# Edges not covered by monochromatic bipartite graphs 

Xiutao Zhu ${ }^{1,2}$, Ervin Győri ${ }^{1}$, Zhen He $^{1,3}$, Zequn Lv ${ }^{* 1,3}$, Nika Salia ${ }^{1,5}$, Casey Tompkins ${ }^{1}$, and Kitti Varga ${ }^{1,4}$<br>${ }^{1}$ Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences.<br>${ }^{2}$ Department of Mathematics, Nanjing University.<br>${ }^{3}$ Department of Mathematical Sciences, Tsinghua University.<br>${ }^{4}$ Department of Computer Science and Information Theory, Budapest University of Technology and Economics.<br>${ }^{5}$ Extremal Combinatorics and Probability Group, Institute for Basic Science, Daejeon, South Korea.


#### Abstract

Let $f_{k}(n, H)$ denote the maximum number of edges not contained in any monochromatic copy of $H$ in a $k$-coloring of the edges of $K_{n}$, and let ex $(n, H)$ denote the Turán number of $H$. In place of $f_{2}(n, H)$ we simply write $f(n, H)$. In [5], Keevash and Sudakov proved that $f(n, H)=\operatorname{ex}(n, H)$ if $H$ is an edge-critical graph or $C_{4}$ and asked if this equality holds for any graph $H$. All known exact values of this question require $H$ to contain at least one cycle. In this paper we focus on acyclic graphs and have the following results: (1) We prove $f(n, H)=\operatorname{ex}(n, H)$ when $H$ is a spider or a double broom. (2) A tail in $H$ is a path $P_{3}=v_{0} v_{1} v_{2}$ such that $v_{2}$ is only adjacent to $v_{1}$ and $v_{1}$ is only adjacent to $v_{0}, v_{2}$ in $H$. We obtain a tight upper bound for $f(n, H)$ when $H$ is a bipartite graph with a tail. This result provides the first bipartite graphs which answer the question of Keevash and Sudakov in the negative. (3) Liu, Pikhurko and Sharifzadeh [6] asked if $f_{k}(n, T)=(k-1) \operatorname{ex}(n, T)$ when $T$ is a tree. We provide an upper bound for $f_{2 k}\left(n, P_{2 k}\right)$ and show it is tight when $2 k-1$ is prime. This provides a negative answer to their question.


## 1 Introduction

Given any graph $H$, the classical theorem of Ramsey asserts that there exists an integer $R(H, H)$ such that every 2 -coloring of the edges of the complete graph $K_{n}$ with $n \geq R(H, H)$ contains a monochromatic copy of $H$. A natural extension of this problem is determining how many monochromatic copies of $H$ there are. For the case of $H=K_{3}$, this question was answered by Goodman [4] and the case of $H=K_{4}$ was settled by Thomason [10].

In a different direction, one can ask how many edges must be contained in some monochromatic copy of $H$ in every 2-coloring of the edges of $K_{n}$ (equivalently how many edges there can be in a 2-coloring which are not contained in any monochromatic copy of $H$ ). The first result about this topic is due to Erdős, Rousseau and Schelp [2]. They considered the maximum number of edges not contained in any monochromatic triangle in a 2 -coloring of the edges of $K_{n}$. Erdős also wrote "many further related questions can be asked" in [2]. In this paper, we will consider problems of this type.

Let $c$ be a 2-coloring of the edges of $K_{n}$ and let $H$ be a graph. If an edge of $K_{n}$ is not contained in any monochromatic copy of $H$, then we say it is NIM- $H$. Let $E(c, H)$ denote the

[^0]set of all NIM- $H$ edges in $K_{n}$ under the 2-edge-coloring $c$ and let
$$
f(n, H)=\max \left\{|E(c, H)|: c \text { is a 2-edge-coloring of } K_{n}\right\} .
$$

Let ex $(n, H)$ be the Turán number of $H$. If one considers a 2-coloring of the edges of $K_{n}$ in which one of the colors yields an extremal graph for $H$, then it is easy to see

$$
\begin{equation*}
f(n, H) \geq \operatorname{ex}(n, H) \tag{1}
\end{equation*}
$$

As observed by Alon, the result on $f\left(n, K_{3}\right)$ by Erdős, Rousseau and Schelp [2] can also be deduced from a result of Pyber [9] (see [5]). In [5], Keevash and Sudakov studied $f(n, H)$ systematically. They proved that if $H$ contains an edge $e$ such that $\chi(H-e)<\chi(H)$ or $H=C_{4}$, then equality holds in (1) for sufficiently large $n$. Furthermore, they asked if the equality holds for all $H$.

Question 1 (Keevash, Sudakov [5]). Is it true that for any graph $H$ we have $f(n, H)=\operatorname{ex}(n, H)$ when $n$ is sufficiently large?

In 2017, Ma [7] provided an affirmative answer to Question 1 for an infinite family of bipartite graphs $H$, including all even cycles and complete bipartite graphs $K_{s, t}$ for $t>s^{2}-3 s+3$ or $(s, t) \in\{(3,3),(4,7)\}$. In 2019, Liu, Pikhurko and Sharifzadeh [6] extended Ma's result by providing a larger family of bipartite graphs for which $f(n, H)=\operatorname{ex}(n, H)$ holds (however, the graphs they construct still contain a cycle). Surprisingly, Yuan [11] recently found an example showing that the assertion in Question 1 does not hold in general.
Theorem 1 (Yuan [11]). Let $p \geq t+1 \geq 4$ and $K_{t}^{p+1}$ denote the graph obtained from $K_{t}$ by replacing each edge of $K_{t}$ with a clique $K_{p+1}$. When $n$ is sufficiently large, then

$$
f\left(n, K_{t}^{p+1}\right)=\operatorname{ex}\left(n, K_{t}^{p+1}\right)+\left(\begin{array}{c}
(t-1 \\
2 \\
2
\end{array}\right)
$$

Based on this result, he conjectured the following.
Conjecture 1 (Yuan [11]). Let $H$ be any graph and $n$ be sufficiently large. Then there exists a constant $C=C(H)$ such that $f(n, H)=\operatorname{ex}(n, H)+C$.

As mentioned earlier, the known results about the exact value of $f(n, H)$ require that $H$ contains a cycle. For acyclic graphs and some other bipartite graphs, the situation is less clear. Thus, in this paper, we will focus on this case. A spider is the graph consisting of $t$ paths with one common end vertex such that all other vertices are distinct. A double broom with parameters $t, s_{1}$ and $s_{2}$ is the graph consisting of a path with $t$ vertices with $s_{1}$ and $s_{2}$ distinct leaves appended to each of its respective end vertices.

Theorem 2. Let $H$ a spider or a double broom with $s_{1}<s_{2}$ and $n$ be sufficiently large, we have

$$
f(n, H)=\operatorname{ex}(n, H)
$$

A tail in a (not necessary acyclic) graph $H$ is a path $P_{3}=v_{0} v_{1} v_{2}$ such that $v_{2}$ is only adjacent to $v_{1}$ and $v_{1}$ is only adjacent to $v_{0}$ and $v_{2}$.
Theorem 3. Let $H=(A, B, E)$ be a bipartite graph containing a tail and $|A| \leq|B|$. When $n$ is sufficiently large, we have

$$
\begin{equation*}
f(n, H) \leq \operatorname{ex}(n, H)+\binom{|A|-1}{2} \tag{2}
\end{equation*}
$$

Furthermore, the upper bound is tight.

Remark 1. In Theorem 3, there are many bipartite graphs $H$ such that $f(n, H)$ achieves an upper bound greater than $\operatorname{ex}(n, H)$. This implies that even for the bipartite case, the answer to Question 1 can be negative. However, the graphs from Theorem 3 satisfy Conjecture 1.

We will also consider the case of edge colorings with 3 or more colors. Let $f_{k}(n, H)$ be the maximum number of edges not contained in any monochromatic copy of $H$ in a $k$-coloring of the edges of $K_{n}$. Thus, $f_{2}(n, H)=f(n, H)$. It appears likely that for $k \geq 3$, the function $f_{k}(n, H)$ has different behavior for bipartite graphs and non-bipartite graphs. For non-bipartite graphs, one can see that $f_{k}(n, H) \neq(k-1) \operatorname{ex}(n, H)$ since $(k-1) \operatorname{ex}(n, H) \geq\binom{ n}{2}$.

For a tree $T$, Ma [7] constructed a lower bound by taking random overlays of $k-1$ copies of extremal $T$-free graphs, and the construction implies $f_{k}(n, T) \geq(k-1-o(1)) \operatorname{ex}(n, T)$. Liu, Pikhurko and Sharifzadeh [6] showed that this lower bound is asymptotically correct.
Theorem 4 (Liu, Pikhurko, Sharifzadeh [6]). Let $T$ be a tree with $h$ vertices. Then there exists a constant $C(k, h)$ such that for all sufficiently large $n$, we have

$$
\left|f_{k}(n, T)-(k-1) \operatorname{ex}(n, T)\right| \leq C(k, h)
$$

For more general bipartite graph $H$, Ma [7] wrote "it may be reasonable to ask if $f_{k}(n, H)=$ $(k-1) \operatorname{ex}(n, H)$ holds for sufficiently large $n "$. However, this is not true for disconnected bipartite graphs. Liu, Pikhurko and Sharifzadeh [6] gave an example and showed $f_{k}\left(n, 2 K_{2}\right)=$ $(k-1) \operatorname{ex}\left(n, 2 K_{2}\right)-\binom{k-1}{2}$. Based on this example, Liu, Pikhurko and Sharifzadeh [6] asked the following question.

Question 2 (Liu, Pikhurko, Sharifzadeh [6]). Is it true that $f_{k}(n, T)=(k-1) \operatorname{ex}(n, T)$ for any tree $T$ and sufficiently large $n$ ?

Our third result concerns the case when $T$ is a path with an even number of vertices and yields a negative answer to Question 2.
Theorem 5. Let $k \geq 1$ and $n \geq(2 k)^{2 k^{2}}$ be integers. We have

$$
f_{2 k}\left(n, P_{2 k}\right) \leq(2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)+(k-1)\binom{2 k-1}{2}
$$

Furthermore, equality holds when $2 k-1$ is a prime and $n \in\{a(2 k-1)+(k-1), a(2 k-1)+k\}$.
Notation and organization. For a given graph $G$, we use $e(G)$ to denote the number of edges of $G$. For a subset of vertices $X$, let $G[X]$ denote the subgraph induced by $X$ and $G-X$ denote the subgraph induced by $V(G) \backslash X$. For two disjoint subset $X, Y$, let $G[X, Y]$ denote the bipartite subgraph of $G$ consisting of the edges of $G$ with one end vertex in $X$ and the other in $Y$. In a red-blue edge-colored complete graph $K_{n}$, we say that $u$ is a red (or blue) neighbor of $v$ if the edge $u v$ is red (or blue). For a set $X$ of vertices, let $N_{r}(v, X)$ and $N_{b}(v, X)$ denote the red and blue neighbors of $v$ in $X$, respectively. Let $d_{r}(v, X)=\left|N_{r}(v, X)\right|$ and $d_{b}(v, X)=\left|N_{b}(v, X)\right|$. If $X=V\left(K_{n}\right)$, then we simply write $d_{r}(v)$ and $d_{b}(v)$. For two graphs $G$ and $H$, we use $G \cup H$ to denote the disjoint union of $G$ and $H$. Let $G+H$ be the graph obtained from $G \cup H$ by adding all edges with one end vertex in $V(G)$ and one end vertex in $V(H)$.

The rest of the paper is organized as follows. In Sections 2 and 3, we study the function $f(n, H)$ and prove Theorems 2 and 3, respectively. In Section 4, we study the general function $f_{k}(n, H)$ and prove Theorem 5.

## 2 Proof of Theorem 2

Let $H$ be a spider or a double broom on $k$ vertices and $c$ be a red-blue edge-coloring of $K_{n}$ with $|E(c, H)|$ being maximum. If $E(c, H)$ contains no $H$, then

$$
f(n, H)=|E(c, H)| \leq \operatorname{ex}(n, H)
$$

and we are done. Hence we may assume there is a non-monochromatic copy of $H$ in $E(c, H)$.
Since we can take $n$ to be larger than the Ramsey number $R\left(k^{2}, k^{2}\right)$, it follows, without loss of generality, that $K_{n}$ contains a blue clique $K$ of size at least $k^{2}$. We partition $V\left(K_{n}\right)$ into two parts $X$ and $Y$ such that $Y$ is maximal with the property that any vertex $v$ in $Y$ has $d_{b}(v, Y) \geq k$ and $X$ consists of the remaining vertices. Note that the large blue clique $K$ is contained in $Y$, and hence $|Y| \geq k^{2}$. Since each vertex in $Y$ has blue degree at least $k$ in $Y$, every blue edge in $Y$ or between $X$ and $Y$ can be extended to a blue copy of $H$. Hence, all blue NIM- $H$ edges are contained in $X$ and $|X| \geq 2$.

For each vertex $u$ in $X$, we have $d_{b}(u, Y) \leq(k-1)$. Thus for each subset $X^{\prime}$ of $X$, the subset $Y^{\prime}=Y \backslash N_{b}\left(X^{\prime}, Y\right)$ is such that $K_{n}\left[X^{\prime}, Y^{\prime}\right]$ is a red complete bipartite graph and $\left|Y^{\prime}\right| \geq$ $|Y|-(k-1)\left|X^{\prime}\right|$. We call $Y^{\prime}$ the corresponding subset of $X^{\prime}$.

First assume $|X| \geq\left\lfloor\frac{k}{2}\right\rfloor+1$. For each red edge $u v$ contained in $X$ or between $X$ and $Y$, we can find a subset $X^{\prime} \subseteq X$ of size $\left\lfloor\frac{k}{2}\right\rfloor$ that contains exactly one of $u$ and $v$. Using the corresponding subset $Y^{\prime}$ of $X^{\prime}$, this red edge $u v$ can be extended to a red copy of $H$. Hence all red NIM- $H$ edges are contained in $Y$ and

$$
|E(c, H)| \leq \operatorname{ex}(|Y|, H)+\operatorname{ex}(|X|, H) \leq \operatorname{ex}(n, H) .
$$

Therefore, in the rest of the proof, we will assume $|X| \leq\left\lfloor\frac{k}{2}\right\rfloor$. Furthermore, each red edge in $Y$ is NIM- $H$, otherwise we replace the color of this edge by blue and since $E(c, H)$ is maximum, it has no changes.

Next we distinguish two cases based on whether $H$ is a spider or a double broom.
The proof when $H$ is a spider. Let $H$ be a spider consisting of $t$ paths with a common initial vertex $v_{0}$. We call each path starting from $v_{0}$ a branch, and we assume that the lengths of these $t$ branches are $\ell_{1}, \ldots, \ell_{t}$ such that $v(H)=k=1+\sum_{i=1}^{t} \ell_{i}$.

Now we choose a copy of $H$ from $E(c, H)$ and denote it by $H^{\prime}$. Let $X^{\prime}=X \cap V\left(H^{\prime}\right)$. Since $H^{\prime}$ contains blue edges and all NIM- $H$ blue edges are contained in $X$, we have $X^{\prime} \neq \emptyset$ and the corresponding subset $Y^{\prime}$ is of size at least

$$
|Y|-(k-1)\left|X^{\prime}\right| \geq k
$$

For every branch of $H^{\prime}$, we apply the following method to replace all blue edges with red edges. First, every branch consisting entirely of blue edges is replaced by a red path of the same length in $K_{n}\left[X^{\prime}, Y^{\prime}\right]$. This can be done since $K_{n}\left[X^{\prime}, Y^{\prime}\right]$ is a complete bipartite graph consisting of only red edges and $Y^{\prime}$ is large enough. For any remaining branch $v_{0} v_{1} \ldots v_{\ell_{m}}$, let $v_{i} v_{i+1}$ be the first red edge on this branch, i.e., every edge in the path $v_{0} v_{1} \ldots v_{i}$ is blue. If $i$ is even, we replace the path $v_{2 j} v_{2 j+1} v_{2 j+2}$ by a new red path $v_{2 j} y_{j} v_{2 j+2}$ with a distinct $y_{j} \in Y^{\prime}$ for all $0 \leq j \leq \frac{i}{2}-1$. If $i$ is odd, we replace the path $v_{2 j} v_{2 j+1} v_{2 j+2}$ by a new red path $v_{2 j} y_{j} v_{2 j+2}$ with a distinct $y_{j} \in Y^{\prime}$ for all $0 \leq j \leq \frac{i-1}{2}-1$ and replace the single edge $v_{i-1} v_{i}$ by a new red path $v_{i-1} y^{\prime} v_{i}$ with a distinct $y^{\prime} \in Y^{\prime}$. For all other blue edges after $v_{i} v_{i+1}$, we replace them by a new red $P_{3}$ with the middle vertices in $Y^{\prime}$. Again, this can be done since $K_{n}\left[X^{\prime}, Y^{\prime}\right]$ is a complete bipartite graph consisting of only red edges and $Y^{\prime}$ is large enough.

After this, the original branch becomes a longer red path and we take the first segment of length $\ell_{m}$ as the new branch. Note that this new branch still contains the original red edge $v_{i} v_{i+1}$ unless $i$ is odd and $i+1=\ell_{m}$. Let $H^{\prime \prime}$ be the resulting copy of $H$.

If $H^{\prime \prime}$ still contains one of the original red edges, then we have a monochromatic copy of $H$, a contradiction since the original edges are NIM- $H$. Otherwise every branch of $H^{\prime}$ is either entirely blue or has even length and is such that only the final edge is red. However, then we have $|X| \geq\left|X^{\prime}\right| \geq\left\lfloor\frac{k}{2}\right\rfloor+1$, a contradiction of our assumption that $|X| \leq\left\lfloor\frac{k}{2}\right\rfloor$ (recall that the blue edges are in $\left.X^{\prime}\right)$. The proof is complete for spiders.

The proof when $H$ is a double broom. Let $H$ be a double broom with parameters $t, s_{1}$ and $s_{2}$ such that $k=t+s_{1}+s_{2}$ and $s_{1}<s_{2}$.

First, assume that $t$ is odd and $|X| \geq\left\lfloor\frac{t}{2}\right\rfloor+1$. For a red edge $u v$ with $u \in X, v \in Y$, there is a subset $X^{\prime} \subseteq X$ of size $\frac{t+1}{2}$ containing $u$. Let $Y^{\prime}$ be the corresponding subset for $X^{\prime}$. Then there is a path $P_{t}$ in $K_{n}\left[X^{\prime}, Y^{\prime}\right]$ which starts from $u$ and ends at another vertex, say $w$ in $X^{\prime}$, and avoids $v$. Since $\left|Y^{\prime}\right| \geq k^{2}-(k-1) \frac{t+1}{2}$, we can select additional red edges incident to $u$ and $w$, which together with the edge $u v$ represent the set of edges incident to the leaves of $H$. It follows that $u v$ is not NIM- $H$. Hence all red NIM- $H$ edges are contained in $X$ and $Y$, and we have

$$
|E(c, H)| \leq \operatorname{ex}(n-|X|, H)+\binom{|X|}{2} \leq \operatorname{ex}(n, H)
$$

where the second inequality holds since $|X| \leq \frac{k}{2}$.
Now assume that $t$ is even and $|X| \geq\left\lfloor\frac{t}{2}\right\rfloor+1$. Let $Y_{1}=\left\{v \in Y: d_{r}(v, X) \geq 1\right\}$ and $Y_{2}=Y \backslash Y_{1}$. Since each vertex in $X$ has at most $k-1$ blue neighbors in $Y$, we have $\left|Y_{2}\right| \leq k-1$.

Now we show that for each vertex $v \in Y_{1}$, there are at most $s_{1}+\frac{t}{2}-1$ NIM- $H$ edges incident to $v$. Suppose by way of contradiction that for a vertex $v \in Y_{1}$, there are at least $s_{1}+\frac{t}{2}$ red NIM- $H$ edges incident to $v$. By the definition of $Y_{1}$, there is a red edge $v u$ with $u \in X$. Let $X^{\prime}=X$ and let $Y^{\prime} \subset Y$ be the corresponding subset of $X^{\prime}$. We extend the red edge $v u$ to a red path $P_{t}$ in such a way that: (1) one of the end vertex is $v$ and the other end vertex $w$ is in $X^{\prime}$, (2) every second vertex of the path is in $X^{\prime}$ and the remaining vertices of the path are in $Y^{\prime}$, (3) there remain at least $s_{1}$ red NIM- $H$ edges incident to $v$ which are not vertices of the path. These conditions can be satisfied since $Y^{\prime}$ is sufficiently large. Now at least $s_{1}$ red NIM- $H$ edges incident to $v$ are not covered by the vertices of the path, which we can view as leaf edges of $H$ incident to $v$. Select another $t$ red (but not necessarily NIM- $H$ ) edges incident to $w$ and to some vertices which have not been used yet. Thus we found a red copy of $H$ containing at least one NIM- $H$ edge, a contradiction.

Therefore, for each vertex $v \in Y_{1}$, there are at most $s_{1}+\frac{t}{2}-1$ NIM- $H$ edges incident to $v$. All other NIM- $H$ edges are contained in $Y_{2}$ and $X$. Hence,

$$
\begin{align*}
|E(c, H)| & \leq\left|Y_{1}\right|\left(s_{1}+\frac{t}{2}-1\right)+\binom{\left|Y_{2}\right|}{2}+\binom{|X|}{2}  \tag{1}\\
& \leq \operatorname{ex}\left(\left|Y_{1}\right|, H\right)+\operatorname{ex}\left(\left|Y_{2}\right|, H\right)+\operatorname{ex}(|X|, H) \\
& \leq \operatorname{ex}(n, H)
\end{align*}
$$

where the second inequality holds since the coefficient of $\left|Y_{1}\right|$ satisfies $s_{1}+\frac{t}{2}-1<\frac{k-2}{2}$ and $\left|Y_{2}\right| \leq k-1,|X| \leq \frac{k}{2}$. Thus, we are done in the case $|X| \geq\left\lfloor\frac{t}{2}\right\rfloor+1$.

Finally, we consider the case when $|X| \leq\left\lfloor\frac{t}{2}\right\rfloor$. Since $|X| \geq 2$, we have $t \geq 4$. Let $Y_{1}=$ $\left\{v \in Y: d_{r}(v, X) \geq 2\right\}$ and $Y_{2}=Y \backslash Y_{1}$. Now we show that there is no red path of length $t-2|X|+1$ in $Y_{1}$. Suppose by way of contradiction that $P$ is a red path of length $t-2|X|+1$ in $Y_{1}$. First, we extend $P$ to a red path of length $t-1$ using vertices in $X$ and the corresponding subset of $X$ in $Y$ such that the two end vertices of this longer path, say $u$ and $v$, are contained in $X$. Since each vertex in $X$ has red degree at least $|Y|-(k-1)$ in $Y$, we can find $s_{1}$ new red neighbors of $u$ and $s_{2}$ new red neighbors of $v$ in $Y$ and view them as the leaf-edges of $H$. That is, we extended the red path $P$ to a red copy of $H$. However, as we assumed all red edges in $Y$ are NIM- $H$, we have a contradiction.

Now we show $\left|Y_{2}\right| \leq s_{1}-1$. Suppose by way of contradiction that $\left|Y_{2}\right| \geq s_{1}$. If there are two vertices $v_{1}, v_{2}$ in $Y_{2}$ such that $N_{b}\left(v_{1}, X\right) \cup N_{b}\left(v_{2}, X\right)=X$, then for any blue edge $u_{1} u_{2}$ in $X$, we have that $v_{1} u_{1} u_{2} v_{2}$ or $v_{1} u_{2} u_{1} v_{2}$ is a blue path. Since $t \geq 4$ and all vertices in $Y$ have large blue degree in $Y$, this blue path can be extended to a blue copy of $H$. Hence there are no blue NIM- $H$ edges, a contradiction. Thus by the definition of $Y_{2}$, there exists a vertex $w \in X$ such that $N_{b}(v, X)=X \backslash\{w\}$ for any $v \in Y_{2}$. Let $u u^{\prime}$ be a blue NIM- $H$ edge in $X$ with $u \neq w$. Using $u u^{\prime}$ and $s_{1}$ blue edges between $u$ and $Y_{2}$, we can find a blue star with $s_{1}+1$ leaves. By the definition of $Y$, we can extend this blue star to a blue copy of $H$ using other vertices in $Y$,
a contradiction. Hence we have $\left|Y_{2}\right| \leq s_{1}-1$. Furthermore, there are at most $\left|Y_{2}\right|$ red NIM- $H$ edges between $X$ and $Y_{2}$.

Therefore, we have

$$
\begin{align*}
|E(c, H)| & \leq \operatorname{ex}\left(\left|Y_{1}\right|, P_{t-2|X|+2}\right)+\left|Y_{1}\right|\left(\left|Y_{2}\right|+|X|\right)+\binom{\left|Y_{2}\right|}{2}+\binom{|X|}{2}+\left|Y_{2}\right| \\
& \leq \frac{t-2|X|}{2}\left|Y_{1}\right|+\left|Y_{1}\right|\left(\left|Y_{2}\right|+|X|\right)+\binom{\left|Y_{2}\right|}{2}+\binom{|X|}{2}+\left|Y_{2}\right| \\
& \leq \frac{t+2\left(s_{1}-1\right)}{2}\left(n-\left(s_{1}-1\right)-|X|\right)+\binom{s_{1}}{2}+\binom{|X|}{2} \\
& \leq \frac{t+2 s_{1}-2}{2} n \leq \operatorname{ex}(n, H), \tag{2}
\end{align*}
$$

where the last inequality holds since $s_{1}<s_{2}$. The proof is complete.
Remark 2. One may note that in inequality (1) and (2), we need the condition $s_{1}<s_{2}$ to ensure that $\frac{t+2 s_{1}-2}{2} n \leq \operatorname{ex}(n, H)$. For the case $s_{1}=s_{2}$, these inequalities still show $f(n, H) \leq \frac{k-2}{2} n$ but this does not imply $f(n, H) \leq \operatorname{ex}(n, H)$ for all $n$. With additional details, one could extend the proof to the case $s_{1}=s_{2}$. But this would make our proof more complicated, so we omit it.

## 3 Proof of Theorem 3

We first construct some bipartite graphs which attain the upper bound in (2). Our idea comes from a theorem of Bushaw and Kettle [1]. Before we present the detailed constructions, we recall some results which we will require.

It is well-known that ex $(n, T) \leq \frac{v(T)-2}{2} n$ when $T$ is a path or star. For a general tree $T$, this is the celebrated Erdős-Sós Conjecture.
Conjecture 2 (Erdős-Sós). For a tree $T$, we have $\operatorname{ex}(n, T) \leq \frac{v(T)-2}{2} n$.
In 2005, McLennan [8] proved that the Erdős-Sós Conjecture holds for trees of diameter at most four.
Theorem 6 (McLennan [8]). Let $T$ be a tree of diameter at most four, then $\operatorname{ex}(n, T) \leq \frac{v(T)-2}{2} n$.
A tree is called balanced if it has the same number of vertices in each color class when the tree is viewed as a bipartite graph. A forest is called balanced if each of its components is a balanced tree. Bushaw and Kettle [1] proved the following theorem.

Theorem 7 (Bushaw and Kettle [1]). Let $H$ be a balanced forest on $2 a$ vertices which comprises at least two trees. If the Erdös-Sós Conjecture holds for each component tree in $H$, then for any $n \geq 3 a^{2}+32 a^{2}\binom{2 a}{a}$, we have

$$
\operatorname{ex}(n, H)= \begin{cases}\binom{a-1}{2}+(a-1)(n-a+1) & \text { if } H \text { admits a perfect matching } \\ (a-1)(n-a+1) & \text { otherwise }\end{cases}
$$

Now, making use of Theorems 6 and 7, we construct some bipartite graphs $H$ which are negative examples for Question 1. Let $\mathcal{H}_{1}$ be the family of all balanced trees on $2 a$ vertices which admit no perfect matching and for which the Erdős-Sós Conjecture holds. One can see that $\mathcal{H}_{1}$ is not empty since a double star $S_{a-1, a-1}$ is a balanced tree on $2 a$ vertices and the Erdős-Sós Conjecture holds for it by Theorem 6. Let $\mathcal{H}_{2}$ be the family of balanced trees on $2 a$ vertices for which the Erdős-Sós Conjecture holds for sufficiently large $n$. Note that $\mathcal{H}_{2}$ is also nonempty, for example a path on $2 a$ vertices belongs to $\mathcal{H}_{2}$.

Let $H_{1} \in \mathcal{H}_{1}, H_{2} \in \mathcal{H}_{2}$ and set $H=H_{1} \cup H_{2}$. We know that $H$ is a balanced forest on $4 a$ vertices. Since $H_{1}$ admits no perfect matching, $H$ admits no perfect matching either. The Erdős-Sós Conjecture holds for each component of $H$, hence by Theorem 7, when $n$ is sufficiently large, we have

$$
\operatorname{ex}(n, H)=(2 a-1)(n-2 a+1)
$$

On the other hand, consider a partition of the vertices of the complete graph $K_{n}$ into parts $X$ and $Y$ with $|X|=2 a-1$ and $|Y|=n-2 a+1$. We color all edges between $X$ and $Y$ red and the remaining edges blue. One can see that the red edges induce a complete bipartite graph $K_{2 a-1, n-2 a+1}$ which contains no red copy of $H$. The blue edges induce a blue $(2 a-1)$-clique and a blue $(n-2 a+1)$-clique which are disjoint with each other. Since each component of $H$ contains $2 a$ vertices, all blue copies of $H$ are contained in the $(n-2 a+1)$-clique. Therefore, all red edges and all the edges in the blue $(2 a-1)$-clique are NIM- $H$, that is,

$$
f(n, H) \geq\binom{ 2 a-1}{2}+(2 a-1)(n-2 a+1)=\binom{2 a-1}{2}+\operatorname{ex}(n, H)
$$

Therefore, such a bipartite graph $H$ attains the upper bound of the inequality (2).
Next we prove that if the bipartite graph $H$ contains a tail $v_{0} v_{1} v_{2}$, then $f(n, H) \leq \operatorname{ex}(n, H)+$ $\binom{|A|-1}{2}$. Note that it is possible that $H$ is disconnected, hence let $H=H_{1} \cup \cdots \cup H_{q}$, where $H_{i}$ are its components (if $H$ is connected, then $H=H_{1}$ ) and we say the tail $v_{0} v_{1} v_{2}$ is contained in $H_{1}$. Let $A_{i}, B_{i}$ be the two color classes of $H_{i}$ with $\left|A_{i}\right| \leq\left|B_{i}\right|$ for any $1 \leq i \leq q$, and let $A=\bigcup_{i=1}^{q} A_{i}, B=\bigcup_{i=1}^{q} B_{i}$. Set $a=|A|$.

Since we take $n$ to be sufficiently large, we may assume $n \geq R\left(K_{v(H)}, K_{v(H)}\right)$. Let $c$ be a red-blue edge-coloring of $K_{n}$. Without loss of generality, there is a blue clique on at least $v(H)$ vertices in $K_{n}$. Let $K_{t}$ be a blue clique in $K_{n}$ such that $t$ is as large as possible. We have $t \geq v(H)$ and every other vertex has a red neighbor in $V\left(K_{t}\right)$. We partition $V\left(K_{n}\right) \backslash V\left(K_{t}\right)$ into two subsets $X, Y$ such that $Y$ consists of the vertices which have blue neighbors in $V\left(K_{t}\right)$ and $X$ consists of the remaining vertices. Hence all edges between $V\left(K_{t}\right)$ and $X$ are red.

The following claims will be used several times.
Claim 1. All blue NIM-H edges are contained in $X$.
Proof. Obviously, the blue edges in $K_{t}$ and $K_{n}\left[V\left(K_{t}\right), Y\right]$ are not NIM- $H$. Let $x y$ be a blue edge with $y \in Y$ and $x \in X \cup Y$. By the definition of $Y$, the vertex $y$ has a blue neighbor, say $v$, in $V\left(K_{t}\right)$. If we embed $V(H) \backslash\left\{v_{1}, v_{2}\right\}$ into $V\left(K_{t}\right)$ and view $v y x$ as the tail of $H$, then we find a blue copy of $H$ containing $x y$. Thus $x y$ is not NIM- $H$. Therefore, all blue NIM- $H$ edges are contained in $X$.

Claim 2. If $|X| \geq a$, then the red edges between $X$ and $V\left(K_{t}\right) \cup Y$ are not NIM-H.
Proof. Since the red edges between $X$ and $V\left(K_{t}\right)$ induce a red complete bipartite graph and $|X| \geq a$ and $t \geq v(H)$, each such edge is contained in a red copy of $H$, thus these edges are not NIM- $H$. Let $x y$ be a red edge with $x \in X, y \in Y$. By the maximality of $K_{t}$, the vertex $y$ has a red neighbor, say $v$, in $V\left(K_{t}\right)$. Actually, $\{x, y, v\}$ induces a red triangle. If the tail $v_{0} v_{1} v_{2}$ of $H$ satisfies $\left\{v_{0}, v_{2}\right\} \subset B$ and $v_{1} \in A$, then embed $B \backslash\left\{v_{2}\right\}$ into $V\left(K_{t}\right)$ so that $v_{0}$ is identified with $v$, embed $A \backslash\left\{v_{1}\right\}$ into $X \backslash\{x\}$ and view $v x y$ as the tail of $H$, thus we find a red copy of $H$ containing $x y$. So in this case, $x y$ is not NIM- $H$. If the tail $v_{0} v_{1} v_{2}$ of $H$ satisfies $\left\{v_{0}, v_{2}\right\} \subset A$ and $v_{1} \in B$, then embed $B \backslash\left\{v_{1}\right\}$ into $V\left(K_{t}\right) \backslash\{v\}$, embed $A \backslash\left\{v_{2}\right\}$ into $X$ so that $v_{0}$ is identified with $x$. View $x y v$ as the tail, we find a red copy of $H$ containing $x y$. So in this case, $x y$ is not NIM- $H$ either.

We distinguish three cases based on the size of $X$.

Case 1: $|X| \geq a+1$. In this case, we first claim that the red edges in $X$ are also not NIM- $H$. Let $x x^{\prime}$ be a red edge contained in $X$ and $v$ be a vertex in $K_{t}$. If the tail $v_{0} v_{1} v_{2}$ in $H$ satisfies $\left\{v_{0}, v_{2}\right\} \subset B$ and $v_{1} \in A$, then since $\left|X \backslash\left\{x, x^{\prime}\right\}\right| \geq a-1=\left|A \backslash\left\{v_{1}\right\}\right|$, we can embed $A \backslash\left\{v_{1}\right\}$ into $X \backslash\left\{x, x^{\prime}\right\}$, embed $B \backslash\left\{v_{2}\right\}$ into $V\left(K_{t}\right)$ so that $v_{0}$ is identified with $v$ and view $v x x^{\prime}$ as the tail $v_{0} v_{1} v_{2}$, thereby finding a red copy of $H$ containing $x x^{\prime}$. So in this case, $x x^{\prime}$ is not NIM- $H$. If the tail $v_{0} v_{1} v_{2}$ in $H$ satisfies $\left\{v_{0}, v_{2}\right\} \subset A$ and $v_{1} \in B$, then we embed $A \backslash\left\{v_{2}\right\}$ into $X \backslash\left\{x^{\prime}\right\}$ so that $v_{0}$ is identified with $x$, embed $B \backslash\left\{v_{1}\right\}$ into $V\left(K_{t}\right) \backslash\{v\}$ and view $x x^{\prime} v$ as the tail, and again we can find a red copy of $H$ containing $x x^{\prime}$. Therefore, $x x^{\prime}$ is not NIM- $H$.

By Claim 2 and the above result, all red NIM- $H$ edges are contained in $V\left(K_{t}\right) \cup Y$. Note that the red NIM- $H$ edges contained in $V\left(K_{t}\right) \cup Y$ induce an $H_{1}$-free graph. Otherwise, such a red copy of $H_{1}$ together with a red copy of $H_{2} \cup \cdots \cup H_{q}$ (if $H$ is disconnected) contained in the complete bipartite graph $K_{n}\left[X, V\left(K_{t}\right)\right]$ yields a red copy of $H$ containing an NIM- $H$ edge, a contradiction. Analogously, the blue NIM- $H$ edges contained in $X$ induce a graph which is $H_{1}$-free. Hence,

$$
\begin{aligned}
|E(c, H)| & \leq \operatorname{ex}\left(|X|, H_{1}\right)+\operatorname{ex}\left(n-|X|, H_{1}\right) \\
& \leq \operatorname{ex}\left(n, H_{1}\right) \leq \operatorname{ex}(n, H)
\end{aligned}
$$

where the second inequality holds since $H_{1}$ is connected. The proof is complete in this case.
Case 2: $|X|=a$. By Claim 2, the set of red NIM- $H$ edges can be partitioned into two parts: the ones contained in $V\left(K_{t}\right) \cup Y$ and the remaining ones which are contained in $X$. Since all blue NIM- $H$ edges are contained in $X$ by Claim 1, the sum of the total number of blue NIM- $H$ edges and the number of red NIM- $H$ edges contained in $X$ is at most $\binom{a}{2}$. The set of red NIM- $H$ edges contained in $V\left(K_{t}\right) \cup Y$ yields an $H_{1}$-free graph. Indeed, otherwise together with a red copy of $H_{2} \cup \cdots \cup H_{q}$ (if $H$ is disconnected) in $K_{n}\left[X, V\left(K_{t}\right)\right]$, we could find a red copy of $H$ containing a red NIM- $H$ edge, a contradiction. Thus the number of red NIM- $H$ edges contained in $V\left(K_{t}\right) \cup Y$ is at most $\operatorname{ex}\left(n-a, H_{1}\right)$.

Therefore, the total number of NIM- $H$ edges is at most $\operatorname{ex}\left(n-a, H_{1}\right)+\binom{a}{2}$. Since $H_{1}$ is connected and contains a tail, it follows that the union of a star $S_{a-1}$ on $a$ vertices and an extremal graph for $\operatorname{ex}\left(n-a, H_{1}\right)$ is still $H_{1}$-free. Hence,

$$
\operatorname{ex}\left(n-a, H_{1}\right)+(a-1) \leq \operatorname{ex}\left(n, H_{1}\right)
$$

Thus, we have

$$
\begin{aligned}
|E(c, H)| & \leq \operatorname{ex}\left(n-a, H_{1}\right)+\binom{a}{2} \leq \operatorname{ex}\left(n, H_{1}\right)+\binom{a-1}{2} \\
& \leq \operatorname{ex}(n, H)+\binom{a-1}{2}
\end{aligned}
$$

and the proof of this case is complete.
Case 3: $|X| \leq a-1$. By Claim 1, the number of blue NIM- $H$ edges is at most $\binom{a-1}{2}$, and the red NIM- $H$ edges yield an $H$-free graph. Hence

$$
|E(c, H)| \leq \operatorname{ex}(n, H)+\binom{a-1}{2}
$$

and the proof is complete.
Remark 3. In [12], the first author and Chen also give a family of examples such that $\chi(H)=3$ and $f(n, H)>\operatorname{ex}(n, H)$.

## 4 Proof of Theorem 5

We first give a $2 k$-edge-coloring of $K_{n}$ with $(2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)+(k-1)\binom{2 k-1}{2}$ NIM- $P_{2 k}$ edges when $2 k-1$ is a prime and $n \in\{a(2 k-1)+(k-1), a(2 k-1)+k\}$. Before showing our construction, we need to recall the exact value of ex $\left(n, P_{\ell}\right)$.
Theorem 8 (Faudree and Schelp [3]). Let $n=a(\ell-1)+b$ with $0 \leq b \leq \ell-2$. Then we have

$$
\operatorname{ex}\left(n, P_{\ell}\right)=a\binom{\ell-1}{2}+\binom{b}{2} .
$$

If $\ell$ is even and $b \in\{\ell / 2, \ell / 2-1\}$, then the extremal graphs are $t K_{\ell-1} \cup\left(K_{\ell / 2-1}+\bar{K}_{n-t(\ell-1)-\ell / 2+1}\right)$ for any $0 \leq t \leq a$. Otherwise $a K_{\ell-1} \cup K_{b}$ is the unique extremal graph.

Therefore, by Theorem 8 , when $n \in\{a(2 k-1)+(k-1), a(2 k-1)+k\}$, the extremal graphs for ex $\left(n, P_{2 k}\right)$ are $t K_{2 k-1} \cup\left(K_{k-1}+\bar{K}_{n-t(2 k-1)-(k-1)}\right)$ for any $0 \leq t \leq a$.

Let $U$ be a subset of size $(2 k-1)^{2}$ of $V\left(K_{n}\right)$ and label the vertices of $U$ by $[i, j]$ where $1 \leq i, j \leq 2 k-1$. We divide $U$ into $2 k-1$ subsets by setting

$$
U_{i}=\{[i, 1],[i, 2], \ldots,[i, 2 k-1]\}, \quad 1 \leq i \leq 2 k-1 .
$$

When it is not confusing, we also let $U$ and $U_{i}$ denote the cliques induced by the vertices in them.

For any $1 \leq i, j \leq 2 k-1$, let $\sigma_{j i}$ denote the clique induced by the vertices $[1, i],[2, i+j], \ldots$, $[2 k-1, i+(2 k-2) j]$, where the indices are taken modulo $2 k-1$. For any $1 \leq j \leq 2 k-1$, let

$$
\mathcal{C}_{j}=\left\{\sigma_{j i}: 1 \leq i \leq 2 k-1\right\} .
$$

Then $\mathcal{C}_{j}$ is a set consisting of $2 k-1$ disjoint $(2 k-1)$-cliques.
Let $c: E\left(K_{n}\right) \rightarrow\left\{c_{1}, \ldots, c_{2 k}\right\}$ be a $2 k$-edge-coloring defined as follows. Let $W=V\left(K_{n}\right) \backslash U$. For any $j \in[2 k-1]$, we assign the color $c_{j}$ to the edges of each clique $\sigma_{j i}$ in $\mathcal{C}_{j}$. Let $\sigma_{j 1}^{\diamond}$ denote the clique induced by the vertices $[k+1,1+k j], \ldots,[2 k-1,1+(2 k-2) j]$. Clearly, we have $\sigma_{j 1}^{\diamond} \subset \sigma_{j 1}$. Now consider the sub-clique $\sigma_{j 1}-\sigma_{j 1}^{\diamond}$ and replace the color $c_{j}$ by $c_{2 k}$ inside it. With this, $\sigma_{j 1}$ decomposes into a copy of $K_{k-1}+\bar{K}_{k}$ colored by $c_{j}$ and a copy of $K_{k}$ colored by $c_{2 k}$. After this, we assign the color $c_{j}$ to all the edges between $\sigma_{j 1}^{\diamond}$ and $V$. Figure 1 shows the subgraph induced by the edges colored by $c_{2 k-1}$. Finally, we assign the color $c_{2 k}$ to the edges which have not been colored yet.


Figure 1: The subgraph induced by the edges of color $c_{2 k-1}$.

In the next two paragraphs, we show that this $2 k$-edge-coloring is well-defined, namely, each edge is assigned exactly one color. Clearly, each edges is assigned at least one color and the edges inside $W$ or between $U_{1} \cup \cdots \cup U_{k}$ and $W$ are assigned exactly one color.

Note that $U$ is a $(2 k-1)^{2}$-clique. Let $1 \leq i, \ell, s, t \leq 2 k-1$. Clearly, the edge $[i, s][i, t]$ is only covered by the clique $U_{i}$. If the edge $[i, s][\ell, t]$ with $i<\ell$ were covered by two cliques, say by one in $\mathcal{C}_{j}$ and by another one in $\mathcal{C}_{j^{\prime}}$ for some $1 \leq j, j^{\prime} \leq 2 k-1$, then

$$
\begin{cases}t \equiv s+(\ell-i) j & (\bmod 2 k-1) \\ t \equiv s+(\ell-i) j^{\prime} & (\bmod 2 k-1)\end{cases}
$$

would hold, and since $2 k-1$ is prime, we would have $j=j^{\prime}$, a contradiction. Thus, each edge inside $U$ is covered by at most one clique in $\mathcal{C}_{j}$ or by the clique $U_{i}$. On the other hand, considering the number of edges in $U$ and the total number of edges of cliques in each $\mathcal{C}_{j}$ and $U_{i}$ yields

$$
e(U)=\sum_{i=1}^{2 k-1} e\left(U_{i}\right)+\sum_{j=1}^{2 k-1} \sum_{\sigma_{j i} \in \mathcal{C}_{j}} e\left(\sigma_{j i}\right)
$$

Therefore, the cliques in each $\mathcal{C}_{j}$ together with the cliques $U_{i}$ for all $1 \leq i, j \leq 2 k-1$ form an edge-decomposition of the large clique $U$. Hence each edge in $U$ is assigned one color.

Now we show that for any $1 \leq j, j^{\prime} \leq 2 k-1$ with $j \neq j^{\prime}$, the sub-cliques $\sigma_{j 1}^{\diamond}$ and $\sigma_{j^{\prime} 1}^{\diamond}$ are vertex-disjoint. Supposing that a vertex $[i, 1+(i-1) j] \in V\left(\sigma_{j 1}^{\diamond}\right)$ is also contained in $\sigma_{j^{\prime} 1}^{\diamond}$ for some $1 \leq i, j, j^{\prime} \leq 2 k-1$, we obtain

$$
1+(i-1) j \equiv 1+(i-1) j^{\prime} \quad(\bmod 2 k-1)
$$

Since $2 k-1$ is a prime number, we get $j=j^{\prime}$, a contradiction. Thus the sub-cliques $\sigma_{j 1}^{\diamond}$ for all $1 \leq j \leq 2 k-1$ form a vertex-decomposition of $U_{k+1} \cup \cdots \cup U_{2 k-1}$. Hence, each edge between $U_{k+1} \cup \cdots \cup U_{2 k-1}$ and $W$ is assigned one color in $\left\{c_{1}, \ldots, c_{2 k-1}\right\}$. Therefore, our $2 k$-edge-coloring $c$ is well-defined.

Note that for any $1 \leq j \leq 2 k-1$, the subgraph induced by the edges of color $c_{j}$ is a copy of $t K_{2 k-1} \cup\left(K_{k-1}+\bar{K}_{n-(k-1)-t(2 k-1)}\right)$ with $t=2 k-2$, and this graph is extremal for $\operatorname{ex}\left(n, P_{2 k}\right)$ when $n \in\{a(2 k-1)+(k-1), a(2 k-1)+k\}$. Now consider the edges colored by $c_{2 k}$. They are in the cliques $U_{i}$ with $1 \leq i \leq 2 k-1$, inside $\sigma_{j 1}-\sigma_{j 1}^{\diamond}$ with $1 \leq j \leq 2 k-1$, inside $W$, and between $U_{1} \cup \cdots \cup U_{k}$ and $W$. Note that for any $k+1 \leq i \leq 2 k-1, U_{i}$ are independent $(2 k-1)$-cliques colored by $c_{2 k}$, hence the edges in $U_{i}$ are also NIM- $P_{2 k}$. For all other $c_{2 k}$-edges, they construct a large connected component such that $W$ is a clique in the component. Hence none of these edges are NIM- $P_{2 k}$.

Therefore,

$$
\left|E\left(c, P_{2 k}\right)\right|=(2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)+(k-1)\binom{2 k-1}{2}
$$

and we are done.
Remark 4. Note that $t K_{2 k-1} \cup\left(K_{k-1}+\bar{K}_{n-(k-1)-t(2 k-1)}\right)$ is not extremal for $\operatorname{ex}\left(n, P_{2 k}\right)$ when $n \notin\{a(2 k-1)+(k-1), a(2 k-1)+k\}$, but we still have

$$
\operatorname{ex}\left(n, P_{2 k}\right)-e\left(t K_{2 k-1} \cup\left(K_{k-1}+\bar{K}_{n-(k-1)-t(2 k-1)}\right)\right)<(k-1)^{2}
$$

Hence in our construction, when $2 k-1$ is prime, the number of NIM- $P_{2 k}$ edges is more than $(2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)$. That is to say, when $2 k-1$ is prime, we have $f_{2 k}\left(n, P_{2 k}\right)>(2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)$ for every sufficiently large $n$.

Next we prove the upper bound of $f_{2 k}\left(n, P_{2 k}\right)$. Let $c: E\left(K_{n}\right) \rightarrow\left\{c_{1}, \ldots, c_{2 k}\right\}$ be a $2 k$ -edge-coloring of $K_{n}$. We call an edge a $c_{i}$-edge if it is of color $c_{i}$ and we let $G_{i}$ denote the subgraph induced by all $c_{i}$-edges, for any $1 \leq i \leq 2 k$. Without loss of generality, we can assume $e\left(G_{2 k}\right) \geq\binom{ n}{2} / 2 k$. By Theorem 8 , there is a path $P$ of at least $\frac{n}{2 k}$ vertices in $G_{2 k}$. Let $G_{2 k}^{\prime}$ be
the component of $G_{2 k}$ which contains the path $P$, and let $X=V\left(G_{2 k}^{\prime}\right)$ and $Y=V\left(K_{n}\right)-X$. Then we have $|X| \geq \frac{n}{2 k}$ and there is no $c_{2 k}$-edge between $X$ and $Y$. Since the component $G_{2 k}^{\prime}$ contains a long path $P$, each edge of $G_{2 k}^{\prime}$ is contained in a monochromatic copy of $P_{2 k}$. Hence, all NIM- $P_{2 k} c_{2 k}$-edges are contained in $Y$.

For each $1 \leq i \leq 2 k-1$, there are at most ex $\left(n, P_{2 k}\right)$ NIM- $P_{2 k} c_{i}$-edges. If $|Y| \leq(k-1)(2 k-1)$, then there are at most ex $\left(|Y|, P_{2 k}\right) \leq(k-1)\binom{2 k-1}{2}$ NIM- $P_{2 k} c_{2 k}$-edges. Hence, the total number of NIM- $P_{2 k}$ edges is at most

$$
(2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)+(k-1)\binom{2 k-1}{2}
$$

so we are done. Therefore, we may assume $|Y| \geq(k-1)(2 k-1)+1$.
Let us define a procedure to find pairs $\left(X_{i}, Y_{i}\right)$ satisfying the following conditions:
(i) $X_{i} \subseteq X$ with $\left|X_{i}\right|=2 k$ and $Y_{i} \subseteq Y$ with $\left|Y_{i}\right|=k$ for any $1 \leq i \leq 2 k$,
(ii) $Y_{i}$ and $Y_{j}$ are disjoint for any $1 \leq i, j \leq 2 k$ with $i \neq j$,
(iii) $K_{n}\left[X_{i}, Y_{i}\right]$ forms a monochromatic copy of complete bipartite graph for any $1 \leq i \leq 2 k$.

Assume that for some $1 \leq i \leq 2 k$, we have found $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{i-1}, Y_{i-1}\right)$ which satisfy the conditions. Let $s=(k-1)(2 k-1)+1$. If

$$
\left|Y \backslash \bigcup_{j=1}^{i-1} Y_{j}\right| \leq s-1
$$

then the procedure terminates. Otherwise we choose a subset $Y_{i}^{\prime}$ of $Y \backslash \bigcup_{j=1}^{i-1} Y_{j}$ with $\left|Y_{i}^{\prime}\right|=s$. Let $Y_{i}^{\prime}=\left\{y_{1}, \ldots, y_{s}\right\}$. For each $x \in X$, we define a vector $\vec{\epsilon}\left(x, Y_{i}^{\prime}\right)=\left(\epsilon_{1}, \ldots, \epsilon_{s}\right)$ as follows: for any $1 \leq j \leq s$, let

$$
\epsilon_{j}=i \text { if and only if the edge } x y_{j} \text { is colored by } c_{i}
$$

Since no edge between $X$ and $Y$ is colored by $c_{2 k}$, we have $\vec{\epsilon}\left(x, Y_{i}^{\prime}\right) \in\{1, \ldots, 2 k-1\}^{s}$ for any $x \in X$. For each $\vec{v} \in\{1, \ldots, 2 k-1\}^{s}$, let $X_{\vec{v}}$ denote the set of vertices $x \in X$ for which $\vec{\epsilon}\left(x, Y_{i}^{\prime}\right)=\vec{v}$. Hence, $X$ is divided into $(2 k-1)^{s}$ subsets and clearly, at least one subset, say $X_{\vec{v}_{i}}$, contains at least $|X| /(2 k-1)^{s}$ vertices. Observe that $K_{n}\left[X_{\vec{v}_{i}}, y_{j}\right]$ is a monochromatic star for any $y_{j} \in Y_{i}^{\prime}$. Since $\left|Y_{i}^{\prime}\right|=(k-1)(2 k-1)+1$ and there are at most $2 k-1$ different colors between $X_{\vec{v}_{i}}$ and $Y_{i}^{\prime}$, by pigeonhole principle, there exists a subset $Y_{i} \subset Y_{i}^{\prime}$ such that $\left|Y_{i}\right|=k$ and the edges between $Y_{i}$ and $X_{\vec{v}_{i}}$ are monochromatic. That is $K_{n}\left[X_{\vec{v}_{i}}, Y_{i}\right]$ is a monochromatic complete bipartite graph. Since $n \geq(2 k)^{2 k^{2}}$,

$$
\left|X_{\vec{v}_{i}}\right| \geq \frac{|X|}{(2 k-1)^{s}} \geq \frac{n}{(2 k)^{s}} \geq 2 k
$$

We can choose a subset $X_{i}$ from $X_{\vec{v}_{i}}$ with $\left|X_{i}\right|=2 k$, thereby finding the pair $\left(X_{i}, Y_{i}\right)$ as we wanted.

Note that since $Y$ is finite, the procedure terminates. Let $t$ denote the number of steps the algorithm took, and let $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{t}, Y_{t}\right)$ be the pairs the algorithm found. Let $Y_{0}=$ $Y \backslash \bigcup_{1}^{t} Y_{i}$. Then we have $\left|Y_{0}\right| \leq(k-1)(2 k-1)$. For any $1 \leq i \leq 2 k-1$, let $t_{i}$ denote the number of the pairs $\left(X_{j}, Y_{j}\right)$ for which the edges of $K_{n}\left[X_{j}, Y_{j}\right]$ are of color $c_{i}$. Without loss of generality, we may assume that $t_{1}, \ldots, t_{h}>0$ for some $1 \leq h \leq 2 k-1$. Then $t=\sum_{i=1}^{h} t_{i}$. Let $1 \leq i \leq h$ and consider the $c_{i}$-edges. Without loss of generality, we can assume that $K_{n}\left[X_{1}, Y_{1}\right], \ldots, K_{n}\left[X_{t_{i}}, Y_{t_{i}}\right]$ are of color $c_{i}$. Then each NIM- $P_{2 k} c_{i}$-edge is contained in $V\left(K_{n}\right) \backslash \bigcup_{j=1}^{t_{i}}\left(X_{j} \cup Y_{j}\right)$. Since the sets $Y_{1}, \ldots, Y_{t_{i}}$ are pairwise disjoint and $X_{1}, \ldots, X_{t_{i}} \subseteq X$, we have

$$
\left|\bigcup_{j=1}^{t_{i}}\left(X_{j} \cup Y_{j}\right)\right| \geq t_{i} k+2 k
$$

thus the number of NIM- $P_{2 k} c_{i}$-edges is at most ex $\left(n-t_{i} k-2 k, P_{2 k}\right)$. Now let $h+1 \leq i \leq 2 k-1$ (if such an index exists). Since $t_{i}=0$, the number of NIM- $P_{2 k} c_{i}$-edges is at most ex $\left(n, P_{2 k}\right)$.

As we have proved, all NIM- $P_{2 k} c_{2 k}$-edges are contained in $Y$ and $|Y| \leq(k-1)(2 k-1)+t k$. Therefore, the total number of NIM- $P_{2 k}$ edges is at most

$$
\begin{equation*}
\operatorname{ex}\left((k-1)(2 k-1)+t k, P_{2 k}\right)+\sum_{i=1}^{h} \operatorname{ex}\left(n-t_{i} k-2 k, P_{2 k}\right)+(2 k-1-h) \operatorname{ex}\left(n, P_{2 k}\right) \tag{3}
\end{equation*}
$$

To prove the final result, we need the following lemma.
Lemma 1. Let $n_{1}, n_{2}$ and $c$ be constants. Then we have

$$
\operatorname{ex}\left(n_{1}, P_{\ell}\right)+\operatorname{ex}\left(n_{2}, P_{\ell}\right)<\operatorname{ex}\left(n_{1}-c, P_{\ell}\right)+\operatorname{ex}\left(n_{2}+c+\ell, P_{\ell}\right)
$$

Proof. Let $n_{1}-c=a_{1}(\ell-1)+b_{1}$ and $n_{2}+c=a_{2}(\ell-1)+b_{2}$, where $0 \leq b_{1}, b_{2} \leq \ell-2$. By Theorem 8, we have

$$
\begin{aligned}
\operatorname{ex}\left(n_{1}-c, P_{\ell}\right) & +\operatorname{ex}\left(n_{2}+c+\ell, P_{\ell}\right) \geq \operatorname{ex}\left(n_{1}-c, P_{\ell}\right)+\operatorname{ex}\left(n_{2}+c, P_{\ell}\right)+\operatorname{ex}\left(\ell, P_{\ell}\right) \\
& >\left(a_{1}+a_{2}\right)\binom{\ell-1}{2}+\binom{b_{1}}{2}+\binom{b_{2}}{2}+\binom{\ell-1}{2}
\end{aligned}
$$

and

$$
\operatorname{ex}\left(n_{1}, P_{\ell}\right)+\operatorname{ex}\left(n_{2}, P_{\ell}\right) \leq \frac{\ell-2}{2}\left(n_{1}+n_{2}\right)=\left(a_{1}+a_{2}\right)\binom{\ell-1}{2}+\left(b_{1}+b_{2}\right) \frac{\ell-2}{2}
$$

Hence we have

$$
\begin{aligned}
\operatorname{ex}\left(n_{1}-c, P_{\ell}\right) & +\operatorname{ex}\left(n_{2}+c+\ell, P_{\ell}\right)-\left(\operatorname{ex}\left(n_{1}, P_{\ell}\right)+\operatorname{ex}\left(n_{2}, P_{\ell}\right)\right) \\
& >\binom{b_{1}}{2}+\binom{b_{2}}{2}+\binom{\ell-1}{2}-\left(b_{1}+b_{2}\right) \frac{\ell-2}{2}>0 .
\end{aligned}
$$

we are done.

When applying the above lemma to (3), we get

$$
\begin{aligned}
\operatorname{ex}\left((k-1)(2 k-1)+t k, P_{2 k}\right) & +\sum_{i=1}^{s} \operatorname{ex}\left(n-t_{i} k-2 k, P_{2 k}\right)+(2 k-1-s) \operatorname{ex}\left(n, P_{2 k}\right) \\
< & (2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)+\operatorname{ex}\left((k-1)(2 k-1), P_{2 k}\right) \\
= & (2 k-1) \operatorname{ex}\left(n, P_{2 k}\right)+(k-1)\binom{2 k-1}{2} .
\end{aligned}
$$

Thus the proof is complete.

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[^0]:    *Correspondoing author. Email: lvzq19@mails.tsinghua.edu.cn

