Two-dimensional sloshing: domains with interior 'high spots'

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Abstract

Considering the two-dimensional sloshing problem, our main focus is to construct domains with interior high spots; that is, points, where the free surface elevation for the fundamental eigenmode attains its critical values. The socalled semi-inverse procedure is applied for this purpose. The existence of high spots is proved rigorously for some domains. Many of the constructed domains have multiple interior high spots and all of them are bulbous at least on one side.

1 Introduction

The sloshing problem (it describes natural frequencies and the corresponding wave eigenmodes in an inviscid, incompressible, heavy fluid bounded above by a restricted free surface) is of great interest to engineers, physicists and mathematicians. Its twodimensional version concerns transversal free oscillations of fluid in an infinitely long canal of uniform cross-section. A historical review of studies in this area going back to the 18th century can be found in [1], whereas various aspects of the problem are presented in the monographs [2], [3] and [4]; the last one provides an advanced mathematical approach to the problem based on spectral theory of operators in a Hilbert space.

Our aim is to consider time-harmonic sloshing in two dimensions and to show that there exist fluid domains in which 'high spots' (points, where the free surface elevation of a fundamental sloshing mode attains its extrema) are located inside the mean free surface. It should be recalled that the free surface elevation of a fluid in the sloshing motion is proportional to the trace of the velocity potential on this horizontal part of domain's boundary. Therefore, at every moment high spots are determined by the trace's maxima and minima provided a time-harmonic factor is removed. The notion of high spot was introduced in [5] by analogy with a hot spot; a conjecture about the latter (formulated by J. Rauch in 1974; see, e.g., the extensive article [6]) says that any eigenfunction, corresponding to the smallest nonzero eigenvalue of the Neumann Laplacian in a bounded domain $F \subset \mathbb{R}^m$, attains its maximum and minimum values on ∂F . The latter has been proven for some domains, most of which are special domains in the plane; there is also a counterexample to the conjecture involving $F \subset \mathbb{R}^2$ with two holes. For highlights of these results, see the recent article [7].

It is well-known that the problem, that describes sloshing in a three-dimensional container having a constant depth and vertical walls, is equivalent to the eigenvalue problem for the Neumann Laplacian in the domain $F \subset \mathbb{R}^2$, which is the container's free surface (indeed, separation of variables immediately yields this). Hence, the absence of interior high spots for such a container follows from the hot spot result for F.

For other container's geometries, the situation with high spots is not as simple and not directly connected to hot spots of the free surface. Indeed, it was proved rigorously [8] and established experimentally [9, Fig. 7], that locations of high spots may vary essentially for convex axisymmetric containers when their form changes slightly. High spots are located inside the free surface of bulbous containers (see [9, Fig. 2]) and on the boundary of the free surface when the container's cross-section decreases with the depth or is constant (see [9, Fig. 1]); see also [10] for further details. The latter (boundary location of high spots) is also proved rigorously [11] for sloshing in troughs of uniform cross-section, whose bottom—the graph of a C^2 -function—forms nonzero angles with the undisturbed free surface. Earlier, the same statement was proved for sloshing in two dimensions [5].

Recently, the first study [12] was published which investigates the effects of surface tension on the location of high spots in two- and three-dimensional icefishing problems. Computational methods are used to demonstrate that the high spot is in the interior of the free surface for large Bond numbers, but for sufficiently small Bond number the high spot is on the boundary of the free surface.

Along with the classical linear sloshing problem considered in the present paper, there are various approaches to nonlinear sloshing phenomena; see, for example, the monograph [13] and references cited in its Chapter 8, in which some typical nonlinear phenomena discovered experimentally long ago are listed; in particular, dependence of the sloshing frequency on wave amplitude, mobility of nodal curves on the free surface, *etc.* In this chapter, special attention is paid to sloshing in cylindrical containers, which allowed the author to obtain some qualitative results.

Another phenomenon similar to nonlinear sloshing is that of standing waves. Their existence on the surface of an infinitely deep perfect fluid under gravity is established in the extensive article [14], where the two-dimensional waves periodic both in space and time are studied in the framework of fully nonlinear model. Also, several approximate approaches developed earlier by a number of authors are reviewed. The plan of the present paper is as follows. Statement of the sloshing problem is formulated in Sect. 2, where general results about it (including the properties of nodal domains) are also described. Rigorously proved results are presented in Sect. 3; the so-called semi-inverse procedure is applied for this purpose. An example of domain with a single interior high spot is investigated in detail in Sect. 3.1, whereas similar results are outlined for another domain in Sect. 3.2. Further examples are obtained numerically and considered in Sect. 4. In Sect. 5, some characteristic features of domains with interior high spots are summarized.

2 Statement of the problem and general results

Let an inviscid, incompressible, heavy fluid occupy an infinitely long canal of uniform cross-section bounded above by a free surface of finite width. The surface tension is neglected and the fluid motion is assumed to be irrotational and of small-amplitude. The latter assumption allows us to linearize boundary conditions on the free surface; see [15, Sect. I.1] for details briefly outlined below. In the case of the two-dimensional motion in planes normal to the generators of canal's bottom, the following relations arise:

$$g\eta(x,t) + \phi_t(x,0,t) = 0, \quad \eta_t(x,t) - \phi_y(x,0,t) = 0.$$
 (1)

In these relations, $\eta(x,t)$ and $\phi(x,y,t)$ are the time-dependent free surface profile and velocity potential, respectively, whereas rectangular Cartesian coordinates (x, y)are taken in the plane of motion so that the x-axis lies in the mean free surface, whereas the y-axis is directed upwards.

The second condition (1) is a kinematic condition; it is a consequence of continuity of the fluid motion and the assumption that the latter is irrotational. The first condition (1) follows from the Hamilton's principle

$$\delta L = 0$$
, where $L = \frac{\rho}{2} \int_0^t \left[\int_W |\nabla \phi|^2 \, \mathrm{d}x \mathrm{d}y - g \int_F \eta^2 \, \mathrm{d}x \right] \mathrm{d}t$

is the standard Lagrangian in which ρ stands for the fluid's density. Here the crosssection W of the canal is a bounded simply connected domain whose piecewise smooth boundary ∂W has no cusps. One of the open arcs forming ∂W is an interval F of the x-axis (the free surface of fluid in equilibrium).

In view of relations (1), if time-harmonic oscillations at the radian frequency ω having the form

$$\eta(x,t) = \zeta(x)\sin\omega(t-a), \quad \phi(x,y,t) = u(x,y)\cos\omega(t-a)$$
(2)

exist for some real a, then the real-valued velocity potential u(x, y) must satisfy the following boundary value problem:

$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \tag{3}$$

$$u_y = \nu u \quad \text{on } F,\tag{4}$$

$$\partial u/\partial n = 0$$
 on B . (5)

Here the bottom $B = \partial W \setminus \overline{F}$ is the union of open arcs lying in the half-plane y < 0and complemented by corner points (if there are any) connecting these arcs, where the normal derivative $\partial/\partial n$ is defined.

The Laplace equation (3) expresses the zero-divergence condition for the velocity; (5) is the no-flow condition on the rigid bottom. Relation (4) arises by excluding η from relations (1), whereas the free surface elevation $\zeta(x)$ is proportional to the trace u(x, 0) of the velocity potential according to (2).

We suppose the orthogonality condition

$$\int_{F} u \, \mathrm{d}x = 0 \tag{6}$$

to hold, thus excluding the eigenvalue $\nu = 0$ of problem (3)–(5), and so the spectral parameter is $\nu = \omega^2/g$, where g is the constant acceleration due to gravity.

It is known since the 1950s that problem (3)-(6) has a discrete spectrum; that is, there exists a sequence of eigenvalues

$$0 < \nu_1 \le \nu_2 \le \dots \le \nu_n \le \dots, \tag{7}$$

each counted according to its finite multiplicity and such that $\nu_n \to \infty$ as $n \to \infty$. The set of corresponding eigenfunctions $\{u_n\}_1^\infty$ belongs to the Sobolev space $H^1(W)$ and forms a complete system in an appropriate Hilbert space. These results can be found in many sources; see, for example, [4] for a comprehensive mathematical treatment.

2.1 Nodal domains and their properties

Let $N(u) = \{(x, y) \in \overline{W} : u(x, y) = 0\}$ be the set of nodal lines of a sloshing eigenfunction u. A connected component of $W \setminus N(u)$ is called a nodal domain for u. Properties of the nodal lined and domains are closely related to our considerations, and so we provide a summary of assertions proved in [16]:

- (i) If R is a nodal domain of u, then $\overline{R} \cap F$ is a subinterval of F.
- (ii) The number of nodal domains corresponding to u_n is less than or equal to n+1.
- (iii) The sloshing eigenfunction u_n cannot change sign more than 2n times on F.

Combining these properties and condition (6), one arrive at the following statement.

Proposition 1. A fundamental sloshing eigenfunction u_1 has a single nodal line which divides W into two nodal domains; this line has one or both ends on F.

2.2 Sloshing in terms of the stream function; auxiliary results

The approach applied in the paper [16] demonstrates that another spectral problem equivalent to (3)–(6) is convenient to deal with; it involves the stream function v

(a harmonic conjugate of u in W defined up to an additive constant):

$$v_{xx} + v_{yy} = 0 \quad \text{in } W, \tag{8}$$

$$-v_{xx} = \nu v_y \quad \text{on } F,\tag{9}$$

$$v = 0 \quad \text{on } B. \tag{10}$$

Notice that condition (10) is obtained from (5) with an appropriate choice of the additive constant; moreover, it implies both conditions (5) and (6). It is obvious that all eigenvalues of problems (8)–(10) and (3)–(6) have the same multiplicity.

Let $N(v) = \{(x, y) \in \overline{W} : v(x, y) = 0\}$ denote the set of nodal lines of a sloshing eigenfunction v. A connected component of $W \setminus N$ is called a nodal domain of v. The following results obtained in [16] are of importance for our considerations.

Proposition 2. Let v be a stream eigenfunction in W corresponding to the eigenvalue ν_1 , then

(i) the single nodal domain of v is W;

(ii) the trace v(x,0) cannot change sign on F and has a single extremum there.

Notice that the first assertion is analogous to the Courant nodal domain theorem for the Dirichlet Laplacian.

3 Fluid domains with interior high spots (rigorous results)

A version of the so-called inverse method is applied here. It is worth mentioning that this method was widely used in continuum mechanics in the pre-computer era; see [17] for a survey. There are two forms of this method that distinguish by the use of boundary conditions. The method is referred to as semi-inverse if some of these conditions, but not all of them, are prescribed at the outset which is convenient for applications in the linear water-wave theory. In particular, Troesch [18] sought a solution of the sloshing problem in the form of a combination of harmonic polynomials satisfying the free-surface boundary condition for a certain frequency. Then the homogeneous Neumann condition was applied for determining the shape of container's bottom, thus giving a family of paraboloids of revolution as bottom surfaces. A similar procedure is applied below for construction of fluid domains with interior high spots.

3.1 Semi-inverse method for $\nu = 3/2$

A particular pair of conjugate harmonic velocity potential/stream functions is used in our version of semi-inverse method, namely:

$$u(x,y) = \int_0^\infty \frac{\cos k(x-\pi) + \cos k(x+\pi)}{k-\nu} e^{ky} dk,$$
(11)

$$v(x,y) = \int_0^\infty \frac{\sin k(x-\pi) + \sin k(x+\pi)}{\nu - k} e^{ky} dk.$$
 (12)

If $\nu = 3/2$ (this pair and similar ones were introduced in [19, Subsection 4.1.1]), then both numerators vanish at $k = \nu = 3/2$, and so the integrals are the usual converging infinite integrals. Similar functions were originally proposed by M. McIver [20], who used them in her construction of modes trapped by a pair of two-dimensional bodies in the water wave problem.

It is easy to verify that u and v are conjugate harmonic functions in \mathbb{R}^2_{-} , and

$$u(-x,y) = u(x,y) \quad \text{and} \quad v(-x,y) = -v(x,y).$$

Moreover, u and v are infinitely smooth up to $\partial \mathbb{R}^2 \setminus \{x = \pm \pi, y = 0\}$, and wellknown facts from theory of distributions imply that $[u_y - \nu u]_{y=0}$ is equal to a linear combination of Dirac's measures at $x = \pi$ and $x = -\pi$. Therefore,

$$u_y = \nu u \quad \text{on } \partial \mathbb{R}^2_- \setminus \{ x = \pm \pi, \, y = 0 \}.$$
(13)

The calculated nodal lines of u and v are shown in Fig. 1(b); the line plotted in solid has the following properties; see [16, Prop. 2.1].

Proposition 3. If $\nu = 3/2$ in (12), then along with $\{x = 0, y < 0\}$, there is only one nodal line of v(x, y) in \mathbb{R}^2_- . It is smooth, symmetric about the y-axis and its both ends are on the x-axis; the right one, say $(x_0, 0)$, lies between the origin and the singularity point $(\pi, 0)$.

Thus, the right half of the curvilinear nodal line together with a part of the y-axis define the bottom $B_{3/2}$ of a fluid domain; it is denoted $W_{3/2}$ in Fig. 1 (b). Indeed, v given by (12) satisfies condition (10) on this line. On the free surface $F_{3/2}$ of this domain, condition (4) holds for u in view of (13), and so u and v with $\nu = 3/2$ satisfy the respective versions of the sloshing problem in $W_{3/2}$.

Notice that we use a half of the domain bounded by the symmetric in x curvilinear nodal line (let us denote this domain by $\mathring{W}_{3/2}$). The point is that although the necessary conditions of Proposition 1 are satisfied for $\mathring{W}_{3/2}$, the sloshing mode uwith $\nu = 3/2$, nevertheless, is not fundamental in this domain. Indeed, the stream function corresponding to ν_1 cannot have two extrema on the free surface in view of Proposition 2. At the same time, u and v restricted to $W_{3/2}$ satisfy necessary conditions of Propositions 1, 2, and so they are good candidates for being the fundamental eigenfunctions corresponding to $\nu = 3/2$.

Indeed, the domain $W_{3/2}$ is nodal for v; see the graph of its trace in Fig. 1 (a). Furthermore, the following assertion was proved in [16, Th. 2.6].

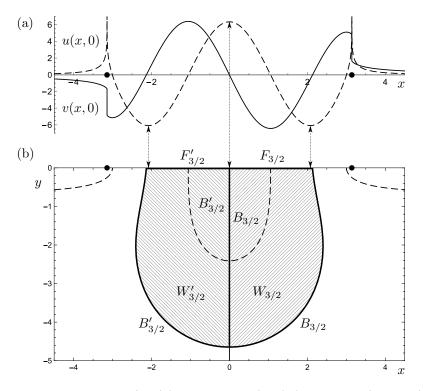


Figure 1: Plotted for $\nu = 3/2$: (a) the traces u(x, 0) (dashed line) and v(x, 0) (solid line); (b) the nodal lines of u (dashed lines) and v (solid lines) given by (11) and (12), respectively. High spots on $\overline{F_{3/2}}$ and $\overline{F'_{3/2}}$ are marked by the arrows connecting them with the extrema of the velocity potential trace.

Theorem 1. In the fluid domain $W_{3/2}$, the sloshing eigenfunction u given by (11) with $\nu = 3/2$ has a single nodal line, whose one endpoint is $(x_n, 0)$ and $x_n \in (0, x_0)$ (here x_0 is the right endpoint of $F_{3/2}$ defined in Proposition 3), is the only minimum point of v(x, 0) on $\{x > 0\}$, whereas the second endpoint is on the y-axis.

Remark 1. The representation

$$u(x,0) = 2\pi \cos \nu x + \int_0^\infty \left[e^{-(\pi - x)k\nu} + e^{-(\pi + x)k\nu} \right] \frac{k \, \mathrm{d}k}{1 + k^2} \tag{14}$$

valid for $x \in [0, \pi)$ (see [16, Form. (2.9)]) implies that $u_x(0, 0) = 0$ and $u_{xx}(0, 0) < 0$. Hence u(x, 0) attains maximum at x = 0, and so the origin is a high spot, but located on the boundary of the free surface $F_{3/2}$.

The following assertion implies that there is an interior high spot on $F_{3/2}$; it corresponds to the right minimum of u(x, 0), which is close to the right endpoint of the free surface $F_{3/2}$; see Remark 2 below.

Theorem 2. The sloshing eigenfunction u given by (11) with $\nu = 3/2$ has an interior high spot on $F_{3/2}$.

Proof. The representation (see [16, Form. (2.7)])

$$v(x,y) = e^{\nu y} \left[v(x,0) + 2x \int_{y}^{0} \frac{k^2 - (\pi^2 - x^2)}{[k^2 + (\pi - x)^2] [k^2 + (\pi + x)^2]} e^{-k\nu} dk \right]$$
(15)

implies that

$$v_y(x_0,0) = u_x(x_0,0) = \frac{2 x_0 (\pi^2 - x_0^2)}{(\pi - x_0)^2 (\pi + x_0)^2} = \frac{2 x_0}{\pi^2 - x_0^2} > 0.$$

Then $u_x(x_h, 0) = 0$ at some $x_h < x_0$ in view of Remark 1. Hence, an interior high spot is located at $(x_h, 0)$ on the left of the endpoint $(x_0, 0)$. By symmetry, $(-x_h, 0)$ is also an interior high spot located on the right of the endpoint $(-x_0, 0)$.

Remark 2. According to computations, $x_h \approx 2.077836$, whereas the endpoint of the free surface $F_{3/2}$ is at $(x_0, 0)$ with $x_0 \approx 2.132704$, that is, the distance from the high spot to the endpoint is approximately 0.054868, which is less than 3% of the distance from the origin to the endpoint of $F_{3/2}$.

Remark 3. The same considerations are valid for the domain $W'_{3/2}$ on the left of the *y*-axis; see Fig. 1 (b). Therefore, $W'_{3/2}$ also provides an example of domain with an interior high spot.

Now we turn to a geometric property intrinsic to domains with interior high spots. According to the following definition, it is evidently holds for $W_{3/2}$; see Fig. 1 (b).

Definition 1. A fluid domain W satisfies John's condition if it is confined to the strip bounded by the straight vertical lines through the endpoints of the free surface F. Domains violating this condition are called bulbous.

Proposition 4. The domain $W_{3/2}$ is bulbous.

Proof. To be specific, let us show that $W_{3/2}$ is bulbous on the right-hand side. Let us consider $B_{3/2}$ as the graph of the implicit function $x \mapsto y$ defined by the equation v(x, y) = 0 in a neighbourhood of $(x_0, 0)$ —the right endpoint of $B_{3/2}$. Therefore, to establish that $W_{3/2}$ is bulbous it is sufficient to prove that

$$y'(x_0) = -v_x(x_0, 0)/v_y(x_0, 0) < 0$$

After some algebra, it follows from (15) that

$$y'(x_0) = \frac{x_0^2 - \pi^2}{2x_0} v_x(x_0, 0) = \frac{\pi^2 - x_0^2}{2x_0} u_y(x_0, 0),$$

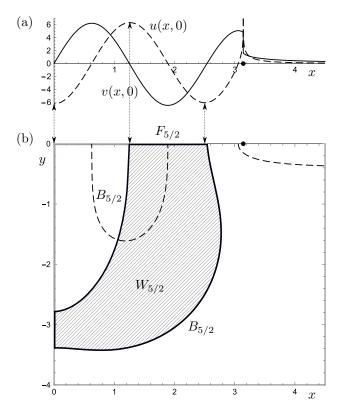


Figure 2: Plotted for $\nu = 5/2$: (a) the traces u(x, 0) (dashed line) and v(x, 0) (solid line); (b) the nodal lines of u (dashed lines) and v (solid lines) given by (11) and (12), respectively. High spots on $F_{5/2}$ are marked by the arrows connecting them with the extrema of the velocity potential trace.

where the last equality is a consequence of the Cauchy–Riemann equations. In view of condition (13) we have for $\nu = 3/2$:

$$y'(x_0) = \frac{3(\pi^2 - x_0^2)}{4x_0} u(x_0, 0).$$
(16)

Since there is only one nodal line of u in $W_{3/2}$, and its right endpoint is $(x_n, 0)$, the trace u(x, 0) is negative for $x \in (x_n, x_0]$. Then, (16) implies that $y'(x_0) < 0$, which completes the proof.

In conclusion of this section, it is worth mentioning an essential property of the harmonic function v defining the domain $W_{3/2}$. This function has a stagnation point in \mathbb{R}^2_{-} , namely, the point, where the curvilinear part of $B_{3/2}$ meets the negative y-axis. Indeed, the derivatives along both these lines vanish because they are nodal lines of v. In what follows, the same property — the presence of a stagnation point — will be used in more complicated cases.

Remark 4. It should be said, that the presence of stagnation points of v and, as a consequence, of corner points on boundaries of all domains constructed in the paper, is rather a matter of convenience for our considerations. For instance, it is easy to observe that any line v = -c in Fig. 2 is located inside $W_{3/2}$ provided $0 < c < -\min\{v(x,0) : x \ge 0\}$, and the domain bounded by this line, say $W_{3/2}^c$, has a smooth bottom. However, $W_{3/2}^c$ can be used as a sloshing domain instead of $W_{3/2}$; moreover, the extrema of u are still located inside the free surface $F_{3/2}^c$ when c is sufficiently small.

3.2 The case $\nu = 5/2$; multiple interior high spots

Rigorous considerations, analogous to those in Sect. 3.1, are applicable to the case when u and v are given by (11) and (12), respectively, but with $\nu = 5/2$. It occurs that the domain similar to $W_{3/2}$ (that is, adjacent to the y-axis), in which this uis an eigenfunction corresponding to $\nu = 5/2$, has no interior high spot. Indeed, it satisfies John's condition which guarantees [5] that high spots are on the boundary of the free surface; see Fig. 2 (b), where this domain left blank. However, this u is also an eigenfunction corresponding to $\nu = 5/2$ in the domain denoted by $W_{5/2}$ in Fig. 2 (b). This follows from Proposition 2 because this domain is nodal for v; indeed, the negative y-axis is a nodal line of v, as well as both lines marked $B_{5/2}$ —the two curved parts forming the bottom of $W_{5/2}$.

There are two interior high spots on the free surface $F_{5/2}$; one of them corresponds to the maximum of u(x,0) attained at $x \approx 1.257429$ between the endpoints of the nodal line of u, which divides $W_{5/2}$ into two nodal domains. It should be noted that the left line $B_{5/2}$ emanates from $x \approx 1.249757$, which is slightly smaller. The second high spot corresponds to the minimum of u(x,0) attained at $x \approx 2.503159$, which is very close to the right endpoint of $F_{5/2}$ at $x \approx 2.539769$.

It is clear that the domain $W_{5/2}$ is bulbous on both sides. It is obvious on the left-hand side, whereas for the right-hand side the reasoning similar to the proof of Proposition 4 is applicable. Of course, the domain obtained by reflection of $W_{5/2}$ in the *y*-axis also provides an example of domain with two interior high spots.

As in the case $\nu = 3/2$, the function v defining the domain $W_{5/2}$ has stagnation points in \mathbb{R}^2_- . However, there are two of them, namely, the points, where both curves $B_{5/2}$ meet the negative y-axis.

4 Further examples of domains with multiple interior high spots (numerical results)

In this section, we present other domains with multiple interior high spots, but they are obtained numerically using the following procedure. For a specified value of ν the bottom is defined by a nonzero level line of v, whose level is chosen so that this

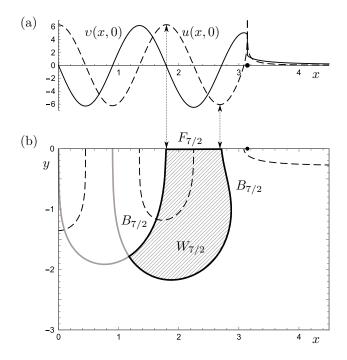


Figure 3: Plotted for $\nu = 7/2$: (a) the traces u(x, 0) (dashed line) and v(x, 0) (solid line); (b) the nodal lines of u (dashed lines), and the level lines $v \approx -0.023145$ (solid lines). Two interior high spots on $F_{7/2}$ are marked by arrows connecting them with the corresponding extrema of the velocity potential's trace.

line has two branches crossing transversally at a stagnation point, thus forming the bottom of the required domain.

This procedure is applicable for integer values of ν as well. However, the following functions

$$u(x,y) = \int_0^\infty \frac{\cos k(x-\pi) - \cos k(x+\pi)}{k-\nu} e^{ky} dk,$$
 (17)

$$v(x,y) = \int_0^\infty \frac{\sin k(x-\pi) - \sin k(x+\pi)}{\nu - k} e^{ky} dk$$
(18)

are used in this case instead of (11) and (12).

4.1 The case $\nu = 7/2$

If v is given by (12) with $\nu = 7/2$, then a stagnation point occurs at the level approximately equal to -0.023145; see Fig. 3 (b), where this point is at the intersection of two solid lines marked $B_{7/2}$. They enclose the domain $W_{7/2}$, in which u given by (11) with $\nu = 7/2$ is an eigenfunction corresponding to $\nu = 7/2$. Indeed, since v is less than -0.023145 in this domain, it is the nodal one for the stream function equal to the difference of v and this value.

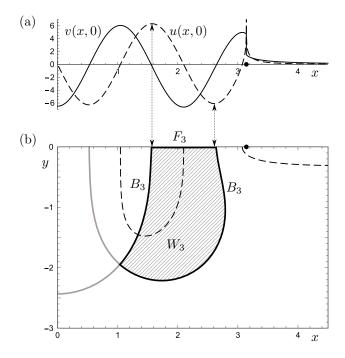


Figure 4: Plotted for $\nu = 3$: (a) the traces u(x, 0) (dashed line) and v(x, 0) (solid line) given by (11) and (12), respectively; (b) the nodal lines of u (dashed lines), and the level lines $v \approx -0.150899$ (solid and dotted lines). Interior high spots on F_3 are marked by arrows connecting them with extrema of the velocity potential trace.

There are two interior high spots on $F_{7/2}$: near its left endpoint at $x \approx 1.795807$, and close to the right endpoint at $x \approx 2.685549$, within approximately 0.026076 from the endpoint. The character of these high spots is the same as those on $F_{5/2}$. Like $W_{5/2}$, the domain $W_{7/2}$ is bulbous on both sides. Of course, the domain obtained by reflection of $W_{7/2}$ in the *y*-axis also provides an example of domain with two interior high spots.

4.2 The case $\nu = 3$

If v is given by (18) with $\nu = 3$, then a stagnation point occurs at the level approximately equal to -0.150899; see Fig. 4 (b), where this point is at the intersection of two solid lines marked B_3 . They enclose the domain W_3 , in which u given by (17) with $\nu = 3$ is an eigenfunction corresponding to $\nu = 3$. Indeed, since v is less than -0.150899 in this domain, it is the nodal one for the stream function equal to the difference of v and this value.

There are two interior high spots on F_3 : near its left endpoint at $x \approx 1.5715649$, and close to the right endpoint at $x \approx 2.6095109$, within approximately 0.029250 from the endpoint. Comparing the fluid domains W_3 and $W_{3/2}$ shown in Figs. 4 and 3, respectively, we see that they have the same structure of defining lines and the

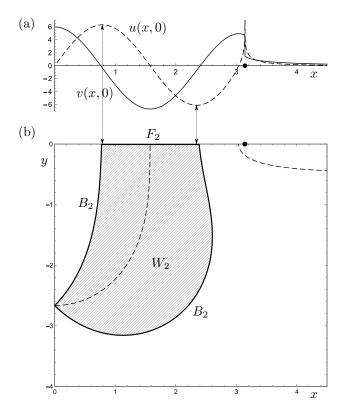


Figure 5: Plotted for $\nu = 2$: (a) the traces u(x, 0) (dashed line) and v(x, 0) (solid line); (b) the nodal lines of u (dashed lines) and the level line $v \approx -0.185125$ (solid line). Interior high spots on F_2 are marked by arrows connecting them with extrema of the velocity potential trace.

same number of high spots. Of course, the domain obtained by reflection of W_3 in the *y*-axis also provides an example of domain with two interior high spots.

4.3 The case $\nu = 2$

Applying the same procedure for this value of ν , one obtains the fluid domain W_2 with two interior high spots on F_2 , which distinguishes essentially from $W_{3/2}$, $W_{5/2}$, $W_{7/2}$ and W_3 ; see Fig. 5 (b). Namely, the nodal line of the eigenfunction u (recall that (17) defines it for $\nu = 2$) connects F_2 with the stagnation point of v on B_2 .

There are two interior high spots on F_2 , both located near its endpoints: on the left at $x \approx 0.786780$ and on the right at $x \approx 2.343392$. The corresponding endpoints of F_2 are at $x \approx 0.774530$ and at $x \approx 2.387143$, respectively. Finally, we notice that the domain W_2 is bulbous on both sides like $W_{5/2}$, $W_{7/2}$ and W_3 . Of course, the domain obtained by reflection of W_2 in the y-axis also provides an example of domain with two interior high spots.

5 Concluding remarks

Two-dimensional sloshing is a common type of fluid oscillations in canals of uniform cross-section and similar troughs with vertical end-walls. Investigating this motion by means of a semi-inverse method, several examples of sloshing domains have been constructed, all of which have the following common property: there is at least one interior high spot, that is, a point on the free surface, where its extremal elevation occurs for a fundamental eigenmode.

In the version of semi-inverse method applied here, the velocity potential satisfies the free-surface boundary condition, involving the spectral parameter proportional to the sloshing frequency squared, whereas a wetted contour is to be determined from the no-flow condition. Using this condition, we restricted ourselves to considering contours that are either adjacent to the y-axis or lying in the fourth quadrant. Along with the latter contours, their symmetric images in the y-axis also provide sloshing domains with interior high spots.

Let us recall some characteristic features of the constructed domains with interior high spots:

• Many of these domains, but not all, have multiple interior high spots.

• All these domains are bulbous on the side, where an interior high spot is located.

• Each found interior high spot is located close to an endpoint of the free surface.

• The bottom profile of every found domain has at least one corner point (domains with smooth bottom profiles can also be constructed; see Remark 4).

• A single nodal line of the velocity potential connects F and B in each example.

It is clear that a sloshing domain $W \subset \mathbb{R}^2_-$ defines a trough $W \times (0, \ell) \subset \mathbb{R}^3_-$ for any $\ell > 0$. Moreover, if u(x, y) is a fundamental eigenmode of sloshing in W, then this function plays the same role for $W \times (0, \ell)$. Therefore, if W has an interior high spot, then there is a straight line (parallel to trough's generators) in the free surface of $W \times (0, \ell)$, each point of which is a high spot interior with respect to the trough's free surface.

In conclusion, we conjecture that the result obtained in [5] for domains satisfying John's condition and having smooth bottom (it bans interior high spots for such domains) is still valid when the bottom is nonsmooth. The domain adjoining $W_{5/2}$ on the left provides a basis for this conjecture as well as similar domains adjoining $W_{7/2}$ and W_3 .

References

- [1] D. W. Fox and J. R. Kuttler, "Sloshing frequencies," ZAMP 34, 668–696 (1983).
- [2] O. M. Faltinsen and A. N. Timokha, *Sloshing*. Cambridge University Press, New York (2009).
- [3] R. A. Ibrahim, *Liquid Sloshing Dynamics*. Cambridge University Press, New York (2005).

- [4] N. D. Kopachevsky and S. G. Krein, Operator Approach to Linear Problems of Hydrodynamics. Birkhäuser, Basel – Boston – Berlin (2001).
- [5] T. Kulczycki and N. Kuznetsov, "'High spots' theorems for sloshing problems," Bull. Lond. Math. Soc. 41, 495–505 (2009).
- [6] R. Bañuelos and K. Burdzy, "On the 'hot spots' conjecture of J. Rauch," J. Funct. Anal. 164, 1–33 (1999).
- [7] C. Judge and S. Mondal, "Euclidean triangles have no hot spots," Ann. Math. 191, 167–211 (2020).
- [8] T. Kulczycki and M. Kwaśnicki, "On high spots of the fundamental sloshing eigenfunctions in axially symmetric domains," Proc. Lond. Math. Soc. (3) 105, 921–952 (2012).
- [9] N. Kuznetsov, T. Kulczycki, M. Kwaśnicki, A. Nazarov, S. Poborchi, I. Polterovich and B. Siudeja, "The legacy of Vladimir Andreevich Steklov," Notices Amer. Math. Soc. 61, 9–22 (2014).
- [10] T. Kulczycki, M. Kwaśnicki and B. Siudeja, "The shape of the fundamental sloshing mode in axisymmetric containers," J. Eng. Math. 99, 157–183 (2016).
- [11] T. Kulczycki and N. Kuznetsov, "On the 'high spots' of fundamental sloshing modes in a trough," Proc. Roy. Soc. London A 467, 1491–1502 (2011).
- [12] N. Willis, C. H. Tan, C. Hohenegger and B. Osting, "High spots for the ice-fishing problem with surface tension," SIAM J. Appl. Math. 82, 1312–1335 (2022).
- [13] I. A. Lukovsky, Nonlinear Dynamics: Mathematical Models for Rigid Bodies with a Liquid. Walter de Gruyter, Berlin-Boston (2015).
- [14] G. Iooss, P. I. Plotnikov, J. F. Toland, "Standing waves on an infinitely deep perfect fluid under gravity," Arch. Ration. Mech. Anal. 177, 367–478 (2005).
- [15] N. N. Moiseev, "Introduction to the theory of oscillations of liquid-containing bodies," Adv. Appl. Mech. 8, 233–289 (1964).
- [16] V. Kozlov, N. Kuznetsov and O. Motygin, "On the two-dimensional sloshing problem," Proc. Roy. Soc. London A 460, 2587–2603 (2004).
- [17] P. F. Neményi, "Recent developments in inverse and semi-inverse methods in the mechanics of continua," Adv. Appl. Mech. 2, 123–148 (1951).
- [18] B. A. Troesch, "Free oscillations of a fluid in a container," Boundary problems in differential equations. University of Wisconsin Press, 279–299 (1960).
- [19] N. Kuznetsov, V. Maz'ya and B. Vainberg, *Linear Water Waves: A Mathematical Approach*. Cambridge University Press, Cambridge (2002).
- [20] M. McIver, "An example of non-uniqueness in the two-dimensional linear water wave problem," J. Fluid Mech. 315, 257–266 (1996).