# On Gupta's Co-density Conjecture 

Yan Cao ${ }^{a}$ Guantao Chen ${ }^{a}$ Guoli Ding ${ }^{b}$ Guangming Jing ${ }^{a *}$ Wenan Zang ${ }^{c \dagger}$<br>${ }^{a}$ Department of Mathematics and Statistics, Georgia State University Atlanta, GA 30303, USA<br>${ }^{b}$ Mathematics Department, Louisiana State University<br>Baton Rouge, LA 70803, USA<br>c Department of Mathematics, The University of Hong Kong Hong Kong, China


#### Abstract

Let $G=(V, E)$ be a multigraph. The cover index $\xi(G)$ of $G$ is the greatest integer $k$ for which there is a coloring of $E$ with $k$ colors such that each vertex of $G$ is incident with at least one edge of each color. Let $\delta(G)$ be the minimum degree of $G$ and let $\Phi(G)$ be the co-density of $G$, defined by $$
\Phi(G)=\min \left\{\frac{2\left|E^{+}(U)\right|}{|U|+1}: U \subseteq V,|U| \geq 3 \text { and odd }\right\}
$$ where $E^{+}(U)$ is the set of all edges of $G$ with at least one end in $U$. It is easy to see that $\xi(G) \leq \min \{\delta(G),\lfloor\Phi(G)\rfloor\}$. In 1978 Gupta proposed the following co-density conjecture: Every multigraph $G$ satisfies $\xi(G) \geq \min \{\delta(G)-1,\lfloor\Phi(G)\rfloor\}$, which is the dual version of the Goldberg-Seymour conjecture on edge-colorings of multigraphs. In this note we prove that $\xi(G) \geq \min \{\delta(G)-1,\lfloor\Phi(G)\rfloor\}$ if $\Phi(G)$ is not integral and $\xi(G) \geq \min \{\delta(G)-2,\lfloor\Phi(G)\rfloor-1\}$ otherwise. We also show that this co-density conjecture implies another conjecture concerning cover index made by Gupta in 1967.


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## 1 Introduction

In this note we consider multigraphs, which may have parallel edges but contain no loops. Let $G=(V, E)$ be a multigraph. The chromatic index $\chi^{\prime}(G)$ of $G$ is the least integer $k$ for which there is a coloring of $E$ with $k$ colors such that each vertex of $G$ is incident with at most one edge of each color. Let $\Delta(G)$ be the maximum degree of $G$ and let $\Gamma(G)$ be the density of $G$, defined by

$$
\Gamma(G)=\max \left\{\frac{2|E(U)|}{|U|-1}: U \subseteq V,|U| \geq 3 \text { and odd }\right\}
$$

where $E(U)$ is the set of all edges of $G$ with both ends in $U$. Clearly, $\chi^{\prime}(G) \geq \max \{\Delta(G), \Gamma(G)\}$; this lower bound, as shown by Seymour [10] using Edmonds' matching polytope theorem [2], is precisely the fractional chromatic index of $G$, which is the optimal value of the fractional edge-coloring problem:

$$
\begin{array}{ll}
\text { Minimize } & \mathbf{1}^{T} \boldsymbol{x} \\
\text { subject to } & A \boldsymbol{x}=\mathbf{1} \\
& \boldsymbol{x} \geq \mathbf{0}
\end{array}
$$

where $A$ is the edge-matching incidence matrix of $G$. In the 1970s Goldberg [3] and Seymour [10] independently made the following conjecture.
Conjecture 1.1. Every multigraph $G$ satisfies $\chi^{\prime}(G) \leq \max \{\Delta(G)+1,\lceil\Gamma(G)\rceil\}$.
Over the past four decades this conjecture has been a subject of extensive research, and has stimulated an important body of work, with contributions from many researchers; see McDonald [7] for a survey on this conjecture and Stiebitz et al. [11] for a comprehensive account of edgecolorings. Recently, three of the authors, Chen, Jing, and Zang, have announced a complete proof of Conjecture 1.1 [1].

The present note is devoted to the study of the dual version of the classical edge-coloring problem (ECP), which asks for a coloring of the edges of $G$ using the maximum number of colors in such a way that at each vertex all colors occur. It is easy to see that each color class induces an edge cover of $G$. (Recall that an edge cover is a subset $F$ of $E$ such that each vertex of $G$ is incident to at least one edge in $F$.) So this problem is actually the edge cover packing problem (ECPP). Let $\xi(G)$ denote the optimal value of ECPP, which we call the cover index of $G$. As it is $N P$-hard [6] in general to determine the chromatic index $\chi^{\prime}(G)$ of a simple cubic graph $G$, determining the cover index $\xi(G)$ is also $N P$-hard.

Let $\delta(G)$ be the minimum degree of $G$, let $E^{+}(U)$ be the set of all edges of $G$ with at least one end in $U$ for each $U \subseteq V$, and let $\Phi(G)$ be the co-density of $G$, defined by

$$
\Phi(G)=\min \left\{\frac{2\left|E^{+}(U)\right|}{|U|+1}: U \subseteq V,|U| \geq 3 \text { and odd }\right\} .
$$

Obviously, $\xi(G) \leq \delta(G)$. Since each edge cover contains at least $(|U|+1) / 2$ edges in $E^{+}(U)$ for any $U \subseteq V$ with $|U| \geq 3$ and odd, $\Phi(G)$ provides another upper bound for $\xi(G)$. So
$\xi(G) \leq \min \{\delta(G), \Phi(G)\}$. Based on a polyhedral description of edge covers (see Theorem 27.3 in Schrijver [9]), Zhao, Chen, and Sang [12] observed that the parameter $\min \{\delta(G), \Phi(G)\}$ is exactly the fractional cover index of $G$, the optimal value of the fractional edge cover packing problem (FECPP):

$$
\begin{array}{ll}
\text { Maximize } & \mathbf{1}^{T} \boldsymbol{x} \\
\text { subject to } & B \boldsymbol{x}=\mathbf{1} \\
& \boldsymbol{x} \geq \mathbf{0},
\end{array}
$$

where $B$ is the edge-edge cover incidence matrix of $G$. They [12] also devised a combinatorial polynomial-time algorithm for finding the co-density $\Phi(G)$ of any multigraph $G$.

In 1978 Gupta 5 proposed the following co-density conjecture, which is the counterpart of Conjecture 1.1 on ECPP.

Conjecture 1.2. Every multigraph $G$ satisfies $\xi(G) \geq \min \{\delta(G)-1,\lfloor\Phi(G)\rfloor\}$.
The reader is referred to Stiebitz et al. [11] for more information about this conjecture. Its validity would imply that, first, there are only two possible values for the cover index $\xi(G)$ of a multigraph $G$ : $\min \{\delta(G)-1,\lfloor\Phi(G)\rfloor\}$ and $\min \{\delta(G),\lfloor\Phi(G)\rfloor\}$; second, any multigraph has a cover index within one of its fractional cover index, so FECPP also has a fascinating integer rounding property (see Schrijver [8, [9]); third, even if $P \neq N P$, the $N P$-hardness of ECPP does not preclude the possibility of designing an efficient algorithm for finding at least $\min \{\delta(G)-1,\lfloor\Phi(G)\rfloor\}$ disjoint edge covers in any multigraph $G$.

To our knowledge, the bound $\xi(G) \geq \min \left\{\left\lfloor\frac{7 \delta(G)+1}{8}\right\rfloor,\lfloor\Phi(G)\rfloor\right\}$ established by Gupta [5] in 1978 remains to be the best approximate version of Conjecture 1.2,

As is well known, the inequality $\chi^{\prime}(G) \leq \Delta(G)+\mu(G)$ holds for any multigraph $G$, where $\mu(G)$ is the maximum multiplicity of an edge in $G$. This result has been successfully dualized by Gupta 4 to packing edge covers: $\xi(G) \geq \delta(G)-\mu(G)$. It is worthwhile pointing out that this dual version follows from Conjecture 1.2 as a corollary, because $\Phi(G) \geq \delta(G)-\mu(G)$. To see this, let $U$ be a subset of $V$ with $|U| \geq 3$ and odd, let $F(U)$ be the set of all edges of $G$ with precisely one end in $U$, and let $G[U]$ be the subgraph of $G$ induced by $U$. Since each vertex in $U$ is adjacent to at most $(|U|-1) \mu(G)$ edges in $G[U]$ and at most $|F(U)|$ edges outside $G[U]$, we have $\delta(G) \leq(|U|-1) \mu(G)+|F(U)|$, which implies that $\delta(G)|U|+|F(U)| \geq(\delta(G)-\mu(G))(|U|+1)$. As $2\left|E^{+}(U)\right|=2|E(U)|+2|F(U)| \geq \delta(G)|U|+|F(U)|$, we obtain $2\left|E^{+}(U)\right| \geq(\delta(G)-\mu(G))(|U|+1)$ and hence $\Phi(G) \geq \delta(G)-\mu(G)$, as desired.

Gupta [4] demonstrated that the lower bound $\delta(G)-\mu(G)$ for $\xi(G)$ is sharp when $\mu(G) \geq 1$ and $\delta(G)=2 p \mu(G)-q$, where $p$ and $q$ are two integers satisfying $q \geq 0$ and $p>\mu(G)+$ $\lfloor(q-1) / 2\rfloor$. This led Gupta [4] to suggest the following conjecture, which aims to give a complete characterization of all values of $\delta(G)$ and $\mu(G)$ for which no multigraph $G$ with $\xi(G)=$ $\delta(G)-\mu(G)$ exists.

Conjecture 1.3. Let $G$ be a multigraph such that $\delta(G)$ cannot be expressed in the form $2 p \mu(G)-$ $q$, for any two integers $p$ and $q$ satisfying $q \geq 0$ and $p>\mu(G)+\lfloor(q-1) / 2\rfloor$. Then $\xi(G) \geq$ $\delta(G)-\mu(G)+1$.

As edge covers are more difficult to manipulate than matchings, it is no surprise that a direct proof of conjecture 1.2 would be more complicated and sophisticated than that of Conjecture 1.1 (see [1], which is under review). One purpose of this note is to establish a slightly weaker version of conjecture 1.2 by using Conjecture 1.1 .

Theorem 1.1. (Assuming Conjecture (1.1) Let $G$ be a multigraph. Then $\xi(G) \geq \min \{\delta(G)-$ $1,\lfloor\Phi(G)\rfloor\}$ if $\Phi(G)$ is not integral and $\xi(G) \geq \min \{\delta(G)-2,\lfloor\Phi(G)\rfloor-1\}$ otherwise.

Remark. Suppose $\Phi(G)<\delta(G)$. By this theorem, we obtain $\xi(G)=\lfloor\Phi(G)\rfloor$ if $\Phi(G)$ is not integral and $\Phi(G)-1 \leq \xi(G) \leq \Phi(G)$ otherwise, because $\xi(G) \leq \min \{\delta(G),\lfloor\Phi(G)\rfloor\}$.

In this note we also show that Conjecture 1.3 is contained in Conjecture 1.2 as a special case.
Theorem 1.2. Conjecture 1.2 implies Conjecture 1.3 .
Throughout this note we shall repeatedly use the following terminology and notations. Let $G=(V, E)$ be a multigraph. A subset $U$ of $V$ is called an odd set if $|U|$ is odd and $|U| \geq 3$. For each $v \in V$, let $d_{G}(v)$ be the degree of $v$ in $G$. For each $U \subseteq V$, let $E_{G}(U)$ be the set of all edges of $G$ with both ends in $U$, let $E_{G}^{+}(U)$ be the set of all edges of $G$ with at least one end in $U$, and let $F_{G}(U)$ be the set of all edges of $G$ with exactly one end in $U$. For any two subsets $X$ and $Y$ of $V$, let $E_{G}(X, Y)$ be the set of all edges of $G$ with one end in $X$ and the other end in $Y$. We write $E_{G}(x, y)$ for $E_{G}(X, Y)$ if $X=\{x\}$ and $Y=\{y\}$. We shall drop the subscript $G$ if there is no danger of confusion.

The proofs of the above two theorems will take up the entire remainder of this note.

## 2 Approximate Version

We present a proof of Theorem 1.1 in this section. Let $G=(V, E)$ be a multigraph and let $Z \subseteq V$. A set $C \subseteq E$ is called a $Z$-cover if every vertex of $Z$ is incident with at least one edge of $C$. Note that if $Z=V$, then $Z$-covers are precisely edge covers of $G$. Let $e \in E(x, y)$ and let $G^{\prime}$ be obtained from $G$ by adding a new vertex $x^{\prime}$ and making $e$ incident with $x^{\prime}$ instead of $x$ (yet still incident with $y$ ); we say that $G^{\prime}$ arises from $G$ by splitting off $e$ from $x$. To prove the theorem, we shall actually establish the following variant.

Theorem 2.1. Let $G=(V, E)$ be a multigraph, let $Z \subseteq V$, let $k$ be a positive integer, and let $\epsilon$ be 0 or 1 . If $d(z) \geq k+1$ for all $z \in Z$ and $\left|E^{+}(U)\right| \geq \frac{|U|+1}{2} k+\epsilon$ for all odd sets $U \subseteq Z$, then $G$ contains $k-1+\epsilon$ disjoint $Z$-covers.

Proof. Splitting off edges from vertices outside $Z$ if necessary, we may assume that all vertices outside $Z$ have degree one. Suppose for a contradiction that Theorem 2.1 is false. We reserve the triple $(G, Z, k)$ for a counterexample with the minimum $\sum_{z \in Z} d(z)$. For convenience, we call an odd set $U \subseteq Z$ optimal if $\left|E^{+}(U)\right|=\frac{|U|+1}{2} k+\epsilon$.

By hypothesis, $d(z) \geq k+1$ for all $z \in Z$, which can be strengthened as follows.
Claim. $d(z)=k+1$ for all $z \in Z$.
Otherwise, $d(z) \geq k+2$ for some $z \in Z$. If $z$ is contained in no optimal odd set $U \subseteq Z$, letting $H$ be obtained from $G$ by splitting off an edge from $z$, then $(H, Z, k)$ would be a smaller counterexample than $(G, Z, k)$, a contradiction. Hence
(1) there exists an optimal odd set $U_{1} \subseteq Z$ containing $z$; subject to this, we assume that $\left|U_{1}\right|$ is minimum.

Since $\left(\left|U_{1}\right|+1\right) k+2 \epsilon=2\left|E^{+}\left(U_{1}\right)\right|=2\left|E\left(U_{1}\right)\right|+2\left|F\left(U_{1}\right)\right| \geq(k+1)\left|U_{1}\right|+\left|F\left(U_{1}\right)\right|$, we have $\left|F\left(U_{1}\right)\right| \leq k-\left|U_{1}\right|+2 \epsilon \leq k<d(z)$. So $z$ is adjacent to some vertex $y \in U_{1}$. Let $H$ be arising from $G$ by splitting off one edge $e \in E(y, z)$ from $z$. We propose to show that
(2) $(H, Z, k)$ is a smaller counterexample than $(G, Z, k)$.

Assume the contrary. Then $\left|E_{H}^{+}\left(U_{2}\right)\right|<\frac{\left|U_{2}\right|+1}{2} k+\epsilon$ for some odd set $U_{2} \subseteq Z$ by the hypothesis of this theorem. Thus
(3) $z \in U_{2}, y \notin U_{2}$, and $\left|E^{+}\left(U_{2}\right)\right|=\frac{\left|U_{2}\right|+1}{2} k+\epsilon$.

Let $T_{1}=U_{1} \backslash U_{2}$ and $T_{2}=U_{2} \backslash U_{1}$. By (3), we have $y \in U_{1} \backslash U_{2}$, so $T_{1} \neq \emptyset$. By the minimality assumption on $\left|U_{1}\right|$ (see (1)), $U_{2}$ is not a proper subset of $U_{1}$, which implies $T_{2} \neq \emptyset$. Since $z \in U_{1} \cap U_{2}$, we obtain $\left|U_{1} \cap U_{2}\right| \geq 1$. Let us consider two cases, according to the parity of $\left|U_{1} \cap U_{2}\right|$.

Case 1. $\left|U_{1} \cap U_{2}\right|$ is odd.
It is a routine matter to check that
(4) $\left|E^{+}\left(U_{1} \cup U_{2}\right)\right|+\left|E^{+}\left(U_{1} \cap U_{2}\right)\right|=\left|E^{+}\left(U_{1}\right)\right|+\left|E^{+}\left(U_{2}\right)\right|-\left|E\left(T_{1}, T_{2}\right)\right|$.

In this case, $U_{1} \cup U_{2}$ is an odd set. So $\left|E^{+}\left(U_{1} \cup U_{2}\right)\right| \geq \frac{\left|U_{1} \cup U_{2}\right|+1}{2} k+\epsilon$ by the hypothesis of this theorem.
(5) $\left|E^{+}\left(U_{1} \cap U_{2}\right)\right| \geq \frac{\left|U_{1} \cap U_{2}\right|+1}{2} k+\epsilon+1$.

To justify this, note that if $\left|U_{1} \cap U_{2}\right|=1$, then $\left|E^{+}\left(U_{1} \cap U_{2}\right)\right|=d(z) \geq k+2$. So (5) holds. If $\left|U_{1} \cap U_{2}\right| \geq 3$, then $U_{1} \cap U_{2}$ is not an optimal odd set by the minimality assumption on $\left|U_{1}\right|$ (see (1)). Thus (5) is also true.

From (4) and (5) we deduce that $\frac{\left|U_{1} \cup U_{2}\right|+1}{2} k+\epsilon \leq\left|E^{+}\left(U_{1} \cup U_{2}\right)\right| \leq\left|E^{+}\left(U_{1}\right)\right|+\left|E^{+}\left(U_{2}\right)\right|-$ $\left|E^{+}\left(U_{1} \cap U_{2}\right)\right| \leq \frac{\left|U_{1}\right|+1}{2} k+\epsilon+\frac{\left|U_{2}\right|+1}{2} k+\epsilon-\frac{\left|U_{1} \cap U_{2}\right|+1}{2} k-\epsilon-1=\frac{\left|U_{1} \cup U_{2}\right|+1}{2} k+\epsilon-1$, a contradiction.

Case 2. $\left|U_{1} \cap U_{2}\right|$ is even.
It is easy to see that $\left|E^{+}\left(U_{1}\right)\right|+\left|E^{+}\left(U_{2}\right)\right|=\left|E^{+}\left(T_{1}\right)\right|+\left|E^{+}\left(T_{2}\right)\right|+2\left|E\left(U_{1} \cap U_{2}\right)\right|+\mid E\left(U_{1} \cap\right.$ $\left.U_{2}, T_{1} \cup T_{2}\right)|+2| E\left(U_{1} \cap U_{2}, \overline{U_{1} \cup U_{2}}\right) \mid$, where $\overline{U_{1} \cup U_{2}}=V-\left(U_{1} \cup U_{2}\right)$. Thus
(6) $\left|E^{+}\left(U_{1}\right)\right|+\left|E^{+}\left(U_{2}\right)\right| \geq\left|E^{+}\left(T_{1}\right)\right|+\left|E^{+}\left(T_{2}\right)\right|+2\left|E\left(U_{1} \cap U_{2}\right)\right|+\left|F\left(U_{1} \cap U_{2}\right)\right|$.

In this case, $\left|T_{i}\right|$ is odd, so $\left|E^{+}\left(T_{i}\right)\right| \geq \frac{\left|T_{i}\right|+1}{2} k+\epsilon$ for $i=1,2$ by the hypothesis of this theorem.

It follows from (3) and (6) that $\frac{\left|U_{1}\right|+1}{2} k+\epsilon+\frac{\left|U_{2}\right|+1}{2} k+\epsilon \geq \frac{\left|T_{1}\right|+1}{2} k+\epsilon+\frac{\left|T_{2}\right|+1}{2} k+\epsilon+2\left|E\left(U_{1} \cap U_{2}\right)\right|+$ $\left|F\left(U_{1} \cap U_{2}\right)\right| \geq \frac{\left|T_{1}\right|+1}{2} k+\epsilon+\frac{\left|T_{2}\right|+1}{2} k+\epsilon+\left|U_{1} \cap U_{2}\right|(k+1)=\frac{\left|U_{1}\right|+1}{2} k+\epsilon+\frac{\left|U_{2}\right|+1}{2} k+\epsilon+\left|U_{1} \cap U_{2}\right|$, a contradiction.

Combining the above two cases, we obtain (2). This contradiction justifies the claim.
For each odd set $U \subseteq Z$, by the above claim, we obtain $|U|(k+1)=2|E(U)|+|F(U)|=$ $|E(U)|+\left|E^{+}(U)\right| \geq|E(U)|+\frac{|U|+1}{2} k+\epsilon$. Thus $|E(U)| \leq \frac{|U|-1}{2}(k+2)+1-\epsilon$. Hence $\frac{2|E(U)|}{|U|-1} \leq k+3$ if $\epsilon=0$ and $\frac{2|E(U)|}{|U|-1} \leq k+2$ if $\epsilon=1$. By Conjecture 1.1, the chromatic index of $G[Z]$ is at most $k+3-\epsilon$. Since all vertices outside $Z$ have degree one, we further obtain $\chi^{\prime}(G) \leq k+3-\epsilon$. So $E$ can be partitioned into $k+3-\epsilon$ matchings $M_{1}, M_{2}, \ldots, M_{k+3-\epsilon}$.

Let us first consider the case when $\epsilon=0$. By the above claim,
(7) each vertex $z \in Z$ is disjoint from precisely two of $M_{1}, M_{2}, \ldots, M_{k+3}$ (as $d(z)=k+1$ ).

Let $H$ be the subgraph of $G$ induced by edges in $M_{k} \uplus M_{k+1} \uplus M_{k+2} \uplus M_{k+3}$, where $\uplus$ is the multiset sum, and let $N$ be an orientation of $H$ such that $\left|d_{N}^{+}(v)-d_{N}^{-}(v)\right| \leq 1$ for each vertex $v$. (It is well known that every multigraph admits such an orientation.) From (7) and this orientation we see that
(8) if a vertex $z \in Z$ is disjoint from precisely one of $M_{1}, M_{2}, \ldots, M_{k-1}$, then $d_{H}(z)=3$ and $d_{N}^{-}(z) \geq 1$; if $z$ is disjoint from precisely two of $M_{1}, M_{2}, \ldots, M_{k-1}$, then $d_{H}(z)=4$ and $d_{N}^{-}(z)=2$.

For each $i=1,2, \ldots, k-1$, let $C_{i}$ be obtained from $M_{i}$ as follows: for each $z \in Z$, if $z$ not covered by $M_{i}$, add an edge from $N$ that is directed to $z$ and has not yet been used in $C_{1} \uplus C_{2} \uplus \ldots \uplus C_{i-1}$, where $C_{0}=\emptyset$. From this construction and (8) we deduce that $C_{1}, C_{2}, \ldots, C_{k-1}$ are pairwise disjoint and each of them is a $Z$-cover in $G$.

It remains to consider the case when $\epsilon=1$. Now
(9) each vertex $z \in Z$ is disjoint from precisely one of $M_{1}, M_{2}, \ldots, M_{k+2}$.

Let $H$ be the subgraph of $G$ induced by edges in $M_{k+1} \uplus M_{k+2}$, and let $N$ be an orientation of $H$ such that $\left|d_{N}^{+}(v)-d_{N}^{-}(v)\right| \leq 1$ for each vertex $v$. From (9) and this orientation we see that
(10) if a vertex $z \in Z$ is disjoint from precisely one of $M_{1}, M_{2}, \ldots, M_{k}$, then $d_{H}(z)=2$ and $d_{N}^{-}(z)=1$.

For each $i=1,2, \ldots, k$, let $C_{i}$ be obtained from $M_{i}$ as follows: for each $z \in Z$, if $z$ not covered by $M_{i}$, add an edge from $N$ that is directed to $z$. From this construction and (10) we deduce that $C_{1}, C_{2}, \ldots, C_{k}$ are pairwise disjoint and each of them is a $Z$-cover in $G$.

## 3 Implication

The purpose of this section is to show that Conjecture 1.3 can be deduced from Conjecture 1.2 .
Proof of Theorem 1.2. We may assume that
(1) $G$ is connected.

To see this, let $G_{1}, G_{2}, \ldots, G_{k}$ be all the components of $G$. For each $i=1,2, \ldots, k$, we aim to establish the inequality $\xi\left(G_{i}\right)>\delta(G)-\mu(G)$. If $\delta\left(G_{i}\right)-\mu\left(G_{i}\right)>\delta(G)-\mu(G)$, then the desired inequality holds, because $\xi\left(G_{i}\right) \geq \delta\left(G_{i}\right)-\mu\left(G_{i}\right)$. So we assume that $\delta\left(G_{i}\right)-\mu\left(G_{i}\right) \leq \delta(G)-\mu(G)$. Since $\delta\left(G_{i}\right) \geq \delta(G)$ and $\mu\left(G_{i}\right) \leq \mu(G)$, from this assumption we deduce that $\delta\left(G_{i}\right)=\delta(G)$ and $\mu\left(G_{i}\right)=\mu(G)$. Thus $G_{i}$ satisfies the hypothesis of Conjecture 1.3. Hence we may assume that $G$ is connected, otherwise we consider its components separately.

By hypothesis, $\delta(G)$ cannot be expressed in the form $2 p \mu(G)-q$, for any two integers $p$ and $q$ satisfying $q \geq 0$ and $p>\mu(G)+\lfloor(q-1) / 2\rfloor$; these two inequalities are equivalent to $0 \leq q \leq 2 p-2 \mu(G)$. Setting $q=0,1, \ldots, 2 p-2 \mu(G)$ respectively, we see that $\delta(G)$ does not belong to the set

$$
\Omega_{p}=\{2(p+1) \mu(G)-2 p, 2(p+1) \mu(G)-2 p+1, \ldots, 2 p \mu(G)\}
$$

where $p \geq \mu(G)$. Note that $2 \mu(G)^{2}$ is the only member of $\Omega_{\mu(G)}$ and that the gap between $\Omega_{p}$ and $\Omega_{p+1}$ consists of all integers $i$ with $2 p \mu(G)+1 \leq i \leq 2(p+2) \mu(G)-(2 p+3)$. So
(2) either $\delta(G) \leq 2 \mu(G)^{2}-1$ or $2 p \mu(G)+1 \leq \delta(G) \leq 2(p+2) \mu(G)-(2 p+3)$ for some $p \geq \mu(G)$.

We may assume that $\delta(G) \geq 1$, for otherwise, $\delta(G)=2 p \mu(G)-q$ for $p=q=0$, contradicting the hypothesis of Conjecture 1.3. Thus $\mu(G) \geq 1$.

To prove the theorem, it suffices to show that for any odd set $U$ of $G$, we have $\frac{2\left|E^{+}(U)\right|}{|U|+1} \geq$ $\delta(G)-\mu(G)+1$, or equivalently,
(3) $2|E(U)|+2|F(U)| \geq(|U|+1)(\delta(G)-\mu(G)+1)$.

Set $k=\mu(G)$ if $\delta(G) \leq 2 \mu(G)^{2}-1$ and set $k=p+1$ if $2 p \mu(G)+1 \leq \delta(G) \leq 2(p+2) \mu(G)-$ $(2 p+3)$ for some $p \geq \mu(G)$. We consider two cases according to the size of $U$.

Case 1. $|U| \geq 2 k+1$.
We divide the present case into two subcases.
Subcase 1.1. Either $U \subsetneq V$ or $U=V$ and $\delta(G)$ is odd. In this subcase,
(4) $2|E(U)|+2|F(U)| \geq|U| \delta(G)+1$.

Indeed, if $U \subsetneq V$, then $|F(U)| \geq 1$ by (1). If $U=V$ and $\delta(G)$ is odd, then $G$ contains at least one vertex of degree at least $\delta(G)+1$, because $|V|=|U|$ is odd and the total number of vertices with odd degree is even. Hence (4) is true.
(5) $|U| \delta(G)+1 \geq(|U|+1)(\delta(G)-\mu(G)+1)$.

Note that (5) amounts to saying that $\delta(G) \leq(|U|+1)(\mu(G)-1)+1$. If $\delta(G) \leq 2 \mu(G)^{2}-1$, then $\delta(G) \leq(2 \mu(G)+2)(\mu(G)-1)+1=(2 k+2)(\mu(G)-1)+1 \leq(|U|+1)(\mu(G)-1)+1$. If $\delta(G) \leq 2(p+2) \mu(G)-(2 p+3)$, then $\delta(G) \leq 2(k+1) \mu(G)-(2 k+1)=(2 k+2)(\mu(G)-1)+1 \leq$ $(|U|+1)(\mu(G)-1)+1$. So (5) is established.

The desired statement (3) follows instantly from (4) and (5).
Subcase 1.2. $U=V$ and $\delta(G)$ is even. In this subcase, we have $\delta(G) \leq 2 \mu(G)^{2}-2$ if $\delta(G) \leq 2 \mu(G)^{2}-1$ and $\delta(G) \leq 2(p+2) \mu(G)-(2 p+4)$ if $\delta(G) \leq 2(p+2) \mu(G)-(2 p+3)$. So $\delta(G) \leq(2 k+2)(\mu(G)-1)$ by the definition of $k$ and hence
(6) $\delta(G) \leq(|U|+1)(\mu(G)-1)$.

From (6) we deduce that $|U| \delta(G) \geq(|U|+1)(\delta(G)-\mu(G)+1)$. Therefore (3) holds, because $2|E(U)|+2|F(U)| \geq|U| \delta(G)$.

Case 2. $|U| \leq 2 k-1$. (So $k \geq 2$ as $|U| \geq 3$.)
By the Pigeonhole Principle, some vertex $v \in U$ is incident with at most $\frac{|F(U)|}{|U|}$ edges in $F(U)$. Note that $v$ is incident with at most $(|U|-1) \mu(G)$ edges in $G[U]$, so $d(v) \leq(|U|-1) \mu(G)+\frac{|F(U)|}{|U|}$. Hence
(7) $\delta(G) \leq(|U|-1) \mu(G)+\frac{|F(U)|}{|U|}$.

We proceed by considering two subcases.
Subcase 2.1. $2 p \mu(G)+1 \leq \delta(G) \leq 2(p+2) \mu(G)-(2 p+3)$, where $p \geq \mu(G)$.
From (7) and the hypothesis of the present subcase, we deduce that $2 p \mu(G)+1 \leq(|U|-$ 1) $\mu(G)+\frac{|F(U)|}{|U|}$. Thus $|F(U)| \geq|U|(2 p+1-|U|) \mu(G)+|U|$. So
(8) $|U| \delta(G)+|F(U)| \geq|U| \delta(G)+|U|(2 p+1-|U|) \mu(G)+|U|$.

Let us show that
(9) $|U| \delta(G)+|U|(2 p+1-|U|) \mu(G)+|U| \geq(|U|+1)(\delta(G)-\mu(G)+1)$.

To justify this, note that (9) is equivalent to
(10) $\delta(G) \leq\{|U|(2 p+2-|U|)+1\} \mu(G)-1$.

By the hypothesis of the present subcase, $\delta(G) \leq 2(p+2) \mu(G)-(2 p+3)$. To establish (10), we turn to proving that $2(p+2) \mu(G)-(2 p+3) \leq\{|U|(2 p+2-|U|)+1\} \mu(G)-1$, or equivalently
(11) $\left\{-|U|^{2}+2(p+1)|U|-(2 p+3)\right\} \mu(G) \geq-(2 p+2)$.

Let $f(x)=-x^{2}+2(p+1) x-(2 p+3)$. Then $f(x)$ is a concave function on $\mathbb{R}$. So on any interval $[a, b], f(x)$ achieves the minimum at $a$ or $b$. By the hypothesis of the present case, $|U| \leq 2 k-1=2 p+1$, so $3 \leq|U| \leq 2 p+1$. By direct computation, we obtain $f(3)=4 p-6 \geq-2$ and $f(2 p+1)=-2$. Thus $f(|U|) \geq-2$ for $3 \leq|U| \leq 2 p+1$, which implies that the LHS of (11) $\geq-2 \mu(G) \geq-(2 p+2)=$ RHS of (11), because $p \geq \mu(G)$. This proves (11) and hence (10) and (9).

Since $2|E(U)|+2|F(U)| \geq|U| \delta(G)+|F(U)|$, the desired statement (3) follows instantly from (8) and (9).

Subcase 2.2. $\delta(G) \leq 2 \mu(G)^{2}-1$.
We may assume that
(12) $\delta(G) \geq(|U|+1)(\mu(G)-1)+1$, for otherwise, $|U| \delta(G) \geq(|U|+1)(\delta(G)-\mu(G)+1)$. So (3) holds, because $2|E(U)|+2|F(U)| \geq|U| \delta(G)$.

By (12) and the hypothesis of the present subcase, either $2 t(\mu(G)-1)+1 \leq \delta(G) \leq$ $2(t+1)(\mu(G)-1)$ for some $t$ with $\frac{|U|+1}{2} \leq t \leq \mu(G)$ or $\delta(G)=2 t(\mu(G)-1)+1$ for $t=\mu(G)+1$.

By $(7)$, we have $2 t(\mu(G)-1)+1 \leq(|U|-1) \mu(G)+\frac{|F(U)|}{|U|}$. So $\frac{|F(U)|}{|U|} \geq(2 t-|U|+1) \mu(G)-2 t+1$, and hence
(13) $|U| \delta(G)+|F(U)| \geq|U|\{\delta(G)+(2 t-|U|+1) \mu(G)-2 t+1\}$.

We propose to show that
(14) $|U|\{\delta(G)+(2 t-|U|+1) \mu(G)-2 t+1\} \geq(|U|+1)(\delta(G)-\mu(G)+1)$.

To justify this, note that (14) is equivalent to
(15) $\delta(G) \leq\{|U|(2 t+2-|U|)+1\} \mu(G)-|U| 2 t-1$.

Suppose $\delta(G)=2 \mu(G)^{2}-1$. Then $t=\mu(G)+1$. So (15) says that $2 \mu(G)^{2}-1 \leq\{|U|(2 \mu(G)+$ $4-|U|)+1\} \mu(G)-|U|(2 \mu(G)+2)-1$, or equivalently, $\{|U|(2 \mu(G)+4-|U|)+1\} \mu(G)-$ $|U|(2 \mu(G)+2) \geq 2 \mu(G)^{2}$. Let $g(x)=\{x(2 \mu(G)+4-x)+1\} \mu(G)-x(2 \mu(G)+2)$. Then $g(x)$ is a concave function on $\mathbb{R}$. So on any interval $[a, b], g(x)$ achieves the minimum at $a$ or $b$. By direct computation, we obtain $g(3)=6 \mu(G)^{2}-2 \mu(G)-6$ and $g(2 \mu(G)-1)=6 \mu(G)^{2}-6 \mu(G)+2$. It is easy to see that $\min \{g(3), g(2 \mu(G)-1)\} \geq 2 \mu(G)^{2}$, because $\mu(G)=k \geq 2$ (see the hypothesis of Case 2). Hence $g(|U|) \geq 2 \mu(G)^{2}$ for $3 \leq|U| \leq 2 \mu(G)-1=2 k-1$. This proves (15) and hence (14) and (13).

So we assume that $\delta(G) \leq 2(t+1)(\mu(G)-1)$ for some $t$ with $\frac{|U|+1}{2} \leq t \leq \mu(G)$. We prove (15) by showing that $2(t+1)(\mu(G)-1) \leq\{|U|(2 t+2-|U|)+1\} \mu(G)-|U| 2 t-1$, or equivalently, $\{|U|(2 t+2-|U|)-2 t-1\} \mu(G)-|U| 2 t \geq-2 t-1$. Let $h(x)=\{x(2 t+2-x)-2 t-1\} \mu(G)-2 t x$. Then $h(x)$ is a concave function on $\mathbb{R}$. So on any interval $[a, b], h(x)$ achieves the minimum at $a$ or $b$. By direct computation, we obtain $h(3)=4(t-1) \mu(G)-6 t$ and $h(2 t-1)=4(t-1) \mu(G)-2 t(2 t-1)$. It is easy to see that $\min \{h(3), h(2 t-1)\} \geq-2 t-1$, because $\mu(G) \geq t \geq \frac{|U|+1}{2} \geq 2$. Hence $h(|U|) \geq-2 t-1$ for $3 \leq|U| \leq 2 t-1$. This proves (15) and hence (14) and (13).

Since $2|E(U)|+2|F(U)| \geq|U| \delta(G)+|F(U)|$, the desired statement (3) follows instantly from (13) and (14), competing the proof of Theorem 1.2,

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[^0]:    *E-mail: gjing1@gsu.edu.
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