Edges not covered by monochromatic bipartite graphs

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Abstract

Let $f_k(n, H)$ denote the maximum number of edges not contained in any monochromatic copy of H in a k-coloring of the edges of K_n , and let ex(n, H) denote the Turán number of H. In place of $f_2(n, H)$ we simply write f(n, H). In [5], Keevash and Sudakov proved that f(n, H) = ex(n, H) if H is an edge-critical graph or C_4 and asked if this equality holds for any graph H. All known exact values of this question require H to contain at least one cycle. In this paper we focus on acyclic graphs and have the following results:

(1) We prove f(n, H) = ex(n, H) when H is a spider or a double broom.

(2) A *tail* in H is a path $P_3 = v_0 v_1 v_2$ such that v_2 is only adjacent to v_1 and v_1 is only adjacent to v_0, v_2 in H. We obtain a tight upper bound for f(n, H) when H is a bipartite graph with a tail. This result provides the first bipartite graphs which answer the question of Keevash and Sudakov in the negative.

(3) Liu, Pikhurko and Sharifzadeh [6] asked if $f_k(n,T) = (k-1)ex(n,T)$ when T is a tree. We provide an upper bound for $f_{2k}(n, P_{2k})$ and show it is tight when 2k - 1 is prime. This provides a negative answer to their question.

1 Introduction

Given any graph H, the classical theorem of Ramsey asserts that there exists an integer R(H, H)such that every 2-coloring of the edges of the complete graph K_n with $n \ge R(H, H)$ contains a monochromatic copy of H. A natural extension of this problem is determining how many monochromatic copies of H there are. For the case of $H = K_3$, this question was answered by Goodman [4] and the case of $H = K_4$ was settled by Thomason [10].

In a different direction, one can ask how many edges must be contained in some monochromatic copy of H in every 2-coloring of the edges of K_n (equivalently how many edges there can be in a 2-coloring which are not contained in any monochromatic copy of H). The first result about this topic is due to Erdős, Rousseau and Schelp [2]. They considered the maximum number of edges not contained in any monochromatic triangle in a 2-coloring of the edges of K_n . Erdős also wrote "many further related questions can be asked" in [2]. In this paper, we will consider problems of this type.

Let c be a 2-coloring of the edges of K_n and let H be a graph. If an edge of K_n is not contained in any monochromatic copy of H, then we say it is NIM-H. Let E(c, H) denote the

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set of all NIM-H edges in K_n under the 2-edge-coloring c and let

$$f(n, H) = \max \{ |E(c, H)| : c \text{ is a 2-edge-coloring of } K_n \}.$$

Let ex(n, H) be the Turán number of H. If one considers a 2-coloring of the edges of K_n in which one of the colors yields an extremal graph for H, then it is easy to see

$$f(n,H) \ge \exp(n,H). \tag{1}$$

As observed by Alon, the result on $f(n, K_3)$ by Erdős, Rousseau and Schelp [2] can also be deduced from a result of Pyber [9] (see [5]). In [5], Keevash and Sudakov studied f(n, H)systematically. They proved that if H contains an edge e such that $\chi(H - e) < \chi(H)$ or $H = C_4$, then equality holds in (1) for sufficiently large n. Furthermore, they asked if the equality holds for all H.

Question 1 (Keevash, Sudakov [5]). Is it true that for any graph H we have f(n, H) = ex(n, H) when n is sufficiently large?

In 2017, Ma [7] provided an affirmative answer to Question 1 for an infinite family of bipartite graphs H, including all even cycles and complete bipartite graphs $K_{s,t}$ for $t > s^2 - 3s + 3$ or $(s,t) \in \{(3,3), (4,7)\}$. In 2019, Liu, Pikhurko and Sharifzadeh [6] extended Ma's result by providing a larger family of bipartite graphs for which f(n, H) = ex(n, H) holds (however, the graphs they construct still contain a cycle). Surprisingly, Yuan [11] recently found an example showing that the assertion in Question 1 does not hold in general.

Theorem 1 (Yuan [11]). Let $p \ge t + 1 \ge 4$ and K_t^{p+1} denote the graph obtained from K_t by replacing each edge of K_t with a clique K_{p+1} . When n is sufficiently large, then

$$f(n, K_t^{p+1}) = \exp(n, K_t^{p+1}) + {\binom{t-1}{2}}{2}.$$

Based on this result, he conjectured the following.

Conjecture 1 (Yuan [11]). Let H be any graph and n be sufficiently large. Then there exists a constant C = C(H) such that f(n, H) = ex(n, H) + C.

As mentioned earlier, the known results about the exact value of f(n, H) require that H contains a cycle. For acyclic graphs and some other bipartite graphs, the situation is less clear. Thus, in this paper, we will focus on this case. A *spider* is the graph consisting of t paths with one common end vertex such that all other vertices are distinct. A *double broom* with parameters t, s_1 and s_2 is the graph consisting of a path with t vertices with s_1 and s_2 distinct leaves appended to each of its respective end vertices.

Theorem 2. Let H a spider or a double broom with $s_1 < s_2$ and n be sufficiently large, we have

$$f(n,H) = \exp(n,H).$$

A *tail* in a (not necessary acyclic) graph H is a path $P_3 = v_0v_1v_2$ such that v_2 is only adjacent to v_1 and v_1 is only adjacent to v_0 and v_2 .

Theorem 3. Let H = (A, B, E) be a bipartite graph containing a tail and $|A| \leq |B|$. When n is sufficiently large, we have

$$f(n,H) \le \exp(n,H) + \binom{|A|-1}{2}.$$
(2)

Furthermore, the upper bound is tight.

Remark 1. In Theorem 3, there are many bipartite graphs H such that f(n, H) achieves an upper bound greater than ex(n, H). This implies that even for the bipartite case, the answer to Question 1 can be negative. However, the graphs from Theorem 3 satisfy Conjecture 1.

We will also consider the case of edge colorings with 3 or more colors. Let $f_k(n, H)$ be the maximum number of edges not contained in any monochromatic copy of H in a k-coloring of the edges of K_n . Thus, $f_2(n, H) = f(n, H)$. It appears likely that for $k \ge 3$, the function $f_k(n, H)$ has different behavior for bipartite graphs and non-bipartite graphs. For non-bipartite graphs, one can see that $f_k(n, H) \ne (k-1)\exp(n, H)$ since $(k-1)\exp(n, H) \ge \binom{n}{2}$.

For a tree T, Ma [7] constructed a lower bound by taking random overlays of k-1 copies of extremal T-free graphs, and the construction implies $f_k(n,T) \ge (k-1-o(1))\exp(n,T)$. Liu, Pikhurko and Sharifzadeh [6] showed that this lower bound is asymptotically correct.

Theorem 4 (Liu, Pikhurko, Sharifzadeh [6]). Let T be a tree with h vertices. Then there exists a constant C(k, h) such that for all sufficiently large n, we have

$$|f_k(n,T) - (k-1)\exp(n,T)| \le C(k,h)$$

For more general bipartite graph H, Ma [7] wrote "it may be reasonable to ask if $f_k(n, H) = (k-1)\exp(n, H)$ holds for sufficiently large n". However, this is not true for disconnected bipartite graphs. Liu, Pikhurko and Sharifzadeh [6] gave an example and showed $f_k(n, 2K_2) = (k-1)\exp(n, 2K_2) - {\binom{k-1}{2}}$. Based on this example, Liu, Pikhurko and Sharifzadeh [6] asked the following question.

Question 2 (Liu, Pikhurko, Sharifzadeh [6]). Is it true that $f_k(n,T) = (k-1)ex(n,T)$ for any tree T and sufficiently large n?

Our third result concerns the case when T is a path with an even number of vertices and yields a negative answer to Question 2.

Theorem 5. Let $k \ge 1$ and $n \ge (2k)^{2k^2}$ be integers. We have

$$f_{2k}(n, P_{2k}) \le (2k - 1) \exp(n, P_{2k}) + (k - 1) \binom{2k - 1}{2}.$$

 $Furthermore, \ equality \ holds \ when \ 2k-1 \ is \ a \ prime \ and \ n \in \big\{a(2k-1)+(k-1), \ a(2k-1)+k\big\}.$

Notation and organization. For a given graph G, we use e(G) to denote the number of edges of G. For a subset of vertices X, let G[X] denote the subgraph induced by X and G - X denote the subgraph induced by $V(G) \setminus X$. For two disjoint subset X, Y, let G[X, Y] denote the bipartite subgraph of G consisting of the edges of G with one end vertex in X and the other in Y. In a red-blue edge-colored complete graph K_n , we say that u is a red (or blue) neighbor of v if the edge uv is red (or blue). For a set X of vertices, let $N_r(v, X)$ and $N_b(v, X)$ denote the red and blue neighbors of v in X, respectively. Let $d_r(v, X) = |N_r(v, X)|$ and $d_b(v, X) = |N_b(v, X)|$. If $X = V(K_n)$, then we simply write $d_r(v)$ and $d_b(v)$. For two graphs G and H, we use $G \cup H$ to denote the disjoint union of G and H. Let G + H be the graph obtained from $G \cup H$ by adding all edges with one end vertex in V(G) and one end vertex in V(H).

The rest of the paper is organized as follows. In Sections 2 and 3, we study the function f(n, H) and prove Theorems 2 and 3, respectively. In Section 4, we study the general function $f_k(n, H)$ and prove Theorem 5.

2 Proof of Theorem 2

Let H be a spider or a double broom on k vertices and c be a red-blue edge-coloring of K_n with |E(c, H)| being maximum. If E(c, H) contains no H, then

$$f(n,H) = |E(c,H)| \le \exp(n,H),$$

and we are done. Hence we may assume there is a non-monochromatic copy of H in E(c, H).

Since we can take n to be larger than the Ramsey number $R(k^2, k^2)$, it follows, without loss of generality, that K_n contains a blue clique K of size at least k^2 . We partition $V(K_n)$ into two parts X and Y such that Y is maximal with the property that any vertex v in Y has $d_b(v, Y) \ge k$ and X consists of the remaining vertices. Note that the large blue clique K is contained in Y, and hence $|Y| \ge k^2$. Since each vertex in Y has blue degree at least k in Y, every blue edge in Y or between X and Y can be extended to a blue copy of H. Hence, all blue NIM-H edges are contained in X and $|X| \ge 2$.

For each vertex u in X, we have $d_b(u, Y) \leq (k-1)$. Thus for each subset X' of X, the subset $Y' = Y \setminus N_b(X', Y)$ is such that $K_n[X', Y']$ is a red complete bipartite graph and $|Y'| \geq |Y| - (k-1)|X'|$. We call Y' the corresponding subset of X'.

First assume $|X| \ge \lfloor \frac{k}{2} \rfloor + 1$. For each red edge uv contained in X or between X and Y, we can find a subset $X' \subseteq X$ of size $\lfloor \frac{k}{2} \rfloor$ that contains exactly one of u and v. Using the corresponding subset Y' of X', this red edge uv can be extended to a red copy of H. Hence all red NIM-H edges are contained in Y and

$$|E(c,H)| \le \exp(|Y|,H) + \exp(|X|,H) \le \exp(n,H).$$

Therefore, in the rest of the proof, we will assume $|X| \leq \lfloor \frac{k}{2} \rfloor$. Furthermore, each red edge in Y is NIM-H, otherwise we replace the color of this edge by blue and since E(c, H) is maximum, it has no changes.

Next we distinguish two cases based on whether H is a spider or a double broom.

The proof when H is a spider. Let H be a spider consisting of t paths with a common initial vertex v_0 . We call each path starting from v_0 a branch, and we assume that the lengths of these t branches are ℓ_1, \ldots, ℓ_t such that $v(H) = k = 1 + \sum_{i=1}^t \ell_i$.

Now we choose a copy of H from E(c, H) and denote it by H'. Let $X' = X \cap V(H')$. Since H' contains blue edges and all NIM-H blue edges are contained in X, we have $X' \neq \emptyset$ and the corresponding subset Y' is of size at least

$$|Y| - (k-1)|X'| \ge k.$$

For every branch of H', we apply the following method to replace all blue edges with red edges. First, every branch consisting entirely of blue edges is replaced by a red path of the same length in $K_n[X', Y']$. This can be done since $K_n[X', Y']$ is a complete bipartite graph consisting of only red edges and Y' is large enough. For any remaining branch $v_0v_1 \ldots v_{\ell_m}$, let v_iv_{i+1} be the first red edge on this branch, i.e., every edge in the path $v_0v_1 \ldots v_i$ is blue. If i is even, we replace the path $v_{2j}v_{2j+1}v_{2j+2}$ by a new red path $v_{2j}y_jv_{2j+2}$ with a distinct $y_j \in Y'$ for all $0 \le j \le \frac{i}{2} - 1$. If i is odd, we replace the path $v_{2j}v_{2j+1}v_{2j+2}$ by a new red path $v_{2j}y_jv_{2j+2}$ with a distinct $y_j \in Y'$ for all $0 \le j \le \frac{i-1}{2} - 1$ and replace the single edge $v_{i-1}v_i$ by a new red path $v_{i-1}y'v_i$ with a distinct $y' \in Y'$. For all other blue edges after v_iv_{i+1} , we replace them by a new red P_3 with the middle vertices in Y'. Again, this can be done since $K_n[X', Y']$ is a complete bipartite graph consisting of only red edges and Y' is large enough.

After this, the original branch becomes a longer red path and we take the first segment of length ℓ_m as the new branch. Note that this new branch still contains the original red edge $v_i v_{i+1}$ unless *i* is odd and $i + 1 = \ell_m$. Let H'' be the resulting copy of *H*.

If H'' still contains one of the original red edges, then we have a monochromatic copy of H, a contradiction since the original edges are NIM-H. Otherwise every branch of H' is either entirely blue or has even length and is such that only the final edge is red. However, then we have $|X| \ge |X'| \ge \lfloor \frac{k}{2} \rfloor + 1$, a contradiction of our assumption that $|X| \le \lfloor \frac{k}{2} \rfloor$ (recall that the blue edges are in X'). The proof is complete for spiders.

The proof when H is a double broom. Let H be a double broom with parameters t, s_1 and s_2 such that $k = t + s_1 + s_2$ and $s_1 < s_2$.

First, assume that t is odd and $|X| \ge \lfloor \frac{t}{2} \rfloor + 1$. For a red edge uv with $u \in X$, $v \in Y$, there is a subset $X' \subseteq X$ of size $\frac{t+1}{2}$ containing u. Let Y' be the corresponding subset for X'. Then there is a path P_t in $K_n[X', Y']$ which starts from u and ends at another vertex, say w in X', and avoids v. Since $|Y'| \ge k^2 - (k-1)\frac{t+1}{2}$, we can select additional red edges incident to u and w, which together with the edge uv represent the set of edges incident to the leaves of H. It follows that uv is not NIM-H. Hence all red NIM-H edges are contained in X and Y, and we have

$$\left|E(c,H)\right| \le \exp\left(n-|X|,H\right) + \binom{|X|}{2} \le \exp(n,H),$$

where the second inequality holds since $|X| \leq \frac{k}{2}$. Now assume that t is even and $|X| \geq \lfloor \frac{t}{2} \rfloor + 1$. Let $Y_1 = \{v \in Y : d_r(v, X) \geq 1\}$ and $Y_2 = Y \setminus Y_1$. Since each vertex in X has at most k-1 blue neighbors in Y, we have $|Y_2| \leq k-1$.

Now we show that for each vertex $v \in Y_1$, there are at most $s_1 + \frac{t}{2} - 1$ NIM-H edges incident to v. Suppose by way of contradiction that for a vertex $v \in Y_1$, there are at least $s_1 + \frac{t}{2}$ red NIM-H edges incident to v. By the definition of Y_1 , there is a red edge vu with $u \in X$. Let X' = X and let $Y' \subset Y$ be the corresponding subset of X'. We extend the red edge vu to a red path P_t in such a way that: (1) one of the end vertex is v and the other end vertex w is in X', (2) every second vertex of the path is in X' and the remaining vertices of the path are in Y', (3) there remain at least s_1 red NIM-H edges incident to v which are not vertices of the path. These conditions can be satisfied since Y' is sufficiently large. Now at least s_1 red NIM-H edges incident to v are not covered by the vertices of the path, which we can view as leaf edges of H incident to v. Select another t red (but not necessarily NIM-H) edges incident to w and to some vertices which have not been used yet. Thus we found a red copy of H containing at least one NIM-H edge, a contradiction.

Therefore, for each vertex $v \in Y_1$, there are at most $s_1 + \frac{t}{2} - 1$ NIM-*H* edges incident to *v*. All other NIM-H edges are contained in Y_2 and X. Hence,

$$\begin{aligned} \left| E(c,H) \right| &\leq |Y_1| \left(s_1 + \frac{t}{2} - 1 \right) + \binom{|Y_2|}{2} + \binom{|X|}{2} \\ &\leq \exp\left(|Y_1|,H\right) + \exp\left(|Y_2|,H\right) + \exp\left(|X|,H\right) \\ &\leq \exp(n,H), \end{aligned}$$
(1)

where the second inequality holds since the coefficient of $|Y_1|$ satisfies $s_1 + \frac{t}{2} - 1 < \frac{k-2}{2}$ and $|Y_2| \le k-1, |X| \le \frac{k}{2}$. Thus, we are done in the case $|X| \ge \lfloor \frac{t}{2} \rfloor + 1$.

Finally, we consider the case when $|X| \leq \lfloor \frac{t}{2} \rfloor$. Since $|X| \geq 2$, we have $t \geq 4$. Let $Y_1 =$ $\{v \in Y : d_r(v,X) \geq 2\}$ and $Y_2 = Y \setminus Y_1$. Now we show that there is no red path of length t-2|X|+1 in Y₁. Suppose by way of contradiction that P is a red path of length t-2|X|+1in Y_1 . First, we extend P to a red path of length t-1 using vertices in X and the corresponding subset of X in Y such that the two end vertices of this longer path, say u and v, are contained in X. Since each vertex in X has red degree at least |Y| - (k-1) in Y, we can find s_1 new red neighbors of u and s_2 new red neighbors of v in Y and view them as the leaf-edges of H. That is, we extended the red path P to a red copy of H. However, as we assumed all red edges in Yare NIM-H, we have a contradiction.

Now we show $|Y_2| \leq s_1 - 1$. Suppose by way of contradiction that $|Y_2| \geq s_1$. If there are two vertices v_1, v_2 in Y_2 such that $N_b(v_1, X) \cup N_b(v_2, X) = X$, then for any blue edge u_1u_2 in X, we have that $v_1u_1u_2v_2$ or $v_1u_2u_1v_2$ is a blue path. Since $t \ge 4$ and all vertices in Y have large blue degree in Y, this blue path can be extended to a blue copy of H. Hence there are no blue NIM-H edges, a contradiction. Thus by the definition of Y_2 , there exists a vertex $w \in X$ such that $N_b(v, X) = X \setminus \{w\}$ for any $v \in Y_2$. Let uu' be a blue NIM-H edge in X with $u \neq w$. Using uu' and s_1 blue edges between u and Y_2 , we can find a blue star with $s_1 + 1$ leaves. By the definition of Y, we can extend this blue star to a blue copy of H using other vertices in Y,

a contradiction. Hence we have $|Y_2| \leq s_1 - 1$. Furthermore, there are at most $|Y_2|$ red NIM-*H* edges between X and Y_2 .

Therefore, we have

$$E(c,H) \Big| \leq \exp(|Y_1|, P_{t-2|X|+2}) + |Y_1| (|Y_2| + |X|) + {|Y_2| \choose 2} + {|X| \choose 2} + |Y_2| \\ \leq \frac{t-2|X|}{2} |Y_1| + |Y_1| (|Y_2| + |X|) + {|Y_2| \choose 2} + {|X| \choose 2} + |Y_2| \\ \leq \frac{t+2(s_1-1)}{2} (n - (s_1 - 1) - |X|) + {s_1 \choose 2} + {|X| \choose 2} \\ \leq \frac{t+2s_1 - 2}{2} n \leq \exp(n, H),$$

$$(2)$$

where the last inequality holds since $s_1 < s_2$. The proof is complete.

Remark 2. One may note that in inequality (1) and (2), we need the condition $s_1 < s_2$ to ensure that $\frac{t+2s_1-2}{2}n \leq ex(n,H)$. For the case $s_1 = s_2$, these inequalities still show $f(n,H) \leq \frac{k-2}{2}n$ but this does not imply $f(n,H) \leq ex(n,H)$ for all n. With additional details, one could extend the proof to the case $s_1 = s_2$. But this would make our proof more complicated, so we omit it.

3 Proof of Theorem 3

We first construct some bipartite graphs which attain the upper bound in (2). Our idea comes from a theorem of Bushaw and Kettle [1]. Before we present the detailed constructions, we recall some results which we will require.

It is well-known that $ex(n,T) \leq \frac{v(T)-2}{2}n$ when T is a path or star. For a general tree T, this is the celebrated Erdős–Sós Conjecture.

Conjecture 2 (Erdős–Sós). For a tree T, we have $ex(n,T) \leq \frac{v(T)-2}{2}n$.

In 2005, McLennan [8] proved that the Erdős–Sós Conjecture holds for trees of diameter at most four.

Theorem 6 (McLennan [8]). Let T be a tree of diameter at most four, then $ex(n,T) \leq \frac{v(T)-2}{2}n$.

A tree is called balanced if it has the same number of vertices in each color class when the tree is viewed as a bipartite graph. A forest is called balanced if each of its components is a balanced tree. Bushaw and Kettle [1] proved the following theorem.

Theorem 7 (Bushaw and Kettle [1]). Let H be a balanced forest on 2a vertices which comprises at least two trees. If the Erdős–Sós Conjecture holds for each component tree in H, then for any $n \ge 3a^2 + 32a^2 \binom{2a}{a}$, we have

$$ex(n,H) = \begin{cases} \binom{a-1}{2} + (a-1)(n-a+1) & \text{if } H \text{ admits a perfect matching,} \\ (a-1)(n-a+1) & \text{otherwise.} \end{cases}$$

Now, making use of Theorems 6 and 7, we construct some bipartite graphs H which are negative examples for Question 1. Let \mathcal{H}_1 be the family of all balanced trees on 2a vertices which admit no perfect matching and for which the Erdős–Sós Conjecture holds. One can see that \mathcal{H}_1 is not empty since a double star $S_{a-1,a-1}$ is a balanced tree on 2a vertices and the Erdős–Sós Conjecture holds for it by Theorem 6. Let \mathcal{H}_2 be the family of balanced trees on 2avertices for which the Erdős–Sós Conjecture holds for sufficiently large n. Note that \mathcal{H}_2 is also nonempty, for example a path on 2a vertices belongs to \mathcal{H}_2 . Let $H_1 \in \mathcal{H}_1$, $H_2 \in \mathcal{H}_2$ and set $H = H_1 \cup H_2$. We know that H is a balanced forest on 4a vertices. Since H_1 admits no perfect matching, H admits no perfect matching either. The Erdős–Sós Conjecture holds for each component of H, hence by Theorem 7, when n is sufficiently large, we have

$$ex(n, H) = (2a - 1)(n - 2a + 1).$$

On the other hand, consider a partition of the vertices of the complete graph K_n into parts X and Y with |X| = 2a - 1 and |Y| = n - 2a + 1. We color all edges between X and Y red and the remaining edges blue. One can see that the red edges induce a complete bipartite graph $K_{2a-1,n-2a+1}$ which contains no red copy of H. The blue edges induce a blue (2a - 1)-clique and a blue (n - 2a + 1)-clique which are disjoint with each other. Since each component of H contains 2a vertices, all blue copies of H are contained in the (n - 2a + 1)-clique. Therefore, all red edges and all the edges in the blue (2a - 1)-clique are NIM-H, that is,

$$f(n,H) \ge \binom{2a-1}{2} + (2a-1)(n-2a+1) = \binom{2a-1}{2} + \exp(n,H).$$

Therefore, such a bipartite graph H attains the upper bound of the inequality (2).

Next we prove that if the bipartite graph H contains a tail $v_0v_1v_2$, then $f(n, H) \leq ex(n, H) + \binom{|A|-1}{2}$. Note that it is possible that H is disconnected, hence let $H = H_1 \cup \cdots \cup H_q$, where H_i are its components (if H is connected, then $H = H_1$) and we say the tail $v_0v_1v_2$ is contained in H_1 . Let A_i, B_i be the two color classes of H_i with $|A_i| \leq |B_i|$ for any $1 \leq i \leq q$, and let $A = \bigcup_{i=1}^q A_i, B = \bigcup_{i=1}^q B_i$. Set a = |A|.

Since we take n to be sufficiently large, we may assume $n \ge R(K_{v(H)}, K_{v(H)})$. Let c be a red-blue edge-coloring of K_n . Without loss of generality, there is a blue clique on at least v(H)vertices in K_n . Let K_t be a blue clique in K_n such that t is as large as possible. We have $t \ge v(H)$ and every other vertex has a red neighbor in $V(K_t)$. We partition $V(K_n) \setminus V(K_t)$ into two subsets X, Y such that Y consists of the vertices which have blue neighbors in $V(K_t)$ and X consists of the remaining vertices. Hence all edges between $V(K_t)$ and X are red.

The following claims will be used several times.

Claim 1. All blue NIM-H edges are contained in X.

Proof. Obviously, the blue edges in K_t and $K_n[V(K_t), Y]$ are not NIM-*H*. Let xy be a blue edge with $y \in Y$ and $x \in X \cup Y$. By the definition of *Y*, the vertex *y* has a blue neighbor, say *v*, in $V(K_t)$. If we embed $V(H) \setminus \{v_1, v_2\}$ into $V(K_t)$ and view vyx as the tail of *H*, then we find a blue copy of *H* containing xy. Thus xy is not NIM-*H*. Therefore, all blue NIM-*H* edges are contained in *X*.

Claim 2. If $|X| \ge a$, then the red edges between X and $V(K_t) \cup Y$ are not NIM-H.

Proof. Since the red edges between X and $V(K_t)$ induce a red complete bipartite graph and $|X| \ge a$ and $t \ge v(H)$, each such edge is contained in a red copy of H, thus these edges are not NIM-H. Let xy be a red edge with $x \in X$, $y \in Y$. By the maximality of K_t , the vertex y has a red neighbor, say v, in $V(K_t)$. Actually, $\{x, y, v\}$ induces a red triangle. If the tail $v_0v_1v_2$ of H satisfies $\{v_0, v_2\} \subset B$ and $v_1 \in A$, then embed $B \setminus \{v_2\}$ into $V(K_t)$ so that v_0 is identified with v, embed $A \setminus \{v_1\}$ into $X \setminus \{x\}$ and view vxy as the tail of H, thus we find a red copy of H containing xy. So in this case, xy is not NIM-H. If the tail $v_0v_1v_2$ of H satisfies $\{v_0, v_2\} \subset A$ and $v_1 \in B$, then embed $B \setminus \{v_1\}$ into $V(K_t) \setminus \{v\}$, embed $A \setminus \{v_2\}$ into X so that v_0 is identified with x. View xyv as the tail, we find a red copy of H containing xy. So in this case, xy is not NIM-H. If the tail $v_0v_1v_2$ of the satisfies $\{v_0, v_2\} \subset A$ and $v_1 \in B$, then embed $B \setminus \{v_1\}$ into $V(K_t) \setminus \{v\}$, embed $A \setminus \{v_2\}$ into X so that v_0 is identified with x. View xyv as the tail, we find a red copy of H containing xy. So in this case, xy is not NIM-H either.

We distinguish three cases based on the size of X.

Case 1: $|X| \ge a+1$. In this case, we first claim that the red edges in X are also not NIM-H. Let xx' be a red edge contained in X and v be a vertex in K_t . If the tail $v_0v_1v_2$ in H satisfies $\{v_0, v_2\} \subset B$ and $v_1 \in A$, then since $|X \setminus \{x, x'\}| \ge a - 1 = |A \setminus \{v_1\}|$, we can embed $A \setminus \{v_1\}$ into $X \setminus \{x, x'\}$, embed $B \setminus \{v_2\}$ into $V(K_t)$ so that v_0 is identified with v and view vxx' as the tail $v_0v_1v_2$, thereby finding a red copy of H containing xx'. So in this case, xx' is not NIM-H. If the tail $v_0v_1v_2$ in H satisfies $\{v_0, v_2\} \subset A$ and $v_1 \in B$, then we embed $A \setminus \{v_2\}$ into $X \setminus \{x'\}$ so that v_0 is identified with x, embed $B \setminus \{v_1\}$ into $V(K_t) \setminus \{v\}$ and view xx'v as the tail, and again we can find a red copy of H containing xx'. Therefore, xx' is not NIM-H.

By Claim 2 and the above result, all red NIM-*H* edges are contained in $V(K_t) \cup Y$. Note that the red NIM-*H* edges contained in $V(K_t) \cup Y$ induce an H_1 -free graph. Otherwise, such a red copy of H_1 together with a red copy of $H_2 \cup \cdots \cup H_q$ (if *H* is disconnected) contained in the complete bipartite graph $K_n[X, V(K_t)]$ yields a red copy of *H* containing an NIM-*H* edge, a contradiction. Analogously, the blue NIM-*H* edges contained in *X* induce a graph which is H_1 -free. Hence,

$$\begin{aligned} \left| E(c,H) \right| &\leq \exp(|X|,H_1) + \exp(n-|X|,H_1) \\ &\leq \exp(n,H_1) \leq \exp(n,H), \end{aligned}$$

where the second inequality holds since H_1 is connected. The proof is complete in this case.

Case 2: |X| = a. By Claim 2, the set of red NIM-*H* edges can be partitioned into two parts: the ones contained in $V(K_t) \cup Y$ and the remaining ones which are contained in *X*. Since all blue NIM-*H* edges are contained in *X* by Claim 1, the sum of the total number of blue NIM-*H* edges and the number of red NIM-*H* edges contained in *X* is at most $\binom{a}{2}$. The set of red NIM-*H* edges contained in $V(K_t) \cup Y$ yields an H_1 -free graph. Indeed, otherwise together with a red copy of $H_2 \cup \cdots \cup H_q$ (if *H* is disconnected) in $K_n[X, V(K_t)]$, we could find a red copy of *H* containing a red NIM-*H* edge, a contradiction. Thus the number of red NIM-*H* edges contained in $V(K_t) \cup Y$ is at most $ex(n - a, H_1)$.

Therefore, the total number of NIM-*H* edges is at most $ex(n - a, H_1) + {a \choose 2}$. Since H_1 is connected and contains a tail, it follows that the union of a star S_{a-1} on *a* vertices and an extremal graph for $ex(n - a, H_1)$ is still H_1 -free. Hence,

$$ex(n-a, H_1) + (a-1) \le ex(n, H_1).$$

Thus, we have

$$\begin{aligned} \left| E(c,H) \right| &\leq \exp(n-a,H_1) + \binom{a}{2} \leq \exp(n,H_1) + \binom{a-1}{2} \\ &\leq \exp(n,H) + \binom{a-1}{2}, \end{aligned}$$

and the proof of this case is complete.

Case 3: $|X| \leq a - 1$. By Claim 1, the number of blue NIM-*H* edges is at most $\binom{a-1}{2}$, and the red NIM-*H* edges yield an *H*-free graph. Hence

$$\left|E(c,H)\right| \le \exp(n,H) + \binom{a-1}{2},$$

and the proof is complete.

Remark 3. In [12], the first author and Chen also give a family of examples such that $\chi(H) = 3$ and $f(n, H) > \exp(n, H)$.

4 Proof of Theorem 5

We first give a 2k-edge-coloring of K_n with $(2k-1)\exp(n, P_{2k}) + (k-1)\binom{2k-1}{2}$ NIM- P_{2k} edges when 2k-1 is a prime and $n \in \{a(2k-1) + (k-1), a(2k-1) + k\}$. Before showing our construction, we need to recall the exact value of $\exp(n, P_{\ell})$.

Theorem 8 (Faudree and Schelp [3]). Let $n = a(\ell - 1) + b$ with $0 \le b \le \ell - 2$. Then we have

$$\operatorname{ex}(n, P_{\ell}) = a \binom{\ell - 1}{2} + \binom{b}{2}.$$

If ℓ is even and $b \in \{\ell/2, \ell/2-1\}$, then the extremal graphs are $tK_{\ell-1} \cup (K_{\ell/2-1} + \overline{K}_{n-t(\ell-1)-\ell/2+1})$ for any $0 \le t \le a$. Otherwise $aK_{\ell-1} \cup K_b$ is the unique extremal graph.

Therefore, by Theorem 8, when $n \in \{a(2k-1) + (k-1), a(2k-1) + k\}$, the extremal graphs for $ex(n, P_{2k})$ are $tK_{2k-1} \cup (K_{k-1} + \overline{K}_{n-t(2k-1)-(k-1)})$ for any $0 \le t \le a$.

Let U be a subset of size $(2k-1)^2$ of $V(K_n)$ and label the vertices of U by [i, j] where $1 \le i, j \le 2k-1$. We divide U into 2k-1 subsets by setting

$$U_i = \{[i, 1], [i, 2], \dots, [i, 2k - 1]\}, 1 \le i \le 2k - 1.$$

When it is not confusing, we also let U and U_i denote the cliques induced by the vertices in them.

For any $1 \le i, j \le 2k-1$, let σ_{ji} denote the clique induced by the vertices $[1, i], [2, i+j], \ldots, [2k-1, i+(2k-2)j]$, where the indices are taken modulo 2k-1. For any $1 \le j \le 2k-1$, let

$$\mathcal{C}_j = \{\sigma_{ji} : 1 \le i \le 2k - 1\}$$

Then C_i is a set consisting of 2k - 1 disjoint (2k - 1)-cliques.

Let $c: E(K_n) \to \{c_1, \ldots, c_{2k}\}$ be a 2k-edge-coloring defined as follows. Let $W = V(K_n) \setminus U$. For any $j \in [2k-1]$, we assign the color c_j to the edges of each clique σ_{ji} in \mathcal{C}_j . Let σ_{j1}^{\diamond} denote the clique induced by the vertices $[k+1, 1+kj], \ldots, [2k-1, 1+(2k-2)j]$. Clearly, we have $\sigma_{j1}^{\diamond} \subset \sigma_{j1}$. Now consider the sub-clique $\sigma_{j1} - \sigma_{j1}^{\diamond}$ and replace the color c_j by c_{2k} inside it. With this, σ_{j1} decomposes into a copy of $K_{k-1} + \overline{K}_k$ colored by c_j and a copy of K_k colored by c_{2k} . After this, we assign the color c_j to all the edges between σ_{j1}^{\diamond} and V. Figure 1 shows the subgraph induced by the edges colored by c_{2k-1} . Finally, we assign the color c_{2k} to the edges which have not been colored yet.

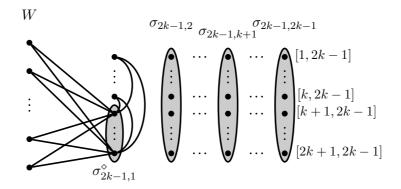


Figure 1: The subgraph induced by the edges of color c_{2k-1} .

In the next two paragraphs, we show that this 2k-edge-coloring is well-defined, namely, each edge is assigned exactly one color. Clearly, each edges is assigned at least one color and the edges inside W or between $U_1 \cup \cdots \cup U_k$ and W are assigned exactly one color.

Note that U is a $(2k-1)^2$ -clique. Let $1 \leq i, \ell, s, t \leq 2k-1$. Clearly, the edge [i, s][i, t] is only covered by the clique U_i . If the edge $[i, s][\ell, t]$ with $i < \ell$ were covered by two cliques, say by one in C_j and by another one in $C_{j'}$ for some $1 \leq j, j' \leq 2k-1$, then

$$\begin{cases} t \equiv s + (\ell - i)j \pmod{2k - 1} \\ t \equiv s + (\ell - i)j' \pmod{2k - 1} \end{cases}$$

would hold, and since 2k - 1 is prime, we would have j = j', a contradiction. Thus, each edge inside U is covered by at most one clique in C_j or by the clique U_i . On the other hand, considering the number of edges in U and the total number of edges of cliques in each C_j and U_i yields

$$e(U) = \sum_{i=1}^{2k-1} e(U_i) + \sum_{j=1}^{2k-1} \sum_{\sigma_{ji} \in \mathcal{C}_j} e(\sigma_{ji}).$$

Therefore, the cliques in each C_j together with the cliques U_i for all $1 \le i, j \le 2k - 1$ form an edge-decomposition of the large clique U. Hence each edge in U is assigned one color.

Now we show that for any $1 \leq j, j' \leq 2k - 1$ with $j \neq j'$, the sub-cliques σ_{j1}^{\diamond} and $\sigma_{j'1}^{\diamond}$ are vertex-disjoint. Supposing that a vertex $[i, 1 + (i - 1)j] \in V(\sigma_{j1}^{\diamond})$ is also contained in $\sigma_{j'1}^{\diamond}$ for some $1 \leq i, j, j' \leq 2k - 1$, we obtain

$$1 + (i-1)j \equiv 1 + (i-1)j' \pmod{2k-1}$$
.

Since 2k - 1 is a prime number, we get j = j', a contradiction. Thus the sub-cliques σ_{j1}^{\diamond} for all $1 \leq j \leq 2k - 1$ form a vertex-decomposition of $U_{k+1} \cup \cdots \cup U_{2k-1}$. Hence, each edge between $U_{k+1} \cup \cdots \cup U_{2k-1}$ and W is assigned one color in $\{c_1, \ldots, c_{2k-1}\}$. Therefore, our 2k-edge-coloring c is well-defined.

Note that for any $1 \leq j \leq 2k - 1$, the subgraph induced by the edges of color c_j is a copy of $tK_{2k-1} \cup (K_{k-1} + \overline{K}_{n-(k-1)-t(2k-1)})$ with t = 2k - 2, and this graph is extremal for $ex(n, P_{2k})$ when $n \in \{a(2k-1) + (k-1), a(2k-1) + k\}$. Now consider the edges colored by c_{2k} . They are in the cliques U_i with $1 \leq i \leq 2k - 1$, inside $\sigma_{j1} - \sigma_{j1}^{\diamond}$ with $1 \leq j \leq 2k - 1$, inside W, and between $U_1 \cup \cdots \cup U_k$ and W. Note that for any $k + 1 \leq i \leq 2k - 1$, U_i are independent (2k-1)-cliques colored by c_{2k} , hence the edges in U_i are also NIM- P_{2k} . For all other c_{2k} -edges, they construct a large connected component such that W is a clique in the component. Hence none of these edges are NIM- P_{2k} .

Therefore,

$$|E(c, P_{2k})| = (2k - 1)\exp(n, P_{2k}) + (k - 1)\binom{2k - 1}{2},$$

and we are done.

Remark 4. Note that $tK_{2k-1} \cup (K_{k-1} + \overline{K}_{n-(k-1)-t(2k-1)})$ is not extremal for $ex(n, P_{2k})$ when $n \notin \{a(2k-1) + (k-1), a(2k-1) + k\}$, but we still have

$$\exp(n, P_{2k}) - e\left(tK_{2k-1} \cup \left(K_{k-1} + \overline{K}_{n-(k-1)-t(2k-1)}\right)\right) < (k-1)^2.$$

Hence in our construction, when 2k - 1 is prime, the number of NIM-P_{2k} edges is more than $(2k-1)ex(n, P_{2k})$. That is to say, when 2k-1 is prime, we have $f_{2k}(n, P_{2k}) > (2k-1)ex(n, P_{2k})$ for every sufficiently large n.

Next we prove the upper bound of $f_{2k}(n, P_{2k})$. Let $c : E(K_n) \to \{c_1, \ldots, c_{2k}\}$ be a 2kedge-coloring of K_n . We call an edge a c_i -edge if it is of color c_i and we let G_i denote the
subgraph induced by all c_i -edges, for any $1 \le i \le 2k$. Without loss of generality, we can assume $e(G_{2k}) \ge {n \choose 2}/2k$. By Theorem 8, there is a path P of at least $\frac{n}{2k}$ vertices in G_{2k} . Let G'_{2k} be

the component of G_{2k} which contains the path P, and let $X = V(G'_{2k})$ and $Y = V(K_n) - X$. Then we have $|X| \ge \frac{n}{2k}$ and there is no c_{2k} -edge between X and Y. Since the component G'_{2k} contains a long path P, each edge of G'_{2k} is contained in a monochromatic copy of P_{2k} . Hence, all NIM- P_{2k} c_{2k} -edges are contained in Y.

For each $1 \leq i \leq 2k-1$, there are at most $ex(n, P_{2k})$ NIM- P_{2k} c_i -edges. If $|Y| \leq (k-1)(2k-1)$, then there are at most $ex(|Y|, P_{2k}) \leq (k-1)\binom{2k-1}{2}$ NIM- P_{2k} c_{2k} -edges. Hence, the total number of NIM- P_{2k} edges is at most

$$(2k-1)$$
ex $(n, P_{2k}) + (k-1)\binom{2k-1}{2}$,

so we are done. Therefore, we may assume $|Y| \ge (k-1)(2k-1)+1$.

Let us define a procedure to find pairs (X_i, Y_i) satisfying the following conditions:

- (i) $X_i \subseteq X$ with $|X_i| = 2k$ and $Y_i \subseteq Y$ with $|Y_i| = k$ for any $1 \le i \le 2k$,
- (ii) Y_i and Y_j are disjoint for any $1 \le i, j \le 2k$ with $i \ne j$,

(iii) $K_n[X_i, Y_i]$ forms a monochromatic copy of complete bipartite graph for any $1 \le i \le 2k$.

Assume that for some $1 \le i \le 2k$, we have found $(X_1, Y_1), \ldots, (X_{i-1}, Y_{i-1})$ which satisfy the conditions. Let s = (k-1)(2k-1) + 1. If

$$\left| Y \setminus \bigcup_{j=1}^{i-1} Y_j \right| \le s-1,$$

then the procedure terminates. Otherwise we choose a subset Y'_i of $Y \setminus \bigcup_{j=1}^{i-1} Y_j$ with $|Y'_i| = s$. Let $Y'_i = \{y_1, \ldots, y_s\}$. For each $x \in X$, we define a vector $\vec{\epsilon}(x, Y'_i) = (\epsilon_1, \ldots, \epsilon_s)$ as follows: for any $1 \leq j \leq s$, let

 $\epsilon_j = i$ if and only if the edge xy_j is colored by c_i .

Since no edge between X and Y is colored by c_{2k} , we have $\vec{\epsilon}(x, Y'_i) \in \{1, \ldots, 2k-1\}^s$ for any $x \in X$. For each $\vec{v} \in \{1, \ldots, 2k-1\}^s$, let $X_{\vec{v}}$ denote the set of vertices $x \in X$ for which $\vec{\epsilon}(x, Y'_i) = \vec{v}$. Hence, X is divided into $(2k-1)^s$ subsets and clearly, at least one subset, say $X_{\vec{v}_i}$, contains at least $|X|/(2k-1)^s$ vertices. Observe that $K_n[X_{\vec{v}_i}, y_j]$ is a monochromatic star for any $y_j \in Y'_i$. Since $|Y'_i| = (k-1)(2k-1) + 1$ and there are at most 2k-1 different colors between $X_{\vec{v}_i}$ and Y'_i , by pigeonhole principle, there exists a subset $Y_i \subset Y'_i$ such that $|Y_i| = k$ and the edges between Y_i and $X_{\vec{v}_i}$ are monochromatic. That is $K_n[X_{\vec{v}_i}, Y_i]$ is a monochromatic complete bipartite graph. Since $n \ge (2k)^{2k^2}$,

$$|X_{\vec{v}_i}| \ge \frac{|X|}{(2k-1)^s} \ge \frac{n}{(2k)^s} \ge 2k.$$

We can choose a subset X_i from $X_{\vec{v}_i}$ with $|X_i| = 2k$, thereby finding the pair (X_i, Y_i) as we wanted.

Note that since Y is finite, the procedure terminates. Let t denote the number of steps the algorithm took, and let $(X_1, Y_1), \ldots, (X_t, Y_t)$ be the pairs the algorithm found. Let $Y_0 = Y \setminus \bigcup_{i=1}^{t} Y_i$. Then we have $|Y_0| \leq (k-1)(2k-1)$. For any $1 \leq i \leq 2k-1$, let t_i denote the number of the pairs (X_j, Y_j) for which the edges of $K_n[X_j, Y_j]$ are of color c_i . Without loss of generality, we may assume that $t_1, \ldots, t_h > 0$ for some $1 \leq h \leq 2k-1$. Then $t = \sum_{i=1}^{h} t_i$. Let $1 \leq i \leq h$ and consider the c_i -edges. Without loss of generality, we can assume that $K_n[X_1, Y_1], \ldots, K_n[X_{t_i}, Y_{t_i}]$ are of color c_i . Then each NIM- P_{2k} c_i -edge is contained in $V(K_n) \setminus \bigcup_{j=1}^{t_i} (X_j \cup Y_j)$. Since the sets Y_1, \ldots, Y_{t_i} are pairwise disjoint and $X_1, \ldots, X_{t_i} \subseteq X$, we have

$$\left|\bigcup_{j=1}^{t_i} (X_j \cup Y_j)\right| \ge t_i k + 2k,$$

thus the number of NIM- P_{2k} c_i -edges is at most $ex(n-t_ik-2k, P_{2k})$. Now let $h+1 \le i \le 2k-1$ (if such an index exists). Since $t_i = 0$, the number of NIM- P_{2k} c_i -edges is at most $ex(n, P_{2k})$.

As we have proved, all NIM- P_{2k} c_{2k} -edges are contained in Y and $|Y| \leq (k-1)(2k-1) + tk$. Therefore, the total number of NIM- P_{2k} edges is at most

$$\exp((k-1)(2k-1)+tk, P_{2k}) + \sum_{i=1}^{h} \exp(n-t_ik-2k, P_{2k}) + (2k-1-h)\exp(n, P_{2k}).$$
(3)

To prove the final result, we need the following lemma.

Lemma 1. Let n_1 , n_2 and c be constants. Then we have

$$ex(n_1, P_\ell) + ex(n_2, P_\ell) < ex(n_1 - c, P_\ell) + ex(n_2 + c + \ell, P_\ell).$$

Proof. Let $n_1 - c = a_1(\ell - 1) + b_1$ and $n_2 + c = a_2(\ell - 1) + b_2$, where $0 \le b_1, b_2 \le \ell - 2$. By Theorem 8, we have

$$\exp(n_1 - c, P_\ell) + \exp(n_2 + c + \ell, P_\ell) \ge \exp(n_1 - c, P_\ell) + \exp(n_2 + c, P_\ell) + \exp(\ell, P_\ell)$$

> $(a_1 + a_2)\binom{\ell - 1}{2} + \binom{b_1}{2} + \binom{b_2}{2} + \binom{\ell - 1}{2}$

and

$$\exp(n_1, P_\ell) + \exp(n_2, P_\ell) \le \frac{\ell - 2}{2}(n_1 + n_2) = (a_1 + a_2)\binom{\ell - 1}{2} + (b_1 + b_2)\frac{\ell - 2}{2}$$

Hence we have

$$\exp(n_1 - c, P_\ell) + \exp(n_2 + c + \ell, P_\ell) - \left(\exp(n_1, P_\ell) + \exp(n_2, P_\ell)\right) \\> {\binom{b_1}{2}} + {\binom{b_2}{2}} + {\binom{\ell - 1}{2}} - (b_1 + b_2)\frac{\ell - 2}{2} > 0.$$

we are done.

When applying the above lemma to (3), we get

$$\exp((k-1)(2k-1) + tk, P_{2k}) + \sum_{i=1}^{s} \exp(n - t_i k - 2k, P_{2k}) + (2k - 1 - s)\exp(n, P_{2k})$$
$$< (2k - 1)\exp(n, P_{2k}) + \exp((k - 1)(2k - 1), P_{2k})$$
$$= (2k - 1)\exp(n, P_{2k}) + (k - 1)\binom{2k - 1}{2}.$$

Thus the proof is complete.

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