## MULTIPLICATIVE PROPERTIES OF HILBERT CUBES

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ABSTRACT. We obtain upper bounds on the cardinality of Hilbert cubes in finite fields, which avoid large product sets and reciprocals of sum sets. In particular, our results replace recent estimates of N. Hegyvári and P. P. Pach (2020), which appear to be void for all admissible parameters. Our approach is different from that of N. Hegyvári and P. P. Pach and is based on some well-known bounds of double character and exponential sums over arbitrary sets, due to A. A. Karatsuba (1991) and N. G. Moshchevitin (2007), respectively.

### 1. Introduction

1.1. **Set-up and background.** Let p be a prime. Given d+1 elements  $a_0, a_1, \ldots, a_d \in \mathbb{F}_p$  of a finite field  $\mathbb{F}_p$  of p elements, the corresponding Hilbert cube of dimension d is defined as the following set

(1.1) 
$$\mathsf{H}(a_0; a_1, \dots, a_d) = \left\{ a_0 + \sum_{i=1}^d \varepsilon_i a_i : \ \varepsilon_i \in \{0, 1\} \right\}.$$

Since Hilbert cubes have a very strong additive structure, it is natural to study how their interact with (for example, avoid) multiplicatively defined sets, such as product sets

$$\mathcal{A} \cdot \mathcal{B} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}$$

and reciprocal sum sets

$$(A + B)^{-1} = \{(a + b)^{-1} : a \in A, b \in B, a \neq -b\}$$

for  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{F}_p$ .

In particular, Hegyvári and Pach [6] have considered the question of estimating the cardinality  $\#\mathcal{H}$  of Hilbert cubes  $\mathcal{H}$  (in terms of  $\#\mathcal{A}$  and  $\#\mathcal{B}$ ), such that  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$  or  $\mathcal{H} \cap (\mathcal{A} + \mathcal{B})^{-1} = \emptyset$  for some sets  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{F}_p$ . However, unfortunately the bounds of [6, Theorems 3.1

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and 3.2] never improve the trivial bound  $\#\mathcal{H} \leq p$ . Here we start with a brief analysis of these results.

First we notice that it appears that in the formulation of [6, Theorem 3.1] the exponent 13/16 has to be 5/16, see the last displayed equation at the end of the proof of [6, Theorem 3.1]. Then the bound of [6, Theorem 3.1] takes the following form. If  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$  for some sets  $\mathcal{A} \subseteq \mathbb{F}_p$  and  $\mathcal{B} \subseteq \mathcal{G}$ , where  $\mathcal{G} \subseteq \mathbb{F}_p^*$  is a multiplicative subgroup of order  $\#\mathcal{G} = O(p^{3/4})$ , then

(1.2) 
$$\#\mathcal{H} \leqslant c \frac{p^{9/4} (\#\mathcal{G})^{1/4}}{(\#\mathcal{A})^{1/2 - \eta/4} (\#\mathcal{B})^{5/8}},$$

with some absolute constant c > 0, where the parameter  $\eta > 0$  controls the additive structure of  $\mathcal{A}$  (its additive energy); we refer to the formulation of [6, Theorem 3.1] for more details.

Since  $\eta > 0$ ,  $\#\mathcal{A} \leq p$ ,  $\#\mathcal{B} \leq \#\mathcal{G} \leq p^{3/4}$  (see [6, Theorem 3.1] for more details on these parameters), we conclude that the right hand side of (1.2) satisfies

$$\frac{p^{9/4} (\#\mathcal{G})^{1/4}}{(\#\mathcal{A})^{1/2-\eta/4} (\#\mathcal{B})^{5/8}} \geqslant \frac{p^{9/4} (\#\mathcal{G})^{1/4}}{p^{1/2} (\#\mathcal{G})^{5/8}} 
= \frac{p^{9/4}}{p^{1/2} (\#\mathcal{G})^{3/8}} \geqslant \frac{p^{9/4}}{p^{1/2+9/32}} = p^{47/32},$$

substantially exceeding the trivial bound  $\#\mathcal{H} \leq p$ .

Furthermore, the bound [6, Theorem 3.2] takes form

Since  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{F}_p$ , the right hand side of (1.3) is at least 32p, which again exceeds the trivial bound  $\#\mathcal{H} \leq p$ .

We also notice that a series of very interesting results on product  $\mathcal{HH}$  and ratio  $\mathcal{HH}^{-1}$  sets of Hilbert cubes have recently been given by Shkredov [12]. Moreover, Shkredov [12] has also given dual results on sum sets of multiplicatively defined analogues of Hilbert cubes.

1.2. **Notation and conventions.** Throughout the paper, the notations U = O(V),  $U \ll V$  and  $V \gg U$  are equivalent to  $|U| \leqslant cV$  for some positive constant c, which may depend on the integer positive parameter r.

To avoid complicated expressions, especially in the exponents, we deviate from the canonical rule and write fractions of the form 1/km to mean 1/(km) (rather than  $(1/k) \cdot m$ ).

1.3. **New results.** Here we obtain new results on the size of Hilbert cubes which avoid product sets and reciprocal sum sets. Our method is different from that of Hegyvári and Pach [6] and is based on some bounds of double character and exponential sums.

In particular, we unify both events  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$  and  $\mathcal{H} \cap (\mathcal{A} + \mathcal{B})^{-1} = \emptyset$  in the following result.

**Theorem 1.1.** Let  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{F}_p^*$  and let  $\mathcal{H} \subseteq \mathbb{F}_p$  be a Hilbert cube with

$$\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$$
 or  $\mathcal{H} \cap (\mathcal{A} + \mathcal{B})^{-1} = \emptyset$ .

Then for any positive integer r we have

$$\#\mathcal{H} \ll \frac{p^{2r+1/2+1/2r}}{\left(\#\mathcal{A}\#\mathcal{B}\right)^r}.$$

Taking Theorem 1.1 with r = 2 we see that for

$$\#\mathcal{A}\#\mathcal{B}\gg p^{15/8+\varepsilon/2}$$

with an arbitrary  $\varepsilon > 0$  we obtain a nontrivial upper bound

$$\#\mathcal{H}\ll p^{1-\varepsilon}$$
.

Furthermore, for an arbitrary  $\varepsilon>0$  and  $r=\lceil 4/\varepsilon \rceil$  we derive from Theorem 1.1 that for

$$\#\mathcal{A}\#\mathcal{B}\gg p^{2-1/(4r^2)}$$

we have

$$\#\mathcal{H}\ll p^{1/2+\varepsilon}.$$

1.4. **Applications.** As we have mentioned, Shkredov [12] has studied the product  $\mathcal{HH}$  and ratio  $\mathcal{HH}^{-1}$  sets of Hilbert cubes and in particular has given lower bounds on  $\#(\mathcal{H} \cdot \mathcal{H})$  and  $\#(\mathcal{H} \cdot \mathcal{H}^{-1})$ , provided that  $\#\mathcal{H}$  is not too large. We now observe that Theorem 1.1 implies the following complementing result, which applies for large Hilbert cubes and shows that their product and ratio sets occupy almost all  $\mathbb{F}_p$ .

Corollary 1.2. Let  $\mathcal{H} \subseteq \mathbb{F}_p$  be a Hilbert cube and let

$$\mathcal{E} = \mathbb{F}_p \backslash (\mathcal{H} \cdot \mathcal{H})$$
 and  $\mathcal{F} = \mathbb{F}_p \backslash (\mathcal{H} \cdot \mathcal{H}^{-1})$ .

Then

$$\#\mathcal{E}, \#\mathcal{F} \ll p^{19/8} \left(\#\mathcal{H}\right)^{-3/2}$$

Indeed, to see Corollary 1.2, it is enough to notice that

$$\mathcal{H} \cap (\mathcal{E} \cdot \mathcal{H}^{-1}) = \varnothing \qquad \text{and} \qquad \mathcal{H} \cap (\mathcal{F} \cdot \mathcal{H}) = \varnothing$$

and apply Theorem 1.1 with r = 2 to

$$(\mathcal{A}, \mathcal{B}) = (\mathcal{E}, \mathcal{H}^{-1})$$
 and  $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{H})$ .

deriving

$$\#\mathcal{H} \ll \frac{p^{4+1/2+1/4}}{\left(\#\mathcal{F}\#\mathcal{H}\right)^2}$$

in each case.

Corollary 1.2 is nontrivial if  $\#\mathcal{H} \gg p^{11/12+\varepsilon}$  with some fixed  $\varepsilon > 0$ . In turn, Corollary 1.2 immediately implies the following.

Corollary 1.3. There is an absolute constant C such that if  $\mathcal{H} \subseteq \mathbb{F}_p$  be a Hilbert cube with

$$\#\mathcal{H} \geqslant Cp^{11/12}$$

then  $\mathcal{H}$  is not contained in a co-set  $\lambda \mathcal{G}$ ,  $\lambda \in \mathbb{F}_p^*$ , of any proper multiplicative subgroup  $\mathcal{G} \subseteq \mathbb{F}_p^*$ .

Indeed, if  $\mathcal{H} \subseteq \lambda \mathcal{G}$  then  $\mathcal{H} \cdot \mathcal{H} \subseteq \lambda^2 \mathcal{G}$ . Hence

$$\# (\mathbb{F}_p \setminus (\mathcal{H} \cdot \mathcal{H})) \geqslant \# (\mathbb{F}_p \setminus \lambda^2 \mathcal{G}) \leqslant p - (p-1)/2 = (p+1)/2$$

and by Corollary 1.2 we obtain the bound of Corollary 1.3.

Corollary 1.3 complements a series of results in [1,4,5,7] on Hilbert cubes  $\mathcal{H}$  avoiding the sets of quadratic non-residues and primitive roots of  $\mathbb{F}_p$ , which characterise  $\mathcal{H}$  in terms of its dimension d of  $\mathcal{H}$  rather than of its size  $\#\mathcal{H}$ .

For example, let f(p) and F(p) be the largest dimension of  $\mathcal{H}$  such that  $\mathcal{H}$  does not contain quadratic non-residues and primitive roots of  $\mathbb{F}_p$ , respectively.

Hegyvári and Sárközy [7, Theorem 2] have proved that  $f(p) \leq 12p^{1/4}$ . This has been improved as  $F(p) \leq p^{1/5+o(1)}$  in [4, Theorem 1.3] and then as  $F(p) \leq p^{3/19+o(1)}$  in [5, Theorem 1.3]. Very recently, Alsetri and Shao [1] have given a further improvement and established the bound  $F(p) \leq p^{1/8+o(1)}$ , which is the best possible estimate until the celebrated bound of Burgess [3] on the smaller primitive roots  $g(p) \leq p^{1/4+o(1)}$  is improved.

# 2. Preliminaries

2.1. Bounds of double character and exponential sums. Let  $\mathcal{X}$  be the set of all p-1 multiplicative characters of  $\mathbb{F}_p$  and let  $\mathcal{X}^* = \mathcal{X} \setminus \{\chi_0\}$  be the set of all non-principal characters, see [8, Chapter 3] for a background.

We first recall the following result of Karatsuba [9] (see also [10, Chapter VIII, Problem 9]):

**Lemma 2.1.** Let  $\mathcal{U}, \mathcal{V} \subseteq \mathbb{F}_p$  be of cardinalities U and V, respectively. Then for any positive integer r, for all  $\chi \in \mathcal{X}^*$ , we have

$$\sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \chi \left( u + v \right) \ll U^{1 - 1/2r} \left( V^{1/2} p^{1/2r} + V p^{1/4r} \right),$$

where the implied constant depends only on r.

Furthermore, Moschevitin [11, Theorem 4] has given an additive analogue of Lemma 2.1 for exponential sums with reciprocals.

We denote  $\mathbf{e}_p(z) = \exp(2\pi i z/p)$ .

**Lemma 2.2.** Let  $\mathcal{U}, \mathcal{V} \subseteq \mathbb{F}_p$  be of cardinalities U and V, respectively. Then for any positive integer r, for all  $\lambda \in \mathbb{F}_p^*$ , we have

$$\sum_{\substack{u \in \mathcal{U}, \ v \in \mathcal{V} \\ u+v \neq 0}} \mathbf{e}_p \left( \frac{\lambda}{u+v} \right) \ll U^{1-1/2r} \left( V^{1/2} p^{1/2r} + V p^{1/4r} \right),$$

where the implied constant depends only on r.

We also note that in some cases the bound of [2, Lemma 6] (which has also been repeated as [6, Lemma 4.3])

$$\sum_{\substack{u \in \mathcal{U}, \ v \in \mathcal{V} \\ u+v \neq 0}} \sum_{\mathbf{e}_p} \mathbf{e}_p \left( \frac{\lambda}{u+v} \right) \leqslant \sqrt{pUV}$$

is stronger, however, it is not useful for the problems of this paper.

2.2. Partitioning Hilbert cubes. We need the following elementary statement, which shows that any Hilbert cube can be represented as a sum set of two sets of essentially any desired size.

**Lemma 2.3.** Let  $\mathcal{H}$  be a Hilbert cube of cardinality H. For any real  $R \in [2, H/2]$  there are two sets  $\mathcal{U}$  and  $\mathcal{V}$  such that

$$\mathcal{H} = \mathcal{U} + \mathcal{V}$$

and

$$\#\mathcal{U}\geqslant H/R \qquad and \qquad \#\mathcal{V}\geqslant R/2.$$

*Proof.* We recall the notation (1.1) and assume that

$$\mathcal{H} = \mathsf{H}(a_0; a_1, \dots, a_d)$$

for some  $a_0, a_1, \ldots, a_d \in \mathbb{F}_p$ . We consider the sequences of Hilbert cubes

$$\mathcal{H}_i = \mathsf{H}\left(a_0; a_1, \dots, a_i\right), \qquad j = 0, \dots, d.$$

Clearly  $\mathcal{H}_0 = \{a_0\}$  and  $\mathcal{H}_d = \mathcal{H}$ . Let *i* be the smallest *j* for which  $\#\mathcal{H}_j \ge H/R$ . In particular  $i \ge 1$ . Since by the choice of *i* we have  $\#\mathcal{H}_{i-1} < H/R$  and also since

$$\mathcal{H}_i = \mathcal{H}_{i-1} \cup (\mathcal{H}_{i-1} + a_i)$$

we trivially have  $\#\mathcal{H}_i \leq 2\#\mathcal{H}_{i-1} < 2H/R \leq H$ . We now set

$$\mathcal{U} = \mathcal{H}_i$$
 and  $\mathcal{V} = \mathsf{H}\left(0; a_{i+1}, \dots, a_d\right)$ .

Thus  $2H/R > \#\mathcal{U} \geqslant H/R$ . We also have  $\mathcal{H} = \mathcal{U} + \mathcal{V}$  and hence  $H \leqslant \#\mathcal{U}\#\mathcal{V} < 2HR^{-1}\#\mathcal{V}$ , which concludes the proof.

### 3. Proof of Theorem 1.1

3.1. Bounding the size of Hilbert cubes avoiding product sets. Assume that for some sets  $\mathcal{A}, \mathcal{B} \subseteq \mathbb{F}_p^*$  and a Hilbert cube  $\mathcal{H} \subseteq \mathbb{F}_p$  we have  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$ .

We fix some r and set  $R = p^{1/2r}$ . We also denote

$$A = \#\mathcal{A}, \qquad B = \#\mathcal{B}, \qquad H = \#\mathcal{H}.$$

Since  $AB \leq p^2$ , the result is trivial for H < 2R. We can also assume that  $p \geq 2^{2r}$  and thus  $R \geq 2$ . Hence, we see that the conditions of Lemma 2.3 are satisfied. Let  $\mathcal{U}$  and  $\mathcal{V}$  be the corresponding sets. The condition  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$  implies that the number of solutions N to the equation

(3.1) 
$$u + v = ab, \quad a \in \mathcal{A}, b \in \mathcal{B}, u \in \mathcal{U}, v \in \mathcal{V},$$
 satisfies

$$(3.2) N = 0.$$

On the other hand, using the orthogonality of characters (and the convention  $\chi(0) = 0$ ) we write

$$N = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \frac{1}{p - 1} \sum_{\chi \in \mathcal{X}} \chi \left( (u + v)a^{-1}b^{-1} \right)$$
$$= \frac{1}{p - 1} \sum_{\chi \in \mathcal{X}} \sum_{a \in \mathcal{A}} \chi \left( a^{-1} \right) \sum_{b \in \mathcal{B}} \chi \left( b^{-1} \right) \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \chi \left( u + v \right).$$

We note that the contribution from the principal character  $\chi_0$  is given by

$$\sum_{\chi \in \mathcal{X}} \sum_{a \in \mathcal{A}} \chi_0 \left( a^{-1} \right) \sum_{b \in \mathcal{B}} \chi_0 \left( b^{-1} \right) \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \chi_0 \left( u + v \right)$$

$$= AB \sum_{\substack{u \in \mathcal{U}, \ v \in \mathcal{V} \\ u + v \neq 0}} 1 = AB \left( UV + O(U) \right) \gg ABUV,$$

which together with the vanishing condition (3.2) implies

$$ABUV \ll \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi\left(a^{-1}\right) \right| \left| \sum_{b \in \mathcal{B}} \chi\left(b^{-1}\right) \right| \left| \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \chi\left(u + v\right) \right|$$
$$= \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi\left(a\right) \right| \left| \sum_{b \in \mathcal{B}} \chi\left(b\right) \right| \left| \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \chi\left(u + v\right) \right|,$$

since  $\chi(z^{-1}) = \overline{\chi}(z)$  for any  $z \in \mathbb{F}_p^*$ . Applying Lemma 2.1 and recalling the choice of R, we derive

$$ABUV \ll U^{1-1/2r} \left( V^{1/2} p^{1/2r} + V p^{1/4r} \right) \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi \left( a \right) \right| \left| \sum_{b \in \mathcal{B}} \chi \left( b \right) \right|$$
$$\ll U^{1-1/2r} V p^{1/4r} \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi \left( a \right) \right| \left| \sum_{b \in \mathcal{B}} \chi \left( b \right) \right|,$$

which we simplify as

(3.3) 
$$U^{1/2r} \leqslant \frac{p^{1/4r}}{AB} \sum_{\gamma \in \mathcal{X}^*} \left| \sum_{a \in A} \chi\left(a\right) \right| \left| \sum_{b \in \mathcal{B}} \chi\left(b\right) \right|.$$

Furthermore, using the orthogonality of characters, we obtain

$$\sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi\left(a\right) \right|^2 \leqslant \sum_{\chi \in \mathcal{X}} \left| \sum_{a \in \mathcal{A}} \chi\left(a\right) \right|^2 = (p-1)A,$$

and similarly for the sum over  $b \in \mathcal{B}$ . Hence, by the Cauchy inequality we have

$$\left(\sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi\left(a\right) \right| \left| \sum_{b \in \mathcal{B}} \chi\left(b\right) \right| \right)^2 \leq \sum_{\chi \in \mathcal{X}^*} \left| \sum_{a \in \mathcal{A}} \chi\left(a\right) \right|^2 \sum_{\chi \in \mathcal{X}^*} \left| \sum_{b \in \mathcal{B}} \chi\left(b\right) \right|^2 \leq p^2 A B.$$

Substituting this inequality in (3.3) and using that

$$U \gg H/R = Hp^{-1/2r},$$

we obtain

$$(Hp^{-1/2r})^{1/2r} \ll \frac{p^{1/4r}}{AB}p(AB)^{1/2} = \frac{p^{1+1/4r}}{(AB)^{1/2}}$$

or

$$Hp^{-1/2r} \ll \frac{p^{2r+1/2}}{(AB)^r},$$

which concludes the proof in the case  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$ .

3.2. Bounding the size of Hilbert cubes avoiding reciprocal sum sets. Let now  $\mathcal{H} \cap (\mathcal{A} + \mathcal{B})^{-1} = \emptyset$ . We proceed exactly as in the case of  $\mathcal{H} \cap (\mathcal{A} \cdot \mathcal{B}) = \emptyset$  with the same parameter R and the sets  $\mathcal{U}$  and  $\mathcal{V}$ . However, instead of (3.1) we arrive to the equation

$$u + v = (a + b)^{-1}, \qquad a \in \mathcal{A}, \ b \in \mathcal{B}, \ u \in \mathcal{U}, \ v \in \mathcal{V},$$

which we transform into

$$(u+v)^{-1} = a+b,$$
  $a \in \mathcal{A}, b \in \mathcal{B}, u \in \mathcal{U}, v \in \mathcal{V}.$ 

This time we express the number N of solutions to this equation, which still satisfies (3.2) via exponential sums

$$N = \sum_{a \in \mathcal{A}} \sum_{b \in \mathcal{B}} \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} \mathbf{e}_p \left( \lambda \left( (u+v)^{-1} - a - b \right) \right)$$
$$= \frac{1}{p} \sum_{\lambda \in \mathbb{F}_p} \sum_{a \in \mathcal{A}} \mathbf{e}_p \left( -\lambda a \right) \sum_{b \in \mathcal{B}} \mathbf{e}_p \left( -\lambda b \right) \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \mathbf{e}_p \left( \lambda (u+v)^{-1} \right).$$

We now separate the contribution from the term corresponding to  $\lambda = 0$  and then use Lemma 2.2 instead of Lemma 2.1 and the orthogonality of exponential functions. Hence, we arrive to the same bound as in Section 3.1. This concludes the proof in the case  $\mathcal{H} \cap (\mathcal{A} + \mathcal{B})^{-1} = \emptyset$ .

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