows $\Lambda_j(w_h) = 0$. Lemma C.3 implies that $\operatorname{Dof}(w_h)$ and $\Lambda_j(w_h)$ uniquely determine $w_h|_{\gamma(j)} = 0$ for all $j = 1, \ldots, J$; whence $w_h|_{\partial P} = 0$. The substitution of $w = \phi = w_h$ in (3.3), the L^2 orthogonality of Π_r , $f \in \mathcal{P}_r(P)$, and $\Pi_r w_h = \Pi_r(Gw_h) = 0$ from (3.4) result in

$$|w_h|_{2,P}^2 = a^P(w_h, w_h) = (f, w_h)_{L^2(P)} = (f, \Pi_r(Gw_h))_{L^2(P)} = 0.$$

Consequently $w_h \in \mathcal{P}_1(P) \cap W_0$, whence $w_h = 0$.

Proof of (H1). The key observation from Step 8 is that $W_h(P)$ has the dimension $2N_P$ and $dof_1, \ldots, dof_{2N_P}$ from (2.1) are linear independent. Consequently $(P, W_h(P), (dof_1, \ldots, dof_{2N_P}))$ is a finite element in the sense of Ciarlet.

C.3 Proof of (H2)

Step 1 defines an HCT interpolation of a nodal basis function of $W_h(P)$. Let $\psi_h \equiv \psi_p$ be a nodal basis function of the finite element $(P, W_h(P), \text{Dof})$ for some $p \in \{1, \dots, 2N_P\}$ and let ψ_{HCT} be its HCT interpolation as in Step 1 from Subsection C.2, namely

$$\psi_{\text{HCT}} = \sum_{k=1}^{N_P} \text{dof}_k(\psi_h) \psi_k^{\text{HCT}} + \sum_{k=N_P+1}^{2N_P} \text{dof}_k(\psi_h) h_{E(k-N_P)}^{-1} \psi_k^{\text{HCT}} + \sum_{k=2N_P+1}^{3N_P} \partial_x(\psi_h) (z_{k-2N_P}) \psi_k^{\text{HCT}} + \sum_{k=3N_P+1}^{4N_P} \partial_y(\psi_h) (z_{k-3N_P}) \psi_k^{\text{HCT}}.$$

Step 2 proves that $|\nabla(\psi_h)(z_k)| \lesssim h_P^{-1}$ for $k=1,\ldots,N_P$. Recall from Step 1 in Subsection C.2 that the two linearly independent tangential derivatives uniquely define $\nabla \psi_h(\zeta)$ at each corner ζ , and the tangential derivative and vanishing normal derivative uniquely define $\nabla \psi_h(z)$ at each vertex z, which is not a corner. Expand $\psi_h|_{\gamma(j)}$ in terms of the finite element $(\gamma(j), S(j), (\text{dof}_{k(j)}, \ldots, \text{dof}_{k(j)+m(j)}, \Lambda_j))$ from Lemma C.3, write $(\cdot)' := (\cdot)_{\tau}$ for the derivative along $\gamma(j)$, and deduce that

$$(\psi_h)_{\tau}|_{\gamma(j)} = \sum_{\ell=0}^m \operatorname{dof}_{\ell}(\psi_h)\psi_{\ell}' + \Lambda_j(\psi_h|_{\gamma(j)})\psi_{m+1}'.$$
 (C.3)

The definition of ψ_h shows that $|\mathrm{dof}_{\ell}(\psi_h)| \leq 1$ for $\ell = 0, \ldots, m$. The definition of $W_h(P)$ from (3.4) and of $\|\Lambda_j\|$ from Subsection 3.2.2 imply

$$|\Lambda_j(\psi_h|_{\gamma(j)})| = |\Lambda_j(G\psi_h|_{\gamma(j)})| \le C_{\Lambda} ||G\psi_h||_{L^{\infty}(\gamma(j))} \le C_{\Lambda} C_{\text{inv}} L^{-1/2} ||G\psi_h||_{L^2(\gamma(j))}$$

with an inverse estimate for $G\psi_h|_{\gamma(j)} \in \mathcal{P}_2(\gamma(j))$ in the last step. This and the trace inequality [7, p. 554] show $|\Lambda_j(\psi_h|_{\gamma(j)})| \leq C_\Lambda C_{\text{inv}} C_T(L^{-1} || G\psi_h ||_{L^2(P)} + |G\psi_h|_{1,P})$. Consequently, $L^{-1} \leq \rho^{-1}h_P^{-1}$ from (M2), Lemma 2.6, and $|\text{Dof}(\psi_h)|_{\ell^2} = 1$ result in

$$|\Lambda_j(\psi_h|_{\gamma(j)})| \le C_{\Lambda} C_{\text{inv}} C_T C_g (1 + \rho^{-1}) |\text{Dof}(\psi_h)|_{\ell^2} \le C_{\Lambda} C_{\text{inv}} C_T C_g (1 + \rho^{-1}).$$
 (C.4)

The inverse inequality for the piecewise quadratic polynomial $\psi_{\ell} \in \mathcal{P}_2(\mathcal{E}(\gamma(j)))$ with $h_E^{-1} \leq \rho^{-1}h_P^{-1}$ and Lemma C.3 lead, for $\ell = 0, \ldots, m+1$, to

$$|\psi_{\ell}|_{1,\infty,E} \le C_{\text{inv}}\rho^{-1}h_P^{-1}\|\psi_{\ell}\|_{L^{\infty}(E)} \le C_{\text{inv}}\rho^{-1}C_{11}h_P^{-1}$$
 for all $E \in \mathcal{E}(\gamma(j))$.

The above estimate for each $\ell = 0, ..., m$, the combination (C.3)-(C.4), and $1+m \le N_P \le M(\rho)$ from Subsection 2.1 prove for a positive constant $C_{12} := C_{\text{inv}}C_{11}\rho^{-1}M(\rho)(1+C_{\Lambda}C_{\text{inv}}C_TC_g)$ (that exclusively depends on ρ) that

$$|(\psi_h)_{\tau}|_{\gamma(i)}(s_k)| \le |\psi_h|_{1,\infty,\gamma(i)} \le C_{12}h_P^{-1} \quad \text{for } k = 0,\dots,m.$$
 (C.5)

Since the normal derivative at a vertex s_k (= $z_{k(j)+k}$), which is not a corner is zero for $k=1,\ldots,m-1$, (C.5) shows that $|\nabla(\psi_h)(s_k)| \leq C_{12}h_P^{-1}$ for $k=1,\ldots,m-1$. The expansion in (C.3) for $(\psi_h)_{\tau}|_{\gamma(j-1)}$ leads to $|(\psi_h)_{\tau}|_{\gamma(j-1)}(\zeta_j)| \leq C_{12}h_P^{-1}$. This and (C.5) prove that $|\nabla(\psi_h)(\zeta_j)| \leq C(\omega_j)h_P^{-1}$ with a positive constant $C(\omega_j)$ that depends on C_{12} and on the interior angle $\omega_j \neq \pi$ at the corner ζ_j . This holds for all $j=1,\ldots,J$ and concludes the proof.

Step 3 provides the scaling of the HCT interpolation. The definition of ψ_h shows $\operatorname{dof}_k(\psi_h) = \delta_{kp}$ for $k, p = 1, \dots, 2N_P$. The scaling of $\psi_k^{\operatorname{HCT}}$ from (C.1) for the first $2N_P$ indices $k = 1, \dots, 2N_P$ and $h_E^{-1} \leq \rho^{-1} h_P^{-1}$ for $E \in \mathcal{E}(P)$ from (M2) show

$$\sum_{k=1}^{N_P} |\operatorname{dof}_k(\psi_h)| |\psi_k^{\operatorname{HCT}}|_{2,P} + \sum_{k=N_P+1}^{2N_P} |\operatorname{dof}_k(\psi_h)| h_{E(k-N_P)}^{-1} |\psi_k^{\operatorname{HCT}}|_{2,P} \le C_{\operatorname{HCT}} (1+\rho^{-1}) h_P^{-1}.$$

The scaling $|\psi_k^{\text{HCT}}|_{2,P} \leq C_{\text{HCT}}$ from (C.1) and from Step 2 for the remaining $2N_P$ indices $k=2N_P+1,\ldots,4N_P$ prove with $C_{13}:=C(\omega_1)+\cdots+C(\omega_J)+C_{12}M(\rho)$ that

$$\sum_{k=2N_P+1}^{3N_P} |\partial_x(\psi_h)(z_{k-2N_P})| |\psi_k^{\text{HCT}}|_{2,P} + \sum_{k=3N_P+1}^{4N_P} |\partial_y(\psi_h)(z_{k-3N_P})| |\psi_k^{\text{HCT}}|_{2,P}$$

$$\leq C_{\text{HCT}} \sum_{\ell=1}^{N_P} |\nabla \psi_h(z_\ell)| \leq C_{\text{HCT}} C_{13} h_P^{-1}.$$

The previous two displayed estimates lead in the representation of ψ_{HCT} from Step 1 to

$$|\psi_{\text{HCT}}|_{2,P} < C_{\text{HCT}}(1+\rho^{-1}+C_{13})h_{P}^{-1}.$$
 (C.6)

Proof of (H2). Step 3 in Subsection C.2 shows for the nodal basis function $\psi_h \in W_h(P)$ that $(\psi_h - \psi_{\text{HCT}})|_{\partial P} = 0$. Hence the test function $\phi = \psi_h - \psi_{\text{HCT}}$ leads in (3.3) to

$$a^{P}(\psi_{h}, \psi_{h} - \psi_{HCT}) = (f, \psi_{h} - \psi_{HCT})_{L^{2}(P)}$$
 (C.7)

and it remains to control $||f||_{L^2(P)} \lesssim h_P^{-2} |\psi_h|_{2,P}$. The analogous arguments in Step 10-11 from Appendix B apply to $\psi_h \in W_h(P)$ and its HCT interpolation $\psi_{\text{HCT}} \in \text{HCT}(\mathcal{T}(P))$ from Step 1. This leads here to $(f, \psi_h - \psi_{\text{HCT}})_{L^2(P)} = (f, \Pi_r G \psi_{\text{HCT}} - \psi_{\text{HCT}})_{L^2(P)}$. The arguments in Step 10 of Appendix B provide $||\Pi_r G \psi_{\text{HCT}} - \psi_{\text{HCT}}||_{L^2(P)} \leq C_{\text{PF}} (1 + 2C_P) h_P^2 |\psi_{\text{HCT}}||_{2,P}$. The sole modification in the arguments concerns the equality $(f, b_P f)_{L^2(P)} = a^P(\psi_h, f)$, which follows from $\phi = b_P f \in H_0^2(P)$ in (3.3). The remaining arguments in Step 11 apply here verbatim and lead to $||f||_{L^2(P)} \leq C_b^2 h_P^{-2} |\psi_h|_{2,P}$. These estimates, Cauchy-Schwarz inequalities, and (C.7) result in

$$|\psi_h|_{2,P} \le (1 + C_b^2(C_{PF}(1 + 2C_P)))|\psi_{HCT}|_{2,P}.$$

The combination with (C.6) shows $|\psi_h|_{2,P} \leq C_{\text{stab}} h_P^{-1}$ with $C_{\text{stab}} := C_{\text{HCT}} (1 + \rho^{-1} + C_{13}) (1 + C_b^2 (C_{\text{PF}} (1 + 2C_{\text{P}})))$. This verifies **(H2)** for a nodal basis function ψ_h in $W_h(P)$ from (3.3)-(3.4) and concludes the proof.

Remark 10 (comparison with [35]). The discrete space in [35] reads

$$V_h^s(P) := \left\{ \begin{array}{c} v_h \in H^2(P) : \Delta^2 v_h \in \mathcal{P}_r(P) \ v_h|_{\partial P} \in \mathcal{P}_2(\mathcal{E}(P)) \ \Delta v_h|_{\partial P} \in \mathcal{P}_0(\mathcal{E}(P)), \\ \forall E \in \mathcal{E}(P) \quad \int_E v_h \ ds = \int_E G v_h \ ds, \quad v_h - G v_h \perp \mathcal{P}_r(P) \ \text{in} \ L^2(P) \right\}. \end{array}$$
 (C.8)

Recall the sides $\gamma(j)$ of a polygonal domain $P \in \mathcal{M}$ for j = 1, ..., J from Subsection 3.2.2. First notice that, for $v_h \in V_h^s(P)$ in (C.8), $v_h|_{\gamma(j)}$ belongs to $C^1(\gamma(j))$ and so $V_h^s(P)$ from (C.8) allows only the polygons without hanging nodes (i.e., all vertices are corner points). Second, as discussed in Remark 8-9, we avoid the strong formulation $\Delta v_h|_{\partial P} \in \mathcal{P}_0(\mathcal{E}(P))$ and solely consider the weak formulation (3.3)-(3.4). Hanging nodes (i.e., vertices that are not corners on ∂P) are important for a more flexible mesh-design to allow obligatory adaptive mesh-refining.

Remark 11 (individual parameters). The selection of the linear functional $\Lambda_j^P := \Lambda_j : S(j) \to \mathbb{R}$ resp. of the parameter $r_P = r = -1, 0, 1, 2$ is individually for each polygon $P \in \mathcal{M}$ and may be labelled with an index P to underline this. Given an interior side $\gamma(j) \subset \partial P_+ \cap \partial P_-$ shared by two polygons $P_+, P_- \in \mathcal{M}$, $\Lambda_j^{P_+}$ and $\Lambda_j^{P_-}$ resp. r_{P_+} and r_{P_-} could be different in general. A single selection $\Lambda_j^{P_+} = \Lambda_j^{P_-}$ is also possible and could be even more appealing; but it does not imply C^0 conformity: The jump $[w]_{\gamma(j)} \in \operatorname{span}\{\psi_j\}$ for $w \in V_{\operatorname{nc}}$ with $w|_{P_\pm} \in W_h(P)$ cannot be expected to vanish; so the schemes are fully nonconforming.