

THE NATURAL MATROID OF AN INTEGER POLYMATROID

JOSEPH E. BONIN, CAROLYN CHUN, AND TARA FIFE

ABSTRACT. The natural matroid of an integer polymatroid was introduced to show that a simple construction of integer polymatroids from matroids yields all integer polymatroids. As we illustrate, the natural matroid can shed much more light on integer polymatroids. We focus on characterizations of integer polymatroids using their bases, their circuits, and their cyclic flats along with the rank of each cyclic flat and each element; we offer some new characterizations and insights into known characterizations.

1. INTRODUCTION

A *polymatroid* is a pair $P = (E, \rho)$ where E is a finite set and the real-valued function $\rho : 2^E \rightarrow \mathbb{R}$, the *rank function* of P , has the following properties:

- (1) ρ is *normalized*, that is, $\rho(\emptyset) = 0$,
- (2) ρ is *non-decreasing*, that is, if $A \subseteq B \subseteq E$, then $\rho(A) \leq \rho(B)$, and
- (3) ρ is *submodular*, that is, $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \subseteq E$.

Less formally, we often talk about a polymatroid ρ on E . A *k-polymatroid*, where $k \in \mathbb{R}$ and $k > 0$, is a polymatroid (E, ρ) for which $\rho(e) \leq k$ for all $e \in E$. For much of this paper, we are concerned with *integer polymatroids* (also called *discrete polymatroids*), that is, polymatroids ρ where the rank $\rho(A)$ of each set A is in the set \mathbb{N} of nonnegative integers.

Intuitively, a matroid (an integer 1-polymatroid) can be thought of as a configuration of points, lines, planes, and so on, in which each of the elements that make up these objects has rank 0 (loops) or 1 (points). An integer polymatroid is the natural generalization in which the elements are not limited to points and loops; we also allow, as the elements, lines (elements of rank 2), planes (elements of rank 3), and so on. Not surprisingly, every integer polymatroid comes from a matroid, as the following result of [14, 18, 20] states.

Theorem 1.1. *A function $\rho : 2^E \rightarrow \mathbb{N}$ is an integer polymatroid if and only if there is a matroid M on a set E' and a function $\phi : E' \rightarrow 2^E$ with $\rho(A) = r_M(\bigcup_{e \in A} \phi(e))$ for all $A \subseteq E$.*

Helgason [14] introduced the natural matroid to prove this result. Geometrically, we get the natural matroid by, for each element e of E , replacing e by a set $\phi(e)$ of $\rho(e)$ points that are placed freely in e ; thus, a line is replaced by two points that are put freely on the line, and a plane by three points that are placed freely on the plane, and so on. (Section 2 has a precise definition of the natural matroid.) Many important properties of integer polymatroids are closely linked to properties of its natural matroid. For instance, Oxley, Semple, and Whittle [22] showed that an integer 2-polymatroid is 3-connected if and only if it has no loops and its natural matroid is 3-connected. We study the natural matroid in its own right.

In Section 2, we review the definition of the natural matroid and prove two results that make it easy to verify that a matroid M is the natural matroid of an integer polymatroid ρ .

We show that, for an integer polymatroid ρ on E that is the sum of the rank functions of matroids M_1, M_2, \dots, M_k on E , the natural matroid of ρ is the matroid union of certain extensions of M_1, M_2, \dots, M_k by loops and elements parallel to those in these matroids.

Herzog and Hibi [15] treat characterizations of integer polymatroids using bases and exchange properties. In Section 3, we show how these results follow easily by observing that the bases of an integer polymatroid are the type vectors of the bases of its natural matroid.

Viewing the bases of an integer polymatroid as the type vectors of the bases of its natural matroid suggests developing an analogous theory for circuits. We do this in Section 4, where, in Theorem 4.3, we introduce circuit axioms for integer polymatroids.

Cyclic flats of matroids, along with their ranks, provide relatively compact descriptions of matroids that allow one to focus on crucial features when, for instance, defining certain matroid constructions (e.g., see [1, 2, 7, 11]); this perspective is also useful in applications such as coding theory (e.g., see [13]). In Section 5, we show that some results about cyclic flats lift from the natural matroid of an integer polymatroid to the polymatroid. In the case of integer polymatroids, this gives another perspective on recent work by Csirmaz [10] characterizing all polymatroids via their cyclic flats and the ranks of these flats and of singleton sets. A key result behind this characterization is the formula that gives the rank function of a polymatroid ρ on E using only its values on cyclic flats and singleton sets, namely,

$$\rho(A) = \min\{\rho(X) + \sum_{i \in A-X} \rho(i) : X \in \mathcal{Z}_\rho\},$$

where \mathcal{Z}_ρ is the lattice of cyclic flats of ρ . For a subset A of E , we consider the set $\mathcal{R}_\rho(A)$ of cyclic flats X that yield this minimum. We show that $\mathcal{R}_\rho(A)$ is a sublattice of \mathcal{Z}_ρ , we identify its least and greatest elements, and we show that each pair of flats in $\mathcal{R}_\rho(A)$ is a modular pair.

Our matroid notation follows Oxley [21]. For a positive integer n , let $[n]$ be the set $\{1, 2, \dots, n\}$. We often take the ground set of an integer polymatroid ρ to be $[n]$ since this provides a natural correspondence between the elements of ρ and the entries in n -tuples. For $n \in \mathbb{N}$, the set of nonnegative integers, let $[n]_0$ be the set $\{0, 1, 2, \dots, n\}$.

For a polymatroid ρ on E and for $A \subseteq E$, the *deletion* $\rho_{\setminus A}$ and *contraction* $\rho_{/A}$, both on $E - A$, are defined by $\rho_{\setminus A}(X) = \rho(X)$ and $\rho_{/A}(X) = \rho(X \cup A) - \rho(A)$ for all $X \subseteq E - A$. The *minors* of ρ are the polymatroids of the form $(\rho_{\setminus A})_{/B}$ (equivalently, $(\rho_{/B})_{\setminus A}$) for disjoint subsets A and B of E . The *k-dual* ρ^* of a k -polymatroid ρ on E is the k -polymatroid that is given by $\rho^*(X) = k|X| - \rho(E) + \rho(E - X)$ for all $X \subseteq E$. The *direct sum* $\rho_1 \oplus \rho_2$ of polymatroids ρ_1 and ρ_2 on disjoint sets E_1 and E_2 is defined by $(\rho_1 \oplus \rho_2)(X) = \rho_1(X \cap E_1) + \rho_2(X \cap E_2)$ for $X \subseteq E_1 \cup E_2$. A polymatroid that is not a direct sum of two polymatroids on nonempty sets is *connected*.

2. THE NATURAL MATROID OF AN INTEGER POLYMATROID

The construction of the natural matroid uses Theorem 2.1 below, due to McDiarmid [20], which strengthens an earlier result of Edmonds and Rota. (Theorem 2.1 is treated in [21, 28].) Consider a collection L of subsets of a set E that includes E and \emptyset , and that is closed under intersection; thus, under inclusion, L is a lattice, and for $A, B \in L$, their meet $A \wedge B$ is $A \cap B$, but their join $A \vee B$ need not be $A \cup B$. For such a lattice L , a function $\sigma : L \rightarrow \mathbb{N}$ is *submodular* if $\sigma(A \vee B) + \sigma(A \cap B) \leq \sigma(A) + \sigma(B)$ for all $A, B \in L$.

Theorem 2.1. *Let L be a lattice of subsets of E that contains \emptyset and E , and is closed under intersection. Let $\sigma : L \rightarrow \mathbb{N}$ be submodular with $\sigma(\emptyset) = 0$. Define $r : 2^E \rightarrow \mathbb{N}$ by*

$$(2.1) \quad r(Y) = \min\{\sigma(S) + |Y - S| : S \in L\},$$

for $Y \subseteq E$. The function r is the rank function of a matroid on E ; its independent sets are the subsets I of E for which $|I \cap S| \leq \sigma(S)$ for all $S \in L$.

Given an integer polymatroid ρ on a set E , its natural matroid M_ρ is defined as follows. For each $i \in E$, let X_i be a set of $\rho(i)$ elements so that the sets X_i , for all $i \in E$, are pairwise disjoint. For $A \subseteq E$, set

$$X_A = \bigcup_{i \in A} X_i$$

and let $E' = X_E$. Let $L = \{X_A : A \subseteq E\}$. Now L is a lattice of subsets of E' with $\emptyset, E' \in L$, $X_A \vee X_B = X_A \cup X_B = X_{A \cup B}$, and $X_A \wedge X_B = X_A \cap X_B = X_{A \cap B}$. Define $\sigma : L \rightarrow \mathbb{N}$ by $\sigma(X_A) = \rho(A)$. Since ρ is submodular, so is σ . The *natural matroid* of ρ , denote M_ρ , is the matroid on E' whose rank function is given by Equation (2.1). The choice of the sets X_i is not unique, but the natural matroid is well-defined up to relabeling the elements in E' .

Corollary 2.2. *A subset I of E' is independent in M_ρ if and only if $|I \cap X_A| \leq \rho(A)$ for every $A \subseteq E$.*

Since ρ is submodular and non-decreasing, if $A, S \subseteq E$, then

$$\rho(A) \leq \rho(A \cap S) + \sum_{i \in A-S} \rho(i) \leq \rho(S) + \sum_{i \in A-S} \rho(i).$$

It follows that $r_{M_\rho}(X_A) = \rho(A)$ for all $A \subseteq E$. Theorem 1.1 follows by letting M be M_ρ and defining $\phi : E \rightarrow 2^{E'}$ by $\phi(i) = X_i$.

The next lemma simplifies proving that a matroid is the natural matroid of ρ . Recall that two elements a and b of a matroid M on E are *clones* if the permutation of E given by the 2-cycle (a, b) (i.e., switching a and b) is an automorphism of M . We say that $X \subseteq E$ is a *set of clones* if $a, b \in X$ are clones whenever $a \neq b$. A *cyclic* set of M is a set X that is a union of circuits, that is, $M|_X$ has no coloops. A *cyclic flat* is a flat that is cyclic. It is easy to prove that, for the set \mathcal{Z}_M of cyclic flats of M , we have, for $Y \subseteq E$,

$$(2.2) \quad r_M(Y) = \min\{r_M(Z) + |Y - Z| : Z \in \mathcal{Z}_M\}.$$

Lemma 2.3. *Let ρ , E , E' , X_i , and X_A be as above. A matroid M on E' is the natural matroid M_ρ of ρ if and only if $\mathcal{Z}_M \subseteq \{X_A : A \subseteq E\}$ and $r_M(X_A) = \rho(A)$ whenever $X_A \in \mathcal{Z}_M$.*

Proof. Above we showed that $r_{M_\rho}(X_A) = \rho(A)$ for all $A \subseteq E$. Also, X_i is a set of clones of M_ρ , so if C is a circuit of M_ρ and $a \in C \cap X_i$, then $(C - a) \cup b$, for each $b \in X_i - C$, is a circuit of M_ρ , and so $X_i \subseteq \text{cl}_{M_\rho}(C)$. Thus, $\mathcal{Z}_{M_\rho} \subseteq \{X_A : A \subseteq E\}$.

To prove the converse, assume that $\mathcal{Z}_M \subseteq \{X_A : A \subseteq E\}$ and $r_M(X_A) = \rho(A)$ whenever $X_A \in \mathcal{Z}_M$. By construction, the rank function of M_ρ is given by

$$\begin{aligned} r_{M_\rho}(Y) &= \min\{\rho(A) + |Y - X_A| : A \subseteq E\} \\ &= \min\{r_M(X_A) + |Y - X_A| : A \subseteq E\} \end{aligned}$$

for all $Y \subseteq E'$. Now

- if $Y, W \subseteq E'$, then $r_M(Y) \leq r_M(W) + |Y - W|$,

- for each Y , some cyclic flat W yields equality in that inequality, and
- $\mathcal{Z}_M \subseteq \{X_A : A \subseteq E\}$.

Thus, Equation (2.2) gives

$$r_M(Y) = \min\{r_M(X_A) + |Y - X_A| : A \subseteq E\}.$$

Thus, M and M_ρ have the same rank function and so are equal, as claimed. \square

The containment $\mathcal{Z}_{M_\rho} \subseteq \{X_A : A \subseteq E\}$ is proper since each set X_i is independent.

Two elements are clones in M if and only if they are in exactly the same cyclic flats of M , so we get the following corollary.

Corollary 2.4. *Let ρ , E , E' , X_i , and X_A be as above. A matroid M on E' is M_ρ if and only if each set X_i is a set of clones and $r_M(X_A) = \rho(A)$ for all $X_A \in \mathcal{Z}_M$.*

With Corollary 2.4, it follows that the natural matroid defined above is the same as that obtained by the construction of iterated principal extensions followed by deletion that is given in the proof of [21, Theorem 11.1.9], and which justifies the geometric view of the natural matroid that is mentioned after Theorem 1.1.

It follows easily from Corollary 2.4, or from the rank functions, that the operations of deletion and taking the natural matroid commute: if $i \in E$, then $M_{\rho_{\setminus i}} = M_\rho \setminus X_i$. The same is not true of contraction. For $i \in E$ and each $j \in E - i$, fix a subset Y_j of any $\rho(\{i, j\}) - \rho(i)$ elements of X_j , and let $E'_{/i}$ be the union of all such sets Y_j . It follows from Corollary 2.4 that $M_{\rho_{/i}} = M_\rho / X_i | E'_{/i}$. From Corollary 2.4, we also get $M_{\rho_1 \oplus \rho_2} = M_{\rho_1} \oplus M_{\rho_2}$ for integer polymatroids ρ_1 and ρ_2 ; so an integer polymatroid ρ on E with $|E| > 1$ is connected if and only if ρ has no loops and M_ρ is connected. The number of elements in the natural matroid is the sum of all terms $\rho(i)$ for $i \in E$, so, for a positive integer k , the natural matroid of the k -dual of an integer k -polymatroid ρ can have fewer, the same number of, or more elements compared to the natural matroid of ρ .

Theorem 2.1, which we used to construct the natural matroid, is the key to defining an important matroid operation, namely, matroid union (see [21, 28]). Let M_1, M_2, \dots, M_k be matroids on E . Their *matroid union*, denoted $M_1 \vee M_2 \vee \dots \vee M_k$, is the matroid on E having the rank function r' where, for $Y \subseteq E$,

$$r'(Y) = \min\{r_{M_1}(X) + r_{M_2}(X) + \dots + r_{M_k}(X) + |Y - X| : X \subseteq Y\}.$$

The independent sets of $M_1 \vee M_2 \vee \dots \vee M_k$ are the sets of the form $I_1 \cup I_2 \cup \dots \cup I_k$ where I_j is independent in M_j . The matroids M_1, M_2, \dots, M_k also give an integer polymatroid on E : the function ρ on 2^E where, for $X \subseteq E$,

$$\rho(X) = r_{M_1}(X) + r_{M_2}(X) + \dots + r_{M_k}(X),$$

is an integer k -polymatroid on E . We write this as $\rho = r_{M_1} + r_{M_2} + \dots + r_{M_k}$ for brevity. We call the multiset $\{M_1, M_2, \dots, M_k\}$ a *decomposition* of ρ and we say that ρ is *decomposable*. Not all integer polymatroids are decomposable. (See [3] for more on this topic.) The next theorem identifies the natural matroid of a decomposable integer polymatroid as a particular matroid union.

Theorem 2.5. *Let $\{M_1, M_2, \dots, M_k\}$ be a decomposition of an integer polymatroid ρ on E . Let the sets E' , X_i , and X_A be as above. For each $j \in [k]$, construct M'_j from M_j by, for each $i \in E$, adding the elements of X_i parallel to i , or as loops if $r_{M_j}(i) = 0$, and then deleting i . Then the natural matroid M_ρ is the matroid union $M'_1 \vee M'_2 \vee \dots \vee M'_k$.*

Proof. For $X, Y \subseteq E'$, note that $r_{M'_j}(Y \cap X) \leq r_{M'_j}(X)$ for each $j \in [k]$, and that $|Y - (Y \cap X)| = |Y - X|$. Given how M'_j is defined, if X_A is the union of all sets X_i such that $X_i \cap X \neq \emptyset$, then $r_{M'_j}(X_A) = r_{M'_j}(X)$, for each $j \in [k]$; also, $|Y - X_A| \leq |Y - X|$. It follows that the rank $r'(Y)$ of Y in $M'_1 \vee M'_2 \vee \cdots \vee M'_k$ is given by

$$r'(Y) = \min\{r_{M'_1}(X_A) + r_{M'_2}(X_A) + \cdots + r_{M'_k}(X_A) + |Y - X_A| : A \subseteq E'\}.$$

Since $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$, we get $r'(Y) = \min\{\rho(A) + |Y - X_A| : A \subseteq E'\}$, that is, $r'(Y) = r_{M_\rho}(Y)$. Thus, M_ρ is $M'_1 \vee M'_2 \vee \cdots \vee M'_k$. \square

An integer polymatroid and its natural matroid may have very different connections to important classes of matroids. For instance, for the binary integer polymatroid on the set of seven lines of the projective plane $PG(2, 2)$ using the construction in Theorem 1.1, the natural matroid is $U_{3,14}$, which is not binary. (The integer 2-polymatroids having natural matroids that are binary are characterized in [8].) In contrast, the next example and result give links between transversal, or Boolean, polymatroids and transversal matroids.

Example 1. If $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ where $r(M_h) \leq 1$ for all $h \in [k]$, then ρ is called a *Boolean* polymatroid. Helgason [14] introduced Boolean polymatroids, calling them *covering hypermatroids*. Some authors call them *transversal polymatroids* [9, 16, 27]. The class of Boolean polymatroids is closed under minors; Matuř [19] found their excluded minors. By Theorem 2.5 and the result that a matroid is transversal if and only if it is a matroid union of rank-1 matroids (see, e.g., [21, Proposition 11.3.7]), it follows that the natural matroid of a Boolean polymatroid is transversal.

Another way to see this is via graphs. A Boolean polymatroid $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ has the following reformulation using a bipartite graph G_ρ . Assume that $E \cap [k] = \emptyset$. The vertex set of G_ρ is $E \cup [k]$, and G_ρ has an edge eh if and only if $r_{M_h}(e) = 1$. The rank $\rho(A)$ of a set $A \subseteq E$ is the cardinality of the set $N(A)$ of neighbors of A . The natural matroid is the transversal matroid that is obtained from G_ρ by replacing each element $e \in E$ by $\rho(e)$ elements, each of which is adjacent to all neighbors of e . (See Figure 1.)

Loopless Boolean 2-polymatroids have received much attention, in part due to another connection with graphs. Given such a polymatroid $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$, the graph G has an edge $e \in E$ incident with a vertex $h \in [k]$ if and only if $r_{M_h}(e) = 1$. Then $\rho(A)$, for $A \subseteq E$, is the number of vertices that are incident with at least one edge in A . The natural matroid of ρ is the bicircular matroid of the graph G' that is obtained from G by putting a new edge parallel to each nonloop edge of G .

Using G_ρ , we see that an integer polymatroid (E, ρ) is Boolean if and only if, for some k , there is a map $N : E \rightarrow 2^{[k]}$ with $\rho(X) = |\bigcup_{x \in X} N(x)|$ for all $X \subseteq E$. (This is Helgason's definition in [14].) Given a set of subsets of $[k]$, there is an isomorphism from the lattice of all unions of those sets onto the lattice of cyclic flats of a transversal matroid so that the size of each union is the rank of its image. This gives the following variant of Theorem 1.1.

Theorem 2.6. *A polymatroid ρ on E is Boolean if and only if there is a transversal matroid M and map $\phi : E \rightarrow \mathcal{Z}_M$ with $\rho(X) = r_M(\bigcup_{i \in X} \phi(i))$ for all $X \subseteq E$.*

Most transversal matroids, such as $U_{2,3}$, are not Boolean polymatroids, so the codomain of the map $\phi : E \rightarrow \mathcal{Z}_M$ cannot be extended to the lattice of flats of M .

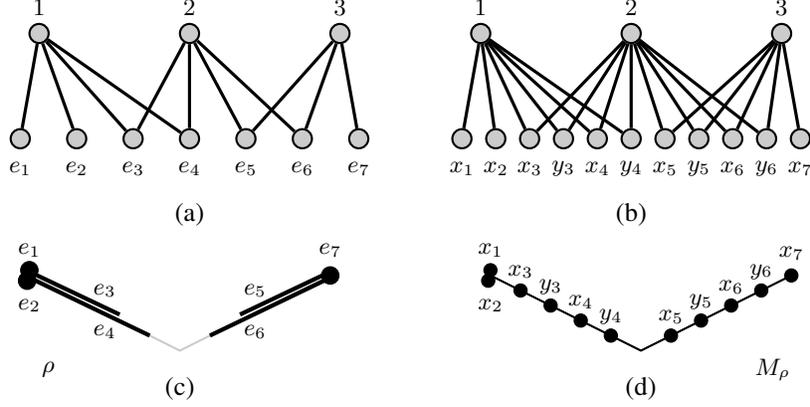


FIGURE 1. For the Boolean polymatroid ρ shown in part (c), part (a) shows its associated bipartite graph G_ρ , as in Example 1. Part (b) shows the bipartite graph that gives the transversal matroid that is the natural matroid M_ρ , which is shown in part (d).

3. BASES OF AN INTEGER POLYMATROID AND ITS NATURAL MATROID

Independent vectors and bases of integer polymatroids are discussed, for instance, by Herzog and Hibi in [15]. In this section, where we focus solely on integer polymatroids, we show how relating the bases of an integer polymatroid to the bases of its natural matroid makes transparent some characterizations of integer polymatroids that use bases.

A basis B of a matroid M on $E = [n]$ is a subset of $[n]$ and so can be represented by its characteristic vector \mathbf{b} , the n -tuple of 0s and 1s in which entry i , denoted b_i , is 1 if and only if $i \in B$. No basis contains a loop, so $b_i \leq r(i)$. Let \mathbf{e}_i be the characteristic vector of the singleton $\{i\}$. For the characteristic vector \mathbf{b} of a basis B and a basis $B' = (B - i) \cup j$ obtained by an exchange, the characteristic vector of B' is $\mathbf{b} - \mathbf{e}_i + \mathbf{e}_j$.

The *norm* of $\mathbf{v} \in \mathbb{N}^n$ is $|\mathbf{v}| = v_1 + v_2 + \cdots + v_n$. For \mathbf{u} and \mathbf{v} in \mathbb{N}^n , we write $\mathbf{u} \leq \mathbf{v}$ if $u_i \leq v_i$ for all $i \in [n]$; also, $\mathbf{u} < \mathbf{v}$ if $\mathbf{u} \leq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$. With this order, \mathbb{N}^n is a lattice; meet and join are given by component-wise min and max, respectively.

A definition of an integer polymatroid that is equivalent to the definition in Section 1 is that an integer polymatroid P is a nonempty finite subset \mathbf{I} of \mathbb{N}^n , for some n , for which

- (I1) if $\mathbf{v} \in \mathbf{I}$ and $\mathbf{u} \in \mathbb{N}^n$ with $\mathbf{u} \leq \mathbf{v}$, then $\mathbf{u} \in \mathbf{I}$, and
- (I2) if $\mathbf{u}, \mathbf{v} \in \mathbf{I}$ with $|\mathbf{u}| < |\mathbf{v}|$, then there is a \mathbf{w} in \mathbf{I} with $\mathbf{u} < \mathbf{w} \leq \mathbf{u} \vee \mathbf{v}$.

(To extend this to all polymatroids, replace \mathbb{N} by $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$ and require \mathbf{I} to be compact, rather than finite.) The vectors in \mathbf{I} are the *independent vectors* of P . A *basis* of P is a vector $\mathbf{v} \in \mathbf{I}$ for which there is no $\mathbf{u} \in \mathbf{I}$ with $\mathbf{v} < \mathbf{u}$. Property (I2) gives $|\mathbf{v}| = |\mathbf{u}|$ for all bases \mathbf{v} and \mathbf{u} of P .

We now relate this notion to the definition given in Section 1. Let $E = [n]$. For $\mathbf{v} \in \mathbb{N}^n$ and $X \subseteq E$, let $|\mathbf{v}|_X$ be $\sum_{i \in X} v_i$, the sum of the entries in \mathbf{v} that are indexed by the elements in X . The rank function $\rho : 2^E \rightarrow \mathbb{N}$ of an integer polymatroid P on E whose set of independent vectors is \mathbf{I} and whose set of bases is \mathbf{B} is given by

$$(3.1) \quad \rho(X) = \max\{|\mathbf{u}|_X : \mathbf{u} \in \mathbf{I}\} = \max\{|\mathbf{u}|_X : \mathbf{u} \in \mathbf{B}\}$$

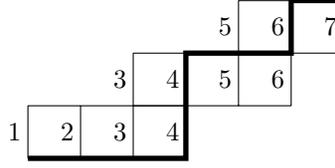


FIGURE 2. A lattice path diagram for the Boolean polymatroid in Figure 1. In the notation of Example 2, we have $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $b_1 = 4$, $b_2 = 6$, and $b_3 = 7$. The highlighted path corresponds to the basis $(0, 0, 0, 2, 0, 1, 0)$.

for $X \subseteq E$. The function ρ satisfies properties (1)–(3) in Section 1 and so is the rank function of an integer polymatroid. Conversely, given an integer polymatroid $\rho : 2^E \rightarrow \mathbb{N}$, the set

$$(3.2) \quad \mathbf{I} = \{\mathbf{u} \in \mathbb{N}^n : |\mathbf{u}|_X \leq \rho(X) \text{ for all } X \subseteq E\}$$

satisfies properties (I1) and (I2). (For a proof, see [28, p. 340, Lemma 5].) Also, the maps $\mathbf{I} \mapsto \rho$ and $\rho \mapsto \mathbf{I}$ are inverses of each other. (See [23, Corollaries 44.3f and 44.3g].)

Given an integer polymatroid ρ on $E = [n]$, let E' , X_i , and X_A , for $i \in E$ and $A \subseteq E$, and the natural matroid M_ρ be defined as above. The *type vector* of a subset V of E' is the vector $\mathbf{v} \in \mathbb{N}^n$ with $v_i = |V \cap X_i|$ for all $i \in E$. We use $\mathbf{T}(V)$ to denote the type vector of V . By Corollary 2.2 and Equation (3.2), a subset V of E' is independent in M_ρ if and only if $\mathbf{T}(V)$ is an independent vector of ρ , and so V is a basis of M_ρ if and only if $\mathbf{T}(V)$ is a basis of ρ .

Example 2. Consider a Boolean polymatroid $\rho = r_{M_1} + r_{M_2} + \cdots + r_{M_k}$ on $[n]$ where each M_h has rank 1 and its rank-1 elements are consecutive integers $a_h, a_h + 1, \dots, b_h$, with $a_1 \leq a_2 \leq \cdots \leq a_k$ and $b_1 \leq b_2 \leq \cdots \leq b_k$. The matroids M_1, M_2, \dots, M_k correspond to the rows in a lattice path diagram, where north steps are labeled by their first coordinate, the lower left corner is $(1, 0)$, and the upper right corner is (n, k) . (See Figure 2.) Bases correspond to lattice paths: entry u_i in a basis \mathbf{u} is the number of north steps in the corresponding path that are labeled i . An elementary argument (as in the proof of [5, Theorem 3.3]) shows that the correspondence between bases and lattice paths is bijective. Schweig [24, 25] introduced these *lattice path polymatroids*. The description of the natural matroid of a Boolean polymatroid in Example 1 along with the ideas in [6, Section 6.1] show that the natural matroid of a lattice path polymatroid is a lattice path matroid (see [5] for these matroids). Like the class of lattice path matroids, that of lattice path polymatroids is closed under minors; the excluded minors for lattice path polymatroids are found in [4]. Unlike the class of lattice path matroids, that of lattice path polymatroids is not closed under duality. Also, most lattice path matroids are not lattice path polymatroids.

The following characterizations of integer polymatroids by bases are known (see, e.g., [15]). We provide a transparent way to see this and similar results using the natural matroid.

Theorem 3.1. *A nonempty set $\mathbf{B} \subseteq \mathbb{N}^n$ is the set of bases of an integer polymatroid on $E = [n]$ if and only if either of the following equivalent conditions holds:*

- (B) *if $\mathbf{u}, \mathbf{v} \in \mathbf{B}$ with $u_i > v_i$ for some $i \in [n]$, then there is a $j \in [n]$ for which $u_j < v_j$ and $\mathbf{u} - \mathbf{e}_i + \mathbf{e}_j$ is in \mathbf{B} ,*
- (B') *if $\mathbf{u}, \mathbf{v} \in \mathbf{B}$ with $u_i > v_i$ for some $i \in [n]$, then there is a $j \in [n]$ for which $u_j < v_j$ and both $\mathbf{u} - \mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{v} - \mathbf{e}_j + \mathbf{e}_i$ are in \mathbf{B} .*

Proof. First let \mathbf{B} be the set of bases of an integer polymatroid ρ . We prove property (B'), which implies property (B). Fix $\mathbf{u}, \mathbf{v} \in \mathbf{B}$ with $u_i > v_i$ for some $i \in [n]$. Let U and V be bases of the natural matroid M_ρ of ρ with $\mathbf{T}(U) = \mathbf{u}$ and $\mathbf{T}(V) = \mathbf{v}$. Since each set X_t is a set of clones, we may assume that $V \cap X_t \subseteq U \cap X_t$ whenever $v_t \leq u_t$. Fix $x \in (U - V) \cap X_i$. By the symmetric basis exchange property for matroids, there is an element $y \in V - U$, say in X_j , so that both $(U - x) \cup y$ and $(V - y) \cup x$ are bases of M_ρ , so $\mathbf{u} - \mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{v} - \mathbf{e}_j + \mathbf{e}_i$ are in \mathbf{B} . Now $V \cap X_j \not\subseteq U \cap X_j$, so, as needed, $u_j < v_j$.

To finish the proof, we show that a nonempty subset \mathbf{B} of \mathbb{N}^n that satisfies property (B) is the set of bases of an integer polymatroid. For each $i \in E$, set $m_i = \max\{u_i : \mathbf{u} \in \mathbf{B}\}$ and let X_i be a set of m_i elements where $X_i \cap X_j = \emptyset$ if $i \neq j$. Set $X_A = \cup_{i \in A} X_i$, for $A \subseteq E$, and $E' = X_E$. Let $\mathcal{B} = \{B : B \subseteq E' \text{ and } \mathbf{T}(B) \in \mathbf{B}\}$. Take $U, V \in \mathcal{B}$ with $x \in U - V$; say $x \in X_i$. If there is an element y in $(V - U) \cap X_i$, then $(U - x) \cup y$ has the same type vector as U and so is in \mathcal{B} . Now assume that $V \cap X_i \subseteq U \cap X_i$. Let $\mathbf{u} = \mathbf{T}(U)$ and $\mathbf{v} = \mathbf{T}(V)$. Thus, $u_i > v_i$. By property (B), there is a $j \in [n]$ for which $u_j < v_j$ and $\mathbf{w} = \mathbf{u} - \mathbf{e}_i + \mathbf{e}_j$ is in \mathbf{B} . Thus, $(V - U) \cap X_j \neq \emptyset$. For any $y \in (V - U) \cap X_j$, the set $W = (U - x) \cup y$ has type vector \mathbf{w} , so $W \in \mathcal{B}$. Thus, \mathcal{B} is the set of bases of a matroid M on E' . Define $\rho : 2^E \rightarrow \mathbb{N}$ by $\rho(A) = r_M(X_A)$. Thus, ρ is an integer polymatroid on E . Also, X_i is a set of clones in M . It now follows from the definition of ρ and Corollary 2.4 that M is the natural matroid of ρ . From the definition of M and the comments before Example 2, we have that \mathbf{B} is the set of bases of ρ , as needed. \square

The strategy we used above adapts to prove integer-polymatroid counterparts of other axiom schemes for matroids that use bases or independent sets. We cite just one example, for the middle basis property.

Theorem 3.2. *A nonempty set $\mathbf{B} \subseteq \mathbb{N}^n$ is the set of bases of an integer polymatroid on $E = [n]$ if and only if the following two conditions hold:*

- if $\mathbf{u}, \mathbf{v} \in \mathbf{B}$ with $\mathbf{u} \neq \mathbf{v}$, then $\mathbf{u} \not\leq \mathbf{v}$ and $\mathbf{v} \not\leq \mathbf{u}$, and
- whenever $\mathbf{x}, \mathbf{y} \in \mathbb{N}^n$ with $\mathbf{x} \leq \mathbf{y}$ and there are $\mathbf{u}, \mathbf{v} \in \mathbf{B}$ with $\mathbf{x} \leq \mathbf{u}$ and $\mathbf{v} \leq \mathbf{y}$, then there is some $\mathbf{w} \in \mathbf{B}$ with $\mathbf{x} \leq \mathbf{w} \leq \mathbf{y}$.

For a positive integer k , certain properties of the k -dual ρ^* of an integer k -polymatroid ρ highlight how natural k -duality is. For instance, generalizing a result of Kung [17] for matroids, Whittle [29] showed that the map $\rho \mapsto \rho^*$ is the only involution on the class of integer k -polymatroids that switches deletion and contraction, i.e., $(\rho_{\setminus i})^* = (\rho^*)_{/i}$ and $(\rho_{/i})^* = (\rho^*)_{\setminus i}$ for all $i \in E$. The set of bases of the dual of a matroid M on E is given by $\{E - B : B \in \mathcal{B}\}$ where \mathcal{B} is the set of bases of M ; the next result generalizes this to the k -dual of an integer k -polymatroid.

Theorem 3.3. *Let k be a positive integer and let ρ be an integer k -polymatroid on $E = [n]$, with \mathbf{B} its set of bases. The set \mathbf{B}^* of bases of the k -dual ρ^* is $\{\mathbf{u}^* : \mathbf{u} \in \mathbf{B}\}$ where $\mathbf{u}^* = (k, k, \dots, k) - \mathbf{u}$.*

Proof. Note that $|\mathbf{u}| = \rho(E)$ if and only if $|\mathbf{u}^*| = k|E| - \rho(E) = \rho^*(E)$. With this, the equivalence of the following statements shows that $\mathbf{u} \in \mathbf{B}$ if and only if $\mathbf{u}^* \in \mathbf{B}^*$:

- $|\mathbf{u}|_A \leq \rho(A)$ for all $A \subseteq E$,
- $|\mathbf{u}|_{E-A} \leq \rho(E - A)$ for all $A \subseteq E$,
- $\rho(E) - |\mathbf{u}|_A \leq \rho(E - A)$ for all $A \subseteq E$,
- $k|A| - |\mathbf{u}|_A \leq k|A| - \rho(E) + \rho(E - A)$ for all $A \subseteq E$,
- $|\mathbf{u}^*|_A \leq \rho^*(A)$ for all $A \subseteq E$. \square

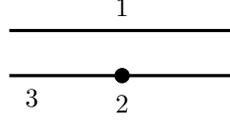


FIGURE 3. The circuits of this rank-3 integer polymatroid are $(2, 0, 2)$, $(2, 1, 1)$, and $(0, 1, 2)$.

4. CIRCUITS OF AN INTEGER POLYMATROID AND ITS NATURAL MATROID

We next develop a theory of circuits for integer polymatroids that is analogous to that for bases in Section 3. Just as the bases of an integer polymatroid ρ are the type vectors of the bases of the natural matroid M_ρ , so the circuits of ρ are the type vectors of the circuits of M_ρ , with one exception: loops of ρ map to the empty set in M_ρ . This is addressed by the ambient set \mathbf{U} that we consider below. Another issue that we must address so that the circuits determine the integer polymatroid is that we need the rank of each element i for which X_i is a set of coloops of M_ρ ; the set \mathbf{U} also takes care of this. (In matroids, such elements have rank one, but in integer polymatroids, the rank could be any positive integer.) Recall that $[n]_0$ denotes the set $\{0, 1, \dots, n\}$.

The *circuits* of an integer polymatroid ρ on $E = [n]$ are the vectors \mathbf{u} in the set

$$\mathbf{U} = [\rho(1)]_0 \times [\rho(2)]_0 \times \cdots \times [\rho(n)]_0$$

that are not independent and each vector \mathbf{w} with $\mathbf{w} < \mathbf{u}$ is independent. Thus, from the set \mathbf{C} of circuits of ρ , the set of independent vectors of ρ is

$$(4.1) \quad \mathbf{I} = \{\mathbf{u} \in \mathbf{U} : \text{there is no } \mathbf{c} \in \mathbf{C} \text{ with } \mathbf{c} \leq \mathbf{u}\}.$$

By the remarks before Example 2, a vector $\mathbf{u} \in \mathbf{U}$ is a circuit of ρ if and only if some (equivalently, every) set C in the natural matroid M_ρ with $\mathbf{u} = \mathbf{T}(C)$ is a circuit of M_ρ . See Figure 3 for an example. The set \mathbf{C} is an antichain in \mathbb{N}^n . Recall that all antichains in \mathbb{N}^n are finite.

While \mathbf{U} gives the rank of each element, the next lemma shows how, from \mathbf{C} alone, to get the rank of any element i for which there is a $\mathbf{c} \in \mathbf{C}$ with $c_i > 0$.

Lemma 4.1. *Let ρ be an integer polymatroid on $E = [n]$. For $i \in E$ with $\rho(i) > 0$, the set X_i is a subset of a circuit of the natural matroid M_ρ if and only if $\rho(E) < \rho(E - i) + \rho(i)$. In this case, $\rho(i) = \max\{c_i : \mathbf{c} \in \mathbf{C}\}$.*

Proof. First assume that $\rho(E) < \rho(E - i) + \rho(i)$. Fix $a \in X_i$. Since X_i is independent in M_ρ , so is $X_i - a$. Extend $X_i - a$ to a basis B of $M_\rho \setminus a$, which, by the assumed inequality, is a basis of M_ρ . Let C be the fundamental circuit of a with respect to the basis B of M_ρ . Then $X_i \subseteq C$, for if $b \in X_i - C$, the subset $(C - a) \cup b$ of the basis B would be a circuit of M_ρ since a and b are clones, but that is a contradiction.

To prove the contrapositive of the converse, assume that $\rho(E) = \rho(E - i) + \rho(i)$. Thus, $r_{M_\rho}(E') = r_{M_\rho}(E' - X_i) + r_{M_\rho}(X_i)$, from which we get $M_\rho = (M_\rho \setminus X_i) \oplus (M_\rho | X_i)$. With this direct sum decomposition and the fact that X_i is independent in M_ρ , it follows that X_i is disjoint from all circuits of M_ρ . \square

The following result will be used in the next section.

Lemma 4.2. *Let ρ be an integer polymatroid on $E = [n]$. For $A \subseteq E$ and $i \in A$, we have $\rho(A) < \rho(A - i) + \rho(i)$ if and only if there is a circuit $\mathbf{u} \in \mathbf{C}$ with $u_i > 0$ and $u_j = 0$ for all $j \in E - A$.*

Proof. Given $\mathbf{u} \in \mathbf{C}$ with $u_i > 0$ and $u_j = 0$ for all $j \in E - A$, for each $a \in X_i$ there is a circuit C of M_ρ with $\mathbf{u} = \mathbf{T}(C)$ and $a \in C$. Thus, a is not a coloop of $M|X_A$, so $\rho(A) < \rho(A - i) + \rho(i)$. The converse follows by applying Lemma 4.1 to $\rho|_{E-A}$. \square

Lemma 4.1 and the remarks before it lead to the following setting for characterizations of integer polymatroids via circuits, as in Theorem 4.3: the set of circuits is a subset of a set of the form $\mathbf{U} = [m_1]_0 \times [m_2]_0 \times \cdots \times [m_n]_0$ where each m_i is a nonnegative integer. Recording m_i is the only way we get $\rho(i)$ when all elements of X_i (if there are any) are coloops of M_ρ . Two circuits in the natural matroid may have different type vectors or the same type vector; therefore there are two circuit elimination properties, (C3) and (C4).

Theorem 4.3. *Let m_1, m_2, \dots, m_n be nonnegative integers and let \mathbf{C} be a subset of $[m_1]_0 \times [m_2]_0 \times \cdots \times [m_n]_0$ where, for each $i \in [n]$, if $u_i > 0$ for some $\mathbf{u} \in \mathbf{C}$, then $m_i = \max\{u_i : \mathbf{u} \in \mathbf{C}\}$. The set \mathbf{C} is the set of circuits of an integer polymatroid on $E = [n]$ if and only if \mathbf{C} satisfies properties (C1)–(C4):*

- (C1) *each vector in \mathbf{C} has at least two positive entries,*
- (C2) *if $\mathbf{u}, \mathbf{v} \in \mathbf{C}$ with $\mathbf{u} \neq \mathbf{v}$, then $\mathbf{u} \not\leq \mathbf{v}$ and $\mathbf{v} \not\leq \mathbf{u}$,*
- (C3) *if $\mathbf{u}, \mathbf{v} \in \mathbf{C}$ with $\mathbf{u} \neq \mathbf{v}$ and if $u_i, v_i > 0$, then there is a $\mathbf{z} \in \mathbf{C}$ so that $\mathbf{z} < \mathbf{u} \vee \mathbf{v}$ and $z_i < \max(u_i, v_i)$, and*
- (C4) *if $\mathbf{u} \in \mathbf{C}$, and $i \in E$ with $0 < u_i < m_i$, and $j \in E - i$ with $0 < u_j$, then there is a $\mathbf{v} \in \mathbf{C}$ with $v_i = u_i + 1$, with $v_h \leq u_h$ for all $h \neq i$, and with $v_j < u_j$.*

Thus, an integer polymatroid on $[n]$ is a pair (\mathbf{U}, \mathbf{C}) where

- (i) $\mathbf{U} = [m_1]_0 \times [m_2]_0 \times \cdots \times [m_n]_0$, for some m_1, m_2, \dots, m_n in \mathbb{N} ,
- (ii) $\mathbf{C} \subseteq \mathbf{U}$ and \mathbf{C} satisfies properties (C1)–(C4), and
- (iii) if $i \in E$ and $u_i > 0$ for some $\mathbf{u} \in \mathbf{C}$, then $m_i = \max\{u_i : \mathbf{u} \in \mathbf{C}\}$.

If ρ is the rank function of the integer polymatroid given by the pair (\mathbf{U}, \mathbf{C}) , then $\rho(i) = m_i$ for all $i \in E$.

Proof. Let \mathbf{C} be the set of circuits of an integer polymatroid ρ on E . By Lemma 4.1, if $\mathbf{u} \in \mathbf{C}$ and $u_i > 0$, then $m_i = \rho(i)$. Property (C1) holds since each set X_i is independent in M_ρ . If $\mathbf{u}, \mathbf{v} \in \mathbf{C}$ and $\mathbf{u} < \mathbf{v}$, then a circuit V of M_ρ with $\mathbf{v} = \mathbf{T}(V)$ and a subset U of V with $\mathbf{u} = \mathbf{T}(U)$ would be comparable circuits of M_ρ ; this contradiction proves property (C2).

For property (C3), take $\mathbf{u}, \mathbf{v} \in \mathbf{C}$ with $\mathbf{u} \neq \mathbf{v}$ and $u_i, v_i > 0$. Let U and V be circuits of M_ρ with $\mathbf{u} = \mathbf{T}(U)$ and $\mathbf{v} = \mathbf{T}(V)$. Each set X_j is a set of clones in M_ρ , so we may assume that if $u_j \leq v_j$, then $U \cap X_j \subseteq V \cap X_j$, and if $v_j \leq u_j$, then $V \cap X_j \subseteq U \cap X_j$. Thus, $U \cap V \cap X_i \neq \emptyset$. For any $a \in U \cap V \cap X_i$, circuit elimination applied to U and V gives a circuit C of M_ρ with $C \subseteq (U \cup V) - a$. The inclusions that we have assumed give $|C \cap X_j| \leq \max(u_j, v_j)$ for all $j \in E$, and $|C \cap X_i| < \max(u_i, v_i)$, as needed.

For property (C4), consider $\mathbf{u} \in \mathbf{C}$ and $i \in E$ with $0 < u_i < \rho(i)$. Let C be a circuit of M_ρ with $\mathbf{T}(C) = \mathbf{u}$. Fix $a \in C \cap X_i$ and $b \in X_i - C$, and set $C' = (C - a) \cup b$, which is a circuit of M_ρ since a and b are clones. For $j \in E - i$ with $u_j > 0$, fix $c \in C \cap X_j$. By circuit elimination, M_ρ has a circuit D with $D \subseteq (C \cup C') - c = (C \cup b) - c$. Property (C2) applied to \mathbf{u} and $\mathbf{T}(D)$ forces $(C \cup b) \cap X_i \subseteq D$. Thus, $|D \cap X_i| = u_i + 1$ and $|D \cap X_h| \leq u_h$ for all $h \neq i$, and the inequality is strict for $h = j$, so property (C4) holds.

For the converse, assume that the pair (\mathbf{U}, \mathbf{C}) satisfies properties (i)–(iii). For each $i \in E$, let X_i be a set of size m_i with $X_i \cap X_j = \emptyset$ whenever $i \neq j$. We use X_A and E' as above. Let $\mathcal{C} = \{C : C \subseteq E' \text{ and } \mathbf{T}(C) \in \mathbf{C}\}$. Now $\emptyset \notin \mathcal{C}$ by property (C1). For two sets C and C' in \mathcal{C} , either (i) $\mathbf{T}(C) = \mathbf{T}(C')$, so for at least one $i \in E$, the subsets

$C \cap X_i$ and $C' \cap X_i$ of X_i are different but have the same size, or (ii) $\mathbf{T}(C) \neq \mathbf{T}(C')$, so by property (C2), there are $i, j \in E$ with $|C \cap X_i| < |C' \cap X_i|$ and $|C' \cap X_j| < |C \cap X_j|$; thus, neither C nor C' contains the other.

We next show that \mathcal{C} satisfies the circuit elimination property. Take two sets $U, V \in \mathcal{C}$ and $a \in U \cap V$; say $a \in X_j$. Let $\mathbf{u} = \mathbf{T}(U)$ and $\mathbf{v} = \mathbf{T}(V)$. If $\mathbf{u} \neq \mathbf{v}$, then by property (C3), there is a $\mathbf{z} \in \mathbf{C}$ with $\mathbf{z} < \mathbf{u} \vee \mathbf{v}$ and $z_j < \max(u_j, v_j)$. Clearly $(U \cup V) - a$ has a subset C with $\mathbf{T}(C) = \mathbf{z}$, as needed. Now assume that $\mathbf{u} = \mathbf{v}$. If $U \cap X_j \neq V \cap X_j$, then there is an element $b \in (V - U) \cap X_j$, and the set $C = (U - a) \cup b$ has $\mathbf{T}(C) = \mathbf{u}$, so $C \in \mathcal{C}$, as needed. If $U \cap X_j = V \cap X_j$, then $U \cap X_i \neq V \cap X_i$ for some $i \in E - j$, and so $0 < u_i < m_i$. By property (C4), since $u_j > 0$, there is an $\mathbf{x} \in \mathbf{C}$ with $x_i = u_i + 1$, with $x_h \leq u_h$ for all $h \neq i$, and with $x_j < u_j$. Clearly $(U \cup V) - a$ has a subset C with $\mathbf{T}(C) = \mathbf{x}$, as needed.

Thus, \mathcal{C} is the set of circuits of a matroid M on E' . As in the proof of Theorem 3.1, from M , we get an integer polymatroid ρ whose natural matroid is M . Since \mathbf{C} is the set of circuits of ρ , this completes the proof. \square

The set $\mathbf{C} = \{(4, 1), (2, 2)\}$ satisfies all properties except (C4), so property (C4) does not follow from properties (C1)–(C3).

Let k be a positive integer. Let \mathbf{H} be the set of the type vectors of the hyperplanes of the natural matroid of an integer k -polymatroid. Let $\mathbf{C}^* = \{(k, k, \dots, k) - \mathbf{u} : \mathbf{u} \in \mathbf{H}\}$. In contrast to Theorem 3.3, \mathbf{C}^* might not be the set of circuits of an integer polymatroid. For instance, for the integer 2-polymatroid ρ in Figure 3, we have

$$\mathbf{C}^* = \{(2, 1, 0), (0, 2, 2), (1, 1, 2), (1, 2, 1)\},$$

and properties (C3) and (C4) fail.

For an integer polymatroid ρ on $E = [n]$ and any $i \in E$, since $\rho_{/i}(j) = \rho(\{i, j\}) - \rho(i)$ for all $j \in E - i$, the circuits of the contraction $\rho_{/i}$ are contained in the Cartesian product

$$\mathbf{U}_{/i} = \prod_{j \in E - i} [\rho(\{i, j\}) - \rho(i)]_0.$$

Let \mathbf{C}' be the set of circuits of ρ with the i th entry deleted from each vector. Since the circuits of a contraction M/Y of a matroid M are the minimal nonempty sets of the form $C - Y$ as C ranges over the circuits of M , and the natural matroid $M_{\rho_{/i}}$ of $\rho_{/i}$ is $M_{\rho}/X_i|E'_{/i}$ (in the notation used after Corollary 2.4), it follows that the circuits of $\rho_{/i}$ are the minimal vectors in $\mathbf{C}' \cap \mathbf{U}_{/i}$ that have at least two positive entries.

We noted in Section 2 that an integer polymatroid ρ on E with $|E| > 1$ is connected if and only if ρ has no loops and M_{ρ} is connected. Thus, an integer polymatroid ρ on $[n]$ is connected if and only if for each pair of distinct integers $i, j \in [n]$, there is a circuit \mathbf{u} of ρ with $u_i > 0$ and $u_j > 0$.

5. FLATS, CYCLIC SETS, AND CYCLIC FLATS IN POLYMATROIDS

While some results in this section apply only to integer polymatroids, many apply to all polymatroids. To describe what we do in this section, we first need some definitions. Flats in a polymatroid ρ on E are defined as in matroids: a subset A of E is a *flat* of ρ if $\rho(A \cup i) > \rho(A)$ for all $i \in E - A$. Let \mathcal{F}_{ρ} denote the set of flats of ρ . Unless we are focusing only on matroids, \mathcal{F}_{ρ} does not determine ρ since, for instance, ρ and $c\rho$, for any positive real c , have the same flats.

There are various equivalent ways to say that a set X in a matroid M is cyclic, including:

- (i) X is a union of circuits;

- (ii) $M|X$ has no coloops;
- (iii) $r(X) < r(X - y) + r(y)$ for each $y \in X$ that is not a loop.

As in [10], we adapt condition (iii) to define cyclic sets in a polymatroid ρ on E : a subset A of E is *cyclic* if $\rho(A) < \rho(A - i) + \rho(i)$ for all $i \in A$ with $\rho(i) > 0$. We let \mathcal{Y}_ρ denote the set of all cyclic sets of ρ .

Of greatest interest are the cyclic flats, that is, the flats that are cyclic. The set of cyclic flats of ρ is denoted \mathcal{Z}_ρ , or \mathcal{Z}_M for a matroid M . As in the case of matroids, \mathcal{Z}_ρ is a lattice under inclusion. (See the comment after Lemma 5.14.) The next result, from [26, 7], characterizes matroids in terms of their cyclic flats and the ranks of those sets.

Theorem 5.1. *For a pair (\mathcal{Z}, r) , where $\mathcal{Z} \subseteq 2^E$ and $r : \mathcal{Z} \rightarrow \mathbb{N}$, there is a matroid M for which $\mathcal{Z} = \mathcal{Z}_M$ and $r(Z) = r_M(Z)$ for all $Z \in \mathcal{Z}$ if and only if*

- (Z0) *ordered by inclusion, \mathcal{Z} is a lattice,*
- (Z1) *$r(\hat{0}_{\mathcal{Z}}) = 0$, where $\hat{0}_{\mathcal{Z}}$ is the least element of \mathcal{Z} ,*
- (Z2) *$0 < r(B) - r(A) < |B - A|$ for all sets A, B in \mathcal{Z} with $A \subsetneq B$, and*
- (Z3) *$r(A \vee B) + r(A \wedge B) + |(A \cap B) - (A \wedge B)| \leq r(A) + r(B)$ for all A, B in \mathcal{Z} .*

Csirmaz [10] extended this theorem. His result, stated next, characterizes polymatroids using cyclic flats and the value of the rank function on each of those flats as well as on each singleton set. The rank of each element must be given since, while in a matroid each element that is not in the least cyclic flat (the set of loops) has rank 1, in a polymatroid, such an element may have any positive rank.

Theorem 5.2. *For a pair (\mathcal{Z}, ρ') , where $\mathcal{Z} \subseteq 2^E$ and $\rho' : \mathcal{Z} \cup E \rightarrow \mathbb{R}_{\geq 0}$, there is a polymatroid ρ on E with $\mathcal{Z} = \mathcal{Z}_\rho$ and $\rho(x) = \rho'(x)$ for all $x \in \mathcal{Z} \cup E$ if and only if*

- (PZ0) *ordered by inclusion, \mathcal{Z} is a lattice,*
- (PZ1) *the least element of \mathcal{Z} , denoted $\hat{0}_{\mathcal{Z}}$, is $\{i \in E : \rho'(i) = 0\}$, and $\rho'(\hat{0}_{\mathcal{Z}}) = 0$,*
- (PZ2) *for all sets A, B in \mathcal{Z} with $A \subsetneq B$,*

$$0 < \rho'(B) - \rho'(A) < \sum_{i \in B - A} \rho'(i),$$

- (PZ3) *for all sets A, B in \mathcal{Z} ,*

$$\rho'(A \vee B) + \rho'(A \wedge B) + \sum_{i \in (A \cap B) - (A \wedge B)} \rho'(i) \leq \rho'(A) + \rho'(B),$$

and

- (PZ4) *if $A \in \mathcal{Z}$ and $i \in A$, then $\rho'(i) \leq \rho'(A)$.*

We will show how, in the case of an integer polymatroid ρ , Theorem 5.2 follows from Theorem 5.1; we do this by relating the flats of ρ to those of its natural matroid, and likewise for cyclic sets and for cyclic flats. The proof of Theorem 5.2 in [10] has the same general outline as the proof of Theorem 5.1 that Sims [26] gave. In particular, for the more involved implication, assuming that the properties above hold for (\mathcal{Z}, ρ') , one defines a function $\rho : 2^E \rightarrow \mathbb{R}$, checks that the defining properties of a polymatroid hold, and shows that its cyclic flats are precisely the sets in \mathcal{Z} , and that ρ and ρ' have the same values on the sets in \mathcal{Z} and the elements of E . The function ρ is defined by

$$\rho(A) = \min\{\rho'(X) + \sum_{i \in A - X} \rho'(i) : X \in \mathcal{Z}\}.$$

This makes it natural to consider, for a polymatroid ρ on E and subset A of E , the set

$$(5.1) \quad \mathcal{R}_\rho(A) = \{B \in \mathcal{Z}_\rho : \rho(A) = \rho(B) + \sum_{i \in A-B} \rho(i)\}.$$

For a matroid M , we write this set as $\mathcal{R}_M(A)$. Our main new result is Theorem 5.17, where we show that $\mathcal{R}_\rho(A)$ is a sublattice of \mathcal{Z}_ρ , we identify its least and greatest elements, and we show that each pair of elements in $\mathcal{R}_\rho(A)$ is a modular pair. To prepare for that, we develop basic results about flats and cyclic sets, and two operators related to them. (While some of these results may be known, we include proofs for completeness.)

We start with flats. The flats of an integer polymatroid are related to those of its natural matroid in the simplest possible way, as the next lemma states.

Lemma 5.3. *For an integer polymatroid ρ on E , let $\hat{0}_\rho$ be $\{i \in E : \rho(i) = 0\}$. A subset A of E is a flat of ρ if and only if $\hat{0}_\rho \subseteq A$ and X_A is a flat of the natural matroid M_ρ .*

Proof. Assume that $\hat{0}_\rho \subseteq A$ and that X_A is a flat of M_ρ . If $i \in E - A$, then there are elements $b \in X_i$, and $r_{M_\rho}(X_{A \cup i}) \geq r_{M_\rho}(X_A \cup b) > r_{M_\rho}(X_A)$, so $\rho(A \cup i) > \rho(A)$, as needed. We now prove the contrapositive of the converse. If $\hat{0}_\rho \not\subseteq A$, then clearly A is not a flat of ρ . Assume that X_A is not a flat of M_ρ , so $r_{M_\rho}(X_A \cup b) = r_{M_\rho}(X_A)$ for some $b \in E' - X_A$; say $b \in X_i$. Then $r_{M_\rho}(X_A \cup c) = r_{M_\rho}(X_A)$ for all $c \in X_i$ since X_i is a set of clones. From this, repeatedly applying submodularity gives $r_{M_\rho}(X_{A \cup i}) = r_{M_\rho}(X_A)$, so $\rho(A \cup i) = \rho(A)$, so A is not a flat of ρ . \square

Corollary 5.4. *The set \mathcal{F}_ρ of flats of an integer polymatroid ρ , ordered by inclusion, is isomorphic to a sublattice of the lattice \mathcal{F}_{M_ρ} . The meet of two flats of ρ is their intersection.*

By [7, Theorem 2.1], every finite lattice is isomorphic to the lattice of cyclic flats of a matroid. With that and the construction in Theorem 1.1, it follows that, in contrast to matroids, every finite lattice is isomorphic to the lattice of flats of an integer polymatroid.

Lemma 5.5. *For a polymatroid ρ on E , the intersection of two flats is a flat, so, ordered by inclusion, \mathcal{F}_ρ is a lattice.*

Proof. Fix $A, B \in \mathcal{F}_\rho$ and $e \in E - (A \cap B)$; say $e \notin A$. From submodularity and these assumptions, $\rho((A \cap B) \cup e) - \rho(A \cap B) \geq \rho(A \cup e) - \rho(A) > 0$, as needed. \square

This lemma justifies extending the definition of the closure operator from matroids to polymatroids. The *closure operator* $\text{cl}_\rho : 2^E \rightarrow 2^E$ of a polymatroid ρ on E is given by

$$(5.2) \quad \text{cl}_\rho(A) = \bigcap \{F : F \in \mathcal{F}_\rho \text{ and } A \subseteq F\}$$

for $A \subseteq E$; equivalently, $\text{cl}_\rho(A)$ is the minimum flat (with respect to inclusion) that is a superset of A . Several results follow immediately: $\text{cl}_\rho(A) \in \mathcal{F}_\rho$ by Lemma 5.5, the image of cl_ρ is \mathcal{F}_ρ , and cl_ρ is a closure operator in the general sense, that is, (i) $A \subseteq \text{cl}_\rho(A)$ for all $A \subseteq E$, (ii) if $A \subseteq B \subseteq E$, then $\text{cl}_\rho(A) \subseteq \text{cl}_\rho(B)$, and (iii) $\text{cl}_\rho(\text{cl}_\rho(A)) = \text{cl}_\rho(A)$ for all $A \subseteq E$. The MacLane-Steinitz exchange property of matroid closure operators fails for most polymatroids; for instance, in the integer polymatroid ρ in Figure 1, (c), we have $e_2 \in \text{cl}_\rho(e_3) - \text{cl}_\rho(\emptyset)$ but $e_3 \notin \text{cl}_\rho(e_2)$.

Lemma 5.6. *Let ρ be a polymatroid on E . If $A \subseteq E$, then $\text{cl}_\rho(A) = \{i : \rho(A \cup i) = \rho(A)\}$ and $\rho(A) = \rho(\text{cl}_\rho(A))$.*

Proof. Let $X = \{i : \rho(A \cup i) = \rho(A)\}$. Now $A \subseteq X$. Repeated use of submodularity gives $\rho(A) = \rho(X)$. If $i \in E - X$, then $\rho(X \cup i) \geq \rho(A \cup i) > \rho(A) = \rho(X)$, so X is a flat. Let F be a flat with $A \subseteq F$. Now $X \subseteq F$ since, for any $i \in X$, from $\rho(A \cup i) = \rho(A)$ we get $\rho(F \cup i) = \rho(F)$ by submodularity. Thus, $\text{cl}_\rho(A) = X$. \square

We next give properties of the closure operator that are special to integer polymatroids.

Lemma 5.7. *For an integer polymatroid ρ on E , let $\hat{0}_\rho$ be $\{i : \rho(i) = 0\}$. For $A \subseteq E$,*

- (1) $\text{cl}_\rho(A) = B$ if and only if $\hat{0}_\rho \subseteq B$ and $\text{cl}_{M_\rho}(X_A) = X_B$, and
- (2) $\text{cl}_\rho(A) = A \cup \hat{0}_\rho \cup C_A$ where C_A is the set of all $i \in E$ for which some circuit \mathbf{u} of ρ has $u_i = 1$ and $u_j = 0$ for all $j \in E - (A \cup i)$.

Proof. Each set X_i is a set of clones of M_ρ , so $\text{cl}_{M_\rho}(X_A)$ is the smallest flat X_B that contains X_A . Part (1) follows from this observation and Lemma 5.3. For part (2), clearly $A \cup \hat{0}_\rho \subseteq \text{cl}_\rho(A)$. Fix $i \in C_A$ and a circuit \mathbf{u} with $u_i = 1$ and $u_j = 0$ for all $j \in E - (A \cup i)$. The circuits C of M_ρ with $\mathbf{T}(C) = \mathbf{u}$ show that $X_i \subseteq \text{cl}_{M_\rho}(X_A)$; thus, $i \in \text{cl}_\rho(A)$, and so $A \cup \hat{0}_\rho \cup C_A \subseteq \text{cl}_\rho(A)$. For the other inclusion, fix a basis D of $M_\rho|_{X_A}$, so D is also a basis of $M_\rho|_{\text{cl}_{M_\rho}(X_A)}$. If $i \in \text{cl}_\rho(A) - (A \cup \hat{0}_\rho)$ and $a \in X_i$, then the type vector of the fundamental circuit of a with respect to D shows that $i \in C_A$. \square

We now turn to cyclic sets. We first focus on integer polymatroids.

Lemma 5.8. *Let ρ be an integer polymatroid on E . For $A \subseteq E$, statements (1)–(3) are equivalent:*

- (1) A is a cyclic set of ρ ,
- (2) X_A is a cyclic set of M_ρ ,
- (3) for each $i \in A$, either $\rho(i) = 0$ or there is a circuit \mathbf{u} of ρ for which $u_i > 0$ and $u_j = 0$ for all $j \in E - A$.

Proof. Assume that statement (1) holds. For any $a \in X_A$, there is an $i \in A$ with $\rho(i) > 0$ and $a \in X_i$. If a were a coloop of $M_\rho|_{X_A}$, then all elements of X_i would be coloops of $M_\rho|_{X_A}$, contrary to having $\rho(A) < \rho(A - i) + \rho(i)$. Thus, statement (2) holds.

Assume that statement (2) holds. Fix $i \in A$ with $\rho(i) > 0$. No $a \in X_i$ is a coloop of $M_\rho|_{X_A}$, so some circuit C of M_ρ has $a \in C \subseteq X_A$. Statement (3) now follows.

By Lemma 4.2, statement (3) implies statement (1). \square

We can expand the list of equivalent conditions for X being a cyclic set of a matroid M (items (i)–(iii) in the second paragraph of this section):

- (iv) X is a union of cocircuits of the dual M^* ,
- (v) $E - X$ is an intersection of hyperplanes of M^* , and
- (vi) $E - X$ is a flat of M^* .

The flats of M^* , ordered by inclusion, form a geometric lattice, so the cyclic sets of M , ordered by inclusion, form a lattice, the order-dual of which is geometric. Thus, we have the following corollary of Lemma 5.8.

Corollary 5.9. *For an integer polymatroid ρ on E , its set \mathcal{Y}_ρ of cyclic sets, ordered by inclusion, is a lattice. The join of two cyclic sets is their union.*

If ρ is not a matroid, then the order dual of \mathcal{Y}_ρ need not be a geometric lattice, as one can see from the 2-polymatroid counterpart of the Vámos matroid shown in Figure 4, where the only sets not in \mathcal{Y}_ρ are the singleton sets and $\{a, d\}$.

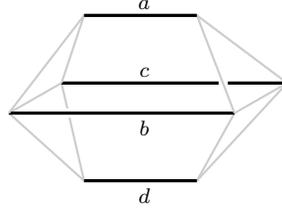


FIGURE 4. A 2-polymatroid counterpart of the Vámos matroid. Each pair of lines is coplanar except a, d .

Lemma 1 of [10] shows that every flat of a polymatroid contains a maximum cyclic flat. By the next result and the discussion below it, a similar statement holds for all sets, and it comes from a property that generalizes Corollary 5.9.

Lemma 5.10. *Let ρ be a polymatroid on E .*

- (1) *If $X, Y \in \mathcal{Y}_\rho$, then $X \cup Y \in \mathcal{Y}_\rho$. Thus, under inclusion, \mathcal{Y}_ρ is a lattice.*
- (2) *If $X \in \mathcal{Y}_\rho$, then $\text{cl}_\rho(X) \in \mathcal{Y}_\rho$, and so $\text{cl}_\rho(X) \in \mathcal{Z}_\rho$.*

Proof. To prove part (1), fix $X, Y \in \mathcal{Y}_\rho$ and $i \in X \cup Y$ with $\rho(i) > 0$; say $i \in X$. By the assumptions and submodularity, $\rho(X \cup Y) - \rho((X \cup Y) - i) \leq \rho(X) - \rho(X - i) < \rho(i)$, so $X \cup Y \in \mathcal{Y}_\rho$. For part (2), take $X \in \mathcal{Y}_\rho$ and $i \in \text{cl}_\rho(X)$ with $\rho(i) > 0$. Then, as needed, $\rho(\text{cl}_\rho(X)) - \rho(\text{cl}_\rho(X) - i) < \rho(i)$ since the left side is 0 if $i \notin X$ (since $\rho(X) = \rho(X \cup i)$), and at most $\rho(X) - \rho(X - i)$ if $i \in X$ (by submodularity). \square

This lemma justifies making the following definition. For a polymatroid ρ on E , its *cyclic operator* $\text{cy}_\rho : 2^E \rightarrow 2^E$ is given by, for $A \subseteq E$,

$$\text{cy}_\rho(A) = \bigcup \{D : D \in \mathcal{Y}_\rho \text{ and } D \subseteq A\}.$$

Thus, $\text{cy}_\rho(A)$ is the maximum cyclic subset of A . If $A \in \mathcal{F}_\rho$, then $\text{cy}_\rho(A) \in \mathcal{F}_\rho$ since $\text{cy}_\rho(A) \subseteq \text{cl}_\rho(\text{cy}_\rho(A)) \subseteq A$ and $\text{cl}_\rho(\text{cy}_\rho(A))$ is cyclic (by part (2) of Lemma 5.10) and so must be $\text{cy}_\rho(A)$. For a matroid M , the cyclic set $\text{cy}_M(A)$ is the union of the circuits that are subsets of A . The operator cy_M plays roles in recent papers, such as [12]. Note that the image of cy_ρ is precisely \mathcal{Y}_ρ . Also, (i) if $A \subseteq E$, then $\text{cy}_\rho(A) \subseteq A$, (ii) if $A \subseteq B \subseteq E$, then $\text{cy}_\rho(A) \subseteq \text{cy}_\rho(B)$, and (iii) if $A \subseteq E$, then $\text{cy}_\rho(\text{cy}_\rho(A)) = \text{cy}_\rho(A)$.

Lemma 5.11. *Let ρ be an integer polymatroid on E . For any set $A \subseteq E$, the set $\text{cy}_\rho(A)$ is the union of all subsets of A of either of the following forms:*

- (i) $\{i\}$ with $\rho(i) = 0$, or
- (ii) $S(\mathbf{u}) = \{i : u_i \neq 0\}$ where \mathbf{u} is a circuit of ρ and $u_j = 0$ for all $j \in E - A$.

Also, $\text{cy}_\rho(A) = B$ if and only if B is the maximum subset of A with $\text{cy}_{M_\rho}(X_A) = X_B$.

Proof. The first assertion follows from Lemma 5.8 and the definition of cy_ρ . That and the connection between the circuits of ρ and those of M_ρ give the second assertion. \square

We state the next lemma, which is basic and well known, so that we can cite it.

Lemma 5.12. *Let ρ be a polymatroid on E . Assume that $A \subseteq E$, that $i \in A$, and that $\rho(A) = \rho(A - i) + \rho(i)$. If $Y \subseteq A$ and $i \in Y$, then $\rho(Y) = \rho(Y - i) + \rho(i)$.*

Proof. By submodularity, $\rho(i) \geq \rho(Y) - \rho(Y - i) \geq \rho(A) - \rho(A - i) = \rho(i)$. \square

The next lemma identifies the elements in $A - \text{cy}_\rho(A)$ as the counterparts of coloops in the deletion $\rho_{\setminus E-A}$.

Lemma 5.13. *Let ρ be a polymatroid on E . For any set $A \subseteq E$,*

- (1) $\text{cy}_\rho(A) = A - \{i \in A : \rho(i) > 0 \text{ and } \rho(A) = \rho(A - i) + \rho(i)\}$, and
- (2) $\rho(A) = \rho(\text{cy}_\rho(A)) + \sum_{i \in A - \text{cy}_\rho(A)} \rho(i)$.

Proof. Let $X = \{i \in A : \rho(i) > 0 \text{ and } \rho(A) = \rho(A - i) + \rho(i)\}$. By Lemma 5.12, no cyclic subset of A contains any $i \in X$, so $\text{cy}_\rho(A) \subseteq A - X$. Part (1) will follow by showing that $A - X$ is cyclic. First note that repeatedly applying Lemma 5.12, adding one element at a time to go from $A - X$ to A , gives

$$(5.3) \quad \rho(A) = \rho(A - X) + \sum_{i \in X} \rho(i).$$

If there were a $j \in A - X$ with $\rho(j) > 0$ and $\rho(A - X) = \rho((A - X) - j) + \rho(j)$, then this equality, Equation (5.3), and submodularity would give

$$\rho(A) - \rho(j) = \rho((A - X) - j) + \sum_{i \in X} \rho(i) \geq \rho(A - j).$$

This inequality is contrary to having $j \notin X$, so $A - X$ is cyclic. Part (2) follows from part (1) and Equation (5.3). \square

The next lemma is like part (2) of Lemma 5.10, but switches flats and cyclic sets.

Lemma 5.14. *For a polymatroid ρ on E , if $A \in \mathcal{F}_\rho$, then $\text{cy}_\rho(A) \in \mathcal{F}_\rho$, so $\text{cy}_\rho(A) \in \mathcal{Z}_\rho$.*

Proof. Fix $A \in \mathcal{F}_\rho$ and $i \notin \text{cy}_\rho(A)$. We must show that $\rho(\text{cy}_\rho(A) \cup i) > \rho(\text{cy}_\rho(A))$. This holds by Lemmas 5.13 and 5.12 if $i \in A - \text{cy}_\rho(A)$. If $i \notin A$, then the assumption $A \in \mathcal{F}_\rho$ and submodularity give $\rho(\text{cy}_\rho(A) \cup i) - \rho(\text{cy}_\rho(A)) \geq \rho(A \cup i) - \rho(A) > 0$. \square

With Lemmas 5.10 and 5.14, we see that \mathcal{Z}_ρ is a lattice: for $A, B \in \mathcal{Z}_\rho$, their meet is $A \wedge B = \text{cy}_\rho(A \cap B)$ and their join is $A \vee B = \text{cl}_\rho(A \cup B)$.

The next lemma, along with Lemma 5.13, is a basic tool for investigating the sets $\mathcal{R}_\rho(A)$, which we defined in Equation (5.1).

Lemma 5.15. *Let ρ be a polymatroid on E . For any subsets A and B of E , the equality*

$$(5.4) \quad \rho(A) = \rho(B) + \sum_{i \in A - B} \rho(i)$$

holds if and only if

- (1) $\rho(A) = \rho(A - i) + \rho(i)$ for all $i \in A - B$, and
- (2) $\rho(A \cap B) = \rho(B)$ (equivalently, $\text{cl}_\rho(A \cap B) = \text{cl}_\rho(B)$).

Proof. First assume that properties (1) and (2) hold. Applying Lemma 5.12 to add one element at a time going from $A \cap B$ to A gives

$$\rho(A) = \rho(A \cap B) + \sum_{i \in A - B} \rho(i)$$

and replacing $\rho(A \cap B)$ by $\rho(B)$, as (2) justifies, yields Equation (5.4).

Now assume that Equation (5.4) holds. Repeated uses of submodularity give

$$\rho(A) \leq \rho(A \cap B) + \sum_{i \in A - B} \rho(i).$$

Also, $\rho(A \cap B) \leq \rho(B)$. These inequalities and Equation (5.4) give $\rho(A \cap B) = \rho(B)$, so property (2) holds. With this, for any $i \in A - B$, we have

$$\rho(A) = \rho(A \cap B) + \left(\sum_{j \in A - B, j \neq i} \rho(j) \right) + \rho(i) \geq \rho(A - i) + \rho(i) \geq \rho(A),$$

from which we get $\rho(A) = \rho(A - i) + \rho(i)$, so property (1) holds. \square

We now consider the operators cl and cy together. Note that if B is a basis of a matroid M that has neither loops nor coloops, then $\text{cl}(\text{cy}(B)) = \emptyset$ but $\text{cy}(\text{cl}(B)) = E(M)$; thus, cl and cy need not commute. Lemmas 5.3 and 5.8 give the following result.

Corollary 5.16. *For an integer polymatroid ρ on E , let $\hat{0}_\rho$ be $\{i \in E : \rho(i) = 0\}$. For $A \subseteq E$, we have $A \in \mathcal{Z}_\rho$ if and only if $\hat{0}_\rho \subseteq A$ and $X_A \in \mathcal{Z}_{M_\rho}$.*

For an integer polymatroid ρ , since all cyclic flats of M_ρ have the form X_A for some $A \subseteq E$ and the map $\phi : \mathcal{Z}_\rho \rightarrow \mathcal{Z}_{M_\rho}$ where $\phi(A) = X_A$ is a bijection, properties that can be described via cyclic flats lift from matroids to integer polymatroids. With these ideas, the case of Theorem 5.2 for integer polymatroids follows from Theorem 5.1.

Not all properties of cyclic flats for matroids extend to polymatroids. For instance, for matroids, the cyclic flats of the dual M^* are the set complements of the cyclic flats of M , so \mathcal{Z}_{M^*} is isomorphic to the order dual of \mathcal{Z}_M . The same is not true for k -polymatroids and their k -duals, as one can check using the example in Figure 3 or 4.

To conclude, we use Lemmas 5.12, 5.13, and 5.15 to show that $\mathcal{R}_\rho(A)$ is a sublattice of \mathcal{Z}_ρ (so the meet and join operations are the same as in \mathcal{Z}_ρ), identify the least and greatest elements of $\mathcal{R}_\rho(A)$, and show that each pair (B, B') of cyclic flats in $\mathcal{R}_\rho(A)$ is a modular pair of flats, that is, $\rho(B) + \rho(B') = \rho(B \cup B') + \rho(B \cap B')$. (That equality can fail if only one of B or B' is in $\mathcal{R}_\rho(A)$.)

Theorem 5.17. *Let ρ be a polymatroid on E . For any subset A of E ,*

- (I) $\text{cl}_\rho(\text{cy}_\rho(A))$ and $\text{cy}_\rho(\text{cl}_\rho(A))$ are in $\mathcal{R}_\rho(A)$,
- (II) if $B \in \mathcal{R}_\rho(A)$, then $\text{cl}_\rho(\text{cy}_\rho(A)) \subseteq B \subseteq \text{cy}_\rho(\text{cl}_\rho(A))$,
- (III) $\mathcal{R}_\rho(A)$ is a sublattice of \mathcal{Z}_ρ , and
- (IV) if $B, B' \in \mathcal{R}_\rho(A)$, then (B, B') is a modular pair of flats.

Proof. When B is $\text{cy}_\rho(A)$, property (1) in Lemma 5.13 holds by Lemma 5.13, as does property (2) since $B \subseteq A$. Those properties then follow when B is $\text{cl}_\rho(\text{cy}_\rho(A))$ since $(\text{cl}_\rho(\text{cy}_\rho(A))) \cap A = \text{cy}_\rho(A)$, so $\text{cl}_\rho(\text{cy}_\rho(A)) \in \mathcal{R}_\rho(A)$. Those properties clearly also hold when B is $\text{cl}_\rho(A)$. From this, when B is $\text{cy}_\rho(\text{cl}_\rho(A))$, we get property (1) by Lemma 5.12, and property (2) by applying Lemma 5.12 as elements of $A - B$ are removed from A and $\text{cl}_\rho(A)$. Thus, $\text{cy}_\rho(\text{cl}_\rho(A)) \in \mathcal{R}_\rho(A)$, so part (I) holds.

Assume that $B \in \mathcal{R}_\rho(A)$. Property (1) of Lemma 5.15 gives $\text{cy}_\rho(A) \subseteq B$, so, since B is a flat, $\text{cl}_\rho(\text{cy}_\rho(A)) \subseteq B$. Property (2) of Lemma 5.15 and the fact that B is a flat give $B = \text{cl}_\rho(A \cap B) \subseteq \text{cl}_\rho(A)$, so, since B is cyclic, $B \subseteq \text{cy}_\rho(\text{cl}_\rho(A))$. Thus, part (II) holds.

For assertion (III), we start with an inequality that we will use below. Let A be any subset of E and let B and B' be in \mathcal{Z}_ρ . We claim that

$$\sum_{i \in (B \cap B') - (B \wedge B')} \rho(i) + \sum_{i \in A - B} \rho(i) + \sum_{i \in A - B'} \rho(i) \geq \sum_{i \in A - (B \vee B')} \rho(i) + \sum_{i \in A - (B \wedge B')} \rho(i).$$

This inequality holds since

- $A - (B \vee B')$ is a subset of each of $A - (B \wedge B')$, $A - B$, and $A - B'$ (so terms $\rho(i)$ coming from its elements appear twice on each side of the inequality), and

- $(A - (B \wedge B')) - (A - (B \vee B')) \subseteq ((B \cap B') - (B \wedge B')) \cup (A - B) \cup (A - B')$
(so terms $\rho(i)$ that appear once on the right side also appear on the left side).

Now assume that $B, B' \in \mathcal{R}_\rho(A)$, so

$$\rho(B) + \sum_{i \in A-B} \rho(i) = \rho(A) = \rho(B') + \sum_{i \in A-B'} \rho(i).$$

Then, using submodularity as formulated in property (PZ3) of Theorem 5.2, along with the inequality above, we have

$$\begin{aligned} 2\rho(A) &= \rho(B) + \rho(B') + \sum_{i \in A-B} \rho(i) + \sum_{i \in A-B'} \rho(i) \\ &\geq \rho(B \vee B') + \rho(B \wedge B') + \sum_{i \in (B \cap B') - (B \wedge B')} \rho(i) + \sum_{i \in A-B} \rho(i) + \sum_{i \in A-B'} \rho(i) \\ &\geq \rho(B \vee B') + \rho(B \wedge B') + \sum_{i \in A - (B \vee B')} \rho(i) + \sum_{i \in A - (B \wedge B')} \rho(i). \end{aligned}$$

Since

$$\rho(B \vee B') + \sum_{i \in A - (B \vee B')} \rho(i) \geq \rho(A) \quad \text{and} \quad \rho(B \wedge B') + \sum_{i \in A - (B \wedge B')} \rho(i) \geq \rho(A),$$

the inequality above forces these inequalities to be equalities, which proves assertion (III). Moreover, all inequalities in the argument above must be equalities, so equality holds in (PZ3) for B and B' . Now $\rho(B \cup B') = \rho(\text{cl}_\rho(B \cup B')) = \rho(B \vee B')$ and

$$\rho(B \cap B') = \rho(B \wedge B') + \sum_{i \in (B \cap B') - (B \wedge B')} \rho(i)$$

by part (I) since $B \cap B' \in \mathcal{F}_\rho$ and $B \wedge B' = \text{cy}_\rho(B \cap B') \in \mathcal{R}_\rho(B \cap B')$, so assertion (IV) follows. \square

While $\mathcal{R}_\rho(A)$ is a sublattice of \mathcal{Z}_ρ , it might not be an interval in \mathcal{Z}_ρ , as taking A to be a basis of the Fano plane shows. The corollary below is immediate from property (II).

Corollary 5.18. *If $A \in \mathcal{Z}_\rho$, then $\mathcal{R}_\rho(A) = \{A\}$.*

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(J. Bonin) DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052, USA

Email address: jbonin@gwu.edu

(C. Chun) UNITED STATES NAVAL ACADEMY, MATHEMATICS DEPARTMENT, ANNAPOLIS, MD, 21402, USA

Email address: chun@usna.edu

(T. Fife) SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY UNIVERSITY OF LONDON, MILE END ROAD, LONDON E1 4NS, UNITED KINGDOM

Email address: fi.tara@gmail.com