FORMULAE FOR MIXED MOMENTS OF WIENER PROCESSES AND A STOCHASTIC AREA INTEGRAL *

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Abstract. This paper deals with the expectation of monomials with respect to the stochastic area integral

$$A_{1,2}(t,t+h) = \int_t^{t+h}\int_t^s \mathrm{d}W_1(r)\mathrm{d}W_2(s) - \int_t^{t+h}\int_t^s \mathrm{d}W_2(r)\mathrm{d}W_1(s)$$

and the increments of two Wiener processes, $\Delta W_i(t,t+h) = W_i(t+h) - W_i(t),\ i = 1,2$. In a monomial, if the exponent of one of the Wiener increments or the stochastic area integral is an odd number, then the expectation of the monomial is zero. However, if the exponent of any of them is an even number, then the expectation is nonzero and its exact value is not known in general. In the present paper, we derive formulae to give the value in general. As an application of the formulae, we will utilize the formulae for a careful stability analysis on a Magnus-type Milstein method. As another application, we will give some mixed moments of the increments of Wiener processes and stochastic double integrals.

Key words. High order moment, stochastic integral, stochastic differential equation, stability analysis, numerical method

MSC codes. 60H10, 60H30, 65C30

1. Introduction. We are concerned with developing and analyzing numerical methods that give strong first order approximations to the solution of noncommutative stochastic differential equations (SDEs). Such methods are usually constructed on the basis of the comparison with the Itô–Taylor expansion or the Stratonovich–Taylor expansion, and as a result, they have one or more terms related to the stochastic area integral

$$A_{i,j}(t,t+h) = \int_t^{t+h}\int_t^s \mathrm{d}W_i(r)\mathrm{d}W_j(s) - \int_t^{t+h}\int_t^s \mathrm{d}W_j(r)\mathrm{d}W_i(s),$$

where $t \geq 0$, $h > 0$, and where $W_i(t), W_j(t)$ are independent Wiener processes for positive integers i, j $(i \neq j)$.

Lévy [13] has studied the stochastic area integral by utilizing the Fourier series of Wiener process. As an example, he has given the probability density function of $A_{i,j}(0,1)$. Gaveau [6] has also studied the stochastic area integral and shown a joint density function in general form, which is related to Wiener increments and stochastic area integrals.

The joint density of the stochastic area integral $A_{1,2}(t,t+h)$ and the Wiener increments $\Delta W_i(t,t+h) = W_i(t+h) - W_i(t),\ i = 1,2$, can be used to generate random numbers which approximate the stochastic area integral. In fact, by utilizing a conditional joint density function of them, Gaines and Lyons [5] have proposed a

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random generator for the stochastic area integral. As the performance of random generators for the stochastic area integral is a very important issue in stochastic simulation, many researchers have struggled to find efficient generators [9, 11, 12, 15, 19].

Let us suppose that we have to calculate the expectation of a function $f(y_{n+1})$ to study properties of a numerical method for approximating the solution of SDEs. Here, y_{n+1} denotes an approximate solution at time $t = t_{n+1} = t_n + h$. If the numerical method contains $I_{i,j}(t_n, t_{n+1}) = (1/2)\Delta W_i(t_n, t_{n+1})\Delta W_j(t_n, t_{n+1}) + (1/2)A_{i,j}(t_n, t_{n+1})$ and if we use the Taylor expansion of $f(y_{n+1})$ centered at y_n, then we will need to calculate the expectation of monomials with respect to $\Delta W_i(t_n, t_{n+1})$, $\Delta W_j(t_n, t_{n+1})$, and $A_{i,j}(t_n, t_{n+1})$. Kloeden and Platen [8] have given the expectation of monomials with respect to $\Delta W_i(t_n, t_{n+1})$, $\Delta W_j(t_n, t_{n+1})$, and $I_{i,j}(t_n, t_{n+1})$ in some restricted cases. However, to the best of the authors' knowledge, the exact value of such expectations is not given in general. Thus, in the present paper we will aim to derive a formula that gives the exact value in general.

Incidentally, Magnus-type expansions for SDEs have recently drawn attention from a number of researchers. Originally, a Magnus expansion has been given for ordinary differential equations by Magnus [14]. However, the Magnus expansion has been extended for SDEs, and its extended expansions have been studied by many researchers [4, 7, 16, 17, 18]. Especially, Yang et al. [20] have proposed a Magnus-type Euler method and a Magnus-type Milstein method for semilinear noncommutative Itô SDEs. When we investigate stability properties of the Magnus-type Milstein method for a noncommutative test SDE, we need the expectation of monomials with respect to $\Delta W_i(t_n, t_{n+1})$, $\Delta W_j(t_n, t_{n+1})$, and $A_{i,j}(t_n, t_{n+1})$ in general. Thus, as an application of the above formula, we will carefully analyse stability properties of the method for a noncommutative test SDE.

The present paper is organized as follows. In section 2, after preliminary discussion we will give our main theorem. In section 3, we will introduce the Magnus-type methods and a noncommutative test equation, and as an application of our formula, we will show that the Magnus-type Milstein method cannot be A-stable, whereas the Magnus-type Euler method is A-stable for this test equation. Finally, we will give concluding remarks.

2. Main theorem.

2.1. Preliminary and motivation. For simplicity, let us take variables a, w_i corresponding to $A_{1,2}(t, t+1)$ and $\Delta W_i(t, t+1)$ $(i = 1, 2)$. Then, the joint density function for $A_{1,2}(t, t+1)$, $\Delta W_1(t, t+1)$ and $\Delta W_2(t, t+1)$ is given as follows [5]:

$$f(a, w_1, w_2) = \frac{1}{2\pi^2} \int_0^\infty \frac{x}{\sinh(x)} \exp\left(\frac{-(w_1^2 + w_2^2)x}{2\tanh(x)}\right) \cos(ax)\mathrm{d}x. \tag{2.1}$$

From this,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_1^{2k_1} w_2^{2k_2} f(a, w_1, w_2)\mathrm{d}w_1\mathrm{d}w_2 \\
&= \frac{1}{\pi}\int_0^\infty \{(2k_1-1)!!\}\{(2k_2-1)!!\}\left(\frac{\tanh(x)}{x}\right)^{k_1+k_2}\frac{\cos(ax)}{\cosh(x)}\mathrm{d}x \\
&= \frac{1}{\pi}\frac{(2k_1)!((2k_2)!)}{(k_1!)(k_2!)2^{k_1+k_2}}\int_0^\infty \left(\frac{\tanh(x)}{x}\right)^{k_1+k_2}\frac{\cos(ax)}{\cosh(x)}\mathrm{d}x \qquad (2.2)
\end{aligned}$$

for any nonnegative integers k_1, k_2, where $(2k-1)!!$ denotes $(2k-1)(2k-3)\cdots 1$ for a positive integer k and $(-1)!! = 1$. Thus, if we denote by $\gamma_{n,k,l}$ the expectation of monomial $E[\{\Delta W_1(t,t+1)\}^{2n}\{A_{1,2}(t,t+1)\}^{2k}\{\Delta W_2(t,t+1)\}^{2l}]$ for nonnegative integers n, k and l, then it can be expressed as

$$\gamma_{n,k,l} = \frac{2}{\pi}\frac{((2n)!)((2l)!)}{(n!)(l!)2^{n+l}}\int_0^\infty\int_0^\infty a^{2k}\left(\frac{\tanh(x)}{x}\right)^{n+l}\frac{\cos(ax)}{\cosh(x)}\mathrm{d}x\mathrm{d}a. \tag{2.3}$$

From this, we have

$$\gamma_{n-2k-l,k,l} = \frac{{}_{n-2k}C_l}{{}_{2(n-2k)}C_{2l}}\gamma_{n-2k,k,0} \tag{2.4}$$

for nonnegative integers k, l and $n \geq 2k+l$, where ${}_nC_l$ denotes a binomial coefficient defined by $n!/((n-l)!(l!))$. When $k=0$, we immediately have

$$\gamma_{n-l,0,l} = \frac{(2(n-l))!((2l)!)}{(n-l)!(l!)2^n},$$

which leads to

$$\int_0^\infty\int_0^\infty\left(\frac{\tanh(x)}{x}\right)^n\frac{\cos(ax)}{\cosh(x)}\mathrm{d}x\mathrm{d}a = \frac{\pi}{2}.$$

When we want to consider $\gamma_{n-2k-l,k,l}$ for $k \neq 0$, we can concentrate on $\gamma_{n-2k,k,0}$ since this gives $\gamma_{n-2k-l,k,l}$ by (2.4), but it is still not trivial to calculate it. One possible way is to utilize the properties of stochastic integrals. Let us denote by $I_{i,j}(t,t+1)$ the double stochastic integral $\int_t^{t+1}\int_t^s \mathrm{d}W_i(r)\mathrm{d}W_j(s)$. Noting that

$$I_{1,2}(t,t+1) = \frac{1}{2}\Delta W_1(t,t+1)\Delta W_2(t,t+1) + \frac{1}{2}A_{1,2}(t,t+1)$$

(see [19]) and $A_{2,1}(t,t+1) = -A_{1,2}(t,t+1)$, we have

$$\begin{aligned} I_{1,2}(t,t+1) + I_{2,1}(t,t+1) &= \Delta W_1(t,t+1)\Delta W_2(t,t+1), \\ I_{1,2}(t,t+1) - I_{2,1}(t,t+1) &= A_{1,2}(t,t+1). \end{aligned}$$

From these,

$$\big(A_{1,2}(t,t+1)\big)^2 = \big(\Delta W_1(t,t+1)\big)^2\big(\Delta W_2(t,t+1)\big)^2 - 4I_{1,2}(t,t+1)I_{2,1}(t,t+1).$$

Utilizing this and

$$E\left[\big(\Delta W_1(t,t+1)\big)^2 I_{1,2}(t,t+1)I_{2,1}(t,t+1)\right] = \frac{1}{3} \tag{2.5}$$

[8, p. 225], we can obtain $\gamma_{1,1,0} = 5/3$, which also gives $\gamma_{0,1,1} = 5/3$ by (2.4) for $n=3$ and $k=l=1$.

In the above calculations, (2.5) is a key point. Such expectations are obtained from the properties of stochastic integrals, but it is not easy to seek them when an exponent is a large number. For example, in order to carry out calculations even for

$$E\left[\big(\Delta W_1(t,t+1)\big)^4 I_{1,2}(t,t+1)I_{2,1}(t,t+1)\right] = 2,$$

we have needed a program code in a symbolic computing package, Mathematica, which utilizes some rules for Stratonovich integrals J_{12}, J_{21} in [10, p. 160]. This fact motivates us to seek for another approach in the next subsection.

Remark 2.1. In contrast to the calculation of $\gamma_{n-2k-l,k,l}$, it is easy to get

$$E\left[\{\Delta W_1(t,t+1)\}^n\{A_{1,2}(t,t+1)\}^k\{\Delta W_2(t,t+1)\})^l\right]=0$$

if one of n, k and l is an odd number. In fact, it is clear that

$$\int_{-\infty}^{\infty} w_1^{2k_1+1} f(a,w_1,w_2)\mathrm{d}w_1=\int_{-\infty}^{\infty} w_2^{2k_2+1} f(a,w_1,w_2)\mathrm{d}w_2=0$$

from (2.1) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_1^{2k_1}w_2^{2k_2}a^{2k_3+1} f(a,w_1,w_2)\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}a=0$$

from (2.2) for any nonnegative integers k_1,k_2,k_3.

2.2. Lemmas and main theorem. As preparation for our main theorem, we begin with a function $r_n(x)$ given as

$$r_n(x)=\left(\frac{\tanh(x)}{x}\right)^n\frac{1}{\cosh(x)}$$

for $n=0,1,\ldots$ and any nonzero real number x. Recalling the series expansion

$$\tanh(x)=x-\frac{x^3}{3}+\frac{2x^5}{15}+\cdots+\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1}+\cdots\quad(x^2<\frac{\pi^2}{4})$$

[21, p. 42], where B_{2n} is the $2n$-th Bernoulli number that is given by

$$B_{2n}=-\frac{1}{2n+1}+\frac{1}{2}-\sum_{k=1}^{n-1}\frac{2n(2n-1)\cdots(2n-2k+2)}{(2k)!}B_{2k}\quad(n=1,2,\ldots)$$

[21, p. 1052], we define the value of $r_n(0)$ by $\lim_{x\to 0} r_n(x)=1$. If we differentiate $r_n(x)$, then

$$\frac{\mathrm{d}r_n}{\mathrm{d}x}(x)=v_n(x)r_n(x),\tag{2.6}$$

where

$$v_n(x)=n\left(\frac{1}{\tanh(x)}-\frac{1}{x}\right)-(n+1)\tanh(x).$$

Recalling the series expansion

$$\frac{1}{\tanh(x)}=\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}+\cdots+\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1}+\cdots\quad(x^2<\pi^2)$$

[21, p. 42], we define the value of $v_n(0)$ by $\lim_{x\to 0} v_n(x)=0$. Similarly, we define the value of $(\mathrm{d}^{2k}v_n/\mathrm{d}x^{2k})(0)$ by $\lim_{x\to 0}(\mathrm{d}^{2k}v_n/\mathrm{d}x^{2k})(x)=0$ for $k=1,2,\ldots$. Now, we can obtain the following lemma regarding $r_n(x)$.

Lemma 2.2. *For $k=1,2,\ldots$, the derivatives of $r_n(x)$ at $x=0$ are given by*

$$\frac{\mathrm{d}^{2k-1}r_n}{\mathrm{d}x^{2k-1}}(0)=0,$$

$$\frac{\mathrm{d}^{2k}r_n}{\mathrm{d}x^{2k}}(0)=\frac{1}{2k}\sum_{j=1}^{k}{}_{2k}C_{2j}2^{2j}B_{2j}\left\{(1-2^{2j})(n+1)+n\right\}\frac{\mathrm{d}^{2(k-j)}r_n}{\mathrm{d}x^{2(k-j)}}(0).\tag{2.7}$$

Proof. As $v_n(0) = 0$ and $r_n(0) = 1$, (2.6) gives $(\mathrm{d}r_n/\mathrm{d}x)(0) = 0$. Next, by applying Leibniz theorem to (2.6) and noting that $(\mathrm{d}^{2k}v_n/\mathrm{d}x^{2k})(0) = 0$ for $k = 1, 2, \ldots$, we have

$$\frac{\mathrm{d}^{2k-1}r_n}{\mathrm{d}x^{2k-1}}(0) = 0, \qquad \frac{\mathrm{d}^{2k}r_n}{\mathrm{d}x^{2k}}(0) = \sum_{j=1}^{k} {}_{2k-1}C_{2j-1}\frac{\mathrm{d}^{2j-1}v_n}{\mathrm{d}x^{2j-1}}(0)\frac{\mathrm{d}^{2(k-j)}r_n}{\mathrm{d}x^{2(k-j)}}(0).$$

Here, as

$$\frac{\mathrm{d}^{2j-1}v_n}{\mathrm{d}x^{2j-1}}(0) = \frac{2^{2j}B_{2j}}{2j}\left\{(1-2^{2j})(n+1)+n\right\},$$

the substitution of this and simplification yield the second equation in the lemma. $\square$

Remark 2.3. If we set $y = 1/\tanh(x)$, then $\mathrm{d}y/\mathrm{d}x = 1 - y^2$. Noting $\mathrm{d}/\mathrm{d}x = (1-y^2)(\mathrm{d}/\mathrm{d}y)$ and $\lim_{x\to\infty} y = 1$, we have $\lim_{x\to\infty}(\mathrm{d}^k y/\mathrm{d}x^k) = 0$ for any positive integer k. We have the same equation also for $y = \tanh(x)$. From these, we have $\lim_{x\to\infty}(\mathrm{d}^k v_n/\mathrm{d}x^k)(x) = 0$. Thus, in a similar way to the proof of Lemma 2.2 we can see that $(\mathrm{d}^k r_n/\mathrm{d}x^k(x)$ exponentially converges to 0 as $x \to \infty$.

As Lemma 2.2 gives a recursive formula for the even-order derivatives of $r_n(x)$ at $x = 0$, it is useful for the fast calculations of them. On the other hand, the following lemma gives an explicit formula, which can be helpful for mathematical analysis.

Lemma 2.4. *The even order derivatives of $r_n(x)$ at $x = 0$ are given by*

$$\frac{\mathrm{d}^{2k}r_n}{\mathrm{d}x^{2k}}(0) = (-1)^k\big((2k)!\big)s_{n,k} \tag{2.8}$$

for $k = 0, 1, \ldots$. Here, $s_{n,k}$ is given by

$$s_{n,0} = 1, \qquad s_{n,k} = \sum_{l_1+2l_2+\cdots+kl_k=k}\left\{\prod_{j=1}^{k}\frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)}\right\} \quad (k = 1, 2, \ldots),$$

where $l_1, l_2, \ldots, l_k$ denote nonnegative integers and where

$$\beta_{n,j} = \frac{2^{2j-1}|B_{2j}|\left\{(2^{2j}-1)(n+1)-n\right\}}{(2j)!}.$$

Proof. The setting of $k = 0$ is a special case and it is trivial that (2.8) holds. In what follows, we shall prove that (2.8) holds by a mathematical induction. When $k = 1$, (2.7) implies

$$\frac{\mathrm{d}^2 r_n}{\mathrm{d}x^2}(0) = 2B_2\left\{(1-2^2)(n+1)+n\right\} = -2\beta_{n,1}.$$

Thus, (2.8) holds for $k = 1$. Next, suppose that (2.8) holds for any k less than or equal to an even number $q = 2m$. From (2.7), we have

$$\begin{aligned}\frac{\mathrm{d}^{2(q+1)}r_n}{\mathrm{d}x^{2(q+1)}}(0) &= \frac{1}{q+1}\sum_{j=1}^{q+1} {}_{2(q+1)}C_{2j}\big((2j)!\big)(-1)^j\beta_{n,j}\frac{\mathrm{d}^{2(q+1-j)}r_n}{\mathrm{d}x^{2(q+1-j)}}(0)\\ &= (-1)^{q+1}\frac{(2(q+1))!}{q+1}(H_1+H_2),\end{aligned} \tag{2.9}$$

where

$$
\begin{aligned}
H_1 = & \sum_{\hat{l}_1+2l_2+\cdots+ql_q=q+1} \left[\hat{l}_1 \frac{\beta_{n,1}^{\hat{l}_1}}{\hat{l}_1!} \prod_{\substack{j=1\\ j\neq 1}}^{q} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right] \\
& + \sum_{l_1+2\hat{l}_2+3l_3+\cdots+(q-1)l_{q-1}=q+1} \left[2\hat{l}_2 \frac{\beta_{n,2}^{\hat{l}_2}}{2\hat{l}_2(\hat{l}_2!)} \prod_{\substack{j=1\\ j\neq 2}}^{q-1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right] + \cdots \\
& + \sum_{l_1+2l_2+\cdots+(m-1)l_{m-1}+m\hat{l}_m+(m+1)l_{m+1}=q+1} \left[m\hat{l}_m \frac{\beta_{n,m}^{\hat{l}_m}}{m^{\hat{l}_m}(\hat{l}_m!)} \prod_{\substack{j=1\\ j\neq m}}^{m+1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right], \\
H_2 = & \sum_{l_1+2l_2+\cdots+ml_m+(m+1)\hat{l}_{m+1}=q+1} \left[\beta_{n,m+1} \prod_{j=1}^{m} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right] \\
& + \sum_{l_1+2l_2+\cdots+(m-1)l_{m-1}+(m+2)\hat{l}_{m+2}=q+1} \left[\beta_{n,m+2} \prod_{j=1}^{m-1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right] + \cdots \\
& + \sum_{l_1+q\hat{l}_q=q+1} \left[\beta_{n,q} \prod_{j=1}^{1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right] + \beta_{n,q+1},
\end{aligned}
$$

and where $\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_q$ denote positive integers. In H_2, for example, we can rewrite the sum of the last two terms as

$$
\begin{aligned}
& \sum_{l_1+q\hat{l}_q=q+1} \left[\beta_{n,q} \prod_{j=1}^{1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right] + \beta_{n,q+1} \\
= & \sum_{l_1+ql_q+(q+1)l_{q+1}=q+1} \left\{ \left(ql_q + (q+1)l_{q+1} \right) \left(\frac{\beta_{n,1}^{l_1}}{l_1!} \right) \left(\frac{\beta_{n,q}^{l_q}}{q^{l_q}(l_q!)} \right) \left(\frac{\beta_{n,q+1}^{l_{q+1}}}{(q+1)^{l_{q+1}}(l_{q+1}!)} \right) \right\}.
\end{aligned}
$$

Noting this, we can see that H_2 is rewritten as

$$
\begin{aligned}
H_2 = \sum_{l_1+2l_2+\cdots+(q+1)l_{q+1}=q+1} \Bigg\{ & \Big((m+1)l_{m+1} + (m+2)l_{m+2} + \cdots \\
& + (q+1)l_{q+1} \Big) \prod_{j=1}^{q+1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \Bigg\}.
\end{aligned}
$$

Similarly, H_1 is rewritten as

$$
\begin{aligned}
H_1 &= \sum_{l_1+2l_2+\cdots+ql_q=q+1} \left\{ (l_1 + 2l_2 + \cdots + ml_m) \prod_{j=1}^{q} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\} \\
&= \sum_{l_1+2l_2+\cdots+(q+1)l_{q+1}=q+1} \left\{ (l_1 + 2l_2 + \cdots + ml_m) \prod_{j=1}^{q+1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\}.
\end{aligned}
$$

The substitution of these into (2.9) implies

$$\frac{\mathrm{d}^{2(q+1)}r_n}{\mathrm{d}x^{2(q+1)}}(0) = (-1)^{q+1}\big((2(q+1))!\big) \sum_{l_1+2l_2+\cdots+(q+1)l_{q+1}=q+1} \left\{ \prod_{j=1}^{q+1} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\}.$$

Thus, (2.8) holds for $k = q+1$.

On the other hand, when we suppose that (2.8) holds for any k less than or equal to an odd number $q = 2m+1$, we can obtain the above equality for $k = q+1$ in a similar way. ∎

Rather than the explicit formula that gives the exact value of even-order derivatives of $r_n(x)$ at $x = 0$, a simpler formula which gives an upper bound of the absolute value of them can be often helpful for mathematical analysis. As preparation, let us consider the summation

$$\hat{s}_{n,L}(k) = \sum_{l_1+2l_2+\cdots+Ll_L=k} \left\{ \prod_{j=1}^{L} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\}$$

for nonnegative integers L and k, which implies $\hat{s}_{n,L}(0) = 1$ and $\hat{s}_{n,0}(k) = 0$ for $k > 0$. When we deal with $\hat{s}_{n,L}(k)$, the following lemma is useful.

Lemma 2.5. *Let L be a nonnegative integer and k a positive integer. Then,*

$$\hat{s}_{n,L}(k) = \begin{cases} \dfrac{1}{k}\displaystyle\sum_{j=1}^{L} \beta_{n,j}\hat{s}_{n,L}(k-j) & (L < k), \\[2ex] \dfrac{1}{k}\displaystyle\sum_{j=1}^{k} \beta_{n,j}\hat{s}_{n,k}(k-j) & (L \geq k). \end{cases}$$

For the proof of this lemma, we refer the reader to Appendix A. Using this, we obtain a function related to $\hat{s}_{n,L}(k)$ as we can see below.

Lemma 2.6. *For a nonnegative integer L, define $M_{n,L}(\theta)$ by*

$$M_{n,L}(\theta) = \exp\left(\sum_{j=1}^{L} \frac{\beta_{n,j}}{j}\theta^j \right), \tag{2.10}$$

where $\theta \in \mathbb{R}$ is a parameter. For any nonnegative integer k, this satisfies

$$\hat{s}_{n,L}(k) = \frac{1}{k!}\frac{\mathrm{d}^k M_{n,L}}{\mathrm{d}\theta^k}(0). \tag{2.11}$$

Proof. When $L = 0$ or $k = 0$, it is trivial that (2.11) holds. When $L > 0$ and $k > 0$, we shall prove this lemma by a mathematical induction. The differentiation of (2.10) and the substitution of $\theta = 0$ into it yield $\frac{\mathrm{d}M_{n,L}}{\mathrm{d}\theta}(0) = \beta_{n,1} = \hat{s}_{n,L}(1)$, which implies that (2.11) holds for $k = 1$. Next, by differentiating (2.10) and applying Leibniz theorem to it, we have

$$\frac{\mathrm{d}^{k+1}M_{n,L}}{\mathrm{d}\theta^{k+1}}(\theta) = \sum_{l=0}^{k} {}_kC_l \left\{ \frac{\mathrm{d}^l}{\mathrm{d}\theta^l} \left(\sum_{j=1}^{L} \beta_{n,j}\theta^{j-1} \right) \right\} \frac{\mathrm{d}^{k-l}M_{n,L}}{\mathrm{d}\theta^{k-l}}(\theta). \tag{2.12}$$

Let us suppose that (2.11) holds for any k less than or equal to a positive integer q, and consider the case of $q < L$. Then, (2.12) gives

$$\begin{aligned}\frac{\mathrm{d}^{q+1}M_{n,L}}{\mathrm{d}\theta^{q+1}}(0) &= \sum_{l=0}^{q} {}_qC_l \beta_{n,l+1}(l!)\frac{\mathrm{d}^{q-l}M_{n,L}}{\mathrm{d}\theta^{q-l}}(0)\\ &= \sum_{l=0}^{q} {}_qC_l \beta_{n,l+1}(l!)((q-l)!)\hat{s}_{n,L}(q-l)\\ &= q!\sum_{l=1}^{q+1}\beta_{n,l}\hat{s}_{n,L}(q+1-l)\\ &= q!\sum_{l=1}^{q+1}\beta_{n,l}\hat{s}_{n,q+1}(q+1-l) = (q+1)!\hat{s}_{n,L}(q+1).\end{aligned}$$

In the last two equalities, we have used $L \geq q+1$ and Lemma 2.5. On the other hand, in the case of $q \geq L$, (2.12) similarly gives

$$\begin{aligned}\frac{\mathrm{d}^{q+1}M_{n,L}}{\mathrm{d}\theta^{q+1}}(0) &= \sum_{l=0}^{L-1} {}_qC_l \beta_{n,l+1}(l!)\frac{\mathrm{d}^{q-l}M_{n,L}}{\mathrm{d}\theta^{q-l}}(0)\\ &= q!\sum_{l=1}^{L}\beta_{n,l}\hat{s}_{n,L}(q+1-l) = (q+1)!\hat{s}_{n,L}(q+1).\end{aligned}$$

Thus, (2.11) holds for $k = q+1$. ◻

Utilizing this lemma, the following lemma gives upper bounds of $s_{n,k}$.

LEMMA 2.7. *Let k_0 be a nonnegative integer. Then,*

$$s_{n,k} \leq \frac{\tan^n(1)}{\cos(1)}\frac{1}{M_{n,k_0}(1)}\max_{0\leq j\leq k}\left\{\frac{1}{j!}\frac{\mathrm{d}^j M_{n,k_0}}{\mathrm{d}\theta^j}(0)\right\} \tag{2.13}$$

for $k \geq k_0 + 1$. Especially, for $k \geq 0$,

$$s_{n,k} \leq \frac{\tan^n(1)}{\cos(1)}. \tag{2.14}$$

Proof. Setting $\hat{s}_{n,L}(k) = 0$ if $k < 0$, we have

$$\hat{s}_{n,L}(k) = \sum_{i=0}^{k}\left\{\frac{\beta_{n,L}^i}{L^i(i!)}\hat{s}_{n,L-1}(k-Li)\right\}$$

for $L \geq 1$. In a similar way, for $L \geq k_0 + 1$,

$$\begin{aligned}\hat{s}_{n,L}(k) = &\sum_{i_1,i_2,\ldots,i_{L-k_0}=0}^{k}\left\{\left(\prod_{j=1}^{L-k_0}\frac{\beta_{n,L-j+1}^{i_j}}{(L-j+1)^{i_j}(i_j!)}\right)\hat{s}_{n,k_0}\left(k-\sum_{j=1}^{L-k_0}(L-j+1)i_j\right)\right\}\\ &\leq \sum_{i_1,i_2,\ldots,i_{L-k_0}=0}^{k}\left(\prod_{j=1}^{L-k_0}\frac{\beta_{n,L-j+1}^{i_j}}{(L-j+1)^{i_j}(i_j!)}\right)\max_{0\leq j\leq k}\hat{s}_{n,k_0}(k-j)\\ &\leq \exp\left(\sum_{i=k_0+1}^{L}\frac{\beta_{n,i}}{i}\right)\max_{0\leq j\leq k}\hat{s}_{n,k_0}(j) \leq \frac{\tan^n(1)}{\cos(1)}\frac{1}{M_{n,k_0}(1)}\max_{0\leq j\leq k}\left\{\frac{1}{j!}\frac{\mathrm{d}^j M_{n,k_0}}{\mathrm{d}\theta^j}(0)\right\}.\end{aligned}$$

In the last inequality, we have utilized Lemma 2.6 and the formulae [21, p. 55]

$$\sum_{j=1}^{\infty} \frac{2^{2j-1}(2^{2j}-1)|B_{2j}|}{j(2j)!} = -\ln\cos(1), \qquad \sum_{j=1}^{\infty} \frac{2^{2j-1}|B_{2j}|}{j(2j)!} = -\ln\sin(1).$$

Thus, when $L = k$ we obtain (2.13). In addition, the substitution of $k_0 = 0$ into (2.13) and $s_{n,0} = 1$ complete the proof. $\square$

As the next step of preparations for our main theorem, we shall derive

$$\int_0^{\infty}\int_0^{\infty} r_n(x)\cos(ax)\mathrm{d}x\mathrm{d}a = \frac{\pi}{2} \tag{2.15}$$

in a different way, although it has already been obtained in subsection 2.1.

When $n = 0$, Levy [13] has shown

$$\int_0^{\infty} r_0(x)\cos(ax)\mathrm{d}x = \frac{\pi}{2\cosh(\pi a/2)}$$

by utilizing an expansion of $1/\cosh(\pi a/2)$ in series of simple fractions [21, p. 44]. This directly gives (2.15) when $n = 0$. Now, let us consider another approach to calculate the double integral when $n = 0$. Noting that we may interchange the integration and differentiation with respect to x and a, respectively in the following, we obtain

$$\begin{aligned}\int_0^{\infty}\int_0^{\infty} r_0(x)\cos(ax)\mathrm{d}x\mathrm{d}a &= \int_0^{\infty}\int_0^{\infty} r_0(x)\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\sin(ax)}{x}\right)\mathrm{d}x\mathrm{d}a \\ &= \lim_{a\to\infty}\int_0^{\infty} r_0(x)\frac{\sin(ax)}{x}\mathrm{d}x. \end{aligned} \tag{2.16}$$

Here, recalling one of the well-known results by the residue theorem:

$$\int_0^{\infty} \frac{\sin(ax)}{x}\mathrm{d}x = \frac{\pi}{2},$$

we consider

$$\int_0^{\infty} r_0(x)\frac{\sin(ax)}{x}\mathrm{d}x - \frac{\pi}{2} = \int_0^{\infty} \frac{r_0(x)-r_0(0)}{x}\sin(ax)\mathrm{d}x,$$

which is interpreted as an improper Riemann integral. Although we cannot directly apply the Riemann-Lebesgue lemma [2, p. 169] to the integral, we can use its basic idea. Taking Lemma 2.2 and Remark 2.3 into account, we have

$$a\int_0^{\infty} \frac{r_0(x)-r_0(0)}{x}\sin(ax)\mathrm{d}x = \int_0^{\infty} \frac{r_0'(x)x - r_0(x) + r_0(0)}{x^2}\cos(ax)\mathrm{d}x < \infty.$$

Thus, as

$$\lim_{a\to\infty}\left\{\int_0^{\infty} r_0(x)\frac{\sin(ax)}{x}\mathrm{d}x - \frac{\pi}{2}\right\} = 0,$$

we obtain (2.15) when $n = 0$. Also for $n \geq 1$, we can show

$$\lim_{a\to\infty}\left\{\int_0^{\infty} r_n(x)\frac{\sin(ax)}{x}\mathrm{d}x - \frac{\pi}{2}\right\} = 0$$

in a similar way. Thus, (2.15) holds for $n = 0, 1, \ldots$.

We have already mentioned that in order to obtain $\gamma_{n-2k-l,k,l}$, we can concentrate on $\gamma_{n-2k,k,0}$ due to (2.4). Our main theorem is useful to get this expression.

THEOREM 2.8.

$$\gamma_{n,k,0} = (-1)^k \frac{(2n)!}{2^n(n!)} \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(0) = \frac{(2n)!((2k)!)}{2^n(n!)} s_{n,k} = \frac{(2n)!}{2^n(n!)} \frac{(2k)!}{k!} \frac{\mathrm{d}^k M_{n,k}}{\mathrm{d}\theta^k}(0) \tag{2.17}$$

for any nonnegative integers n, k.

Proof. The others except for the first equality are immediately given by Lemma 2.4 and Lemma 2.6 if the first equality holds, all that remains is its proof. For a positive integer k, we obtain

$$\int_0^\infty \int_0^\infty a^{2k} r_n(x) \cos(ax) \mathrm{d}x \mathrm{d}a = -\int_0^\infty \int_0^\infty a^{2k-1} \frac{\mathrm{d}r_n}{\mathrm{d}x}(x) \sin(ax) \mathrm{d}x \mathrm{d}a,$$

using integration by parts with respect to x and noting Remark 2.3. Its repeated applications and the interchange of the integration and differentiation lead to

$$\int_0^\infty \int_0^\infty a^{2k} r_n(x) \cos(ax) \mathrm{d}x \mathrm{d}a = \lim_{a \to \infty} (-1)^k \int_0^\infty \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(x) \frac{\sin(ax)}{x} \mathrm{d}x$$

in a similar way to (2.16). Here, taking Lemma 2.2 and Remark 2.3 into account, we have

$$\begin{aligned} & a \int_0^\infty \frac{\frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(x) - \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(0)}{x} \sin(ax) \mathrm{d}x \\ &= \int_0^\infty \frac{\frac{\mathrm{d}^{2k+1} r_n}{\mathrm{d}x^{2k}}(x)x - \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(x) + \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(0)}{x^2} \cos(ax) \mathrm{d}x < \infty. \end{aligned}$$

Thus,

$$\lim_{a \to \infty} \left\{ \int_0^\infty \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(x) \frac{\sin(ax)}{x} \mathrm{d}x - \frac{\pi}{2} \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(0) \right\} = 0.$$

Consequently,

$$\int_0^\infty \int_0^\infty a^{2k} r_n(x) \cos(ax) \mathrm{d}x \mathrm{d}a = (-1)^k \frac{\pi}{2} \frac{\mathrm{d}^{2k} r_n}{\mathrm{d}x^{2k}}(0).$$

The substitution of the equation and $l = 0$ into (2.3) completes the proof. ∎

Remark 2.9. In the interpretation of (2.17), $M_{n,k}(\theta)$ is the mixed moment generating function of $\{\Delta W_1(t, t+1)\}^2$ and $\{A_{1,2}(t, t+1)\}^2$, whereas $M_{n,2}(\theta)$ is the moment generating function of a normal random variable Y_n with mean $\beta_{n,1}$ and variance $\beta_{n,2}$ in the usual meaning.

This theorem and Lemma 2.7 immediately give the following corollary.

COROLLARY 2.10. *Let k_0 be a nonnegative integer. Then,*

$$\gamma_{n,k,0} \leq \frac{(2n)!((2k)!)}{2^n(n!)} \frac{\tan^n(1)}{\cos(1)} \frac{1}{M_{n,k_0}(1)} \max_{0 \leq j \leq k} \left\{ \frac{1}{j!} \frac{\mathrm{d}^j M_{n,k_0}}{\mathrm{d}\theta^j}(0) \right\}$$

for $k \geq k_0 + 1$. Especially, for $k \geq 0$,

$$\gamma_{n,k,0} \leq \frac{(2n)!((2k)!)}{2^n(n!)} \frac{\tan^n(1)}{\cos(1)}.$$

3. Applications. We give two examples as applications of our main theorem. The first example is the stability analysis for Magnus-type methods. The second example is the mixed moments of Wiener increments and stochastic double integrals.

3.1. Stability analysis for Magnus-type methods. Using a noncommutative test SDE, we analyse stability properties of Magnus-type methods. Then, a very careful treatment is necessary especially for the Magnus-type Milstein method. For this, first we derive polynomials that play an important role in the analysis.

3.1.1. Important polynomials in the analysis. As a new type of method for SDEs, Magnus-type methods [20] have been recently proposed for semilinear SDEs given by

$$\mathrm{d}\boldsymbol{y}(t) = \left\{F_0\boldsymbol{y}(t) + \boldsymbol{g}_0(\boldsymbol{y}(t))\right\}\mathrm{d}t + \sum_{j=1}^{m}\left\{F_j\boldsymbol{y}(t) + \boldsymbol{g}_j(\boldsymbol{y}(t))\right\}\mathrm{d}W_j(t), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{3.1}$$

where $t \in [0,T]$ and where $\boldsymbol{g}_j$, $j = 0,1,\ldots,m$ are $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$, the $W_j(t)$, $j = 1,2,\ldots,m$ are independent Wiener processes on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and $\boldsymbol{y}_0$ is independent of $W_j(t) - W_j(0)$. Especially here, note that F_j, $j = 0,1,\ldots,m$, are constant matrices and they are noncommutative.

Let $\boldsymbol{y}_n$ denote a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (3.1) for an equidistant grid point $t_n \stackrel{\text{def}}{=} nh$ $(n = 1,2,\ldots,M)$ with step size $h = T/M < 1$ (M is a positive integer). In addition, let us introduce the notations

$$\Omega^{[1]}(t_n,t_{n+1}) = \left(F_0 - \frac{1}{2}\sum_{j=1}^{m}F_j^2\right)h + \sum_{j=1}^{m}F_j\Delta W_j(t_n,t_{n+1}),$$
$$\Omega^{[2]}(t_n,t_{n+1}) = \Omega^{[1]}(t_n,t_{n+1}) + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=i+1}^{m}\left(F_iF_j - F_jF_i\right)\left(I_{j,i}(t_n,t_{n+1}) - I_{i,j}(t_n,t_{n+1})\right).$$

Then, the Magnus-type Euler and Milstein methods are respectively given as follows [20]:

$$\boldsymbol{y}_{n+1} = \exp\left(\Omega^{[1]}(t_n,t_{n+1})\right)\left\{\boldsymbol{y}_n + \tilde{\boldsymbol{g}}_0(\boldsymbol{y}_n)h + \sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{y}_n)\Delta W_j(t_n,t_{n+1})\right\}, \tag{3.2}$$

$$\begin{aligned}\boldsymbol{y}_{n+1} = \exp\left(\Omega^{[2]}(t_n,t_{n+1})\right)\Bigg\{&\boldsymbol{y}_n + \tilde{\boldsymbol{g}}_0(\boldsymbol{y}_n)h + \sum_{j=1}^{m}\boldsymbol{g}_j(\boldsymbol{y}_n)\Delta W_j(t_n,t_{n+1})\\ &+ \sum_{i,j=1}^{m}\boldsymbol{H}_{ij}(\boldsymbol{y}_n)I_{i,j}(t_n,t_{n+1})\Bigg\},\end{aligned} \tag{3.3}$$

where $\tilde{\boldsymbol{g}}_0(\boldsymbol{y}) = \boldsymbol{g}_0(\boldsymbol{y}) - \sum_{j=1}^{m}F_j\boldsymbol{g}_j(\boldsymbol{y})$ and $\boldsymbol{H}_{i,j}(\boldsymbol{y}) = \boldsymbol{g}_i'(\boldsymbol{y})\bigl(F_j\boldsymbol{y} + \boldsymbol{g}_j(\boldsymbol{y})\bigr) - F_j\boldsymbol{g}_j(\boldsymbol{y})$.

For the linear stability analysis of our methods, suppose that $\boldsymbol{g}_j(\boldsymbol{y}) = \boldsymbol{0}$, $j = 0,1,\ldots,m$, in (3.1). If we set

$$F_0 = \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}, \qquad F_1 = \begin{bmatrix}\sigma_1 & 0\\ 0 & -\sigma_1\end{bmatrix}, \qquad F_2 = \begin{bmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{bmatrix} \tag{3.4}$$

for $d = m = 2$ and nonzero real numbers λ, σ_1 and σ_2, then we have the noncommutative test SDE that Buckwar and Sickenberger [3] proposed:

$$\mathrm{d}\boldsymbol{y}(t) = F_0\boldsymbol{y}(t)\mathrm{d}t + F_1\boldsymbol{y}(t)\mathrm{d}W_1(t) + F_2\boldsymbol{y}(t)\mathrm{d}W_2(t), \quad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{3.5}$$

in which the zero solution is asymptotically mean square (MS) stable if and only if

$$2\lambda + \sigma_1^2 + \sigma_2^2 < 0. \tag{3.6}$$

We use the test SDE in our analysis.

In the rest of this subsection, for simplicity we will use the notations $\Delta W_j = \Delta W_j(t_n, t_{n+1})$ and $A_{i,j} = A_{i,j}(t_n, t_{n+1}) = I_{i,j}(t_n, t_{n+1}) - I_{j,i}(t_n, t_{n+1})$ without indicating the dependence of t_n and t_{n+1}, if it is obvious from the context. From (3.4),

$$\Omega^{[1]}(t_n, t_{n+1}) = \tilde{F}_0 h + F_1\Delta W_1 + F_2\Delta W_2, \qquad \Omega^{[2]}(t_n, t_{n+1}) = \Omega^{[1]}(t_n, t_{n+1}) + GA_{1,2},$$

where

$$\tilde{F}_0 = \begin{bmatrix} \lambda - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) & 0 \\ 0 & \lambda - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & -\sigma_1\sigma_2 \\ \sigma_1\sigma_2 & 0 \end{bmatrix}.$$

Thus, (3.2) and (3.3) give the amplification factors

$$\begin{aligned} R_E &= \exp(\tilde{F}_0 h + F_1\Delta W_1 + F_2\Delta W_2), \\ R_M &= \exp(\tilde{F}_0 h + F_1\Delta W_1 + F_2\Delta W_2 + GA_{1,2}), \end{aligned}$$

respectively, as the nonlinear functions $\boldsymbol{g}_j(\boldsymbol{y})$, $j = 0, 1, \ldots, m$, vanish.

First, let us start with the linear stability analysis for the Magnus-type Euler method. We have the following lemma.

LEMMA 3.1. *When* (3.2) *is applied to* (3.5)*, the stability matrix is expressed as*

$$E\left[R_E^\top R_E\right] = \mathrm{e}^{2p-q_1-q_2}\left(\sum_{n=0}^{\infty}\varphi_E(n)\right) I_d, \tag{3.7}$$

where $p = \lambda h$, $q_i = \sigma_i^2 h$ $(i = 1, 2)$ *and* I_d *stands for the identity matrix, and where*

$$\varphi_E(n) = 2^n \sum_{k=0}^{n} \frac{{}_nC_k}{{}_{2n}C_{2k}} \frac{q_1^{n-k} q_2^k}{(n-k)!(k!)}. \tag{3.8}$$

Proof. Noting that $\tilde{F}_0^\top = \tilde{F}_0$, $F_1^\top = F_1$ and $F_2^\top = F_2$ as well as $\tilde{F}_0 h$ and $F_1\Delta W_1 + F_2\Delta W_2$ are commutative, we have

$$R_E^\top R_E = \exp(2\tilde{F}_0 h)\exp(2(F_1\Delta W_1 + F_2\Delta W_2)).$$

As $(2(F_1\Delta W_1 + F_2\Delta W_2))^{2n} = 2^{2n}(\sigma_1^2\Delta W_1^2 + \sigma_2^2\Delta W_2^2)^n I_d$ for any nonnegative integer n, we obtain

$$E\left[(2(F_1\Delta W_1 + F_2\Delta W_2))^{2n}\right] = 2^{2n} E\left[(\sigma_1^2\Delta W_1^2 + \sigma_2^2\Delta W_2^2)^n\right] I_d,$$

whereas $E[(2(F_1\Delta W_1 + F_2\Delta W_2))^{2n+1}] = O$. Here,

$$\begin{aligned} E\left[(\sigma_1^2\Delta W_1^2 + \sigma_2^2\Delta W_2^2)^n\right] &= \sum_{k=0}^{n} {}_nC_{n-k}\sigma_1^{2(n-k)} E\left[\Delta W_1^{2(n-k)}\right] \sigma_2^{2k} E\left[\Delta W_2^{2k}\right] \\ &= \frac{n!}{2^n} \sum_{k=0}^{n} {}_{2(n-k)}C_{n-k}\, q_1^{n-k} ({}_{2k}C_k) q_2^k. \end{aligned}$$

From these,

$$E\left[\exp(2(F_1\Delta W_1+F_2\Delta W_2))\right]=\sum_{n=0}^{\infty}\frac{1}{(2n)!}E\left[(2(F_1\Delta W_1+F_2\Delta W_2))^{2n}\right]=\varphi_E(n).$$

∎

Let us define A-stability of numerical methods for the noncommutative SDE (3.5) with (3.4) and nonzero $\lambda,\sigma_1,\sigma_2$.

DEFINITION 3.2. *A numerical method is said to be A-stable in the MS for* (3.5) *if* $E[\boldsymbol{y}_n^\top\boldsymbol{y}_n]\to 0$ *as* $n\to\infty$ *for any step size* h *whenever the nonzero parameters* $\lambda,\sigma_1,\sigma_2$ *satisfy* (3.6) *and* $E[\boldsymbol{y}_0^\top\boldsymbol{y}_0]<\infty$, *where* $\boldsymbol{y}_n$ *is a numerical solution given by the method when applied to* (3.5).

As (3.8) is a symmetric polynomial in q_1,q_2, let us suppose that $q_1\geq q_2>0$ without loss of generality and denote q_2/q_1 by x. Then, (3.8) is expressed as

$$\varphi_E(n)=(2q_1)^n\sum_{k=0}^{n}\frac{{}_nC_k}{{}_{2n}C_{2k}}\frac{x^k}{(n-k)!(k!)},$$

whereas (3.6) is expressed as $2p+q_1(1+x)<0$. Regarding the stability of (3.2), we have the following theorem.

THEOREM 3.3. *The Magnus-type Euler method is A-stable in the MS for* (3.5).

Proof. As $\Delta W_1(t_n,t_{n+1})$ and $\Delta W_2(t_n,t_{n+1})$ are independent of $\mathcal{F}_{t_n}$, we have

$$E[\boldsymbol{y}_{n+1}^\top\boldsymbol{y}_{n+1}]=E[\boldsymbol{y}_n^\top E[R_E^\top R_E\mid\mathcal{F}_{t_n}]\boldsymbol{y}_n]=E[\boldsymbol{y}_n^\top E[R_E^\top R_E]\boldsymbol{y}_n].$$

From this and (3.7),

$$E[\boldsymbol{y}_{n+1}^\top\boldsymbol{y}_{n+1}]=\mathrm{e}^{2p-q_1(1+x)}\left(\sum_{n=0}^{\infty}\varphi_E(n)\right)E[\boldsymbol{y}_n^\top\boldsymbol{y}_n]. \tag{3.9}$$

Here, this equation is admissible for any p,q_1,x since $\mathrm{e}^{2p-q_1(1+x)}(\sum_{n=0}^{\infty}\varphi_E(n))\geq 0$ holds for them clearly. As ${}_nC_k\leq{}_{2n}C_{2k}$, $\varphi_E(n)$ is bounded by $(2q_1)^nV_n(x)$, where

$$V_n(x)=\sum_{l=0}^{n}b(n,l)x^l,\qquad b(n,l)=\frac{1}{(n-l)!(l!)}.$$

Thus, since

$$\sum_{n=0}^{\infty}\varphi_E(n)\leq\sum_{n=0}^{\infty}(2q_1)^nV_n(x)=\sum_{n=0}^{\infty}\frac{(2q_1)^n}{n!}(1+x)^n=\mathrm{e}^{2q_1(1+x)},$$

we have

$$\mathrm{e}^{2p-q_1(1+x)}\left(\sum_{n=0}^{\infty}\varphi_E(n)\right)\leq\mathrm{e}^{2p+q_1(1+x)}<1$$

for any $h>0$ in (3.7) due to $2p+q_1(1+x)<0$ from (3.6). ∎

Remark 3.4. The authors in [3] have chosen another approach on their stability analysis. They use the following form:

$$E\left[\mathrm{vec}\left(\boldsymbol{y}_{n+1}\boldsymbol{y}_{n+1}^\top\right)\right]=E\left[R_E\otimes R_E\right]E\left[\mathrm{vec}\left(\boldsymbol{y}_n\boldsymbol{y}_n^\top\right)\right],$$

where vec(A) denotes a vectorisation of a matrix A and $A \otimes B$ denotes the Kronecker product of matrices A and B, and investigate the spectral radius of $E[R_E \otimes R_E]$. On the other hand, in our proof we have avoided dealing with $E[R_E \otimes R_E]$, and have mentioned that (3.9) is admissible.

Remark 3.5. In the theorem, we have dealt with $x \in (0, 1]$ for noncommutative noise. If $x = 0$, then (3.5) reduces to an SDE with one scalar noise. The Magnus-type Euler method is clearly A-stable also for the SDE with one scalar noise.

Next, let us analyse stability properties for the Magnus-type Milstein method. We have the following lemma.

LEMMA 3.6. *When* (3.3) *is applied to* (3.5), *the stability matrix is expressed as*

$$E\left[R_M^\top R_M\right] = \mathrm{e}^{2p-q_1-q_2}\left(1 + \sum_{n=1}^{\infty} \varphi_M(n)\right) I_d,$$

where

$$\varphi_M(n) = \sum_{k=0}^{\tilde{n}} (-1)^k \frac{2^{2(n-k)}}{(2(n-k))!} \frac{n-2k}{n-k} {}_{n-k}C_k \sum_{l=k}^{n-k} {}_{n-2k}C_{l-k}\, q_1^{n-l} q_2^l \gamma_{n-k-l,k,l-k} \tag{3.10}$$

for which $\tilde{n}$ *is given by*

$$\tilde{n} = \begin{cases} r-1 & (n = 2r), \\ r & (n = 2r+1). \end{cases} \tag{3.11}$$

Here, r *stands for an integer.*

For the proof of this lemma, we refer the reader to Appendix B.

Due to $\gamma_{n-k-l,k,l-k} = \gamma_{l-k,k,n-k-l}$, (3.10) is a symmetric polynomial in q_1, q_2. By using (2.4) and (2.17), we can rewrite (3.10) as $\varphi_M(n) = (2q_1)^n U_n(x)$, where

$$U_n(x) = \sum_{k=0}^{\tilde{n}} (-1)^k \frac{n-2k}{n-k} \frac{{}_{n-k}C_k}{{}_{2(n-k)}C_{2k}} \frac{s_{n-2k,k}}{(n-2k)!} \sum_{l=k}^{n-k} \frac{({}_{n-2k}C_{l-k})^2}{{}_{2(n-2k)}C_{2(l-k)}} x^l.$$

In a similar way to (3.9), we have

$$E[\boldsymbol{y}_{n+1}^\top \boldsymbol{y}_{n+1}] = \mathrm{e}^{2p-q_1(1+x)} \left(1 + \sum_{n=1}^{\infty} (2q_1)^n U_n(x)\right) E[\boldsymbol{y}_n^\top \boldsymbol{y}_n]. \tag{3.12}$$

Remark 3.7. For a fixed l, the monomials with respect to x^l and x^{n-l} appear only for $k = 0, 1, \ldots, \min(l, \tilde{n})$ in $U_n(x)$. Noting this, we can express $U_n(x)$ in a usual form with coefficients $a(n,l)$, $l = 0, 1, \ldots, n$: $U_n(x) = \sum_{l=0}^{n} a(n,l) x^l$, where

$$a(n,l) = \sum_{k=0}^{l} \delta_{n,k,l} \quad (l = 0, 1, \ldots, \tilde{n}), \quad a(n,l) = \sum_{k=0}^{n-l} \delta_{n,k,l} \quad (l = \tilde{n}+2, \tilde{n}+3, \ldots, n),$$

$$a(n, \tilde{n}+1) = \sum_{k=0}^{\tilde{n}} \delta_{n,k,\tilde{n}+1}, \quad \delta_{n,k,l} = (-1)^k \frac{n-2k}{n-k} \frac{{}_{n-k}C_k}{{}_{2(n-k)}C_{2k}} \frac{s_{n-2k,k}}{(n-2k)!} \frac{({}_{n-2k}C_{l-k})^2}{{}_{2(n-2k)}C_{2(l-k)}}.$$

If the coefficients are given, the usual form with Horner's rule is helpful for the faster calculation of the value of $U_n(x)$ at each given x.

For the stability analysis of (3.3) when it is applied to (3.5), we investigate $U_n(x)$ in detail.

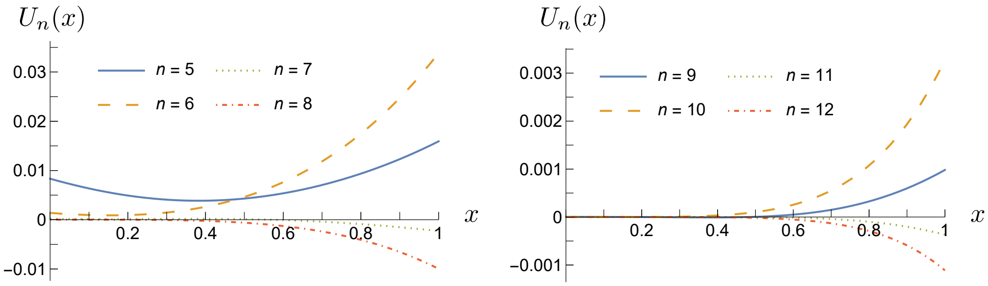

FIG. 1. *Plots of* $U_n(x)$ *for* $n = 5, 6, \ldots, 12$

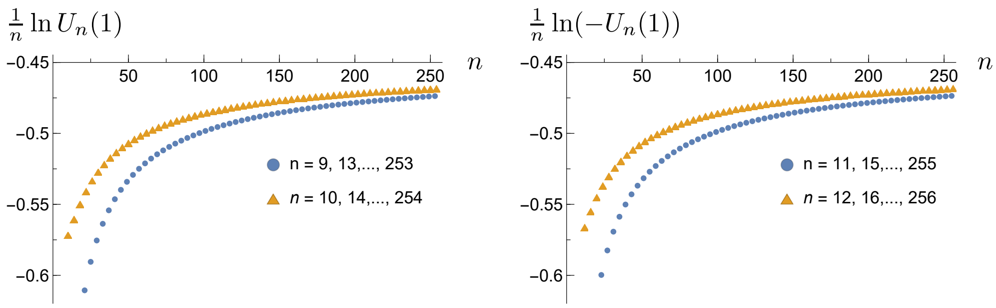

FIG. 2. $\frac{1}{n} \ln |U_n(1)|$ *versus* n

3.1.2. Analysis for the Magnus-type Milstein method. Let us give the first four examples of $U_n(x)$

$$U_1(x) = 1 + x, \qquad U_2(x) = \frac{1}{2} + \frac{1}{3}x + \frac{1}{2}x^2, \qquad U_3(x) = \frac{1}{6} - \frac{17}{180}x - \frac{17}{180}x^2 + \frac{1}{6}x^3,$$
$$U_4(x) = \frac{1}{24} - \frac{17}{315}x - \frac{23}{756}x^2 - \frac{17}{315}x^3 + \frac{1}{24}x^4$$

and show others in Figure 1. In addition, let us investigate $(1/n) \ln |U_n(1)|$ separately for $n = 4k + i$, where $i = 1, 2, 3, 4$ and $k = 2, 3, \ldots, 63$, and show them in Figure 2. More concretely, for example, $(1/n) \ln(U_n(1)) = -0.473732$ and $(1/n) \ln(-U_n(1)) = -0.468831$ for $n = 253, 256$, respectively. These lead to $U_{253}(1) = (0.622674)^{253}$ and $U_{256}(1) = -(0.625734)^{256}$ as $\exp(-0.473732) = 0.622674$ and $\exp(-0.468831) = 0.625734$. Taking these into account, when $x = 1$, we can see that the series in (3.12) does not even converge for large q_1 such as $q_1 = 1$. That is, (3.12) is inadmissible for large q_1 when $x = 1$.

Remark 3.8. If $x = 0$, then (3.5) reduces to an SDE with one scalar noise. The Magnus-type Milstein method is clearly A-stable for the SDE with one scalar noise.

Now, let us plot stability regions for the Magnus-type Milstein method by

$$0 \leq \mathrm{e}^{2p - q_1(1+x)} \left(1 + \sum_{n=1}^{\infty} (2q_1)^n U_n(x) \right) < 1. \tag{3.13}$$

We utilize a truncated expansion for the series in the inequality. Figure 3 indicates the stability regions by colored part, when $U_n(x)$, $n = 1, 2, \ldots, 256$, are used with $x = 1/64, 1$. The other parts enclosed by the mesh indicate the region that satisfies $2p + q_1(1 + x) < 0$ for $x = 1/64, 1$.

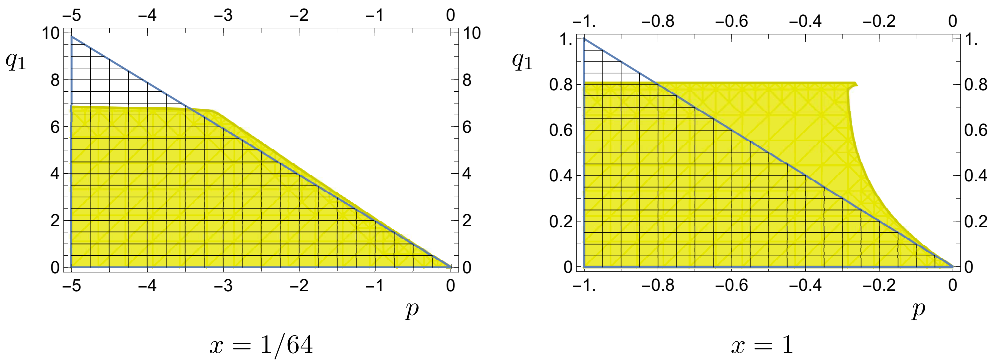

FIG. 3. *Stability regions for the Magnus-type Milstein method when $x = 1/64, 1$*

TABLE 1
Logarithm of the error of $\boldsymbol{y}_n$ at $t_n = 1$ to base 2

$\log_2 h$	-1	-2	-3	-4	-5	-6
Magnus-type Euler	-7.49	-7.91	-8.42	-8.94	-9.39	-9.88
Magnus-type Milstein	-8.80	-9.78	-10.7	-11.7	-12.7	-13.6

Remark 3.9. The results obtained so far allow us to infer the following: there might exist a positive real number q_0 such that (3.13) holds for any $x \in [0, 1]$ and $0 < q_1 \leq q_0$ under the condition $2p + q_1(1 + x) < 0$, but q_0 would not be very large.

If q_1 is not very large, it is not difficult to confirm our theoretical results numerically (see also https://github.com/yosh-komori/mag-simul). In what follows, let us use a fixed initial condition $\boldsymbol{y}(0) = [1\ 1]^\top$. When the Magnus-type methods are applied to (3.5) with $\lambda = -1/4, \sigma_1 = 1/2, \sigma_2 = 2/5$ and 1000 independent trajectories are simulated for a given h, Table 1 shows errors $\langle \|\boldsymbol{y}_n - \boldsymbol{y}_n^{\text{ref}}\|^2 \rangle$ at $t_n = 1$, which stands for the arithmetic mean of $\|\boldsymbol{y}_n - \boldsymbol{y}_n^{\text{ref}}\|^2$, and where $\boldsymbol{y}_n^{\text{ref}}$ is a reference solution computed by the Milstein method with $h = 2^{-9}$. From the table, we can see that the methods achieve the theoretical order of convergence. When the Magnus-type Milstein method with $h = 2^{-1}$ is applied to (3.5) with $\lambda = -1/5, -2/5, -4/5, -6/5, \sigma_1 = 1$ and $\sigma_2 = 1$, and 10 batches of the 10^5 independent trajectories are simulated for each value of λ, Table 2 shows the arithmetic mean and standard deviation (SD) of $\|\boldsymbol{y}_n\|^2$ at $t_n = 5$. From the table, we can see that the method numerically reproduces the stability properties indicated by the right-hand plot in Figure 3.

On the other hand, as q_1 becomes larger, situations change. When we replace $\sigma_1 = \sigma_2 = 1$ with $\sigma_1 = \sigma_2 = 3$ and set $\lambda = -0.01$ in the last case, the setting makes numerical solutions of the method unstable, but the numerical value of the second moment by the method with $h = 2^{-1}$ is 3.17×10^{-14} (9.32×10^{-14}), where the value in parentheses indicates SD. It is much smaller than the values in Table 2. One of the reasons for this phenomena can be considered as follows. The MS instability of the method is determined by rare exploding trajectories. The standard Monte Carlo approach is bound to miss those rare events [1], and $I_{i,j}(t_n, t_{n+1})$ is approximated [19]. The authors in [1] have tackled such a problem in a multi-dimensional linear SDE with diagonal diffusion matrices, and they have pointed out that noncommutative cases are more challenging. We leave this problem as a future work.

3.2. Mixed moments of Wiener increments and stochastic double integrals. In subsection 2.1, we mentioned that mixed moments of $\Delta W_1(t, t+1), \Delta W_2(t, t+$

Table 2
Arithmetic mean and standard deviation of $\|\boldsymbol{y}_n\|^2$ at $t_n = 5$ by the Magnus-type Milstein method when $q_1 = 1/2$ and $x = 1$

p	-0.1	-0.2	-0.4	-0.6
$\langle \|\boldsymbol{y}_n\|^2 \rangle$	5.17×10	6.99	1.28×10^{-1}	2.35×10^{-3}
SD	2.64×10	3.57	6.55×10^{-2}	1.20×10^{-3}

1), $I_{1,2}(t,t+1)$ and $I_{2,1}(t,t+1)$ are obtained from the properties of stochastic integrals, but it is not easy to seek them when an exponent is large number. As another application of our main theorem, we give such mixed moments. In the rest of this subsection, for simplicity we will use the notations $\Delta W_j = \Delta W_j(t,t+1)$, $A_{i,j} = A_{i,j}(t,t+1)$ and $I_{i,j} = I_{i,j}(t,t+1)$.

Noting $A_{2,1} = -A_{1,2}$, we have $I_{1,2} + I_{2,1} = \Delta W_1 \Delta W_2$. This and $I_{1,2} - I_{2,1} = A_{1,2}$ give

$$A_{1,2}^2 = (\Delta W_1 \Delta W_2)^2 - 4 I_{1,2} I_{2,1}.$$

Thus, we have $E\left[\Delta W_1^2 A_{1,2}^2\right] = E\left[\Delta W_1^4 \Delta W_2^2\right] - 4E\left[\Delta W_1^2 I_{1,2} I_{2,1}\right]$. As our main theorem gives

$$E\left[\Delta W_1^2 A_{1,2}^2\right] = \gamma_{1,1,0} = \frac{5}{3},$$

we have $E\left[\Delta W_1^2 I_{1,2} I_{2,1}\right] = \frac{1}{3}$. Note that this expectation is replaced with $h^3/3$ if we change $t+1$ to $t+h$ in $\Delta W_1, I_{1,2}$ and $I_{2,1}$ [8, p. 225].

Similarly, we have

$$\begin{aligned} E\left[\Delta W_1^4 A_{1,2}^2\right] &= E\left[\Delta W_1^6 \Delta W_2^2\right] - 4E\left[\Delta W_1^4 I_{1,2} I_{2,1}\right], \\ E\left[\Delta W_1^2 A_{1,2}^2 \Delta W_2^2\right] &= E\left[\Delta W_1^4 \Delta W_2^4\right] - 4E\left[\Delta W_1^2 \Delta W_2^2 I_{1,2} I_{2,1}\right]. \end{aligned}$$

On the other hand, our main theorem and (2.4) give

$$E\left[\Delta W_1^4 A_{1,2}^2\right] = \gamma_{2,1,0} = 7, \qquad E\left[\Delta W_1^2 A_{1,2}^2 \Delta W_2^2\right] = \gamma_{1,1,1} = \frac{{}_2C_1}{{}_4C_2} \gamma_{2,1,0} = \frac{7}{3}.$$

Thus, we have $E\left[\Delta W_1^4 I_{1,2} I_{2,1}\right] = 2$ and $\left[\Delta W_1^2 \Delta W_2^2 I_{1,2} I_{2,1}\right] = 5/3$. In a similar way, we can easily obtain the following mixed moments and others.

$$\begin{aligned} &E\left[\Delta W_1^6 I_{1,2} I_{2,1}\right] = 15, \quad E\left[\Delta W_1^4 \Delta W_2^2 I_{1,2} I_{2,1}\right] = 9, \quad E\left[\Delta W_1^2 I_{1,2}^2 I_{2,1}^2\right] = \frac{49}{20}, \\ &E\left[\Delta W_1^8 I_{1,2} I_{2,1}\right] = 140, \quad E\left[\Delta W_1^6 \Delta W_2^2 I_{1,2} I_{2,1}\right] = 65, \quad E\left[\Delta W_1^4 \Delta W_2^4 I_{1,2} I_{2,1}\right] = 48, \\ &E\left[\Delta W_1^4 I_{1,2}^2 I_{2,1}^2\right] = \frac{339}{20}, \quad E\left[\Delta W_1^2 \Delta W_2^2 I_{1,2}^2 I_{2,1}^2\right] = \frac{679}{60}. \end{aligned}$$

4. Concluding remarks. We have derived a way of obtaining the expectation of monomial $\gamma_{n,k,l} = E\left[\{\Delta W_1(t,t+1)\}^{2n} \{A_{1,2}(t,t+1)\}^{2k} \{\Delta W_2(t,t+1)\}^{2l}\right]$ for non-negative integers n, k and l. Due to (2.4), it is essential to obtain $\gamma_{n,k,0}$. We have derived the three types of formulae which give it: a recursive formula, an explicit formula, and a simpler formula that includes the mixed moment generating function of $\{\Delta W_1(t,t+1)\}^2$ and $\{A_{1,2}(t,t+1)\}^2$.

If a numerical method contains $I_{i,j}(t_n,t_{n+1}) = (1/2)\Delta W_i(t_n,t_{n+1})\Delta W_j(t_n,t_{n+1}) + (1/2)A_{i,j}(t_n,t_{n+1})$ and if we use the Taylor expansion of a function $f(y_{n+1})$ centered at y_n for the analysis of the method, then we often need to calculate $\gamma_{n,k,l}$. One of such methods is the Magnus-type Milstein method. Utilizing the formulae, we have analysed the stability properties of the method when it is applied to the noncommutative test SDE (3.5). As a result, our analysis has shown that the method cannot achieve A-stability for the test SDE.

Finally, we make the following remarks. If $\sigma_1 h$ or $\sigma_2 h$ is not very large for the parameters σ_1, σ_2 in (3.5), it is not difficult to confirm our theoretical results numerically. On the other hand, if they are large, situations change. Due to the term $\exp\left(\Omega^{[2]}(t_n, t_{n+1})\right)$ in the method, the MS instability of the method is determined by rare exploding trajectories. The standard Monte Carlo approach is bound to miss those rare events. This is another challenging issue in noncommutative cases. Thus, we will consider this issue in future work.

Appendix A. Proof of Lemma 2.5.

When $L = 0$, it is trivial that the lemma holds. As $\hat{s}_{n,L}(k) = \hat{s}_{n,k}(k)$ for $L > k$, we can suppose that $L \leq k$ without loss of generality. Then, similarly to a part of the proof of Lemma 2.4,

$$\begin{aligned}
&\sum_{j=1}^{L} \beta_{n,j} \hat{s}_{n,L}(k-j) \\
&= \sum_{\hat{l}_1 + 2l_2 + \cdots + Ll_L = k} \left\{ \hat{l}_1 \frac{\beta_{n,1}^{\hat{l}_1}}{\hat{l}_1!} \prod_{\substack{j=1 \\ j \neq 1}}^{L} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\} + \sum_{l_1 + 2\hat{l}_2 + \cdots + Ll_L = k} \left\{ 2\hat{l}_2 \frac{\beta_{n,2}^{\hat{l}_2}}{2^{\hat{l}_2}(\hat{l}_2!)} \prod_{\substack{j=1 \\ j \neq 2}}^{L} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\} \\
&\quad + \cdots + \sum_{l_1 + 2l_2 + \cdots + (L-1)l_{L-1} + L\hat{l}_L = k} \left\{ L\hat{l}_L \frac{\beta_{n,L}^{\hat{l}_L}}{L^{\hat{l}_L}(\hat{l}_L!)} \prod_{\substack{j=1 \\ j \neq L}}^{L} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\} \\
&= \sum_{l_1 + 2l_2 + \cdots + Ll_L = k} \left\{ (l_1 + 2l_2 + \cdots + Ll_L) \prod_{j=1}^{L} \frac{\beta_{n,j}^{l_j}}{j^{l_j}(l_j!)} \right\} = k \hat{s}_{n,L}(k),
\end{aligned}$$

where $\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_L$ denote positive integers. This completes the proof.

Appendix B. Proof of Lemma 3.6.

Noting that $\tilde{F}_0^\top = \tilde{F}_0, F_1^\top = F_1, F_2^\top = F_2$ and $G^\top = -G$ as well as $\tilde{F}_0 h$ and $F_1 \Delta W_1 + F_2 \Delta W_2 + G A_{1,2}$ are commutative, we have

$$R_M^\top R_M = \exp(2\tilde{F}_0 h) \mathrm{e}^P \mathrm{e}^Q,$$

where $P = F_1 \Delta W_1 + F_2 \Delta W_2 - G A_{1,2}$ and $Q = F_1 \Delta W_1 + F_2 \Delta W_2 + G A_{1,2}$. Although P and Q are noncommutative, they satisfy the equations

$$\begin{aligned}
&PQ = (\sigma_1^2 \Delta W_1^2 + \sigma_2^2 \Delta W_2^2 + (\sigma_1 \sigma_2)^2 A_{1,2}^2) I_d + 2F_1 G A_{1,2} \Delta W_1 + 2F_2 G A_{1,2} \Delta W_2. \\
&P^{2k} = Q^{2k} = (\sigma_1^2 \Delta W_1^2 + \sigma_2^2 \Delta W_2^2 - (\sigma_1 \sigma_2)^2 A_{1,2}^2)^k I_d
\end{aligned}$$

for any positive integer k. From these and the properties of the random variables, we have

$$\begin{aligned}
&E[PQ] = E\left[(\sigma_1^2 \Delta W_1^2 + \sigma_2^2 \Delta W_2^2 + (\sigma_1 \sigma_2)^2 A_{1,2}^2)\right] I_d = E\left[P^2 + 2(\sigma_1 \sigma_2)^2 A_{1,2}^2 I_d\right], \\
&E\left[P^{2k}\right] = E\left[Q^{2k}\right] = E\left[(\sigma_1^2 \Delta W_1^2 + \sigma_2^2 \Delta W_2^2 - (\sigma_1 \sigma_2)^2 A_{1,2}^2)^k\right] I_d, \\
&E\left[P^{2k+1}\right] = E\left[Q^{2k+1}\right] = O, \qquad E\left[P^k Q^l\right] = O \quad (\text{if } k+l \text{ is an odd number}), \\
&E\left[P^{2k} PQ Q^{2l}\right] = E\left[P^{2k} PQ P^{2l}\right] = E\left[P^{2(k+l+1)} + 2(\sigma_1 \sigma_2)^2 A_{1,2}^2 P^{2(k+l)}\right].
\end{aligned}$$

The application of these relationships to

$$\begin{aligned}
&E\left[\mathrm{e}^P\mathrm{e}^Q\right] \\
&= E\left[I_d + \frac{1}{2!}(P^2+2PQ+Q^2) + \frac{1}{4!}(P^4+4P^3Q+6P^2Q^2+4PQ^3+Q^4)\right. \\
&\qquad \left. + \frac{1}{6!}(P^6+6P^5Q+15P^4Q^2+20P^3Q^3+15P^2Q^4+6PQ^5+Q^6)+\cdots\right]
\end{aligned}$$

leads to

$$\begin{aligned}
E\left[\mathrm{e}^P\mathrm{e}^Q\right] = E\Bigg[I_d &+ \frac{2^2}{2!}(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2)I_d + \frac{2^4}{4!}(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2)P^2 \\
&+ \frac{2^6}{6!}(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2)P^4+\cdots\Bigg].
\end{aligned}$$

The substitution of $P^2 = \left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2 - (\sigma_1\sigma)^2A_{1,2}^2\right)I_d$ into this and simplification yield

$$E\left[\mathrm{e}^P\mathrm{e}^Q\right] = E\left[1+\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}\sum_{k=0}^{n-1}{}_{n-1}C_k\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^{n-k}\left\{-(\sigma_1\sigma_2)^2A_{1,2}^2\right\}^k\right]I_d.$$

Noting that

$$E\left[\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^{n-k}\right] = O(h^{n-k}), \qquad E\left[\left\{-(\sigma_1\sigma_2)^2A_{1,2}^2\right\}^k\right] = O(h^{2k}),$$

we can see that the coefficient of $O(h^n)$ in $E\left[\mathrm{e}^P\mathrm{e}^Q\right]$, say $\varphi_M(n)$, is given by

$$\begin{aligned}
&E\left[\frac{2^{2n}}{(2n)!}{}_{n-1}C_0\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^n\right. \\
&\quad + \frac{2^{2(n-1)}}{(2(n-1))!}{}_{n-2}C_1\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^{n-2}\left\{-(\sigma_1\sigma_2)^2A_{1,2}^2\right\} \\
&\quad \left. +\cdots+\frac{2^{2(n-k)}}{(2(n-k))!}{}_{n-k-1}C_k\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^{n-2k}\left\{-(\sigma_1\sigma_2)^2A_{1,2}^2\right\}^k\right], \qquad \text{(B.1)}
\end{aligned}$$

where $n-2k\geq 1$. Here, since

$$\begin{aligned}
&\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^{n-2k}\left\{-(\sigma_1\sigma_2)^2A_{1,2}^2\right\}^k \\
&= (-1)^k\sum_{l=0}^{n-2k}{}_{n-2k}C_l\sigma_1^{2(n-k-l)}\sigma_2^{2(k+l)}\Delta W_1^{2(n-2k-l)}\Delta W_2^{2l}A_{1,2}^{2k}
\end{aligned}$$

and

$$E\left[\{\Delta W_1(t_n,t_{n+1})\}^{2(n-2k-l)}\{\Delta W_2(t_n,t_{n+1})\}^{2l}\{A_{1,2}(t_n,t_{n+1})\}^{2k}\right] = \gamma_{n-2k-l,k,l}h^n,$$

we have

$$\begin{aligned}
&\frac{2^{2(n-k)}}{(2(n-k))!}{}_{n-k-1}C_kE\left[\left(\sigma_1^2\Delta W_1^2+\sigma_2^2\Delta W_2^2\right)^{n-2k}\left\{-(\sigma_1\sigma_2)^2A_{1,2}^2\right\}^k\right] \\
&= (-1)^k\frac{2^{2(n-k)}}{(2(n-k))!}\frac{n-2k}{n-k}{}_{n-k}C_k\sum_{l=k}^{n-k}{}_{n-2k}C_{l-k}q_1^{n-l}q_2^l\gamma_{n-k-l,k,l-k}
\end{aligned}$$

in (B.1). Here, remember that $q_i = \sigma_i^2 h$ $(i = 1, 2)$. Thus, $\varphi_M(n)$ is rewritten as

$$\varphi_M(n) = \sum_{k=0}^{\tilde{n}} (-1)^k \frac{2^{2(n-k)}}{(2(n-k))!} \frac{n-2k}{n-k} {}_{n-k}C_k \sum_{l=k}^{n-k} {}_{n-2k}C_{l-k} q_1^{n-l} q_2^l \gamma_{n-k-l,k,l-k}.$$

As k must satisfy $n - 2k \geq 1$, $\tilde{n}$ is given by (3.11).

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