ENHANCED DISSIPATION AND BLOW-UP SUPPRESSION IN A CHEMOTAXIS-FLUID SYSTEM

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ABSTRACT. In this paper, we investigate a coupled Patlak-Keller-Segel-Navier-Stokes (PKS-NS) system. We show that globally regular solutions with arbitrary large cell populations exist. The primary blowup suppression mechanism is the shear flow mixing induced enhanced dissipation phenomena.

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1. INTRODUCTION

We consider the coupled Patlak-Keller-Segel-Navier-Stokes (PKS-NS) system modeling the chemotaxis phenomenon in a moving fluid:

(1.1)
$$\begin{cases} \partial_t n + v \cdot \nabla n + \kappa \nabla \cdot (n \nabla c) = \kappa \Delta n, & (1 - \Delta)c = n, \\ \partial_t v + (v \cdot \nabla)v + \nabla p = \nu \Delta v + \kappa n \nabla c, & \nabla \cdot v = 0, \\ n(t = 0) = n_{\rm in}, & v(t = 0) = v_{\rm in}, & (x, y) \in \mathbb{T} \times \mathbb{R}. \end{cases}$$

Here *n* denotes the cell density, and *c* is the chemoattractant density. The divergence-free vector field *v* indicates the ambient fluid velocity. The first equation (Patlak-Keller-Segel equation) describes the time evolution of the cell density subject to transportation by ambient fluid flow *v*, aggregation trigged by chemotaxis, and diffusion in the media. The aggregation and diffusion take effects on a time scale $\mathcal{O}(\kappa^{-1})$, $\kappa \in (0, \infty)$. The cells move towards higher concentrations of the chemoattractants. In the meantime, they secrete the chemoattractants to re-enhance this aggregation effect. By assuming that the secretion and redistribution of chemoattractants happen at a fast timescale, we establish an elliptic-type partial differential relation between the density distributions, *n* and *c*. The equation (Navier-Stokes equation) on the divergence-free vector field *v* describes the fluid motion subject to force. The parameter ν is the inverse Reynold number, and the scalar function *p* denotes the pressure that ensures the divergence-free condition. The fluid exerts friction force on the moving cells to guarantee that they move without acceleration. Hence, Newton's law predicts that there exists a reaction force from the cells to the fluid. The coupling $n\nabla c$ in the Navier-Stokes equation models this interaction. The same forcing appears in the Nernst-Planck-Navier-Stokes system, see, e.g., [23].

If no ambient fluid flows are present, i.e., $v \equiv 0$, the coupled system (1.1) simplifies to a variant of the classical parabolic-elliptic Patlak-Keller-Segel (PKS) equation

(1.2)
$$\partial_t n + \kappa \nabla \cdot (n \nabla c) = \kappa \Delta n, \quad -\Delta c = n.$$

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The equation (1.2) is derived by C. Patlak [60], and E. Keller and L. Segel [42]. Simplified models are proposed by V. Nanjundiah, [58]. The literature on the PKS model is extensive, and we refer the interested readers to the representative works, [2, 9, 12–14, 16, 17, 29, 38, 41, 56, 57], and the references therein. We summarize the main results on the plane \mathbb{R}^2 as follows. Thanks to the divergence structure of the PKS equations (1.2), the total cell population/mass is conserved over time, i.e., $M := ||n(t)||_{L^1} = ||n_{in}||_{L^1}$. The long-time behavior of the equation (1.2) hinges on the total mass M. Suppose the initial cell density has a finite second moment and a total mass M strictly less than 8π . In that case, the unique solutions to (1.2) become smooth instantly and exist for all time, see, e.g., [9, 14, 17, 29, 39, 67]. A key observation in deriving sharp regularity results is that the Patlak-Keller-Segel equations have natural dissipative free energy

$$E := \int n \log n - \frac{1}{2} n c dV.$$

On the contrary, if the initial mass is strictly larger than 8π , the solutions with finite second moment blow up in finite time, e.g., [14–16, 41]. The refined description of the singularities is provided in work [21, 22, 37, 61, 64–66]. In the borderline case, $M = 8\pi$, the solutions with finite second moments form Dirac-type singularities as the time approaches infinity, [13, 26, 31].

If there is ambient fluid flow, the long-time dynamics of the systems (1.1) are delicate. In the pioneering work, [44], A. Kiselev and X. Xu show that if the fluid vector field v is relaxation enhancing in the sense of [24], then by choosing a large enough amplitude $(||v||_{\infty})$, the chemotactic blowups are suppressed. Their analysis is later generalized in [40] to a broader class of fluid vector fields. Furthermore, in work, [7, 33], the authors show that strong shear flows can suppress the blowups through a fast dimension reduction process. Last but not least, the authors of [35, 36] exploit the fast-spreading scenario of the hyperbolic and quenching shear flows to reach the same goal. In work mentioned above, the ambient fluid velocity fields v are passive because the dynamics of the cell density do not alter the fluid itself. If there is active coupling between the cell dynamics and the fluid motion, the only known result is [76]. In this work, the authors prove that if the underlying fluid flow is close to the Couette flow, strong enough shear suppresses the blowup of a specific type of PKS-NS system.

One can regard the system (1.1) as one among many attempts to model the chemotaxis phenomena in a fluid. The literature on coupled chemotaxis-fluid equations is vast, and we refer the interested readers to the papers [19, 28, 30, 45, 50–52, 62, 63, 70, 74] and the references therein. Many of these works investigate coupled systems involving fully parabolic Patlak-Keller-Segel and Navier-Stokes equations. The parabolic nature of the chemical equations complicates the analysis. For example, I. Tuval et al. proposed the following model [63],

$$\begin{cases} \partial_t n + v \cdot \nabla n + \nabla \cdot (n \nabla c) = \Delta n, \\ \partial_t c + v \cdot \nabla c = \Delta c - n f(c), \\ \partial_t v + (v \cdot \nabla) v + \nabla p = \Delta v + n \nabla \phi, \quad \nabla \cdot v = 0 \end{cases}$$

Here a parabolic equation governs the dynamics of the chemicals (oxygen), and the coupling $n\nabla\phi$ in the fluid equation models the buoyancy effect. The regularity and long-time behaviors of the system are explored in [71–73, 75]. On the other hand, A. Lorz [52], and H. Kozono et al. [45] proposed models whose chemical densities are determined through elliptic-type relations.

A new feature of the coupled system (1.1) is that it retains dissipative free energy

$$F = \int n \log n - \frac{1}{2}nc + \frac{1}{2}|v|^2 dV.$$

The dissipative free energy, together with the logarithmic Hardy-Littlewood-Sobolev inequality, yields global regularity of the solutions to a variant of (1.1) in the entire subcritical mass regime, i.e., $M < 8\pi$ ([32]). The critical mass case is analyzed in [46]. In the supercritical case $M > 8\pi$, there exists a solution with finite-time blow-up ([32]).

We consider the system perturbed around the Couette flow v(x, y) = (y, 0), a stationary solution to the Naver-Stokes equation. By decomposing the velocity field as v = y + u and writing the fluid equation in vorticity form, we end up with the system:

(1.3)
$$\begin{cases} \partial_t n + y \partial_x n + u \cdot \nabla n + \kappa \nabla \cdot (n \nabla c) = \kappa \Delta n, & (1 - \Delta)c = n, \\ \partial_t \omega + y \partial_x \omega + u \cdot \nabla \omega = \nu \Delta \omega + \kappa \nabla^{\perp} \cdot (n \nabla c), & u = \nabla^{\perp} \Delta^{-1} \omega, \\ n(t = 0) = n_{\rm in}, & \omega(t = 0) = \omega_{\rm in}, & (x, y) \in \mathbb{T} \times \mathbb{R}. \end{cases}$$

Here $\omega = \nabla^{\perp} \cdot u = -\partial_y u^{(1)} + \partial_x u^{(2)}$ and $\nabla^{\perp} = (-\partial_y, \partial_x)$. In this formulation, we can view the coupled system as a nonlinear perturbation of the Naver-Stokes solution (y, 0). The problem of suppressing the chemotactic blow-up is now equivalent to a nonlinear stability problem of the Couette flow. The fundamental question in the study of hydrodynamic stability is to determine the functional space X and the parameter $\alpha \in [0, \infty)$ such that

$$\|\omega_{\rm in}\|_X \lesssim \nu^{\alpha} \quad \Rightarrow \quad {\rm Stability.}$$

Here ν^{α} is the stability threshold of the flow associated with the spaces X. The nonlinear stability threshold of the Couette flow has attracted much attention in the last decade, see, e.g., [4–6, 10, 11, 20, 54, 55]. A complete survey of the literature is out of the scope of this paper. Therefore, we highlight some of the work focusing on a 2-dimensional setting. In [10], J. Bedrossian, N. Masmoudi, and V. Vicol showed that the stability threshold of the 2*D*-Couette flow is $\mathcal{O}(1)$ in the Gevrey spaces. The stability thresholds associated with the Sobolev spaces are shown to be $\mathcal{O}(\nu^{1/2})$ on the cylinder $\mathbb{T} \times \mathbb{R}$ (J. Bedrossian, V. Vicol and F. Wang, [11]) and in the channel (Q. Chen, T. Li, D. Wei, and Z. Zhang, [20]). Later, N. Masmoudi and W. Zhao considered higher-order Sobolev norms (H^{49}) and improved the threshold to $\mathcal{O}(\nu^{1/3})$, [55]. The enhanced dissipation phenomenon of the Couette flow plays an essential role in the above works. We also refer the interested readers to the work [27, 49, 53, 78, 79] for the detailed stability analysis of Couette flow in MHD, Boussinesq equations. For the enhanced dissipation phenomena associated with other shear flows, we refer the interested readers to the work [1, 3, 25, 34, 47, 48, 68, 69], and the references therein.

Our main result is that global-in-time regular solutions with arbitrarily large mass M to the system (1.1) exist.

Theorem 1.1. Consider the equation (1.3) subject to initial conditions $n_{\text{in}} \geq 0$, $n_{\text{in}} \in L^1 \cap H^s(\mathbb{T} \times \mathbb{R})$, $n_{\text{in}}|y|^2 \in L^1(\mathbb{T} \times \mathbb{R})$, $\omega_{\text{in}} \in H^s(\mathbb{T} \times \mathbb{R})$, $5 \leq s \in \mathbb{N}$. Assume that the parameters κ , ν are in the regime $0 < \kappa \leq \nu \leq 1$. There exists a threshold $\epsilon_0(\|n_{\text{in}}\|_{L^1 \cap H^s}) \in (0, 1)$ such that if the following relations hold

(1.4)
$$\|\omega_{\rm in}\|_{H^s} \le \epsilon \nu^{1/2}, \quad \kappa = \epsilon \nu, \quad 0 < \epsilon \le \epsilon_0,$$

the regular solutions to (1.3) exist for all time. Moreover, there exists a universal constant $\delta \in (0,1)$ such that the following enhanced dissipation estimate holds

$$\left\| e^{\delta \kappa^{1/3} |\partial_x|^{2/3} t} \left(n - \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} n dx \right) \right\|_{L^2} + \left\| e^{\delta \kappa^{1/3} |\partial_x|^{2/3} t} \left(\omega - \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \omega dx \right) \right\|_{L^2} \le \mathcal{B}(\|n_{\mathrm{in}}\|_{L^1 \cap H^s}), \quad \forall t \in [0, \infty).$$

Here the bound \mathcal{B} only depends on the initial data.

Remark 1.1. We compare our result with that of [76]. In [76], the authors considered a similar system with buoyancy force coupling between the fluid and cell-density equations and showed suppression of blowup results. However, here our system (1.1) involves a nonlinear coupling in the fluid equation, which complicates the analysis. Moreover, the methods we employ here are different from that of [76].

Remark 1.2. The stability threshold in Theorem 1.1 matches that of the paper [11]. However, we expect that by using more complicated techniques developed in [55], one can improve the stability threshold to $\epsilon \nu^{1/3}$. We will leave the analysis in later work.

Remark 1.3. This paper focuses on the parameter regime where $\kappa \leq \nu$. The method does not seem applicable in the parameter regime $\nu \gg \kappa$. If the viscosity ν is much greater than κ , the biological phenomena take place on time scale $\mathcal{O}(\kappa^{-1})$, which can be much shorter than the fluid dynamics time scale. As a result, the dynamics of the cell evolution have a nontrivial impact on the fluid motion. Thanks to this nontrivial interaction, the fluid flow might no longer be stable. Hence it is a great problem to understand the long time dynamics of the system in this regime.

The remaining part of the paper is organized as follows. In Section 2, we sketch the proof of Theorem 1.1. In Section 3, we provide estimates of the fluid equation. In Section 4, the estimates of the cell density are derived. Finally, we collect technical lemmas in the appendix.

2. Sketch of the Proof

In this section, we present the main idea of the proof of Theorem 1.1. Similar to the works [8, 43, 59, 77], we first consider a new coordinate system

$$z = x - ty, \quad y = y.$$

we have the following:

(2.1)
$$\partial_t N + U \cdot \nabla_L N + \kappa \nabla_L \cdot (N \nabla_L C) = \kappa \Delta_L N, \quad C = (1 - \Delta_L)^{-1} N; \\ \partial_t \Omega + U \cdot \nabla_L \Omega = \nu \Delta_L \Omega + \kappa \nabla_L^{\perp} \cdot (N \nabla_L C), \quad U = -\nabla_L^{\perp} (-\Delta_L)^{-1} \Omega.$$

Here the following notations are adopted:

$$\nabla_L := \begin{pmatrix} \partial_z \\ \partial_y^t \end{pmatrix} := \begin{pmatrix} \partial_z \\ \partial_y - t\partial_z \end{pmatrix}, \quad \nabla_L^{\perp} := \begin{pmatrix} -\partial_y^t \\ \partial_z \end{pmatrix}, \quad \Delta_L = \nabla_L \cdot \nabla_L = \partial_{zz} + (\partial_y - t\partial_z)^2.$$

Since the enhanced dissipation phenomenon is heterogeneous, we consider the z-average f_0 and the remainder f_{\neq} of functions on $\mathbb{T} \times \mathbb{R}$:

$$f_0(t,y) = \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} f(t,z,y) dz, \quad f_{\neq}(t,z,y) = f(t,z,y) - f_0(t,y).$$

We decompose the solutions N, Ω into the z-average and remainder:

(2.2a)
$$\partial_t N_{\neq} + (U \cdot \nabla_L N)_{\neq} = \kappa \Delta_L N_{\neq} - \kappa (\nabla_L \cdot (N \nabla_L C))_{\neq};$$

(2.2b)
$$\partial_t N_0 + (U \cdot \nabla_L N)_0 = \kappa \partial_{yy} N_0 - \kappa (\nabla_L \cdot (N \nabla_L C))_0;$$

(2.2c)
$$\partial_t \Omega_{\neq} + (U \cdot \nabla_L \Omega)_{\neq} = \nu \Delta_L \Omega_{\neq} + \kappa (\nabla_L^{\perp} \cdot (N \nabla_L C))_{\neq},$$

(2.2d)
$$\partial_t \Omega_0 + (U \cdot \nabla_L \Omega)_0 = \nu \partial_{yy} \Omega_0 + \kappa (\nabla_L^{\perp} \cdot (N \nabla_L C))_0$$

To analyze the above equations, we apply the spacial Fourier transform $f(t, z, y) \xrightarrow{\mathcal{F}} \widehat{f}(t, k, \eta)$. Next we introduce the following multipliers W_{κ} , W_{ν} , W, which are motivated by [11]:

(2.3)
$$W_{\kappa}(t,k,\eta) = \pi - \arctan\left(\kappa^{1/3}|k|^{2/3}\left(t-\frac{\eta}{k}\right)\right) \mathbf{1}_{0<|k|\leq\kappa^{-1/2}};$$

(2.4)
$$W_{\nu}(t,k,\eta) = \pi - \arctan\left(\nu^{1/3}|k|^{2/3}\left(t-\frac{\eta}{k}\right)\right) \mathbf{1}_{0<|k|\leq\nu^{-1/2}};$$

(2.5)
$$\mathcal{W}(t,k,\eta) = \pi - \arctan\left(t - \frac{\eta}{k}\right) \mathbf{1}_{k \neq 0}$$

We observe that these multiplier functions take values in $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Moreover, they are monotonically decreasing in time. We further define the following multipliers associated with the cell density N and the vorticity Ω :

(2.6)
$$M_{\kappa} = W_{\kappa} \mathcal{W}, \quad M_{\nu} = W_{\nu} \mathcal{W}.$$

Further define that

(2.7)
$$A_{\iota}(t,k,\eta) = M_{\iota}(t,k,\eta) e^{\delta \kappa^{1/3} |k|^{2/3} t} (1+|k|^2+|\eta|^2)^{s/2}, \quad \iota \in \{\nu,\kappa\}, \quad s \ge 0.$$

Here $0 < \delta < 1$ is a universal constant. We note that the A_{ν} -multiplier and A_{κ} -multiplier share the same exponential factor $e^{\delta \kappa^{1/3}|k|^{2/3}t}$. The multipliers $\{A_{\kappa}, M_{\kappa}\}$ will act on the cell density and chemical density N, C and $\{A_{\nu}, M_{\nu}\}$ will act on the vorticity and velocity of the fluid Ω, U . The properties of these multipliers are collected in the appendix.

Next, we present a local well-posedness result, which can be proven through standard argument.

Theorem 2.1. Consider solutions N, Ω to the equation (2.1) subject to initial data $0 \leq N_{\text{in}} \in L^1 \cap H^s(\mathbb{T} \times \mathbb{R})$, $N_{\text{in}}|y|^2 \in L^1(\mathbb{T} \times \mathbb{R})$, $\Omega_{\text{in}} \in H^s(\mathbb{T} \times \mathbb{R})$, $3 \leq s \in \mathbb{N}$. There exists a small constant $T_{\varepsilon}(||N_{\text{in}}||_{L^1 \cap H^s}, ||\Omega_{\text{in}}||_{H^s})$ such that the unique solution exists on the time interval $[0, T_{\varepsilon}]$. Moreover, $N \geq 0$ on $[0, T_{\varepsilon}]$.

To prove Theorem 1.1, we use a bootstrap argument. Assume that $[0, T_{\star}]$ is the largest time interval on which the following **Hypotheses** hold:

(2.8a)
$$||A_{\kappa}N_{\neq}(t)||_{L^{2}}^{2} + \int_{0}^{t} \left\| \sqrt{\frac{-\partial_{\tau}M_{\kappa}}{M_{\kappa}}} A_{\kappa}N_{\neq} \right\|_{L^{2}}^{2} + \kappa ||A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}||_{L^{2}}^{2} d\tau \leq 2\mathcal{B}_{N_{\neq}}^{2};$$

(2.8b)
$$||N_0(t)||^2_{H^s} \le 2\mathcal{B}^2_{N_0};$$

(2.8c)
$$||A_{\nu}\Omega_{\neq}(t)||_{L^{2}}^{2} + \int_{0}^{t} \left\| \sqrt{\frac{-\partial_{\tau}M_{\nu}}{M_{\nu}}} A_{\nu}\Omega_{\neq} \right\|_{L^{2}}^{2} + \nu ||A_{\nu}\sqrt{-\Delta_{L}}\Omega_{\neq}||_{L^{2}}^{2} d\tau \leq 2\mathcal{B}_{\Omega_{\neq}}^{2}\epsilon^{2}\nu;$$

(2.8d)
$$\|\Omega_0(t)\|_{H^s}^2 + \nu \int_0^t \|\partial_y \Omega_0\|_{H^s}^2 d\tau \le 2\mathcal{B}_{\Omega_0}^2 \epsilon^2 \nu, \quad \forall t \in [0, T_*]$$

Without loss of generality, we set $\mathcal{B}_{N_{\neq}}, \mathcal{B}_{N_0}, \mathcal{B}_{\Omega_{\neq}}, \mathcal{B}_{\Omega_0} \geq 1$. Moreover, they only depend on the initial data $\|N_{\text{in}}\|_{L^1 \cap H^s}$ and the regularity level s.

Proposition 2.1. Consider the system (2.1) subject to initial condition $0 \le N_{\text{in}} \in L^1 \cap H^s(\mathbb{T} \times \mathbb{R})$, $N_{\text{in}}|y|^2 \in L^1(\mathbb{T} \times \mathbb{R})$, $\Omega_{\text{in}} \in H^s(\mathbb{T} \times \mathbb{R})$, $5 \le s \in \mathbb{N}$. Assume that $0 < \kappa \le \nu \le 1$. Let $[0, T_\star]$ be the largest interval on which the hypotheses (2.8) hold. There exists a threshold $\epsilon_0 = \epsilon_0(||N_{\text{in}}||_{L^1 \cap H^s})$ such that if the condition (1.4) is satisfied, the following stronger estimates hold

$$(2.9a) \quad \|A_{\kappa}N_{\neq}(t)\|_{L^{2}}^{2} + \int_{0}^{t} \left\|\sqrt{\frac{-\partial_{\tau}M_{\kappa}}{M_{\kappa}}}A_{\kappa}N_{\neq}\right\|_{L^{2}}^{2} + \kappa\|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}}^{2}d\tau \leq \mathcal{B}_{N_{\neq}}^{2};$$

(2.9b)
$$\|N_0(t)\|_{H^s}^2 \leq \mathcal{B}_{N_0}^2;$$

(2.9c)
$$||A_{\nu}\Omega_{\neq}(t)||_{L^{2}}^{2} + \int_{0}^{t} \left\| \sqrt{\frac{-\partial_{\tau}M_{\nu}}{M_{\nu}}} A_{\nu}\Omega_{\neq} \right\|_{L^{2}}^{2} + \nu ||A_{\nu}\sqrt{-\Delta_{L}}\Omega_{\neq}||_{L^{2}}^{2} d\tau \leq \mathcal{B}_{\Omega_{\neq}}^{2} \epsilon^{2}\nu;$$

(2.9d)
$$\|\Omega_0(t)\|_{H^s}^2 + \nu \int_0^t \|\partial_y \Omega_0\|_{H^s}^2 d\tau \leq \mathcal{B}_{\Omega_0}^2 \epsilon^2 \nu, \quad \forall t \in [0, T_\star].$$

Here the bounds $\mathcal{B}_{N_{\neq}}$, \mathcal{B}_{N_0} , $\mathcal{B}_{\Omega_{\neq}}$ and \mathcal{B}_{Ω_0} depend only on the initial data $\|N_{\mathrm{in}}\|_{L^1 \cap H^s}$.

Remark 2.1. The explicit choice of the constants $\mathcal{B}_{N_{\neq}}$, \mathcal{B}_{N_0} , $\mathcal{B}_{\Omega_{\neq}}$ and \mathcal{B}_{Ω_0} are listed in (4.25). The choice of the threshold ϵ_0 can be found in (4.26).

Now we can conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Combining Theorem 2.1 and Proposition 2.1, we see that $[0, T_*]$ is both open and closed on $[0, \infty)$. Hence $T_* = \infty$ and the solutions with estimates (2.9) exist for all time. The enhanced dissipation estimate (1.5) is a consequence of the bounds (2.9a), (2.9c),

$$\begin{aligned} \left\| e^{\delta \kappa^{1/3} |\partial_x|^{2/3} t} \left(n - \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} n dx \right) \right\|_{L^2_{x,y}} + \left\| e^{\delta \kappa^{1/3} |\partial_x|^{2/3} t} \left(\omega - \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} \omega dx \right) \right\|_{L^2_{x,y}} \\ &\leq C \|A_{\kappa} N_{\neq}(t)\|_{L^2_{x,y}} + C \|A_{\nu} \Omega_{\neq}(t)\|_{L^2_{x,y}} \leq C \mathcal{B}^2_{N_{\neq}}(\|N_{\mathrm{in}}\|_{L^1 \cap H^s}) + C \mathcal{B}^2_{\Omega_{\neq}}(\|N_{\mathrm{in}}\|_{L^1 \cap H^s}), \quad \forall t \in [0,\infty). \end{aligned}$$

This concludes the proof.

2.1. Notations. Throughout the paper, the constant C, which can only depend on the regularity level s, will change from line to line. Constants with subscript, i.e., $\mathcal{B}_{N\neq}$, will be fixed. For $A, B \geq 0$, we use the notation $A \approx B$ to highlight that there exists a strictly positive constant C such that $B/C \leq A \leq CB$. We also use the notation $A \lesssim B$ to represent that there exists a constant C > 0 such that $A \leq CB$.

Recall the classical L^p norms and Sobolev H^s , $s \in \mathbb{N}_+$ norms:

$$\|f\|_{L^{p}} = \|f\|_{p} = \left(\int |f|^{p} dV\right)^{1/p}; \quad \|f\|_{L^{q}_{t}([0,T];L^{p})} = \left(\int_{0}^{T} \|f(t)\|_{L^{p}}^{q} dt\right)^{1/q};$$
$$\|f\|_{H^{s}} = \left(\sum_{i+j\leq s} \|\partial_{z}^{i}\partial_{y}^{j}f\|_{L^{2}}^{2}\right)^{1/2}; \quad \|f\|_{\dot{H}^{s}} = \left(\sum_{i+j=s} \|\partial_{z}^{i}\partial_{y}^{j}f\|_{L^{2}}^{2}\right)^{1/2}.$$

Here dV = dzdy = dxdy is the volume element. If p or q is ∞ , then we use the classical L^{∞} -norm.

We use \hat{f} to denote Fourier transform of function f in the (z, y) variables. The frequency variables corresponding z and y are denoted by $\{k, \ell\}$ and $\{\eta, \xi\}$, respectively. The Fourier multipliers are defined as follows:

$$(\widehat{\mathfrak{M}(\nabla)f})(k,\eta) = \mathfrak{M}(ik,i\eta)\widehat{f}(k,\eta)$$

Given a set $S \in \mathbb{Z}$, we define the projection $\mathbb{P}_{k \in S}$

$$\widehat{\mathbb{P}_{k\in\mathcal{S}}f} = \mathbf{1}_{k\in\mathcal{S}}\widehat{f}(k,\eta).$$

We apply the following notations

$$\langle k \rangle = \sqrt{1+k^2}, \quad \langle k,\eta \rangle = (1+k^2+\eta^2)^{1/2}, \quad |k,\eta| = |k|+|\eta|.$$

We recall from classical literature that $||f||_{H^s} \approx ||\langle \partial_z, \partial_y \rangle^s f||_{L^2}$.

3. VORTICITY ESTIMATES

In this section, we derive the estimates associated with the fluid motion. We organize the proof into two subsections. In Subsection 3.1, we prove the enhanced dissipation estimate of the vorticity remainder Ω_{\neq} (2.9c). In Subsection 3.2, we prove the z-average (Ω_0) estimate (2.9d).

3.1. **Remainder Estimates.** In this section, we consider the remainder of the vorticity, which solves the equation (2.2c). We will prove the conclusion (2.9c).

Application of the energy estimate yields that

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|A_{\nu}\Omega_{\neq}\|_{L^{2}}^{2} = & \delta \kappa^{1/3} \sum_{k \neq 0} |k|^{2/3} \int |A_{\nu}(t,k,\eta)\widehat{\Omega}(t,k,\eta)|^{2} d\eta - \sum_{k \neq 0} \int \frac{-\partial_{t}M_{\nu}}{M_{\nu}} |A_{\nu}(t,k,\eta)\widehat{\Omega}(t,k,\eta)|^{2} d\eta \\ & -\nu \sum_{k \neq 0} \int |A_{\nu}(t,k,\eta)(\sqrt{|k|^{2} + |\eta - kt|^{2}})\widehat{\Omega}(t,k,\eta)|^{2} d\eta \\ & -\int A_{\nu}(U \cdot \nabla_{L}\Omega)_{\neq} A_{\nu}\Omega_{\neq} dV + \kappa \int A_{\nu}(\nabla_{L}^{\perp} \cdot (N\nabla_{L}C))_{\neq} A_{\nu}\Omega_{\neq} dV. \end{aligned}$$

The relation $\kappa \leq \nu$, together with the property of the multiplier M_{ν} (A.4) yields that if $\delta \leq \frac{1}{16\pi^2}$, then the time evolution of the norm is bounded as follows:

$$\frac{d}{dt}\frac{1}{2}\|A_{\nu}\Omega_{\neq}\|_{L^{2}}^{2} \leq -\frac{1}{2}\left\|\sqrt{\frac{-\partial_{t}M_{\nu}}{M_{\nu}}}A_{\nu}\Omega_{\neq}\right\|_{L^{2}}^{2} - \frac{1}{2}\nu\|A_{\nu}\sqrt{-\Delta_{L}}\Omega_{\neq}\|_{L^{2}}^{2} - \int A_{\nu}(U\cdot\nabla_{L}\Omega)_{\neq} A_{\nu}\Omega_{\neq}dV + \kappa \int A_{\nu}(\nabla_{L}^{\perp}\cdot(N\nabla_{L}C))_{\neq} A_{\nu}\Omega_{\neq}dV.$$

Integration in time yields that

$$\begin{split} \|A_{\nu}\Omega_{\neq}(t)\|_{L^{2}}^{2} + \int_{0}^{t} \left\|\sqrt{\frac{-\partial_{\tau}M_{\nu}}{M_{\nu}}}A_{\nu}\Omega_{\neq}\right\|_{L^{2}}^{2} d\tau + \nu \int_{0}^{t} \|A_{\nu}\sqrt{-\Delta_{L}}\Omega_{\neq}\|_{L^{2}}^{2} d\tau \\ \leq \|A_{\nu}\Omega_{\mathrm{in};\neq}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}(U \cdot \nabla_{L}\Omega)_{\neq} A_{\nu}\Omega_{\neq}dVd\tau\right| + 2\left|\kappa \int_{0}^{t} \int A_{\nu}(\nabla_{L}^{\perp} \cdot (N\nabla_{L}C))_{\neq} A_{\nu}\Omega_{\neq}dVd\tau \\ (3.1) =: \|A_{\nu}\Omega_{\mathrm{in};\neq}\|_{L^{2}}^{2} + T_{\Omega_{\neq};1} + T_{\Omega_{\neq};2}. \end{split}$$

The estimates of the terms $T_{\Omega_{\neq};1}$ and $T_{\Omega_{\neq};2}$ are summarized in Lemma 3.1 and Lemma 3.2.

Lemma 3.1. Assume all the conditions in Proposition 2.1. The $T_{\Omega_{\neq;1}}$ -term in (3.1) is bounded as follows

(3.2)
$$T_{\Omega_{\neq};1} \leq C(\mathcal{B}_{\Omega_{\neq}} + \mathcal{B}_{\Omega_{0}})\mathcal{B}_{\Omega_{\neq}}^{2}\epsilon^{3}\nu.$$

Here the constant C depends only on the regularity level s.

Lemma 3.2. Assume all the conditions in Proposition 2.1. There exists a constant C = C(s) such that the following estimate of the $T_{\Omega_{\neq};2}$ -term holds

$$(3.3) T_{\Omega_{\neq};2} \le \frac{1}{4}\nu \int_{0}^{t} \|A_{\nu}\sqrt{-\Delta_{L}}\Omega_{\neq}\|_{L^{2}}^{2}d\tau + \frac{1}{4}\int_{0}^{t} \left\|A_{\nu}\sqrt{\frac{-\partial_{\tau}M_{\nu}}{M_{\nu}}}\Omega_{\neq}\right\|_{L^{2}}^{2}d\tau + C(\mathcal{B}_{N_{\neq}}^{4} + \mathcal{B}_{N_{0}}^{4})\epsilon^{2}\nu.$$

With these two lemmas, we are ready to prove the improved bound (2.9c).

Proof of the conclusion (2.9c). First, we recall the initial condition (1.4), which ensures that $||A_{\nu}\Omega_{\text{in};\neq}||_{L^{2}_{z,y}} \leq C||\Omega_{\text{in};\neq}||_{H^{s}_{x,y}} \leq C\epsilon\nu^{1/2}$. Combining it with the relation (3.1) and the estimates (3.2), (3.3), we obtain that

$$\begin{split} \|A_{\nu}\Omega_{\neq}(t)\|_{L^{2}}^{2} + \int_{0}^{t} \left\|\sqrt{\frac{-\partial_{\tau}M_{\nu}}{M_{\nu}}}\Omega_{\neq}(\tau)\right\|_{L^{2}}^{2} d\tau + \nu \int_{0}^{t} \|A_{\nu}\sqrt{-\Delta_{L}}\Omega_{\neq}(\tau)\|_{L^{2}}^{2} d\tau \\ \leq C\|\Omega_{\mathrm{in};\neq}\|_{H^{s}}^{2} + C\left(\epsilon(\mathcal{B}_{\Omega_{\neq}} + \mathcal{B}_{\Omega_{0}})\mathcal{B}_{\Omega_{\neq}}^{2} + \mathcal{B}_{N_{\neq}}^{4} + \mathcal{B}_{N_{0}}^{4}\right)\epsilon^{2}\nu \leq C\left(\epsilon(\mathcal{B}_{\Omega_{\neq}} + \mathcal{B}_{\Omega_{0}})\mathcal{B}_{\Omega_{\neq}}^{2} + 1 + \mathcal{B}_{N_{\neq}}^{4} + \mathcal{B}_{N_{0}}^{4}\right)\epsilon^{2}\nu. \end{split}$$

Here the constant C depends only on the regularity level s. As a result the following choices of constants yields (2.9c)

(3.4)
$$\frac{1}{2}\mathcal{B}_{\Omega_{\neq}}^{2} \geq C(1+\mathcal{B}_{N_{\neq}}^{4}+\mathcal{B}_{N_{0}}^{4}), \quad \epsilon \leq \frac{1}{2C(\mathcal{B}_{\Omega_{\neq}}+\mathcal{B}_{\Omega_{0}})}.$$

This concludes the proof.

In the remaining part of this subsection, we prove Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1. The estimate of the $T_{\Omega_{\neq};1}$ -term is in the same vein as the one in [11]. We carry out the details for the sake of completeness. We recall that the Biot-Savart law yields that $U_0^{(2)} = \partial_z \Delta_y^{-1} \Omega_0 = 0$. Hence we can expand the $T_{\Omega_{\neq};1}$ term as follows:

$$(3.5) \quad T_{\Omega_{\neq};1} \leq 2 \left| \int_0^t \int A_{\nu} (U_{\neq} \cdot \nabla_L \Omega) \ A_{\nu} \Omega_{\neq} dV d\tau \right| + 2 \left| \int_0^t \int A_{\nu} (U_0^{(1)} \partial_z \Omega) \ A_{\nu} \Omega_{\neq} dV d\tau \right| =: \sum_{i=1}^2 T_{\Omega_{\neq};1i}.$$

For the $T_{\Omega_{\neq};11}$ -term, we first invoke the Biot-Savart law to rewrite the velocity as $U_{\neq} = \nabla_L^{\perp} \Delta_L^{-1} \Omega_{\neq}$, and then apply the product estimate associated with the multiplier A_{ν} (A.10) to obtain the following bound

$$T_{\Omega_{\neq};11} \leq C \int_{0}^{t} \|A_{\nu}(\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq} \cdot \nabla_{L}\Omega)\|_{L^{2}} \|A_{\nu}\Omega_{\neq}\|_{L^{2}} d\tau \leq C \int_{0}^{t} \|A_{\nu}\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq}\|_{L^{2}} \|A_{\nu}\nabla_{L}\Omega\|_{L^{2}} d\tau \|A_{\nu}\Omega_{\neq}\|_{L^{\infty}_{t}L^{2}} d\tau \leq C \int_{0}^{t} \|A_{\nu}\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq}\|_{L^{2}} \|A_{\nu}\nabla_{L}\Omega\|_{L^{2}} d\tau \|A_{\nu}\Omega_{\neq}\|_{L^{\infty}_{t}L^{2}} d\tau \leq C \int_{0}^{t} \|A_{\nu}\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq}\|_{L^{2}} \|A_{\nu}\nabla_{L}\Omega\|_{L^{2}} d\tau \|A_{\nu}\Omega_{\neq}\|_{L^{\infty}_{t}L^{2}} d\tau \leq C \int_{0}^{t} \|A_{\nu}\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq}\|_{L^{2}} \|A_{\nu}\nabla_{L}\Omega\|_{L^{2}} d\tau \|A_{\nu}\Omega_{\neq}\|_{L^{2}} d\tau \leq C \int_{0}^{t} \|A_{\nu}\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq}\|_{L^{2}} d\tau \|A_{\nu}\nabla_{L}\Omega\|_{L^{2}} d\tau \|A_{\nu}\nabla_{L}\Omega\|_{L^{2}}$$

Next we observe that the M_{ν} -properties (A.3b), (A.3c) imply that $\|A_{\nu}\nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq}\|_{L^{2}} \leq C\|A_{\nu}\sqrt{\frac{-\partial_{t}M_{\nu}}{M_{\nu}}}\Omega_{\neq}\|_{L^{2}}$. Combining this with the bootstrap hypothesis (2.8c), (2.8d) yields that

$$T_{\Omega_{\neq};11} \leq \frac{C}{\sqrt{\nu}} \left\| A_{\nu} \sqrt{\frac{-\partial_t M_{\nu}}{M_{\nu}}} \Omega_{\neq} \right\|_{L^2_t L^2} (\nu^{1/2} \| A_{\nu} \sqrt{-\Delta_L} \Omega_{\neq} \|_{L^2_t L^2} + \nu^{1/2} \| \partial_y \Omega_0 \|_{L^2_t H^s}) \| A_{\nu} \Omega_{\neq} \|_{L^\infty_t L^2}$$

$$(3.6) \qquad \leq C \epsilon (\mathcal{B}_{\Omega_{\neq}} + \mathcal{B}_{\Omega_0}) \mathcal{B}^2_{\Omega_{\neq}} \epsilon^2 \nu.$$

This is consistent with the estimate (3.2).

For the $T_{\Omega_{\neq};12}$ term in (3.5), we first observe there is a cancellation relation

$$\int \partial_y \Delta_y^{-1} \Omega_0 \ \partial_z A_\nu \Omega_\neq A_\nu \Omega_\neq dV = \int \partial_y \Delta_y^{-1} \Omega_0 \partial_z \left(\frac{A_\nu \Omega_\neq}{2}\right)^2 dV = 0.$$

Hence we can rewrite the term $T_{\Omega_{\neq};12}$ using the Biot-Savart law $U_0^{(1)} = -\partial_y \Delta_y^{-1} \Omega_0$ as follows:

$$(3.7) T_{\Omega_{\neq};12} = 2 \left| \int_{0}^{t} \int \left(A_{\nu}(-\partial_{y}\Delta_{y}^{-1}\Omega_{0} \ \partial_{z}\Omega_{\neq}) + \partial_{y}\Delta_{y}^{-1}\Omega_{0} \ \partial_{z}A_{\nu}\Omega_{\neq} \right) A_{\nu}\Omega_{\neq} \ dVd\tau$$
$$= C \left| \sum_{k\neq 0} \int_{0}^{t} \iint \left(M_{\nu}(\tau,k,\eta)\langle k,\eta \rangle^{s} - M_{\nu}(\tau,k,\xi)\langle k,\xi \rangle^{s} \right) \ \frac{i(\eta-\xi)}{(\eta-\xi)^{2}} \widehat{\Omega}_{0}(\eta-\xi) \right| \times \left(ke^{\delta\kappa^{1/3}|k|^{2/3}\tau} \widehat{\Omega}(\tau,k,\xi) \right) \ \overline{A_{\nu}\widehat{\Omega}(\tau,k,\eta)} \ d\eta d\xi d\tau \right|.$$

Now we invoke the commutator estimate (A.11) and the Young's convolution inequality to obtain that

 $T_{\Omega_{\neq};12}$

$$\leq C \bigg| \int_0^t \sum_{k \neq 0} \iint \left(\langle k, \xi \rangle^s + \langle \eta - \xi \rangle^s \right) \left| \widehat{\Omega}_0(\eta - \xi) \right| \bigg| e^{\delta \kappa^{1/3} |k|^{2/3} \tau} \widehat{\Omega}(\tau, k, \xi) \bigg| \left| A_\nu \widehat{\Omega}(\tau, k, \eta) \right| d\eta d\xi d\tau \bigg|$$

$$\leq C \bigg| \int_0^t \sum_{k \neq 0} \|A_\nu \widehat{\Omega}_k(\cdot)\|_{L^2_\eta} \left(\| \langle \cdot \rangle^s \widehat{\Omega}_0(\cdot)\|_{L^2_\eta} \| e^{\delta \kappa^{1/3} |k|^{2/3} \tau} \widehat{\Omega}_k(\cdot)\|_{L^1_\eta} + \| \widehat{\Omega}_0(\cdot)\|_{L^1_\eta} \| e^{\delta \kappa^{1/3} |k|^{2/3} \tau} \langle k, \cdot \rangle^s \widehat{\Omega}_k(\cdot)\|_{L^2_\eta} \bigg) d\tau \bigg|$$

Combining the M_{ν} -properties (A.3b), (A.4), the definition of A_{ν} (2.7) and the inequality $\|\widehat{f}(\cdot)\|_{L^{1}_{\eta}} \leq C \|\langle \cdot \rangle^{s} \widehat{f}(\cdot)\|_{L^{2}_{\nu}}, s \geq 1$ yields that

$$T_{\Omega_{\neq};12} \leq C \int_{0}^{t} \sum_{k \neq 0} \|A_{\nu} \widehat{\Omega}_{k}(\cdot)\|_{L_{\eta}^{2}}^{2} d\tau \|\Omega_{0}\|_{L_{t}^{\infty} H^{s}}$$
$$\leq C \nu^{-1/3} \left(\int_{0}^{t} \left\|A_{\nu} \sqrt{\frac{-\partial_{\tau} M_{\nu}}{M_{\nu}}} \Omega_{\neq} \right\|_{L^{2}}^{2} + \nu \|A_{\nu} \sqrt{-\Delta_{L}} \Omega_{\neq}\|_{L^{2}}^{2} d\tau \right) \|\Omega_{0}\|_{L_{t}^{\infty} H^{s}}$$

Hence the bootstrap hypotheses (2.8c), (2.8d) implies that

(3.8)
$$T_{\Omega_{\neq};12} \le C \epsilon^3 \nu \mathcal{B}_{\Omega_{\neq}}^2 \mathcal{B}_{\Omega_0}$$

Combining the decomposition (3.5) and the estimates (3.6), (3.8), we have obtained the result (3.2). \Box

Proof of Lemma 3.2. We further decompose the $T_{\Omega_{\neq};2}$ term in (3.1) as follows:

$$(3.9) \qquad T_{\Omega_{\neq};2} \leq 2\kappa \left| \int_{0}^{t} \int A_{\nu}(\partial_{z}(N_{\neq}\partial_{y}C_{0})) A_{\nu}\Omega_{\neq}dVd\tau \right| + 2\kappa \left| \int_{0}^{t} \int A_{\nu}(\nabla_{L}^{\perp} \cdot (N\nabla_{L}C_{\neq})) A_{\nu}\Omega_{\neq}dVd\tau \right|$$
$$=:T_{\Omega_{\neq};21} + T_{\Omega_{\neq};22}.$$

Before analyzing the terms, we make a comment about the multipliers. Thanks to the $\{M_{\nu}, M_{\kappa}\}$ -property (A.3b) and the definitions of $\{A_{\nu}, A_{\kappa}\}$ (2.7), we have that the A_{ν}, A_{κ} multipliers are comparable, i.e.,

(3.10)
$$\frac{1}{16\pi^4} A_{\kappa}(t,k,\eta) \le A_{\nu}(t,k,\eta) \le 16\pi^4 A_{\kappa}(t,k,\eta)$$

As a result, we have the freedom to adjust the multipliers $\{A_{\iota}\}_{\iota \in \{\kappa,\nu\}}$ when considering different objects.

With the multiplier properties explained, we start the estimate. To estimate the $T_{\Omega_{\neq};21}$ -term, we apply the relation $(1 - \partial_{yy})C_0 = N_0$, the definition (2.7), and the $\partial_t M_{\kappa}$ -estimate (A.3c) to get the following

$$\begin{split} T_{\Omega_{\neq};21} \leq & C\kappa \sum_{k \neq 0} \int_{0}^{t} \iint \left| \sqrt{|k|^{2} + |\eta - k\tau|^{2}} A_{\nu} \widehat{\Omega}(\tau, k, \eta) \right| \\ & \times \left| A_{\kappa}(\tau, k, \eta) \sqrt{\frac{|k|^{2}}{|k|^{2} + |\eta - k\tau|^{2}}} \left(|\widehat{N}(\tau, k, \xi)| \frac{|\eta - \xi|}{1 + |\eta - \xi|^{2}} |\widehat{N}_{0}(\tau, \eta - \xi)| \right) \right| d\eta d\xi d\tau \\ \leq & C\kappa \sum_{k \neq 0} \int_{0}^{t} \iint \left| \sqrt{|k|^{2} + |\eta - k\tau|^{2}} A_{\nu} \widehat{\Omega}(\tau, k, \eta) \right| \\ & \times \left| M_{\kappa}(\tau, k, \eta) \langle k, \eta \rangle^{s} \sqrt{-\partial_{\tau} M_{\kappa}(\tau, k, \eta)} \left(e^{\delta \kappa^{1/3} |k|^{2/3} \tau} |\widehat{N}(\tau, k, \xi)| \frac{|\eta - \xi|}{1 + |\eta - \xi|^{2}} |\widehat{N}_{0}(\tau, \eta - \xi)| \right) \right| d\eta d\xi d\tau. \end{split}$$

Now we invoke the M_{κ} properties (A.3b), (A.3e) to obtain

 $T_{\Omega_{\neq};21}$

$$\leq C\kappa \sum_{k\neq 0} \int_0^t \iint \left| \sqrt{|k|^2 + |\eta - k\tau|^2} A_{\nu} \widehat{\Omega}(\tau, k, \eta) \right|$$

$$\times \left(M_{\kappa}(\tau, k, \xi) \langle k, \xi \rangle^s + \langle \eta - \xi \rangle^s \right) e^{\delta \kappa^{1/3} |k|^{2/3} \tau} \sqrt{-\partial_{\tau} M_{\kappa}(\tau, k, \xi)} |\widehat{N}(\tau, k, \xi)| \; \frac{|\eta - \xi| \langle \eta - \xi \rangle}{1 + |\eta - \xi|^2} |\widehat{N}_0(\tau, \eta - \xi)| d\eta d\xi d\tau.$$

Now we apply similar argument as in (3.8) to estimate the term. Application of Young's convolution inequality, the M_{κ} -bound (A.3b), the A_{κ} -definition (2.7), and the fact that $\|\widehat{f}(\cdot)\|_{L^{1}_{\eta}} \leq C \|\langle \cdot \rangle^{s} \widehat{f}(\cdot)\|_{L^{2}_{\eta}}$, $s \geq 1$ yields that

$$(3.11) T_{\Omega_{\neq};21} \leq \frac{1}{8} \nu \int_{0}^{t} \left\| A_{\nu} \sqrt{-\Delta_{L}} \Omega_{\neq} \right\|_{L^{2}}^{2} d\tau + C \epsilon^{2} \nu \int_{0}^{t} \left\| A_{\kappa} \sqrt{-\frac{\partial_{\tau} M_{\kappa}}{M_{\kappa}}} N_{\neq} \right\|_{L^{2}}^{2} \| N_{0} \|_{H^{s}}^{2} d\tau \\ \leq \frac{1}{8} \nu \int_{0}^{t} \left\| A_{\nu} \sqrt{-\Delta_{L}} \Omega_{\neq} \right\|_{L^{2}}^{2} d\tau + C \epsilon^{2} \nu \mathcal{B}_{N_{\neq}}^{2} \mathcal{B}_{N_{0}}^{2}.$$

Here in the last line, the hypotheses (2.8a) and (2.8b) are employed.

For the $T_{\Omega_{\neq};22}$ term in (3.9), we apply integration by parts, and then estimate it with the product estimate (A.10), the elliptic estimate (A.8), and the bootstrap hypotheses (2.8a), (2.8b) as follows

$$\begin{split} T_{\Omega_{\neq};22} &\leq \kappa \int_{0}^{t} \|A_{\nu} \nabla_{L}^{\perp} \Omega_{\neq}\|_{L^{2}} (\|N_{0}\|_{H^{s}} + \|A_{\kappa}N_{\neq}\|_{L^{2}}) \|A_{\kappa} \nabla_{L} C_{\neq}\|_{L^{2}} d\tau \\ &\leq \frac{1}{16} \nu \int_{0}^{t} \|A_{\nu} \sqrt{-\Delta_{L}} \Omega_{\neq}\|_{L^{2}}^{2} d\tau + C \epsilon^{2} \nu (\|N_{0}\|_{L_{t}^{\infty}H^{s}}^{2} + \|A_{\kappa}N_{\neq}\|_{L_{t}^{\infty}L^{2}}^{2}) \|A_{\kappa} \nabla_{L} (1 - \Delta_{L})^{-1} N_{\neq}\|_{L_{t}^{2}L^{2}}^{2} \\ &\leq \frac{1}{16} \nu \int_{0}^{t} \|A_{\nu} \sqrt{-\Delta_{L}} \Omega_{\neq}\|_{L^{2}}^{2} d\tau + C \epsilon^{2} \nu (\|N_{0}\|_{L_{t}^{\infty}H^{s}}^{2} + \|A_{\kappa}N_{\neq}\|_{L_{t}^{2}L^{2}}^{2}) \left\|A_{\kappa} \sqrt{\frac{-\partial_{t}M_{\kappa}}{M_{\kappa}}} N_{\neq}\right\|_{L_{t}^{2}L^{2}}^{2} \\ &\leq \frac{1}{16} \nu \int_{0}^{t} \|A_{\nu} \sqrt{-\Delta_{L}} \Omega_{\neq}\|_{L^{2}}^{2} d\tau + C \epsilon^{2} \nu (\mathcal{B}_{N_{0}}^{2} + \mathcal{B}_{N_{\neq}}^{2}) \mathcal{B}_{N_{\neq}}^{2}. \end{split}$$

Combining this with the decomposition (3.9) and the bound (3.11) yields the estimate (3.3).

3.2. The z-average Estimates. In this subsection, we prove the estimate (2.9d). First of all, we recall the equation (2.2d), and decompose the nonlinearity as follows

$$\partial_t \Omega_0 + (U_{\neq} \cdot \nabla_L \Omega_{\neq})_0 + (U_0 \cdot \nabla \Omega_0)_0 = \nu \Delta \Omega_0 + \kappa \left(\nabla_L^{\perp} \cdot (N_0 \nabla_L C_0) \right)_0 + \kappa \left(\nabla_L^{\perp} \cdot (N_{\neq} \nabla_L C_{\neq}) \right)_0.$$

Here we observe two null-structures which lead to simplification. First, we observe that by the Biot-Savart law, the vertical velocity field $U_0^{(2)} = \partial_z \Delta_y^{-1} \Omega_0 = 0$. Hence,

$$(U_0 \cdot \nabla \Omega_0)_0 = (U_0^{(1)} \partial_z \Omega_0)_0 = 0.$$

On the other hand, the following term vanishes,

$$\kappa \left(\nabla_L^{\perp} \cdot (N_0 \nabla_L C_0) \right)_0 = -\kappa \left((\partial_y - t \partial_z) (N_0 \partial_z C_0) \right)_0 + \kappa \left(\partial_z (N_0 \partial_y C_0) \right)_0 = 0$$

Hence the Ω_0 -equation can be simplified to the following,

$$\partial_t \Omega_0 - \nu \Delta \Omega_0 = -(U_{\neq} \cdot \nabla_L \Omega_{\neq})_0 + \kappa \left(\nabla_L^{\perp} \cdot (N_{\neq} \nabla_L C_{\neq}) \right)_0$$

Recalling that $A_{\nu}(t, k = 0, \eta) = \pi^2 \langle \eta \rangle^s$, we calculate the time evolution of the $||A_{\nu}\Omega_0||_{L^2}^2$ as follows:

$$\frac{d}{dt}\frac{1}{2}\|A_{\nu}\Omega_{0}\|_{L^{2}}^{2} = -\nu\|A_{\nu}\partial_{y}\Omega_{0}\|_{L^{2}}^{2} - \int A_{\nu}((U_{\neq}\cdot\nabla_{L}\Omega_{\neq})_{0}) A_{\nu}\Omega_{0}dV + \kappa \int A_{\nu}\left(\nabla_{L}^{\perp}\cdot(N_{\neq}\nabla_{L}C_{\neq})\right)_{0} A_{\nu}\Omega_{0}dV$$

Now integration in time yields

(3.12)

$$\begin{aligned} \|A_{\nu}\Omega_{0}(t)\|_{L^{2}}^{2} + 2\nu \int_{0}^{t} \|A_{\nu}\partial_{y}\Omega_{0}\|_{L^{2}}^{2} d\tau \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}((U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0}) A_{\nu}\Omega_{0}dVd\tau\right| + 2\kappa \left|\int_{0}^{t} \int A_{\nu}\left(\nabla_{L}^{\perp} \cdot (N_{\neq}\nabla_{L}C_{\neq})\right)_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}((U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0}) A_{\nu}\Omega_{0}dVd\tau\right| + 2\kappa \left|\int_{0}^{t} \int A_{\nu}\left(\nabla_{L}^{\perp} \cdot (N_{\neq}\nabla_{L}C_{\neq})\right)_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}((U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0}) A_{\nu}\Omega_{0}dVd\tau\right| + 2\kappa \left|\int_{0}^{t} \int A_{\nu}\left(\nabla_{L}^{\perp} \cdot (N_{\neq}\nabla_{L}C_{\neq})\right)_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}((U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0}) A_{\nu}\Omega_{0}dVd\tau\right| + 2\kappa \left|\int_{0}^{t} \int A_{\nu}\left(\nabla_{L}^{\perp} \cdot (N_{\neq}\nabla_{L}C_{\neq})\right)_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}(U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0} A_{\nu}\Omega_{0}dVd\tau\right| + 2\kappa \left|\int_{0}^{t} \int A_{\nu}\left(\nabla_{L}^{\perp} \cdot (N_{\neq}\nabla_{L}C_{\neq})\right)_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}(U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}(U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|\int_{0}^{t} \int A_{\nu}(U_{\neq} \cdot \nabla_{L}\Omega_{\neq})_{0} A_{\nu}\Omega_{0}dVd\tau\right| \\ \leq \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|A_{\nu}\Omega_{0}(U_{\neq})\right|_{L^{2}}^{2} + 2\left|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2} + 2\left|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^{2}}^{2}$$

 $=: \|A_{\nu}\Omega_{\mathrm{in};0}\|_{L^2}^2 + T_{\Omega_0;1} + T_{\Omega_0;2}.$

We rewrite the $T_{\Omega_0;1}$ -term in (3.12) with the Biot-Savart law, and then estimate it with the product estimate (A.10), the M_{ν} -bound (A.3b), the elliptic estimate (A.8), and the hypotheses (2.8c), (2.8d) as follows

$$T_{\Omega_{0};1} = \int_{0}^{t} \int A_{\nu} (\nabla_{L}^{\perp} \Delta_{L}^{-1} \Omega_{\neq} \cdot \nabla_{L} \Omega_{\neq})_{0} A_{\nu} \Omega_{0} dV \leq C \int_{0}^{t} \|A_{\nu} \nabla_{L}^{\perp} \Delta_{L}^{-1} \Omega_{\neq}\|_{L^{2}} \|A_{\nu} \nabla_{L} \Omega_{\neq}\|_{L^{2}} \|A_{\nu} \Omega_{0}\|_{L^{2}} d\tau$$

$$(3.13) \leq C \left\|A_{\nu} \sqrt{\frac{-\partial_{t} M_{\nu}}{M_{\nu}}} \Omega_{\neq}\right\|_{L^{2}_{t}L^{2}} \|A_{\nu} \nabla_{L} \Omega_{\neq}\|_{L^{2}_{t}L^{2}} \|\Omega_{0}\|_{L^{\infty}_{t}H^{s}_{y}} \leq \epsilon^{2} \nu \left(\epsilon C \mathcal{B}_{\Omega_{0}} \mathcal{B}^{2}_{\Omega_{\neq}}\right).$$

Next we estimate $T_{\Omega_0;2}$ -term in (3.12). By observing $(\partial_z F)_0 \equiv 0$ and integration by parts, we rewrite the term as follows,

$$T_{\Omega_0;2} = 2 \left| \kappa \int_0^t \int A_\nu \Omega_0 A_\nu ((\partial_y - t\partial_z)(N_{\neq}\partial_z C_{\neq}))_0 dV d\tau \right| = 2 \left| \kappa \int_0^t \int A_\nu \partial_y \Omega_0 A_\nu (N_{\neq}\partial_z C_{\neq})_0 dV d\tau \right|.$$

Application of the product estimate (A.10) and the fact that $A_{\nu} \approx A_{\kappa}$ (3.10) yields that

$$T_{\Omega_0;2} \leq \frac{1}{2} \nu \|A_{\nu} \partial_y \Omega_0\|_{L^2_t L^2}^2 + C \epsilon^2 \nu \|A_{\kappa} N_{\neq}\|_{L^\infty_t L^2}^2 \|A_{\kappa} \partial_z C_{\neq}\|_{L^2_t L^2}^2.$$

After invoking the relation $\partial_z C_{\neq} = \partial_z (1 - \Delta_L)^{-1} N_{\neq}$, the M_{κ} -estimate (A.3b) and the elliptic estimate (A.8), we estimate the $T_{\Omega_{0;2}}$ -term with the hypotheses (2.9a) as follows

$$T_{\Omega_{0};2} \leq \frac{1}{2}\nu \|A_{\nu}\partial_{y}\Omega_{0}\|_{L^{2}_{t}L^{2}}^{2} + C\epsilon^{2}\nu \|A_{\kappa}N_{\neq}\|_{L^{\infty}_{t}L^{2}}^{2} \|A_{\kappa}\partial_{z}(1-\Delta_{L})^{-1}N_{\neq}\|_{L^{2}_{t}L^{2}}^{2}$$

$$(3.14) \leq \frac{1}{2}\nu \|A_{\nu}\partial_{y}\Omega_{0}\|_{L^{2}_{t}L^{2}}^{2} + C\epsilon^{2}\nu \|A_{\kappa}N_{\neq}\|_{L^{\infty}_{t}L^{2}}^{2} \left\|A_{\kappa}\sqrt{\frac{-\partial_{t}M_{\kappa}}{M_{\kappa}}}N_{\neq}\right\|_{L^{2}_{t}L^{2}}^{2} \leq \frac{1}{2}\nu \|A_{\nu}\partial_{y}\Omega_{0}\|_{L^{2}_{t}L^{2}}^{2} + C\mathcal{B}_{N_{\neq}}^{4}\epsilon^{2}\nu.$$

Combining the decomposition (3.12), the estimates (3.13), (3.14), and the initial constraint (1.4), we have that

(3.15)

$$\|A_{\nu}\Omega_0(t)\|_{L^2}^2 + \nu \|A_{\nu}\partial_y\Omega_0\|_{L^2_tL^2}^2 \leq C \|\Omega_{\mathrm{in};0}\|_{H^s}^2 + \epsilon^2 \nu C(\epsilon \mathcal{B}_{\Omega_0}\mathcal{B}_{\Omega_{\neq}}^2 + \mathcal{B}_{N_{\neq}}^4) \leq \epsilon^2 \nu C(1 + \epsilon \mathcal{B}_{\Omega_0}\mathcal{B}_{\Omega_{\neq}}^2 + \mathcal{B}_{N_{\neq}}^4).$$

Here $C \ge 1$ is a constant depending only on the regularity level s. Hence the following choice of constants guarantees the conclusion (2.9d)

(3.16)
$$\mathcal{B}_{\Omega_0}^2 \ge 4C(1+\mathcal{B}_{\Omega_{\neq}}^2+\mathcal{B}_{N_{\neq}}^4), \quad \epsilon \le \frac{1}{1+\mathcal{B}_{\Omega_0}}$$

Here the constant C is the one in (3.15).

4. Cell Density Estimates

In this section, we derive the estimates associated with the cell dynamics. We organize the proof into two subsections. In Subsection 4.1, we prove the enhanced dissipation estimate of the cell density's remainder N_{\neq} (2.9a). In Subsection 4.2, we prove the z-average (N_0) estimate (2.9b).

4.1. The Remainder of the Cell Density. In this section, we prove the estimate (2.9a). First we calculate the time evolution of $||A_{\kappa}N_{\neq}||_{L^2}^2$ using the equation (2.2a):

$$\frac{1}{2} \|A_{\kappa}N_{\neq}(t)\|_{L^{2}}^{2} = \frac{1}{2} \|A_{\kappa}N_{\mathrm{in};\neq}\|_{L^{2}}^{2} + \delta\kappa^{1/3} \int_{0}^{t} \||\partial_{x}|^{1/3} A_{\kappa}N_{\neq}\|_{L^{2}}^{2} d\tau - \int_{0}^{t} \left\|\sqrt{\frac{-\partial_{t}M_{\kappa}}{M_{\kappa}}} A_{\kappa}N_{\neq}\right\|_{L^{2}}^{2} d\tau - \kappa \int_{0}^{t} \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}}^{2} d\tau - \int_{0}^{t} \int A_{\kappa}N_{\neq} A_{\kappa}(U \cdot \nabla_{L}N)_{\neq} dV d\tau - \kappa \int_{0}^{t} \int A_{\kappa}N_{\neq} A_{\kappa}(\nabla_{L} \cdot (N\nabla_{L}C))_{\neq} dV d\tau.$$

Recalling the relation (A.4) and the null condition $U_0^{(2)} = \partial_z \Delta_y^{-1} \Omega_0 = 0$, we have that if $\delta \leq \frac{1}{16\pi^2}$, then

$$\frac{1}{2} \|A_{\kappa}N_{\neq}(t)\|_{L^{2}}^{2} \leq \frac{1}{2} \|A_{\kappa}N_{\mathrm{in};\neq}\|_{L^{2}}^{2} - \frac{1}{2} \int_{0}^{t} \left\|\sqrt{\frac{-\partial_{\tau}M_{\kappa}}{M_{\kappa}}} A_{\kappa}N_{\neq}\right\|_{L^{2}}^{2} d\tau - \frac{\kappa}{2} \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}L^{2}}^{2} \\
+ \left|\int_{0}^{t} \int A_{\kappa}(U_{0}^{(1)} \partial_{z}N_{\neq}) A_{\kappa}N_{\neq}dVd\tau\right| + \left|\int_{0}^{t} \int A_{\kappa}N_{\neq} A_{\kappa}\nabla_{L} \cdot (N \nabla_{L}^{\perp}\Delta_{L}^{-1}\Omega_{\neq})dVd\tau\right| \\
+ \left|\kappa \int_{0}^{t} \int A_{\kappa}\partial_{y}^{\tau} (N_{\neq}\partial_{y}C_{0}) A_{\kappa}N_{\neq}dVd\tau\right| + \left|\kappa \int_{0}^{t} \int A_{\kappa}\nabla_{L} \cdot (N\nabla_{L}C_{\neq}) A_{\kappa}N_{\neq}dVd\tau\right| \\
= :\frac{1}{2} \|A_{\kappa}N_{\mathrm{in};\neq}\|_{L^{2}}^{2} - \frac{1}{2} \int_{0}^{t} \left\|\sqrt{\frac{-\partial_{\tau}M_{\kappa}}{M_{\kappa}}} A_{\kappa}N_{\neq}\right\|_{L^{2}}^{2} d\tau - \frac{\kappa}{2} \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}L^{2}}^{2} \\$$
(4.1)

The estimates of the terms in (4.1) are collected in the following lemmas.

Lemma 4.1. The $T_{N_{\neq};11}$, $T_{N_{\neq};12}$ terms in (4.1) are bounded as follows

(4.2)
$$T_{N_{\neq};11} + T_{N_{\neq};12} \le C\epsilon^{1/2} (\mathcal{B}_{\Omega_{\neq}}^2 + \mathcal{B}_{\Omega_0}^2 + \mathcal{B}_{N_{\neq}}^2 + \mathcal{B}_{N_0}^2) \mathcal{B}_{N_{\neq}}.$$

Here C is a constant depending only on the regularity level s.

Lemma 4.2. The $T_{N_{\neq};21}$ and $T_{N_{\neq};22}$ terms are bounded as follows

(4.3)
$$T_{N_{\neq};21} + T_{N_{\neq};22} \le \frac{\kappa}{8} \int_0^t \|A_{\kappa} \sqrt{-\Delta_L} N_{\neq}\|_{L^2}^2 d\tau + C\kappa^{2/3} (\mathcal{B}_{N_{\neq}}^2 + \mathcal{B}_{N_0}^2) \mathcal{B}_{N_{\neq}}^2.$$

Here C is a constant depending only on the regularity level s.

Now we complete the proof of (2.9a).

Proof of (2.9a). Combining the decomposition (4.1), Lemma 4.1, Lemma 4.2 and the choice of parameters $0 < \kappa \leq \epsilon = \frac{\kappa}{\nu} \leq 1$, we obtain

$$\begin{aligned} \|A_{\kappa}N_{\neq}(t)\|_{L^{2}}^{2} + \left\|\sqrt{\frac{-\partial_{t}M_{\kappa}}{M_{\kappa}}}A_{\kappa}N_{\neq}\right\|_{L^{2}_{t}L^{2}}^{2} + \kappa\|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} \\ \leq C\|N_{\mathrm{in};\neq}\|_{H^{s}}^{2} + C\epsilon^{1/2}(\mathcal{B}_{\Omega_{\neq}}^{2} + \mathcal{B}_{\Omega_{0}}^{2} + \mathcal{B}_{N_{\neq}}^{2} + \mathcal{B}_{N_{0}}^{2})\mathcal{B}_{N_{\neq}} + C\epsilon^{1/2}(\mathcal{B}_{N_{\neq}}^{2} + \mathcal{B}_{N_{0}}^{2})\mathcal{B}_{N_{\neq}}^{2}. \end{aligned}$$

Here C is universal constant depending only on s. The following choices of parameters yields (2.9a):

(4.4)
$$\mathcal{B}_{N\neq}^{2} \ge 4C \|N_{\text{in};\neq}\|_{H^{s}}^{2}, \quad \epsilon \le \frac{1}{16C^{2}(\mathcal{B}_{\Omega\neq}^{2} + \mathcal{B}_{\Omega_{0}}^{2} + \mathcal{B}_{N\neq}^{2} + \mathcal{B}_{N_{0}}^{2})^{2}}.$$

The remaining part of the subsection is devoted to the proof of Lemma 4.1 and Lemma 4.2.

Proof of Lemma 4.1. The estimate of $T_{N\neq;11}$ is similar to the estimate of $T_{\Omega\neq;12}$ in (3.5). Hence we will only sketch the estimate. First we note that by the velocity law and the null condition,

$$\int \partial_y \Delta_y^{-1} \Omega_0 \partial_z A_\kappa N_{\neq} A_\kappa N_{\neq} dV = \frac{1}{2} \int \partial_y \Delta_y^{-1} \Omega_0 \ \partial_z \left(A_\kappa N_{\neq} \right)^2 dV = 0,$$

the $T_{N_{\neq};11}$ -term can be rewritten as follows

$$\begin{split} T_{N_{\neq};11} = & \left| \int_{0}^{t} \int \left(-A_{\kappa}(\partial_{y}\Delta_{y}^{-1}\Omega_{0} \ \partial_{z}N_{\neq}) + \partial_{y}\Delta_{y}^{-1}\Omega_{0} \ \partial_{z}A_{\kappa}N_{\neq} \right) \ A_{\kappa}N_{\neq}dVd\tau \right| \\ = & C \bigg| \sum_{k\neq 0} \int_{0}^{t} \iint (M_{\kappa}(\tau,k,\eta)\langle k,\eta\rangle^{s} - M_{\kappa}(\tau,k,\xi)\langle k,\xi\rangle^{s}) \frac{i(\eta-\xi)}{(\eta-\xi)^{2}} \widehat{\Omega}(\tau,0,\eta-\xi) \\ & \times (ike^{\delta\kappa^{1/3}|k|^{2/3}\tau} \widehat{N}(\tau,k,\xi)) \ \overline{A_{\kappa}\widehat{N}}(\tau,k,\eta)d\xi d\eta d\tau \bigg|. \end{split}$$

Now we observe that this is in the same form as (3.7). Hence combining the application of an identical argument as in (3.8), the enhanced dissipation relation (A.7) and the bootstrap hypotheses (2.8a), (2.8d) yields that

 $T_{N_{\neq};11} \le C \|A_{\nu}\Omega_0\|_{L^{\infty}_t L^2} \|A_{\kappa}N_{\neq}\|^2_{L^2_t L^2} \le C \mathcal{B}_{\Omega_0} \mathcal{B}^2_{N_{\neq}} \epsilon^{1/2} \kappa^{1/6}.$

We note that this is consistent with (4.2).

For the $T_{N_{\neq};12}$ term in (4.1), we apply integration by parts, the Biot-Savart law, and Hölder inequality to obtain the following:

$$T_{N_{\neq};12} = \left| \int_0^t \int \nabla_L A_{\kappa} N_{\neq} \cdot A_{\kappa} (\nabla_L^{\perp} \Delta_L^{-1} \Omega_{\neq} N) dV d\tau \right| \le C \|A_{\kappa} \sqrt{-\Delta_L} N_{\neq}\|_{L^2_t L^2} \|A_{\kappa} (\nabla_L^{\perp} \Delta_L^{-1} \Omega_{\neq} N)\|_{L^2_t L^2}.$$

Now we invoke the product estimate (A.10), and then the M_{κ} -estimate (A.3b) and the elliptic estimate (A.8) to derive the following bound

$$T_{N_{\neq};12} \leq \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}_{t}L^{2}}\|A_{\nu}\nabla^{\perp}_{L}\Delta^{-1}_{L}\Omega_{\neq}\|_{L^{2}_{t}L^{2}}\|A_{\kappa}N\|_{L^{\infty}_{t}L^{2}}$$
$$\leq \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}_{t}L^{2}}\left\|A_{\nu}\sqrt{\frac{-\partial_{t}M_{\nu}}{M_{\nu}}}\Omega_{\neq}\right\|_{L^{2}_{t}L^{2}}(\|A_{\kappa}N_{0}\|_{L^{\infty}_{t}L^{2}}+\|A_{\kappa}N_{\neq}\|_{L^{\infty}_{t}L^{2}}).$$

Now the bootstrap hypotheses (2.8a), (2.8b), (2.8c) yields

$$T_{N_{\neq};12} \leq C\kappa^{-1/2} \mathcal{B}_{N_{\neq}} \mathcal{B}_{\Omega_{\neq}} \epsilon \nu^{1/2} (\mathcal{B}_{N_0} + \mathcal{B}_{N_{\neq}}) \leq C \mathcal{B}_{N_{\neq}} (\mathcal{B}_{\Omega_{\neq}}^2 + \mathcal{B}_{N_0}^2 + \mathcal{B}_{N_{\neq}}^2) \sqrt{\epsilon}.$$

Combining the above estimates of $T_{N_{\neq};11}$ and $T_{N_{\neq};12}$ yields the result (4.2).

Proof of Lemma 4.2. First we estimate the $T_{N\neq;21}$ -term with integration by parts and the product estimate (A.10) as follows:

$$T_{N_{\neq};21} = \left| \kappa \int_{0}^{t} \int A_{\kappa} \nabla_{L} N_{\neq} \cdot A_{\kappa} (N_{\neq} \partial_{y} C_{0}) dV d\tau \right| \leq \frac{\kappa}{16} \|A_{\kappa} \sqrt{-\Delta_{L}} N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} + C\kappa \|A_{\kappa} N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} \|A_{\kappa} \partial_{y} C_{0}\|_{L^{\infty}_{t}L^{2}}^{2}.$$

Now we invoke the chemical gradient estimate (A.2), the enhanced dissipation relation (A.7) and the hypotheses (2.8a), (2.8b) to get

$$T_{N_{\neq};21} \leq \frac{\kappa}{16} \int_0^t \|A_{\kappa} \sqrt{-\Delta_L} N_{\neq}\|_2^2 d\tau + C \kappa^{2/3} \mathcal{B}_{N_{\neq}}^2 \mathcal{B}_{N_0}^2.$$

To estimate the $T_{N_{\neq};22}$ -term, we use the integration by parts and the product estimate (A.10) to obtain that

$$T_{N_{\neq};22} = \left| \kappa \int_{0}^{t} \int A_{\kappa} \nabla_{L} N_{\neq} \cdot A_{\kappa} (N \nabla_{L} C_{\neq}) dV d\tau \right| \leq \frac{\kappa}{16} \|A_{\kappa} \sqrt{-\Delta_{L}} N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} + C\kappa \|A_{\kappa} N\|_{L^{\infty}_{t}L^{2}}^{2} \|A_{\kappa} \nabla_{L} C_{\neq}\|_{L^{2}_{t}L^{2}}^{2}.$$

We use the M_{κ} -estimate (A.3b), (A.3c) and the hypotheses (2.8a), (2.8b) to obtain the following

$$T_{N_{\neq};22} \leq \frac{\kappa}{16} \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} + C\kappa\|A_{\kappa}N\|_{L^{\infty}_{t}L^{2}}^{2} \|A_{\kappa}\nabla_{L}(1-\Delta_{L})^{-1}N_{\neq}\|_{L^{2}_{t}L^{2}}^{2}$$
$$\leq \frac{\kappa}{16} \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} + C\kappa\|A_{\kappa}N\|_{L^{\infty}_{t}L^{2}}^{2} \left\|A_{\kappa}\sqrt{\frac{-\partial_{t}M_{\kappa}}{M_{\kappa}}}N_{\neq}\right\|_{L^{2}_{t}L^{2}}^{2}$$
$$\leq \frac{\kappa}{16} \|A_{\kappa}\sqrt{-\Delta_{L}}N_{\neq}\|_{L^{2}_{t}L^{2}}^{2} + C\kappa(\mathcal{B}_{N_{\neq}}^{2} + \mathcal{B}_{N_{0}}^{2})\mathcal{B}_{N_{\neq}}^{2}.$$

Combining the above estimates of $T_{N_{\neq};21}$ and $T_{N_{\neq};22}$ yields the result (4.3).

4.2. The z-average of Cell Density. In this section, we prove (2.9b). The strategy we adopt is to derive an estimate of the L^2 -norm of N_0 , and inductively derive higher Sobolev norm bound.

First we write down the time evolution of the L^2 and H^s energy. We consider multiplier $\mathfrak{M} \in \{1, \langle \partial_y \rangle, ..., \langle \partial_y \rangle^s\}$. Recalling the equation (2.2b), we have that

$$\frac{1}{2} \frac{d}{dt} \|\mathfrak{M}N_0\|_2^2 = -\kappa \|\partial_y \mathfrak{M}N_0\|_2^2 - \kappa \int \mathfrak{M}(\nabla_L \cdot (N\nabla_L C))_0 \,\mathfrak{M}N_0 dV - \int \mathfrak{M}\left(\nabla_L \cdot (N\nabla_L^{\perp}\Delta_L^{-1}\Omega)\right)_0 \,\mathfrak{M}N_0 dV \\ = -\kappa \|\partial_y \mathfrak{M}N_0\|_2^2 - \kappa \int \mathfrak{M}(\partial_y (N_0 \partial_y C_0)) \,\mathfrak{M}N_0 dV \\ -\kappa \int \mathfrak{M}(\nabla_L \cdot (N_{\neq} \nabla_L C_{\neq}))_0 \,\mathfrak{M}N_0 dV - \int \mathfrak{M}\left(\nabla_L \cdot (N\nabla_L^{\perp}\Delta_L^{-1}\Omega)\right)_0 \,\mathfrak{M}N_0 dV.$$

Next we observe the following relations

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$$(\nabla_L \cdot (N_{\neq} \nabla_L C_{\neq}))_0 = (\partial_y^t (N_{\neq} \partial_y^t C_{\neq}))_0 = \partial_y (N_{\neq} \partial_y^t C_{\neq})_0, (\nabla_L \cdot (N \nabla_L^\perp \Delta_L^{-1} \Omega))_0 = (\partial_y^t (N \partial_z \Delta_L^{-1} \Omega))_0 = \partial_y (N_{\neq} \partial_z \Delta_L^{-1} \Omega_{\neq})_0$$

With these, we can rewrite the time evolution of $\|\mathfrak{M}N_0\|_2^2$ as follows

$$\frac{1}{2} \frac{d}{dt} \|\mathfrak{M}N_0\|_2^2 = -\kappa \|\partial_y \mathfrak{M}N_0\|_2^2 + \kappa \int \mathfrak{M}\partial_y N_0 \ \mathfrak{M}(N_0 \partial_y C_0) dV
+ \kappa \int \mathfrak{M}\partial_y N_0 \ \mathfrak{M}(N_{\neq} \partial_y^t C_{\neq})_0 dV + \int \mathfrak{M}\partial_y N_0 \ \mathfrak{M}\left(N_{\neq} \partial_z \Delta_L^{-1} \Omega_{\neq}\right)_0 dV
=: -\kappa \|\partial_y \mathfrak{M}N_0\|_2^2 + T_{N_0;0} + T_{N_0;1} + T_{N_0;2}$$

The remaining part of the proof is subdivided into several lemmas. The first lemma provides estimates for the contributions from the non-zero modes.

Lemma 4.3. There exists a constant C, which only depends on the regularity level s, such that the following estimates hold

(4.6)
$$\int_0^t \|(N_{\neq}\partial_y^{\tau}C_{\neq})_0\|_{H^s_y}^2 d\tau \leq C\mathcal{B}_{N_{\neq}}^4;$$

(4.7)
$$\int_0^t \|(N_{\neq}\partial_z \Delta_L^{-1}\Omega_{\neq})_0\|_{H^s_y}^2 d\tau \leq C\mathcal{B}_{N_{\neq}}^2 \mathcal{B}_{\Omega_{\neq}}^2 \epsilon^2 \nu.$$

Proof. To prove (4.6), we use Lemma A.1, the M_{κ} -multiplier bound (A.3b), the definition of A_{κ} (2.7), the product estimate (A.10), the elliptic estimate (A.8), and the bootstrap hypothesis (2.8a) as follows

(4.8)

$$\int_{0}^{\circ} \|(N_{\neq}\partial_{y}^{\tau}C_{\neq})_{0}\|_{H_{y}^{s}}^{2}d\tau \leq C\|\langle\partial_{z},\partial_{y}\rangle^{s}(\partial_{y}^{t}C_{\neq}N_{\neq})\|_{L_{t}^{2}L^{2}}^{2} \leq C\|A_{\kappa}(\partial_{y}^{t}C_{\neq}N_{\neq})\|_{L_{t}^{2}L^{2}}^{2}$$

$$\leq C\|A_{\kappa}\partial_{y}^{t}\Delta_{L}^{-1}N_{\neq}\|_{L_{t}^{2}L^{2}}^{2}\|A_{\kappa}N_{\neq}\|_{L_{t}^{\infty}L^{2}}^{2}$$

$$\leq C\|A_{\kappa}\sqrt{\frac{-\partial_{t}M_{\kappa}}{M_{\kappa}}}N_{\neq}\|_{L_{t}^{2}L^{2}}^{2}\|A_{\kappa}N_{\neq}\|_{L_{t}^{\infty}L^{2}}^{2} \leq C\mathcal{B}_{N_{\neq}}^{4}.$$

This concludes the proof of (4.6). Next we prove the estimate (4.7). The idea is identical to the proof (4.8). The adjustment is that we apply the fact $A_{\kappa} \approx A_{\nu}$ (3.10), the bootstrap hypotheses (2.8a) and (2.8c) during the estimate

$$\begin{split} \int_{0}^{t} \| (N_{\neq} \partial_{z} \Delta_{L}^{-1} \Omega_{\neq})_{0} \|_{H_{y}^{s}}^{2} d\tau \leq C \| \langle \partial_{z}, \partial_{y} \rangle^{s} (N_{\neq} \partial_{z} \Delta_{L}^{-1} \Omega_{\neq}) \|_{L_{t}^{2}L^{2}}^{2} \leq C \| A_{\kappa} (N_{\neq} \partial_{z} \Delta_{L}^{-1} \Omega_{\neq}) \|_{L_{t}^{2}L^{2}}^{2} \\ \leq C \| A_{\kappa} N_{\neq} \|_{L_{t}^{2}L^{2}}^{2} \| A_{\kappa} \partial_{z} \Delta_{L}^{-1} \Omega_{\neq} \|_{L_{t}^{\infty}L^{2}}^{2} \leq C \| A_{\kappa} N_{\neq} \|_{L_{t}^{\infty}L^{2}}^{2} \| A_{\nu} \partial_{z} \Delta_{L}^{-1} \Omega_{\neq} \|_{L_{t}^{2}L^{2}}^{2} \\ \leq C \| A_{\kappa} N_{\neq} \|_{L_{t}^{\infty}L^{2}}^{2} \left\| A_{\nu} \sqrt{\frac{-\partial_{t} M_{\nu}}{M_{\nu}}} \Omega_{\neq} \right\|_{L_{t}^{2}L^{2}}^{2} \leq C \mathcal{B}_{N_{\neq}}^{2} \mathcal{B}_{\Omega_{\neq}}^{2} \epsilon^{2} \nu. \end{split}$$

This concludes the proof of (4.7).

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Next, we prove the following L^2 estimate for N_0 .

Lemma 4.4. If the hypotheses (2.8a), (2.8b) hold on $[0, T_{\star}]$, then the solution N_0 is bounded uniformly in L^2

(4.9)
$$\|N_0\|_{L^2_y(\mathbb{R})} \le C_{L^2} := \|N_{\mathrm{in};0}\|_2 + CM^{3/2} + C(\mathcal{B}^2_{N_{\neq}} + \mathcal{B}_{N_{\neq}}\mathcal{B}_{\Omega_{\neq}}\epsilon^{1/2}), \quad \forall t \in [0, T_{\star}].$$

Here $M = ||n||_{L^1} = |\mathbb{T}| \int_{\mathbb{R}} N_{\mathrm{in};0}(y) dy$ is the conserved total mass of the cell density.

Proof. Before estimating the norm $||N_0||_2$, we collect the $L^1(\mathbb{R})$ bound of N_0 . Since the solution N(t, z, y) to the equation (2.1) is positive by Theorem 2.1, we have that the function $N_0(y) = \frac{1}{2\pi} \int_{\mathbb{T}} N(t, z, y) dz$ is positive. Moreover, the total integral of N_0 is preserved due to the divergence form of the equation (2.2b),

$$\int N_0(t)dy = \int N_0(0,y)dy,$$

which, together with the positivity of N_0 , implies that

(4.10)
$$||N_0(t)||_{L^1(\mathbb{R})} \equiv \frac{M}{2\pi}, \quad \forall t \in [0, T_\star].$$

Next we estimate each component in $(4.5)_{\mathfrak{M}=1}$, which describes the time evolution of $||N_0(t)||_2^2$. Combining the information (4.10), (A.1), the $T_{N_0;0}$ -term in (4.5) can be estimated as follows

$$(4.11)T_{N_0;0} \le \kappa \int |\partial_y N_0| \ |\partial_y C_0 N_0| dy \le \frac{1}{4} \kappa \|\partial_y N_0\|_2^2 + \kappa \|\partial_y C_0\|_\infty^2 \|N_0\|_2^2 \le \frac{1}{4} \kappa \|\partial_y N_0\|_2^2 + C\kappa M^2 \|N_0\|_2^2.$$

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The $T_{N_0;1}$, $T_{N_0;2}$ terms can be estimated as follows

$$T_{N_0;1}(t) + T_{N_0;2}(t) \leq \frac{1}{2} \kappa \|\partial_y N_0\|_{L^2_y}^2(t) + C \left(\kappa \|N_{\neq} \partial_y^t C_{\neq}\|_{L^2_{z,y}}^2(t) + \frac{1}{\kappa} \|N_{\neq} \partial_z \Delta_L^{-1} \Omega_{\neq}\|_{L^2_{z,y}}^2(t)\right)$$
$$=: \frac{1}{2} \kappa \|\partial_y N_0\|_{L^2_y}^2(t) + \frac{d}{dt} G_{L^2}(t), \quad G_{L^2}(0) = 0.$$

By Lemma 4.3,

(4.12)
$$G_{L^2}(t) \le C\mathcal{B}^4_{N_{\neq}} + C\mathcal{B}^2_{N_{\neq}}\mathcal{B}^2_{\Omega_{\neq}}\epsilon, \quad \forall t \in [0, T_{\star}].$$

Combining the above estimates (4.11), (4.12) and the relation (4.5), we have

(4.13)
$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}N_0^2dy \le -\frac{1}{4}\kappa \|\partial_y N_0\|_2^2 + C\kappa M^2 \|N_0\|_2^2 + \frac{d}{dt}G_{L^2}(t).$$

Now we try to get an estimate on the L^2 norm from (4.13). Applying the following Nash inequality

$$||f||_{L^{2}(\mathbb{R})}^{3} \leq C ||f||_{L^{1}(\mathbb{R})}^{2} ||\partial_{y}f||_{L^{2}(\mathbb{R})}$$

yields

$$-\|\partial_y f\|_2^2 \le -\frac{\|f\|_2^6}{C\|f\|_1^4}.$$

Combining this with the estimate (4.13), we have that

$$(4.14) \qquad \frac{d}{dt} \left(\|N_0\|_2^2 - G_{L^2}(t) \right) \leq -\frac{1}{2C} \kappa \|\partial_y N_0\|_2^2 + \kappa C M^2 \|N_0\|_2^2 \leq -\kappa \frac{\|N_0\|_2^6}{CM^4} + \kappa C M^2 \|N_0\|_2^2 \\ \leq -\frac{1}{CM^4} \kappa \|N_0\|_2^2 \left(\|N_0\|_2^4 - C M^6 \right) \\ \leq -\frac{1}{CM^4} \kappa \|N_0\|_2^2 \left(\|N_0\|_2^2 - G_{L^2}(t) - C M^3 \right) \left(\|N_0\|_2^2 + C M^3 \right).$$

We can see that the $||N_0||_2$ is bounded uniformly in $[0, T_*]$ in the sense that

$$\|N_0(t)\|_2 \le C_{L^2} = \|N_{\text{in};0}\|_2 + CM^{3/2} + C(\mathcal{B}^2_{N_{\neq}} + \mathcal{B}_{N_{\neq}}\mathcal{B}_{\Omega_{\neq}}\epsilon^{1/2}), \quad \forall t \in [0, T_\star]$$

which is (4.9).

Next we try to use the information on $||N_0||_2$ to get the bound on higher H^s norm.

Lemma 4.5. Assuming the hypotheses (2.8a), (2.8c) hold on $[0, T_{\star}]$, then the H^{s} norm of the solution N_{0} to (2.2b) is bounded:

$$\|N_0(t)\|_{H^s} \le C_{H^s}(\|N_{\mathrm{in}}\|_{H^s}, M, \mathcal{B}_{N_{\neq}}, \sqrt{\epsilon}\mathcal{B}_{\Omega_{\neq}}), \quad \forall t \in [0, T_\star]$$

Here $M = \|n\|_{L^1} = |\mathbb{T}| \int_{\mathbb{R}} N_{in;0}(y) dy$ is the conserved total mass of the cell density. Moreover, if

(4.15)
$$\epsilon \le \frac{1}{\mathcal{B}_{\Omega_{\neq}}^2},$$

then

(4.16)
$$\|N_0\|_{H^s} \le C_{H^s}(\|N_{\rm in}\|_{H^s}, M, \mathcal{B}_{N_{\neq}}).$$

Proof. We estimate the H^i -norms $(i \in \{1, 2, ..., s\})$ in the inductive fashion. Assume that we have the following estimate for some $1 \le i \le s$,

$$||N_0||_{H^{i-1}} \le C_{H^{i-1}}(||N_{\text{in};0}||_{H^s}, M, \mathcal{B}_{N_{\neq}}, \sqrt{\epsilon}\mathcal{B}_{\Omega_{\neq}}).$$

We would like to prove that

(4.17)
$$\|N_0\|_{H^i} \le C_{H^i}(\|N_{\text{in};0}\|_{H^s}, M, \mathcal{B}_{N_{\neq}}, \sqrt{\epsilon}\mathcal{B}_{\Omega_{\neq}})$$

Similar to the L^2 case, we estimate each term in $(4.5)_{\mathfrak{M}=\langle \partial_u \rangle^i}$.

The $T_{N_0;1}$, $T_{N_0;2}$ terms can be estimated as follows

(4.18)
$$T_{N_{0};1}(t) + T_{N_{0};2}(t) \leq \frac{1}{2} \kappa \|\partial_{y} \langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}^{2}(t) + C \left(\kappa \|N_{\neq} \partial_{y}^{t} C_{\neq}\|_{H^{i}}^{2}(t) + \frac{1}{\kappa} \|N_{\neq} \partial_{z} \Delta_{L}^{-1} \Omega_{\neq}\|_{H^{i}}^{2}(t)\right) = \frac{1}{2} \kappa \|\partial_{y} \langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}^{2}(t) + \frac{d}{dt} G_{H^{i}}(t), \quad G_{H^{i}}(0) = 0.$$

By Lemma 4.3,

(4.19)
$$G_{H^i}(t) \le C\mathcal{B}^4_{N_{\neq}} + C\mathcal{B}^2_{N_{\neq}}\mathcal{B}^2_{\Omega_{\neq}}\epsilon, \quad \forall t \in [0, T_{\star}].$$

The $T_{N_0;0}$ term can be estimated using Hölder's inequality, product estimate of Sobolev functions $f, g \in H^i(\mathbb{R}), i \geq 1$, Gagliardo-Nirenberg inequality, and the chemical gradient estimates (A.1), (A.2), as follows,

$$T_{N_{0};0} \leq \kappa C \|\partial_{y} \langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}} \|\langle \partial_{y} \rangle^{i} (N_{0} \partial_{y} C_{0})\|_{L^{2}} \leq \kappa C \|\partial_{y} \langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}} \|\langle \partial_{y} \rangle^{i} \partial_{y} C_{0}\|_{L^{2}} \|\langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}$$

$$(4.20) \leq \frac{1}{4} \kappa \|\partial_{y} \langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}^{2} + C \kappa \|\langle \partial_{y} \rangle^{i-1} N_{0}\|_{L^{2}}^{2} \|\langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}^{2} \leq \frac{1}{4} \kappa \|\partial_{y} \langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}^{2} + C \kappa C_{H^{i-1}}^{2} \|\langle \partial_{y} \rangle^{i} N_{0}\|_{L^{2}}^{2}.$$

Applying the following Gagliardo-Nirenberg inequality

$$\|\partial_y^i f\|_2 \le C \|f\|_2^{\frac{1}{i+1}} \|\partial_y^{i+1} f\|_2^{\frac{i}{i+1}}$$

yields that

(4.21)
$$-\|\partial_y^{i+1}N_0\|_2^2 \le -\frac{\|\partial_y^i N_0\|_2^{\frac{2i+2}{i}}}{C\|N_0\|_2^{\frac{2}{i}}} \le -\frac{\|\partial_y^i N_0\|_2^{\frac{2i+2}{i}}}{CC_{L^2}(M, \mathcal{B}_{N_{\neq}}, \sqrt{\epsilon}\mathcal{B}_{\Omega_{\neq}})^{\frac{2}{i}}}$$

Combining (4.18), (4.20) and (4.21) yields

(4.22)
$$\frac{d}{dt} \|\langle \partial_y \rangle^i N_0 \|_2^2 \le -\kappa \frac{\|\partial_y^i N_0\|_2^{2+\frac{2}{i}}}{4CC_{L^2}^{\frac{2}{i}}} + \kappa CC_{H^{i-1}}^2 \|\langle \partial_y \rangle^i N_0 \|_2^2 + \frac{d}{dt} G_{H^i}(t)$$

Here C_{L^2} and $C_{H^{i-1}}$ only depend on $\|N_{\text{in}}\|_{H^s}$, M, $\mathcal{B}_{N_{\neq}}$ and $\sqrt{\epsilon}\mathcal{B}_{\Omega_{\neq}}$. Now we apply an ODE argument similar to the one in (4.14) to derive that the quantity $\|N_0\|_{H^i}$ is uniformly bounded on the time interval $[0, T_*]$, i.e., (4.17). We first recall the relation $C_i^{-1}(\|\partial_y^i N_0\|_{L^2}^2 + \|N_0\|_{H^{i-1}}^2) \leq \|\langle\partial_y\rangle^i N_0\|_{L^2}^2 \leq C_i(\|\partial_y^i N_0\|_{L^2}^2 + \|N_0\|_{H^{i-1}}^2)$ with $C_i \geq 1$, and the G_{H^i} -estimate (4.19). Then we distinguish between two scenarios:

$$(4.23) a) \|\partial_y^i N_0(t)\|_{L^2}^2 < 4C_i(G_{H^i}(t) + C_{H^{i-1}}^2); b) \|\partial_y^i N_0(t)\|_{L^2}^2 \ge 4C_i(G_{H^i}(t) + C_{H^{i-1}}^2).$$

In the first scenario, the estimate (4.17) is direct. Assume that on some open time intervals, the estimate b) in (4.23) holds. Then the time evolution (4.22) implies that there exists a constant C such that

$$\frac{d}{dt}(\|\langle \partial_y \rangle^i N_0(t)\|_2^2 - G_{H^i}(t)) \le -\kappa \frac{(\|\langle \partial_y \rangle^i N_0(t)\|_2^2 - G_{H^i}(t))^{\frac{i+1}{i}}}{CC_{L^2}^{2/i}} + \kappa CC_{H^{i-1}}^2(\|\langle \partial_y \rangle^i N_0(t)\|_2^2 - G_{H^i}(t)).$$

Hence by (4.19), we have that

$$\|\langle \partial_y \rangle^i N_0(t)\|_{L^2}^2 \le C(\|N_{\rm in}\|_{H^s}, M, C_{L^2}, C_{H^{i-1}}, \mathcal{B}_{N_{\neq}}, \sqrt{\epsilon} \mathcal{B}_{\Omega_{\neq}}).$$

This is consistent with (4.17). Hence, the induction estimate (4.17) is established. Since *i* ranges in $\{1, ..., s\}$, we end up with the following

$$\|N_0(t)\|_{H^s} \le C_{H^s}(\|N_{\mathrm{in}}\|_{H^s}, M, \mathcal{B}_{N_{\neq}}, \sqrt{\epsilon}\mathcal{B}_{\Omega_{\neq}}), \quad \forall t \in [0, T_\star]$$

This concludes the proof.

Proof of (2.9b). To prove (2.9b), we choose (4.15), and

(4.24)
$$\mathcal{B}_{N_0} \ge 4C_{H^s}(\|N_{\rm in}\|_{H^s}, M, \mathcal{B}_{N_{\neq}}).$$

Here C_{H^s} is defined in (4.16).

Proof of Proposition 2.1. It is enough to show that the choice of parameters (3.4), (3.16), (4.4), (4.15), and (4.24) are consistent. We choose the parameters

(4.25)
$$\begin{aligned} \mathcal{B}_{N_{\neq}}^{2} &:= C_{1} \|N_{\mathrm{in}}\|_{H^{s}}^{2}, \quad \mathcal{B}_{N_{0}}^{2} := C_{2} C_{H^{s}}^{2} (\|N_{\mathrm{in}}\|_{L^{1} \cap H^{s}}, \mathcal{B}_{N_{\neq}}), \\ \mathcal{B}_{\Omega_{\neq}}^{2} &:= C_{3} (\mathcal{B}_{N_{\neq}}^{4} + \mathcal{B}_{N_{0}}^{4}), \quad \mathcal{B}_{\Omega_{0}}^{2} := C_{4} (\mathcal{B}_{\Omega_{\neq}}^{2} + \mathcal{B}_{N_{\neq}}^{4}). \end{aligned}$$

Here C_1, C_2, C_3, C_4 are constants depending only on the universal constants appeared in (3.4), (3.16), (4.4), (4.15), and (4.24). Now we summarize the choice of $\epsilon = \kappa/\nu$:

(4.26)
$$\epsilon \leq \epsilon_0 := \frac{1}{C_5(\mathcal{B}_{\Omega_{\neq}}^4 + \mathcal{B}_{\Omega_0}^4 + \mathcal{B}_{N_{\neq}}^4 + \mathcal{B}_{N_0}^4)}$$

Here C_5 is a constant depending only on the universal constants appeared in (3.4), (3.16), (4.4), (4.15). This concludes the proof of Proposition 2.1.

APPENDIX A.

A.1. Miscellaneous.

Lemma A.1. Let $F \in L^p(\mathbb{T} \times \mathbb{R})$, $G \in H^s(\mathbb{T} \times \mathbb{R})$, and $F_0(y) := \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} F(z, y) dz$, $G_0(y) := \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} G(z, y) dz$. The following estimates hold:

$$||F_0||_{L^p(\mathbb{R})} \le ||F||_{L^p(\mathbb{T}\times\mathbb{R})}, \quad 1 \le p \le \infty,$$

$$\langle \partial_y \rangle^s G_0||_{L^2(\mathbb{R})} \le C ||\langle \partial_z, \partial_y \rangle^s G||_{L^2(\mathbb{T}\times\mathbb{R})}.$$

Proof. Applying the Hölder's inequality yields that for $p \in (1, \infty)$,

$$\begin{split} \|F_0\|_{L^p(\mathbb{R})} &= \left(\int_{\mathbb{R}} \left|\frac{1}{2\pi} \int_{\mathbb{T}} Fdz\right|^p dy\right)^{1/p} \leq \left(\int_{\mathbb{R}} \left(\left(\int_{\mathbb{T}} |F|^p dz\right)^{1/p} \left(\int_{\mathbb{T}} (2\pi)^{-p'} dz\right)^{1/p'}\right)^p dy\right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}} \int_{\mathbb{T}} |F|^p dz dy\right)^{1/p} = \|F\|_{L^p(\mathbb{T} \times \mathbb{R})}. \end{split}$$

The proof in the $p = 1, \infty$ cases are variants of the argument above. Applying the Fourier transform and the Plancherel equality yields

$$\|\langle \partial_y \rangle^s F_0\|_{L^2(\mathbb{R})}^2 = C \int_{\mathbb{R}} \langle \eta \rangle^{2s} |\widehat{F_0}|^2(\eta) d\eta \le C \sum_k \int_{\mathbb{R}} \langle k, \eta \rangle^{2s} |\widehat{F}|^2(k, \eta) d\eta = C \|\langle \partial_z, \partial_y \rangle^s F\|_{L^2(\mathbb{T} \times \mathbb{R})}^2.$$

This finishes the proof of the lemma.

Lemma A.2 (Chemical gradient estimates). Consider solution C_0 to the equation $(1 - \partial_{yy})C_0 = N_0$. The following estimates hold

(A.1)
$$\|\partial_y C_0\|_{L^\infty_u} \le \|N_0\|_{L^1_u};$$

(A.2)
$$\|\partial_y C_0\|_{H^s_y} \le \|N_0\|_{H^{s-1}_y}, s \in \mathbb{N}_+$$

Proof. To prove the (A.1), we use the explicit solution formula to get

$$\|\partial_y C_0\|_{L^{\infty}_y} = C \left\| \int_{\mathbb{R}} \frac{y - y'}{|y - y'|} e^{-|y - y'|} N_0(y') dy' \right\|_{L^{\infty}_y} \le C \|N_0\|_{L^1_y}$$

To prove the (A.2), we use the Fourier transform and the Plancherel equality

$$\|\partial_y C_0\|_{H^s}^2 = C \int \langle \eta \rangle^{2s} \left| \frac{i\eta}{1+|\eta|^2} \widehat{N}_0 \right|^2 d\eta \le C \int \langle \eta \rangle^{2(s-1)} |\widehat{N}_0|^2 d\eta \le C \|N_0\|_{H^{s-1}}^2.$$

Next we provide a sketch of the proof of Theorem 2.1.

 \square

Proof of Theorem 2.1. Standard energy argument yields the local-in-time $\|N\|_{H^s_{z,y}}, \|\Omega\|_{H^s_{z,y}}$ -estimates. We omit further details for the sake of brevity.

The justification of the positivity of the cell density, i.e., $N \ge 0$, is as follows. We use the technique from [18]. The explicit argument is as follows. We consider the family of convex positive functions j_{ϵ} which approximates $j(s) = \max\{-s, 0\}$. We can choose the j_{ϵ} 's to be monotonically increasing to j. These functions $j'_{\epsilon} = j'$ in $[-\epsilon, 0]^c$ and $0 \le j''_{\epsilon} \le 2\epsilon^{-1}$ in $[-\epsilon, 0]$. So $J_{\epsilon}(s) := \int_0^s j''_{\epsilon}(\sigma)\sigma d\sigma$ satisfies $|J_{\epsilon}(s)| \le 2|s|$ for $\epsilon \in (0,1)$, and $\lim_{\epsilon \to 0^+} J_{\epsilon}(s) = 0$ for all s. Now we have that

$$\begin{split} \frac{d}{dt} \int_{\mathbb{T}\times\mathbb{R}} j_{\epsilon}(N(t,z,y)) dV &= \int_{\mathbb{T}\times\mathbb{R}} j_{\epsilon}'(N)(\kappa\Delta_L N - U \cdot \nabla_L N - \kappa\nabla_L \cdot (N\nabla_L C)) dV \\ &= -\kappa \int_{\mathbb{T}\times\mathbb{R}} j_{\epsilon}''(N)(|\partial_z N|^2 + |\partial_y^t N|^2) dV - \int_{\mathbb{T}\times\mathbb{R}} U \cdot \nabla_L(j_{\epsilon}(N)) dV \\ &+ \kappa \int_{\mathbb{T}\times\mathbb{R}} j_{\epsilon}''(N)(N\partial_z N\partial_z C + N\partial_y^t N\partial_y^t C) dV \\ &\leq \kappa \int_{\mathbb{T}\times\mathbb{R}} \partial_z \left(\int_0^N j_{\epsilon}''(s) s ds \right) \partial_z C dV + \kappa \int_{\mathbb{T}\times\mathbb{R}} \partial_y^t \left(\int_0^N j_{\epsilon}''(s) s ds \right) \partial_y^t C dV \\ &= \kappa \int_{\mathbb{T}\times\mathbb{R}} \left(\int_0^N j_{\epsilon}''(s) s ds \right) (-\Delta_L C) dV = \int_{\mathbb{T}\times\mathbb{R}} \left(\int_0^N j_{\epsilon}''(s) s ds \right) (N - C) dV \end{split}$$

Therefore,

$$\int_{\mathbb{T}\times\mathbb{R}} j_{\epsilon}(N(t))dV - \int_{\mathbb{T}\times\mathbb{R}} j_{\epsilon}(N(0))dV = \int_{0}^{t} \int_{\mathbb{T}\times\mathbb{R}} \left(\int_{0}^{N(\tau)} j_{\epsilon}''(s)sds \right) (N-C)dVd\tau.$$

Since $|J_{\epsilon}(s)| \leq 2|s|$, we can use $4(N^2 + C^2)$ as the dominator and invoke the Dominated convergence theorem and Monotone convergence theorem to get for all [0, t] on which L^2 is bounded,

$$\begin{split} \|N_{-}(t)\|_{L^{1}_{z,y}} &= \int_{\mathbb{T}\times\mathbb{R}} j(N(t))dV = \lim_{\epsilon \to 0^{+}} \int_{\mathbb{R}^{2}} j_{\epsilon}(N(t))dV = \lim_{\epsilon \to 0^{+}} \int_{0}^{t} \int_{\mathbb{T}\times\mathbb{R}} \left(\int_{0}^{N(\tau)} j_{\epsilon}''(s)sds \right) (N-C)dVd\tau \\ &= \int_{0}^{t} \int_{\mathbb{T}\times\mathbb{R}} \lim_{\epsilon \to 0^{+}} \left(\int_{0}^{N(\tau,X)} j_{\epsilon}''(s)sds \right) (N-C)dVd\tau = 0. \end{split}$$

As a result, $N \ge 0$.

A.2. Fourier Multipliers. In this section, we summarize the properties of the Fourier multipliers that we employ. First, we collect some basic properties of the multipliers M_{κ}, M_{ν} defined in (2.6).

Lemma A.3. For $\iota \in \{\kappa, \nu\}$, the following properties for M_{ι} hold

(A.3a)
$$M_{\iota}(t,k,\eta) = \pi^2, \quad |k| \notin (0,\iota^{-1/2}]$$

(A.3b)
$$\frac{9}{4}\pi^2 \ge M_\iota(t,k,\eta) \ge \frac{\pi^2}{4};$$

(A.3c)
$$-\partial_t M_{\iota}(t,k,\eta) \ge \frac{\pi}{2} \frac{|k|^2}{|k|^2 + |\eta - kt|^2}, \quad k \neq 0;$$

(A.3d)
$$|\partial_{\eta}M_{\iota}(t,k,\eta)| \leq \frac{4\pi}{|k|}, \quad k \neq 0,$$

(A.3e)
$$\frac{\sqrt{-\partial_t M_\iota(t,k,\eta)}}{\sqrt{-\partial_t M_\iota(t,k,\xi)}} \le 2(1+|\eta-\xi|^2)^{\frac{1}{2}}, \quad |k| \in (0,\iota^{-1/2}]$$

Moreover, the multipliers M_{κ}, M_{ν} have the enhanced dissipation properties

(A.4)
$$\frac{1}{3\pi}\iota^{1/3}|k|^{2/3} \le -\frac{\partial_t M_\iota}{M_\iota}(t,k,\eta) + \iota(|k|^2 + |\eta - kt|^2), \quad \iota \in \{\kappa,\nu\}.$$

Proof. First of all, the first two inequalities (A.3a), (A.3b) are consequences of the definitions (2.3), (2.4), (2.5) and the boundedness of the arctan-function. The time derivative of the multiplier \mathcal{W} reads as follows

(A.5)
$$\partial_t \mathcal{W}(t,k,\eta) = -\frac{|k|^2}{|k|^2 + |\eta - kt|^2}, \quad k \neq 0$$

Hence combining this expression, and the bounds $W_{\iota} \geq \pi/2$, $\partial_t W_{\iota} \leq 0$, we have that

$$-\partial_t M_\iota = -\mathcal{W}\partial_t W_\iota - W_\iota \partial_t \mathcal{W} \ge \frac{\pi}{2} \frac{|k|^2}{|k|^2 + |\eta - kt|^2}, \quad k \neq 0.$$

This completes the proof of (A.3c).

To prove (A.3d), we observe that

$$\begin{aligned} |\partial_{\eta} M_{\iota}(t,k,\eta)| \leq & |\mathcal{W}\partial_{\eta} W_{\iota}| + |W_{\iota}\partial_{\eta} \mathcal{W}| \leq 2\pi \ \mathbf{1}_{0 < |k| \leq \iota^{-1/2}} \frac{1}{|k|} \frac{\iota^{1/3} |k|^{2/3}}{1 + \iota^{2/3} |k|^{4/3} |t - \frac{\eta}{k}|^2} + 2\pi \ \mathbf{1}_{k \neq 0} \frac{1}{|k|} \frac{|k|^2}{|k|^2 + |\eta - kt|^2} \\ \leq & \frac{4\pi}{|k|} \mathbf{1}_{k \neq 0}. \end{aligned}$$

This concludes the proof of (A.3d).

Next, we prove (A.3e). Direct computation yields that

(A.6)
$$\partial_t W_{\iota}(t,k,\eta) = -\frac{\iota^{1/3}|k|^{2/3}}{1+\iota^{2/3}|k|^{4/3}|t-\frac{\eta}{k}|^2} \mathbf{1}_{|k|\in(0,\iota^{-1/2}]}, \quad \iota \in \{\kappa,\nu\}.$$

For the wave number ranging in $|k| \in (0, \iota^{-1/2}]$, we invoke the expressions (A.5), (A.6) to obtain the following estimates of the quotient

$$\begin{split} \frac{-\partial_t M_{\iota}(t,k,\eta)}{-\partial_t M_{\iota}(t,k,\xi)} &\leq & \frac{1+|t-\frac{\xi}{k}|^2}{1+|t-\frac{\eta}{k}|^2} + \frac{1+\iota^{2/3}|k|^{4/3}|t-\frac{\xi}{k}|^2}{1+\iota^{2/3}|k|^{4/3}|t-\frac{\eta}{k}|^2} \\ &\leq & \frac{1+2|t-\frac{\eta}{k}|^2+2|\frac{\eta-\xi}{k}|^2}{1+|t-\frac{\eta}{k}|^2} + \frac{1+2\iota^{2/3}|k|^{4/3}|t-\frac{\eta}{k}|^2+2\iota^{2/3}|k|^{4/3}|\frac{\eta-\xi}{k}|^2}{1+\iota^{2/3}|k|^{4/3}|t-\frac{\eta}{k}|^2} \\ &\leq & 4+4\frac{|\eta-\xi|^2}{|k|^2}. \end{split}$$

This yields (A.3e).

Finally, we prove (A.4). We recall the expressions (A.5), (A.6) and the bound $W_{\iota} \in [\pi/2, 3\pi/2]$. Combining these ingredients yields that

$$\begin{aligned} \frac{-\partial_t M_\iota}{M_\iota} + \iota(|k|^2 + |\eta - kt|^2) &\geq \frac{-\mathcal{W}\partial_t W_\iota}{M_\iota} + \iota(|k|^2 + |\eta - kt|^2) \\ &\geq \mathbf{1}_{0 < |k| \le \iota^{-1/2}} \frac{\iota^{1/3} |k|^{2/3}}{1 + \iota^{2/3} |k|^{4/3} |t - \frac{\eta}{k}|^2} \frac{2}{3\pi} + \iota |k|^2 \left(1 + \left| t - \frac{\eta}{k} \right|^2 \right). \end{aligned}$$

There are two regimes for the wave number k: $|k| \in (0, \iota^{-1/2}]$ or $|k| \notin (0, \iota^{-1/2}]$. If $|k| \notin (0, \iota^{-1/2}]$, then it can be checked that $\iota |k|^2 \ge \iota^{1/3} |k|^{2/3}$. Hence the result (A.4) is ensured. On the other hand, if $|k| \in (0, \iota^{-1/2}]$, we estimate the above expression as follows

$$\frac{-\partial_t M_{\iota}}{M_{\iota}} + \iota(|k|^2 + |\eta - kt|^2) \\ \ge \mathbf{1}_{|t - \frac{\eta}{k}| \le \iota^{-1/3}|k|^{-2/3}} \frac{\iota^{1/3}|k|^{2/3}}{1 + \iota^{2/3}|k|^{4/3}|t - \frac{\eta}{k}|^2} \frac{2}{3\pi} + \mathbf{1}_{|t - \frac{\eta}{k}| > \iota^{-1/3}|k|^{-2/3}} \iota|k|^2 \left(1 + \left|t - \frac{\eta}{k}\right|^2\right) \ge \frac{1}{3\pi} \iota^{1/3}|k|^{2/3}.$$

This concludes the proof of the lemma.

The following lemma is a natural consequence of Lemma A.3.

Lemma A.4. For any function $f_{\neq} \in H^s$ with vanishing mean $\frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} f_{\neq} dz \equiv 0$, the following estimates hold for $\iota \in \{\kappa, \nu\}$,

(A.7)
$$\int_{0}^{t} \|A_{\iota}f_{\neq}\|_{L^{2}}^{2} d\tau \leq C\iota^{-1/3} \int_{0}^{t} \left\|A_{\iota}\sqrt{\frac{-\partial_{\tau}M_{\iota}}{M_{\iota}}}f_{\neq}\right\|_{L^{2}}^{2} + \iota\|A_{\iota}\sqrt{-\Delta_{L}}f_{\neq}\|_{L^{2}}^{2} d\tau;$$

(A.8)
$$\|A_{\iota}\nabla_{L}\Delta_{L}^{-1}f_{\neq}\|_{L^{2}} + \|A_{\iota}\nabla_{L}(1-\Delta_{L})^{-1}f_{\neq}\|_{L^{2}} \le C \left\|A_{\iota}\frac{1}{|\partial_{z}|}\sqrt{\frac{-\partial_{t}M_{\iota}}{M_{\iota}}}f_{\neq}\right\|_{L^{2}}.$$

Here the constant C is universal.

Proof. The inequality (A.7) follows from the property (A.4) and the Plancherel equality. The inequality (A.8) follows from the properties (A.3b), (A.3c) and the Plancherel equality. \Box

The product rule for the multiplier M_{ι} is contained in the next lemma:

Lemma A.5. Consider the multipliers M_{ι} , $\iota \in \{\kappa, \nu\}$. For two functions $f, g \in H^{s}(\mathbb{T} \times \mathbb{R})$, s > 1, we have the following product rule

(A.9)
$$\|M_{\iota}(fg)\|_{H^{s}} \leq C \|M_{\iota}f\|_{H^{s}} \|M_{\iota}g\|_{H^{s}}$$

Similarly, we have the following product rule for A_{ι} , $\iota \in \{\kappa, \nu\}$, s > 1,

(A.10)
$$\|A_{\iota}(fg)\|_{L^{2}} \leq C \|A_{\iota}f\|_{L^{2}} \|A_{\iota}g\|_{L^{2}}.$$

Proof. First recall the product rule for the usual Sobolev functions on $\mathbb{T} \times \mathbb{R}$:

 $||fg||_{H^s} \le C ||f||_{H^s} ||g||_{H^s}, \quad s > 1.$

On the other hand, we recall that the bound (A.3b) yields that

$$\|f\|_{H^s} \approx \|M_{\iota}f\|_{H^s}, \quad \|g\|_{H^s} \approx \|M_{\iota}g\|_{H^s}, \ \iota \in \{\kappa, \nu\}.$$

Combining these estimates yields the result (A.9).

To prove the product estimate (A.10), we observe the following relation for $x, y \ge 0$,

$$|x+y|^{2/3} \le |x|^{2/3} + |y|^{2/3}.$$

As a result, we have that

$$e^{\delta\kappa^{1/3}|k|^{2/3}t} \le e^{\delta\kappa^{1/3}|k-\ell|^{2/3}t}e^{\delta\kappa^{1/3}|\ell|^{2/3}t}, \quad \forall k, \ell \in \mathbb{Z}$$

Next we combine the above relation and the bounds of M_{ι} (A.3b) to derive the following,

$$\begin{split} \|A_{\iota}(fg)\|_{L^{2}}^{2} \\ &\leq \sum_{k\in\mathbb{Z}} \int e^{2\delta\kappa^{1/3}|k|^{2/3}t} M_{\iota}^{2}(k,\eta) \Big| (1+|k|^{2}+|\eta|^{2})^{\frac{s}{2}} \sum_{\ell\in\mathbb{Z}} \int |\widehat{f}(k-\ell,\eta-\xi)| \ |\widehat{g}(\ell,\xi)| d\xi \Big|^{2} d\eta \\ &\leq C \sum_{k\in\mathbb{Z}} \int \Big| \sum_{\ell\in\mathbb{Z}} \int \left(e^{\delta\kappa^{1/3}|k-\ell|^{2/3}t} M_{\iota}(k-\ell,\eta-\xi)(1+|k-\ell|^{2}+|\eta-\xi|^{2})^{\frac{s}{2}} |\widehat{f}(k-\ell,\eta-\xi)| \right) \\ & \qquad \times \left(e^{\delta\kappa^{1/3}|\ell|^{2/3}t} |\widehat{g}(\ell,\xi)| \right) d\xi \Big|^{2} d\eta \\ &+ C \sum_{k\in\mathbb{Z}} \int \Big| \sum_{\ell\in\mathbb{Z}} \int \left(e^{\delta\kappa^{1/3}|\ell|^{2/3}t} M_{\iota}(\ell,\xi)(1+|\ell|^{2}+|\xi|^{2})^{\frac{s}{2}} |\widehat{g}(\ell,\xi)| \right) \left(e^{\delta\kappa^{1/3}|k-\ell|^{2/3}t} |\widehat{f}(k-\ell,\eta-\xi)| \right) d\xi \Big|^{2} d\eta \end{split}$$

Now we apply the Young's convolution inequality, Hölder's inequality and the inequality $\sum_{k \in \mathbb{Z}} \|\widehat{F}_k(\cdot)\|_{L^1_\eta} \leq C(\sum_{k \in \mathbb{Z}} \|\langle \cdot \rangle^s \widehat{F}_k(\cdot)\|_{L^2_\eta}^2)^{1/2}$, s > 1 to derive that

$$\begin{aligned} \|A_{\iota}(fg)\|_{L^{2}} &\leq C \left(\|A_{\iota}f\|_{L^{2}} \sum_{k \in \mathbb{Z}} \left\| e^{\delta \kappa^{1/3} |k|^{2/3} t} \widehat{g}_{k}(\cdot) \right\|_{L^{1}_{\eta}} + \|A_{\iota}g\|_{L^{2}} \sum_{k \in \mathbb{Z}} \left\| e^{\delta \kappa^{1/3} |k|^{2/3} t} \widehat{f}_{k}(\cdot) \right\|_{L^{1}_{\eta}} \right) \\ &\leq C \|A_{\iota}f\|_{L^{2}} \|A_{\iota}g\|_{L^{2}}. \end{aligned}$$

This concludes the proof of the estimate (A.10) and completes the proof of the lemma.

In the proof, the following commutator estimate is needed:

Lemma A.6 (Commutator estimates). The following commutator estimate concerning M_{ι} is satisfied

(A.11)
$$|M_{\iota}(t,k,\eta)\langle k,\eta\rangle^{s} - M_{\iota}(t,k,\xi)\langle k,\xi\rangle^{s}| \le C\frac{|\eta-\xi|}{|k|}(\langle \eta-\xi\rangle^{s} + \langle k,\xi\rangle^{s}), \quad k\neq 0.$$

Proof. Here the difference $|M_{\iota}(t,k,\eta)\langle k,\eta\rangle^s - M_{\iota}(t,k,\xi)\langle k,\xi\rangle^s|$ can be decomposed as follows:

$$\begin{aligned} (A.12) \, M_{\iota}(t,k,\eta) \langle k,\eta \rangle^{s} &- M_{\iota}(t,k,\xi) \langle k,\xi \rangle^{s} \\ &= M_{\iota}(t,k,\eta) \left((1+k^{2}+\eta^{2})^{s/2} - (1+k^{2}+\xi^{2})^{s/2} \right) + (M_{\iota}(t,k,\eta) - M_{\iota}(t,k,\xi))(1+k^{2}+\xi^{2})^{s/2} \mathbf{1}_{k\neq 0} \\ &=: \mathcal{T}_{1} + \mathcal{T}_{2}. \end{aligned}$$

For the first term in (A.12), one applies the mean value theorem to obtain that there exists $\theta \in [0, 1]$, such that the following estimate holds

$$\begin{aligned} |\mathcal{T}_1| &= \left| M_{\iota}(k,\eta) \frac{s}{2} (1+k^2 + ((1-\theta)\eta + \theta\xi)^2)^{\frac{s}{2}-1}) 2((1-\theta)\eta + \theta\xi)(\eta-\xi) \right| \\ &\leq C \bigg((1+k^2+\xi^2)^{\frac{s-1}{2}} + (1+k^2+\eta^2)^{\frac{s-1}{2}} \bigg) |\eta-\xi| \leq C \frac{|\eta-\xi|}{|k|} (\langle k,\xi \rangle^s + \langle \eta-\xi \rangle^s). \end{aligned}$$

To estimate the \mathcal{T}_2 term in (A.12), we apply the property (A.3d) and the mean value theorem to obtain that

$$|\mathcal{T}_2| \le \frac{C|\eta - \xi|}{|k|} (1 + k^2 + \xi^2)^{\frac{s}{2}} \mathbf{1}_{k \neq 0}$$

Combining the two estimates and (A.12), we obtain (A.11). The proof of the lemma is finished.

References

- D. Albritton, R. Beekie, and M. Novack. Enhanced dissipation and Hörmander's hypoellipticity. J. Funct. Anal., 283(3):Paper No. 109522, 38, 2022.
- J. Bedrossian. Large mass global solutions for a class of L¹-critical nonlocal aggregation equations and parabolic-elliptic Patlak-Keller-Segel models. Comm. Partial Differential Equations, 40(6):1119–1136, 2015.
- J. Bedrossian and M. Coti Zelati. Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows. Arch. Ration. Mech. Anal., 224(3):1161–1204, 2017.
- [4] J. Bedrossian, P. Germain, and N. Masmoudi. On the stability threshold for the 3D Couette flow in Sobolev regularity. Ann. of Math. (2), 185(2):541–608, 2017.
- [5] J. Bedrossian, P. Germain, and N. Masmoudi. Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold. Mem. Amer. Math. Soc., 266(1294):v+158, 2020.
- [6] J. Bedrossian, P. Germain, and N. Masmoudi. Dynamics near the subcritical transition of the 3D Couette flow II: Above threshold. Mem. Amer. Math. Soc., 279(1377):v+135, 2022.
- [7] J. Bedrossian and S. He. Suppression of blow-up in Patlak-Keller-Segel via shear flows. SIAM Journal on Mathematical Analysis, 50(6):6365-6372, 2018.
- [8] J. Bedrossian and N. Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. Publications mathématiques de l'IHÉS, pages 1–106, 2013.
- [9] J. Bedrossian and N. Masmoudi. Existence, uniqueness and Lipschitz dependence for Patlak-Keller-Segel and Navier-Stokes in R² with measure-valued initial data. Arch. Rat. Mech. Anal., 214(3):717–801, 2014.
- [10] J. Bedrossian, N. Masmoudi, and V. Vicol. Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the 2D Couette flow. Arch. Rat. Mech. Anal., 216(3):1087–1159, 2016.
- J. Bedrossian, V. Vicol, and F. Wang. The Sobolev stability threshold for 2D shear flows near Couette. J. Nonlinear Sci., 28(6):2051–2075, 2018.
- [12] P. Biler, G. Karch, P. Laurençot, and T. Nadzieja. The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane. Math. Meth. Appl. Sci, 29:1563–1583, 2006.
- [13] A. Blanchet, J. Carrillo, and N. Masmoudi. Infinite time aggregation for the critical Patlak-Keller-Segel model in ℝ². Comm. Pure Appl. Math., 61:1449–1481, 2008.
- [14] A. Blanchet, J. Dolbeault, and B. Perthame. Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions. E. J. Diff. Eqn, 2006(44):1–32, 2006.
- [15] V. Calvez and J. Carrillo. Volume effects in the Keller-Segel model: energy estimates preventing blow-up. J. Math. Pures Appl., 86:155–175, 2006.
- [16] V. Calvez and L. Corrias. The parabolic-parabolic Keller-Segel model in \mathbb{R}^2 . Commun. Math. Sci., 6(2):417–447, 2008.
- [17] J. Carrillo and J. Rosado. Uniqueness of bounded solutions to aggregation equations by optimal transport methods. Proc. 5th Euro. Congress of Math. Amsterdam, 2008.
- [18] J. A. Carrillo, D. Gómez-Castro, Y. Yao, and C. Zeng. Asymptotic simplification of aggregation-diffusion equations towards the heat kernel. arXiv:2105.13323, 2021.

- [19] M. Chae, K. Kang, and J. Lee. Existence of smooth solutions to coupled chemotaxis-fluid equations. Discrete Contin. Dyn. Syst., 33(6):2271–2297, 2013.
- [20] Q. Chen, T. Li, D. Wei, and Z. Zhang. Transition threshold for the 2-D Couette flow in a finite channel. Arch. Ration. Mech. Anal., 238(1):125–183, 2020.
- [21] C. Collot, T.-E. Ghoul, N. Masmoudi, and V. T. Nguyen. Spectral analysis for singularity formation of the two dimensional Keller-Segel system. arXiv:1911.10884, 2019.
- [22] C. Collot, T.-E. Ghoul, N. Masmoudi, and V. T. Nguyen. Refined description and stability for singular solutions of the 2D Keller-Segel system. Comm. Pure Appl. Math., 75(7):1419–1516, 2022.
- [23] P. Constantin and M. Ignatova. On the Nernst-Planck-Navier-Stokes system. Arch. Ration. Mech. Anal., 232(3):1379–1428, 2019.
- [24] P. Constantin, A. Kiselev, L. Ryzhik, and A. Zlatoš. Diffusion and mixing in fluid flow. Ann. of Math. (2), 168(2):643–674, 2008.
- [25] M. Coti-Zelati and T. D. Drivas. A stochastic approach to enhanced diffusion. arXiv:1911.09995v1, 2019.
- [26] J. Davila, M. del Pino, J. Dolbeault, M. Musso, and J. Wei. Infinite time blow-up in the Patlak-Keller-Segel system: existence and stability. arXiv:1911.12417, 2019.
- [27] W. Deng, J. Wu, and P. Zhang. Stability of Couette flow for 2D Boussinesq system with vertical dissipation. J. Funct. Anal., 281(12):Paper No. 109255, 40, 2021.
- [28] R.-J. Duan, A. Lorz, and P. Markowich. Global solutions to the coupled chemotaxis-fluid equations. Comm. Partial Differential Equations, Vol. 35, pages 1635–1673, 2010.
- [29] G. Egaña and S. Mischler. Uniqueness and long time asymptotic for the keller-segel equation: the parabolic-elliptic case. Arch. Ration. Mech. Anal. 220 (2016), no. 3.
- [30] M. D. Francesco, A. Lorz, and P. Markowich. Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior. Discrete Contin. Dyn. Syst. Ser. A, 28:1437–1453, 2010.
- [31] T.-E. Ghoul and N. Masmoudi. Minimal mass blowup solutions for the Patlak-Keller-Segel equation. Comm. Pure Appl. Math., 71(10):1957–2015, 2018.
- [32] Y. Gong and S. He. On the 8π-critical-mass threshold of a Patlak-Keller-Segel-Navier-Stokes system. SIAM J. Math. Anal., 53(3):2925–2956, 2021.
- [33] S. He. Suppression of blow-up in parabolic-parabolic Patlak-Keller-Segel via strictly monotone shear flows. Nonlinearity, 31(8):3651–3688, 2018.
- [34] S. He. Enhanced dissipation, hypoellipticity for passive scalar equations with fractional dissipation. J. Funct. Anal., 282(3):Paper No. 109319, 28, 2022.
- [35] S. He and E. Tadmor. Suppressing chemotactic blow-up through a fast splitting scenario on the plane. Arch. Ration. Mech. Anal., 232(2):951–986, 2019.
- [36] S. He, E. Tadmor, and A. Zlatos. On the fast spreading scenario. Comm. Amer. Math. Soc., 2:149–171, 2022.
- [37] M. Herrero and J. Velázquez. Singularity patterns in a chemotaxis model. Math. Ann., 306:583–623, 1996.
- [38] D. Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein, 105(3):103–165, 2003.
- [39] C.-Y. Hsieh and Y. Yu. Long-time dynamics of classical Patlak-Keller-Segel equation. arXiv:2009.01550, 2020.
- [40] G. Iyer, X. Xu, and A. Zlatoš. Convection-induced singularity suppression in the Keller-Segel and other non-linear PDEs. Trans. Amer. Math. Soc., 374(9):6039–6058, 2021.
- [41] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. Trans. Amer. Math. Soc., 329(2):819–824, 1992.
- [42] E. F. Keller and L. Segel. Model for chemotaxis. J. Theor. Biol., 30:225–234, 1971.
- [43] L. Kelvin. Stability of fluid motion-rectilinear motion of viscous fluid between two parallel plates. Phil. Mag., (24):188, 1887.
- [44] A. Kiselev and X. Xu. Suppression of chemotactic explosion by mixing. Arch. Ration. Mech. Anal., 222(2):1077–1112, 2016.
- [45] H. Kozono, M. Miura, and Y. Sugiyama. Time global existence and finite time blow-up criterion for solutions to the Keller-Segel system coupled with the Navier-Stokes fluid. J. Differential Equations, 267(9):5410–5492, 2019.
- [46] C.-C. Lai, J. Wei, and Y. Zhou. Global existence of free-energy solutions to the 2D Patlak–Keller–Segel–Navier–Stokes system with critical and subcritical mass. arXiv:2101.08306, 2021.
- [47] H. Li and W. Zhao. Metastability for the dissipative quasi-geostrophic equation and the non-local enhancement. arXiv:2107.10594, 2021.
- [48] T. Li, D. Wei, and Z. Zhang. Pseudospectral bound and transition threshold for the 3D Kolmogorov flow. Comm. Pure Appl. Math., 73(3):465–557, 2020.
- [49] K. Liss. On the Sobolev stability threshold of 3D Couette flow in a uniform magnetic field. Comm. Math. Phys., 377(2):859–908, 2020.
- [50] J.-G. Liu and A. Lorz. A coupled chemotaxis-fluid model: global existence. Annales de l'Institut Henri Poincaré. Analyse Non Linéaire, Vol. 28, pages 643–652, 2011.
- [51] A. Lorz. Coupled chemotaxis fluid model. Math. Models Methods Appl. Sci., Vol. 20, 2010.
- [52] A. Lorz. A coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay. Communications in Mathematical Sciences, Vol. 10, pages 555–574, 2012.
- [53] N. Masmoudi, B. Said-Houari, and W. Zhao. Stability of Couette flow for 2D Boussinesq system without thermal diffusivity. arXiv:2010.01612, 2020.

- [54] N. Masmoudi and W. Zhao. Enhanced dissipation for the 2D Couette flow in critical space. Comm. Partial Differential Equations, 45(12):1682–1701, 2020.
- [55] N. Masmoudi and W. Zhao. Stability threshold of two-dimensional Couette flow in Sobolev spaces. Ann. Inst. H. Poincaré C Anal. Non Linéaire, 39(2):245–325, 2022.
- [56] T. Nagai. Blow-up of radially symmetric solutions to a chemotaxis system. Adv. Math. Sci. Appl., 5(2):581–601, 1995.
- [57] T. Nagai, T. Senba, and K. Yoshida. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40(3):411–433, 1997.
- [58] V. Nanjundiah. Chemotaxis, signal relaying and aggregation morphology. Journal of Theoretical Biology, 42(1):63–105, 1973.
- [59] W. Orr. The stability or instability of steady motions of a perfect liquid and of a viscous liquid, Part I: a perfect liquid. Proc. Royal Irish Acad. Sec. A: Math. Phys. Sci., 27:9–68, 1907.
- [60] C. S. Patlak. Random walk with persistence and external bias. Bull. Math. Biophys., 15:311–338, 1953.
- [61] P. Raphaël and R. Schweyer. On the stability of critical chemotactic aggregation. Math. Ann., 359(1-2):267–377, 2014.
- [62] Y. Tao and M. Winkler. Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion. Ann. Inst. H. Poincaré Anal. Non Linéaire, 30(1):157–178, 2013.
- [63] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler, and R. E. Goldstein. Bacterial swimming and oxygen transport near contact lines. *Proceedings of the National Academy of Sciences*, 102(7):2277–2282, 2005.
- [64] J. Velázquez. Stability of some mechanisms of chemotactic aggregation. SIAM J. Appl. Math., 62(5):1581–1633, 2002.
- [65] J. Velázquez. Point dynamics in a singular limit of the Keller-Segel model i: motion of the concentration regions. SIAM J. Appl. Math., 64(4):1198–1223, 2004.
- [66] J. Velázquez. Point dynamics in a singular limit of the Keller-Segel model ii: formation of the concentration regions. SIAM J. Appl. Math., 64(4):1224–1248, 2004.
- [67] D. Wei. Global well-posedness and blow-up for the 2-D Patlak-Keller-Segel equation. J. Funct. Anal., 274(2):388–401, 2018.
- [68] D. Wei. Diffusion and mixing in fluid flow via the resolvent estimate. Science China Mathematics, pages 1–12, 2019.
- [69] D. Wei, Z. Zhang, and W. Zhao. Linear inviscid damping and enhanced dissipation for the Kolmogorov flow. Adv. Math., 362:106963, 103, 2020.
- [70] M. Winkler. Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. Comm. Partial Differential Equations, 37(2):319–351, 2012.
- [71] M. Winkler. Stabilization in a two-dimensional chemotaxis-Navier-Stokes system. Arch. Ration. Mech. Anal., 211(2):455–487, 2014.
- [72] M. Winkler. Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system. Ann. Inst. H. Poincaré Anal. Non Linéaire, 33(5):1329–1352, 2016.
- [73] M. Winkler. How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system? Trans. Amer. Math. Soc., 369(5):3067–3125, 2017.
- [74] M. Winkler. Small-mass solutions in the two-dimensional Keller-Segel system coupled to the Navier-Stokes equations. SIAM J. Math. Anal., 52(2):2041–2080, 2020.
- [75] M. Winkler. Does Leray's structure theorem withstand buoyancy-driven chemotaxis-fluid interaction? J. Eur. Math. Soc. (JEMS), 2021.
- [76] L. Zeng, Z. Zhang, and R. Zi. Suppression of blow-up in Patlak-Keller-Segel-Navier-Stokes system via the Couette flow. J. Funct. Anal., 280(10):Paper No. 108967, 40, 2021.
- [77] C. Zillinger. Linear inviscid damping for monotone shear flows. Trans. Amer. Math. Soc., 369(12):8799-8855, 2017.
- [78] C. Zillinger. On enhanced dissipation for the Boussinesq equations. J. Differential Equations, 282:407–445, 2021.
- [79] C. Zillinger. On the Boussinesq equations with non-monotone temperature profiles. J. Nonlinear Sci., 31(4):Paper No. 64, 38, 2021.

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