# TWO CONJECTURES ON THE STOKES COMPLEX IN THREE DIMENSIONS ON FREUDENTHAL MESHES 

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#### Abstract

In recent years a great deal of attention has been paid to discretizations of the incompressible Stokes equations that exactly preserve the incompressibility constraint. These are of substantial interest because these discretizations are pressure-robust, i.e. the error estimates for the velocity do not depend on the error in the pressure. Similar considerations arise in nearly incompressible linear elastic solids. Conforming discretizations with this property are now well understood in two dimensions, but remain poorly understood in three dimensions. In this work we state two conjectures on this subject. The first is that the Scott-Vogelius element pair is inf-sup stable on uniform meshes for velocity degree $k \geq 4$; the best result available in the literature is for $k \geq 6$. The second is that there exists a stable space decomposition of the kernel of the divergence for $k \geq 5$. We present numerical evidence supporting our conjectures.


1. Introduction. We consider two closely related problems for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^{d}, d \in\{2,3\}$. The first is the incompressible Stokes equations: given $f \in L^{2}\left(\Omega ; \mathbb{R}^{d}\right)$, find the velocity $u: \Omega \rightarrow \mathbb{R}^{d}$ and pressure $p: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{align*}
-\nabla \cdot \varepsilon u+\nabla p=f & \text { in } \Omega  \tag{1.1a}\\
\nabla \cdot u=0 & \text { in } \Omega  \tag{1.1b}\\
u=0 & \text { on } \partial \Omega \tag{1.1c}
\end{align*}
$$

where $\varepsilon u=1 / 2\left(\nabla u+(\nabla u)^{T}\right)$ is the symmetric gradient of $u$. The second is the Navier-Cauchy equation of linear elasticity: given $f \in L^{2}\left(\Omega ; \mathbb{R}^{d}\right)$ and $\gamma>0$, find the displacement $u: \Omega \rightarrow \mathbb{R}^{d}$ that satisfies

$$
\begin{align*}
-\nabla \cdot \varepsilon u-\gamma \nabla \nabla \cdot u & =f & & \text { in } \Omega,  \tag{1.2a}\\
u & =0 & & \text { on } \partial \Omega . \tag{1.2b}
\end{align*}
$$

Here $\gamma=2 \lambda / \mu$, where $\lambda$ and $\mu$ are the Lamé parameters. As $\gamma \rightarrow \infty$, the material is said to be nearly incompressible. The term $-\nabla \nabla \cdot u$ in (1.2a) is connected to the incompressibility constraint (1.1b); it arises in the Stokes momentum equation (1.1a) when employing an augmented Lagrangian approach [6] to enforcing the divergence-zero constraint (1.1b).

When discretizing (1.1), it is highly desirable to choose spaces $V_{h}$ for the velocity and $\Pi_{h}$ for the pressure such that all discretely divergence free functions are pointwise divergence free, i.e. the incompressibility constraint (1.1b) is satisfied exactly on the discrete level [27]. Achieving this is difficult; no element pair for exact enforcement is known that is simultaneously inf-sup stable, low-order, conforming, has polynomial basis functions, and is effective on general meshes. On simplicial grids with special mesh structure, it is possible to use the conforming Scott-Vogelius finite element pair $\left[\mathrm{CG}_{k}\right]^{d_{-}}$ $\mathrm{DG}_{k-1}[48,49]$ for $k \geq d$ (for Alfeld splits [39, 22, 17, 52]) or $k \geq d-1$ (for Powell-Sabin splits [20, 53, 55, 21]). The approach of Guzmán and Neilan [22] is conforming, works for arbitrary degree and on general meshes, but requires the use of piecewise polynomial

[^0]

Fig. 2.1. Unit cube for a structured mesh in three dimensions used for computational experiments. Given $N \in \mathbb{N}$, a $N \times N \times N$ mesh of cubes is generated, each of which is subdivided into 6 tetrahedra as shown. Vertex labels correspond to the vertices enumerated in Appendix A.
basis functions on each cell (instead of standard polynomials). Non-conforming divergence-free discretizations are reviewed in John et al. [27, §4.4]. Finally, another approach is to consider the use of high-order discretizations, which are attractive for their advantageous computational properties on modern architectures [49, 48, 37, 54]. Another alternative is to modify the right-hand side of the problem with an operator that maps discretely divergence-free test functions to exactly divergence-free ones [30].

In this paper we state two conjectures regarding the discretization and multigrid solution of (1.1) and (1.2) on structured uniform tetrahedral meshes (Freudenthal meshes [7]). We focus on Freudenthal meshes since some theoretical results are known in this case. For concreteness we briefly describe the Freudenthal triangulation of the unit cube in Appendix A. The first is that the Scott-Vogelius element pair is inf-sup stable on Freudenthal meshes for $k \geq 4$; the best available result is that of Zhang, who proved that the pair is stable for $k \geq 6$ [54]. We conjecture this on the basis of numerical calculations of the inf-sup constant for varying $k$. These rely on a new algorithm that can compute the inf-sup constant for elements that are divergence-free when an exact characterization of the pressure space is not known. The second is that on the same meshes the subspace $Z_{h} \subset V_{h}$ of divergence-free functions admits a local basis defined on the vertex-centred patches for $k \geq 5$, and that the associated space decomposition is stable. This is significant because identifying a local basis for the kernel of the divergence operator is crucial for multigrid algorithms applied to (1.2). We conjecture this on the basis of observed $\gamma$-robustness of a multigrid solution algorithm for (1.2), for which the local kernel decomposition is essential [43]. The existence of a local basis is known in three dimensions for Alfeld splits [17] and Worsey-Farin splits [21], but remains an open question for general meshes, and in particular for the Freudenthal meshes considered here.
2. Inf-sup stability of the Scott-Vogelius element. The mixed formulation of (1.1) is to find $(u, p) \in V \times \Pi:=H_{0}^{1}\left(\Omega ; \mathbb{R}^{d}\right) \times L_{0}^{2}(\Omega)$ such that

$$
\begin{equation*}
(\varepsilon u, \varepsilon v)_{L^{2}}-(p, \nabla \cdot v)_{L^{2}}-(q, \nabla \cdot u)_{L^{2}}=(f, v)_{L^{2}(\Omega)} \quad \text { for all }(v, q) \in V \times \Pi . \tag{2.1}
\end{equation*}
$$

Define $(T, U)_{L^{2}(\Omega)}=(T, U)_{L^{2}}$ for any scalar-, vector-, or tensor-valued functions $T, U$.
The inf-sup condition determines whether or not a pair of spaces $V_{h} \subset V, \Pi_{h} \subset \Pi$ in a mixed finite element method provide a compatible discretization [9], since the symmetric gradient term is coercive on $V$. In the context of (2.1), this condition encapsulates a constraint between the divergence of the velocity space $V_{h}$ and the
pressure space $\Pi_{h}$ : there exists $\beta>0$ such that

$$
\begin{equation*}
\beta \sqrt{(q, q)_{L^{2}(\Omega)}} \leq \sup _{\substack{v \in V_{h} \\ v \neq 0}} \frac{(\nabla \cdot v, q)_{L^{2}(\Omega)}}{\sqrt{(\nabla v, \nabla v)_{L^{2}(\Omega)}}} \quad \forall q \in \Pi_{h} \tag{2.2}
\end{equation*}
$$

In this section we make the following conjecture:
Conjecture 1. Let $V_{h}$ be constructed with continuous Lagrange elements of degree $k$, and choose $\Pi_{h}=\nabla \cdot V_{h}$. For $k \geq 4$, the inf-sup condition (2.2) holds on structured tetrahedral meshes of the form in Figure 2.1, known as the Freudenthal triangulation [7], with a constant that depends only on $k$.

We conjecture this on the basis of numerical computations with a new algorithm for calculating the inf-sup constant, which we now describe.
2.1. Computing the inf-sup constant. There are several methods to estimate computationally the inf-sup constant $\beta$ for various spaces and variational problems [39, 41, 4]. In [18], approximation of the corresponding Ladyzhenskaya inf-sup constant for the continuous problem is studied. All of these are based on solving eigenproblems.

In [41], an automated system ASCoT was introduced for computing inf-sup constants. It takes as input $V_{h}$ and $\Pi_{h}$. However, when $\Pi_{h}=\nabla \cdot V_{h}$, it is advantageous to exploit this structure in the calculation of the inf-sup constant. More importantly, the space $\nabla \cdot V_{h}$ may not be known a priori, and so a different algorithm is needed that does not require $\Pi_{h}$ as input.

A technique was developed by one of the authors in [46] that deals specifically with the $\Pi_{h}=\nabla \cdot V_{h}$ case and casts this as an eigenvalue problem, as follows. Define the bilinear forms

$$
\begin{equation*}
a(u, v)=\int_{\Omega} \nabla u: \nabla v d x, \quad b(v, q)=\int_{\Omega}(\nabla \cdot v) q d x \tag{2.3}
\end{equation*}
$$

First define

$$
\begin{equation*}
Z_{h}=\left\{v \in V_{h}: b(v, q)=0 \forall q \in \Pi_{h}\right\} \tag{2.4}
\end{equation*}
$$

which is the set of divergence-free functions in $V_{h}$ since $\Pi_{h}=\nabla \cdot V_{h}$. Define $\kappa$ by

$$
\begin{equation*}
\kappa=\min _{0 \neq v \in V_{h}, v \perp_{a} Z_{h}} \frac{(\nabla \cdot v, \nabla \cdot v)_{L^{2}}}{a(v, v)}=\min _{0 \neq v \in Z_{h}^{\perp}} \frac{(\nabla \cdot v, \nabla \cdot v)_{L^{2}}}{a(v, v)} \tag{2.5}
\end{equation*}
$$

where $v \perp_{a} Z_{h}$ means that $a(v, w)=0$ for all $w \in Z_{h}$ and

$$
\begin{equation*}
Z_{h}^{\perp}=\left\{v \in V_{h}: a(v, w)=0 \forall w \in Z_{h}\right\} \tag{2.6}
\end{equation*}
$$

We recall the following lemma from [46, Lemma 26.1].
Lemma 2.1. Suppose that $\|v\|_{V}=\sqrt{a(v, v)}$ and $\Pi_{h}=\nabla \cdot V_{h}$. Then

$$
\begin{equation*}
\beta=\inf _{\substack{q \in \Pi_{h} \\ q \neq 0}} \sup _{\substack{v \in V_{h} \\ v \neq 0}} \frac{b(v, q)}{\|v\|_{V}\|q\|_{L^{2}}}=\inf _{\substack{q \in \Pi_{h} \\ q \neq 0}} \sup _{\substack{v \in Z_{h}^{\perp} \\ v \neq 0}} \frac{b(v, q)}{\|v\|_{V}\|q\|_{L^{2}}} \geq \sqrt{\kappa} \geq \frac{1}{2} \beta \tag{2.7}
\end{equation*}
$$

where $\kappa$ is defined in (2.5).

This result holds also for the bilinear form

$$
\begin{equation*}
a(u, v)=(\varepsilon u, \varepsilon v)_{L^{2}(\Omega)}, \tag{2.8}
\end{equation*}
$$

with possibly different constants.
Computing $\kappa$ is equivalent to finding the smallest eigenvalue $\lambda$ of the following eigenproblem: find $0 \neq u_{h} \in Z_{h}^{\perp}$ such that

$$
\begin{equation*}
\left(\nabla \cdot u_{h}, \nabla \cdot v_{h}\right)_{L^{2}(\Omega)}=\lambda a\left(u_{h}, v_{h}\right) \quad \forall v_{h} \in Z_{h}^{\perp}, \tag{2.9}
\end{equation*}
$$

which is equivalent to the Rayleigh quotient minimization (2.5). Note that $\kappa>0$ since $\kappa=0$ leads to the contradiction $\nabla \cdot u_{h}=0$, that is, $u_{h} \in Z_{h} \cap Z_{h}^{\perp}$. Thus there are no spurious modes when $\Pi_{h}=\nabla \cdot V_{h}$.

We can write (2.9) in operator form as

$$
\begin{equation*}
B u_{h}=\lambda A u_{h} \tag{2.10}
\end{equation*}
$$

The equation (2.10) is a symmetric generalized eigenvalue problem [50], and its eigenvalues are all real. There are many algorithms for solving symmetric generalized eigenvalue problems [19, 42, 50]. However, we do not have an explicit basis for the space $Z_{h}^{\perp}$, so we choose to use matrix-free methods. Here we focus on a simple method related to the power method.

Since $A$ is invertible on all of $V_{h}$, it would be attractive to utilize an iteration in which we invert $A$, and not $B$. We introduce a shift $\sigma$ :

$$
\begin{equation*}
(B-\sigma A) u_{h}=\lambda^{\sigma} A u_{h} \tag{2.11}
\end{equation*}
$$

If $\lambda_{1} \leq \lambda_{2} \leq \cdots \leq \lambda_{N}$ are the eigenvalues of the symmetric problem (2.10), that is, for the operator $A^{-1} B$, and $\lambda_{1}^{\sigma} \leq \lambda_{2}^{\sigma} \leq \cdots \leq \lambda_{N}^{\sigma}$ are the eigenvalues of the shifted problem (2.11), then $\lambda_{i}^{\sigma}=\lambda_{i}-\sigma$ for all $i=1, \ldots, N$. Moreover, $\lambda_{i}^{\sigma}$ are the eigenvalues for the operator $A^{-1} B-\sigma I$, and the eigenvectors $u_{h}^{i}$ for $\lambda_{i}^{\sigma}$ and $\lambda_{i}$ are the same, for all $i=1, \ldots, N$.
2.2. Using the power method. We can solve for certain eigenvalues and eigenvectors $u_{h}$ via the power method [45] for (2.11), namely to find $u^{n} \in Z_{h}^{\perp}$ such that

$$
\begin{align*}
a\left(u^{n}, v_{h}\right) & =\left(\nabla \cdot \hat{u}^{n-1}, \nabla \cdot v_{h}\right)_{L^{2}(\Omega)}-\sigma a\left(\hat{u}^{n-1}, v_{h}\right) \quad \forall v_{h} \in Z_{h}^{\perp} \\
\lambda^{n+1} & =\frac{\left(\nabla \cdot u^{n}, \nabla \cdot u^{n}\right)_{L^{2}(\Omega)}}{a\left(u^{n}, u^{n}\right)}, \quad \hat{u}^{n}=\left(\left\|u^{n}\right\|_{V}\right)^{-1} u^{n} . \tag{2.12}
\end{align*}
$$

The choices of initial iterate and termination criterion are discussed subsequently.
We now consider how to compute this, despite the fact that the space $Z_{h}^{\perp}$ is not explicitly known. Suppose that $\hat{u}^{0} \in Z_{h}^{\perp}$ is given and that we solve for $u^{1} \in V_{h}$ via

$$
a\left(u^{1}, v_{h}\right)=\left(\nabla \cdot \hat{u}^{0}, \nabla \cdot v_{h}\right)_{L^{2}(\Omega)}-\sigma a\left(\hat{u}^{0}, v_{h}\right) \quad \forall v_{h} \in V_{h}
$$

which we can do since $a(\cdot, \cdot)$ is coercive on $V_{h}$. Then for all $v_{h} \in Z_{h}$

$$
a\left(u^{1}, v_{h}\right)=\left(\nabla \cdot \hat{u}^{0}, \nabla \cdot v_{h}\right)_{L^{2}(\Omega)}-\sigma a\left(\hat{u}^{0}, v_{h}\right)=0
$$

since both terms vanish. Thus $u^{1} \in Z_{h}^{\perp}$. Moving on, we can solve for $u^{n} \in V_{h}$ via

$$
\begin{equation*}
a\left(u^{n}, v_{h}\right)=\left(\nabla \cdot \hat{u}^{n-1}, \nabla \cdot v_{h}\right)_{L^{2}(\Omega)}-\sigma a\left(\hat{u}^{n-1}, v_{h}\right) \quad \forall v_{h} \in V_{h} . \tag{2.13}
\end{equation*}
$$

By induction, $a\left(u^{n}, v_{h}\right)=0$ for all $v_{h} \in Z_{h}$, so that $u^{n} \in Z_{h}^{\perp}$ for all $n>0$.
To start the process, we can solve for $u^{0} \in V_{h}$ via

$$
\begin{equation*}
a\left(u^{0}, v\right)=(\nabla \cdot w, \nabla \cdot v)_{L^{2}(\Omega)} \quad \forall v \in V_{h} \tag{2.14}
\end{equation*}
$$

where we must pick some $w$ where $\nabla \cdot w \neq 0$. In practice, we chose

$$
\begin{equation*}
w=(\sin (\omega x), \cos (\omega y)) \tag{2.15}
\end{equation*}
$$

for $\omega \in \mathbb{Z}$, although other initializations worked as well. We then set

$$
\begin{equation*}
\hat{u}^{0}=\left(\left\|u^{0}\right\|_{V}\right)^{-1} u^{0} \tag{2.16}
\end{equation*}
$$

Note that $a\left(u^{0}, v\right)=0$ for all $v \in Z_{h}$, which means that $u^{0} \in Z_{h}^{\perp}$. The division in (2.16) provides a natural check that $u^{0} \neq \mathbf{0}$.
2.3. Eigenvalue bounds. The eigenvalues $\lambda_{i}$ are bounded above since

$$
\left\|\nabla \cdot v_{h}\right\|_{L^{2}}^{2} \leq C a\left(v_{h}, v_{h}\right) \quad \forall v_{h} \in V
$$

This is obvious for the gradient form (2.3) with $C \leq d$, where $d$ is the dimension of $\Omega$. For the $\boldsymbol{\varepsilon}$ form (2.8), it follows by the Korn inequality [9, §11.2]. A good estimate of $C$ can be obtained computationally since it is the largest eigenvalue of (2.9). In all computational examples here, using the gradient form (2.3), $C=1$, apparently due to the homogeneous Dirichlet boundary conditions in $V$. Using the $\varepsilon$ form (2.8), which is equivalent to (2.3) by Korn's inequality, would change the computational results here by at most a constant factor. When homogeneous Dirichlet boundary conditions are enforced, it is known [46, (13.12)] that the forms (2.8) and (2.3) produce identical values for divergence-free functions.

Thus we can shift by a constant $\sigma$ independent of $h$ to ensure that $\lambda_{i}^{\sigma}<0$ for all $i$. The algorithm (2.12) is the power method for $A^{-1} B-\sigma I$, and it will generically converge to the eigenvector associated with the most negative eigenvalue, provided $\sigma>\frac{1}{2} \lambda_{N}$, and so it will generically converge to $\lambda_{1}^{\sigma}$. Taking $\sigma=0$ will converge to the largest eigenvalue $\lambda_{N}$, since $0<\lambda_{1}$. We found that taking $\sigma=0.6$ gave acceptable convergence.
2.4. Effect of round-off error. It is essential to start the iteration (2.12) with $\hat{u}^{0} \in Z_{h}^{\perp}$. To see what goes wrong otherwise, write the algorithm in (2.12) as

$$
u^{k+1}=c_{k}\left(A^{-1} B-\sigma I\right) u^{k}
$$

where $c_{k}=1 /\left\|u^{k}\right\|_{V}$. Suppose that $u^{0}=v^{0}+w^{0}$ where $v^{0} \in Z_{h}^{\perp}$ and $w^{0} \in Z_{h}$. Then $u^{k}=v^{k}+w^{k}$ where $v^{k} \in Z_{h}^{\perp}$ and $w^{k} \in Z_{h}$, and

$$
v^{k+1}=c_{k}\left(A^{-1} B-\sigma I\right) v^{k}, \quad w^{k+1}=c_{k}(-\sigma I) w^{k} .
$$

The reason is that $A^{-1} B$ maps $Z_{h}^{\perp}$ into itself and $Z_{h}$ to the zero vector. Define $C_{k}=\prod_{i=0}^{k-1} c_{i}$. Then

$$
v^{k}=C_{k}\left(A^{-1} B-\sigma I\right)^{k} v^{0}, \quad w^{k}=C_{k}(-\sigma I)^{k} w^{0}
$$

In seeking the smallest eigenvalue $\lambda_{1}$, we will need to take $\sigma$ as large as at least half of the largest eigenvalue $\lambda_{N}$. Thus $w^{k}$ can become significant even if $w^{0}$ is on the order of round-off error. Once $w^{k}$ becomes dominant, the Rayleigh quotient in (2.12)


FIG. 2.2. Two types of regular meshes in two dimensions. The Type II mesh is also known as the Malkus split.
defining $\lambda^{k}$ will go to zero. Thus it is necessary to monitor the projection $z^{k}$ of $u^{k}$ onto $Z_{h}$ and to project $u^{k}$ onto $Z_{h}^{\perp}$ (by simply subtracting $z^{k}$ ) when the divergence-zero component is too large. This may need to be done for $k=0$ as well.

The projection $z^{k} \in Z_{h}$ satisfies

$$
a\left(z^{k}, v\right)=a\left(u^{k}, v\right) \forall v \in Z_{h} .
$$

The projection $z^{k}$ can be computed via the iterated penalty method. Some care is required in using the iterated penalty method to do this, since it will be slow to converge if the inf-sup constant is small. But this appears to work well in practice, albeit with a large number of iterations required when the inf-sup constant is smaller.
2.5. Full algorithm. We now summarize the full algorithm. First we compute $\hat{u}^{0}$ via (2.14) and (2.16), where $w$ is given in (2.15). Next, we solve for $u^{n}$ via (2.13), $n \geq 1$. Then we project $u^{n}$ onto $Z_{h}$ using the iterated penalty method [9, §13.1]. To compute the projection of $u^{n}$ onto $Z_{h}$, we solve for $z^{\ell}, w^{\ell} \in V_{h}$ such that

$$
\begin{align*}
& a\left(z_{\ell}, v\right)+\rho\left(\nabla \cdot z_{\ell}, \nabla \cdot v\right)=a\left(u^{n}, v\right)-\left(\nabla \cdot w_{\ell}, \nabla \cdot v\right) \quad \forall v \in V_{h}  \tag{2.17}\\
& w_{\ell+1}=w_{\ell+1}+\rho z_{\ell}
\end{align*}
$$

where we start with $w_{0}=0$. The parameter $\rho$ is the penalty parameter enforcing the incompressibility constraint; in our computations we set $\rho=10^{4}$. We terminate the iteration on $\ell$ when

$$
\begin{equation*}
\left\|\nabla \cdot z_{\ell}\right\|_{L^{2}(\Omega)} \leq \tau \tag{2.18}
\end{equation*}
$$

where we picked $\tau=10^{-14}$ in our computations. Then $z_{\ell} \approx \Pi_{Z_{h}} u^{n}$, and if

$$
\left\|\nabla z_{\ell}\right\|_{L^{2}(\Omega)} \geq \zeta\left\|\nabla u^{n}\right\|_{L^{2}(\Omega)}
$$

we update $u^{n}=u^{n}-z_{\ell}$. In our computations, we picked $\zeta=10^{-12}$. Once $u^{n}$ has been computed, we define

$$
\begin{equation*}
\lambda^{n+1}=\frac{\left(\nabla \cdot u^{n}, \nabla \cdot u^{n}\right)}{a\left(u^{n}, u^{n}\right)}, \quad \hat{u}^{n}=\left(\left\|u^{n}\right\|_{V}\right)^{-1} u^{n} \tag{2.19}
\end{equation*}
$$

This iteration is continued while $\left|\lambda^{n+1}-\lambda^{n}\right|>\epsilon$, where $\epsilon$ is a pre-specified tolerance, with $\epsilon=10^{-8}$ in our computations.
2.6. 2D tests. To verify the algorithm, we compare with known results. We summarize some known results in two and three dimensions in Table 2.1. In Table 2.2, we give results for Lagrange elements of degree $k=1,2$ in two dimensions on Malkus

Table 2.1
Mesh restrictions for $V_{h}$ constructed with continuous piecewise polynomials of degree $k$, and $\Pi_{h}=\nabla \cdot V_{h}$. Here $d$ is the dimension of $\Omega$. In the lowest-order case $k=1$ in two dimensions, the velocity approximation is optimal order despite the fact that the inf-sup constant deteriorates.

| $d$ | $k$ | inf-sup | mesh restrictions |
| :--- | ---: | :---: | :--- |
| 2 | 1 | no, but | optimal velocity approximation on Malkus splits [33, 40] |
| 2 | 1 | yes | Powell-Sabin splits [13, 20, 53] |
| 2 | 2 | yes | some crossed triangles required [39] or Alfeld splits [22, 17] |
| 2 | 3 | yes | under certain conditions [23] |
| 2 | $\geq 4$ | yes | $p$-robust when no nearly singular vertices [48, 49, 24, 1] |
| 3 | $\geq 1$ | yes | Worsey-Farin splits [13] |
| 3 | $\geq 3$ | yes | Alfeld splits [52, 22, 17] |
| 3 | $\geq 6$ | yes | only one family of meshes known [54] |

TABLE 2.2
Computation of inf-sup constants on Malkus splits (Type II meshes) in two dimensions. The shift parameter in (2.11) was set to $\sigma=0.6$. The mesh size $N$ refers to the $N \times N$ mesh of quadrilaterals before subdivision. The parameter $\omega$ determines the initial eigenvector approximation defined in (2.15). Iterations were continued until the change in eigenvalue was less than $10^{-7}$.

| degree $k$ | $N$ | $\omega$ | iterations | $\inf$-sup $\lambda_{1}$ | restarts |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 5 | 5 | 47 | $4.08 \times 10^{-1}$ | 0 |
| 1 | 5 | 10 | 40 | $4.08 \times 10^{-1}$ | 0 |
| 1 | 5 | 100 | 51 | $4.08 \times 10^{-1}$ | 1 |
| 1 | 10 | 10 | 258 | $1.13 \times 10^{-2}$ | 1 |
| 1 | 20 | 10 | 642 | $2.98 \times 10^{-3}$ | 1 |
| 2 | 10 | 10 | 193 | $1.49 \times 10^{-1}$ | 13 |
| 2 | 20 | 10 | 189 | $1.48 \times 10^{-1}$ | 14 |
| 2 | 40 | 10 | 187 | $1.48 \times 10^{-1}$ | 19 |

splits (Type II meshes in the nomenclature of [28] as shown in Figure 2.2), for various mesh sizes. A Malkus split mesh starts with a quadrilateral subdivision and creates a triangulation by subdividing each quadrilateral by adding the two diagonals connecting opposite vertices. On Malkus splits, for $k \geq 2$, the inf-sup constant is bounded and converges relatively rapidly in our tests, but for $k=1$, the inf-sup constant goes to zero with a rate equal to one over the mesh size parameter [39]. The column "iterations" lists the number of iterations of the power method required to achieve a change in $\lambda$ less than $10^{-8}$. The column "restarts" lists the number of times the projection $z_{\ell}$ was subtracted from $u^{n}$.

In Table 2.3, we give results for Lagrange elements of degree $k=2,3,4$ in two dimensions on Type I meshes, that is, meshes consisting of 45-degree right triangles (Figure 2.2). The results match the theory indicated in Table 2.1: the inf-sup constant degenerates for $k=2,3$, but does not for $k=4$. There is renewed interest in the low order cases due to the emergence of the grad-div penalized Taylor-Hood method $[10,31,32]$. The approximations from these methods tend to divergence free functions as the penalty is increased, and the low-order Taylor-Hood methods are

Table 2.3
Computation of inf-sup constants on Type I meshes (see Figure 2.2) in two dimensions. The shift parameter in (2.11) was set to $\sigma=0.6$. The mesh size $N$ refers to the $N \times N$ mesh of quadrilaterals before subdivision. The parameter $\omega$ determines the initial eigenvector approximation defined in (2.15). Iterations were continued until the change in eigenvalue was less than $10^{-7}$ for $k=3$ and $k=4$, but we had to continue iterations until the change in eigenvalue was less than $10^{-10}$ for $k=2$ to obtain a reliable second digit.

| degree $k$ | $N$ | $\omega$ | iterations | inf-sup $\lambda_{1}$ | restarts |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 8 | 10 | 13577 | $1.60 \times 10^{-3}$ | 29 |
| 2 | 16 | 10 | 50690 | $4.08 \times 10^{-4}$ | 129 |
| 2 | 32 | 10 | 19851 | $1.07 \times 10^{-4}$ | 19397 |
| 2 | 64 | 10 | 54650 | $2.75 \times 10^{-5}$ | 54649 |
| 3 | 3 | 10 | 790 | $8.46 \times 10^{-3}$ | 11 |
| 3 | 5 | 10 | 768 | $3.52 \times 10^{-3}$ | 8 |
| 3 | 10 | 10 | 1858 | $9.50 \times 10^{-4}$ | 15 |
| 4 | 5 | 10 | 585 | $2.59 \times 10^{-2}$ | 12 |
| 4 | 10 | 10 | 789 | $2.60 \times 10^{-2}$ | 17 |
| 4 | 20 | 10 | 1265 | $2.60 \times 10^{-2}$ | 28 |

widely used. It has been known that divergence-free quadratics on Type I meshes have reduced approximation order (this is equivalent to reduced approximation for $C^{1}$ piecewise cubic approximation for scalar functions [12]). But so far we know of no estimates for controlled degeneration of the inf-sup constants on general meshes.
2.7. 3D tests. Mesh restrictions in three dimensions for Scott-Vogelius elements are not fully understood. Our computations for Freudenthal meshes are summarized in Table 2.4. They reveal a surprising fact: on this mesh family, the inf-sup constant is bounded for degrees $k \geq 4$, whereas Zhang [54] was able to prove inf-sup stability only for $k \geq 6$. Neilan [36, Proposition 6.5] extends this to more general meshes that satisfy a special condition, which he states as a conjecture, but still for $k \geq 6$. The results in Table 2.4 lead us to make Conjecture 1.
2.8. Size of $\nabla \cdot V_{h}^{k}$. In general, $\nabla \cdot V_{h}^{k} \subset D G_{h}^{k-1}$, the latter space being all discontinuous piecewise polynomials of degree $k-1$. In two dimensions, these spaces can be very close in size, differing because of singular vertices and the mean zero constraint due to the homogeneous boundary conditions on $V_{h}^{k}$. But in three dimensions, much less is known. The difference between $\nabla \cdot V_{h}^{k}$ and $D G_{h}^{k-1}$ is known to be quite large for Freudenthal meshes. The constraints on the latter space required to be satisfied to be in the former space are listed in [54, page 691]. For a single cube subdivided by 6 tetrahedra, the dimension of the quotient space is 67 for $k=6$. We should note that there is a typographical error in equation (3.67) in [54], which should read instead [56]

$$
\operatorname{dim} P_{h}=n^{3}(k+2)(k+1) k-3 k n\left(n^{2}+n+2\right)+5 .
$$

Note that $\operatorname{dim} D G_{h}^{k-1}=n^{3}(k+2)(k+1) k$, so the number of constraints (the dimension of the quotient space) is $3 k n\left(n^{2}+n+2\right)-5$. The algorithm proposed by Rognes [41] can be used to compute the number of constraints on general meshes.
2.9. More general elements. The algorithm described and tested here can be used more generally for computing inf-sup constants. The only restriction is that

TABLE 2.4
Computation of inf-sup constants for various polynomial degrees $k$ in three dimensions on the Freudenthal meshes depicted in Figure 2.1. The shift parameter in (2.11) was set to $\sigma=0.6$. The mesh size $N$ refers to the $N \times N \times N$ mesh of cubes before subdivision. Iterations were continued until the change in eigenvalue was less than $10^{-7}$ for $k=4$ and $k=5$, but we had to continue iterations until the change in eigenvalue was less than $10^{-8}$ for $k=3$.

| degree $k$ | $N$ | iterations | inf-sup $\lambda_{1}$ | restarts |
| :---: | :---: | :---: | :---: | :---: |
| 3 | 2 | 1026 | $5.75 \times 10^{-4}$ | 4 |
| 3 | 3 | 3400 | $4.26 \times 10^{-4}$ | 14 |
| 3 | 4 | 4445 | $2.93 \times 10^{-4}$ | 29 |
| 3 | 5 | 6044 | $2.11 \times 10^{-4}$ | 2048 |
| 3 | 6 | 7732 | $1.59 \times 10^{-4}$ | 4863 |
| 4 | 2 | 802 | $4.28 \times 10^{-3}$ | 7 |
| 4 | 3 | 2067 | $4.13 \times 10^{-3}$ | 16 |
| 4 | 4 | 1961 | $4.22 \times 10^{-3}$ | 16 |
| 4 | 5 | 1325 | $4.27 \times 10^{-3}$ | 12 |
| 4 | 6 | 1380 | $4.26 \times 10^{-3}$ | 12 |
| 5 | 2 | 1004 | $6.46 \times 10^{-3}$ | 12 |
| 5 | 3 | 1003 | $6.59 \times 10^{-3}$ | 13 |
| 5 | 4 | 877 | $6.34 \times 10^{-3}$ | 12 |
| 5 | 5 | 938 | $6.26 \times 10^{-3}$ | 13 |
| 5 | 6 | 978 | $6.23 \times 10^{-3}$ | 13 |

$\Pi_{h}=\nabla \cdot V_{h}$.
3. Space decompositions for the kernel of the divergence. Our second conjecture relates to multigrid solvers for (1.2). In the nearly incompressible regime, the equation becomes nearly singular, and standard multigrid methods break down. A key breakthrough for such problems was made by Schöberl [43], who devised conditions on the relaxation and prolongation operators that guarantee that a multigrid method is parameter-robust in the nearly-singular regime.
3.1. Background. The key condition for the relaxation is best stated in terms of space decompositions and subspace corrections [51]. The multigrid relaxation method we employ will be induced by a space decomposition

$$
\begin{equation*}
V_{h}=\sum_{i=1}^{J} V_{i} \tag{3.1}
\end{equation*}
$$

where an equation for an approximation to the error is solved over each subspace $V_{i}$ and the updates combined additively or multiplicatively. Each $V_{i}$ is assumed small enough so that direct solvers can be afforded. For example, if $V_{h}=\operatorname{span}\left\{\phi_{1}, \phi_{2}, \ldots, \phi_{N}\right\}$, and each $V_{i}$ is chosen to be $V_{i}=\operatorname{span}\left\{\phi_{i}\right\}$, then combining the updates additively would yield the Jacobi relaxation, while combining multiplicatively would yield the GaussSeidel relaxation. In a domain decomposition approach, each $V_{i}$ could be taken as the functions supported on one subdomain of a given parallel decomposition of the mesh, combined with a suitable global coarse space. Another important example is to define each $V_{i}$ as the functions with support on the patch of cells surrounding each vertex,
the so-called vertex-star space decomposition: this arises in $k$-robust preconditioners for symmetric and coercive problems [38, 44], as the Arnold-Falk-Winther (AFW) relaxation for $H$ (div) and $H$ (curl) [3], and in Reynolds-robust preconditioners for the Navier-Stokes equations [6, 16].

Let us consider an abstract nearly singular problem, following Lee et al. [29]. For $\varepsilon>0$, consider the finite-dimensional linear variational problem: find $u \in V_{h}$ such that

$$
\begin{equation*}
a_{0}(u, v)+\varepsilon^{-1} a_{1}(u, v)=(f, v) \text { for all } v \in V_{h} \tag{3.2}
\end{equation*}
$$

where $a_{0}$ is symmetric and coercive, and $a_{1}$ is symmetric but only positive semi-definite. In the context of $(1.2), a_{1}(u, v)=(\nabla \cdot u, \nabla \cdot v)_{L^{2}}$. Define the kernel

$$
\begin{equation*}
\mathcal{N}=\left\{u \in V_{h}: a_{1}(u, v)=0 \text { for all } v \in V_{h}\right\} . \tag{3.3}
\end{equation*}
$$

In our context, these are divergence-free functions in the finite element space. A key condition that must be satisfied by the space decomposition for parameter-robustness in $\varepsilon$ is [29, Assumption (A1)]:

$$
\begin{equation*}
\mathcal{N}=\sum_{i=1}^{J}\left(\mathcal{N} \cap V_{i}\right) \tag{3.4}
\end{equation*}
$$

In other words, when challenged with a divergence-free function, it must be possible to decompose this as the sum of functions with the $i^{\text {th }}$ summand drawn from the divergence-free functions in $V_{i}{ }^{1}$. The space decomposition that induces Jacobi or Gauss-Seidel relaxation does not generally satisfy (3.4), because the intersection $\mathcal{N} \cap V_{i}=\{0\}$ for typical finite element methods where the standard bases used are not divergence-free.
3.2. Space decompositions from de Rham complexes. One way to devise space decompositions that satisfy (3.4) is by inspecting discrete subcomplexes of a suitable underlying Hilbert complex. For concreteness, consider $\Omega \subset \mathbb{R}^{2}$, with $\Omega$ simply connected. The Stokes complex [27] is given by

$$
\begin{equation*}
\mathbb{R} \xrightarrow{\text { id }} H^{2}(\Omega) \xrightarrow{\text { curl }}\left[H^{1}(\Omega)\right]^{2} \xrightarrow{\text { div }} L^{2}(\Omega) \xrightarrow{\text { null }} 0 . \tag{3.5}
\end{equation*}
$$

This complex is discretized with a discrete subcomplex

$$
\begin{equation*}
\mathbb{R} \xrightarrow{\text { id }} \Sigma_{h} \xrightarrow{\text { curl }} V_{h} \xrightarrow{\text { div }} Q_{h} \xrightarrow{\text { null }} 0, \tag{3.6}
\end{equation*}
$$

where $\Sigma_{h} \subset H^{2}(\Omega), V_{h} \subset\left[H^{1}(\Omega)\right]^{2}$, and $Q_{h} \subset L^{2}(\Omega)$. These complexes have the property that the kernel of an operator curl, div, or null is a subspace of the range of the preceding operator, e.g. that $\operatorname{ker}(\operatorname{div}) \subset$ range (curl) [2]. The complex is called exact if the kernel of an operator is precisely the range of the preceding operator. The Stokes complex is exact if the domain $\Omega$ is simply connected, and this property is inherited by (3.6) if constructed appropriately (with bounded cochain projections).

In our context, these complexes are useful because they offer a crisp characterization of $\mathcal{N}=\operatorname{ker}($ div $)$. Let $\Sigma_{h}=\operatorname{span}\left\{\psi_{1}, \ldots, \psi_{M}\right\}$ and $V_{h}=\operatorname{span}\left\{\phi_{1}, \ldots, \phi_{N}\right\}$. If $u_{h} \in \mathcal{N}$, then $u_{h}=\operatorname{curl} \Psi_{h}$ for $\Psi_{h} \in \Sigma_{h}$ by exactness of (3.6). Expanding $\Psi_{h}$ in terms of its basis functions

$$
\begin{equation*}
\Psi_{h}=\sum_{i=1}^{M} c_{i} \psi_{i} \tag{3.7}
\end{equation*}
$$

[^1]yields an expression for $u_{h}$
\[

$$
\begin{equation*}
u_{h}=\sum_{i=1}^{M} c_{i} \operatorname{curl} \psi_{i} \tag{3.8}
\end{equation*}
$$

\]

Suppose the space decomposition (3.1) is chosen as $J=N+M$ with

$$
V_{i}= \begin{cases}\operatorname{span}\left(\operatorname{curl} \psi_{i}\right) & i=1, \ldots, M  \tag{3.9}\\ \operatorname{span}\left(\phi_{i-M}\right) & i=M+1, \ldots, M+N .\end{cases}
$$

This decomposition would satisfy (3.4), with $V_{i} \cap \mathcal{N}=V_{i}$ for $i \leq M$ or $V_{i} \cap \mathcal{N}=\{0\}$ for $i>M^{2}$.

Solving the problem (3.2) over span $\left\{\operatorname{curl} \psi_{i}\right\}$ reduces to solving a problem in a locally-supported subspace of $\Sigma_{h}$, as described in Hiptmair [25] and Hiptmair-Xu [26] for the $L^{2}$ de Rham complex. However, for the Stokes complex in three dimensions, explicit constructions of $\Sigma_{h}$ are poorly understood. The alternative approach (which we refer to as Pavarino-Arnold-Falk-Winther, PAFW) is to construct the space decomposition using only knowledge of the supports of the basis functions $\psi_{i}$. For example, if $\Sigma_{h}$ exists with the property that each basis function $\psi_{i}$ is a polynomial of degree $k+1$ supported on a certain region of the domain $\operatorname{supp}\left(\psi_{i}\right)$, then we may choose $\left\{V_{j}\right\}$ so that each $V_{j}$ captures all polynomials of degree $k$ on one $\operatorname{supp}\left(\psi_{i}\right)$. In this way, $\forall i=1, \ldots, M \exists j$ s.t. curl $\psi_{i} \in V_{j}$, and (3.4) will also be satisfied, without needing to explicitly construct $\Sigma_{h}$ or $\left\{\psi_{i}\right\}_{i=1}^{M}$. Only knowledge (or conjecture) of the supports is required.

To illustrate the PAFW construction, consider the case of the Scott-Vogelius element pair with $k \geq 4$ and $d=2$. This element constructs $V_{h}$ with $\left[\mathrm{CG}_{k}\right]^{d}$ and $Q_{h}$ with $\mathrm{DG}_{k-1}$ on a mesh $\mathcal{T}_{h}$. The corresponding $C^{1}$-conforming finite element for $\Sigma_{h}$ is the Morgan-Scott element of degree $k+1$ [34], which is not implemented in general purpose finite element software, preventing the application of (3.9). The Morgan-Scott element employs (among others) degrees of freedom at the vertices of the mesh, and the basis function associated with such a degree of freedom will have support over the patch of cells sharing that vertex. This suggests the following vertex-star space decomposition, which we now describe. Let $\nu_{1}, \ldots, \nu_{J}$ be the vertices of $\mathcal{T}_{h}$. The subspaces are given by

$$
\begin{equation*}
V_{j}=\left\{v_{h} \in V_{h}: \operatorname{supp}\left(v_{h}\right) \subset \operatorname{star}\left(\nu_{j}\right)\right\} \tag{3.10}
\end{equation*}
$$

where the star operation of a simplex $p$ returns the union of all simplices containing $p$ as a subsimplex $[35, \S 2]$. In other words, when applied to a vertex $\nu_{j}$, the star is the union of the cells and edges (and faces in $d=3$ ) containing $\nu_{j}$, as well as $\nu_{j}$ itself. In [34], it is proven that the Morgan-Scott basis functions are each supported in star $\left(\nu_{j}\right)$ for some $j$, provided $k \geq 4$. Thus (3.10) also satisfies the kernel decomposition (3.4). Numerical experiments indicate that (3.10) does indeed yield $\varepsilon$-robust convergence for $k \geq 4$, as expected from the Morgan-Scott theory, while it does not for $k<4$ [14, §4.2].

There are other examples where the vertex-star space decomposition does not yield $\varepsilon$-robust convergence. For $k=1$ on Malkus splits, the basis with smallest possible support has support that is larger than $\operatorname{star}\left(\nu_{j}\right)$, although it is in

[^2]star(closure $\left.\left(\operatorname{star}\left(\nu_{j}\right)\right)\right)$ [47], where the star of a set of simplices is the union of their stars, and closure is defined in [14].
3.3. Second conjecture. The preceding discussion indicates that one may experimentally investigate whether $\Sigma_{h}$ permits a basis with support captured by the stars of the vertices, by applying the multigrid algorithm and observing whether the convergence is $\varepsilon$-robust or not.

The continuous Stokes complexes in 3D [15] is given by

$$
\begin{equation*}
\mathbb{R} \xrightarrow{\text { id }} H^{2}(\Omega) \xrightarrow{\text { grad }} H^{1}(\operatorname{curl}, \Omega) \xrightarrow{\text { curl }}\left[H^{1}(\Omega)\right]^{3} \xrightarrow{\text { div }} L^{2}(\Omega) \xrightarrow{\text { null }} 0 . \tag{3.11}
\end{equation*}
$$

Here

$$
\begin{equation*}
H^{1}(\operatorname{curl}, \Omega):=\left\{v \in\left[H^{1}(\Omega)\right]^{3}: \operatorname{curl} v \in\left[H^{1}(\Omega)\right]^{3}\right\} \tag{3.12}
\end{equation*}
$$

This complex is discretized with a discrete subcomplex

$$
\begin{equation*}
\mathbb{R} \xrightarrow{\text { id }} S_{h} \xrightarrow{\text { grad }} \Sigma_{h} \xrightarrow{\text { curl }} V_{h} \xrightarrow{\text { div }} \Pi_{h} \xrightarrow{\text { null }} 0 . \tag{3.13}
\end{equation*}
$$

The space $S_{h}$ consists of $C^{1}$ piecewise polynomials of degree $k+2$. The space $\Sigma_{h}$ consists of continuous vector-valued piecewise polynomials of degree $k+1$ which have a continuous curl. Note that $\left(S_{h}\right)^{3} \subset \Sigma_{h}$.

The potential space $\Sigma_{h}$ is known on special meshes (Alfeld or Worsey-Farin splits) in three dimensions $[17,21,8]$. We are not aware of any results that characterize $\Sigma_{h}$ on Freudenthal meshes.

Based on the numerical experiments we will report, we make the following conjecture:

Conjecture 2. Let $V_{h} \subset\left[H^{1}(\Omega)\right]^{3}$ be constructed with continuous Lagrange elements of degree $k$, and let $\Sigma_{h} \subset H^{1}$ (curl, $\Omega$ ) be the space preceding $V_{h}$ in a subcomplex of the three-dimensional Stokes complex.
(a) For $k=4$, there does not exist a local basis for $\Sigma_{h}$, supported on the stars of vertices, on the Freudenthal meshes depicted in Figure 2.1.
(b) For $k \geq 5$, there exists a local basis for $\Sigma_{h}$, supported on the stars of vertices, on the Freudenthal meshes depicted in Figure 2.1.
Our evidence for Conjecture 2(b) is not as strong as for Conjecture 2(a). For example, the results of Schöberl [43] are not necessary and sufficient, so robustness could occur for some other reason. But lack of robustness implies lack of a suitable local basis.
3.4. Numerical experiments. We provide numerical evidence for Conjecture 2. We solve the problem: find $u \in V_{h} \subset\left[H_{0}^{1}(\Omega)\right]^{3}$ such that:

$$
\begin{equation*}
(\nabla u, \nabla v)_{L^{2}(\Omega)}+\gamma(\nabla \cdot u, \nabla \cdot v)_{L^{2}(\Omega)}=(1, v)_{L^{2}(\Omega)} \quad \forall v \in V_{h} \tag{3.14}
\end{equation*}
$$

The problem becomes nearly singular as $\gamma \rightarrow \infty$. Here $V_{h}$ is constructed with continuous Lagrange elements of degree $k$. We employ the solver denoted in Figure 3.1. The relaxation on each level is one application of damped Richardson iteration preconditioned by the additive Schwarz method with vertex-star subspaces chosen as (3.10). The damping factor is set to $1 / M$ where $M$ is the maximum number of patches that a given vertex is contained in. Here $M=3$ for $d=2$ and $M=4$ for $d=3$. The coarse grid solve was computed with CHOLMOD [11] via PETSc [5]. The code was run in serial.


Fig. 3.1. Solver diagram for (3.14).
Table 3.1
Iteration counts for (3.14) on a uniform refinement of a Type $I \times 4$ mesh. The iteration counts are $\gamma$-robust for $k \geq 4$, but not for $k<4$.

| $k \backslash \gamma$ | $10^{0}$ | $10^{1}$ | $10^{2}$ | $10^{3}$ | $10^{4}$ | $10^{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 11 | 17 | 35 | 54 | 87 | 100 |
| 3 | 10 | 13 | 23 | 49 | 62 | 83 |
| 4 | 9 | 10 | 13 | 13 | 13 | 12 |
| 5 | 9 | 10 | 11 | 12 | 11 | 10 |

3.5. Two dimensions. The convergence results for the solver depicted in Figure 3.1 are shown in Table 3.1, as a function of $k$ and $\gamma$. The Krylov method was terminated when the unpreconditioned residual was reduced by 8 orders of magnitude. The coarse mesh was a Type I mesh [28] of size $4 \times 4$. The main observation is that $k \geq 4$ is required for $\gamma$-robust performance. This matches well with the existing theory described above: the key property that changes between these two cases is the existence of the Morgan-Scott element for $\Sigma_{h}$ with a basis supported on vertex-stars for $k \geq 4$.
3.6. Three dimensions. In three dimensions we again choose $V_{h}=\left[\mathrm{CG}_{k}\right]^{d}$. We employ an analog of a Type I coarse mesh of size $4 \times 4 \times 4$, with one refinement. We show iteration counts for the solution of the problem of (3.14) in Table 3.2. As in two dimensions, the solver was terminated when the unpreconditioned residual was reduced by 8 orders of magnitude. These results lead us to Conjecture 2.

Table 3.2
Iteration counts for (3.14) on 2 uniform refinements of a Type $I 4 \times 4 \times 4$ mesh. The iteration counts are $\gamma$-robust for $k \geq 5$, but not for $k<5$.

| $k \backslash \gamma$ | $10^{0}$ | $10^{1}$ | $10^{2}$ | $10^{3}$ | $10^{4}$ | $10^{5}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 14 | 23 | 52 | 118 | 300 | 849 |
| 3 | 11 | 16 | 29 | 66 | 173 | 458 |
| 4 | 11 | 13 | 19 | 26 | 54 | 110 |
| 5 | 11 | 12 | 16 | 19 | 20 | 19 |
| 6 | 10 | 11 | 14 | 15 | 16 | 15 |
| 7 | 10 | 11 | 13 | 14 | 14 | 13 |

Note that our Conjecture 1 is that the global inf-sup condition holds on this mesh family for $k \geq 4$. Thus there is a gap for $k=4$ : we conjecture that the global inf-sup condition holds, but that there is not a basis for the potential space supported on vertex-stars.
4. Conclusions. Our computational experiments have suggested two conjectures regarding the three-dimensional Stokes complex on Freudenthal meshes. We conjecture that the inf-sup condition holds for a velocity space $V_{h}$ consisting of continuous piecewise polynomials of degree $k$ and pressure space $\nabla \cdot V_{h}$ for $k \geq 4$, and that there is a local basis for the associated potential space $\Sigma_{h}$ for $k \geq 5$.
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## Appendix A. Freudenthal triangulation of the unit cube.

For concreteness we describe the Freudenthal triangulation of the unit cube $[0,1]^{3}$. Define vertices with coordinates

$$
\begin{array}{llll}
\mathbf{v}_{1}=(0,0,0), & \mathbf{v}_{2}=(1,0,0), & \mathbf{v}_{3}=(1,1,0), & \mathbf{v}_{4}=(1,1,1) \\
\mathbf{v}_{5}=(0,1,0), & \mathbf{v}_{6}=(1,0,1), & \mathbf{v}_{7}=(0,1,1), & \mathbf{v}_{8}=(0,0,1)
\end{array}
$$

Then the six cells (tetrahedra) $\mathbf{c}_{i}$ have vertices

$$
\begin{aligned}
& \mathbf{c}_{1}=\left[\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}, \mathbf{v}_{4}\right], \quad \mathbf{c}_{2}=\left[\mathbf{v}_{1}, \mathbf{v}_{3}, \mathbf{v}_{4}, \mathbf{v}_{5}\right], \quad \mathbf{c}_{3}=\left[\mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{4}, \mathbf{v}_{6}\right], \\
& \mathbf{c}_{4}=\left[\mathbf{v}_{1}, \mathbf{v}_{4}, \mathbf{v}_{5}, \mathbf{v}_{7}\right], \quad \mathbf{c}_{5}=\left[\mathbf{v}_{1}, \mathbf{v}_{4}, \mathbf{v}_{6}, \mathbf{v}_{8}\right], \quad \mathbf{c}_{6}=\left[\mathbf{v}_{1}, \mathbf{v}_{4}, \mathbf{v}_{7}, \mathbf{v}_{8}\right] .
\end{aligned}
$$

It is well known [54, Figure 2] that a standard multigrid subdivision of a Freudenthal triangulation of the unit cube $[0,1]^{3}$ yields eight subcubes, each subdivided by a Freudenthal triangulation. In particular, the subdivision of the Freudenthal triangulation of the unit cube $[0,1]^{3}$ consists of cutting by the three planes $x=1 / 2, y=1 / 2$, and $z=1 / 2$. See [54, section 4] for more information regarding multigrid for these meshes.

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[^1]:    ${ }^{1}$ This decomposition must also be stable, but we shall not elaborate here.

[^2]:    ${ }^{2}$ This space decomposition relates to that proposed by Hiptmair [25]. As written it is not robust in $k$, but (experimentally) can be made so by applying suitable block Jacobi methods in $V_{h}$ and $\Sigma_{h}$.

