

Refined list version of Hadwiger's Conjecture

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Abstract

Assume $\lambda = \{k_1, k_2, \dots, k_q\}$ is a partition of $k_\lambda = \sum_{i=1}^q k_i$. A λ -list assignment of G is a k_λ -list assignment L of G such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into $|\lambda| = q$ sets C_1, C_2, \dots, C_q such that for each i and each vertex v of G , $|L(v) \cap C_i| \geq k_i$. We say G is λ -choosable if G is L -colourable for any λ -list assignment L of G . The concept of λ -choosability is a refinement of choosability that puts k -choosability and k -colourability in the same framework. If $|\lambda|$ is close to k_λ , then λ -choosability is close to k_λ -colourability; if $|\lambda|$ is close to 1, then λ -choosability is close to k_λ -choosability. This paper studies Hadwiger's Conjecture in the context of λ -choosability. Hadwiger's Conjecture is equivalent to saying that every K_t -minor-free graph is $\{1 \star (t-1)\}$ -choosable for any positive integer t . We prove that for $t \geq 5$, for any partition λ of $t-1$ other than $\{1 \star (t-1)\}$, there is a K_t -minor-free graph G that is not λ -choosable. We then construct several types of K_t -minor-free graphs that are not λ -choosable, where $k_\lambda - (t-1)$ gets larger as $k_\lambda - |\lambda|$ gets larger. In particular, for any q and any $\epsilon > 0$, there exists t_0 such that for any $t \geq t_0$, for any partition λ of $\lfloor (2-\epsilon)t \rfloor$ with $|\lambda| = q$, there is a K_t -minor-free graph that is not λ -choosable. The $q = 1$ case of this result was recently proved by Steiner, and our proof uses a similar argument. We also generalize this result to (a, b) -list colouring.

Keywords: Hadwiger's Conjecture, λ -choosability, (a, b) -list colouring.

1 Introduction

Given graphs H and G , we say H is a *minor* of G (or G has an H -minor) if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. Let K_t be the t -vertex complete graph. A graph G is K_t -minor-free if G has no K_t -minor. In 1943, Hadwiger [8] conjectured the following upper bound on the chromatic number of K_t -minor-free graphs:

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Conjecture 1 (Hadwiger’s Conjecture). *For every integer $t \geq 1$, every K_t -minor-free graph is $(t - 1)$ -colourable.*

This conjecture is a deep generalization of the Four Colour Theorem, and has motivated many developments in graph colouring and graph minor theory. Hadwiger [8] and Dirac [6] independently showed that Hadwiger’s Conjecture holds for $t \leq 4$. Wagner [27] proved that for $t = 5$ the conjecture is equivalent to the Four Colour Theorem, which was subsequently proved by Appel, Haken and Koch [2, 3] and Robertson, Sanders, Seymour and Thomas [20], both using extensive computer assistance. Robertson, Seymour and Thomas [21] went one step further and proved Hadwiger’s Conjecture for $t = 6$, also by reducing it to the Four Colour Theorem. The conjecture for $t \geq 7$ is open and seems to be extremely challenging. For more on Hadwiger’s Conjecture, see the survey of Seymour [23].

The evident difficulty of Hadwiger’s Conjecture has inspired many researchers to study the following natural weakening (cf. [9, 10, 19]):

Conjecture 2 (Linear Hadwiger’s Conjecture). *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every K_t -minor-free graph is Ct -colourable.*

For many decades, the best general upper bound on the chromatic number of K_t -minor-free graphs was $O(t\sqrt{\log t})$, which was proved independently by Kostochka [12, 13] and Thomason [24] in the 1980s. In 2019, Norine, Postle and Song [15] broke this barrier, and proved that the maximum chromatic number of K_t -minor-free graphs is in $O(t(\log t)^{1/4+o(1)})$. Following a series of improvements [14, 16–18], the best known bound is $O(t \log \log t)$ due to Delcourt and Postle [5].

A list assignment of a graph G is a mapping L that assigns to each vertex v of G a set $L(v)$ of permissible colours. An L -colouring of G is a proper colouring f of G such that for each vertex v of G , $f(v) \in L(v)$. We say G is L -colourable if G has an L -colouring. A k -list assignment of G is a list assignment L with $|L(v)| \geq k$ for each vertex v . We say G is k -choosable if G is L -colourable for any k -list assignment L of G . The choice-number of G is the minimum integer k such that G is k -choosable.

Hadwiger’s Conjecture is also widely considered in the setting of list colourings. Voigt [26] constructed planar graphs (hence K_5 -minor-free) with choice-number 5. Hence the list version of Hadwiger’s Conjecture is false. Nevertheless, the list version of Linear Hadwiger’s Conjecture, proposed by Kawarabayashi and Mohar [10] in 2007, remains open.

Conjecture 3 (List Hadwiger’s Conjecture). *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every K_t -minor-free graph is Ct -choosable.*

The current state-of-the-art upper bound on the choice-number of K_t -minor-free graphs is $O(t(\log \log t)^2)$ [5].

If Conjecture 3 is true, then a natural problem is to determine the minimum value of C . Barát, Joret and Wood [4] constructed K_t -minor-free graphs that are not $4(t - 3)/3$ -choosable, implying $C \geq \frac{4}{3}$ in Conjecture 3. Improving upon this result, Steiner [22]

recently proved that the maximum choice-number of K_t -minor-free graphs is at least $2t - o(t)$, and hence $C \geq 2$ in Conjecture 3.

1.1 λ -Choosability

In general, k -colourability and k -choosability behave very differently. Indeed, bipartite graphs can have arbitrary large choice-number. Zhu [28] introduced a refinement of the concept of choosability, λ -choosability, that puts k -choosability and k -colourability in the same framework and considers a more complex hierarchy of colouring parameters.

Definition 1. Let $\lambda = \{k_1, k_2, \dots, k_q\}$ be a multiset of positive integers. Let $k_\lambda = \sum_{i=1}^q k_i$ and $|\lambda| = q$. A λ -list assignment of G is a list assignment L such that the colour set $\bigcup_{v \in V(G)} L(v)$ can be partitioned into q sets C_1, C_2, \dots, C_q such that for each i and each vertex v of G , $|L(v) \cap C_i| \geq k_i$. We say G is λ -choosable if G is L -colourable for any λ -list assignment L of G .

Note that for each vertex v , $|L(v)| \geq \sum_{i=1}^q k_i = k_\lambda$. So a λ -list assignment L is a k_λ -list assignment with some restrictions on the set of possible lists.

For a positive integer a , let $m_\lambda(a)$ be the multiplicity of a in λ . If $m_\lambda(a) = m$, then instead of writing m times the integer a , we write $a \star m$. For example, $\lambda = \{1 \star k_1, 2 \star k_2, 3\}$ means that λ is the multiset consisting of k_1 copies of 1, k_2 copies of 2 and one copy of 3. If $\lambda = \{k\}$, then λ -choosability is the same as k -choosability; if $\lambda = \{1 \star k\}$, then λ -choosability is equivalent to k -colourability. So the concept of λ -choosability puts k -choosability and k -colourability in the same framework.

For $\lambda = \{k_1, k_2, \dots, k_q\}$ and $\lambda' = \{k'_1, k'_2, \dots, k'_p\}$, we say λ' is a *refinement* of λ if $p \geq q$ and there is a partition I_1, I_2, \dots, I_q of $\{1, 2, \dots, p\}$ such that $\sum_{j \in I_t} k'_j = k_t$ for $t = 1, 2, \dots, q$. We say λ' is obtained from λ by *increasing* some parts if $p = q$ and $k_t \leq k'_t$ for $t = 1, 2, \dots, q$. We write $\lambda \leq \lambda'$ if λ' is obtained from a refinement of λ by increasing some parts. It follows from the definitions that if $\lambda \leq \lambda'$, then every λ -choosable graph is λ' -choosable. Conversely, Zhu [28] proved that if $\lambda \not\leq \lambda'$, then there is a λ -choosable graph that is not λ' -choosable. In particular, λ -choosability implies k_λ -colourability, and if $\lambda \neq \{1 \star k_\lambda\}$, then there are k_λ -colourable graphs that are not λ -choosable.

All the partitions λ of a positive integer k are sandwiched between $\{k\}$ and $\{1 \star k\}$ in the above order. As observed above, $\{k\}$ -choosability is the same as k -choosability, and $\{1 \star k\}$ -choosability is equivalent to k -colourability. By considering other partitions λ of k , λ -choosability provides a complex hierarchy of colouring parameters that interpolate between k -colourability and k -choosability.

The framework of λ -choosability provides room to explore strengthenings of colourability and choosability results. For example, Kermnitz and Voigt [11] proved that there are planar graphs that are not $\{1, 1, 2\}$ -choosable. This result strengthens Voigt's result that there are non-4-choosable planar graphs, and shows that the Four Colour Theorem is sharp in the sense that for any partition λ of 4 other than $\{1 \star 4\}$, there is a planar graph that is not λ -choosable.

This paper considers Hadwiger's Conjecture in the context of λ -choosability. Conjectures 1, 2 and 3 can be restated in the language of λ -choosability as follows:

Conjecture 1'. *For every integer $t \geq 1$, every K_t -minor-free graph is $\{1 \star (t-1)\}$ -choosable.*

Conjecture 2'. *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every K_t -minor-free graph is $\{1 \star Ct\}$ -choosable.*

Conjecture 3'. *There exists a constant $C > 0$ such that for every integer $t \geq 1$, every K_t -minor-free graph is $\{Ct\}$ -choosable.*

1.2 Results

This paper constructs several examples of K_t -minor-free graphs that are not λ -choosable where $k_\lambda \geq t-1$ and q is close to k_λ . In particular, if the multiplicity of 1 in λ is large enough, then the number of parts of λ will be close to k_λ .

First we strengthen the above-mentioned result of Kermnitz and Voigt to K_t -minor-free graphs for $t \geq 5$ as follows:

Theorem 1. *For every integer $t \geq 5$, there exists a K_t -minor-free graph that is not $\{1 \star (t-3), 2\}$ -choosable.*

If λ is a partition of $t-1$ other than $\{1 \star (t-1)\}$, then $\{1 \star (t-3), 2\}$ is a refinement of λ . Hence we have the following corollary.

Corollary 2. *If λ is a partition of $t-1$ other than $\{1 \star (t-1)\}$, then there is a K_t -minor-free graph that is not λ -choosable.*

For a multiset λ of positive integers, let $h(\lambda)$ be the maximum t such that every K_t -minor-free graph is λ -choosable. Since $K_{k_\lambda+1}$ is not k_λ -colourable and hence not λ -choosable, we know that $h(\lambda) \leq k_\lambda + 1$.

For a multiset λ of positive integers, $k_\lambda - |\lambda|$ measures the “distance” of λ -choosability from k_λ -colourability. Hadwiger's Conjecture says that if $k_\lambda - |\lambda| = 0$, then $h(\lambda) = k_\lambda + 1$. By Theorem 1, if $k_\lambda - |\lambda| \geq 1$, then $h(\lambda) \leq k_\lambda$, provided that $k_\lambda \geq 5$. It seems natural that if $k_\lambda - |\lambda|$ gets bigger, then $k_\lambda - h(\lambda)$ also gets bigger, provided that k_λ is sufficiently large. The next result shows this is true for various λ .

Theorem 3. *For each integer $a \geq 0$, there exists an integer $t_1 = t_1(a)$ such that for every integer $t \geq t_1$, there exists a K_t -minor-free graph that is not $\{1 \star (t-2a-6), 3a+6\}$ -choosable.*

For the λ in Theorem 3, $k_\lambda = t+a$, $h(\lambda) \leq t-1$ and $|\lambda| = t - (2a+5)$. As $k_\lambda - |\lambda| = 3a+5$ tends to infinity, the difference $k_\lambda - h(\lambda) \geq a+1$ also tends to infinity, provided that $k_\lambda \geq \phi(k_\lambda - |\lambda|)$, where ϕ is a certain given function. It remains open whether such a conclusion holds for all λ . We conjecture a positive answer.

Conjecture 4. *There are functions $\phi, \psi : \mathbb{N} \rightarrow \mathbb{N}$ for which the following hold:*

- $\lim_{n \rightarrow \infty} \psi(n) = \infty$.
- *For any multiset λ of positive integers, if $k_\lambda \geq \phi(k_\lambda - |\lambda|)$, then $k_\lambda - h(\lambda) \geq \psi(k_\lambda - |\lambda|)$.*

It is easy to see that if $k_\lambda - |\lambda| = b$, then $\{1 \star (k_\lambda - 2b'), 2 \star b'\}$ is a refinement of λ , where $b \geq b' \geq b/2$. Thus to prove Conjecture 4, it suffices to prove it for λ of the form $\{1 \star k_1, 2 \star k_2\}$.

Theorem 4 below shows that Conjecture 4 holds for any λ of the form $\{1 \star k_1, 3 \star k_2\}$.

Theorem 4. *For each integer $a \geq 0$, there exists an integer $t_2 = t_2(a)$ such that for every integer $t \geq t_2$, there exists a K_t -minor-free graph that is not $\{1 \star (t - 5a - 9), 3 \star (2a + 3)\}$ -choosable.*

As $|\lambda|$ becomes very small compared to k_λ , say $|\lambda|$ is constant and k_λ tends to infinity, then λ -choosability becomes very close to k_λ -choosability. The following result, which generalizes the main result of Steiner [22], deals with such λ .

Theorem 5. *For every $\varepsilon \in (0, 1)$ and $q \in \mathbb{N}$, there exists an integer $t_3 = t_3(q, \varepsilon)$ such that for every integer $t \geq t_3$ and $k_1, k_2, \dots, k_q \in \mathbb{N}$ satisfying*

$$\sum_{j=1}^q k_j \leq (2 - \varepsilon)t,$$

there exists a K_t -minor-free graph G that is not $\{k_1, k_2, \dots, k_q\}$ -choosable.

The $q = 1$ case of Theorem 5 was proved by Steiner [22].

1.3 Fractional Colouring

Next we consider the fractional version of Hadwiger's Conjecture. A b -fold colouring of a graph G is a mapping ϕ that assigns to each vertex v of G a set $\phi(v)$ of b colours, so that adjacent vertices receive disjoint colour sets. An (a, b) -colouring of G is a b -fold colouring ϕ of G such that $\phi(v) \subseteq \{1, 2, \dots, a\}$ for each vertex $v \in V(G)$. The *fractional chromatic number* of G is

$$\chi_f(G) := \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-colourable} \right\}.$$

The fractional version of Hadwiger's Conjecture was studied by Reed and Seymour [19], who proved that every K_t -minor-free graph G has fractional chromatic number at most $2t$.

An a -list assignment of G is a mapping L that assigns to each vertex v a set $L(v)$ of a permissible colours. A b -fold L -colouring of G is a b -fold colouring ϕ of G such that

$\phi(v) \subseteq L(v)$ for each vertex v . We say G is (a, b) -choosable if for any a -list assignment L of G , there is a b -fold L -colouring of G . The *fractional choice-number* of G is

$$\text{ch}_f(G) := \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-choosable} \right\}.$$

Alon, Tuza and Voigt [1] proved that for any graph G , if G is (a, b) -colourable, then G is (am, bm) -choosable for some integer m . So for any graph G , $\chi_f(G) = \text{ch}_f(G)$, and moreover the infimum in the definition of $\text{ch}_f(G)$ is attained and hence can be replaced by minimum.

We prove the following result by an argument parallel to the proofs in [22].

Theorem 6. *Let $\varepsilon \in (0, 1)$ be fixed. For every positive integer m , there exists $t_0 = t_0(\varepsilon)$ such that for every integer $t \geq t_0$ there exists a K_t -minor-free graph G that is not $((2 - \varepsilon)tm, m)$ -choosable.*

Note that the graph G in Theorem 6 depends on m (as well as on ε). The result of Reed and Seymour implies that for every K_t -minor-free graph G , there is a constant m such that G is $(2tm, m)$ -choosable. Here the integer m depends on G . Theorem 6 has no implication for the fractional choice number of G .

2 A key lemma

Let G_1 and G_2 be graphs, and S_i be a k -clique in G_i for $i = 1, 2$. We say a graph G is a k -clique-sum of G_1 and G_2 (on S_1 and S_2), if G is obtained from the disjoint union of G_1 and G_2 by identifying pairs of the vertices of S_1 and S_2 to form a single shared clique and then possibly deleting some of the clique edges. The lemma is well-known and easily proved.

Lemma 7. *Let G_1 and G_2 be K_t -minor-free graphs. If G is a k -clique-sum of G_1 and G_2 (on cliques S_1 of G_1 and S_2 of G_2), then G is K_t -minor-free.*

Definition 2. Assume $\lambda = \{k_1, k_2, \dots, k_q\}$ is a multiset of positive integers, G is a graph and $K = \{v_1, v_2, \dots, v_p\}$ is a clique in G , and $\mathcal{C} = (C_1, C_2, \dots, C_q)$ is a q -tuple of disjoint colour sets. A (λ, \mathcal{C}) -list assignment of G is a list assignment L of G such that for each vertex v , $|L(v) \cap C_i| \geq k_i$.

Assume $K' = \{v'_1, v'_2, \dots, v'_p\}$ is a p -clique (disjoint from G), and ψ' is a proper colouring of K' . A (λ, \mathcal{C}) -list assignment L of G is a ψ' -obstacle for (G, K, \mathcal{C}) if the colouring ψ of K defined as $\psi(v_i) = \psi'(v'_i)$ is an L -colouring of K that cannot be extended to a proper L -colouring of G .

The following lemma will be used in some of our proofs.

Lemma 8. *Let t be a positive integer and $\lambda = \{k_1, k_2, \dots, k_q\}$ be a multiset of positive integers. Assume there are K_t -minor-free graphs H_1 and H_2 , a clique $K = \{v_1, v_2, \dots, v_p\}$ in H_1 , a q -tuple of disjoint colour sets $\mathcal{C} = (C_1, C_2, \dots, C_q)$ and a (λ, \mathcal{C}) -list assignment L of H_2 , for which the following holds:*

- for any L -colouring ψ of H_2 , there is a p -clique $K_\psi = \{v_{\psi,1}, v_{\psi,2}, \dots, v_{\psi,p}\}$ in H_2 , such that there exists a $\psi|_{K_\psi}$ -obstacle L_ψ for (H_1, K, \mathcal{C}) .

Then there is a K_t -minor-free graph G that is not λ -choosable.

Proof. We shall construct a graph G and a λ -list assignment L' of G so that G is K_t -minor-free and G is not L' -colourable. Let H_1, H_2 be graphs, K be a p -clique in H_1 , and L be a (λ, \mathcal{C}) -list assignment of H_2 , satisfying the assumption of the lemma.

Now we start the construction of G and L' . First we take a copy of H_2 , and let L be the (λ, \mathcal{C}) -list assignment of H_2 as above.

For each proper L -colouring ψ of H_2 , choose a p -clique $K_\psi = \{v_{\psi,1}, v_{\psi,2}, \dots, v_{\psi,p}\}$ in H_2 , for which there is a $\psi|_{K_\psi}$ -obstacle L_ψ for (H_1, K, \mathcal{C}) . Take a copy H_ψ of H_1 . Let $K'_\psi = \{v'_{\psi,1}, v'_{\psi,2}, \dots, v'_{\psi,p}\}$ be the copy of K in H_ψ (where $v'_{\psi,i}$ is the copy of $v_i \in V(K)$ in K'_ψ). Identify K_ψ with K'_ψ in such a way that $v_{\psi,i}$ is identified with $v'_{\psi,i}$. Extend the list assignment L to $V(H_\psi) - K'_\psi$ by letting $L'(v_\psi) = L_\psi(v)$, where v_ψ is the copy of $v \in H_1$ in H_ψ .

This completes the construction of the graph G and the list assignment L' of G . Observe that for each proper L -colouring ψ of H_2 , we have chosen a p -clique $K_\psi = \{v_{\psi,1}, v_{\psi,2}, \dots, v_{\psi,p}\}$ in H_2 . A vertex v of H_2 may be contained in many copies of p -cliques, say v is contained in $K_{\psi_1}, K_{\psi_2}, \dots, K_{\psi_s}$. Then v has different names in these copies of cliques. The name for v in K_ψ is only used to find its partner vertex (the vertex to be identified with v) in H_ψ . So this leads to no confusion.

It follows from Lemma 7 that G is K_t -minor-free, and L' is a (λ, \mathcal{C}) -list assignment of G .

Now we show that G is not L' -colourable. Assume to the contrary that there is a proper L' -colouring ϕ of G . Let ψ be the restriction of ϕ to H_2 . The restriction of ϕ to H_ψ is an L_ψ -colouring of H_ψ . But L_ψ is a $\psi|_{K_\psi}$ -obstacle for (H_1, K, \mathcal{C}) , a contradiction. \square

3 Proofs of the theorems

Theorem 1. *For every integer $t \geq 5$, there exists a K_t -minor-free graph that is not $\{1 \star (t-3), 2\}$ -choosable.*

Proof. The proof is by induction on t . For $t = 5$, a non- $\{1, 1, 2\}$ -choosable planar graph (hence a K_5 -minor-free graph) was constructed in [11]. Assume $t \geq 6$ and there exists a K_{t-1} -minor-free graph G_{t-1} that is not λ -choosable, where $\lambda = \{1 \star (t-4), 2\}$.

Let L be a λ -list assignment of G_{t-1} , such that G_{t-1} is not L -colourable. Let $\mathcal{C} = (C_1, C_2, \dots, C_{t-3})$ be a $(t-3)$ -tuple of disjoint colour sets so that for each vertex v of G_{t-1} , $|L(v) \cap C_i| = 1$ for $1 \leq i \leq t-4$ and $|L(v) \cap C_{t-3}| = 2$. Zhu [28] showed that we may assume $C_i = \{c_i\}$ for $i = 1, 2, \dots, t-4$.

Let H_1 be the graph obtained from G_{t-1} by adding a vertex u adjacent to every vertex of G_{t-1} . Let

$$\mathcal{C}' = (C'_1, C'_2, \dots, C'_{t-2})$$

where $C'_i = C_i$ for $i = 1, 2, \dots, t-4$, $C'_{t-3} = \{c_{t-3}\}$ and $C'_{t-2} = C_{t-3}$. Let a, b be two colours from C_{t-3} . Let $\lambda' = \{1 \star (t-3), 2\}$ and let L' be the (λ', \mathcal{C}') -list assignment of H_1 defined as

$$L'(v) = \begin{cases} L(v) \cup \{c_{t-3}\}, & \text{if } v \in V(G_{t-1}), \\ \{c_1, c_2, \dots, c_{t-3}, a, b\}, & \text{if } v = u. \end{cases}$$

Let $K = \{u\}$ be the 1-clique in H_1 . If ψ is an L' -colouring of a copy of $K_1 = \{u\}$ with $\psi(u') = c_i$ for some $1 \leq i \leq t-3$, then L' is a ψ -obstacle for (H_1, K, \mathcal{C}') .

Let H_2 be a triangle and L'' be the (λ', \mathcal{C}') -list assignment of H_2 , defined as $L''(v) = \{c_1, c_2, \dots, c_{t-3}, a, b\}$ for each vertex v of H_2 . Then for any proper L'' -colouring ψ of H_2 , there is a vertex v (a copy of K_1) such that $\psi(v) = c_i$ for some $1 \leq i \leq t-3$. Hence L' is a $\psi|_{\{v\}}$ -obstacle for (H_1, K, \mathcal{C}') .

By Lemma 8, there is a K_t -minor-free graph G_t that is not λ' -choosable. \square

Theorem 3. *For each integer $a \geq 0$, there exists an integer $t_1 = t_1(a)$ such that for every integer $t \geq t_1$, there exists a K_t -minor-free graph that is not $\{1 \star (t-2a-6), 3a+6\}$ -choosable.*

Proof. Assume a is a positive integer. Let

$$m := \binom{2a+5}{a+3} \quad \text{and} \quad t_1 := (2a+5)m + 2.$$

Assume $t \geq t_1$. We shall construct a K_t -minor-free graph G that is not $\{1 \star (t-2a-6), 3a+6\}$ -choosable by using Lemma 8.

First, let H_1 be a graph with vertex set $A \cup B$ such that $A \cap B = \emptyset$ and

- A induces a $(2a+5)$ -clique, B induces a $(t-2)$ -clique,
- each vertex in B has exactly $a+3$ neighbours in A , and
- for each $(a+3)$ -subset X of A , if $B_X := \{v \in B : N_{H_1}(v) \cap A = X\}$, then

$$|B_X| \geq \left\lfloor \frac{t-2}{m} \right\rfloor.$$

It is easy to see that such a graph H_1 exists.

Claim 1. The graph H_1 is K_t -minor-free.

Proof. Assume that H_1 has a K_t -minor. Then there exists a collection \mathcal{Z} of t non-empty and pairwise disjoint subsets of $V(H_1)$ such that for each $Z \in \mathcal{Z}$, $H_1[Z]$ is connected, and for any two distinct $Z, Z' \in \mathcal{Z}$, there exists at least one edge in H_1 joining a vertex in Z to a vertex in Z' . In particular, for any $Z \in \mathcal{Z}$, there are at least $(t-1)$ vertices in $V(H_1) - Z$ adjacent to vertices in Z .

Since $|B| = t-2$, there are at least two subsets $Z \in \mathcal{Z}$ that are contained in A . As $|A| = 2a+5$, there exists $Z \in \mathcal{Z}$ such that $Z \subseteq A$ and $|Z| \leq a+2$.

Let X be an $(a+3)$ -subset of $A-Z$. Then

$$\begin{aligned}
|N_{H_1}(Z)| &\leq |V(H_1)| - |B_X| \leq (2a+5) + (t-2) - \left\lfloor \frac{t-2}{m} \right\rfloor \\
&< t + 2a + 4 - \frac{t-2}{m} \\
&= t + 2a + 4 - \frac{t-2}{\binom{2a+5}{a+3}} \\
&\leq t + 2a + 4 - (2a+5) \\
&= t - 1,
\end{aligned}$$

a contradiction. \square

Label the vertices in A as $v_1, v_2, \dots, v_{2a+5}$. Let

$$\{a_i : i \in [3a+6]\}, \{b_i : i \in [t-2a-6]\}, \{c_i : i \in [2a+3]\}$$

be pairwise disjoint colour sets. Let $\psi : A \rightarrow \{a_i : i \in [3a+6]\}$ be an injective mapping. Let L_ψ be the list assignment of H_1 defined as follows:

$$(LA) \quad L_\psi(v) = \{b_i : i \in [t-2a-6]\} \cup \{a_i : i \in [3a+6]\} \text{ for } v \in A.$$

$$(LB) \quad L_\psi(v) = \psi(N_A(v)) \cup \{b_i : i \in [t-2a-6]\} \cup \{c_i : i \in [2a+3]\} \text{ for } v \in B.$$

Let

$$\mathcal{C} = (C_1, C_2, \dots, C_{t-2a-5})$$

where $C_i = \{b_i\}$, for $i = 1, 2, \dots, t-2a-6$, $C_{t-2a-5} = \{a_i : i \in [3a+6]\} \cup \{c_i : i \in [2a+3]\}$. Let $\lambda = \{1 \star (t-2a-6), 3a+6\}$. Then L_ψ is a (λ, \mathcal{C}) -list assignment of H_1 : For each vertex v of H_1 , $|L_\psi(v) \cap C_i| = 1$ for $i = 1, 2, \dots, t-2a-6$, and $|L_\psi(v) \cap C_{t-2a-5}| = 3a+6$. Moreover, ψ is an L_ψ -colouring of A .

Claim 2. ψ cannot be extended to an L_ψ -colouring of H_1 .

Proof. Assume that H_1 has an L -colouring ϕ_ψ which is an extension of ψ . Then $\phi(v) = \psi(v)$ for each $v \in A$ and $\phi(v) \in \{b_i : i \in [t-2a-6]\} \cup \{c_i : i \in [2a+3]\}$ for every vertex $v \in B$. Thus the vertices of the $(t-2)$ -clique induced by B are coloured by $(t-2a-6) + (2a+3) = t-3$ colours, a contradiction. \square

Let H_2 be a $(t-1)$ -clique and L' be the $\{1 \star (t-2a-6), 3a+6\}$ -list assignment defined as

$$L'(v) = \{b_i : i \in [t-2a-6]\} \cup \{a_i : i \in [3a+6]\},$$

for each vertex v of H_2 . Then for any proper L' -colouring ψ of H_2 , there is a $(2a+5)$ -clique $K_\psi = \{v_{\psi,1}, v_{\psi,2}, \dots, v_{\psi,2a+5}\}$ in H_2 such that $\psi(v_{\psi,i}) \in \{a_j : j \in [3a+6]\}$ for $i \in [2a+5]$.

By Claim 2, L_ψ is a $\psi|_{K_\psi}$ -obstacle for $(H_1, H_1[A], \mathcal{C})$.

By Lemma 8, there is a K_t -minor-free graph G that is not $\{1 \star (t-2a-6), 3a+6\}$ -choosable. \square

Theorem 4. *For each integer $a \geq 0$, there exists an integer $t_2 = t_2(a)$ such that for every integer $t \geq t_2$, there exists a K_t -minor-free graph that is not $\{1 \star (t - 5a - 9), 3 \star (2a + 3)\}$ -choosable.*

Proof. Assume a is a positive integer. Let

$$m := 3^{a+2} \quad \text{and} \quad t_2 := (2a + 5)m + a + 3.$$

Assume $t \geq t_2$. We shall construct a K_t -minor-free graph G that is not $\{1 \star (t - 4a - 9), 3 \star (2a + 3)\}$ -choosable by using Lemma 8.

Let H_1 be a graph with vertex set $(A \cup B)$ such that $A \cap B = \emptyset$ and

- A induces a $3(a + 2)$ -clique, B induces a $(t - a - 3)$ -clique.
- $\{A_1, A_2, \dots, A_{a+2}\}$ is a partition of A with $|A_i| = 3$ for $i \in [a + 2]$ and $T = \{X \subseteq A : |X \cap A_i| = 2, \text{ for each } i \in [a + 2]\}$. For each vertex $v \in B$, $N_A(v) \in T$, and for each $X \in T$,

$$|\{v \in B : N_A(v) = X\}| \geq \lfloor \frac{t - a - 3}{|T|} \rfloor = \lfloor \frac{t - a - 3}{m} \rfloor.$$

It is easy to see that such a graph H_1 exists.

Claim 3. The graph H_1 is K_t -minor-free.

Proof. Assume that H_1 has a K_t -minor. Then there exists a collection \mathcal{Z} of non-empty and pairwise disjoint subsets of $V(H_1)$ such that for each $Z \in \mathcal{Z}$, $H_1[Z]$ is connected, and for any two distinct $Z, Z' \in \mathcal{Z}$, there exists at least one edge in H_1 joining a vertex in Z to a vertex in Z' . In particular, for any $Z \in \mathcal{Z}$, there are at least $(t - 1)$ vertices in $V(H_1) - Z$ adjacent to vertices in Z .

Since $|B| = t - a - 3$, there are at least $(a + 3)$ subsets $Z \in \mathcal{Z}$ that are contained in A . As the partition of A has $(a + 2)$ parts A_1, A_2, \dots, A_{a+2} and $|A_i| = 3$ for $i \in [a + 2]$, there exists $Z \in \mathcal{Z}$ such that $|Z \cap A_i| \leq 1$ for each $i \in [a + 2]$.

Let X be a $2(a + 2)$ -subset of $A - Z$ such that $|X \cap A_i| = 2$ for each $i \in [a + 2]$. Let $B_X = \{v \in B : N_A(v) = X\}$. Then

$$\begin{aligned} |N_{H_1}(Z)| &\leq |V(H_1)| - |B_X| \leq 3(a + 2) + (t - a - 3) - \lfloor \frac{t - a - 3}{m} \rfloor \\ &< t + 2a + 4 - \frac{t - a - 3}{m} \\ &= t + 2a + 4 - \frac{t - a - 3}{3^{a+2}} \\ &\leq t + 2a + 4 - (2a + 5) \\ &= t - 1, \end{aligned}$$

a contradiction. □

Label the vertices in A_i as $A_i = \{u_1^i, u_2^i, u_3^i\}$ for each $i \in [a+2]$. Let

$$\bigcup_{i \in [2a+3]} \{d_1^i, d_2^i, d_3^i\}, \{b_i : i \in [t-5a-9]\}, \{c_i : i \in [2a+3]\}, \bigcup_{i \in [2a+3]} \{c_1^i, c_2^i, c_3^i\}$$

be pairwise disjoint colour sets.

Let $\psi : A \rightarrow \bigcup_{i \in [2a+3]} \{d_1^i, d_2^i, d_3^i\}$ be an injective mapping such that for each $i \in [a+2]$ there exists $i_0 \in [2a+3]$, $\psi(u_j^i) = d_{j_0}^{i_0}$ for $j \in [3]$. Let

$$I(\psi) = \{i_0 \in [2a+3] : \text{there exists } i \in [a+2] \text{ such that } \psi(u_j^i) = d_{j_0}^{i_0} \text{ for } j \in [3]\}.$$

Note that $|I(\psi)| = a+2$. Let L_ψ be the list assignment of H_1 defined as follows:

$$(LA') \quad L_\psi(v) = \bigcup_{j \in [2a+3]} \{d_1^j, d_2^j, d_3^j\} \cup \{b_i : i \in [t-5a-9]\} \text{ for } v \in A;$$

$$(LB') \quad L_\psi(v) = \psi(N_A(v)) \cup \{b_i : i \in [t-5a-9]\} \cup \{c_i : i \in I(\psi)\} \cup \bigcup_{i \in [2a+3] \setminus I(\psi)} \{c_1^i, c_2^i, c_3^i\}, \\ \text{for } v \in B.$$

Let

$$\mathcal{C} = (C_1, C_2, \dots, C_{t-3a-6})$$

where $C_i = \{b_i\}$, for $i = 1, 2, \dots, t-5a-9$, $C_{t-5a-9+j} = \{d_1^j, d_2^j, d_3^j, c_j, c_1^j, c_2^j, c_3^j\}$, for $j = 1, 2, \dots, 2a+3$. Let $\lambda = \{1 \star (t-4a-9), 3 \star (2a+3)\}$. Then L_ψ is a (λ, \mathcal{C}) -list assignment of H_1 : For each vertex v of B , if $i = 1, 2, \dots, t-5a-9$, $|L_\psi(v) \cap C_i| = 1$; If $j \in I(\psi)$, $|L_\psi(v) \cap C_{t-5a-9+j}| = |\bigcup_{u_j^i \in N_A(v)} \psi(u_j^i) \cup \{c_j\}| = 3$; If $j \in [2a+3] \setminus I(\psi)$, $|L_\psi(v) \cap C_{t-5a-9+j}| = |\{c_1^j, c_2^j, c_3^j\}| = 3$. Moreover, ψ is an L_ψ -colouring of A .

Claim 4. ψ cannot be extended to an L_ψ -colouring of H_1

Proof. Assume that H_1 has an L_ψ -colouring ϕ which is an extension of ψ . Then $\phi(v) = \psi(v)$, for $v \in A$, and hence $\phi(v) \in \{b_i : i \in [t-5a-9]\} \cup \{c_i : i \in I(\psi)\} \cup \bigcup_{i \in [2a+3] \setminus I(\psi)} \{c_1^i, c_2^i, c_3^i\}$ for every vertex $v \in B$. Thus the $(t-a-3)$ -clique induced by B are coloured by $(t-5a-9) + (a+2) + 3(a+1) = t-a-4$ colours, a contradiction. \square

Let H_2 be a $(t-1)$ -clique and L' be the $\{1 \star (t-5a-9), 3 \star (2a+3)\}$ -list assignment defined as

$$L'(v) = \bigcup_{j \in [2a+3]} \{d_1^j, d_2^j, d_3^j\} \cup \{b_i : i \in [t-5a-9]\},$$

for each vertex v of H_2 .

Assume ψ is a proper L' -colouring of H_2 . At least $5a+8$ vertices of H_2 are coloured by colours from $\bigcup_{j \in [2a+3]} \{d_1^j, d_2^j, d_3^j\}$. Hence there is a $3(a+2)$ -clique $K_\psi = \bigcup_{i \in [a+2]} \{u_{\psi,1}^i, u_{\psi,2}^i, u_{\psi,3}^i\}$ in H_2 such that for each $i \in [a+2]$ there exists $i_0 \in [2a+3]$, $\psi(u_{\psi,j}^i) = d_{j_0}^{i_0}$ and for $j \in [3]$.

By Claim 4, L_ψ is a $\psi|_{K_\psi}$ -obstacle for $(H_1, H_1[A], \mathcal{C})$.

By Lemma 8, there is a K_t -minor-free graph G that is not $\{1 \star (t-5a-9), 3 \star (2a+3)\}$ -choosable. \square

Next, we prove Theorems 5 and 6 by using a construction similar to that used by Steiner [22], who proved the following lemma using a probabilistic approach.

Lemma 9. *For every $\varepsilon \in (0, 1)$, there is $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$, there exists a graph H whose vertex set $V(H)$ can be partitioned into two disjoint sets A and B of size n , and such that the following properties hold:*

1. *Both A and B are cliques of H ;*
2. *Every vertex in H has at most εn non-neighbors in H ;*
3. *For $t = \lceil (1 + 2\varepsilon)n \rceil$, H does not contain K_t as a minor.*

Theorem 5. *For every $\varepsilon \in (0, 1)$ and $q \in \mathbb{N}$, there exists an integer $t_3 = t_3(q, \varepsilon)$ such that for every integer $t \geq t_3$ and $k_1, k_2, \dots, k_q \in \mathbb{N}$ satisfying*

$$\sum_{j=1}^q k_j \leq (2 - \varepsilon)t,$$

there exists a K_t -minor-free graph G that is not $\{k_1, k_2, \dots, k_q\}$ -choosable.

Proof. Let $\varepsilon \in (0, 1)$ and $q \in \mathbb{N}$ be given. Pick some $\varepsilon' \in (0, 1)$ such that $\frac{2 - q\varepsilon'}{1 + 2\varepsilon'} \geq 2 - \frac{\varepsilon}{2}$. Let $n_0 = n_0(\varepsilon') \in \mathbb{N}$ be as in Lemma 9, and define $t_0 := \max\{\lceil (1 + 2\varepsilon')n_0 \rceil, \lceil \frac{6}{\varepsilon} \rceil\}$. Let $t \geq t_0$ be any given integer. Define $n := \lfloor \frac{t}{1 + 2\varepsilon'} \rfloor \geq n_0$ and then $t \geq (1 + 2\varepsilon')n$.

Applying Lemma 9, there exists a graph H whose vertex set is partitioned into two sets A and B of size n , such that both A and B form cliques in H , every vertex in H has at most $\varepsilon'n$ non-neighbors, and H is K_t -minor-free.

Let $X_1, X_2, \dots, X_q, Y_1, Y_2, \dots, Y_q$ be pairwise disjoint subsets of \mathbb{N} , with $|X_j| = k_j, |Y_j| = \varepsilon'n$ for each $1 \leq j \leq q$. For each injection c from vertices in A to $X_1 \cup X_2 \cup \dots \cup X_q$, let H_c be a copy of H with the vertex set $A_c \cup B_c$ and G be a graph obtained from all copies of H by identifying the different copies of $v \in A$ into a single vertex for each vertex $v \in A$. Denote the vertex set of G by $A \cup \bigcup_c B_c$. Since H is K_t -minor-free and the set A forms a clique of size n , G is K_t -minor-free by repeated application of Lemma 7.

Consider an assignment $L : V(G) \rightarrow 2^{\mathbb{N}}$ as follows: For every vertex $x \in A$, we define $L(x) := \bigcup_{j=1}^q X_j$, and for every vertex $y \in B_c$ for some injection c from vertices in A to $X_1 \cup X_2 \cup \dots \cup X_q$, define

$$L(y) := \bigcup_{j=1}^q (X_j \cup Y_j) \setminus \bigcup_{x \in A, xy \notin E(G)} c(x).$$

Let C_1, C_2, \dots, C_q where $C_j = X_j \cup Y_j$ for $1 \leq j \leq q$, and $\mathcal{C} = \{C_1, C_2, \dots, C_q\}$. Let $\lambda = \{k_1, k_2, \dots, k_q\}$. Now we show that L is a (λ, \mathcal{C}) -list assignment of G . For each $1 \leq j \leq q$, $|L(v) \cap C_j| = k_j$ if $v \in A$, and $|L(v) \cap C_j| \geq k_j - \varepsilon'n + \varepsilon'n = k_j$ if $v \in V(G) \setminus A$.

It remains to prove that G is not L -colourable. Assume to the contrary that there exist an L -colouring ϕ of G . Let c be the restriction of ϕ to A . Then c is an injection

c from vertices in A to $X_1 \cup X_2 \cup \dots \cup X_q$. Consider the colouring restricted to H_c . Note that $|\bigcup_{v \in V(H_c)} L(v)| = |\bigcup_{j=1}^q X_j \cup Y_j| = \sum_{j=1}^q k_j + q \cdot \varepsilon' n$. Since $\frac{2-q\varepsilon'}{1+2\varepsilon'} \geq 2 - \frac{\varepsilon}{2}$ and $n = \lfloor \frac{t}{1+2\varepsilon'} \rfloor \geq \frac{t}{1+2\varepsilon'} - 1$,

$$\sum_{j=1}^q k_j \leq (2 - \varepsilon)t = (2 - \frac{\varepsilon}{2})t - \frac{\varepsilon}{2}t \leq \frac{2-q\varepsilon'}{1+2\varepsilon'}t - \frac{\varepsilon}{2}t \leq (2 - q\varepsilon')(n+1) - \frac{\varepsilon}{2}t.$$

Since $t \geq t_0 \geq \frac{6}{\varepsilon}$,

$$|\bigcup_{v \in V(H_c)} L(v)| = \sum_{j=1}^q k_j + q\varepsilon'n \leq (2 - q\varepsilon')(n+1) - \frac{\varepsilon}{2}t + q\varepsilon'n < 2(n+1) - \frac{\varepsilon}{2}t \leq 2n - 1.$$

Since $|V(H_c)| = 2n$ and $|\bigcup_{v \in V(H_c)} L(v)| \leq 2n - 1$, there are two vertices x, y in H_c for which $\phi(x) = \phi(y)$. Since A and B_c form cliques in H_c , we may assume that $x \in A$ and $y \in B_c$. Now $\phi(x) = \phi(y)$ implies that $xy \notin E(G)$. But then $\phi(x) = c(x) \notin L(y)$, a contradiction. \square

Theorem 6. *Let $\varepsilon \in (0, 1)$ be fixed. For every positive integer m , there exists $t_0 = t_0(\varepsilon)$ such that for every integer $t \geq t_0$ there exists a K_t -minor-free graph G that is not $((2 - \varepsilon)tm, m)$ -choosable.*

Proof. Let $\varepsilon \in (0, 1)$ be given. Pick some $\varepsilon' \in (0, 1)$ such that $\frac{2-\varepsilon'}{1+2\varepsilon'} \geq 2 - \frac{\varepsilon}{2}$. Let $n_0 = n_0(\varepsilon') \in \mathbb{N}$ be as in Lemma 9, and define $t_0 := \max\{\lceil (1 + 2\varepsilon')n_0 \rceil, \lfloor \frac{6}{\varepsilon} \rfloor\}$. Now, let $t \geq t_0$ be any given integer. Define $n := \lfloor \frac{t}{1+2\varepsilon'} \rfloor \geq n_0$ and then $t \geq (1 + 2\varepsilon')n$.

By Lemma 9, there exists a graph H whose vertex set is partitioned into two non-empty sets A and B of size n , such that both A and B form cliques in H , every vertex in H has at most $\varepsilon'n$ non-neighbors, and H is K_t -minor-free.

For any fixed positive integer m , let D be the family of all m -subsets of $[2nm - 1] = \{1, 2, \dots, 2nm - 1\}$. For each injection c from vertices in A to D , let H_c be a copy of H with the vertex set $A_c \cup B_c$ and G be a graph obtained from all copies of H by identifying the different copies of $v \in A$ into a single vertex for each vertex $v \in A$. Denote the vertex set of G by $A \cup \bigcup_c B_c$. Since H is K_t -minor-free and the set A forms a clique of size n , G is K_t -minor-free by repeated application of Lemma 7.

Consider an assignment $L : V(G) \rightarrow 2^{\mathbb{N}}$ to vertices in G as follows: For every vertex $x \in A$, we define $L(x) := [2nm - 1]$, and for every vertex $y \in B_c$ for some injection c from vertices in A to D , define

$$L(y) := [2nm - 1] \setminus \bigcup_{x \in A, xy \notin E(G)} c(x).$$

Recall that every vertex in H has at most $\varepsilon'n$ non-neighbors. So $|L(v)| \geq 2nm - 1 - \varepsilon'nm$ for every vertex $v \in V(G)$.

It remains to prove that G does not admit a m -fold L -colouring, which will then prove that G is not $((2 - \varepsilon')nm - 1, m)$ -choosable. Assume to the contrary that there exists an

m -fold L -colouring ϕ of G . Let c be the restriction of ϕ to A , and then c is an injection from vertices in A to D .

Consider the colouring restricted to the subgraph induced by $H_c := A \cup B_c$ in G . Since $|V(H_c)| = 2n$ and $\bigcup_{v \in V(H_c)} L(v) = [2nm - 1]$, there are two vertices x, y in H_c which have $\phi(x) \cap \phi(y) \neq \emptyset$. Since both of A and B_c form a clique in H_c , there exists $x \in A$, $y \in B_c$ and a colour $i \in [2nm - 1]$ such that $xy \notin E(G)$ and $i \in \phi(x) \cap \phi(y)$. Thus $i \in c(x)$ and hence $i \notin L(y)$, a contradiction.

Since $t \geq t_0 \geq \frac{6}{\varepsilon}$,

$$\begin{aligned} (2 - \varepsilon')nm - 1 &= (2 - \varepsilon') \left\lfloor \frac{t}{1+2\varepsilon'} \right\rfloor m - 1 \\ &> (2 - \varepsilon') \left(\frac{t}{1+2\varepsilon'} - 1 \right) m - 1 \\ &\geq (2 - \frac{\varepsilon}{2})tm - (2m - \varepsilon'm + 1) \\ &\geq (2 - \varepsilon)tm. \end{aligned}$$

Hence, we conclude that G is a K_t -minor-free graph that is not $((2 - \varepsilon)tm, m)$ -choosable. \square

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