Regularisation and separation for evolving surface Cahn-Hilliard equations

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Abstract

We consider the Cahn-Hilliard equation with constant mobility and logarithmic potential on a two-dimensional evolving closed surface embedded in $\mathbb{R}^3$, as well as a related weighted model. The well-posedness of weak solutions for the corresponding initial value problems on a given time interval $[0, T]$ have already been established by the first two authors. Here we first prove some regularisation properties of weak solutions in finite time. Then, we show the validity of the strict separation property for both the problems. This means that the solutions stay uniformly away from the pure phases ± 1 from any positive time on. This property plays an essential role to achieve higher-order regularity for the solutions. Also, it is a rigorous validation of the standard double-well approximation. The present results are a twofold extension of the well-known ones for the classical equation in planar domains.

1 Introduction

The Cahn-Hilliard equation in an evolving surface setting has recently been studied in [13]. More precisely, having fixed $T > 0$, considering a family of closed, connected, oriented surfaces $\Gamma(t) \subset \mathbb{R}^3$ such that its evolution is given a priori as a flow determined by the (sufficiently smooth) velocity field $\mathbf{V}$, the evolving surface Cahn-Hilliard equation reads

$$\begin{cases} \dot{u} + u\nabla_\Gamma \cdot \mathbf{V} - \nabla_\Gamma \cdot (u\mathbf{V}_a^\tau) + \Delta_\Gamma^2 u - \Delta_\Gamma F'(u) = 0, & \text{in } \mathcal{G}_T, \\ u(0) = u_0, & \text{in } \Gamma(0), \end{cases} \tag{1.1}$$

where $\mathcal{G}_T := \bigcup_{t\in[0,T]}\{t\} \times \Gamma(t)$. Here $\mathbf{V}_a^\tau$ corresponds to the difference between the tangential component of $\mathbf{V}$ and an advective velocity $\mathbf{V}_a$ on the surface. The quantity u can be interpreted as the difference between the concentration of two immiscible substances which are present on the surface. In the same contribution, the following related weighted Cahn-Hilliard system has also been analysed

$$\begin{cases} \dot{\rho} + \rho\nabla_\Gamma \cdot \mathbf{V} = 0, & \text{in } \mathcal{G}_T, \\ \rho\dot{c} - \nabla_\Gamma \cdot \left(\rho\nabla_\Gamma\left(-\frac{1}{\rho}\Delta_\Gamma c + F'(c)\right)\right) = 0, & \text{in } \mathcal{G}_T, \\ c(0) = c_0, \ \rho(0) \equiv 1, & \text{in } \Gamma(0), \end{cases} \tag{1.2}$$

where ρ is a suitable weight function transported by the surface evolution (in particular, it can be interpreted as the total mass density) and c can be seen as the relative (i.e., dimensionless) concentration difference between the two substances on the surface. In the equations above, equipped with suitable initial conditions, ∇_Γ denotes the tangential gradient on the surface $\Gamma(\cdot)$, Δ_Γ is the Laplace-Beltrami operator and $\dot{u}$ denotes the material time derivative of u (see Section 2 for more details). The functional framework in this work is the same as in [13]. Model (1.2) is a simplified version of the one presented in [49, 56], which also includes a coupling with an equation for the surface deformation, i.e., the Kirchhoff-Love thin shell equation. We decided to concentrate on the single Cahn-Hilliard equation in the same formulation arising from the model in [56], so that this analysis, although being interesting *per se*, could also be exploited (and extended) to take the evolution of the surface into account as in [49, 56]; in our case, this evolution is given *a priori*. In both models, the main physical

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property is the conservation of total mass. In particular (see also Remark 4.1 for more details), in the case of model (1.1) this implies, u denoting the difference density, that we have

$$\int_{\Gamma(t)} u \equiv \int_{\Gamma_0} u_0, \tag{1.3}$$

whereas, for model (1.2), it holds

$$\int_{\Gamma(t)} \rho c \equiv \int_{\Gamma_0} c_0, \tag{1.4}$$

ρc being this time the difference density between the two substances and having assumed $\rho(0) \equiv 1$.

This work aims at proving that weak solutions to equations (1.1) and (1.2) regularise in finite time and enjoy the instantaneous strict separation property from the pure phases, i.e., for each $\tau > 0$, there exists $\delta > 0$ depending on τ, T and the data such that

$$\|u(t)\|_{L^\infty(\Gamma(t))} \leq 1 - \delta, \qquad \text{for a.a. } t \in [\tau, T]. \tag{1.5}$$

The present results answer to some regularity issues left open (see [13, 5.1.3]).

The classical Cahn-Hilliard equation was introduced in [15] (see also [14]) for the study of spinodal decomposition in binary alloys. More precisely, it models the phenomenon of phase separation of an alloy of two components when the temperature of the system is quenched to a critical temperature, resulting on a spatially separated two-phase structure. On a smooth bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 1, 2, 3$, it reads as

$$\begin{cases} \dot{u} = \Delta w, & \text{in } \Omega \times (0,T), \\ w = -\Delta u + F'(u), & \text{in } \Omega \times (0,T), \end{cases} \tag{1.6}$$

where w is the so called chemical potential and is complemented with the (no-flux) boundary and initial conditions

$$\partial_n u = \partial_n w = 0 \ \text{ on } \partial\Omega \times (0,T) \quad \text{ and } \quad u(0) = u_0 \ \text{ in } \Omega.$$

Note that we have assumed constant mobility and set it equal to 1. It has since then inspired numerous works in other areas of science; to name a few, versions of the Cahn-Hilliard equation have been used to study population dynamics [18], tumour growth models [38] and they have been exploited in image processing analysis [11, 12] (see also the recent book [42] for other examples). It is also worth recalling that phase separation has recently become a paradigm in cell biology (see, for instance, [21, 22]). For example, Cahn-Hilliard equations have been used to model solid tumour growth (see, e.g., [17, 52, 53] and the book [20, Part I, Chap. 5]), dynamics of plasma membranes and multicomponent vesicles [8–10, 16, 39, 41, 50, 54]. In some of these cases, such as sorting in biological membranes, phase separation and coarsening take place in a thin, evolving layer of self-organising molecules, which in continuum-based approach can be modelled as a material surface. This justifies the recent interest in the Cahn-Hilliard equation posed on evolving surfaces. In particular, one of the most interesting phenomenon which has been modelled by evolving-surface Cahn-Hilliard equations (see, e.g., [31, 56]) is the lipid rafts formation on cell membranes, which are composed of lipids, proteins, and cholesterol. Whereas proteins mediate traffic and serve as signalling devices, lipids provide a fluid matrix within which transmembrane proteins are free to move. The separation of lipids into two immiscible liquid phases is often linked to the formation of rafts in cell membranes. These rafts are heterogeneous, highly dynamic, sterol, and sphingolipid-enriched domains that compartmentalise cellular processes and they are thought to be in the liquid-ordered phase. Rafts are believed to play an important role in regulating protein activity ([37]) that may in turn affect biological processes such as trafficking and signalling and are known to be central to the replication of viruses ([48]). All of the above phenomena involve elastic bending of cell membranes being fully coupled with the irreversible processes of lipid flow, the diffusion of lipids and proteins, and the surface binding of proteins. Comprehensive membrane models which include these effects are needed to fully understand the complex physical behavior of biological membranes. In our work we make a first stride in the direction of the analysis of these highly complex models. In particular, as already noticed, taking inspiration from such models (see, e.g., [49, 56]), we do not consider the elasticity of the surface, that would determine an evolution equation for the surface itself, which we assume to be given, but we only study the Cahn-Hilliard equation arising from such models. The natural direction for future work is then to consider the fully coupled system, where the evolution of the surface is itself part of problem, see for instance [1, 2, 49, 56]. In equation (1.6), u is again to be thought of as the difference between the concentrations of the two components in the mixture. The function F is the homogeneous free energy (potential) of the system, and is defined as follows

$$F(r) = \frac{\theta}{2}\left((1+r)\ln(1+r) + (1-r)\ln(1-r)\right) - \frac{\theta_0}{2} r^2, \quad r \in [-1,1], \tag{1.7}$$

where θ, θ_0 are absolute temperatures and satisfy $0 < \theta < \theta_0$. This ensures that F has a double-well shape. From the modelling point of view, the logarithmic terms are related to the entropy of the system, while the quadratic term accounts for demixing effects. It is the competition between the two terms that gives rise to the spatially distributed phase separation. The potential defined in this way is called *singular*, whereas many authors considered a proper approximation, which avoids the fact that F' is unbounded at the pure phases ± 1. The most common choice is a polynomial of fourth degree, typically of the form $F(r) = \frac{1}{4}(r^2 - u_\theta^2)^2$, where $u_\theta > 0$ and $-u_\theta < 0$ are the minima of (1.7); this guarantees that F has a symmetric double-well shape also with minima at $\pm u_\theta$. This is usually referred as a regular potential. This case was also taken into account in [13]. However, the polynomial approximation does not ensure the existence of physical solutions, that is, solutions whose values are in $[-1, 1]$, due to the lack of comparison principles for the Cahn-Hilliard equation. We refer to the original proof of well-posedness with (1.7) in [27], as well as the survey articles [26, 45] and the recent book [42] for an overview of the mathematical results for (1.6).

As far as the evolving surface version is concerned, both systems (1.1) and (1.2) are treated in [13], where the authors establish existence, uniqueness and stability of solutions for the different cases where F is a smooth, logarithmic or double obstacle potential, respectively. Some regularity results are also proved. We refer also to [28, 47, 55] for different derivations of the equation and some numerical results, and to [56] for a rigorous modelling source for the weighted system (1.2). Nevertheless, for the sake of completeness we will give a short derivation of each model in Sections 3.1 and 4.1, respectively. Interest in these equations is part of the more general problem of considering partial differential equations on domains or surfaces that evolve in time which is presently being vastly studied, since these have been seen to provide more realistic models for physical and biological phenomena. Some examples are [7, 25, 30, 32, 35, 51]. Not only are these relevant for applications, but they also raise interesting modelling, numerical and computational questions, as well as challenging problems from the point of view of mathematical analysis. We refer in particular to [29] for a detailed exposition of the numerical analysis of such problems and to [4–6] for an abstract functional framework well-suited for the treatment of such problems.

The importance of establishing the strict separation property is twofold.

- First it is essential when one considers higher-order regularity of weak solutions, due to the behaviour of F and its derivatives close to ± 1. Indeed, note that

$$F'(r) = \frac{\theta}{2} \ln \frac{1+r}{1-r} - \theta_0 r \quad \text{and} \quad F''(r) = \frac{\theta}{2} \left(\frac{1}{1+r} + \frac{1}{1-r} \right) - \theta_0$$

are both singular when $r \to \pm 1$, and even though the structure of F' can be exploited in order to obtain some estimates, it is more challenging to do so for F''. As noted in, e.g., [36], we have

$$F''(r) \leq C e^{C|F'(r)|}, \tag{1.8}$$

which precludes us from controlling F'' in L^p-spaces in terms of the L^p-norms of F'. It is then important to study conditions that ensure the integrability of F''. It is clear that establishing strict separation from the pure phases as in (1.5) is crucial to achieve this. Furthermore, the strict separation property is a fundamental ingredient in the study of longtime behavior of solutions (see [36, 43]).

- Secondly, if the strict separation property holds then, being the solution away from ± 1, the logarithmic potential F is smooth and can be dominated by a polynomial. Therefore, this result can be viewed as a rigorous justification of the usual aforementioned polynomial double-well approximation.

Separation from the pure phases for dimensions $d \geq 3$ is unknown even in the fixed domain setting, in the case of constant mobility. This apparently technical restriction is related to the growth condition (1.8) (see [34] for a detailed analysis). As a consequence, our results are restricted to two-dimensional surfaces. The strict separation for $d = 2$ was first established for (1.1) in [43]. Then, a more general argument was introduced in [36]. For an up-to-date picture of the state-of-the-art, the reader is referred to [34], where new proofs and further generalizations are given.

Here, not only we extend the result for two-dimensional planar domains for equation (1.1), but we also prove the first separation result for the weighted model (1.2).

This article on one hand complements [13] by proving instantaneous regularisation of the weak solutions and on the other hand extends the validity of the strict separation property for the local Cahn-Hilliard equation with constant mobility to the setting of evolving surfaces in $\mathbb{R}^3$ in two cases. It is also worth observing that our approach to the estimates for solutions to the approximate problems allows us to forgo some of the assumptions made in [13] (see, e.g., [13, Assumption A_P]), generalising the results therein. For the sake of completeness, here we also include the proof of continuous dependence on the initial data, entailing uniqueness for the general system (1.2), which was omitted in [13].

The paper is structured as follows. In Section 2 we briefly recall the setting from [13] on the evolution of the surfaces and state some additional regularity assumptions. Section 3 is devoted to the analysis of the first system (1.1) and we establish higher-order regularity for the solution u and prove that it satisfies the strict separation property. Finally, in Section 4 we introduce a better suited Galerkin approximation for the alternative weighted model (1.2) and prove analogous regularity results as well as the strict separation property. We also include three appendices for the sake of completeness: in Appendix A we present the proofs of some propositions which are only stated in the main body of the paper. Appendix B collects some preliminary results essential to obtain the control of higher-order derivatives of the logarithmic potential (in particular, we report a Moser-Trudinger type inequality valid on compact Riemannian manifolds and a generalized Young's inequality), whereas in Appendix C we show the validity of a well-known embedding result for Bochner spaces also in the evolving space setting.

2 Surface motion: assumptions

We refer to the setting and the notation of [23]. To be more precise, we consider $T > 0$ and a C^2-evolving surface $\{\Gamma(t)\}_{t\in[0,T]}$ in $\mathbb{R}^3$, i.e. a closed, connected, orientable C^2-surface Γ_0 in $\mathbb{R}^3$ together with a smooth *flow map*

$$\Phi\colon [0,T]\times\Gamma_0 \to \mathbb{R}^3$$

such that

(i) denoting $\Gamma(t) := \Phi_t^0(\Gamma_0)$, the map

$$\Phi_t^0 := \Phi(t,\cdot)\colon \Gamma_0 \to \Gamma(t)$$

is a C^2-diffeomorphism, with inverse map

$$\Phi_0^t\colon \Gamma(t) \to \Gamma_0;$$

(ii) $\Phi_0^0 = \mathrm{id}_{\Gamma_0}$.

It follows from the definition above that, for each $t \in [0,T]$, $\Gamma(t)$ is also a closed, connected, orientable C^2-surface. In addition to the assumptions on the surface motion, we first recall the same hypothesis as [13, $\mathbf{A}_\Phi$]:

Assumption $\mathbf{A}_\Phi$: There exists a velocity field $\mathbf{V} : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ with regularity

$$\mathbf{V} \in C^0([0,T]; C^2(\mathbb{R}^3;\mathbb{R}^3)),$$

such that, for any $t \in [0,T]$ and every $x \in \Gamma_0$,

$$\frac{d}{dt}\,\Phi_t^0(x) = \mathbf{V}\left(t, \Phi_t^0(x)\right), \quad \text{in } [0,T] \tag{2.1}$$

$$\Phi_0^0(x) = x. \tag{2.2}$$

In addition to $\mathbf{A}_\Phi$ we will also suppose, where necessary, that the velocity field $\mathbf{V}\colon [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ satisfies

Assumptions $\mathbf{B}_\Phi$:

1. $\mathbf{V}$ is such that

$$\mathbf{V} \in C^1([0,T]; C^2(\mathbb{R}^3,\mathbb{R}^3)); \tag{2.3}$$

2. The advective tangential velocity $\mathbf{V}_a$ is such that

$$\mathbf{V}_a \in C^1([0,T]; C^1(\mathbb{R}^3,\mathbb{R}^3)). \tag{2.4}$$

Note that the above assumptions $\mathbf{B}_\Phi$ require more regularity in time for the map $\Phi_0^{(\cdot)}$ than $\mathbf{A}_\Phi$. Denoting by $\mathbf{V}_\tau$ and $\mathbf{V}_\nu$ the tangential and normal components of $\mathbf{V}$, respectively, Assumptions $\mathbf{B}_\Phi$ imply, in particular, for $t \in [0,T]$,

$$\begin{aligned} &\|\mathbf{V}_\tau(t)\|_{C^2(\Gamma(t))}, \|\mathbf{V}_\nu(t)\|_{C^2(\Gamma(t))} \le \|\mathbf{V}(t)\|_{C^2(\Gamma(t))} \le C_{\mathbf{V}}, \\ &\|\mathbf{V}_a(t)\|_{C^0(\Gamma(t))}, \|\partial^\bullet \mathbf{V}_a^\tau(t)\|_{C^0(\Gamma(t))} \le C_{\mathbf{V}}, \end{aligned} \tag{2.5}$$

for $t \in [0,T]$ and for some $C_{\mathbf{V}} > 0$ independent of time, where (see [13]) $\mathbf{V}_a^\tau := \mathbf{V}_\tau - \mathbf{V}_a$. whereas the single extra Assumption $\mathbf{B}_\Phi$.2. implies

$$\|\partial^\bullet \mathbf{V}_a(t)\|_{C^0(\Gamma(t))} \leq C_{\mathbf{V}}, \tag{2.6}$$

for $t \in [0,T]$ and for some $C_{\mathbf{V}} > 0$ independent of time.

In this setting we can define the normal material time derivative of a scalar quantity u on $\Gamma(t)$ by

$$\partial^\circ u := \hat{u}_t + \nabla \hat{u} \cdot \mathbf{V}_\nu,$$

where $\hat{u}$ denotes any extension of u to a (space-time) neighbourhood of $\Gamma(t)$, and its full material time derivative, or strong time derivative, as

$$\partial^\bullet u := \partial^\circ u + \nabla_\Gamma u \cdot \mathbf{V}_\tau = \hat{u}_t + \nabla \hat{u} \cdot \mathbf{V}, \tag{2.7}$$

These definitions take into account not only the evolution of the quantity u but also the movement of points in the surface. The definition in (2.7) can be abstracted to a weaker sense as follows. Let $u \in L^2_{H^1}$. A functional $v \in L^2_{H^{-1}}$ is said to be the weak time derivative of u, and we write $v = \partial^\bullet u$, if, for any $\eta \in \mathcal{D}_{H^1}(0,T)$, we have

$$\int_0^T \langle v(t), \eta(t)\rangle_{H^{-1}\times H^1} = -\int_0^T (u(t), \partial^\bullet \eta(t))_{L^2} - \int_0^T \int_{\Gamma(t)} u(t)\eta(t)\nabla_{\Gamma(t)} \cdot \mathbf{V}(t). \tag{2.8}$$

Observe that $\partial^\bullet \eta$ is the strong material derivative of η. We will use $\partial^\bullet u$ for both the strong and weak material derivatives.

Thanks to [24, Lemma 2.6], we observe that, for a sufficiently regular vector field $\mathbf{f}$, there holds

$$\partial^\bullet(\nabla_\Gamma \cdot \mathbf{f}) = \partial^\bullet \underline{D}_l^{\Gamma(t)} \mathbf{f}_l = \underline{D}_l^{\Gamma(t)} \partial^\bullet \mathbf{f}_l - A_{lr}(\mathbf{V}) \underline{D}_r^{\Gamma(t)} \mathbf{f}_l, \quad l,r = 1,2,3.$$

Here $\underline{D}_i^{\Gamma(t)}$, $i = 1,2,3$, is the i^{th} component of the tangential gradient and

$$A_{lr}(\mathbf{V}) = \underline{D}_l^{\Gamma(t)} \mathbf{V}_r - \nu_s \nu_l \underline{D}_r^{\Gamma(t)} \mathbf{V}_s, \quad s = 1,2,3,$$

with ν standing for the normal vector field to the surface. Therefore, by (2.3) we can set $\mathbf{f} = \mathbf{V}$ in the previous equality and deduce that, due to the regularity of $\Gamma(t)$, we also have

$$\|\partial^\bullet \nabla_\Gamma \cdot \mathbf{V}(t)\|_{C^0(\Gamma(t))} \leq C_{\mathbf{V}}, \quad \text{for any } t \in [0,T]. \tag{2.9}$$

Moreover, denoting by J_t^0 and J_0^t, the change of area element from Γ_0 to $\Gamma(t)$ and the one from $\Gamma(t)$ to Γ_0, respectively, we have, for any $\eta : \Gamma(t) \to \mathbb{R}$,

$$\int_{\Gamma(t)} \eta d\Gamma = \int_{\Gamma_0} \widetilde{\eta} J_t^0 d\Gamma_0,$$

with $\widetilde{\eta}(p) = \eta(\Phi_t^0(p))$, for any $p \in \Gamma_0$. Note that $J_t^0(\cdot) = \left|\mathrm{det} D_{\Gamma_0} \Phi_t^0(\cdot)\right|$. Then, due to the previous assumptions, we deduce

$$\frac{1}{C_J} \leq \|J_t^0\|_{C^0(\Gamma(t))} \leq C_J, \quad \text{for any } t \in [0,T], \tag{2.10}$$

where C_J can be chosen to be independent of t. This also implies that there exists a constant $C_\Gamma > 0$, independent of t, such that

$$\frac{|\Gamma_0|}{C_\Gamma} \leq |\Gamma(t)| \leq C_\Gamma |\Gamma_0|, \tag{2.11}$$

for any $t \in [0,T]$. Here $|\Gamma_0|$ stands for the Lebesgue surface measure. For any integrable function v over a surface Γ of positive measure, we set

$$(v)_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma v.$$

General agreement. The symbol $C > 0$ will denote a generic constant, depending only on the structural parameters of the problem and T, but independent of time t and of the approximating indices δ, M (unless otherwise specified).

3 The first model

In this section we consider system (1.1), which is a model proposed, e.g., in [46] and [13, Problem 4.1]. We start by briefly recalling its derivation.

3.1 Derivation

Fix $t \in [0,T]$ and consider a scalar quantity $u = u(t)\colon \Gamma(t) \to \mathbb{R}$, to be thought of, in our context, as the concentration difference between two immiscible substances present in a mixture on the surface. The main property for u is that its total mass is conserved, so that (1.3) holds. Starting from the balance law

$$\frac{d}{dt}\int_{P(t)} u = -\int_{\partial P(t)} q \cdot \mu,$$

on every portion $P(t) \subset \Gamma(t)$ evolving under the purely normal velocity $\mathbf{V}_\nu$, where q is a flux to be defined later on, we are led to

$$\int_{P(t)} \partial^\circ u + u \operatorname{div}_\Gamma \mathbf{V}_\nu + \operatorname{div}_\Gamma q = 0.$$

Since this holds for every region we obtain the equation

$$\partial^\circ u + u \operatorname{div}_\Gamma \mathbf{V}_\nu + \operatorname{div}_\Gamma q = 0 \quad \text{on } \Gamma(t).$$

Now, the flux q consists of an advective term $q_a = u\mathbf{V}_a$, where $\mathbf{V}_a$ is tangential and represents the particle velocity in the fluid, and a diffusive part $q_d = -\nabla_\Gamma(-\Delta_\Gamma u + F'(u))$. Substituting in the equation above leads to

$$\partial^\circ u + u \operatorname{div}_\Gamma \mathbf{V}_\nu + \operatorname{div}_\Gamma (u\mathbf{V}_a) = \Delta_\Gamma \left(-\Delta_\Gamma u + F'(u)\right). \tag{3.1}$$

This is the *physical* equation that we wish to consider. In order to apply the functional framework developed in [4], we work under Assumption $\mathbf{A}_\Phi$. To incorporate the tangential component $\mathbf{V}_\tau$, which is to be interpreted as an *arbitrary parametrisation* of the surface $\Gamma(t)$, we add and subtract a term $\operatorname{div}_\Gamma(u\mathbf{V}_\tau)$ to (3.1), resulting in

$$\partial^\bullet u + u \operatorname{div}_\Gamma \mathbf{V} + \operatorname{div}_\Gamma (u(\mathbf{V}_a - \mathbf{V}_\tau)) = \Delta_\Gamma \left(-\Delta_\Gamma u + F'(u)\right), \tag{3.2}$$

or equivalently

$$\begin{aligned} \partial^\bullet u + u \operatorname{div}_\Gamma \mathbf{V} + \operatorname{div}_\Gamma (u(\mathbf{V}_a - \mathbf{V}_\tau)) &= \Delta_\Gamma w, \\ -\Delta_\Gamma u + F'(u) &= w. \end{aligned}$$

3.2 Weak formulation

In order to introduce the variational formulation of the Cahn-Hilliard systems we need to introduce the abstract framework of [4, 5] for the definition of the time-dependent function spaces. We do it in a summarised way and refer the reader to [4, 5] for a more rigorous and detailed explanation of how these are constructed and how they can be abstracted to a more generalised setting.

We use the flow map Φ_t^0 to define pullback and pushforward operators

$$\begin{aligned} \phi_{-t} u &= u \circ \Phi_t^0 \colon \Gamma_0 \to \mathbb{R} \quad \text{for } u\colon \Gamma(t) \to \mathbb{R}, \\ \phi_t v &= v \circ \Phi_0^t \colon \Gamma(t) \to \mathbb{R} \quad \text{for } v\colon \Gamma_0 \to \mathbb{R} \end{aligned}$$

and suppose, for $t \in [0,T]$, $X(t)$ to be a Banach space of functions over $\Gamma(t)$; typically $L^2(\Gamma(t))$ or some higher-order Sobolev space $H^k(\Gamma(t))$, but also other $L^p(\Gamma(t))$ or even the dual space $H^{-1}(\Gamma(t)) := (H^1(\Gamma(t))^*$. We consider functions of the form

$$u\colon [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\}, \quad t \mapsto (\overline{u}(t), t)$$

and identify $u(t) \equiv \overline{u}(t)$; in practice, we want to see $u(t)$ as an element of the space $X(t)$. The function spaces are defined as follows.

(i) For $p \in [1,\infty]$, $u \in L^p_X$ if $t \mapsto \phi_{-(\cdot)}\overline{u}(\cdot) \in L^p(0,T;X_0)$ with the norm

$$\|u\|_{L^p_X} := \begin{cases} \left(\int_0^T \|u(t)\|^p_{X(t)}\right)^{1/p} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_{X(t)} & \text{if } p = \infty \end{cases};$$

if $X(t) = H(t)$ are Hilbert spaces, then so is L^2_H with the inner product

$$(u,v)_{L^2_H} := \int_0^T (u(t), v(t))_{H(t)}.$$

(ii) For $k \in \mathbb{N} \cup \{0\}$, $u \in C^k_X$ if $t \mapsto \phi_{-t} u(t) \in C^k([0,T]; X_0)$ and we define its time derivatives as

$$\partial^{\bullet,s} u(t) = \phi_t \frac{d^s}{dt^s} \phi_{-t} u(t), \quad s = 1, \dots, k.$$

We will denote $\partial^{\bullet,1} = \partial^\bullet$; it is easy to see that the latter produces the same definition as in (2.7). These spaces are endowed with the norm

$$\|u\|_{C^k_X} := \sup_{t\in[0,T]} \|u(t)\|_{X(t)} + \sum_{s=1}^k \sup_{t\in[0,T]} \|\partial^{\bullet,s} u(t)\|_{X(t)}.$$

(iii) A test function $u \in \mathcal{D}_X(0,T)$ if $t \mapsto \phi_{-t} u(t) \in C^\infty_c((0,T); X_0)$.

(iv) We define the Banach space

$$H^1_{H^{-1}} := \left\{ u \in L^2_{H^1} : \partial^\bullet u \in L^2_{H^{-1}} \right\} \text{ with } \|u\|_{H^1_{H^{-1}}} := \|u\|_{L^2_{H^1}} + \|\partial^\bullet u\|_{L^2_{H^{-1}}}.$$

(v) Similarly, $H^1_{L^2}$ denotes the space of those $u \in L^2_{H^1}$ which have a more regular weak time derivative $\partial^\bullet u \in L^2_{L^2}$.

Remark 3.1. *We point out that in order to define the spaces L^p_X we are implicitly assuming that the family $(\phi_t, X(t))_{t\in[0,T]}$ is compatible in the sense of [13, Assumption 3.1]. Moreover, in the cases where $X(t)$ is a space of functions with some regularity, it is also assumed that the differentiation makes sense on $\Gamma(t)$. In our setting, these assumptions require appropriate regularity of the flow map Φ and the velocity field $\mathbf{V}$. More precisely, if for some $k \in \mathbb{N}$ we have*

$$\Phi^0_{(\cdot)}, \Phi^{(\cdot)}_0 \in C^1([0,T]; C^k(\mathbb{R}^3, \mathbb{R}^3)) \quad \text{and} \quad \mathbf{V} \in C^0([0,T]; C^k(\mathbb{R}^3, \mathbb{R}^3)),$$

then:

(i) $\Gamma(t)$ is a C^k-surface, and it makes sense to define strong and weak derivatives up to order k for functions on $\Gamma(t)$ (in our case we have $k = 2$);

(ii) the pair $(\phi_t, X(t))_t$ is compatible where $X = L^p$, C^0, H^{-1}, $W^{r,p}$, C^r for $p \in [1,\infty]$ and $1 \le r \le k$ (see e.g. [13, Section 7] for examples).

Notice that the assumptions in Sec.2 are enough to guarantee that all the function spaces involved are well defined, as well as to ensure compatibility with the evolution.

Let us now explicitly recall some notation introduced in [13, Section 3.2]. In particular, for $t \in [0,T]$, we define the following bilinear forms:

1. for $\eta, \phi \in L^2(\Gamma(t))$, the zero order terms

$$m(t;\eta,\phi) := \int_{\Gamma(t)} \eta\phi, \quad g(t;\eta,\phi) := \int_{\Gamma(t)} \eta\phi \nabla_\Gamma \cdot \mathbf{V}(t);$$

2. for $\eta \in L^2(\Gamma(t))$, $\phi \in H^1(\Gamma(t))$, the first order term

$$a_N(t;\eta,\phi) := \int_{\Gamma(t)} \eta \mathbf{V}^\tau_a(t) \cdot \nabla_\Gamma \phi;$$

3. for $\eta, \phi \in H^1(\Gamma(t))$, the second order terms

$$a_S(t;\eta,\phi) := \int_{\Gamma(t)} \nabla_\Gamma \eta \cdot \nabla_\Gamma \phi, \quad b(t;\eta,\phi) := \int_{\Gamma(t)} B(\mathbf{V}(t)) \nabla_\Gamma \eta \cdot \nabla_\Gamma \phi,$$

with $B(\mathbf{V}) = (\nabla_\Gamma \cdot \mathbf{V})\mathbf{I} - 2\mathbf{D}(\mathbf{V})$, where $\mathbf{D}(\mathbf{V})$ stands for the symmetrized rate tensor;

4. for $\eta \in H^{-1}(\Gamma(t))$, $\phi \in H^1(\Gamma(t))$, the duality pairing

$$m_\star(t;\eta,\phi) := \langle \eta, \phi \rangle_{H^{-1}(\Gamma(t)), H^1(\Gamma(t))}.$$

For the sake of simplicity, from now on we will omit the explicit dependence on time and we will denote $\|\cdot\|_{L^2(\Gamma(t))}$ only by $\|\cdot\|$. In conclusion, we recall [13, Prop.2.8], see also [23, Sec.8.2]) in

Proposition 3.2. *The following transport formulas hold:*

(a) For $\eta, \phi \in H^1_{H^{-1}}$,

$$\frac{d}{dt} m(\eta, \phi) = m_*(\partial^\bullet \eta, \phi) + m_*(\partial^\bullet \phi, \eta) + g(\eta, \phi).$$

(b) If additionally $\nabla_\Gamma \partial^\bullet \eta, \nabla_\Gamma \partial^\bullet \phi \in L^2_{L^2}$, then

$$\frac{d}{dt} a_S(\eta, \phi) = a_S(\partial^\bullet \eta, \phi) + a_S(\eta, \partial^\bullet \phi) + b(\eta, \phi).$$

(c) For $\eta, \phi \in H^1_{L^2}$, with $\nabla_\Gamma \partial^\bullet \phi \in L^2_{L^2}$, the following identity holds

$$\begin{aligned}\frac{d}{dt} a_N(\eta, \phi) &= a_N(\partial^\bullet \eta, \phi) + a_N(\eta, \partial^\bullet \phi) \\ &\quad + \int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t), \mathbf{V}(t)) \eta \cdot \nabla_\Gamma \phi,\end{aligned} \tag{3.3}$$

for almost any $t \in [0, T]$, where B_{adv} is a vector field given by

$$B_{adv}(\mathbf{V}_a^\tau, \mathbf{V})_i := (\partial^\bullet \mathbf{V}_a^\tau)_i + (\nabla_\Gamma \cdot \mathbf{V})(\mathbf{V}_a^\tau)_i - \sum_{j=1}^{3} (\mathbf{V}_a^\tau)_j \underline{D}_j^{\Gamma(t)} \mathbf{V}_i, \quad i = 1, 2, 3.$$

In the above we use the same notation as in [29]. We postpone the proof of this result to Appendix A.

Let us now recall the compatibility condition on the initial datum u_0 in order to obtain the existence of a (weak) solution to the Cahn-Hilliard equation with logarithmic potential. As noted in [13], there is an interplay between the evolution of the surfaces $\Gamma(t)$ and the admissible initial conditions. In effect, setting

$$m_\eta(t) := \frac{1}{|\Gamma(t)|} \left| \int_{\Gamma_0} \eta \right|,$$

we require that the initial datum u_0 satisfies

$$\max_{t \in [0,T]} m_{u_0}(t) < 1. \tag{3.4}$$

Although it might seem unnatural to prescribe a condition involving the initial data and the area of the surfaces at all future times, this assumption has an interesting physical meaning. Let us introduce the maximum shrinkage ratio S_R on $[0, T]$, which is *a priori* prescribed by the evolution of the surfaces $\Gamma(t)$, as:

$$S_R := \max_{t \in [0,T]} \frac{|\Gamma_0|}{|\Gamma(t)|}.$$

Condition (3.4) then means that

$$|(u_0)_{\Gamma_0}| S_R < 1. \tag{3.5}$$

Since at time $t = 0$ we already assume $|(u_0)_{\Gamma_0}| < 1$ for the initial mass (meaning that we do not start with a one-phase mixture), condition (3.5) shows that the absolute value of the initial mass, $|(u_0)_{\Gamma_0}|$, compensates for the maximum shrinkage ratio S_R of the evolving surface $\Gamma(t)$ on $[0, T]$. Therefore, the higher the value of S_R (i.e., the smaller the surface $\Gamma(t)$ becomes over $[0, T]$), the further u_0 must be from the pure phases ± 1 in Γ_0.

Moreover, recalling that the Cahn-Hilliard dynamics implies that the total mass of the solution is preserved, namely,

$$\int_{\Gamma(t)} u(t) \equiv \int_{\Gamma_0} u_0, \quad \text{for all } t \in [0, T],$$

we can write

$$|(u(t))_{\Gamma(t)}| = m_{u_0}(t)$$

and thus the condition $m_{u_0}(t) < 1$ for every $t \in [0, T]$ is the dynamic counterpart of

$$|(u(t))_\Omega| = \left| \frac{1}{|\Omega|} \int_\Omega u(t)\, dx \right| = \left| \frac{1}{|\Omega|} \int_\Omega u_0\, dx \right| = |(u_0)_\Omega| < 1$$

which holds in the case of a static bounded domain Ω.

For future use, let us then define the set $\mathcal{I}_0$ of admissible initial conditions as

$$\mathcal{I}_0 := \left\{ \eta \in H^1(\Gamma_0) : |\eta| \le 1 \text{ a.e. on } \Gamma_0, \text{ and } |(\eta)_{\Gamma_0}|\, S_R < 1 \right\}.$$

We recall that the free energy is given by

$$E^{CH}[u] = \int_{\Gamma(t)} \left(\frac{|\nabla u|^2}{2} + F(u) \right),$$

with F logarithmic potential. Note that, if $u_0 \in \mathcal{I}_0$, then u_0 has finite energy. Indeed $\|u_0\|_{L^\infty(\Gamma_0)} \le 1$ implies $F(u_0) \in L^1(\Gamma_0)$. We can now recall the result obtained in [13, Theorems 5.14, 5.15] for

$$F(s) = \frac{\theta}{2}((1+r)\ln(1+r) + (1-r)\ln(1-r)) + \frac{1-r^2}{2}, \quad r \in (-1,1), \tag{3.6}$$

where $0 < \theta < 1$ (F is extended by continuity at $r = \pm 1$). We set $F_{ln}(r) := (1+r)\ln(1+r) + (1-r)\ln(1-r)$ for the sake of simplicity and denote $\varphi(s) = F'_{ln}(s)$. Note that this choice corresponds to the singular potential (1.7) with $\theta_0 = 1$, up to a translation to get directly $F \ge 0$ in $[-1,1]$.

Remark 3.3. *Concerning the logarithmic potential, there exists a constant $C > 0$ such that*

$$\frac{\theta}{2}\varphi'(s) \le e^{C\left|\frac{\theta}{2}\varphi(s)\right| + C}, \quad \forall s \in (-1,1). \tag{3.7}$$

We now let Assumption $\mathbf{A}_\Phi$ hold: we have (see [13])

Theorem 3.4. *Let $u_0 \in \mathcal{I}_0$ and $F : [-1,1] \to \mathbb{R}$ be given by (3.6). Then there exists a unique pair (u,w) with*

$$u \in L^\infty_{H^1} \cap H^1_{H^{-1}} \quad \text{and} \quad w \in L^2_{H^1},$$

such that, for almost any $t \in (0,T]$, $|u(t)| < 1$ almost everywhere on $\Gamma(t)$ and u satisfies, for all $\eta \in L^2_{H^1}$ and almost any $t \in [0,T]$,

$$m_\star(\partial^\bullet u, \eta) + g(u,\eta) + a_N(u,\eta) + a_S(w,\eta) = 0, \tag{3.8}$$

$$a_S(u,\eta) + \frac{\theta}{2} m(\varphi(u),\eta) - m(u,\eta) - m(w,\eta) = 0, \tag{3.9}$$

where $u(0) = u_0$ almost everywhere in Γ_0. The solution u also satisfies the additional regularity

$$u \in C^0_{L^2} \cap L^\infty_{L^p} \cap L^2_{H^2},$$

for all $p \in [1,+\infty)$. Furthermore, if $u_0, v_0 \in \mathcal{I}_0$ are such that $(u_0)_{\Gamma_0} = (v_0)_{\Gamma_0}$, and u, v are the solutions of the system with $u(0) = u_0$ and $v(0) = v_0$, then there exists a constant $C > 0$ independent of t, such that, for almost any $t \in [0,T]$,

$$\|u(t) - v(t)\|_{H^{-1}(\Gamma(t))} \le e^{Ct}\|u_0 - v_0\|_{H^{-1}(\Gamma(t))}.$$

Remark 3.5. *We notice that the regularity stated in Theorem 3.4 can be slightly improved. In particular, since $u \in L^2_{H^2}$ solves the problem, for almost all $t \in [0,T]$,*

$$-\Delta_\Gamma u(t) = w(t) - F'(u(t)) \in L^2(\Gamma(t)),$$

we are allowed to multiply by $-\Delta_\Gamma u \in L^2(\Gamma(t))$ for almost any $t \in [0,T]$. Recalling that $\varphi' > 0$, after an integration by parts, being $\Gamma(t)$ closed and $u \in L^\infty_{H^1}$, we obtain

$$\|\Delta_\Gamma u\|^2 \le \|\Delta_\Gamma u\|^2 + \frac{\theta}{2} m(\varphi'(u), |\nabla_\Gamma u|^2) \le \|\nabla_\Gamma w\|\|\nabla_\Gamma u\| + \|\nabla_\Gamma u\|^2 \le C(1 + \|\nabla_\Gamma w\|),$$

and knowing that $w \in L^2_{H^1}$, we infer $u \in L^4_{H^2}$.

3.3 Regularisation and strict separation property

In this subsection we need Assumptions $\mathbf{B}_\Phi$. We first show that the approximating solution given by the Galerkin scheme is consequently more regular. This allows us to perform higher-order regularisation estimates and to establish the strict separation property.

3.3.1 Galerkin approximation

Motivated by the approach in [13], we start by considering the Galerkin approximation with the approximated potential $F^\delta \in C^2(\mathbb{R})$ defined by

$$F^\delta(r) := F^\delta_{ln}(r) + \frac{1-r^2}{2}, \quad \forall r \in \mathbb{R},$$

where

$$F^\delta_{ln}(r) = \frac{\theta}{2}\begin{cases} (1-r)\ln(\delta) + (1+r)\ln(2-\delta) + \frac{(1-r)^2}{2\delta} + \frac{(1+r)^2}{2(2-\delta)} - 1, & r \geq 1-\delta \\ (1+r)\ln(1+r) + (1-r)\ln(1-r), & |r| \leq 1-\delta \\ (1+r)\ln(\delta) + (1-r)\ln(2-\delta) + \frac{(1+r)^2}{2\delta} + \frac{(1-r)^2}{2(2-\delta)} - 1, & r \leq -1+\delta \end{cases}$$

and denote $\varphi^\delta = (F^\delta_{ln})'$. Let us recall [13, Sec.4.1] and consider a basis $\{\chi^0_j : j \in \mathbb{N}\}$ orthonormal in $L^2(\Gamma_0)$ and orthogonal in $H^1(\Gamma_0)$ consisting of smooth functions such that χ^0_1 is constant (for example consider the eigenfunctions of the Laplace-Beltrami operator). We then transport this basis using the flow map. This gives $\{\chi^t_j := \phi_t(\chi^0_j) : j \in \mathbb{N}\} \subset H^1(\Gamma(t))$ and we define the finite-dimensional spaces

$$V_M(t) := \{\chi^t_j \colon j = 1, \dots, M\}. \tag{3.10}$$

The goal in this section is to find an approximating solution pair (u^M, w^M) in these spaces; more precisely, for each $M \in \mathbb{N}$, we aim to find functions $u^M, w^M \in L^2_{V_M}$ with $\partial^\bullet u^M \in L^2_{V_M}$ such that, for any $\eta \in L^2_{V_M}$ and all $t \in [0,T]$,

$$m(\partial^\bullet u^M, \eta) + g(u^M, \eta) + a_N(u^M, \eta) + a_S(w^M, \eta) = 0, \tag{3.11}$$

$$a_S(u^M, \eta) + m((F^\delta)'(u^M), \eta) - m(w^M, \eta) = 0, \tag{3.12}$$

and $u^M(0) = P^0_M u_0$ almost everywhere in Γ_0 (P^0_M is the L^2 orthogonal projector operator at $t = 0$ (see [13, Sec.4.1]). We first prove a refinement of [13, Prop.4.4], namely,

Proposition 3.6. *Let Assumptions $\boldsymbol{B}_\Phi$ hold. Then there exists a unique local solution pair to (3.11)-(3.12). In particular there exist functions (u^M, w^M) satisfying (3.11)-(3.12) on an interval $[0, t^\star)$, $0 \leq t^\star \leq T$, together with the initial condition $u^M(0) = P^0_M u_0$. The functions are of the form*

$$u^M(t) = \sum_{i=1}^M u^M_i(t)\chi^t_i, \qquad w^M(t) = \sum_{i=1}^M w^M_i(t)\chi^t_i, \qquad t \in [0, t^\star),$$

with $u^M_i \in C^2([0, t^\star))$ and $w^M_i \in C^2([0, t^\star))$, for every $i \in \{1, \dots, M\}$.

Proof. We consider the matrix form of the equations, where we denote $\mathbf{u}^M(t) = (u^M_1(t), \dots, u^M_M(t))$ and $\mathbf{w}^M(t) = (w^M_1(t), \dots, w^M_M(t))$,

$$\begin{aligned} &M(t)\dot{\mathbf{u}}^M(t) + G(t)\mathbf{u}^M(t) + A_N(t)\mathbf{u}^M(t) + A_S(t)\mathbf{w}^M(t) = 0, \\ &A_S(t)\mathbf{u}^M(t) + (\mathbf{F}^\delta)'(\mathbf{u}^M(t)) - M(t)\mathbf{w}^M(t) = 0. \end{aligned}$$

Here

$$\begin{aligned} &(M(t))_{ij} = m(t; \chi^t_i, \chi^t_j), \qquad (G(t))_{ij} = g(t; \chi^t_j, \chi^t_i), \\ &(A_S(t))_{ij} = a_S(t; \chi^t_i, \chi^t_j), \qquad (A_N(t))_{ij} = a_N(t; \chi^t_j, \chi^t_i), \end{aligned}$$

and

$$(\mathbf{F}^\delta)'(\mathbf{u}^M(t))_j = m(t; (F^\delta)'(u^M(t)), \chi^t_j).$$

We now observe that actually these matrices enjoy more regularity than noted in [13]. Indeed, let us consider for example M_{ij}. By [13, Prop. 2.8], recalling that $\partial^\bullet \chi^t_i \equiv 0$ for any $i = 1, \dots, M$, we have

$$\frac{d}{dt} M_{ij} = g(\chi^t_i, \chi^t_j) \in C^0([0,T]),$$

due to the regularity assumptions on $\Gamma(t)$, on the corresponding flow map and on the velocity field. We also have

$$\frac{d}{dt}(M^{-1})_{ij} = -\left(M^{-1}\left(\frac{d}{dt}M\right)M^{-1}\right)_{ij} \in C^0([0,T]).$$

Similarly, we get

$$\frac{d}{dt}(A_S(t))_{ij} = b(\chi_i^t, \chi_j^t) \in C^0([0,T]),$$

and, by (3.3),

$$\frac{d}{dt}(A_N(t))_{ij} = \int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t), \mathbf{V}(t))\chi_j^t \cdot \nabla_\Gamma \chi_i^t \in C^0([0,T]).$$

Moreover, by Proposition 3.2 point (a),

$$\frac{d}{dt}(G(t))_{ij} = m(\chi_i^t \partial^\bullet(\nabla_\Gamma \cdot \mathbf{V}), \chi_j^t) + g\left(\chi_i^t \nabla_\Gamma \cdot \mathbf{V}, \chi_j^t\right) \in C^0([0,T]),$$

thanks to (2.9). Furthermore, we get

$$\frac{d}{dt}(\mathbf{F}^\delta)'(\mathbf{y})_j = g\left((F^\delta)'\left(\sum_{i=1}^M y_i \chi_i^t\right), \chi_j^t\right) \in C^0([0,T]).$$

Recalling then that $(F^\delta)'$ is $C^{1,1}(\mathbb{R})$, ensuring the spatial $C^{1,1}$-continuity of the $\mathbf{y}$-Jacobian of $\mathbf{y} \mapsto (\mathbf{F}^\delta)'(\mathbf{y})$, the result follows from the general ODE theory. $\square$

We now obtain the energy estimate for the Galerkin approximation, without letting $M \to \infty$ (cf. [13]). This is essential in order to maintain the necessary regularity to perform higher-order estimates.

Proposition 3.7. *Denoting by (u_δ^M, w_δ^M) the solution pair to (3.11)-(3.12), we have the energy estimate*

$$\sup_{t\in[0,T]} \tilde{E}^{CH}[t; u_\delta^M(t)] + \int_0^T \|\nabla_\Gamma w_\delta^M\|^2 dt \leq C_\delta(T), \tag{3.13}$$

for some $C_\delta(T)$ depending on δ and T, but not on M, where $\tilde{E}^{CH} := E^{CH} + \tilde{C}$ for some $\tilde{C} > 0$ chosen so that $\tilde{E}^{CH} \geq 0$. Moreover, it holds

$$\sup_{t\in[0,T]} \tilde{E}^{CH}[u_\delta^M(t)] + \frac{1}{4}\int_0^T \|\nabla_\Gamma w_\delta^M\|^2 \leq C(T)\left(1 + \int_0^T |m(w_\delta^M, 1)|^2 dt\right), \tag{3.14}$$

for some $C(T) > 0$ independent of M, δ but possibly depending on T.

Remark 3.8. *We observe that, even though the energy E^{CH} is not necessarily non-negative, it is bounded from below. Indeed, there exists $C = C(\theta)$ such that $F \geq C(\theta)$, and therefore*

$$E^{CH}[u] \geq C(\theta)|\Gamma(t)| \geq \frac{C(\theta)}{C_\Gamma}|\Gamma_0|, \tag{3.15}$$

where C_Γ is defined in (2.11). Note that $\tilde{C}$ (see the previous statement) depends on T, Γ_0 and θ, as well as on an upper bound for the C^1-norm of the flow map Φ.

Proof. Keep M, δ fixed and denote by C a generic positive constant independent of M and δ. We consider the Galerkin approximation and the results given by Proposition 3.6. Here we denote the solution by (u_δ^M, w_δ^M) to emphasize the dependence on δ. We then note that the total mass of u_δ^M is preserved in time, since $\eta \equiv 1 \in V_M$ for any $M \in \mathbb{N}$. We then come back to the energy estimate as in the proof of [13, Prop.5.7, a)]:

$$\frac{d}{dt}\tilde{E}^{CH}[u_\delta^M] + \frac{1}{2}\|\nabla_\Gamma w_\delta^M\|^2 \leq -g(u_\delta^M, w_\delta^M) + C_0 + C_1 \tilde{E}^{CH}[u_\delta^M], \tag{3.16}$$

where $\tilde{E}_{CH} := E_{CH} + \tilde{C}$, with $\tilde{C} > 0$ a suitable constant so that $\tilde{E}_{CH} \geq 0$. Thanks to the bound on $\mathbf{V}$, by Young and Poincaré's inequalities we have

$$\begin{aligned}
-g(u_\delta^M, w_\delta^M) &\leq C\|u_\delta^M\|\|w_\delta^M\| \leq C\|u_\delta^M\|(|m(w_\delta^M, 1)| + \|\nabla_\Gamma w_\delta^M\|) \\
&\leq \frac{1}{4}\|\nabla_\Gamma w_\delta^M\|^2 + C(\|u_\delta^M\|^2 + |m(w_\delta^M, 1)|^2) \\
&\leq \frac{1}{4}\|\nabla_\Gamma w_\delta^M\|^2 + C(1 + \|\nabla_\Gamma u_\delta^M\|^2 + |m(w_\delta^M, 1)|^2),
\end{aligned}$$

where we exploited the fact that, by mass conservation, Poincaré's inequality and (2.11),

$$\|u_\delta^M\| \leq \frac{1}{|\Gamma(t)|}|m(u_\delta^M, 1)| + \|\nabla_\Gamma u_\delta^M\| \leq C(|m(P_M^0 u_0, 1)| + \|\nabla_\Gamma u_\delta^M\|) \leq C(1 + \|\nabla_\Gamma u_\delta^M\|), \tag{3.17}$$

recalling that $\|P_M^0 u_0\| \le \|u_0\| \le C$. For further use, this gives

$$\frac{d}{dt}\tilde{E}^{CH}[u_\delta^M] + \frac{1}{4}\|\nabla_\Gamma w_\delta^M\|^2 \le C_0 + C_1\tilde{E}^{CH}[u_\delta^M] + C_2|m(w_\delta^M,1)|^2, \tag{3.18}$$

with $C_0, C_1, C_2 > 0$ independent of M, δ. Since estimate (3.46) cannot be performed in the Galerkin context (note that, differently from the static domain case, here it is not ensured that $\Delta_\Gamma u_\delta^M \in L^2_{V_M}$), we now first obtain an estimate depending on δ but not on M, and in a second time, after passing to limit as $M \to \infty$, we obtain estimates uniform in δ, thanks to (3.46). Therefore, we now exploit $|F_\delta'(u_\delta^M)| \le C_\delta |u_\delta^M|$, being $|F_\delta''| \le C_\delta$ and $F_\delta'(0) = F'(0) = 0$, to deduce

$$|m(w_\delta^M,1)| \le |m(F_\delta'(u_\delta^M),1)| \le C_\delta \|u_\delta^M\|_{L^1(\Gamma(t))} \le C_\delta \|u_\delta^M\| \tag{3.19}$$

with $C_\delta > 0$ depending on δ, so that, by Young and Poincaré's inequalities, we find

$$-g(u_\delta^M, w_\delta^M) \le C\|u_\delta^M\|\|w_\delta^M\| \le C_\delta\left(1 + \|u_\delta^M\|^2\right) + \frac{3}{8}\|\nabla_\Gamma w_\delta^M\|^2,$$

and thus, from (3.16) and (3.17),

$$\frac{d}{dt}\tilde{E}^{CH}[u_\delta^M] + \frac{1}{8}\|\nabla_\Gamma w_\delta^M\|^2 \le C_\delta \tilde{E}^{CH}[u_\delta^M] + C_\delta.$$

Therefore Gronwall's Lemma, together with (3.17) and [13, Lemma 5.6] give

$$\sup_{t\in[0,T]} \tilde{E}^{CH}[t; u_\delta^M(t)] + \sup_{t\in[0,T]} \|u_\delta^M(t)\|^2 + \int_0^T \|\nabla_\Gamma w_\delta^M\|^2 dt \le C_\delta, \tag{3.20}$$

independently of M. Moreover, by Gronwall's Lemma and [13, Lemma 5.6], applied to (3.18), we also get (3.14). $\square$

3.3.2 Regularisation and strict separation property

The main result of this section is the following, letting Assumptions $\mathbf{B}_\Phi$ hold.

Theorem 3.9. *Let the assumptions of Theorem 3.4 and Assumptions $\mathbf{B}_\Phi$ hold. Denote by (u, w) the (unique) weak solution to (3.8)-(3.9) with $u(0) = u_0$.*

(i) There exists a constant $C = C(T, E^{CH}(u_0)) > 0$ such that, for almost any $t \in [0,T]$,

$$t\|w\|^2_{H^1(\Gamma(t))} + \int_0^t s\|\partial^\bullet u\|^2_{H^1(\Gamma(s))} ds \le C(T, E^{CH}(u_0)). \tag{3.21}$$

(ii) For any $0 < \tau \le T$, there exist constants $C = C(T, \tau, E^{CH}(u_0)) > 0$ and $C_p = C_p(T, \tau, p, E^{CH}(u_0)) > 0$ such that, for almost any $t \in [\tau, T]$,

$$\|w\|_{H^1(\Gamma(t))} \le C(T, \tau, E^{CH}(u_0)), \tag{3.22}$$

$$\|\varphi(u)\|_{L^p(\Gamma(t))} + \|\varphi'(u)\|_{L^p(\Gamma(t))} \le C_p(T, \tau, p, E^{CH}(u_0)), \quad \forall p \in [2, \infty),$$

$$\|u\|_{H^2(\Gamma(t))} \le C(T, \tau, E^{CH}(u_0)).$$

(iii) There exists $\xi = \xi(T, \tau, E^{CH}(u_0)) > 0$ such that

$$\|u\|_{L^\infty(\Gamma(t))} \le 1 - \xi, \quad \text{for a.a. } t \in [\tau, T].$$

Remark 3.10. *Notice that actually we could say more about the separation property, if we assume the regularity on the flow map and the velocity field as in Lemma C.1. Indeed, for any $\tau > 0$, we have $u \in L^\infty_{H^2}(\tau, T)$ and $\partial^\bullet u \in L^2_{H^1}(\tau, T)$ (we use the symbol $L^q_X(\tau, T)$ to mean that the set of times is $[\tau, T]$ instead of $[0, T]$, typical of L^q_X). By the embedding result shown in Appendix C, we infer that $u \in C^0_{H^{3/2}}(\tau, T)$, thus by the embedding $H^{3/2}(\Gamma(t)) \hookrightarrow C^0(\Gamma(t))$, $u \in C^0_{C^0}(\tau, T)$, implying*

$$\sup_{t\in[\tau,T]} \|u\|_{C^0(\Gamma(t))} \le 1 - \xi.$$

Proof. From now on, we will omit the dependence on $E^{CH}(u_0)$, since it is considered understood due to Prop. 3.7.

Part (i). **Step 1. Limit as** $M \to \infty$. We test (3.11) with $\eta = \partial^\bullet w_\delta^M \in L^2_{V_M}$, obtaining

$$m(\partial^\bullet u_\delta^M, \partial^\bullet w_\delta^M) + g(u_\delta^M, \partial^\bullet w_\delta^M) + a_N(u_\delta^M, \partial^\bullet w_\delta^M) + a_S(w_\delta^M, \partial^\bullet w_\delta^M) = 0. \tag{3.23}$$

Observe now that, by Proposition 3.2, for $\eta \in L^2_{V_M}$ such that $\partial^\bullet \eta \in L^2_{V_M}$, we have

$$\frac{d}{dt} m(w_\delta^M, \eta) = m(\partial^\bullet w_\delta^M, \eta) + m(w_\delta^M, \partial^\bullet \eta) + g(w_\delta^M, \eta), \tag{3.24}$$

but also (see (3.12))

$$\frac{d}{dt} m(w_\delta^M, \eta) = \frac{d}{dt}\left(a_S(u_\delta^M, \eta) + m((F^\delta)'(u_\delta^M), \eta)\right) \tag{3.25}$$

Moreover, again by Proposition 3.2, we infer

$$\frac{d}{dt} a_S(u_\delta^M, \eta) = a_S(\partial^\bullet u_\delta^M, \eta) + a_S(u_\delta^M, \partial^\bullet \eta) + b(u_\delta^M, \eta).$$

On the other hand, exploiting the chain rule, we get

$$\frac{d}{dt} m((F^\delta)'(u_\delta^M), \eta) = m((F^\delta)''(u_\delta^M)\partial^\bullet u_\delta^M, \eta) + m((F^\delta)'(u_\delta^M), \partial^\bullet \eta) + g((F^\delta)'(u_\delta^M), \eta).$$

Therefore, comparing (3.24) with (3.25), we obtain

$$\begin{aligned} m(\partial^\bullet w_\delta^M, \eta) + m(w_\delta^M, \partial^\bullet \eta) + g(w_\delta^M, \eta) &= m((F^\delta)''(u_\delta^M)\partial^\bullet u_\delta^M, \eta) \\ &+ m((F^\delta)'(u_\delta^M), \partial^\bullet \eta) + g((F^\delta)'(u_\delta^M), \eta) + a_S(\partial^\bullet u_\delta^M, \eta) + a_S(u_\delta^M, \partial^\bullet \eta) + b(u_\delta^M, \eta), \end{aligned}$$

but since $\partial^\bullet \eta$ is still an admissible function in (3.12), we infer

$$m(\partial^\bullet w_\delta^M, \eta) = -g(w_\delta^M, \eta) + m((F^\delta)''(u_\delta^M)\partial^\bullet u_\delta^M, \eta) + g((F^\delta)'(u_\delta^M), \eta) + a_S(\partial^\bullet u_\delta^M, \eta) + b(u_\delta^M, \eta).$$

Thus, setting $\eta = \partial^\bullet u_\delta^M \in L^2_{V_M}$ (note that $\partial^{\bullet,2} u_\delta^M \in L^2_{V_M}$ by Proposition 3.6), we get

$$\begin{aligned} m(\partial^\bullet w_\delta^M, \partial^\bullet u_\delta^M) &= -g(w_\delta^M, \partial^\bullet u_\delta^M) + m((F^\delta)''(u_\delta^M)\partial^\bullet u_\delta^M, \partial^\bullet u_\delta^M) \\ &+ g((F^\delta)'(u_\delta^M), \partial^\bullet u_\delta^M) + a_S(\partial^\bullet u_\delta^M, \partial^\bullet u_\delta^M) + b(u_\delta^M, \partial^\bullet u_\delta^M). \end{aligned} \tag{3.26}$$

Using once more Proposition 3.2, on account of the regularity given by Proposition 3.6, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla_\Gamma w_\delta^M\|^2 = a_S(w_\delta^M, \partial^\bullet w_\delta^M) + \frac{1}{2} b(w_\delta^M, w_\delta^M). \tag{3.27}$$

Furthermore, consider the term g in (3.23). By Proposition 3.2, we have

$$\begin{aligned} \frac{d}{dt} g(u_\delta^M, w_\delta^M) &= g(u_\delta^M, \partial^\bullet w_\delta^M) + m(\partial^\bullet (u_\delta^M \nabla_\Gamma \cdot \mathbf{V}), w_\delta^M) + g(u_\delta^M \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M) \\ &= g(u_\delta^M, \partial^\bullet w_\delta^M) + m(\partial^\bullet u_\delta^M \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M) + m(u_\delta^M \partial^\bullet \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M) + g(u_\delta^M \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M). \end{aligned}$$

Therefore we get

$$g(u_\delta^M, \partial^\bullet w_\delta^M) = \frac{d}{dt} g(u_\delta^M, w_\delta^M) - m(\partial^\bullet u_\delta^M \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M) - m(u_\delta^M \partial^\bullet \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M) - g(u_\delta^M \nabla_\Gamma \cdot \mathbf{V}, w_\delta^M). \tag{3.28}$$

Recalling (3.3), we have

$$\frac{d}{dt} a_N(u_\delta^M, w_\delta^M) = a_N(\partial^\bullet u_\delta^M, w_\delta^M) + a_N(u_\delta^M, \partial^\bullet w_\delta^M) + \int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t), \mathbf{V}(t)) u_\delta^M \cdot \nabla_\Gamma w_\delta^M,$$

that is,

$$a_N(u_\delta^M, \partial^\bullet w_\delta^M) = \frac{d}{dt} a_N(u_\delta^M, w_\delta^M) - a_N(\partial^\bullet u_\delta^M, w_\delta^M) - \int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t), \mathbf{V}(t)) u_\delta^M \cdot \nabla_\Gamma w_\delta^M. \tag{3.29}$$

Thanks to Proposition 3.2, we deduce

$$\frac{d}{dt}\frac{\theta}{2}g(F_{ln}^{\delta}(u_\delta^M),1) = \frac{\theta}{2}g(\varphi_\delta(u_\delta^M),\partial^\bullet u_\delta^M) + \frac{\theta}{2}\int_{\Gamma(t)} F_{ln}^{\delta}(u_\delta^M)\partial^\bullet\nabla_\Gamma\cdot\mathbf{V} + \frac{\theta}{2}g(F_{ln}^{\delta}(u_\delta^M),\nabla_\Gamma\cdot\mathbf{V}). \tag{3.30}$$

This last equality is necessary, since in the Galerkin setting we are not able to retrieve a uniform estimate for $\|\varphi_\delta(u_\delta^M)\|_{L^2_{L^2}}$ but only for its L^2-orthogonal projection over $V_M(\mathrm{t})$, whereas, thanks to (3.20), we are able to uniformly estimate $\sup_{t\in[0,T]}\|F_{ln}^{\delta}(u_\delta^M)\|_{L^1(\Gamma(t))}$. Collecting (3.23)-(3.29), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla_\Gamma w_\delta^M\|^2 + a_N(u_\delta^M,w_\delta^M) + g(u_\delta^M,w_\delta^M)\right) + \|\nabla_\Gamma\partial^\bullet u_\delta^M\|^2\\
&\qquad + m((F^\delta)''(u_\delta^M)\partial^\bullet u_\delta^M,\partial^\bullet u_\delta^M)\\
&\quad = g(w_\delta^M,\partial^\bullet u_\delta^M) - g((F^\delta)'(u_\delta^M),\partial^\bullet u_\delta^M) - b(u_\delta^M,\partial^\bullet u_\delta^M) + \frac{1}{2}b(w_\delta^M,w_\delta^M)\\
&\qquad + m(\partial^\bullet u_\delta^M\nabla_\Gamma\cdot\mathbf{V},w_\delta^M) + m(u_\delta^M\partial^\bullet\nabla_\Gamma\cdot\mathbf{V},w_\delta^M)\\
&\qquad + g(u_\delta^M\nabla_\Gamma\cdot\mathbf{V},w_\delta^M) + a_N(\partial^\bullet u_\delta^M,w_\delta^M)\\
&\qquad + \int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t),\mathbf{V}(t))u_\delta^M\cdot\nabla_\Gamma w_\delta^M.
\end{aligned}$$

In particular, substituting the explicit value of F^δ, setting $\varphi_\delta := (F_{ln}^\delta)'$, and exploiting (3.30), we find

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla_\Gamma w_\delta^M\|^2 + a_N(u_\delta^M,w_\delta^M) + g(u_\delta^M,w_\delta^M) + \frac{\theta}{2}g(F_{ln}^\delta(u_\delta^M),1)\right)\\
&+ \|\nabla_\Gamma\partial^\bullet u_\delta^M\|^2 + \frac{\theta}{2}m(\varphi_\delta'(u_\delta^M)\partial^\bullet u_\delta^M,\partial^\bullet u_\delta^M)\\
&= g(w_\delta^M,\partial^\bullet u_\delta^M) + m(\partial^\bullet u_\delta^M,\partial^\bullet u_\delta^M) + g(u_\delta^M,\partial^\bullet u_\delta^M) - b(u_\delta^M,\partial^\bullet u_\delta^M)\\
&+ \frac{1}{2}b(w_\delta^M,w_\delta^M) + m(\partial^\bullet u_\delta^M\nabla_\Gamma\cdot\mathbf{V},w_\delta^M) + m(u_\delta^M\partial^\bullet\nabla_\Gamma\cdot\mathbf{V},w_\delta^M)\\
&+ g(u_\delta^M\nabla_\Gamma\cdot\mathbf{V},w_\delta^M) + a_N(\partial^\bullet u_\delta^M,w_\delta^M) + \int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t),\mathbf{V}(t))u_\delta^M\cdot\nabla_\Gamma w_\delta^M\\
&+ \frac{\theta}{2}\int_{\Gamma(t)} F_{ln}^\delta(u_\delta^M)\partial^\bullet\nabla_\Gamma\cdot\mathbf{V} + \frac{\theta}{2}g(F_{ln}^\delta(u_\delta^M),\nabla_\Gamma\cdot\mathbf{V}).
\end{aligned}$$

Recalling that $\varphi_\delta' = (F_{ln}^\delta)'' \geq 0$, we have $\frac{\theta}{2}m(\varphi_\delta'(u_\delta^M)\partial^\bullet u_\delta^M,\partial^\bullet u_\delta^M) \geq 0$. Therefore, setting

$$\mathcal{Q}_\delta^M := \frac{1}{2}\|\nabla_\Gamma w_\delta^M\|^2 + a_N(u_\delta^M,w_\delta^M) + g(u_\delta^M,w_\delta^M) + \frac{\theta}{2}g(F_{ln}^\delta(u_\delta^M),1), \tag{3.31}$$

and multiplying by $t\in(0,T]$ the above inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\left(t\mathcal{Q}_\delta^M(t)\right) + t\|\nabla_\Gamma\partial^\bullet u_\delta^M\|^2 + t\frac{\theta}{2}m(\varphi_\delta'(u_\delta^M)\partial^\bullet u_\delta^M,\partial^\bullet u_\delta^M)\\
&= \mathcal{Q}_\delta^M(t) + g(w_\delta^M,t\partial^\bullet u_\delta^M) + tm(\partial^\bullet u_\delta^M,\partial^\bullet u_\delta^M)\\
&+ g(u_\delta^M,t\partial^\bullet u_\delta^M) - b(u_\delta^M,t\partial^\bullet u_\delta^M) + \frac{t}{2}b(w_\delta^M,w_\delta^M)\\
&+ m(\partial^\bullet u_\delta^M\nabla_\Gamma\cdot\mathbf{V},tw_\delta^M) + m(u_\delta^M\partial^\bullet\nabla_\Gamma\cdot\mathbf{V},tw_\delta^M)\\
&+ g(u_\delta^M\nabla_\Gamma\cdot\mathbf{V},tw_\delta^M) + a_N(\partial^\bullet u_\delta^M,tw_\delta^M) + t\int_{\Gamma(t)} B_{adv}(\mathbf{V}_a^\tau(t),\mathbf{V}(t))u_\delta^M\cdot\nabla_\Gamma w_\delta^M\\
&+ \frac{\theta t}{2}\int_{\Gamma(t)} F_{ln}^\delta(u_\delta^M)\partial^\bullet\nabla_\Gamma\cdot\mathbf{V} + \frac{\theta t}{2}g(F_{ln}^\delta(u_\delta^M),\nabla_\Gamma\cdot\mathbf{V}).
\end{aligned}$$

We now recall that, by Poincaré's inequality,

$$\|w_\delta^M\| \leq C(\|\nabla_\Gamma w_\delta^M\| + |m(w_\delta^M,1)|). \tag{3.32}$$

Then, on account of (2.3) and using Hölder's and Young's inequalities, we deduce

$$\begin{aligned}
&g(w_\delta^M,t\partial^\bullet u_\delta^M) + m(\partial^\bullet u_\delta^M,t\partial^\bullet u_\delta^M)\\
&\quad + g(u_\delta^M,t\partial^\bullet u_\delta^M) - b(u_\delta^M,t\partial^\bullet u_\delta^M) + \frac{t}{2}b(w_\delta^M,w_\delta^M)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta t}{2}\int_{\Gamma(t)} F^\delta_{ln}(u^M_\delta)\partial^\bullet\nabla_\Gamma\cdot\mathbf{V}+\frac{\theta t}{2}g(F^\delta_{ln}(u^M_\delta),\nabla_\Gamma\cdot\mathbf{V})\\
&\le Ct\|w^M_\delta\|\|\partial^\bullet u^M_\delta\|+t\|\partial^\bullet u^M_\delta\|^2\\
&+t\|u^M_\delta\|\|\partial^\bullet u^M_\delta\|+Ct\|\nabla_\Gamma u^M_\delta\|\|\nabla_\Gamma\partial^\bullet u^M_\delta\|+Ct\|\nabla_\Gamma w^M_\delta\|^2\\
&+Ct\|F^\delta_{ln}(u^M_\delta)\|_{L^1(\Gamma(t))}+Ct\|F^\delta_{ln}(u^M_\delta)\|_{L^1(\Gamma(t))}\\
&\le Ct\|\partial^\bullet u^M_\delta\|^2+Ct\|\nabla_\Gamma w^M_\delta\|^2+\frac{t}{2}\|\nabla_\Gamma\partial^\bullet u^M_\delta\|^2\\
&+Ct(\|u^M_\delta\|^2+|m(w^M_\delta,1)|^2+\|F^\delta_{ln}(u^M_\delta)\|_{L^1(\Gamma(t))}),
\end{aligned}
$$

where we exploited (2.9) and (3.32). Analogously, recalling (2.5), (2.9) and (3.32), thanks to Cauchy-Schwarz and Young's inequalities, we find

$$
\begin{aligned}
&m(\partial^\bullet u^M_\delta\nabla_\Gamma\cdot\mathbf{V},tw^M_\delta)+m(u^M_\delta\partial^\bullet\nabla_\Gamma\cdot\mathbf{V},tw^M_\delta)\\
&+g(u^M_\delta\nabla_\Gamma\cdot\mathbf{V},tw^M_\delta)+a_N(\partial^\bullet u^M_\delta,tw^M_\delta)+t\int_{\Gamma(t)}B_{adv}(\mathbf{V}^\tau_a(t),\mathbf{V}(t))u^M_\delta\cdot\nabla_\Gamma w^M_\delta\\
&\le Ct\|\partial^\bullet u^M_\delta\|(|m(w^M_\delta,1)|+\|\nabla_\Gamma w^M_\delta\|)+Ct\|u^M_\delta\|(|m(w^M_\delta,1)|+\|\nabla_\Gamma w^M_\delta\|)\\
&+Ct\|u^M_\delta\|(|m(w^M_\delta,1)|+\|\nabla_\Gamma w^M_\delta\|)+Ct\|\partial^\bullet u^M_\delta\|\|\nabla_\Gamma w^M_\delta\|+Ct\|u^M_\delta\|\|\nabla_\Gamma w^M_\delta\|\\
&\le Ct\|\nabla_\Gamma w^M_\delta\|^2+Ct\|\partial^\bullet u^M_\delta\|^2+Ct(1+|m(w^M_\delta,1)|^2+\|u^M_\delta\|^2)
\end{aligned}
$$

Collecting the above inequalities, we eventually obtain

$$
\begin{aligned}
\frac{d}{dt}\left(t\mathcal{Q}^M_\delta(t)\right)+\frac{t}{2}\|\nabla_\Gamma\partial^\bullet u^M_\delta\|^2&\le\mathcal{Q}^M_\delta(t)+Ct\|\nabla_\Gamma w^M_\delta\|^2+Kt\|\partial^\bullet u^M_\delta\|^2\\
&+Ct(\|u^M_\delta\|^2+|m(w^M_\delta,1)|^2+\|F^\delta_{ln}(u^M_\delta)\|_{L^1(\Gamma(t))}),
\end{aligned}\tag{3.33}
$$

where we explicitly denoted by $K>0$, for further use, the constant in front of $\|\partial^\bullet u^M_\delta\|$, which has to be controlled in a uniform way. Since in the Galerkin setting we are not able to find a uniform estimate for $\|\partial^\bullet u^M_\delta\|_{L^2_{H^{-1}}}$, we cannot exploit the classical interpolation inequality $\|\partial^\bullet u^M_\delta\|\le C\|\partial^\bullet u^M_\delta\|^{1/2}_{H^{-1}(\Gamma(t))}\|\partial^\bullet u^M_\delta\|^{1/2}_{H^1(\Gamma(t))}$ as in the case of nonevolving surfaces. Therefore, we test (3.11) with $\eta=t\partial^\bullet u^M_\delta\in L^2_{V_M}$, obtaining

$$
t\|\partial^\bullet u^M_\delta\|^2+tg(u^M_\delta,\partial^\bullet u^M_\delta)+ta_N(u^M_\delta,\partial^\bullet u^M_\delta)+ta_S(w^M_\delta,\partial^\bullet u^M_\delta)=0,
$$

from which, by standard estimates, for some κ sufficiently small to be chosen later on, we obtain

$$
t\|\partial^\bullet u^M_\delta\|^2\le\frac{t}{2}\|\partial^\bullet u^M_\delta\|^2+t\kappa\|\nabla_\Gamma\partial^\bullet u^M_\delta\|^2+Ct\|\nabla_\Gamma w^M_\delta\|^2+Ct\|u^M_\delta\|^2,
$$

that is,

$$
\frac{t}{2}\|\partial^\bullet u^M_\delta\|^2\le t\kappa\|\nabla_\Gamma\partial^\bullet u^M_\delta\|^2+Ct\|\nabla_\Gamma w^M_\delta\|^2+Ct\|u^M_\delta\|^2.
$$

We can now add this inequality multiplied by $\omega=4K$ to (3.33), choosing $\kappa=\frac{1}{16K}$. This gives, for all $t\in[0,T]$,

$$
\begin{aligned}
\frac{d}{dt}\left(t\mathcal{Q}^M_\delta(t)\right)+\frac{t}{4}\|\nabla_\Gamma\partial^\bullet u^M_\delta\|^2+Kt\|\partial^\bullet u^M_\delta\|^2&\le\mathcal{Q}^M_\delta(t)+Ct\|\nabla_\Gamma w^M_\delta\|^2+Ct(\|u^M_\delta\|^2\\
&+|m(w^M_\delta,1)|^2+\|F^\delta_{ln}(u^M_\delta)\|_{L^1(\Gamma(t))}).
\end{aligned}\tag{3.34}
$$

Recalling Proposition 3.7 and (3.19), from (3.34) we get, for all $t\in[0,T]$

$$
\frac{d}{dt}\left(t\mathcal{Q}^M_\delta(t)\right)+\frac{t}{4}\|\nabla_\Gamma\partial^\bullet u^M_\delta\|^2+Kt\|\partial^\bullet u^M_\delta\|^2\le\mathcal{Q}^M_\delta(t)+Ct\|\nabla_\Gamma w^M_\delta\|^2+C_\delta t,\tag{3.35}
$$

where $C_\delta>0$ depends on δ. Observe now that

$$
|a_N(u^M_\delta,w^M_\delta)|\le C\|u^M_\delta\|\|\nabla_\Gamma w^M_\delta\|\le C\|u^M_\delta\|^2+\frac{1}{8}\|\nabla_\Gamma w^M_\delta\|^2,\tag{3.36}
$$

and, by Poincaré's inequality,

$$
|g(u^M_\delta,w^M_\delta)|\le C\|u^M_\delta\|\|w^M_\delta\|\le C(\|u^M_\delta\|^2+|m(w^M_\delta,1)|^2)+\frac{1}{8}\|\nabla_\Gamma w^M_\delta\|^2,\tag{3.37}
$$

so that (see Proposition 3.7, (3.19) and (3.32))

$$|g(u_\delta^M, w_\delta^M)| \le C_\delta + \frac{1}{8}\|\nabla_\Gamma w_\delta^M\|^2,$$

Moreover, owing to (3.20), we have

$$\frac{\theta}{2}|g(F_{ln}^\delta(u_\delta^M),1)| \le C\|F_{ln}^\delta(u_\delta^M)\|_{L^1(\Gamma(t))} \le C_\delta.$$

implying that there exists $\widehat{C}_\delta > 0$, depending on δ, such that

$$\frac{1}{4}\|\nabla_\Gamma w_\delta^M\|^2 - \widehat{C}_\delta \le \mathcal{Q}_\delta^M(t) \le C_\delta(1+\|\nabla_\Gamma w_\delta^M\|^2), \quad \forall t \in [0,T].$$

Note now that, due to the energy estimate (3.20), we have that $\mathcal{Q}_\delta^M$ is uniformly bounded in $L^1(0,T)$. Thus, defining $\widetilde{\mathcal{Q}}_\delta^M := \mathcal{Q}_\delta^M + \widehat{C}_\delta \ge 0$, we obtain from (3.35)

$$\frac{d}{dt}\left(t\widetilde{\mathcal{Q}}_\delta^M(t)\right) + Kt\|\partial^\bullet u_\delta^M\|^2 + \frac{t}{4}\|\nabla_\Gamma \partial^\bullet u_\delta^M\|^2 \le \widetilde{\mathcal{Q}}_\delta^M(t) + C_\delta t\widetilde{\mathcal{Q}}_\delta^M(t) + C_\delta t. \tag{3.38}$$

Therefore, by Gronwall's Lemma applied to $y(t) = t\widetilde{\mathcal{Q}}_\delta^M(t)$, recalling that $\widetilde{\mathcal{Q}}_\delta^M$ is uniformly (in M) bounded in $L^1(0,T)$, we infer

$$t\widetilde{\mathcal{Q}}_\delta^M(t) \le C_\delta(T), \quad \forall t \in [0,T],$$

entailing, recalling (3.32),

$$t\|w_\delta^M\|^2_{H^1(\Gamma(t))} + \int_0^t s\|\partial^\bullet u_\delta^M\|^2_{H^1(\Gamma(s))}ds \le C_\delta(T), \quad \forall t \in [0,T], \tag{3.39}$$

where $C_\delta(T) > 0$ depends on δ but is independent of M. For further use we also integrate in time (3.34), to get, for all $t \in [0,T]$,

$$\begin{aligned} &t\mathcal{Q}_\delta^M(t) + \int_0^t \frac{s}{4}\|\nabla_\Gamma \partial^\bullet u_\delta^M\|^2 ds + \int_0^t Ks\|\partial^\bullet u_\delta^M\|^2 ds \\ &\le \int_0^t \mathcal{Q}_\delta^M(s)ds + C\int_0^t s\|\nabla_\Gamma w_\delta^M\|^2 ds \\ &+ C\int_0^t s(\|u_\delta^M\|^2 + |m(w_\delta^M,1)|^2 + \|F_{ln}^\delta(u_\delta^M)\|_{L^1(\Gamma(t))})ds. \end{aligned} \tag{3.40}$$

Then by the definition (3.31) of $\mathcal{Q}_\delta^M$, (3.36) and (3.37), we also deduce that, for any $t \in [0,T]$,

$$\begin{aligned} \int_0^t \mathcal{Q}_\delta^M(s)ds &\le C(T)\int_0^T \left(\tilde{E}^{CH}[t; u_\delta^M(t)] + \|u_\delta^M\|^2\right)dt + C(T)\int_0^T \|\nabla_\Gamma w_\delta^M\|^2 dt \\ &\le C(T)\left(1 + \int_0^T |m(w_\delta^M,1)|^2 dt\right), \end{aligned}$$

having applied (3.14), (3.17) and the conservation of mass. This, together with (3.40), gives

$$t\mathcal{Q}_\delta^M(t) + \int_0^t \frac{s}{4}\|\nabla_\Gamma \partial^\bullet u_\delta^M\|^2 ds + \int_0^t Ks\|\partial^\bullet u_\delta^M\|^2 ds \le C(T)\left(1 + \int_0^T |m(w_\delta^M,1)|^2 dt\right), \tag{3.41}$$

where again we applied (3.14) and (3.17) for the last summands in the right-hand side of (3.40). Note also that, by the definition (3.31) of $\mathcal{Q}_\delta^M$, from (3.41) we infer

$$\begin{aligned} &\frac{t}{2}\|\nabla_\Gamma w_\delta^M\|^2 + \int_0^t \frac{s}{4}\|\nabla_\Gamma \partial^\bullet u_\delta^M\|^2 ds + \int_0^t Ks\|\partial^\bullet u_\delta^M\|^2 ds \\ &\le C(T)\left(1 + \int_0^T |m(w_\delta^M,1)|^2 dt\right) \\ &- ta_N(u_\delta^M, w_\delta^M) - tg(u_\delta^M, w_\delta^M) - \frac{\theta t}{2}g(F_{ln}^\delta(u_\delta^M),1) \end{aligned}$$

$$\leq C(T)\left(1+\int_0^T|m(w_\delta^M,1)|^2dt\right)+\frac{t}{4}\|\nabla_\Gamma w_\delta^M\|^2+C(T)(1+\tilde{E}^{CH}[t;u_\delta^M]),$$

due to (3.17), (3.36) and (3.37). This implies, by (3.14),

$$\frac{t}{4}\|\nabla_\Gamma w_\delta^M\|^2+\int_0^t\frac{s}{4}\|\nabla_\Gamma\partial^\bullet u_\delta^M\|^2ds+\int_0^t Ks\|\partial^\bullet u_\delta^M\|^2ds\leq C(T)\left(1+\int_0^T|m(w_\delta^M,1)|^2dt\right), \tag{3.42}$$

for any $t\in[0,T]$, where $C(T)$ is independent of M,δ. By standard compactness arguments, exploiting the regularity we have just obtained, arguing as in [13, Sec.4.2], we can pass to the limit as $M\to\infty$ and obtain the existence of a couple (u_δ,w_δ) satisfying for any $\eta\in L^2_{H^1}$ and all $t\in[0,T]$,

$$m_\star(\partial^\bullet u_\delta,\eta)+g(u_\delta,\eta)+a_N(u_\delta,\eta)+a_S(w_\delta,\eta)=0, \tag{3.43}$$

$$a_S(u_\delta,\eta)+m((F^\delta)'(u_\delta),\eta)-m(w_\delta,\eta)=0, \tag{3.44}$$

and $u_\delta(0)=u_0$ almost everywhere in Γ_0. Moreover, it holds

$$\sup_{t\in[0,T]}\tilde{E}^{CH}[t;u_\delta(t)]+\sup_{t\in[0,T]}\|u_\delta(t)\|^2+\int_0^T\|w_\delta\|^2_{H^1(\Gamma(t))}dt\leq C_\delta(T),$$

$$t\|w_\delta\|^2_{H^1(\Gamma(t))}+\int_0^t s\|\partial^\bullet u_\delta\|^2_{H^1(\Gamma(s))}ds\leq C_\delta(T),\quad\forall\,t\in[0,T],$$

for some $C_\delta(T)>0$ possibly depending on δ. Now note that, since clearly (see [13, Sec.4.2]) we have $u_\delta^M\to u_\delta$ in $L^2_{L^2}$ as $M\to\infty$, and being $|F'_\delta(c_\delta^M)|\leq C_\delta|c_\delta^M|$ and $|F''_\delta|\leq C_\delta$, we have, since $m(w_\delta^M,1)=m(F'(u_\delta^M),1)$ and $m(w_\delta,1)=m(F'(u_\delta),1)$,

$$\begin{aligned}\left|\int_0^T\left(|m(w_\delta^M,1)|^2-|m(w_\delta,1)|^2\right)dt\right|&\leq\int_0^T\left(|m(F'(u_M^\delta),1)|+|m(F'(u^\delta),1)|\right)\left|m(F'(u_M^\delta)-F'(u^\delta),1)\right|dt\\&\leq C_\delta(T)\|u_\delta^M-u_\delta\|_{L^2_{L^2}}\to 0\quad\text{as }M\to\infty,\end{aligned}$$

having also exploited the uniform controls on u_δ^M and u_δ in $L^\infty_{L^2}$. Therefore, these result, together with the bound (3.39) (which gives weak lower sequential semicontinuity of the norms on the left-hand side of (3.42)), is enough to pass to the limit as $M\to\infty$ in (3.42), so that it holds

$$\operatorname*{ess\,sup}_{t\in[0,T]}\left(\frac{t}{4}\|\nabla_\Gamma w_\delta\|^2\right)+\int_0^T\frac{s}{4}\|\nabla_\Gamma\partial^\bullet u_\delta\|^2ds+\int_0^T Ks\|\partial^\bullet u_\delta\|^2ds\leq C(T)\left(1+\int_0^T|m(w_\delta,1)|^2ds\right), \tag{3.45}$$

where this time $C(T)>0$ is a constant *independent* of δ.

Step 2. Limit as $\delta\to 0$. In order to pass to the limit in δ we need to find a more refined energy estimate. In particular, let us first observe that again, from (3.43), it holds the conservation of total mass, by choosing $\eta\equiv 1$. Then we test (3.43) by $\eta=u_\delta$, inferring

$$\frac{1}{2}\frac{d}{dt}\|u_\delta\|^2=m_\star(\partial_t u_\delta,u_\delta)+\frac{1}{2}g(u_\delta,u_\delta)=-\frac{1}{2}g(u_\delta,u_\delta)-a_N(u_\delta,u_\delta)-a_S(w_\delta,u_\delta).$$

Integrating by parts, exploiting also (3.44), we get, by the definition of F''_δ,

$$\begin{aligned}a_S(w_\delta,u_\delta)=-\int_{\Gamma(t)}w_\delta\Delta_\Gamma u_\delta&=\|\Delta_\Gamma u_\delta\|^2+\int_{\Gamma(t)}F''_\delta(u_\delta)|\nabla_\Gamma u_\delta|^2\\&=\|\Delta_\Gamma u_\delta\|^2+\int_{\Gamma(t)}\varphi'_\delta(u_\delta)|\nabla_\Gamma u_\delta|^2-\|\nabla_\Gamma u_\delta\|^2.\end{aligned} \tag{3.46}$$

Notice that by [13, Prop. 4.12] the solution to (3.43)-(3.44) is unique and thus it enjoys the regularity $L^2_{H^2}$ as in [13, Thm. 4.14]. As a consequence, the second equation (i.e. the one for w_δ) holds pointwise almost everywhere, and therefore the regularity of u_δ is sufficient to perform rigorously the calculation above. Recalling now that $\varphi'_\delta\geq 0$ and by the regularity of $\mathbf{V}$, we end up with

$$\frac{1}{2}\frac{d}{dt}\|u_\delta\|^2+\|\Delta_\Gamma u_\delta\|^2\leq-\frac{1}{2}g(u_\delta,u_\delta)-a_N(u_\delta,u_\delta)+\|\nabla_\Gamma u_\delta\|^2\leq C(\|u_\delta\|^2+\|\nabla_\Gamma u_\delta\|^2).$$

We can now apply the following basic interpolation inequality, which can be obtained after an integration by parts and an application of Cauchy-Schwarz and Young's inequalities:

$$\|\nabla_\Gamma u_\delta\|^2 = -\int_{\Gamma(t)} u_\delta \Delta_\Gamma u_\delta \le \frac{1}{2}\|u_\delta\|^2 + \frac{1}{2}\|\Delta_\Gamma u_\delta\|^2,$$

allowing to infer, by Young's inequality,

$$\frac{1}{2}\frac{d}{dt}\|u_\delta\|^2 + \frac{1}{2}\|\Delta_\Gamma u_\delta\|^2 \le C\|u_\delta\|^2,$$

so that, in the end, by Gronwall's Lemma, we deduce

$$\sup_{t\in[0,T]} \|u_\delta\|^2 + \int_0^T \|\Delta_\Gamma u_\delta\|^2 ds \le C(T), \tag{3.47}$$

uniformly in δ. Concerning the energy estimate, by the same arguments as for (3.16) we get:

$$\frac{d}{dt}\tilde{E}^{CH}[u_\delta] + \frac{1}{2}\|\nabla_\Gamma w_\delta^M\|^2 \le -g(u_\delta, w_\delta) + C_0 + C_1\tilde{E}^{CH}[u_\delta]. \tag{3.48}$$

Thanks to (3.47) and the bound on $\mathbf{V}$, by Poincaré's inequality we have

$$-g(u_\delta, w_\delta) \le C\|u_\delta\|\|w_\delta\| \le C\|w_\delta\| \le \widetilde{C}(|m(w_\delta,1)| + \|\nabla_\Gamma w_\delta\|), \tag{3.49}$$

with $\widetilde{C} > 0$ a suitable constant independent of δ. Then, using known arguments together with the conservation of total mass (see also the proof of [13, Prop.5.7, a)]) we deduce, from [13, (5.1.7)],

$$|m(w_\delta, u_\delta)| \le \frac{1-\alpha}{8\widetilde{C}}\|\nabla_\Gamma w_\delta\|^2 + C\|\nabla_\Gamma u_\delta\|^2 + \alpha|m(w_\delta,1)|, \tag{3.50}$$

where $\alpha \in [0,1)$ is a constant such that

$$0 \le m_{u_0}(t) \le \alpha < 1, \tag{3.51}$$

for any $t \in [0,T]$, whose existence is guaranteed being $\sup_{[0,T]} m_{u_0}(t) < 1$ by assumption. Now, from [13, (5.1.5)],

$$|m(w_\delta,1)| \le C\left(1 + \|\nabla_\Gamma u_\delta\|^2\right) + |m(w_\delta,u_\delta)|, \tag{3.52}$$

thus by (3.50) we infer

$$|m(w_\delta,1)| \le C\left(1 + \|\nabla_\Gamma u_\delta\|^2\right) + \frac{1}{8\widetilde{C}}\|\nabla_\Gamma w_\delta\|^2, \tag{3.53}$$

so that, together with (3.48) and (3.49), we deduce

$$\frac{d}{dt}\tilde{E}^{CH}[u_\delta] + \frac{3}{8}\|\nabla_\Gamma w_\delta^M\|^2 \le C(1 + \tilde{E}^{CH}[u_\delta]),$$

entailing, by Gronwall's Lemma,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \tilde{E}^{CH}[u_\delta] + \int_0^T \|\nabla_\Gamma w_\delta\|^2 dt \le C(T), \tag{3.54}$$

uniformly in δ. Observe now that by (3.54) we can deduce a better estimate of $|m(w_\delta,u_\delta)|$. In particular, from [13, (5.1.6)], (3.54) and the conservation of mass, by Poincaré's inequality we infer

$$\begin{aligned}|m(w_\delta,u_\delta)| &\le C\|\nabla_\Gamma w_\delta\|\|u_\delta - (u_\delta)_{\Gamma(t)}\| + m_{u_0}(t)|m(w_\delta,1)|\\ &\le C\|\nabla_\Gamma w_\delta\| + \alpha|m(w_\delta,1)|,\end{aligned}$$

for α already defined in (3.51). On account of (3.47), (3.54), we have $\|u_\delta\|_{H^1(\Gamma(t))} \le C(T)$ for almost any $t \in [0,T]$, so that, since $|\varphi_\delta(r)| \le \varphi_\delta(r)r + 1$ and $m(w_\delta,1) = -m(u_\delta,1) + m(\varphi_\delta(u_\delta),1)$,

$$\begin{aligned}|m(w_\delta,1)| &\le |m(u_\delta,1)| + |m(\varphi_\delta(u_\delta),1)|\\ &\le |m(u_\delta,1)| + |m(\varphi_\delta(u_\delta),u_\delta)| + |\Gamma(t)|\end{aligned}$$

$$\le C + \|u_\delta\|^2 + \|\nabla_\Gamma u_\delta\|^2 + |m(w_\delta, u_\delta)|$$
$$\le C(1 + \|\nabla_\Gamma w_\delta\|) + \alpha |m(w_\delta, 1)|,$$

recalling $m(\varphi_\delta(u_\delta), u_\delta) = m(w_\delta, u_\delta) - \|\nabla u_\delta\|^2$. This implies, being $\alpha < 1$, by Poincaré's inequality,

$$|m(w_\delta, 1)| \le C(1 + \|\nabla_\Gamma w_\delta\|), \tag{3.55}$$

from which we infer, together with (3.54),

$$\int_0^T |m(w_\delta, 1)|^2 \le C(T),$$

entailing, by (3.45),

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left(\frac{t}{4} \|\nabla_\Gamma w_\delta\|^2 \right) + \int_0^T \frac{s}{4} \|\nabla_\Gamma \partial^\bullet u_\delta\|^2 ds + \int_0^T K s \|\partial^\bullet u_\delta\|^2 ds \le C(T). \tag{3.56}$$

Following [13, Sec.4.2], [13, Propositions 5.8-5.10] and [13, Lemma 5.11], the uniform estimates (3.47), (3.54) and (3.56) are then enough, by standard compactness arguments, to pass to the limit as $\delta \to 0$, obtaining a weak solution with the same properties as the one of Theorem 3.4. Therefore, this solution coincides with the one of Theorem 3.4 by uniqueness, and it also enjoys (3.21) and (3.22). This concludes the proof of Part (i).

Part (ii). Let us fix $\tau > 0$. We clearly have that $\|w\|_{H^1(\Gamma(t))} \le C$ for almost any $t \ge \tau$. Let then introduce the cutoff function

$$h_k(r) = \begin{cases} 1 - \frac{1}{k}, & r > 1 - \frac{1}{k}, \\ r, & -1 + \frac{1}{k} \le r \le 1 - \frac{1}{k}, \\ -1 + \frac{1}{k}, & r < -1 + \frac{1}{k}, \end{cases} \tag{3.57}$$

which is Lipschitz continuous. Then we set $u_k = h_k(u)$. Being u in $L^\infty_{H^1}$, we have that the chain rule holds giving

$$\nabla_\Gamma u_k = \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(u) \nabla_\Gamma u.$$

Accordingly, for any $k > 1$ and $p \ge 2$, $f_k = \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi(u_k)$ is well defined and belongs to $L^\infty_{H^1}$ and satisfies

$$\nabla_\Gamma \left(\left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi(u_k) \right) = (p-1) \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi'(u_k) \nabla_\Gamma u_k.$$

If we now set $\eta = f_k$ in (3.9), we infer that

$$(p-1) \int_{\Gamma(t)} \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi'(u_k) \nabla_\Gamma u \cdot \nabla_\Gamma u_k + \int_{\Gamma(t)} \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi(u_k) \frac{\theta}{2}\varphi(u) = \int_{\Gamma(t)} \widehat{w} \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi(u_k),$$

where $\widehat{w} = w + u$. Being F_{ln} strictly convex, the first term in the left-hand side is nonnegative. Moreover, since φ is increasing, by the definition of u_k, we immediately infer

$$\varphi(u_k)^2 \le \varphi(u)\varphi(u_k), \quad \forall k > 1 \tag{3.58}$$

and thus for the second term we have

$$\int_{\Gamma(t)} \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi(u_k) \frac{\theta}{2}\varphi(u) \ge \int_{\Gamma(t)} \left|\frac{\theta}{2}\varphi(u_k)\right|^p$$

Regarding the right-hand side, we easily get, by the Sobolev embedding $H^1(\Gamma(t)) \hookrightarrow L^p(\Gamma(t))$,

$$\int_{\Gamma(t)} \widehat{w} \left|\frac{\theta}{2}\varphi(u_k)\right|^{p-2} \frac{\theta}{2}\varphi(u_k) \le \frac{1}{2} \left\|\frac{\theta}{2}\varphi(u_k)\right\|^p_{L^p(\Gamma(t))} + C\|\widehat{w}\|^p_{L^p(\Gamma(t))}$$
$$\le \frac{1}{2} \left\|\frac{\theta}{2}\varphi(u_k)\right\|^p_{L^p(\Gamma(t))} + C_p\|\widehat{w}\|^p_{H^1(\Gamma(t))},$$

with $C_p > 0$ depending on p. Collecting the above estimates, being $u \in L^\infty_{H^1}$ and $\|w\|_{H^1(\Gamma(t))} \le C$ for almost any $t \ge \tau$, we immediately deduce

$$\|\varphi(u_k)\|_{L^p(\Gamma(t))} \le C_p(T, \tau, p), \quad \forall p \in [2, \infty), \tag{3.59}$$

for almost any $t \in [\tau, T]$. Therefore, there exists $\zeta \in L^\infty_{L^p}(\tau, T)$ such that, up to subsequences $\varphi(u_k) \overset{*}{\rightharpoonup} \zeta$ in $L^\infty_{L^p}(\tau, T)$ as $k \to \infty$. Now observe that, being $|u| < 1$ almost everywhere on $\Gamma(t)$ (for almost any $t \in (0, T)$), and since, for almost any $t \in (0, T)$, $u_k \to u$ as $k \to \infty$ almost everywhere on $\Gamma(t)$, we get $\varphi(u_k) \to \varphi(u)$ almost everywhere. It is now immediate to deduce by (3.59) that, for $p = 2$, $\|\varphi(u_k)\|_{L^2_{L^2}} \leq C(\tau, T)$, therefore we can apply [13, Thm.B.2] on $[\tau, T]$ to infer $\varphi(u_k) \rightharpoonup \varphi(u)$ in $L^2_{L^2}(\tau, T)$. By uniqueness of the weak limit we therefore infer $\zeta = \varphi(u)$. Then, by weak* sequential lower semicontinuity we get

$$\operatorname*{ess\,sup}_{t\in[\tau,T]} \|\varphi(u)\|_{L^p(\Gamma(t))} \leq C_p(T, \tau, p), \quad \forall p \in [2, \infty). \tag{3.60}$$

Concerning φ', we consider $g_k = \frac{\theta}{2}\varphi(u_k)e^{L\frac{\theta}{2}|\varphi(u_k)|} \in L^\infty_{H^1}$, for some arbitrary $L > 0$, and we observe that

$$\nabla_\Gamma \left(\frac{\theta}{2}\varphi(u_k)e^{L\frac{\theta}{2}|\varphi(u_k)|}\right) = \frac{\theta}{2}\varphi'(u_k)\left(1 + L\frac{\theta}{2}|\varphi(u_k)|\right)e^{L\frac{\theta}{2}|\varphi(u_k)|}\nabla_\Gamma u_k.$$

Therefore, considering again (3.9) with $\eta = g_k$, we get

$$\begin{aligned}\int_{\Gamma(t)} \nabla_\Gamma u \cdot \nabla_\Gamma u_k \frac{\theta}{2}\varphi'(u_k)\left(1 + L\frac{\theta}{2}|\varphi(u_k)|\right)e^{L\frac{\theta}{2}|\varphi(u_k)|} + \int_{\Gamma(t)} \frac{\theta}{2}\varphi(u)\frac{\theta}{2}\varphi(u_k)e^{L\frac{\theta}{2}|\varphi(u_k)|} \\ = \int_{\Gamma(t)} \widehat{w}\frac{\theta}{2}\varphi(u_k)e^{L\frac{\theta}{2}|\varphi(u_k)|}.\end{aligned}$$

Again the first term in the left-hand side is nonnegative, whereas, exploiting again (3.58), we obtain in the end

$$\int_{\Gamma(t)} \left(\frac{\theta}{2}\varphi(u_k)\right)^2 e^{L\frac{\theta}{2}|\varphi(u_k)|} \leq \int_{\Gamma(t)} \widehat{w}\frac{\theta}{2}\varphi(u_k)e^{L\frac{\theta}{2}|\varphi(u_k)|}$$

By Lemma B.4, we get

$$\int_{\Gamma(t)} |\widehat{w}|\left|\frac{\theta}{2}\varphi(u_k)\right| e^{L\left|\frac{\theta}{2}\varphi(u_k)\right|} \leq \frac{1}{2}\int_{\Gamma(t)} \left|\frac{\theta}{2}\varphi(u_k)\right|^2 e^{L\left|\frac{\theta}{2}\varphi(u_k)\right|} + \int_{\Gamma(t)} e^{N|\widehat{w}|}, \tag{3.61}$$

implying

$$\frac{1}{2}\int_{\Gamma(t)} \left|\frac{\theta}{2}\varphi(u_k)\right|^2 e^{L\left|\frac{\theta}{2}\varphi(u_k)\right|} \leq \int_{\Gamma(t)} e^{N|\widehat{w}|} \tag{3.62}$$

for any $L > 0$ and some $N = N(L)$. Now we exploit (2.10), and then apply Lemma B.3 with $u = N\widetilde{\widehat{w}}$ and the manifold $\mathcal{M} = \Gamma_0$ (with the corresponding metric) to infer

$$\int_{\Gamma(t)} e^{N|\widehat{w}|} = \int_{\Gamma_0} e^{N|\widetilde{\widehat{w}}|} J^0_t d\Gamma_0 \leq C\int_{\Gamma_0} e^{N|\widetilde{\widehat{w}}|} d\Gamma_0 \leq Ce^{CN^2\|\widetilde{\widehat{w}}\|^2_{H^1(\Gamma_0)}} \leq Ce^{CN^2\|\widehat{w}\|^2_{H^1(\Gamma(t))}},$$

where in the last estimate we exploited the property that $(H^1(\Gamma(t)), \phi_t)_{t\in[0,T]}$ is a compatible space, where $\phi_{-t}v = \widetilde{v}$. Recalling now property (3.7), we infer that, for some $\tilde{C} > 0$ sufficiently large,

$$\left(\frac{\theta}{2}\varphi'(s)\right)^p \leq e^{pC}\left(\tilde{C} + \left|\frac{\theta}{2}\varphi(s)\right|^2 e^{pC\left|\frac{\theta}{2}\varphi(s)\right|}\right), \quad \forall s \in (-1, 1), \tag{3.63}$$

thus, taking $L = pC$ in (3.62) and recalling that $\|\widehat{w}\|_{H^1(\Gamma(t))} \leq C$ for almost any $t \geq \tau$, we end up with

$$\|\varphi'(u_k)\|_{L^p(\Gamma(t))} \leq C_p(T, \tau, p),$$

implying, by the same arguments used for $\varphi(u_k)$, applied in this case to $\varphi'(u_k)$, that

$$\operatorname*{ess\,sup}_{t\in[\tau,T]} \|\varphi'(u)\|_{L^p(\Gamma(t))} \leq C_p(T, \tau, p), \quad \forall p \in [2, \infty). \tag{3.64}$$

Therefore, exploiting elliptic regularity and recalling that $u \in L^\infty_{H^1}$, we obtain

$$\|u\|_{H^2(\Gamma(t))} \leq \left(C + \|\Delta_{\Gamma(t)}u\|\right) \leq C\left(1 + \|w\| + \|\varphi(u)\|\right) \leq C(T, \tau),$$

for almost any $t \in [\tau, T]$.

Part (iii). If we now apply the chain rule to $\varphi(u)$ (which is possible, e.g., by approximation with the truncated functions u_k and then passing to the limit as $k \to \infty$) we obtain

$$\nabla_\Gamma \varphi(u) = \varphi'(u)\nabla_\Gamma u, \qquad \text{for a.a. } \ t \in [\tau, T].$$

Then, for almost any $t \in [\tau, T]$, we have that

$$\|\nabla_\Gamma \varphi(u)\|_{L^p(\Gamma(t))} \le \|\varphi'(u)\|_{L^{2p}(\Gamma(t))}\|\nabla_\Gamma u\|_{L^{2p}(\Gamma(t))} \le C_p(T, \tau, p),$$

by the Sobolev embedding $H^2(\Gamma(t)) \hookrightarrow W^{1,q}(\Gamma(t))$ for every $q \ge 2$. Therefore we get

$$\|\varphi(u)\|_{W^{1,p}(\Gamma(t))} \le C_p(T, \tau, p),$$

for every $p \ge 2$. This implies, choosing, e.g., $p = 3$, that

$$\|\varphi(u)\|_{L^\infty(\Gamma(t))} \le C(T, \tau), \qquad \text{for a.a. } t \in [\tau, T], \tag{3.65}$$

by the embedding $W^{1,3}(\Gamma(t)) \hookrightarrow L^\infty(\Gamma(t))$. Therefore, being $u(t) \in H^2(\Gamma(t)) \hookrightarrow C^0(\Gamma(t))$ for almost any $t \in [0, T]$, it follows from the singularities of φ at ± 1 and the estimate (3.65) that we can find $\xi = \xi(T, \tau) > 0$ such that

$$\|u\|_{L^\infty(\Gamma(t))} \le 1 - \xi, \qquad \text{for a.a. } t \in [\tau, T],$$

that is, the strict separation property holds. This concludes the proof. □

4 The second model

We now consider the alternative weighted model (1.2), which is analysed in [13, Sec.6] (and references therein). In particular, as noticed in the Introduction, this is a simplified version of the model presented in [49, 56], in which the problem is governed by two coupled fourth-order nonlinear PDEs that live on an evolving two-dimensional manifold. For the phase transitions, the PDE is the Cahn-Hilliard equation for curved surfaces, which can be derived from surface mass balance in the framework of irreversible thermodynamics. For the surface deformation, the PDE is the (vector-valued) Kirchhoff–Love thin shell equation. In our work, we study only the Cahn-Hilliard equation in the same formulation arising from the model in [56], so that this analysis could, in a future work, be extended to consider the complete model as [49, 56], in which the evolution of the surface is part of the model itself. We now show a sketch of the derivation of our model (1.2), taken directly from [56].

4.1 Derivation

For consistency with the literature, see e.g. [49, 56], to derive this system we start with a description of the surfaces $\{\Gamma(t)\}_t$ given by the flow $\Phi^0_t \colon \Gamma_0 \to \Gamma(t)$ as in $\mathbf{A}_\Phi$. Assume that the surface $\Gamma(t)$ consists of two species with the mass densities per unit area ρ_1 and ρ_2. The total mass of each species is assumed to be conserved. This entails, for the total density $\rho := \rho_1 + \rho_2$, the balance law

$$\frac{d}{dt}\int_{P(t)} \rho \equiv 0,$$

for any surface patch $P(t) \subset \Gamma(t)$ evolving under the full velocity $\mathbf{V}$. By the Reynolds transport theorem, see also Proposition 3.2, and the arbitrariness of $P(t)$ we easily deduce

$$\partial^\bullet \rho + \rho \nabla_\Gamma \cdot \mathbf{V} = 0, \quad \rho(0) = \widehat{\rho},$$

where $\widehat{\rho}$ is the initial density. This equation shows in particular that we have the identity

$$\rho(t, \Phi^0_t(p)) = \frac{\widehat{\rho}}{J^0_t(p)}, \quad \forall p \in \Gamma_0, \tag{4.1}$$

where J^0_t is the area change defined in (2.10).

For the process on the surface we now introduce the dimensionless concentrations $c_i = \rho_i/\rho$, for $i = 1, 2$. Since $c_1 + c_2 = 1$, it is sufficient to consider c_1 to model the local density fractions of the two species. The mass of species 1 in the membrane patch $P(t)$, evolving under $\mathbf{V}$, may only change due to a diffusive mass flux q_d at the boundary, so that

$$\int_{P(t)} \rho \partial^\bullet c_1 = \frac{d}{dt}\int_{P(t)} \rho c_1 = -\int_{\partial P(t)} q_d \cdot \boldsymbol{\mu} = -\int_{P(t)} \nabla_\Gamma \cdot q_d, \tag{4.2}$$

where $\boldsymbol{\mu}$ denotes the outer unit conormal of $\partial P(t)$. The last identity comes from the fact that we can directly consider, without loss of generality, q_d to be purely tangential to $\Gamma(t)$. The diffusive flux q_d is related to the chemical potential in the following sense. In analogy to 3D problems ([15]), the Cahn-Hillard energy per reference area takes the form

$$\Psi_{CH} = \Psi_{mix}(c_1, T) + \Psi_i(J_t^0, \nabla_\Gamma c_1),$$

where T is the absolute temperature and, up to setting some constants to 1 for simplicity,

$$\Psi_{mix}(c_1, T) = T\,(c_1 \ln(c_1) + (1 - c_1)\ln(1 - c_1)) + c_1(1 - c_1),$$
$$\Psi_i(J_t^0, \nabla_\Gamma c_1) = \frac{\lambda}{2} J_t^0 |\nabla_\Gamma c_1|^2,$$

with $\lambda > 0$ a coefficient related to the width of the phase interface. As noted in [56, (35)], the area change J_t^0 is included in Ψ_i, since Ψ_{CH} is postulated to be an energy w.r.t. the reference configuration Γ_0, while $\nabla_\Gamma c_1$ refers to the current configuration (it can be viewed as having units of 1/(current length)). If we then want a Cahn-Hilliard energy per current area instead of one per reference area we simply need to multiply Ψ_{CH} by $J_0^t = (J_t^0)^{-1}$; in effect, we have (current area)/(reference area)= J_t^0, thus, heuristically,

$$\frac{\text{energy}}{\text{current area}} = \frac{\text{reference area}}{\text{current area}}\,\frac{\text{energy}}{\text{reference area}} = \frac{1}{J_t^0}\,\frac{\text{energy}}{\text{reference area}},$$

so that the total energy on the current configuration $\Gamma(t)$ can be defined as

$$E^\rho_{CH}(c_1) := \int_{\Gamma(t)} J_0^t \Psi_{CH} = \int_{\Gamma(t)} J_0^t \Psi_{mix} + \frac{\lambda}{2}\int_{\Gamma(t)} |\nabla_\Gamma c_1|^2. \tag{4.3}$$

Then, following the thermodynamical derivation in [56, Appendix A] one can obtain that the chemical potential w is defined by [1]

$$w = J_t^0 \frac{\delta E^\rho_{CH}}{\delta c_1} = -J_t^0 \lambda \Delta_\Gamma c_1 + \Psi'_{mix}(c_1), \tag{4.4}$$

and the diffusive flux is (see [56, (121)])

$$q_d = -\frac{M}{J_t^0}\nabla_\Gamma w,$$

where $M > 0$ is a mobility coefficient. The presence of the term $1/J_t^0$ in q_d can again be heuristically explained by the fact that w is defined per reference area, whereas q_d is defined per current area. All in all, by setting for simplicity $M = 1$, $\lambda = 1$ and $\widehat{\rho} \equiv 1$, so that $J_t^0 = 1/\rho$, from (4.2) we are led to

$$\rho \dot{c}_1 - \nabla_\Gamma \cdot \left(\rho \nabla_\Gamma \left(-\frac{1}{\rho}\Delta_\Gamma c_1 + \Psi'_{mix}(c_1)\right)\right) = 0.$$

Problem (1.2) can be retrieved by rewriting the equation for the dimensionless concentration difference $c := \frac{\rho_1 - \rho_2}{\rho} = 2c_1 - 1$ and making a proper rescaling on Ψ_{mix} so that it coincides with the potential F defined in (1.7).

Remark 4.1. *Two observations are timely regarding the relation between models (1.1) and (1.2).*

1. *As observed in [55], model (1.2) is similar to model (1.1), but neither is a particular case of the other. In model (1.1), the equation is written for the conserved variable $u = \rho c$ and under the assumption that the free energy functional depends on the concentration difference u rather than on the relative concentration difference c as in model (1.2). Formally, systems (1.1) and (1.2) are the same problem for inextensible membranes (i.e., $\nabla_\Gamma \cdot \boldsymbol{V} \equiv 0$) and if one assumes $\rho = const$. Here the situation partially resembles the coupling of a two-component compressible fluid flow with dissipative Ginzburg–Landau interface dynamics discussed in [40], where, depending on the choice of the variables to define the energy functional, the effects of compressibility have to be considered in the definition of the chemical potential. These differences between the two models explain why the conserved quantities (1.3)-(1.4) are in principle different.*

2. *In the derivation of (1.2), we think of the tangential component $\mathbf{V}_\tau$ as fixing a parametrisation of $\Gamma(t)$, as well as describing advection on the surface; note that the balance laws are considered on portions evolving under the full velocity $\mathbf{V}$. This is consistent with the framework of the physics literature, namely [49, 56], in which this tangential component $\boldsymbol{V}_\tau$ is part of the unknowns. We can also see this model in the light of (1.1) by heuristically taking $\mathbf{V}_\tau = \mathbf{V}_a$, so that the term involving both velocities vanishes and the notion of a material time derivative coincides with both the one in our analytical framework, i.e. involving the parametrisation, and the usual physical notion of including the advection of material points on the surface.*

[1] As already noticed, this is performed in [56, Appendix A] in the case of an elastic surface, but we keep the same derivation as a first step in the analysis of the more complex model.

4.2 Weak formulation

Let Assumption $\mathbf{A}_\Phi$ hold. Then the problem reads: find a pair (c, w) such that, for all $\eta \in L^2_{H^1}$,

$$m_\star(\rho\partial^\bullet c, \eta) + \int_{\Gamma(t)} \rho\nabla_\Gamma w \cdot \nabla_\Gamma \eta = 0, \tag{4.5}$$

$$\int_{\Gamma(t)} \nabla_\Gamma c \cdot \nabla_\Gamma \eta + \int_{\Gamma(t)} \rho F'(c)\eta = \int_{\Gamma(t)} \rho w\eta, \tag{4.6}$$

$$c(0) = c_0, \qquad \text{a.e. on } \Gamma_0. \tag{4.7}$$

Here the weight function ρ, i.e., the total density, is determined by the transport equation

$$\partial^\bullet \rho + \rho\nabla_\Gamma \cdot \mathbf{V} = 0, \tag{4.8}$$

with $\rho(0) \equiv 1$ on Γ_0. As noticed in Section 4.1, we have an interesting characterisation of ρ:

$$\rho(t, \Phi(t, p)) = (J^0_t(p))^{-1}, \quad \forall p \in \Gamma_0. \tag{4.9}$$

In particular, we also have the bounds

$$0 < \frac{1}{C_\rho} \le \rho \le C_\rho, \quad |\nabla_\Gamma \rho| \le C_\rho, \tag{4.10}$$

with $C_\rho > 1$. Note that the total energy of the system reads (see also (4.3), but with respect to the relative concentration difference c)

$$E^\rho_{CH}(c) := \int_{\Gamma(t)} \left(\frac{|\nabla_\Gamma c|^2}{2} + \rho F(c) \right).$$

We recall the following result (see [13, Thm.6.2]). The statement below and its proof involve the *weighted inverse Laplacian* operator, which we denote as A^{-1}_ρ, define to be, for any

$$f \in H^{-1}(\Gamma(t)) \quad \text{such that} \quad m_\star(\rho f, 1) = 0,$$

the unique solution $\zeta := A^{-1}_\rho f$ to the problem

$$\int_{\Gamma(t)} \rho\nabla_\Gamma \zeta \cdot \nabla_\Gamma \eta = m_\star(\rho f, \eta) \quad \text{with} \quad \int_{\Gamma(t)} \zeta = 0. \tag{4.11}$$

We define also, for such elements f, the weighted norm

$$\|f\|_{\rho,-1} := \|\sqrt{\rho}\,\nabla_\Gamma \zeta\| = \langle \rho f, \zeta\rangle. \tag{4.12}$$

Note that there exist $C_1, C_2 > 0$ such that $C_1\|\nabla_\Gamma \zeta\| \le \|f\|_{\rho,-1} \le C_2\|\nabla_\Gamma \zeta\|$.

Theorem 4.2. *Let $c_0 \in H^1(\Gamma_0)$, $|c_0| \le 1$, $|(c_0)_{\Gamma_0}| < 1$ and $F : [-1, 1] \to \mathbb{R}$ be given by (3.6). Then there exists a unique pair (c, w) with*

$$c \in L^\infty_{H^1} \cap H^1_{H^{-1}}, \quad w \in L^2_{H^1},$$

such that, for almost any $t \in [0, T]$, $|c(t)| < 1$ almost everywhere in $\Gamma(t)$ and (c, w) satisfies, for almost any $t \in [0, T]$, (4.5)-(4.6), with $c(0) = c_0$ almost everywhere in Γ_0. The solution c also satisfies the additional regularity

$$c \in C^0_{L^2} \cap L^\infty_{L^p} \cap L^2_{H^2},$$

for all $p \in [1, +\infty)$. Furthermore, if $c_{0,1}, c_{0,2} \in H^1(\Gamma_0)$, satisfying the existence assumptions, are such that $(c_{0,1})_{\Gamma_0} = (c_{0,2})_{\Gamma_0}$, and c_1, c_2 are the solutions of the system with $c_1(0) = c_{0,1}$ and $c_2(0) = c_{0,2}$, then there exists a constant $C > 0$ independent of t, such that, for almost any $t \in [0, T]$,

$$\|c_1(t) - c_2(t)\|_{\rho,-1} \le e^{Ct}\|c_{0,1} - c_{0,2}\|_{\rho,-1}. \tag{4.13}$$

Remark 4.3. *The assumption $|(c_0)_{\Gamma_0}| < 1$ is necessary and standard when dealing with Cahn-Hilliard equations, since it simply excludes that the initial datum is a pure phase, i.e, $c_0 \equiv 1$ or $c_0 \equiv -1$. Indeed, in these cases, the phase separation phenomenon would not take place, due to the presence of one single substance.*

The proof of existence follows from the uniform estimates obtained in the proof of Theorem 4.6 below. For completeness, we now include a proof of continuous dependence on the initial data, entailing uniqueness, which was omitted in [13].

Proof. (Stability of weak solutions to (4.5)-(4.7)) The density ρ is determined by a single ordinary differential equation which can be solved explicitly, and it is thus unique. Now suppose (c_1, w_1) and (c_2, w_2) both solve (4.5), (4.6), with initial conditions $c_1(0) = c_{0,1}$ and $c_2(0) = c_{0,2}$, such that $(c_{0,1})_{\Gamma_0} = (c_{0,2})_{\Gamma_0}$. Denote $\xi^c = c_1 - c_2$, $\xi^w = w_1 - w_2$. Subtracting the corresponding equations leads to

$$m_\star(\rho\partial^\bullet\xi^c, \eta) + \int_{\Gamma(t)} \rho\nabla_\Gamma\xi^w \cdot \nabla_\Gamma\eta = 0 \tag{4.14}$$

$$\int_{\Gamma(t)} \nabla_\Gamma\xi^c \cdot \nabla_\Gamma\eta + \int_{\Gamma(t)} \rho(F'(c_1) - F'(c_2))\eta = \int_{\Gamma(t)} \rho\xi^w\eta. \tag{4.15}$$

Using the ODE (4.8) for ρ and (4.5), noting that $\xi^c(0) = c_{0,1} - c_{0,2}$, we obtain

$$\frac{d}{dt}\int_{\Gamma(t)} \rho\xi^c = m_\star(\rho\partial^\bullet\xi^c, 1) = 0 \quad \Longrightarrow \quad \int_{\Gamma(t)} \rho\xi^c = \int_{\Gamma_0} c_{0,1} - c_{0,2} \ \equiv 0.$$

Therefore the weighted inverse Laplacian $(\Delta_{\Gamma,\rho}^{-1})\xi^c$ is well defined, and taking $\eta = (\Delta_{\Gamma,\rho}^{-1})\xi^c$ in (4.14) we get

$$m_\star(\rho\partial^\bullet\xi^c, (\Delta_{\Gamma,\rho}^{-1})\xi^c) + \int_{\Gamma(t)} \rho\nabla_\Gamma\xi^w \cdot \nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c = 0. \tag{4.16}$$

Now note:

$$\begin{aligned}
m_\star(\rho\partial^\bullet\xi^c, (\Delta_{\Gamma,\rho}^{-1})\xi^c) &= \frac{d}{dt}\left[\int_{\Gamma(t)} \rho\xi^c(\Delta_{\Gamma,\rho}^{-1})\xi^c\right] - \int_{\Gamma(t)} \rho\xi^c\partial^\bullet(\Delta_{\Gamma,\rho}^{-1})\xi^c \\
&= \frac{d}{dt}\left[\int_{\Gamma(t)} \rho|\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c|^2\right] - \int_{\Gamma(t)} \rho\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c \cdot \nabla_\Gamma\partial^\bullet(\Delta_{\Gamma,\rho}^{-1})\xi^c \\
&= \frac{d}{dt}\|\xi^c\|_{\rho,-1}^2 - \int_{\Gamma(t)} \rho\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c \cdot \nabla_\Gamma\partial^\bullet(\Delta_{\Gamma,\rho}^{-1})\xi^c
\end{aligned}$$

and

$$\int_{\Gamma(t)} \rho\nabla_\Gamma\xi^w \cdot \nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c = \int_{\Gamma(t)} \rho\xi^w\xi^c$$

so that (4.16) becomes

$$\frac{d}{dt}\|\xi^c\|_{\rho,-1}^2 + \int_{\Gamma(t)} \rho\xi^w\xi^c = \int_{\Gamma(t)} \rho\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c \cdot \nabla_\Gamma\partial^\bullet(\Delta_{\Gamma,\rho}^{-1})\xi^c. \tag{4.17}$$

We now also test (4.15) with $\eta = \xi^c$ to obtain

$$\|\nabla_\Gamma\xi^c\|^2 + \int_{\Gamma(t)} \rho(F'(c_1) - F'(c_2))\xi^c = \int_{\Gamma(t)} \rho\xi^w\xi^c. \tag{4.18}$$

Due to the structure of the logarithmic potential F, being F_{ln} strictly convex, we have the estimate

$$\int_{\Gamma(t)} \rho(F'(c_1) - F'(c_2))\xi^c \geq -C\|\sqrt{\rho}\xi^c\|^2$$

from where (4.18) becomes

$$\|\nabla_\Gamma\xi^c\|^2 \leq \int_{\Gamma(t)} \rho\xi^w\xi^c + C\|\sqrt{\rho}\xi^c\|^2. \tag{4.19}$$

Adding (4.17) and (4.19) the terms involving the product $\xi^w\xi^c$ cancel out and we are led to

$$\frac{d}{dt}\|\xi^c\|_{\rho,-1}^2 + \|\nabla_\Gamma\xi^c\|^2 \leq C\|\sqrt{\rho}\xi^c\|^2 + \int_{\Gamma(t)} \rho\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c \cdot \nabla_\Gamma\partial^\bullet(\Delta_{\Gamma,\rho}^{-1})\xi^c. \tag{4.20}$$

We now estimate the first term on the right-hand side as

$$C\|\sqrt{\rho}\xi^c\|^2 = C\int_{\Gamma(t)} \rho|\xi^c|^2 = C\int_{\Gamma(t)} \rho\nabla_\Gamma\xi^c \cdot \nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c \leq \frac{1}{4}\|\nabla_\Gamma\xi^c\|^2 + C\|\xi^c\|_{\rho,-1}^2.$$

For the second term, we note that

$$\int_{\Gamma(t)} \rho\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c \cdot \nabla_\Gamma\partial^\bullet(\Delta_{\Gamma,\rho}^{-1})\xi^c \le \frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)} \rho|\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c|^2 + C\int_{\Gamma(t)} \rho|\nabla_\Gamma(\Delta_{\Gamma,\rho}^{-1})\xi^c|^2$$
$$\le \frac{1}{2}\frac{d}{dt}\|\xi^c\|_{\rho,-1}^2 + C\|\xi^c\|_{\rho,-1}^2.$$

All in all, we obtain from (4.20) the estimate

$$\frac{d}{dt}\|\xi^c\|_{\rho,-1}^2 + \|\nabla_\Gamma\xi^c\|^2 \le C\|\xi^c\|_{\rho,-1}^2,$$

and an application of Gronwall's Lemma implies the continuous dependence estimate stated in (4.13). Then uniqueness follows by setting $c_{0,1} \equiv c_{0,2}$. □

Remark 4.4. *Notice that the regularity stated in Theorem 4.2 can be slightly improved as in the case of the first model (see Remark 3.5). In particular, since $c \in L^2_{H^2}$ solves the problem, for almost all $t \in [0,T]$,*

$$-\frac{1}{\rho}\Delta_\Gamma c(t) = w(t) - F'(c(t)) \in L^2(\Gamma(t)),$$

we are allowed to multiply by $-\Delta_\Gamma c \in L^2(\Gamma(t))$ for almost any $t \in [0,T]$. Recalling that $\varphi' > 0$, after an integration by parts, being $\Gamma(t)$ closed and $c \in L^\infty_{H^1}$, we obtain, by (4.10),

$$C\|\Delta_\Gamma c\|^2 \le m(\frac{1}{\rho}\Delta_\Gamma c, \Delta_\Gamma c) + \frac{\theta}{2}m(\varphi'(c), |\nabla_\Gamma c|^2) \le \|\nabla_\Gamma w\|\|\nabla_\Gamma c\| + \|\nabla_\Gamma c\|^2 \le C(1 + \|\nabla_\Gamma w\|),$$

and knowing that $w \in L^2_{H^1}$, we infer $c \in L^4_{H^2}$.

We make use of the following weighted L^2 and H^1 products, whose induced norms are equivalent norms on $L^2(\Gamma(t))$ and $H^1(\Gamma(t))$, respectively:

$$(f,g)_\rho := m(\rho f, g), \qquad (f,g)_{1,\rho} := (f,g)_\rho + \int_{\Gamma(t)} \rho\nabla_\Gamma f \cdot \nabla_\Gamma g. \tag{4.21}$$

Notice that for $t = 0$ these definitions coincide with the natural norms on these spaces, being $\rho(0,x) \equiv 1$ for any $x \in \Gamma_0$. We now give an extension of Proposition 3.2, whose proof is shown in Appendix A:

Proposition 4.5. *For $\eta, \phi \in H^1_{L^2}$, with $\nabla_\Gamma\partial^\bullet\phi \in L^2_{L^2}$ and $\nabla_\Gamma\partial^\bullet\eta \in L^2_{L^2}$, the following identity holds*

$$\frac{d}{dt}(\nabla_\Gamma\eta, \nabla_\Gamma\phi)_\rho = m(\partial^\bullet\rho\nabla_\Gamma\eta, \nabla_\Gamma\phi) + (\nabla_\Gamma\partial^\bullet\eta, \nabla_\Gamma\phi)_\rho + (\nabla_\Gamma\eta, \nabla_\Gamma\partial^\bullet\phi)_\rho + \int_{\Gamma(t)} \rho B(\mathbf{V})\nabla_\Gamma\eta \cdot \nabla_\Gamma\phi$$
$$= (\nabla_\Gamma\partial^\bullet\eta, \nabla_\Gamma\phi)_\rho + (\nabla_\Gamma\eta, \nabla_\Gamma\partial^\bullet\phi)_\rho + \int_{\Gamma(t)} \rho\widetilde{B}(\mathbf{V})\nabla_\Gamma\eta \cdot \nabla_\Gamma\phi, \tag{4.22}$$

for almost any $t \in [0,T]$, where B is the vector field given in Section 3.2 and $\widetilde{B}(\mathbf{V}) := -2\mathbf{D}(\mathbf{V})$.

In conclusion, being $\partial^\bullet\rho = -\rho\nabla_\Gamma \cdot \mathbf{V}$, we infer by Proposition 3.2

$$\frac{d}{dt}(\eta,\phi)_\rho = (\partial^\bullet\eta, \phi)_\rho + (\eta, \partial^\bullet\phi)_\rho, \quad \forall\, \eta, \phi \in H^1_{L^2}, \tag{4.23}$$

retrieving the classical integration by parts formula as in the case of a fixed manifold. This is not surprising, in the sense that the term ρ accounts for local stretching or compressing of the surfaces and somehow annihilates the effects of the evolving surface.

4.3 Regularisation and strict separation property

For this model we only need to assume $\mathbf{A}_\Phi$, without any extra regularity hypothesis. Our main result is the following

Theorem 4.6. *Let the assumptions of Theorem 4.2 hold. Denote by (c,w) the (unique) weak solution to (4.5)-(4.7).*

(i) There exists a constant $C = C(T, E^\rho_{CH}(u_0)) > 0$ such that, for almost any $t \in [0,T]$,

$$t\|w\|^2_{H^1(\Gamma(t))} + \int_0^t s\|\partial^\bullet c\|^2_{H^1(\Gamma(s))}ds \le C(T, E^\rho_{CH}(u_0)). \tag{4.24}$$

(ii) For any $0 < \tau \leq T$, there exist constants $C = C(T, \tau, E^\rho_{CH}(u_0)) > 0$ and $C_p = C_p(T, \tau, p, E^\rho_{CH}(u_0)) > 0$ such that, for almost any $t \in [\tau, T]$

$$\|w\|_{H^1(\Gamma(t))} \leq C(T, \tau, E^\rho_{CH}(u_0)), \tag{4.25}$$

$$\|\varphi(c)\|_{L^p(\Gamma(t))} + \|\varphi'(c)\|_{L^p(\Gamma(t))} \leq C_p(T, \tau, p, E^\rho_{CH}(u_0)), \quad \forall p \in [2, \infty),$$

$$\|c\|_{H^2(\Gamma(t))} \leq C(T, \tau, E^\rho_{CH}(u_0)).$$

(iii) There exists $\xi = \xi(T, \tau, E^\rho_{CH}(u_0)) > 0$, such that

$$\|c\|_{L^\infty(\Gamma(t))} \leq 1 - \xi, \qquad \text{for almost any } t \in [\tau, T].$$

Remark 4.7. *For this second model the simple assumption $\boldsymbol{A}_\Phi$ is enough. Indeed, due to the presence of ρ we do not need, e.g., (2.9). For this reason, it is our belief that the second model seems more natural in the context of evolving surfaces.*

Remark 4.8. *As in Remark 3.10, assuming the regularity stated in Lemma C.1, also in this case we actually obtain that, for any $\tau > 0$, $c \in C^0_{C^0}(\tau, T)$ and thus*

$$\sup_{t \in [\tau, T]} \|c\|_{C^0(\Gamma(t))} \leq 1 - \xi.$$

4.3.1 Galerkin approximation

We need a slight revision of the Galerkin approximation scheme. Let us recall [13, Sec.4.1] and consider a basis $\{\chi^0_j : j \in \mathbb{N}\}$ orthonormal in $L^2(\Gamma_0)$ and orthogonal in $H^1(\Gamma_0)$ consisting of smooth functions such that χ^0_1 is constant (for example consider the eigenfunctions of the Laplace-Beltrami operator). We then transport this basis using the flow map. This gives $\{\chi^t_j := \phi_t(\chi^0_j) : j \in \mathbb{N}\} \subset H^1(\Gamma(t))$. Observe that this basis is still an orthonormal basis in $L^2(\Gamma(t))$ endowed with the norm induced by (4.21). Indeed, thanks to (4.9), we have

$$(\chi^t_i, \chi^t_j)_\rho = m(\rho \chi^t_i, \chi^t_j) = \int_{\Gamma_0} \chi^0_i \chi^0_j = \delta_{ij}, \quad \forall\, i, j \in \mathbb{N}.$$

Notice that the same basis is such that $\overline{\text{span}\{\chi^t_j : j \in \mathbb{N}\}}^{H^1(\Gamma(t))} = H^1(\Gamma(t))$, but we are not able to show any orthogonality property in this space, even endowing the space with the norm induced by (4.21). We can then introduce the finite dimensional spaces

$$V_M(t) = \text{span}\{\chi^t_j : 1 \leq j \leq M\} \subset H^1(\Gamma(t)),$$

and, exploiting the orthogonality with respect to the equivalent L^2 inner product $(4.21)_1$, we can give an explicit expression to the L^2-orthogonal projector $P^\rho_M(t) : L^2(\Gamma(t)) \to V_M(t)$ as

$$P^\rho_M(t) f := \sum_{j=1}^M (f, \chi^t_j)_\rho \chi^t_j = \sum_{j=1}^M \left(\int_{\Gamma_0} \widetilde{f} \chi^0_j \right) \chi^t_j = \phi_t(P^0_M \widetilde{f}),$$

where $P^0_M = P^\rho_M(0)$ is the orthogonal projector over $V_M(0)$ (endowed with the canonical L^2-norm). Clearly this implies $\|P^\rho_M(t) f\|_\rho \leq \|f\|_\rho$ uniformly in M, for any $t > 0$. Following [4, Sec.7], we now introduce the matrix

$$\mathbf{A}^0_t := (D\Phi^0_t)^T D\Phi^0_t + \nu_0 \otimes \nu_0,$$

where ν_0 is the normal to Γ_0. This matrix is invertible in $\mathbb{R}^3$ thanks to the extension in the normal direction. We then have that (see [4, (7.5)])

$$\phi_{-t}(\nabla_\Gamma h) = D\Phi^0_t (\mathbf{A}^0_t)^{-1} \nabla_{\Gamma_0}(\phi_{-t}(h)) = D\Phi^0_t (\mathbf{A}^0_t)^{-1} \nabla_{\Gamma_0} \widetilde{h}. \tag{4.26}$$

Note also that, for any $t \in [0, T]$,

$$\|D\Phi^0_t (\mathbf{A}^0_t)^{-1}\|_{C^0(\Gamma(t))} \leq C, \tag{4.27}$$

being, thanks to assumption $\mathbf{A}_\Phi$, $\Phi^0_{(\cdot)} \in C^1([0, T]; C^2(\mathbb{R}^3; \mathbb{R}^3))$. Therefore, again by (4.9), the properties of the orthogonal basis $\{\chi^0_j : j \in \mathbb{N}\}$ in $H^1(\Gamma_0)$, and the compatibility of the space $(H^1(\Gamma(t)), \phi_t)_{t \in [0,T]}$,

$$\int_{\Gamma(t)} \rho \nabla_\Gamma w \cdot \nabla_\Gamma P^\rho_M(t) f = \int_{\Gamma_0} \nabla_{\Gamma_0} \widetilde{w} (D\Phi^0_t)^T (\mathbf{A}^0_t)^{-T} D\Phi^0_t (\mathbf{A}^0_t)^{-1} \nabla_{\Gamma_0} P^0_M \widetilde{f}$$

$$
\begin{aligned}
&\le C\|\nabla_{\Gamma_0}\widetilde{w}\|_{L^2(\Gamma_0)}\|\nabla_{\Gamma_0}P_M^0\widetilde{f}\|_{L^2(\Gamma_0)}\\
&\le C\|\nabla_{\Gamma_0}\widetilde{w}\|_{L^2(\Gamma_0)}\|\nabla_{\Gamma_0}\widetilde{f}\|_{L^2(\Gamma_0)}\\
&\le C\|\nabla_\Gamma w\|\|\nabla_\Gamma f\|, \qquad (4.28)
\end{aligned}
$$

for any $w, f \in H^1(\Gamma(t))$. We can also prove that the time derivative $\partial^\bullet$ commutes with the projector $P_M^\rho(t)$. Indeed, since $\partial^\bullet \chi_j^t \equiv 0$ for any $j \in \mathbb{N}$, we have

$$
\begin{aligned}
\partial^\bullet(P_M^\rho(t)f) &= \sum_{i=1}^M \partial^\bullet(f,\chi_i^t)_\rho \chi_i^t\\
&= \sum_{i=1}^M \left((\partial^\bullet f,\chi_i^t)_\rho + m(f\partial^\bullet\rho,\chi_i^t) + g(\rho f,\chi_i^t)\right)\chi_i^t\\
&= \sum_{i=1}^M (\partial^\bullet f,\chi_i^t)_\rho \chi_i^t\\
&= P_M^\rho(t)\partial^\bullet f, \quad \forall f \in H^1_{L^2},
\end{aligned} \tag{4.29}
$$

where we exploited the fact that $\partial^\bullet\rho = -\rho\nabla_\Gamma\cdot\mathbf{V}$.

We then consider the Galerkin approximation with the approximated potential F^δ ($\varphi^\delta = (F^\delta_{ln})'$) of the original problem. More precisely, given the spaces V_M as in (3.10), for each $M \in \mathbb{N}$, find functions $c_\delta^M, w_\delta^M \in L^2_{V_M}$ with $\partial^\bullet c_\delta^M \in L^2_{V_M}$ such that, for any $\eta \in L^2_{V_M}$ and all $t \in [0,T]$,

$$
(\partial^\bullet c_\delta^M,\eta)_\rho + (\nabla_\Gamma w_\delta^M,\nabla_\Gamma\eta)_\rho = 0, \tag{4.30}
$$

$$
w_\delta^M = P_M^\rho(t)(-\frac{1}{\rho}\Delta_{\Gamma(t)}c_\delta^M + (F^\delta)'(c_\delta^M)), \tag{4.31}
$$

$$
c_\delta^M(0) = P_M^0 c_0, \qquad \text{a.e. on } \Gamma_0. \tag{4.32}
$$

Notice that this formulation, written with respect to the ρ inner product, is very similar the classical weak formulation of CH equation in bounded domains of $\mathbb{R}^2$ (or $\mathbb{R}^3$). For this problem we then have

Proposition 4.9. *Let Assumption $\boldsymbol{A}_\Phi$ hold. Then there exists a unique local solution (c_δ^M, w_δ^M) to (4.30)-(4.32). In particular there exist functions (c^M, w^M) satisfying (4.30)-(4.31) on an interval $[0,t^\star)$, $0 \le t^\star \le T$, together with (4.32). The functions are of the form (omitting for simplicity the dependence on δ)*

$$
c_\delta^M(t) = \sum_{i=1}^M c_i^M(t)\chi_i^t, \qquad w_\delta^M(t) = \sum_{i=1}^M w_i^M(t)\chi_i^t, \qquad t\in[0,t^\star),
$$

with $c_i^M \in C^2([0,t^\star))$ and $w_i^M \in C^2([0,t^\star))$, for every $i \in \{1,\dots,M\}$.

Proof. We consider the matrix form of the equations, where, as before, we set $\mathbf{c}^M(t) = (c_1^M(t),\dots,c_M^M(t))$ and $\mathbf{w}^M(t) = (w_1^M(t),\dots,w_M^M(t))$,

$$
\begin{aligned}
&M^\rho(t)\dot{\mathbf{c}}^M(t) + A_S^\rho(t)\mathbf{w}^M(t) = 0,\\
&A_S(t)\mathbf{c}^M(t) + (\mathbf{F}_\rho^\delta)'(\mathbf{c}^M(t)) - M^\rho(t)\mathbf{w}^M(t) = 0.
\end{aligned}
$$

Here

$$
\begin{aligned}
(M^\rho(t))_{ij} &= (\chi_i^t,\chi_j^t)_\rho = \delta_{ij},\\
(A_S(t))_{ij} &= a_S(t;\chi_i^t,\chi_j^t),\\
(A_S^\rho(t))_{ij} &= (\nabla_\Gamma\chi_i^t,\nabla_\Gamma\chi_j^t)_\rho,\\
(\mathbf{F}_\rho^\delta)'(\mathbf{c}^M(t))_j &= ((F^\delta)'(c^M(t)),\chi_j^t)_\rho.
\end{aligned}
$$

We now observe that again these matrices are more regular. Indeed, M_{ij}^ρ is actually the identity matrix. Similarly, we get

$$
\frac{d}{dt}(A_S(t))_{ij} = b(\chi_i^t,\chi_j^t) \in C^0([0,T]),
$$

and the same goes for A_S^ρ, by (4.22). Recalling then that $(F^\delta)'$ is $C^{1,1}(\mathbb{R})$, the result follows from the general theory of ODEs. ☐

We establish some a priori estimates for the solutions of the Galerkin approximation.

Proposition 4.10. *For the approximating solution pair (u_δ^M, w_δ^M) we have:*

$$\sup_{t\in[0,T]} \tilde{E}_{CH}^\rho(c_\delta^M) + \frac{1}{2}\int_0^T \int_{\Gamma(t)} |\nabla_\Gamma w_\delta^M|^2 dt \le C(T) \tag{4.33}$$

$$\|\partial^\bullet c_\delta^M\|_{L^2_{H^{-1}}} \le C(T), \quad \text{and} \quad \|w_\delta^M\|_{L^2_{H^1}} \le C_\delta, \tag{4.34}$$

where as before $\tilde{E}_{CH}^\rho := E_{CH}^\rho + \tilde{C}$ for some $\tilde{C} > 0$ chosen so that $\tilde{E}_{CH}^\rho \ge 0$.

Proof. We consider the Galerkin approximation of Proposition 4.9. First notice that we have the conservation of the total mass, by choosing $\eta \equiv 1$:

$$\frac{d}{dt}\int_{\Gamma(t)} \rho c_\delta^M = \int_{\Gamma(t)} \rho \partial^\bullet c_\delta^M \equiv 0. \tag{4.35}$$

Arguing as in [13, (6.2.1)] we can obtain

$$\frac{d}{dt}E_{CH}^\rho(c_\delta^M) + \int_{\Gamma(t)} \rho|\nabla_\Gamma w_\delta^M|^2 dt = b(c_\delta^M, c_\delta^M). \tag{4.36}$$

Observe also that, by the conservation of total mass, see (1.4) and (4.35),

$$(\tilde{c}_\delta^M)_{\Gamma_0} \equiv \frac{\int_{\Gamma(t)} \rho c_\delta^M}{|\Gamma_0|} \equiv (c_0)_{\Gamma_0}.$$

In the end we easily get

$$\frac{d}{dt}\tilde{E}_{CH}^\rho(c_\delta^M) + \frac{1}{2}\int_{\Gamma(t)} \rho|\nabla_\Gamma w_\delta^M|^2 dt \le C\tilde{E}_{CH}^\rho(c_\delta^M), \tag{4.37}$$

where $\tilde{E}_{CH}^\rho(c_\delta^M) := E_{CH}^\rho(c_\delta^M) + C$, so that $\tilde{E}_{CH}^\rho(c_\delta^M) \ge 0$. Therefore, an application of Gronwall's Lemma gives

$$\sup_{t\in[0,T]} \tilde{E}_{CH}^\rho(c_\delta^M) + \frac{1}{2}\int_0^T \int_{\Gamma(t)} |\nabla_\Gamma w_\delta^M|^2 dt \le C(T)\tilde{E}_{CH}^\rho(P_M^0 c_0), \tag{4.38}$$

and, recalling the properties of P_M^0 (see, e.g., [13, Lemma 5.6]), we have

$$\tilde{E}_{CH}^\rho(P_M^0 c_0) \le CT. \tag{4.39}$$

This clearly allows us to extend the maximal time from $t^\star$ to T. Concerning the estimate of the mean value of c, since in this model this quantity is not conserved, being conserved the product ρc, we can observe, as in [13], that

$$\frac{d}{dt}\int_{\Gamma(t)} c_\delta^M = \int_{\Gamma(t)} \partial^\bullet c_\delta^M + \int_{\Gamma(t)} c_\delta^M \nabla_\Gamma \cdot \mathbf{V}.$$

If we now take $\eta = P_M^\rho(t)\frac{1}{\rho}$ in (4.30), we have, being $\partial^\bullet c_\delta^M \in L^2_{V_M}$ and $P_M^\rho(t)$ self-adjoint,

$$\left(\partial^\bullet c_\delta^M, P_M^\rho(t)\frac{1}{\rho}\right)_\rho = m(\partial^\bullet c_\delta^M, 1) = -\left(\nabla_\Gamma w_\delta^M, \nabla_\Gamma P_M^\rho(t)\frac{1}{\rho}\right)_\rho.$$

Therefore, integrating by parts $\int_{\Gamma(t)} c_\delta^M \nabla_\Gamma \cdot \mathbf{V}$, recalling that $\Gamma(t)$ is closed, we get

$$\frac{d}{dt}\int_{\Gamma(t)} c_\delta^M = -\left(\nabla_\Gamma w_\delta^M, \nabla_\Gamma P_M^\rho(t)\frac{1}{\rho}\right)_\rho - \int_{\Gamma(t)} \nabla_\Gamma c_\delta^M \cdot \mathbf{V},$$

so that

$$\int_{\Gamma(t)} c_\delta^M = \int_{\Gamma_0} P_M^0 c_0 - \int_0^t \left(\nabla_{\Gamma(s)} w_\delta^M, \nabla_{\Gamma(s)} P_M^\rho(s)\frac{1}{\rho}\right)_\rho ds - \int_0^t \int_{\Gamma(s)} \nabla_\Gamma c_\delta^M \cdot \mathbf{V} ds.$$

Thus, by (4.10), (4.39), the properties of P_M^0, and (4.28), we deduce

$$\left|\int_{\Gamma(t)} c_\delta^M\right| \le C + C\int_0^t \|\nabla_{\Gamma(s)} w_\delta^M\| \left\|\nabla_{\Gamma(s)}\frac{1}{\rho}\right\| ds + C(T)$$

$$\leq C\left(1+\int_0^t \|\nabla_\Gamma w_\delta^M\| ds\right) + C(T) \leq C(T).$$

Combining this result with (4.38) and Poincaré's inequality, we infer that

$$\|c_\delta^M\|_{L^\infty_{H^1}} \leq C(T), \tag{4.40}$$

independently of M and δ. Observe now that, by (4.28), we can find a uniform estimate for $\partial^\bullet c_\delta^M$. Indeed we can write, for any $\eta \in L^2_{H^1}$, being $P_M^\rho(t)$ self-adjoint (with respect to $(4.21)_1$) and by (4.28) and (4.40),

$$\begin{aligned}
m_\star(\partial^\bullet c_\delta^M, \eta) = (P_M^\rho(t)\partial^\bullet c_\delta^M, \eta) &= \left(P_M^\rho(t)\partial^\bullet c_\delta^M, \frac{1}{\rho}\eta\right)_\rho \\
&= \left(\partial^\bullet c_\delta^M, P_M^\rho(t)\left(\frac{1}{\rho}\eta\right)\right)_\rho \\
&= -\left(\nabla_\Gamma w_\delta^M, \nabla_\Gamma P_M^\rho(t)\left(\frac{1}{\rho}\eta\right)\right)_\rho \\
&\leq C\|\nabla_\Gamma w_\delta^M\| \left\|\nabla_\Gamma \frac{\eta}{\rho}\right\| + C\|c_\delta^M\| \left\|\nabla_\Gamma \frac{\eta}{\rho}\right\| \\
&\leq C\left(\|\nabla_\Gamma w_\delta^M\| + 1\right) \|\eta\|_{H^1(\Gamma(t))},
\end{aligned}$$

which implies, by (4.38),

$$\|\partial^\bullet c_\delta^M\|_{L^2_{H^{-1}}} \leq C(T), \tag{4.41}$$

independently of M and δ. We now need a control over the mean value of w_δ^M. This can be obtained by testing (4.31) with $\eta = P_M^\rho(t)\frac{1}{\rho}$ being $w \in L^2_{V_M}$. On account of (4.28), we have

$$\begin{aligned}
\left(w_\delta^M, \frac{1}{\rho}\right)_\rho = \left(w_\delta^M, P_M^\rho(t)\frac{1}{\rho}\right)_\rho &= \left(-\frac{1}{\rho}\Delta c_\delta^M, P_M^\rho(t)\frac{1}{\rho}\right)_\rho + \left(F_\delta'(c_\delta^M), P_M^\rho(t)\frac{1}{\rho}\right)_\rho \\
&= m\left(\nabla_\Gamma c_\delta^M, \nabla_\Gamma P_M^\rho(t)\frac{1}{\rho}\right) + \left(F_\delta'(c_\delta^M), P_M^\rho(t)\frac{1}{\rho}\right)_\rho \\
&\leq C\|\nabla_\Gamma c_\delta^M\| \left\|\nabla_\Gamma\left(P_M^\rho(t)\frac{1}{\rho}\right)\right\| + C\|F_\delta'(c_\delta^M)\| \left\|P_M^\rho(t)\frac{1}{\rho}\right\| \\
&\leq C\|\nabla_\Gamma c_\delta^M\| \left\|\nabla_\Gamma \frac{1}{\rho}\right\| + C_\delta\|c_\delta^M\| \leq C_\delta,
\end{aligned}$$

by (4.10) and (4.38). Here we have also applied the fact that $|F_\delta'(c_\delta^M)| \leq C_\delta|c_\delta^M|$, being $|F_\delta''| \leq C_\delta$ and $F_\delta'(0) = F'(0) = 0$. In the Galerkin scheme we are not able to retrieve a uniform-in-δ estimate for the mean value of w_δ^M. Indeed, we should be able to control the $L^\infty(\Gamma(t))$ norm of $\rho P_M^\rho(t)\frac{1}{\rho}$ and then control $\int_{\Gamma(t)} |F_\delta'(c_\delta^M)|$, but this does not seem feasible. Therefore, we will need to pass to the limit in M first. Then this control will be obtained independently of δ. The above bound entails, thanks to Poincaré's inequality combined with (4.33),

$$\|w_\delta^M\|_{L^2_{H^1}} \leq C_\delta. \tag{4.42}$$

□

4.3.2 Proof of Theorem 4.6

Part (i). We now need to find higher-order estimates. In particular, we set $\eta = \partial^\bullet w_\delta^M \in L^2_{V_M}$ in (4.30), to get

$$(\partial^\bullet c_M^\delta, \partial^\bullet w_\delta^M)_\rho + (\nabla_\Gamma w_M^\delta, \nabla_\Gamma \partial^\bullet w_M^\delta)_\rho = 0. \tag{4.43}$$

Observe that, by (4.22),

$$\frac{1}{2}\frac{d}{dt}(\nabla_\Gamma w_\delta^M, \nabla_\Gamma w_\delta^M)_\rho = (\nabla_\Gamma w_M^\delta, \nabla_\Gamma \partial^\bullet w_M^\delta)_\rho + \int_{\Gamma(t)} \rho \widetilde{B}(\mathbf{V}) \nabla_\Gamma w_\delta^M \cdot \nabla_\Gamma w_\delta^M. \tag{4.44}$$

On the other hand, by Proposition 3.2, we have, for any $\eta \in L^2_{V_M}$ such that $\partial^\bullet \eta \in L^2_{V_M}$,

$$\frac{d}{dt} m(\nabla_\Gamma c_\delta^M, \nabla_\Gamma \eta) = m(\nabla_\Gamma \partial^\bullet c_\delta^M, \nabla_\Gamma \eta) + m(\nabla_\Gamma c_\delta^M, \nabla_\Gamma \partial^\bullet \eta) + \int_{\Gamma(t)} B(\mathbf{V}) \nabla_\Gamma c_\delta^M \cdot \nabla_\Gamma \eta$$

and, by (4.23),

$$\frac{d}{dt}(w_\delta^M,\eta)_\rho = (\partial^\bullet w_\delta^M,\eta)_\rho + (w,\partial^\bullet\eta)_\rho.$$

Furthermore, we have, using again (4.23),

$$\frac{d}{dt}(F_\delta'(c_\delta^M),\eta)_\rho = (F_\delta''(c_\delta^M),\partial^\bullet c_\delta^M\eta)_\rho + (F_\delta'(c_\delta^M),\partial^\bullet\eta)_\rho.$$

We now recall that

$$\frac{d}{dt}(w_\delta^M,\eta)_\rho = \frac{d}{dt}m(\nabla_\Gamma c_\delta^M,\nabla_\Gamma\eta) + \frac{d}{dt}(F_\delta'(c_\delta^M),\eta)_\rho,$$

thus

$$\begin{aligned}(\partial^\bullet w_\delta^M,\eta)_\rho + (w,\partial^\bullet\eta)_\rho &= m(\nabla_\Gamma\partial^\bullet c_\delta^M,\nabla_\Gamma\eta) + m(\nabla_\Gamma c_\delta^M,\nabla_\Gamma\partial^\bullet\eta)\\ &\quad+\int_{\Gamma(t)} B(\mathbf{V})\nabla_\Gamma c_\delta^M\cdot\nabla_\Gamma\eta\\ &\quad+(F_\delta''(c_\delta^M),\partial^\bullet c_\delta^M\eta)_\rho + (F_\delta'(c_\delta^M),\partial^\bullet\eta)_\rho.\end{aligned}$$

By noticing that, being $\partial^\bullet\eta\in L^2_{V_M}$,

$$(w,\partial^\bullet\eta)_\rho = m(\nabla_\Gamma c_\delta^M,\nabla_\Gamma\partial^\bullet\eta) + (F_\delta'(c_\delta^M),\partial^\bullet\eta)_\rho,$$

we infer, choosing $\eta=\partial^\bullet c_\delta^M$ (which is sufficiently regular thanks to Proposition 4.9),

$$(\partial^\bullet c_\delta^M,\partial^\bullet w_\delta^M)_\rho = (F_\delta''(c_\delta^M)\partial^\bullet c_\delta^M,\partial^\bullet c_\delta^M)_\rho + \|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2 + b(\mathbf{V};c_\delta^M,\partial^\bullet c_\delta^M)$$

From these results, together with (4.43) and (4.44), we eventually get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\nabla_\Gamma w_\delta^M,\nabla_\Gamma w_\delta^M)_\rho + (F_\delta''(c_\delta^M)\partial^\bullet c_\delta^M,\partial^\bullet c_\delta^M)_\rho + \|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2\\ &= -\int_{\Gamma(t)} B(\mathbf{V})\nabla_\Gamma c_\delta^M\cdot\nabla_\Gamma\partial^\bullet c_\delta^M + \int_{\Gamma(t)}\rho\widetilde{B}(\mathbf{V})\nabla_\Gamma w_\delta^M\cdot\nabla_\Gamma w_\delta^M,\end{aligned}$$

and, recalling the definition of F_δ'' and $\varphi_\delta'\geq 0$, we infer

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\nabla_\Gamma w_\delta^M,\nabla_\Gamma w_\delta^M)_\rho + \|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2 &\leq \frac{1}{2}\frac{d}{dt}(\nabla_\Gamma w_\delta^M,\nabla_\Gamma w_\delta^M)_\rho + \frac{\theta}{2}(\varphi_\delta'(c_\delta^M),\partial^\bullet c_\delta^M\partial^\bullet c_\delta^M)_\rho + \|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2\\ &= (\partial^\bullet c_\delta^M,\partial^\bullet c_\delta^M)_\rho - \int_{\Gamma(t)} B(\mathbf{V})\nabla_\Gamma c_\delta^M\cdot\nabla_\Gamma\partial^\bullet c_\delta^M\\ &\quad+\int_{\Gamma(t)}\rho\widetilde{B}(\mathbf{V})\nabla_\Gamma w_\delta^M\cdot\nabla_\Gamma w_\delta^M.\end{aligned}\tag{4.45}$$

By (4.10), (4.38), interpolation and standard inequalities, we then have

$$\begin{aligned}&(\partial^\bullet c_\delta^M,\partial^\bullet c_\delta^M)_\rho - \int_{\Gamma(t)} B(\mathbf{V})\nabla_\Gamma c_\delta^M\cdot\nabla_\Gamma\partial^\bullet c_\delta^M + \int_{\Gamma(t)}\rho\widetilde{B}(\mathbf{V})\nabla_\Gamma w_\delta^M\cdot\nabla_\Gamma w_\delta^M\\ &\leq C_1\|\partial^\bullet c_\delta^M\|^2 + C + \frac{1}{4}\|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2 + C\|\nabla_\Gamma w_\delta^M\|^2.\end{aligned}$$

Notice that we have only exploited assumption $\mathbf{A}_\Phi$ (see also Remark 4.7). Here $C_1>0$ is a positive constant independent of δ, M. We now test (4.30) by $\eta=\partial^\bullet c_\delta^M$, obtaining, for κ suitably small to be chosen later on,

$$C_2\|\partial^\bullet c_\delta^M\|^2 \leq \kappa\|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2 + C(1+\|\nabla_\Gamma w_\delta^M\|^2),$$

where $C_2>0$ is independent of δ, M. Adding this inequality, multiplied by $\omega = 2\frac{C_1}{C_2}$, and (4.45) together, choosing $\kappa=\frac{C_2}{8C_1}$, and recalling (4.10), we find

$$\frac{d}{dt}\mathcal{Q}_\rho + \frac{1}{2}\|\nabla_\Gamma\partial^\bullet c_\delta^M\|^2 + C_1\|\partial^\bullet c_\delta^M\|^2 \leq C(1+(\nabla_\Gamma w_\delta^M,\nabla_\Gamma w_\delta^M)_\rho),\tag{4.46}$$

where C and C_1 in this estimate do not depend on δ (they rely only on (4.38)), and

$$\mathcal{Q}_\rho := \frac{1}{2}(\nabla_\Gamma w_\delta^M, \nabla_\Gamma w_\delta^M)_\rho \geq 0.$$

We then multiply (4.46) by t. This gives

$$\frac{d}{dt}(t\mathcal{Q}_\rho) + \frac{t}{2}\|\nabla_\Gamma \partial^\bullet c_\delta^M\|^2 + C_1 t\|\partial^\bullet c_\delta^M\|^2 \leq C(t + t\mathcal{Q}_\rho(t)) + \mathcal{Q}_\rho(t). \tag{4.47}$$

Then, thanks to (4.38), we have $\mathcal{Q}_\rho \in L^1(0,T)$, so that we can apply Gronwall's Lemma and infer

$$\|\sqrt{t}\nabla_\Gamma w_\delta^M\|_{L^\infty_{L^2}} + \|\sqrt{t}\partial^\bullet u_\delta^M\|_{L^2_{H^1}} \leq C(T). \tag{4.48}$$

It is crucial to stress again that also the above constant does not depend on δ.

Having this uniform (in M) regularity, we can easily pass to the limit as $M \to \infty$ and obtain, by compactness arguments, the existence of a solution (c_δ, w_δ) such that, for any $\eta \in L^2_{H^1}$,

$$(\partial^\bullet c_\delta, \eta)_\rho + (\nabla_\Gamma w_\delta, \nabla_\Gamma \eta)_\rho = 0, \tag{4.49}$$

$$(w_\delta, \eta)_\rho = m(\nabla_\Gamma c_\delta, \nabla_\Gamma \eta) + ((F^\delta)'(c_\delta), \eta)_\rho, \tag{4.50}$$

and $c_\delta(0) = c_0$ almost everywhere in Γ_0. In particular, we have the following convergences (see (4.38), (4.41), and (4.42))

$$\begin{aligned} &c_\delta^M \overset{*}{\rightharpoonup} c_\delta, \quad \text{in } L^\infty_{H^1},\\ &\partial^\bullet c_\delta^M \rightharpoonup \partial^\bullet c_\delta, \quad \text{in } L^2_{H^{-1}},\\ &w_\delta^M \rightharpoonup w_\delta, \quad \text{in } L^2_{H^1}, \end{aligned}$$

which also imply, by (4.48),

$$\begin{aligned} &\sqrt{t}\partial^\bullet c_\delta^M \rightharpoonup \sqrt{t}\partial^\bullet c_\delta, \quad \text{in } L^2_{H^1},\\ &\sqrt{t}\nabla_\Gamma w_\delta^M \overset{*}{\rightharpoonup} \sqrt{t}\nabla_\Gamma w_\delta, \quad \text{in } L^\infty_{L^2}. \end{aligned}$$

As a consequence, from (4.38), (4.40), (4.41), and (4.48) we can obtain the following bounds, which are still independent of δ,

$$\|c_\delta\|_{L^\infty_{H^1}} + \|\nabla_\Gamma w_\delta\|_{L^2_{L^2}} + \|\partial^\bullet c_\delta\|_{L^2_{H^{-1}}} + \|\sqrt{t} w_\delta\|_{L^\infty_{L^2}} + \|\sqrt{t}\partial^\bullet c_\delta\|_{L^2_{H^1}} \leq C(T). \tag{4.51}$$

We are left to find an estimate for $(w_\delta)_{\Gamma_0}$ which is independent of δ. Being now $\eta = \frac{1}{\rho} \in L^2_{H^1}$, we can use it as a test function in (4.49), to get

$$\int_{\Gamma(t)} w_\delta = \left(w_\delta, \frac{1}{\rho}\right)_\rho = \left(\nabla_\Gamma c_\delta, \nabla_\Gamma \frac{1}{\rho}\right) + \left(F_\delta'(c_\delta), \frac{1}{\rho}\right)_\rho.$$

At this point we can repeat word by word the proof in [13, Sec.1.6], exploiting the fact that $(\widetilde{c})_{\Gamma_0} = (c_0)_{\Gamma_0}$ and $|(c_0)_{\Gamma_0}| < 1$, and obtaining

$$\left|\int_{\Gamma(t)} w_\delta\right| \leq C(1 + \|\nabla_\Gamma w_\delta\|).$$

This result finally allows, by Poincaré's inequality, to infer

$$\|w_\delta\|_{L^2_{H^1}} + \|\sqrt{t} w_\delta\|_{L^\infty_{H^1}} \leq C(T). \tag{4.52}$$

Therefore, we can pass to the limit with respect to δ. Again by standard compactness arguments, we obtain a solution (c, w) to (4.5)-(4.7). In particular, we can retrieve, exactly as done for the first model in the proof of [13, Thm.5.14], the bound $|c| < 1$ almost everywhere on $\Gamma(t)$, for almost any $t \in [0,T]$. Clearly, again by sequential lower semicontinuity, we have the following bounds on the final solution (which is also unique by Theorem 4.2):

$$\|c\|_{L^\infty_{H^1}} + \|w\|_{L^2_{H^1}} + \|\partial^\bullet c\|_{L^2_{H^{-1}}} + \|\sqrt{t} w\|_{L^\infty_{H^1}} + \|\sqrt{t}\partial^\bullet c\|_{L^2_{H^1}} \leq C(T). \tag{4.53}$$

Part (ii). Concerning the strict separation property, we use an argument similar to the one used in the proof of Theorem 3.9. Let us fix $\tau > 0$. We have that $\|w\|_{H^1(\Gamma(t))} \leq C$ for almost any $t \geq \tau$. Then we set $c_k = h_k(c)$, where h_k is defined in (3.57). Being c in $L^\infty_{H^1}$, we have

$$\nabla_\Gamma c_k = \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(c)\nabla_\Gamma c.$$

Accordingly, for any $k > 1$ and $p \geq 2$, $f_k = \left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi(c_k)$ is well defined and belongs to $L^\infty_{H^1}$ and satisfies

$$\nabla_\Gamma\left(\left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi(c_k)\right) = (p-1)\left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi'(c_k)\nabla_\Gamma c_k.$$

If we now set $\eta = \left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi(c_k)$ in (4.6) then we infer that

$$(p-1)\int_{\Gamma(t)}\left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi'(c_k)\nabla_\Gamma c\cdot\nabla_\Gamma c_k + \int_{\Gamma(t)}\rho\left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi(c_k)\frac{\theta}{2}\varphi(c) = \int_{\Gamma(t)}\rho\widehat{w}\left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi(c_k),$$

where $\widehat{w} = w + c$. Being F_{ln} strictly convex, the first term in the left-hand side is nonnegative. Since φ is increasing we infer

$$\varphi(c_k)^2 \leq \varphi(c)\varphi(c_k), \quad \forall k > 1. \tag{4.54}$$

Regarding the right-hand side, by the Sobolev embedding $H^1(\Gamma(t)) \hookrightarrow L^p(\Gamma(t))$ and (4.10), we easily get

$$\int_{\Gamma(t)}\rho\widehat{w}\left|\frac{\theta}{2}\varphi(c_k)\right|^{p-2}\frac{\theta}{2}\varphi(c_k) \leq \frac{1}{2C_\rho}\left\|\frac{\theta}{2}\varphi(c_k)\right\|^p_{L^p(\Gamma(t))} + C\|\widehat{w}\|^p_{L^p(\Gamma(t))} \leq \frac{1}{2C_\rho}\left\|\frac{\theta}{2}\varphi(c_k)\right\|^p_{L^p(\Gamma(t))} + C_p\|\widehat{w}\|^p_{H^1(\Gamma(t))},$$

with $C_p > 0$ depending on p. Now, collecting the above estimates, being $c \in L^\infty_{H^1}$ and $\|w\|_{H^1(\Gamma(t))} \leq C$ for almost any $t \geq \tau$, and recalling, by (4.10), that $0 < \frac{1}{C_\rho} \leq \rho$, we immediately deduce (see (3.59))

$$\operatorname*{ess\,sup}_{t\in[\tau,T]}\|\varphi(c_k)\|_{L^p(\Gamma(t))} \leq C_p(T,\tau,p), \quad \forall p \in [2,\infty). \tag{4.55}$$

Consider now $g_k = \frac{\theta}{2}\varphi(c_k)e^{L\frac{\theta}{2}|\varphi(c_k)|}$, for some arbitrary $L > 0$, and observe that

$$\nabla_\Gamma\left(\frac{\theta}{2}\varphi(c_k)e^{L\frac{\theta}{2}|\varphi(c_k)|}\right) = \frac{\theta}{2}\varphi'(c_k)\left(1 + L\frac{\theta}{2}|\varphi(c_k)|\right)e^{L\frac{\theta}{2}|\varphi(c_k)|}\nabla_\Gamma c_k.$$

Therefore, using (4.6) with $\eta = g_k$, we find

$$\int_{\Gamma(t)}\nabla_\Gamma c\cdot\nabla_\Gamma c_k\frac{\theta}{2}\varphi'(c_k)\left(1 + L\frac{\theta}{2}|\varphi(c_k)|\right)e^{L\frac{\theta}{2}|\varphi(c_k)|} + \int_{\Gamma(t)}\rho\frac{\theta}{2}\varphi(c)\frac{\theta}{2}\varphi(c_k)e^{L\frac{\theta}{2}|\varphi(c_k)|} = \int_{\Gamma(t)}\rho\widehat{w}\frac{\theta}{2}\varphi(c_k)e^{L\frac{\theta}{2}|\varphi(c_k)|}.$$

Observe that the first term on the left-hand side is nonnegative. Hence, exploiting again (4.54) and (4.10), we obtain in the end

$$\frac{1}{C_\rho}\int_{\Gamma(t)}\left(\frac{\theta}{2}\varphi(c_k)\right)^2 e^{L\frac{\theta}{2}|\varphi(c_k)|} \leq \int_{\Gamma(t)}\rho\widehat{w}\frac{\theta}{2}\varphi(c_k)e^{L\frac{\theta}{2}|\varphi(c_k)|}$$

By Lemma B.4, with $\rho_\star = \frac{1}{C_\rho}$, we get

$$\int_{\Gamma(t)}\rho|\widehat{w}|\left|\frac{\theta}{2}\varphi(c_k)\right|e^{L\left|\frac{\theta}{2}\varphi(c_k)\right|} \leq \frac{1}{2C_\rho}\int_{\Gamma(t)}\left|\frac{\theta}{2}\varphi(c_k)\right|^2 e^{L\left|\frac{\theta}{2}\varphi(c_k)\right|} + \int_{\Gamma(t)}e^{N\rho|\widehat{w}|}, \tag{4.56}$$

implying

$$\frac{1}{2C_\rho}\int_{\Gamma(t)}\left|\frac{\theta}{2}\varphi(c_k)\right|^2 e^{L\left|\frac{\theta}{2}\varphi(c_k)\right|} \leq \int_{\Gamma(t)}e^{\rho N|\widehat{w}|}, \tag{4.57}$$

for any $L > 0$ and some $N = N(L, C_\rho)$. Exploiting (4.10), the control over J^0_t and then applying Lemma B.3 with $u = C_\rho N\widetilde{\widehat{w}}$ and the manifold $\mathcal{M} = \Gamma_0$ (with the corresponding metric) we deduce

$$\int_{\Gamma(t)}e^{\rho N|\widehat{w}|} \leq \int_{\Gamma(t)}e^{C_\rho N|\widehat{w}|} = \int_{\Gamma_0}e^{C_\rho N|\widetilde{\widehat{w}}|}J^0_t d\Gamma_0 \leq C\int_{\Gamma_0}e^{C_\rho N|\widetilde{\widehat{w}}|}d\Gamma_0 \leq Ce^{CN^2\|\widetilde{\widehat{w}}\|^2_{H^1(\Gamma_0)}} \leq Ce^{CN^2\|\widehat{w}\|^2_{H^1(\Gamma(t))}},$$

being $(H^1(\Gamma(t)), \phi_t)_{t\in[0,T]}$ a compatible space. On account of (3.63), taking $L = pC$ in (4.57) and recalling that $\|\widehat{w}\|_{H^1(\Gamma(t))} \leq C$ for almost any $t \geq \tau$, we end up with

$$\|\varphi'(c_k)\|_{L^p(\Gamma(t))} \leq C_p(T,\tau,p),$$

which implies (see proof of Theorem 3.9)

$$\operatorname*{ess\,sup}_{t\in[\tau,T]} \|\varphi'(c)\|_{L^p(\Gamma(t))} \leq C_p(T,\tau,p), \quad \forall p \in [2,\infty). \tag{4.58}$$

Therefore, thanks to elliptic regularity, being $c \in L^\infty_{H^1}$ we get (see also (4.10))

$$\|c\|_{H^2(\Gamma(t))} \leq \left(C + \|\Delta_{\Gamma(t)} c\|\right) \leq C\left(1 + \|w\| + \|\varphi(c)\|\right) \leq C(T,\tau),$$

for almost any $t \in [\tau, T]$.

Part (iii). If we now apply the chain rule to $\varphi(c)$ (again we can obtain this by a truncation argument) we obtain

$$\nabla_\Gamma \varphi(c) = \varphi'(c)\nabla_\Gamma c, \quad \text{for a.a. } t \in [\tau,T].$$

Then, for almost any $t \in [\tau, T]$, we have that

$$\|\nabla_\Gamma \varphi(c)\|_{L^p(\Gamma(t))} \leq \|\varphi'(c)\|_{L^{2p}(\Gamma(t))}\|\nabla_\Gamma c\|_{L^{2p}(\Gamma(t))} \leq C_p(T,\tau,p),$$

by the Sobolev embedding $H^2(\Gamma(t)) \hookrightarrow W^{1,q}(\Gamma(t))$ which holds for every $q \geq 2$. Therefore we infer

$$\|\varphi(c)\|_{W^{1,p}(\Gamma(t))} \leq C_p(T,\tau,p), \quad \forall\, p \geq 2,$$

so that, choosing, e.g. $p = 3$,

$$\|\varphi(c)\|_{L^\infty(\Gamma(t))} \leq C(T,\tau), \quad \text{for a.a. } t \in [\tau,T].$$

Therefore, being $c(t) \in H^2(\Gamma(t)) \hookrightarrow C^0(\Gamma(t))$ for almost any $t \in [0,T]$, we can find $\xi = \xi(T,\tau) > 0$ such that

$$\|c\|_{L^\infty(\Gamma(t))} \leq 1 - \xi, \quad \text{for a.a. } t \in [\tau,T],$$

that is, the strict separation property holds. This concludes the proof.

A Proofs of some technical results

In this Appendix, we present the proofs of some results which were only stated in the main body of the paper.

A.1 Proof of Proposition 3.2

Even though it is a straightforward result, we give here a short proof of the formula (3.3). We consider only sufficiently smooth functions η, ϕ, since the general result can be obtained by a density argument. It is enough to prove this relation locally. Let $\Omega \subset \mathbb{R}^2$ be an open set and let $X = X(\theta,t)$, $\theta \in \Omega$, $X(\cdot,t) : \Omega \to U \cap \Gamma(t)$ be a local regular parametrization of the open set $U \cap \Gamma(t)$ (w.r.t. the induced metric) of the surface $\Gamma(t)$ which evolves so that $X_t = \mathbf{V}(X(\theta,t),t)$. The induced metric $(g_{ij}), i,j = 1,2$, is given by $g_{ij} = X_{\theta_i} \cdot X_{\theta_j}$ with $g = \det(g_{ij})$. Note that, as usual, $g^{ij} = (g_{ij})^{-1}$. Define then $\mathbf{f} = \mathbf{V}_a^\tau \eta$ and set

$$F(\theta,t) = \mathbf{f}(X(\theta,t),t), \qquad \Phi(\theta,t) = \phi(X(\theta,t),t), \qquad \mathcal{V}(\theta,t) = \mathbf{V}(X(\theta,t),t).$$

For the sake of simplicity we will omit the dependence on θ, t. We have

$$\begin{aligned}
\frac{d}{dt}\int_{U\cap\Gamma(t)} \mathbf{f}\cdot\nabla_\Gamma\phi &= \frac{d}{dt}\int_\Omega F_l g^{ij}\Phi_{\theta_j} X^l_{\theta_i}\sqrt{g}d\theta \\
&= \int_\Omega F_{l,t} g^{ij}\Phi_{\theta_j} X^l_{\theta_i}\sqrt{g}d\theta + \int_\Omega F_l g^{ij}_t \Phi_{\theta_j} X^l_{\theta_i}\sqrt{g}d\theta \\
&+ \int_\Omega F_l g^{ij}\Phi_{\theta_j,t} X^l_{\theta_i}\sqrt{g}d\theta + \int_\Omega F_l g^{ij}\Phi_{\theta_j}\mathcal{V}^l_{\theta_i}\sqrt{g}d\theta \\
&+ \int_\Omega F_l g^{ij}\Phi_{\theta_j} X^l_{\theta_i}\partial_t\sqrt{g}d\theta.
\end{aligned}$$

Recalling that $g^{ij}_t = -g^{ik}g^{jl}(\mathcal{V}_{\theta_k}\cdot X_{\theta_l} + X_{\theta_k}\cdot\mathcal{V}_{\theta_l})$ and

$$\partial_t\sqrt{g} = \sqrt{g}g^{ij}X_{\theta_i}\cdot\mathcal{V}_{\theta_j},$$

we obtain

$$\begin{aligned}\int_\Omega F_l g_t^{ij}\Phi_{\theta_j}X^l_{\theta_i}\sqrt{g} &= -\int_\Omega F_l\Phi_{\theta_j}X^l_{\theta_i}g^{ik}g^{jl}\mathcal{V}^r_{\theta_k}X^r_{\theta_l}\sqrt{g} - \int_\Omega F_l\Phi_{\theta_j}X^l_{\theta_i}g^{ik}g^{jl}\mathcal{V}^r_{\theta_l}X^r_{\theta_k}\sqrt{g}\\ &= -\int_\Omega F_l(g^{ik}X^l_{\theta_i}\mathcal{V}^r_{\theta_k})(g^{jl}X^r_{\theta_l}\Phi_{\theta_j})\sqrt{g} - \int_\Omega F_l(g^{ik}X^l_{\theta_i}X^r_{\theta_k})(g^{jl}\Phi_{\theta_j}\mathcal{V}^r_{\theta_l})\sqrt{g}\\ &= -\int_{U\cap\Gamma(t)}\mathbf{f}_l\underline{D}^{\Gamma(t)}_l\mathbf{V}_r\underline{D}^{\Gamma(t)}_r\phi - \int_{U\cap\Gamma(t)}\mathbf{f}_l(\delta_{lr}-\nu_l\nu_r)\nabla_\Gamma\phi\cdot\nabla_\Gamma\mathbf{V}_r,\end{aligned}$$

exploiting $\underline{D}^{\Gamma(t)}_l x^r = \delta_{lr}-\nu_l\nu_r$ and

$$g^{jl}\Phi_{\theta_j}\mathcal{V}^r_{\theta_l} = g^{jl}g_{jk}g^{km}\Phi_{\theta_m}\mathcal{V}^r_{\theta_l} = (g^{km}\Phi_{\theta_m}X^n_{\theta_k})(g^{jl}\mathcal{V}^r_{\theta_l}X^n_{\theta_j}). \tag{A.1}$$

Then, on account of $g^{ij}X_{\theta_i}\cdot\mathcal{V}_{\theta_j} = (\nabla_\Gamma\cdot\mathbf{V})(X,\cdot)$, we obtain

$$\int_\Omega F_l g^{ij}\Phi_{\theta_j}X^l_{\theta_i}\partial_t\sqrt{g} = \int_{U\cap\Gamma(t)}\mathbf{f}\cdot\nabla_\Gamma\phi(\nabla_\Gamma\cdot\mathbf{V}).$$

Moreover, again by (A.1), we have

$$\int_\Omega F_l g^{ij}\Phi_{\theta_j}\mathcal{V}^l_{\theta_i}\sqrt{g} = \int_{U\cap\Gamma(t)}\mathbf{f}_l\nabla_\Gamma\phi\cdot\nabla_\Gamma\mathbf{V}_l. \tag{A.2}$$

In conclusion, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{U\cap\Gamma(t)}\mathbf{f}\cdot\nabla_\Gamma\phi &= \int_{U\cap\Gamma(t)}\partial^\bullet\mathbf{f}\cdot\nabla_\Gamma\phi + \int_{U\cap\Gamma(t)}\mathbf{f}\cdot\nabla_\Gamma\partial^\bullet\phi\\ &+ \int_{U\cap\Gamma(t)}\mathbf{f}\cdot\nabla_\Gamma\phi(\nabla_\Gamma\cdot\mathbf{V}) - \int_{U\cap\Gamma(t)}\mathbf{f}_l\underline{D}^{\Gamma(t)}_l\mathbf{V}_r\underline{D}^{\Gamma(t)}_r\phi,\end{aligned}$$

where we have exploited the fact that $\mathbf{f}\cdot\nu = 0$, being a tangential vector. Indeed, setting $\mathbf{G}_r := \nabla_\Gamma\phi\cdot\nabla_\Gamma\mathbf{V}_r$, then $\mathbf{G}_{\tau,l} := (\delta_{lr}-\nu_l\nu_r)\mathbf{G}_r$ is the projection of $\mathbf{G}$ on the tangent space to $\Gamma(t)$. This entails that $\mathbf{f}\cdot\mathbf{G} = \mathbf{f}\cdot\mathbf{G}_\tau$. Therefore, we deduce

$$-\int_{U\cap\Gamma(t)}\mathbf{f}_l(\delta_{lr}-\nu_l\nu_r)\nabla_\Gamma\phi\cdot\nabla_\Gamma\mathbf{V}_r = -\int_{U\cap\Gamma(t)}\mathbf{f}_l\nabla_\Gamma\phi\cdot\nabla_\Gamma\mathbf{V}_l,$$

and this term simplifies with (A.2). To conclude the proof it suffices to note that

$$\partial^\bullet\mathbf{f} = \partial^\bullet\eta\mathbf{V}^\tau_a + \eta\partial^\bullet\mathbf{V}^\tau_a.$$

A.2 Proof of Proposition 4.5

Exploiting the same notation as in the proof of Proposition 3.3 (Section A.1) and considering, for simplicity, the case $\int_{\Gamma(t)}\rho\,|\nabla_\Gamma f|^2$, for f sufficiently regular, we can obtain

$$\hat\rho|\nabla F|^2 = \hat\rho g^{ij}F_{\theta_j}F_{\theta_i},$$

with $\hat\rho(\theta,t) = \rho(X(\theta,t),t)$. Therefore, we find

$$\begin{aligned}\frac{d}{dt}\int_{U\cap\Gamma(t)}\rho|\nabla_\Gamma f|^2 &= \frac{d}{dt}\int_\Omega\hat\rho F_l g^{ij}\Phi_{\theta_j}X^l_{\theta_i}\sqrt{g}d\theta\\ &= \int_\Omega\hat\rho_{,t}F_l g^{ij}\Phi_{\theta_j}X^l_{\theta_i}\sqrt{g}d\theta + \int_\Omega\hat\rho(F_l g^{ij}\Phi_{\theta_j}X^l_{\theta_i}\sqrt{g})_{,t}d\theta\\ &= \int_{U\cap\Gamma(t)}\partial^\bullet\rho\,|\nabla_\Gamma f|^2 + \int_{U\cap\Gamma(t)}\rho D_i\mathbf{V}_j D_i f D_j f\\ &+ 2\int_{U\cap\Gamma(t)}\rho\nabla_\Gamma f\cdot\nabla_\Gamma\partial^\bullet f + \int_{U\cap\Gamma(t)}\rho\nabla_\Gamma\cdot\mathbf{V}|\nabla_\Gamma f|^2,\end{aligned}$$

where we have argued as in the proof of Proposition 3.2 (see, e.g., [23, Sec.5.1]). In conclusion, to obtain the last identity in (4.22), it is enough to recall that $\partial^\bullet\rho = -\rho\nabla_\Gamma\cdot\mathbf{V}$. The general case follows by polarization with respect to the inner product (4.21). The proof is ended.

B Two basic inequalities

B.1 Moser-Trudinger inequality

The Moser-Trudinger inequality for manifolds is given by (see [33])

Lemma B.1. *Let $(\mathcal{M}, r)$ be a compact $n-$dimensional Riemannian manifold, with $n \geq 2$ and r as a metric. Then there exists a constant C depending only on $(\mathcal{M}, r)$ and $\beta_0 = \beta_0(n) > 0$ such that*

$$\sup_{\int_{\mathcal{M}} u dV = 0,\ \int_{\mathcal{M}} |\nabla_{\mathcal{M}} u|^n dV \leq 1} \int_{\mathcal{M}} e^{\beta_0 |u|^p} dV \leq C,$$

where $p = \frac{n}{n-1}$.

Remark B.2. *Notice that the constant C depends not only on the volume of $\mathcal{M}$, $Vol(\mathcal{M})$, but also on its metric r. Therefore, in the case of $\mathcal{M} = \Gamma(t)$ it does not seem easy to find a constant C independent of time.*

Adapting the proof proposed in [44], we can easily obtain the following

Lemma B.3. *Let $(\mathcal{M}, r)$ be a compact $n-$dimensional Riemannian manifold with metric r and $n \geq 2$. Let $u \in W^{1,n}(\mathcal{M})$. Then*

$$\int_{\mathcal{M}} e^{|u|} dV \leq C_1 e^{C_2 \|u\|^n_{W^{1,n}(\mathcal{M})}},$$

where the constant $C_1 > 0$ does not depend on u, but depends on n and on $(\mathcal{M}, r)$, whereas $C_2 > 0$ depends only on n.

Proof. Let us first consider $u \in W^{1,n}(\mathcal{M})$ with $\int_{\mathcal{M}} u dV = 0$. We then define $v = \frac{u}{\|\nabla u\|_n}$. Clearly $\|\nabla v\|_n = 1$, therefore by Lemma B.1 we get, for $p = \frac{n}{n-1}$,

$$\int_{\mathcal{M}} \mathrm{e}^{\beta_0 |v|^p} dV \leq C. \tag{B.1}$$

Now, recalling (B.1), by Young's inequality we obtain

$$\int_{\mathcal{M}} \mathrm{e}^{|u|} dV = \int_{\mathcal{M}} e^{(p\beta_0)^{1/p}|v|(p\beta_0)^{-1/p}\|\nabla u\|_n} dV \leq \int_{\mathcal{M}} e^{\beta_0 |v|^p + \frac{1}{n}(p\beta_0)^{-n/p}\|\nabla u\|^n_n} dV \leq C e^{\frac{1}{\beta_n}\|\nabla u\|^n_n},$$

having set $\beta_n = \left(\frac{1}{n}(p\beta_0)^{-n/p}\right)^{-1} = n\left(\frac{n\beta_0}{n-1}\right)^{(n-1)}$. To conclude the proof, let us fix $u \in W^{1,n}(\mathcal{M})$. Then, setting $w = u - (u)_{\mathcal{M}}$, so that $(w)_{\mathcal{M}} = 0$, we can apply the result we have just proved, to infer

$$\int_{\mathcal{M}} \mathrm{e}^{|u|} dV \leq \mathrm{e}^{|(u)_{\mathcal{M}}|} \int_{\mathcal{M}} \mathrm{e}^{|w|} dV \leq C \mathrm{e}^{|(u)_{\mathcal{M}}| + \frac{1}{\beta_n}\|\nabla u\|^n_n} \leq C_1 e^{C_2 \|u\|^n_{W^{1,n}(\mathcal{M})}},$$

concluding the proof. □

B.2 Generalized Young's inequality

We will use the following version of Young's inequality (see [19, Appendix A] and [36]).

Lemma B.4. *Let $L > 0$ and $\rho_\star > 0$ be given. Then, there exists $N = N(L, \rho_\star) > 0$ such that*

$$xye^{Ly} \leq e^{Nx} + \frac{\rho_\star}{2} y^2 e^{Ly}, \quad \forall x, y \geq 0.$$

Proof. We recall the generalized Young's inequality (see, e.g., [3, Sec.8.2]): for any $a, b \geq 0$,

$$ab \leq \Phi(a) + \Psi(b),$$

with, given $s \geq 0$,

$$\Phi(s) = e^s - s - 1, \qquad \Psi(s) = (1+s)\ln(1+s) - s.$$

Then, we choose $a = Nx$ and $b = N^{-1}ye^{Ly}$. We obtain, recalling that $\ln(1+s) \leq s$ for any $s \geq 0$,

$$xye^{Ly} \leq e^{Nx} - Nx - 1 + (1 + N^{-1}ye^{Ly})\ln(1 + N^{-1}ye^{Ly}) - N^{-1}ye^{Ly}$$

$$\begin{aligned}
&= e^{Nx} - Nx - 1 + \ln(1 + N^{-1}ye^{Ly}) - N^{-1}ye^{Ly} + N^{-1}ye^{Ly}\ln(1 + N^{-1}ye^{Ly}) \\
&\le e^{Nx} + N^{-1}ye^{Ly}[\ln(e^{Ly}) + \ln(e^{-Ly} + N^{-1}y)] \\
&\le e^{Nx} + N^{-1}ye^{Ly}[Ly + \ln(1 + N^{-1}y)] \\
&\le e^{Nx} + N^{-1}(L + N^{-1})y^2e^{Ly},
\end{aligned}$$

and if we choose $N = N(L, \rho_\star) > \dfrac{L + \sqrt{L^2 + 2\rho_\star}}{\rho_\star}$ we finally obtain

$$xye^{Ly} \le e^{Nx} + \frac{\rho_\star}{2}y^2e^{Ly},$$

concluding the proof. □

C The embedding $\mathbb{W}^{\infty,2}(H^2, H^1) \hookrightarrow C^0_{H^{3/2}}$

In this Appendix we aim to sketch a proof of the following result, which we make use of to obtain the extra regularity in Remark 3.10.

Lemma C.1. *Assuming the following extra regularity for* $\mathbf{V}$ *and* Φ *(with respect to Assumption* $\boldsymbol{A}_\Phi$*)*

$$\mathbf{V} \in C^0([0,T]; C^3(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \quad \text{and} \quad \Phi_0^{(\cdot)} \in C^1([0,T]; C^3(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})):$$

(i) The families $(H^2(\Gamma(t)), \phi_t)_t$, $(H^1(\Gamma(t)), \phi_t)_t$ *and* $(H^{3/2}(\Gamma(t)), \phi_t)_t$ *are compatible.*

(ii) The spaces $\mathbb{W}^{\infty,2}(H^2, H^1)$ *and* $\mathcal{W}^{\infty,2}(H^2(\Gamma_0), H^1(\Gamma_0))$ *satisfy the evolving space equivalence.*

(iii) We have the embedding $\mathbb{W}^{\infty,2}(H^2, H^1) \hookrightarrow C^0_{H^{3/2}}$.

Before showing the proof we briefly recall some definitions. We denote

$$H^{3/2}(\Gamma(t)) = \left\{ u \in H^1(\Gamma(t)) \colon \int_{\Gamma(t)}\int_{\Gamma(t)} \frac{\left(\underline{D}_i^{\Gamma(t)}u(x) - \underline{D}_i^{\Gamma(t)}u(y)\right)^2}{|x-y|^{n+1}} < \infty,\ \forall i \right\}$$

with the norm

$$\begin{aligned}
\|u\|^2_{H^{3/2}} &= \|u\|^2_{L^2} + \|\nabla_\Gamma u\|^2_{L^2} + \sum_{i=1}^{n+1}\int_{\Gamma(t)}\int_{\Gamma(t)} \frac{(\underline{D}_i u(x) - \underline{D}_i u(y))^2}{|x-y|^{n+1}} \\
&= \|u\|^2_{L^2} + \|\nabla_\Gamma u\|^2_{L^2} + \int_{\Gamma(t)}\int_{\Gamma(t)} \frac{|\nabla_\Gamma u(x)|^2 - 2\nabla_\Gamma u(x)\cdot\nabla_\Gamma u(y) + |\nabla_\Gamma u(y)|^2}{|x-y|^{n+1}}.
\end{aligned}$$

Denote

$$[u]_{3/2} := \int_{\Gamma(t)}\int_{\Gamma(t)} \frac{|\nabla_\Gamma u(x)|^2 - 2\nabla_\Gamma u(x)\cdot\nabla_\Gamma u(y) + |\nabla_\Gamma u(y)|^2}{|x-y|^{n+1}}.$$

Proof. (i). Compatibility of the first two pairs is established in [4, Lemmas 7.2, 7.5]. For the third pair, we need to show:

a) $\phi_t \colon H^{3/2}(\Gamma_0) \to H^{3/2}(\Gamma(t))$ and its inverse $\phi_{-t} \colon H^{3/2}(\Gamma(t)) \to H^{3/2}(\Gamma_0)$ are linear maps satisfying $\phi_0 = \mathrm{Id}$ and are also bounded: there exists a constant $C_X > 0$ s.t.

$$\|\phi_t u\|_{H^{3/2}} \le C_X\|u\|_{H^{3/2}}, \quad \|\phi_{-t}u\|_{H^{3/2}} \le C_X\|u\|_{H^{3/2}}.$$

Proof. Linearity and the initial condition $\phi_0 = \mathrm{Id}$ are immediate. Let $u \in H^{3/2}(\Gamma_0)$, then since $u \in H^1(\Gamma_0)$ we have from [4, Lemma 7.2] that $\phi_t u \in H^1(\Gamma(t))$ as well as the bounds

$$\|\phi_t u\|^2_{L^2} \le C\|u\|^2_{L^2} \quad \text{and} \quad \|\phi_t u\|^2_{H^1} \le C\|u\|^2_{H^1}$$

where the constant C depends only on an upper bound for the $C^1([0,T]\times\Gamma_0)$-norm of $\Phi_0^{(\cdot)}$. Now we aim to estimate

$$[\phi_t u]_{3/2} = \int_{\Gamma(t)}\int_{\Gamma(t)} \frac{|\nabla_\Gamma\phi_t u(x)|^2 - 2\nabla_\Gamma\phi_t u(x)\cdot\nabla_\Gamma\phi_t u(y) + |\nabla_\Gamma\phi_t u(y)|^2}{|x-y|^{n+1}}.$$

We treat each term separately. Below $M > 0$ is a general constant depending only on an upper bound on the $C^1([0,T] \times \Gamma_0)$-norm of $\Phi_0^{(\cdot)}$. We have by (4.26)

$$\int_{\Gamma(t)} \int_{\Gamma(t)} \frac{|\nabla_\Gamma \phi_t u(x)|^2}{|x-y|^{n+1}} = \int_{\Gamma_0} \int_{\Gamma_0} \frac{|(\mathbf{D}_\Gamma \Phi_t^0)(p)(\mathbf{A}_t^0)^{-1}(p)\nabla_{\Gamma_0} u(p)|^2}{|\Phi_t^0(p) - \Phi_t^0(s)|^{n+1}} J_t^0(p) J_t^0(s)$$
$$\leq M \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\nabla_{\Gamma_0} u(p)|^2}{|p-s|^{n+1}},$$

in which we applied the bi-Lipschitz property of Φ_t^0, which is ensured by its regularity, so that

$$|\Phi_t^0(p) - \Phi_t^0(s)| \geq C_L |p - s|, \tag{C.1}$$

with $C_L > 0$ a constant independent of time. Similarly

$$\int_{\Gamma(t)} \int_{\Gamma(t)} \frac{|\nabla_\Gamma \phi_t u(y)|^2}{|x-y|^{n+1}} \leq M \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\nabla_{\Gamma_0} u(s)|^2}{|p-s|^{n+1}},$$

and for the remaining term

$$\int_{\Gamma(t)} \int_{\Gamma(t)} \frac{\nabla_\Gamma \phi_t u(x) \cdot \nabla_\Gamma \phi_t u(y)}{|x-y|^{n+1}}$$
$$= \int_{\Gamma_0} \int_{\Gamma_0} \frac{(\mathbf{D}_\Gamma \Phi_t^0)(p)(\mathbf{A}_t^0)^{-1}(p)\nabla_{\Gamma_0} u(p) \cdot (\mathbf{D}_\Gamma \Phi_t^0)(s)(\mathbf{A}_t^0)^{-1}(s)\nabla_{\Gamma_0} u(s)}{|\Phi_t^0(p) - \Phi_t^0(s)|^{n+1}} J_t^0(p) J_t^0(s)$$
$$\leq M \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\nabla_{\Gamma_0} u(p)|\, |\nabla_{\Gamma_0} u(s)|}{|p-s|^{\frac{n+1}{2}}\, |p-s|^{\frac{n+1}{2}}}$$
$$\leq \frac{M}{2} \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\nabla_{\Gamma_0} u(p)|^2}{|p-s|^{n+1}} \,\mathrm{d}\Gamma(p)\,\mathrm{d}\Gamma(s) + \frac{M}{2} \int_{\Gamma_0} \int_{\Gamma_0} \frac{|\nabla_{\Gamma_0} u(s)|^2}{|p-s|^{n+1}}$$

Combining the above leads to $[\phi_t u]_{3/2} \leq M[u]_{3/2}$, proving that ϕ_t indeed maps $H^{3/2}(\Gamma_0)$ into $H^{3/2}(\Gamma(t))$ and is bounded. The calculations for ϕ_{-t} are analogous, since all the properties exploited are shared by Φ_0^t as well.

b) for all $u \in H^{3/2}(\Gamma_0)$, the map $t \mapsto \|\phi_t u\|_{H^{3/2}}$ is measurable.

Proof. It remains to show that $t \mapsto [\phi_t u]_{H^{3/2}}$ is measurable. From the calculations above it follows that the seminorm $[\phi_t u]_{H^{3/2}}$ is the sum of three components:

- The first term is the integral of

$$g_1(t,p,s) := \frac{|(\mathbf{D}_\Gamma \Phi_t^0)(p)(\mathbf{A}_t^0)^{-1}(p)\nabla_{\Gamma_0} u(p)|^2}{|\Phi_t^0(p) - \Phi_t^0(s)|^{n+1}} J_t^0(p) J_t^0(s)$$

  which is a continuous function of $t \in [0,T]$ for almost every $(p,s) \in \Gamma_0 \times \Gamma_0$ and can be (uniformly in time) dominated, thanks to (C.1), as

$$|g_1(t,p,s)| \leq M \frac{|\nabla_{\Gamma_0} u(p)|^2}{|p-s|^{n+1}} \in L^1(\Gamma_0 \times \Gamma_0).$$

  Therefore, by Lebesgue's dominated convergence Theorem,

$$t \mapsto \int_{\Gamma_0} \int_{\Gamma_0} g_1(t,p,s)$$

  is continuous and thus measurable.
- Similarly for the last term, the function

$$g_3(t,p,s) := \frac{|(\mathbf{D}_\Gamma \Phi_t^0)(p)(\mathbf{A}_t^0)^{-1}(p)\nabla_{\Gamma_0} u(s)|^2}{|\Phi_t^0(p) - \Phi_t^0(s)|^{n+1}} J_t^0(p) J_t^0(s)$$

  is continuous for almost every $(p,s) \in \Gamma_0 \times \Gamma_0$ and can be (uniformly in time) dominated, thanks again to (C.1), as

$$|g_3(t,p,s)| \leq M \frac{|\nabla_{\Gamma_0} u(s)|^2}{|p-s|^{n+1}} \in L^1(\Gamma_0 \times \Gamma_0),$$

  giving that

$$t \mapsto \int_{\Gamma_0} \int_{\Gamma_0} g_3(t,p,s)$$

  is continuous and then measurable.

- Finally, the integrand in the middle term, say $g_2(t,p,s)$, is also a continuous function of $t \in [0,T]$ for almost every $(p,s) \in \Gamma_0 \times \Gamma_0$ and can now be (uniformly in time) dominated, again by (C.1), as
$$|g_2(t,p,s)| \le \frac{M}{2}\left(\frac{|\nabla_{\Gamma_0} u(p)|^2}{|p-s|^{n+1}} + \frac{|\nabla_{\Gamma_0} u(s)|^2}{|p-s|^{n+1}}\right) \in L^1(\Gamma_0 \times \Gamma_0),$$
therefore
$$t \mapsto \int_{\Gamma_0}\int_{\Gamma_0} g_2(t,p,s)$$
is continuous and hence measurable.

This proves that also $(H^{3/2}(\Gamma(t)), \phi_t)_t$ is compatible.

(ii). The conditions verified in [4, Proposition 7.7] show that the improved result in [4, Theorem 5.10] holds true, giving the evolving space equivalence between the spaces $\mathbb{W}^{\infty,2}(H^2,H^1)$ and $\mathcal{W}^{\infty,2}(H^2(\Gamma_0),H^1(\Gamma_0))$.

(iii). This follows from the previous result together with the classical embedding
$$\mathcal{W}^{\infty,2}(H^2(\Gamma_0),H^1(\Gamma_0)) \hookrightarrow C^0([0,T];H^{3/2}(\Gamma_0)).$$
Indeed, if $u \in W^{\infty,2}(H^2,H^1)$, then by the evolving space equivalence in (ii) we obtain
$$\phi_{-(\cdot)}u \in \mathcal{W}^{\infty,2}(H^2(\Gamma_0),H^1(\Gamma_0)) \hookrightarrow C^0([0,T];H^{3/2}(\Gamma_0)),$$
and since $(H^{3/2}(\Gamma(t)), \phi_t)_t$ is compatible we deduce by definition $u \in C^0_{H^{3/2}}$, which concludes the proof. $\square$

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