# The Spectrum of Triangle-free Graphs

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#### Abstract

Denote by  $q_n(G)$  the smallest eigenvalue of the signless Laplacian matrix of an *n*-vertex graph G. Brandt conjectured in 1997 that for regular triangle-free graphs  $q_n(G) \leq \frac{4n}{25}$ . We prove a stronger result: If G is a triangle-free graph then  $q_n(G) \leq \frac{15n}{94} < \frac{4n}{25}$ . Brandt's conjecture is a subproblem of two famous conjectures of Erdős:

(1) Sparse-Half-Conjecture: Every *n*-vertex triangle-free graph has a subset of vertices of size  $\left\lceil \frac{n}{2} \right\rceil$  spanning at most  $n^2/50$  edges.

(2) Every *n*-vertex triangle-free graph can be made bipartite by removing at most  $n^2/25$ edges.

In our proof we use linear algebraic methods to upper bound  $q_n(G)$  by the ratio between the number of induced paths with 3 and 4 vertices. We give an upper bound on this ratio via the method of flag algebras.

#### Introduction 1

We prove a result on eigenvalues of triangle-free graphs which is motivated by the following two famous conjectures of Erdős.

**Conjecture 1.1** (Erdős' Sparse Half Conjecture [9, 10]). Every triangle-free graph on n vertices has a subset of vertices of size  $\lceil \frac{n}{2} \rceil$  vertices spanning at most  $n^2/50$  edges.

Erdős offered a \$250 reward for proving this conjecture. There has been progress on this conjecture in various directions [4, 12, 14, 15, 17]. Most recently, Razborov [17] proved that every

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triangle-free graph on n vertices has an induced subgraph on n/2 vertices with at most  $(27/1024)n^2$  edges.

For a graph G, denote by  $D_2(G)$  the minimum number of edges which have to be removed to make G bipartite.

**Conjecture 1.2** (Erdős [9]). Let G be a triangle-free graph on n vertices. Then  $D_2(G) \leq n^2/25$ .

There also has been work on this conjecture [1,3,11,13,18], most recently, Balogh, Clemen and Lidický [3] proved  $D_2(G) \leq n^2/23.5$ .

Brandt [5] found a surprising connection between these two conjectures and the eigenvalues of triangle-free graphs. Denote by  $\lambda_n(G) \leq \ldots \leq \lambda_1(G)$  the eigenvalues of the adjacency matrix of an *n*-vertex graph *G*. Brandt [5] proved that

$$D_2(G) \ge \frac{\lambda_1(G) + \lambda_n(G)}{4} \cdot n \tag{1}$$

for regular graphs and conjectured the following.

**Conjecture 1.3** (Brandt [5]). Let G be a triangle-free regular n-vertex graph. Then

$$\lambda_1(G) + \lambda_n(G) \le \frac{4}{25} \cdot n.$$

Towards this conjecture, Brandt [5] proved a bound  $\lambda_1(G) + \lambda_n(G) \leq (3 - 2\sqrt{2})n \approx 0.1715n$  for regular triangle-free graphs, which was very recently shown to hold also in the non-regular setting by Csikvári [7]. Brandt also noted that  $\lambda_1(G_{HS}) + \lambda_n(G_{HS}) = 0.14n$  for the so-called Higman-Sims graph  $G_{HS}$ , which is the unique strongly regular graph with parameters (n, d, t, k) = (100, 22, 0, 6). Recall that an (n, d, t, k)-strongly regular graph is an n-vertex d-regular graph, where the number of common neighbors of every pair of adjacent vertices is t and the number of common neighbors of a non-adjacent pair of vertices is k.

The value 4/25 is motivated by the fact that if either of Conjectures 1.1 or 1.2 were true, it would imply Conjecture 1.3. As observed by Brandt [5], Conjecture 1.1 implies Conjecture 1.3 by applying the following version of the Expander Mixing Lemma for a set  $S \subset V(G)$  of size n/2 with  $e(S) \leq n^2/50$ .

**Lemma 1.4** (Bussemaker-Cvetković-Seidel [6], Alon-Chung [2]). Let G be an n-vertex d-regular graph. Then, for every  $S \subseteq V(G)$ , we have

$$e(S) \ge |S| \cdot \frac{|S|d + (n - |S|)\lambda_n(G)}{2n}$$

Given a graph G, denote by Q = A + D the signless Laplacian matrix of G, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G. Let  $q_n(G) \leq \ldots \leq q_1(G)$  be the eigenvalues of Q. By considering the signless Laplacian matrix, De Lima, Nikiforov and Olivera [8] extended (1) beyond regular graphs as follows.

**Theorem 1.5** (De Lima, Nikiforov and Olivera [8]). For every n-vertex graph G we have

$$D_2(G) \ge \frac{q_n(G)}{4} \cdot n.$$

By Theorem 1.5, if Conjecture 1.2 holds then  $q_n(G) \leq \frac{4n}{25}$  for every triangle-free *n*-vertex graph G. Motivated by this observation De Lima, Nikiforov and Olivera [8] proposed investigating upper bounds on  $q_n(G)$ , and proved  $q_n(G) \leq \frac{2n}{9}$  for *n*-vertex triangle-free graphs G. Our main result is an improvement of this bound, which solves Conjecture 1.3.

**Theorem 1.6.** If G is a triangle-free n-vertex graph, then

$$q_n(G) \le \frac{15}{94} \cdot n < 0.1596n.$$

Note that, if G is d-regular, then  $\lambda_1(G) = d$  and  $q_n(G) = \lambda_n(G) + d = \lambda_n(G) + \lambda_1(G)$ . Thus Theorem 1.6 implies that  $\lambda_1(G) + \lambda_n(G) < 0.1596n < \frac{4n}{25}$  for every regular triangle-free *n*-vertex graph G, confirming Conjecture 1.3 in strong form.

It remains open to determine a sharp upper bound for  $q_n(G)/n$  for triangle-free *n*-vertex graph G. While we only prove Theorem 1.6 with the constant  $\frac{15}{94} \approx 0.1596$ , a larger flag algebra computation yields  $q_n(G) < 0.15467n$ . Also, one can additionally assume that G is regular and use flag algebras to show a slightly stronger bound  $q_n(G) = \lambda_1(G) + \lambda_n(G) < 0.15442n$ . As we believe neither of these two bounds are sharp (see Section 3), we omit presenting their proofs.

## 2 Proof of Theorem 1.6

Our proof is based on bounding the ratio between the number of induced paths with 3 and 4 vertices in triangle-free graphs. On one hand, we upper bound  $q_n(G)$  in terms of this ratio in Lemma 2.1 and Corollary 2.2. On the other hand, Lemma 2.3, which is proved using flag algebras, gives a sufficiently good bound on the ratio.

For an edge e = xy of a graph G, let  $m_{xy}$  be the number of edges  $uv \in E(G)$  such that  $ux, vy \in E(G)$ . For a vertex  $x \in V(G)$ , let  $w_x$  to be the number of walks of length two starting in x, i.e.  $w_x$  is the number of edges  $uv \in E(G)$  such that  $xu \in E(G)$ .

**Lemma 2.1.** If G is an n-vertex triangle-free graph and  $xy \in E(G)$ , then

$$\left(\deg(x) + \deg(y)\right) \cdot q_n(G) \le w_x + w_y - 2m_{xy}.$$
(2)

*Proof.* Define a vector  $z = (z_v)_{v \in V(G)} \in \mathbb{R}^{V(G)}$  by

$$z_v = \begin{cases} +1, & \text{if } xv \in E(G), \\ -1, & \text{if } yv \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The vector z is well-defined since G is triangle-free. Also note that  $||z||^2 = \deg(x) + \deg(y)$ . Let Q be the signless Laplacian matrix of G. We have

$$z^{T}Qz = \sum_{u,v \in V(G)} Q_{uv} z_{u} z_{v} = \sum_{u \in V(G)} (z_{u})^{2} \deg(u) + 2 \cdot \sum_{uv \in E(G)} z_{u} z_{v}$$
$$= w_{x} + w_{y} + 2 \cdot \sum_{uv \in E(G)} z_{u} z_{v} = w_{x} + w_{y} - 2m_{xy},$$

where in the last equality we used that G is triangle-free. Since Q is symmetric,  $q_n(G)$  is upper bounded by the Rayleigh-Ritz quotient of z, i.e.

$$q_n(G) \le \frac{z^T Q z}{\|z\|^2} = \frac{w_x + w_y - 2m_{xy}}{\deg(x) + \deg(y)},$$

as desired.

A map  $\varphi: V(H) \to V(G)$  is a strong homomorphism from a graph H to a graph G if for every pair of vertices  $u, v \in V(H)$  we have  $uv \in E(H)$  if and only if  $\varphi(u)\varphi(v) \in E(G)$ . Let  $\hom_s(H, G)$ denote the number of strong homomorphisms from H to G. Let  $P_k$  denote the k-vertex path. Summing the bound from Lemma 2.1 over all the edges of G yields the following.

Corollary 2.2. If G is an n-vertex triangle-free graph, then

$$\hom_s(P_3, G) \cdot q_n(G) \le \hom_s(P_4, G).$$
(3)

*Proof.* First, note that

$$\sum_{xy \in E(G)} (\deg(x) + \deg(y)) = \sum_{x \in V(G)} \deg^2(x) = \hom_s(P_3, G),$$
(4)

where in the last equality we used that G is triangle-free. Meanwhile,  $\sum_{xy\in E(G)}(w_x+w_y)$  is equal to the number of walks of length three in G, i.e. the number of maps  $\phi: \{1,2,3,4\} \to V(G)$  such that  $\{\phi(1)\phi(2),\phi(2)\phi(3),\phi(3)\phi(4)\} \subset E(G)$ . Similarly, the expression  $2\sum_{xy\in E(G)} m_{xy}$  is equal to the number of maps  $\phi: \{1,2,3,4\} \to V(G)$  such that  $\{\phi(1)\phi(2),\phi(2)\phi(3),\phi(3)\phi(4),\phi(4)\phi(1)\} \subset E(G)$ . It follows that  $\sum_{xy\in E(G)}(w_x+w_y-2m_{xy})$  counts the maps  $\psi: \{1,2,3,4\} \to V(G)$  such that  $\{\psi(1)\psi(2),\psi(2)\psi(3),\psi(3)\psi(4)\} \subset E(G)$  and  $\psi(4)\psi(1) \notin E(G)$ , i.e.,

$$\sum_{xy \in E(G)} (w_x + w_y - 2m_{xy}) = \hom_s(P_4, G).$$
(5)

Summing (2) over all  $xy \in E(G)$  and using (4) and (5), we obtain (3).

Theorem 1.6 is an immediate consequence of the above corollary and the following lemma which is proved using standard, albeit computer-assisted flag-algebra calculation.

**Lemma 2.3.** If G is an n-vertex triangle-free graph, then

$$\hom_s(P_4, G) \le \frac{15n}{94} \cdot \hom_s(P_3, G). \tag{6}$$

*Proof.* Suppose the lemma is false, and let G be an *n*-vertex triangle-free graph such that

$$\hom_s(P_4, G) = \frac{15n}{94} \cdot \hom_s(P_3, G) + \varepsilon n^4, \qquad (7)$$

for some  $\varepsilon > 0$ . Let  $G^{(b)}$  be the *b*-blowup of *G*, obtained by replacing every vertex of *G* by *b* pairwise non-adjacent vertices. Then  $\hom_s(P_k, G^{(b)}) = \hom_s(P_k, G) \cdot b^k$  for k = 3, 4. In particular, for every  $b \in \mathbb{N}$ , the graph  $G^{(b)}$  satisfies the analogue of (7) as well.

Let us now reformulate (7) in the flag algebra language [16]. Given a graph H, let  $p(\Lambda, H)$  be the probability that a 3-vertex subset of V(H) chosen uniformly at random induces exactly

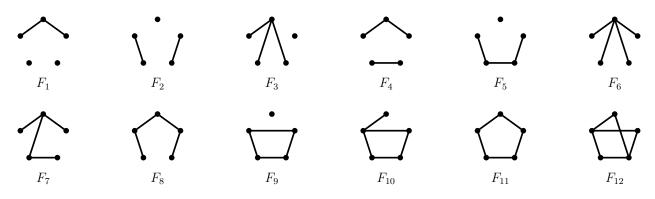


Figure 1: The set  $\mathcal{F}$  of 5-vertex triangle-free graphs with at least 2 edges.

two edges. Analogously, let  $p(\coprod, H)$  be the probability that a randomly chosen 4-vertex subset induces a path of length 3.

For every fixed  $\ell$ -vertex graph F and a k-vertex graph H, only  $O(k^{\ell-1})$  maps  $V(F) \to V(H)$ are non-injective. Therefore,  $\hom_s(F, H) = |\operatorname{Aut}(F)| \cdot p(F, H) \cdot \binom{k}{\ell} + O(k^{\ell-1})$ , so in particular every k-vertex triangle-free graph H satisfies

$$\hom_{s}(P_{4},H) = \frac{k^{4}}{12} \cdot p\left(\coprod,H\right) + O\left(k^{3}\right) \quad \text{and} \quad \hom_{s}(P_{3},H) = \frac{k^{3}}{3} \cdot p\left(\bigwedge,H\right) + O\left(k^{2}\right).$$

Therefore, the fact  $G^{(b)}$  satisfies (7), after multiplying by  $564/(bn)^4$  and rearranging, translates to

$$\lim_{b \to \infty} 30 \cdot p\left(\bigwedge, G^{(b)}\right) - 47 \cdot p\left(\coprod, G^{(b)}\right) = -564\varepsilon^4$$

In order to derive a contradiction, we present a flag algebra computation proving that an inequality

$$30 \cdot \bigwedge - 47 \cdot \bigsqcup \ge 0 \tag{8}$$

asymptotically holds in the theory of triangle-free graphs. To see that, consider the following 6 flag-algebra expressions, which are all non-negative:

$$1) \qquad \left(13 \cdot \left(\bigwedge_{123}^{\prime} + \bigwedge_{123}^{\prime} + \bigwedge_{123}^{\prime}\right) - 52 \cdot \left(\bigwedge_{123}^{\prime} + \bigwedge_{123}^{\prime} + \bigwedge_{123}^{\prime}\right) + 84 \cdot \bigwedge_{123}^{\prime}\right)^{2}$$

$$2) \qquad \left(31 \cdot \left(\begin{smallmatrix}1 \cdot & & \\ 2 & & \\ 2 & & \\ 1 & & \\ 2 & & \\ 1 & &$$

Let  $\mathcal{F}$  be the set of all the 5-vertex triangle-free graphs with at least 2 edges. A case analysis yields  $|\mathcal{F}| = 12$ ; see Figure 1. Now observe that averaging over all choices of the labelled vertices in each of the 6 expressions yields a linear combination of subgraph densities, where every term has

5 vertices and at least 2 edges. Thus a flag algebra argument yields that the average of the *i*-th expression is equal to the *i*-th coordinate of  $M \cdot (v_{\mathcal{F}})^T$ , where  $v_{\mathcal{F}} = (F_1, \ldots, F_{12})$  and

$$M = \frac{1}{30} \times \begin{pmatrix} 507 & 2028 & 0 & -4056 & -3549 & 0 & 1248 & 8112 & 16224 & -13104 & 0 & 21168 \\ 0 & 0 & 2883 & 381 & 961 & 0 & -3906 & -4098 & 3844 & 63 & 19845 & 0 \\ 12100 & -23688 & -19140 & -23620 & 12172 & 20184 & -37248 & 17486 & 47664 & 2956 & 86730 & -7392 \\ 0 & 0 & 6 & 140 & 0 & 0 & -48 & -100 & 0 & 358 & -1200 & 0 \\ 196 & 0 & 798 & 196 & -420 & 2166 & 762 & -1036 & -2464 & -702 & -3080 & 792 \\ 81 & 0 & -378 & 81 & 54 & 1176 & -165 & 27 & -108 & -87 & -135 & 279 \end{pmatrix}$$

On the other hand, another flag algebra argument yields that the left-hand side of (8) is equal to

$$3 \cdot F_1 + 9 \cdot F_3 + 3 \cdot F_4 - \frac{17}{5} \cdot F_5 + 18 \cdot F_6 - \frac{34}{5} \cdot F_7 - \frac{49}{5} \cdot F_8 + 12 \cdot F_9 - \frac{4}{5} \cdot F_{10} - 32 \cdot F_{11} + 27 \cdot F_{12}.$$

A tedious yet straightforward calculation reveals the following coordinate-wise inequality

$$\left(\frac{1}{33}, \frac{12}{209}, \frac{3}{1147}, \frac{231}{163}, \frac{17}{84}, \frac{12}{293}\right) \cdot M < \left(3, 0, 9, 3, -\frac{17}{5}, 18, -\frac{34}{5}, -\frac{49}{5}, 12, -\frac{4}{5}, -32, 27\right)$$

which in turn shows that (8) asymptotically holds in the theory of triangle-free graphs.

The flag algebra calculations used in the proof of Lemma 2.3 can be independently verified by a SAGE script, which is available as an ancillary file of the arXiv version of this manuscript.

## 3 Concluding remarks

As we have already mentioned in the introduction, a significantly larger flag algebra computation than the one used in our proof yields that  $q_n(G) < 0.15467n$  for every triangle-free *n*-vertex graph. Similarly, assuming that G is regular allows us to show  $\lambda_1(G) + \lambda_n(G) < 0.15442n$ . On the other hand, our method will be able to get neither of the coefficients below  $42/275 = 0.15\overline{27}$ .

Indeed, consider the Higman-Sims graph  $G_{HS}$ . It is edge-transitive so  $m_{xy} = 21 \cdot 6 + 22$  for every  $xy \in E(G_{HS})$ , and  $w_x = 22^2$  for every  $x \in V(G)$ , where  $m_{xy}$  and  $w_x$  are defined as before Lemma 2.1. Therefore,

$$\frac{w_x + w_y - 2m_{xy}}{(\deg(x) + \deg(y)) \cdot |V(G_{HS})|} = \frac{2(22^2 - 21 \cdot 6 - 22)}{2 \cdot 22 \cdot 100} = \frac{42}{275}$$

for every  $xy \in E(G_{HS})$ , and so Lemma 2.1 only yields  $q_n(G_{HS}) \leq \frac{42}{275} \cdot |V(G_{HS})|$ . However, we have  $q_n(G_{HS}) = \lambda_1(G_{HS}) + \lambda_n(G_{HS}) = 0.14 \cdot |V(G_{HS})|$ . It might be that  $q_n(G) \leq 0.14n$  holds for every triangle-free graph G on n vertices.

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