GRÖBNER BASES OF RADICAL LI-LI TYPE IDEALS ASSOCIATED WITH PARTITIONS

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ABSTRACT. For a partition λ of n, the Specht ideal $I_{\lambda} \subset K[x_1, \ldots, x_n]$ is the ideal generated by all Specht polynomials of shape λ . In their unpublished manuscript, Haiman and Woo showed that I_{λ} is a radical ideal, and gave its universal Gröbner basis (Murai et al. published a quick proof of this result). On the other hand, an old paper of Li and Li studied analogous ideals, while their ideals are not always radical. The present paper introduces a class of ideals generalizing both Specht ideals and *radical* Li-Li ideals, and studies their radicalness and Gröbner bases.

1. INTRODUCTION

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field K. For a subset $A = \{a_1, a_2, \ldots, a_m\}$ of $[n] := \{1, 2, \ldots, n\}$, let

$$\Delta(A) := \prod_{1 \le i < j \le m} (x_{a_i} - x_{a_j}) \in S$$

be the difference product. For a sequence of subsets $\mathcal{Y} = (Y_1, Y_2, \dots, Y_{k-1})$ with $[n] \supset Y_1 \supset Y_2 \supset \dots \supset Y_{k-1}$, Li and Li [8] studied the ideal

(1.1)
$$I_{\mathcal{Y}} := \left(\prod_{i=1}^{k-1} \Delta(X_i) \middle| X_i \supset Y_i \text{ for all } i, \bigcup_{i=1}^{k-1} X_i = [n]\right)$$

of S (more precisely, the polynomial ring in [8] is $\mathbb{Z}[x_1, \ldots, x_n]$). Among other things, they showed the following.

Theorem 1.1 (c.f. Li-Li [8, Theorem 2]). With the above notation, $I_{\mathcal{Y}}$ is a radical ideal if and only if $\#Y_2 \leq 1$.

A partition of a positive integer n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_p)$ with $\lambda_1 + \cdots + \lambda_p = n$. Let P_n be the set of all partitions of n. A partition λ is frequently represented by its Young diagram. For example, (4, 2, 1) is represented as \square . A (Young) tableau of shape $\lambda \in P_n$ is a bijective filling of the

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squares of the Young diagram of λ by the integers in [n]. For example,



is a tableau of shape (4, 2, 1). Let $\operatorname{Tab}(\lambda)$ be the set of all tableaux of shape λ . Recall that the Specht polynomial f_T of $T \in \operatorname{Tab}(\lambda)$ is $\prod_{j=1}^{\lambda_1} \Delta(T(j))$, where T(j) is the set of the entries of the *j*-th column of T (here the entry in the *i*-th row is the *i*-th element of T(j)). For example, if T is the above tableau, then $f_T = (x_4 - x_5)(x_4 - x_6)(x_5 - x_6)(x_3 - x_2)$.

We call the ideal

 $I_{\lambda} := (f_T \mid T \in \operatorname{Tab}(\lambda)) \subset S$

the Specht ideal of λ . These ideals have been studied from several points of view (and under several names and characterizations), see for example, [1, 9, 10, 13]. The following is an unpublished result of Haiman and Woo ([6]), to which Murai, Ohsugi and the second author ([11]) published a quick proof.

Theorem 1.2 (Haiman-Woo [6], see also [11]). If $\mathcal{F} \subset P_n$ is a lower filter with respect to the dominance order \trianglelefteq , then $I_{\mathcal{F}} := \sum_{\lambda \in \mathcal{F}} I_{\lambda}$ is a radical ideal, for which $\{f_T \mid T \in \text{Tab}(\mu), \mu \in \mathcal{F}\}$ forms a universal Gröbner basis (i.e., a Gröbner basis with respect to all monomial orders). In particular, I_{λ} is a radical ideal, for which $\{f_T \mid T \in \text{Tab}(\mu), \mu \trianglelefteq \lambda\}$ forms a universal Gröbner basis.

Let us explain why the second assertion follows form the first. Since $\lambda \succeq \mu$ for $\lambda, \mu \in P_n$ implies $I_{\lambda} \supset I_{\mu}$ (c.f. Lemma 2.5), we have $I_{\lambda} = I_{\mathcal{F}}$ for the lower filter $\mathcal{F} := \{\mu \in P_n \mid \mu \leq \lambda\}.$

The Li-Li ideals $I_{\mathcal{Y}}$ and the Specht ideals I_{λ} share common examples. In fact, for $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_{k-1})$ with $\#Y_1 \leq 1, Y_2 = \cdots = Y_{k-1} = \emptyset$ and $\lambda = (\lambda_1, \ldots, \lambda_p) \in P_n$ with $\lambda_1 = \cdots = \lambda_{p-1} = k - 1$, we have $I_{\mathcal{Y}} = I_{\lambda}$ by [8, Corollary 3.2].

In this paper, we study a common generalization of the *radical* Li-Li ideals and the Specht ideals, for which *almost* direct analogs of Theorem 1.2 hold. For example, in Sections 2 and 3, we take a positive integer l, and a partition $\lambda \in P_{n+l-1}$ with $\lambda_1 \geq l$, and consider tableaux like

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(l = 4 in this case). Using these tableau, we define the ideal $I_{l,\lambda}$.

The symmetric group S_{n-1} of the set $\{2, \ldots, n\}$ still acts on $I_{l,\lambda}$, so our ideals have representation theoretic interest. The following are other motivations of the present paper.

(1) Recently, the defining ideals of subspace arrangements have been intensely studied (c.f. [2, 4, 14]). Our $I_{l,\lambda}$ and its generalization $\sqrt{I_{l,m,\lambda}}$ introduced in Section 4 give new classes of these ideals. Note that $I_{l,m,\lambda}$ is not a radical ideal in general, while Corollary 4.5 gives the generators of its radical explicitly.

- (2) A universal Gröbner basis is very important, since it is closely related to the Gröbner fan. While we can use a computer for explicit examples, it is extremely difficult to construct universal Gröbner bases for some (infinite) family of ideals. Theorem 1.2 gives universal Gröbner bases of Specht ideals I_{λ} . However, since I_{λ} are symmetric, this case is exceptional. So it must be very interesting, if the Gröbner bases of non-symmetric ideals $I_{l,\lambda}$ given in Theorem 2.6 are universal. Corollary 3.11 is an affirmative evidence.
- (3) One of the motivations of the paper [8] of Li and Li is an application to graph theory (see [5] for further connection to Gröbner bases theory). We expect that the present paper gives a new inslight to this direction.

In the present paper, for the convention and notation of the Gröbner bases theory, we basically follow [7, Chapter 1].

2. A generalization of the case $\#Y_1 = \cdots = \#Y_l = 1$

We keep the same notation as Introduction, and fix a positive integer l. For $\lambda \in [P_{n+l-1}]_{\geq l} := \{\lambda \in P_{n+l-1} \mid \lambda_1 \geq l\}$, we consider a bijective filling of the squares l copies

of the Young diagram of λ by the multiset $\{1, \ldots, 1, 2, \ldots, n\}$ such that no two copies of 1 are contained in the same column. Let $\operatorname{Tab}(l, \lambda)$ be the set of such tableaux. For example, the tableau (1.2) above is an element of $\operatorname{Tab}(4, \lambda)$ for $\lambda = (6, 3, 2)$ (moreover, this is a *standard* tableau defined below). The Specht polynomial f_T of $T \in \operatorname{Tab}(l, \lambda)$ is defined by the same way as in the classical case. For example, if T is the one in (1.2), then $f_T = (x_1 - x_4)(x_1 - x_6)(x_4 - x_6)(x_1 - x_5)(x_1 - x_7)(x_5 - x_7)(x_1 - x_8)$. For $\lambda \in [P_{n+l-1}]_{\geq l}$, consider the ideal

$$I_{l,\lambda} := (f_T \mid T \in \operatorname{Tab}(l,\lambda))$$

of S. Clearly, $\operatorname{Tab}(1, \lambda) = \operatorname{Tab}(\lambda)$ and $I_{1,\lambda} = I_{\lambda}$.

For $\lambda = (\lambda_1, \ldots, \lambda_p), \mu = (\mu_1, \ldots, \mu_q) \in P_m$, we write $\lambda \geq \mu$ if λ is equal to or larger than μ with respect to the *dominance order*, that is,

$$\lambda_1 + \dots + \lambda_i \ge \mu_1 + \dots + \mu_i \quad \text{for } i = 1, 2, \dots, \min\{p, q\}.$$

In what follows, we regard $[P_{n+l-1}]_{\geq l}$ as a subposet of P_{n+l-1} .

For $\lambda \in P_m$ and j with $1 \leq j \leq \lambda_1$, let λ_j^{\perp} be the length of the j-th column of the Young diagram of λ . Then $\lambda^{\perp} = (\lambda_1^{\perp}, \lambda_2^{\perp}, \ldots)$ is a partition of m again. It is a classical result that $\lambda \geq \mu$ if and only if $\lambda^{\perp} \leq \mu^{\perp}$.

Remark 2.1. By [3, Proposition 2.3], if λ covers μ (i.e., $\lambda \triangleright \mu$, and there is no other partition between them), then there are two integers i, i' with i < i' such that $\mu_i = \lambda_i - 1$, $\mu_{i'} = \lambda_{i'} + 1$, and $\mu_k = \lambda_k$ for all $k \neq i, i'$, equivalently, there are two integers j, j' with j < j' such that $\mu_j^{\perp} = \lambda_j^{\perp} + 1$, $\mu_{j'}^{\perp} = \lambda_{j'}^{\perp} - 1$, and $\mu_k^{\perp} = \lambda_k^{\perp}$ for all $k \neq j, j'$. Clearly, $\mu_j^{\perp} \ge \mu_{j'}^{\perp} + 2$ in this case. Here, we allow the case i' is larger than the length p of λ , where we set $\lambda_{i'} = 0$. Similarly, the case $\mu_{j'}^{\perp} = 0$ might occur.

Remark 2.2. By a similar argument to the proof of Lemma 2.3, to generate $I_{l,\lambda}$, it suffices to use $T \in \text{Tab}(l,\lambda)$ such that the left most l squares in the first row are

filled by 1. So, in manner of (1.1), the ideal $I_{l,\lambda}$ can be represented as follows.

$$I_{l,\lambda} = \left(\prod_{i=1}^{\lambda_1} \Delta(X_i) \middle| 1 \in X_i \text{ for } 1 \le i \le l, \ \#X_i = \lambda_i^{\perp} \text{ for all } i, \ \bigcup_{i=1}^{\lambda_1} X_i = [n]\right)$$

Convention. Throughout this paper, when we consider the Gröbner bases, we use the lexicographic order with $x_1 < \cdots < x_n$ unless otherwise specified (see Lemma 3.10 below, which states that only the order among the variables x_1, \ldots, x_n matters for our Gröbner bases), and the initial monomial $in_{\leq}(f)$ of $0 \neq f \in S$ will be simply denoted by in(f).

For $T \in \text{Tab}(l, \lambda)$, recall that T(j) is the set of the entries of the *j*-th column of T. If σ is a permutation on T(j), we have $f_{\sigma T} = \text{sgn}(\sigma)f_T$ for each *j*. In this sense, to consider f_T , we may assume that T is *column standard*, that is, all columns are increasing from top to bottom (in particular, all 1's appear in the 1st row).

If T is column standard and the number i is in the d_i -th row of T, we have

(2.1)
$$in(f_T) = \prod_{i=1}^n x_i^{d_i - 1}$$

(recall our convention on the monomial order).

If a column standard tableau $T \in \text{Tab}(l, \lambda)$ is also row semi-standard (i.e., all rows are non-decreasing from left to right), we say T is *standard*. Let $\text{STab}(l, \lambda)$ be the set of standard tableaux in $\text{Tab}(l, \lambda)$. We simply denote $\text{STab}(1, \lambda)$ by $\text{STab}(\lambda)$. The next result is very classical when l = 1.

Lemma 2.3. For $\lambda \in [P_{n+l-1}]_{\geq l}$, $\{f_T \mid T \in \operatorname{STab}(l, \lambda)\}$ forms a basis of the vector space V spanned by $\{f_T \mid T \in \operatorname{Tab}(l, \lambda)\}$. Hence $\{f_T \mid T \in \operatorname{STab}(l, \lambda)\}$ is a minimal system of generators of $I_{l,\lambda}$.

Proof. In the classical case (i.e., when l = 1), we can rewrite f_T for $T \in \operatorname{Tab}(\lambda)$ as a linear combination of f_{T_i} 's for $T_i \in \operatorname{STab}(\lambda)$ repeatedly using the relations given by *Garnir elements* (see [12, §2.6]). Such a relation concerns the *j*-th and the (j + 1)-st columns of T. The classical argument directly works in our case unless both of these columns contain 1. So we assume that both columns have 1. Since $f_T = \prod_{j=1}^{\lambda_1} \Delta(T(j))$, we can concentrate on the *j*-th and (j+1)-st columns of T, and may assume that T consists of two columns (i.e., λ is of the form $(2, \lambda_2, \ldots, \lambda_p) \in$ $P_{n+2-1} = P_{n+1}$) and l = 2. Set $\tilde{\lambda} := (\lambda_2, \ldots, \lambda_p) \in P_{n-1}$. Removing the first row from $T \in \operatorname{Tab}(2, \lambda)$, we have $\tilde{T} \in \operatorname{Tab}(\tilde{\lambda})$ (the set of the entries of \tilde{T} is $\{2, \ldots, n\}$). The converse operation $\operatorname{Tab}(\tilde{\lambda}) \ni \tilde{T} \longmapsto T \in \operatorname{Tab}(2, \lambda)$ also makes sense. Clearly, $f_T = (\prod_{i=2}^n (x_1 - x_i)) \cdot f_{\tilde{T}}$. Multiplying $\prod_{i=2}^n (x_1 - x_i)$ to both sides of a Garnir relation $f_{\tilde{T}} = \sum_{i=1}^k \pm f_{\tilde{T}_i} (\tilde{T}, \tilde{T}_i \in \operatorname{Tab}(\tilde{\lambda}))$, we have the relation $f_T = \sum_{i=1}^k \pm f_{T_i}$ $(T, T_i \in \operatorname{Tab}(2, \lambda))$. As in the classical case, T_i need not to be standard, but is closer to standard than T. Using these relations, the argument in [12, §2.6] is applicable to our case, and we can show that $\{f_T \mid T \in \operatorname{STab}(l, \lambda)\}$ spans V.

As we have seen in (2.1), $in(f_T) \neq in(f_{T'})$ holds for distinct $T, T' \in STab(l, \lambda)$. So $\{f_T \mid T \in STab(l, \lambda)\}$ is linearly independent. For $\boldsymbol{a} = (a_1, \ldots, a_n) \in K^n$, there are distinct $\alpha_1, \ldots, \alpha_p \in K$ with $\{\alpha_1, \ldots, \alpha_p\} = \{a_1, \ldots, a_n\}$ as sets. Now we can define the partition $\mu = (\mu_1, \ldots, \mu_p) \in P_n$ such that α_i appears μ_i times in (a_1, \ldots, a_n) for each *i*. This partition μ will be denoted by $\Lambda(\boldsymbol{a})$. For example, $\Lambda((1, 0, 2, 1, 2, 2)) = (3, 2, 1)$.

For $\mathbf{a} \in K^n$, set $\mathbf{a}_{(l)} := (\overline{a_1, \ldots, a_1}, a_2, \ldots, a_n) \in K^{n+l-1}$ and $\Lambda_l(\mathbf{a}) := \Lambda(\mathbf{a}_{(l)}) \in [P_{n+l-1}]_{\geq l}$. For example, if $\mathbf{a} = (1, 0, 2, 1, 2, 2)$, then $\mathbf{a}_{(3)} = (1, 1, 1, 0, 2, 1, 2, 2)$ and $\Lambda_3(\mathbf{a}) = (4, 3, 1)$. When l = 1, the following result is classical.

Lemma 2.4 (c.f. [11, Lemma 2.1.]). Let $\lambda \in [P_{n+l-1}]_{\geq l}$ and $T \in \operatorname{Tab}(l, \lambda)$. For $a \in K^n$ with $\Lambda_l(a) \not \leq \lambda$, we have $f_T(a) = 0$.

Proof. For $\mathbf{a} = (a_1, \ldots, a_n) \in K^n$, replacing i with a_i for each i in T, we have a tableau $T(\mathbf{a})$, whose entries are elements in K. It is easy to see that $f(\mathbf{a}) \neq 0$ if and only if the entries in each column of $T(\mathbf{a})$ are all distinct. So the assertion follows from the same argument as [11, Lemma 2.1].

Lemma 2.5 (c.f. [10, Theorem 1.1]). For $\lambda, \mu \in [P_{n+l-1}]_{\geq l}$ with $\lambda \supseteq \mu$, we have $I_{l,\lambda} \supset I_{l,\mu}$.

Proof. The proof is essentially same as the classical case, while we have to care about one point. First, we will recall a basic property of difference products. For subsets $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_{k'}\}$ of [n] with $k \ge k' + 2$, we have (2.2)

$$\Delta(A) \cdot \Delta(B) = \sum_{k-k' \le i \le k} (-1)^{i-k+k'} \left[\Delta(A \setminus \{a_i\}) \cdot \Delta(B \cup \{a_i\}) \cdot \prod_{1 \le i' < k-k'} (x_{a_{i'}} - x_{a_i}) \right]$$

by [8, Proposition 3.1], where we regard a_i as the last element of $B \cup \{a_i\}$.

Let us start with the main body of the proof. To prove the assertion, we may assume that λ covers μ . By Remark 2.1, there are j, j' with j < j' such that $\mu_j^{\perp} = \lambda_j^{\perp} + 1, \ \mu_{j'}^{\perp} = \lambda_{j'}^{\perp} - 1, \ \text{and} \ \mu_i^{\perp} = \lambda_i^{\perp}$ for all $i \neq j, j'$. Take $T \in \text{Tab}(l, \mu)$, and let $A = \{a_1, \ldots, a_k\}$ (resp. $B = \{b_1, \ldots, b_{k'}\}$) be the set of the contents of the *j*-th (resp. *j'*-th) column of *T*. For *i* with $k - k' \leq i \leq k$, consider the tableau T_i whose *j*-th (resp. *j'*-th) column consists of the elements of $A \setminus \{a_i\}$ (resp. $B \cup \{a_i\}$) and the other columns are same as those of *T*. Since $a_i \geq 2$ for $i \geq 2$, we have $T_i \in \text{Tab}(l, \lambda)$. By (2.2), we have

(2.3)
$$f_T = \sum_{k-k' \le i \le k} (-1)^{i-k+k'} \left[f_{T_i} \cdot \prod_{1 \le i' < k'-k} (x_{a_{i'}} - x_{a_i}) \right] \in I_{l,\lambda},$$

and it means that $I_{l,\lambda} \supset I_{l,\mu}$.

We say that $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$ is a *lower (resp. upper) filter* if $\lambda \in \mathcal{F}, \mu \in [P_{n+l-1}]_{\geq l}$ and $\mu \leq \lambda$ (resp. $\mu \geq \lambda$) imply $\mu \in \mathcal{F}$. For a *lower* filter $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$, set

$$G_{l,\mathcal{F}} := \{ f_T \mid T \in \operatorname{STab}(l,\lambda) \text{ for } \lambda \in \mathcal{F} \},\$$

and let $I_{l,\mathcal{F}} \subset S$ be the ideal generated by $G_{l,\mathcal{F}}$, equivalently,

$$I_{l,\mathcal{F}} := \sum_{\lambda \in \mathcal{F}} I_{l,\lambda}.$$

In particular, for $\lambda \in [P_{n+l-1}]_{\geq l}$, $\mathcal{F}_{\lambda} := \{\mu \in [P_{n+l-1}]_{\geq l} \mid \mu \leq \lambda\}$ is a lower filter, and we have $I_{l,\lambda} = I_{l,\mathcal{F}_{\lambda}}$ by Lemma 2.5. For convenience, set $G_{l,\emptyset} = \emptyset$ and $I_{l,\emptyset} = (0)$.

For an upper filter $\emptyset \neq \mathcal{F} \subset [P_{n+l-1}]_{\geq l}$, we consider the ideal

$$J_{l,\mathcal{F}} := (f \in S \mid f(\boldsymbol{a}) = 0 \text{ for } \forall \boldsymbol{a} \in K^n \text{ with } \Lambda_l(\boldsymbol{a}) \in \mathcal{F}).$$

Clearly, $J_{l,\mathcal{F}}$ is a radical ideal.

Theorem 2.6. Let $\mathcal{F} \subseteq [P_{n+l-1}]_{\geq l}$ be a lower filter, and $\mathcal{F}^{\mathsf{c}} := [P_{n+l-1}]_{\geq l} \setminus \mathcal{F}$ its complement (note that \mathcal{F}^{c} is an upper filter). Then $G_{l,\mathcal{F}}$ is a Gröbner basis of $J_{l,\mathcal{F}^{\mathsf{c}}}$.

The following corollary is immediate from the theorem.

Corollary 2.7. With the above situation, we have $I_{l,\mathcal{F}} = J_{l,\mathcal{F}^c}$, and $I_{l,\mathcal{F}}$ is a radical ideal. In particular, $I_{l,\lambda}$ is a radical ideal with

$$I_{l,\lambda} = (f \in S \mid f(\boldsymbol{a}) = 0 \text{ for } \forall \boldsymbol{a} \in K^n \text{ with } \Lambda_l(\boldsymbol{a}) \not\leq \lambda),$$

for which $\{f_T \mid T \in \operatorname{STab}(l, \mu), \mu \trianglelefteq \lambda\}$ forms a Gröbner basis.

The strategy of the proof of Theorem 2.6 is essentially same as that of [11, Theorem 1.1], but we repeat it here for the reader's convenience. For a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in P_m$ and a positive integer *i*, we write $\lambda + \langle i \rangle$ for the partition of m + 1 obtained by rearranging the sequence $(\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_p)$, where we set $\lambda + \langle i \rangle = (\lambda_1, \ldots, \lambda_p, 1)$ when i > p. For example $(4, 2, 2, 1) + \langle 2 \rangle = (4, 2, 2, 1) + \langle 3 \rangle = (4, 3, 2, 1)$, and $(4, 2, 2, 1) + \langle i \rangle = (4, 2, 2, 1, 1)$ for all $i \ge 5$. Since $\lambda \le \mu$ implies $\lambda + \langle i \rangle \le \mu + \langle i \rangle$ for all *i*, if $\mathcal{F} \subset P_m$ is an upper (resp. lower) filter, then so is

$$\mathcal{F}_i := \{ \mu \in P_{m-1} \mid \mu + \langle i \rangle \in \mathcal{F} \}.$$

Example 2.8. Even if a lower filter \mathcal{F} has a unique maximal element, \mathcal{F}_i does not in general. For example, if $\mathcal{F} := \{\lambda \in [P_7]_{\geq l} \mid \lambda \leq (3, 2, 2)\}$ for l = 1, 2, then \mathcal{F}_2 has two maximal elements (3, 1, 1, 1) and (2, 2, 2).

Lemma 2.9 (c.f. [11, Lemma 3.3]). Let $\emptyset \neq \mathcal{F} \subset [P_{n+l-1}]_{\geq l}$ be an upper filter, and let f be a polynomial in $J_{l,\mathcal{F}}$ of the form

$$f = g_d x_n^d + \dots + g_1 x_n + g_0,$$

where $g_0, \ldots, g_d \in K[x_1, \ldots, x_{n-1}]$ and $g_d \neq 0$. Then g_0, \ldots, g_d belong to $J_{l, \mathcal{F}_{d+1}}$.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathcal{F}_{d+1}$, and take $\boldsymbol{a} = (a_1, \ldots, a_{n-1}) \in K^{n-1}$ with $\Lambda_l(\boldsymbol{a}) = \lambda$. Then there are distinct elements $\alpha_1, \ldots, \alpha_p \in K$ such that α_i appears λ_i times in $\boldsymbol{a}_{(l)}$ for $i = 1, \ldots, p$. Since \mathcal{F} is an upper filter, we have $\lambda + \langle i \rangle \in \mathcal{F}$ for $i = 1, 2, \ldots, d+1$. We will consider two cases as follows (in the sequel, for $\alpha \in K$, (\boldsymbol{a}, α) means the point in K^n whose coordinate is $(a_1, \ldots, a_{n-1}, \alpha)$): (i) If p < d+1, then $\lambda + \langle d+1 \rangle = (\lambda_1, \ldots, \lambda_p, 1)$. Thus, for any $\alpha \in K \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$, we have $\Lambda_l(\boldsymbol{a}, \alpha) = \lambda + \langle d+1 \rangle \in \mathcal{F}$, and hence $f(\boldsymbol{a}, \alpha) = 0$. (ii) If $p \ge d+1$, then we have

 $\Lambda_l(\boldsymbol{a}, \alpha_i) = \lambda + \langle i \rangle \in \mathcal{F} \text{ for any } i = 1, \dots, d+1 \text{ (note that } \lambda + \langle i \rangle \supseteq \lambda + \langle d+1 \rangle \in \mathcal{F} \text{ for these } i \text{), and hence } f(\boldsymbol{a}, \alpha_i) = 0.$

In both cases, it follows that the polynomial $f(\boldsymbol{a}, x_n) = \sum_{i=0}^d g_i(\boldsymbol{a}) x_n^i \in K[x_n]$ has at least d+1 zeros. Since the degree of $f(\boldsymbol{a}, x_n)$ is $d, f(\boldsymbol{a}, x_n)$ is the zero polynomial in $K[x_n]$. Thus, $g_i(\boldsymbol{a}) = 0$ for $i = 0, 1, \ldots, d$. Hence, $g_0, \ldots, g_d \in J_{l, \mathcal{F}_{d+1}}$. \Box

The proof of Theorem 2.6. First, we show that $G_{l,\mathcal{F}} \subset J_{l,\mathcal{F}^c}$. Take $T \in \mathrm{STab}(l,\lambda)$ for $\lambda \in \mathcal{F}$, and $\boldsymbol{a} \in K^n$ with $\Lambda_l(\boldsymbol{a}) \in \mathcal{F}^c$ (i.e., $\Lambda_l(\boldsymbol{a}) \notin \mathcal{F}$). Since \mathcal{F} is a lower filter, we have $\Lambda_l(\boldsymbol{a}) \not \leq \lambda$, and hence $f_T(\boldsymbol{a}) = 0$ by Lemma 2.4. So $f_T \in J_{l,\mathcal{F}^c}$.

For $\mu \in [P_{n+l-2}]_{\geq l}$, it is easy to see that

$$\mu \notin (\mathcal{F}^{\mathsf{c}})_i \Longleftrightarrow \mu + \langle i \rangle \notin \mathcal{F}^{\mathsf{c}} \Longleftrightarrow \mu + \langle i \rangle \in \mathcal{F} \Longleftrightarrow \mu \in \mathcal{F}_i,$$

so we have $[P_{n+l-2}]_{\geq l} \setminus (\mathcal{F}^{\mathsf{c}})_i = \mathcal{F}_i$.

To prove the theorem, it suffices to show that the initial monomial $\operatorname{in}(f)$ for all $0 \neq f \in J_{l,\mathcal{F}^c}$ can be divided by $\operatorname{in}(f_T)$ for some $f_T \in G_{l,\mathcal{F}}$. We will prove this by induction on n. The case n = 1 is trivial. For $n \geq 2$, let $f = g_d x_n^d + \cdots + g_1 x_n + g_0 \in J_{l,\mathcal{F}^c}$, where $g_i \in K[x_1, \ldots, x_{n-1}]$ and $g_d \neq 0$. By Lemma 2.9, one has $g_d \in J_{l,(\mathcal{F}^c)_{d+1}}$. By the induction hypothesis, we have $G_{l,\mathcal{F}_{d+1}}(=G_{l,[P_{n+l-2}]\geq l\setminus(\mathcal{F}^c)_{d+1}})$ is a Gröbner basis of $J_{l,(\mathcal{F}^c)_{d+1}}$. Then there is $T \in \operatorname{STab}(l,\mu)$ for $\mu \in \mathcal{F}_{d+1}$ such that $\operatorname{in}(f_T)$ divides $\operatorname{in}(g_d)$. Set $\lambda := \mu + \langle d+1 \rangle \in \mathcal{F}$. Let us consider the tableau $T' \in \operatorname{STab}(l,\lambda)$ such that the image of each $i = 1, 2, \ldots, n-1$ is same for T and T'. So n is in the newly added square. Since $\lambda = \mu + \langle d+1 \rangle$, n is in the (q+1)-st row of T' for some $q \leq d$. Since we have $\operatorname{in}(f) = \operatorname{in}(g_d x_n^d) = x_n^d \cdot \operatorname{in}(g_d)$ and $\operatorname{in}(f_{T'}) = x_n^q \cdot \operatorname{in}(f_T)$ by (2.1), $\operatorname{in}(f_{T'})$ divides $\operatorname{in}(f)$. Hence, the proof is completed.

For $\lambda = (\lambda_1, \ldots, \lambda_p) \in [P_{n+l-1}]_{\geq l}$, set $H_{l,\lambda} := \{ \boldsymbol{a} \in K^n \mid \Lambda_l(\boldsymbol{a}) = \lambda \}$. Then we have the decomposition $K^n = \bigsqcup_{\lambda \in [P_{n+l-1}]_{\geq l}} H_{l,\lambda}$, and the dimension of $H_{l,\lambda}$ equals the length p of λ . For an upper filter $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}, S/J_{l,\mathcal{F}} (= S/I_{l,\mathcal{F}^c})$ is the coordinate ring of $\bigsqcup_{\lambda \in \mathcal{F}} H_{l,\lambda}$.

Proposition 2.10. The codimension of the ideal $I_{l,\lambda}$ is $\lambda_1 - l + 1$.

Proof. By the above remark, the algebraic set defined by $I_{l,\lambda}$ is the union of $H_{l,\mu}$ for all $\mu \in [P_{n+l-1}]_{\geq l}$ with $\mu \not\leq \lambda$. Among these partitions, $\mu' = (\lambda_1 + 1, 1, 1, ...)$ has the largest length $n + l - 1 - \lambda_1$, and hence codim $I_{l,\lambda} = n - \dim S/I_{l,\lambda} = n - (n + l - 1 - \lambda_1) = \lambda_1 - l + 1$.

Example 2.11. For $\lambda = (3, 3, 1)$, STab $(2, \lambda)$ consists of the following 11 elements



so $I_{2,\lambda}$ is minimally generated by 11 elements. For a non-empty subset $F \subset [n]$, consider the ideal $P_F = (x_i - x_j \mid i, j \in F)$. Clearly, P_F is a prime ideal of

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codimension #F - 1. By Corollary 2.7, $I_{3,\lambda}$ is a radical ideal whose minimal primes are P_F for $F \subset [n]$ either (i) $1 \in F$ and #F = 3, or (ii) $1 \notin F$ and #F = 4.

3. Under the opposite monomial order

Philosophically, we next treat the Gröbner basis of $I_{l,\lambda}$ with respect to the lexicographic order with $x_1 > x_2 > \cdots > x_n$, which is opposite to the one used in the previous section. However, for notational simplicity, we keep using the lexicographic order with $x_1 < \cdots < x_n$, but we consider tableaux whose squares are bijectively l copies

filled by the multiset $\{1, \ldots, n-1, n, \ldots, n\}$. For $\lambda \in [P_{n+l-1}]_{\geq l}$, let $\operatorname{Tab}(\lambda, l)$ be the set of such tableaux of shape λ . As in the previous section, we can define the standard-ness of $T \in \operatorname{Tab}(\lambda, l)$. For example, the tableau T in (3.2) below is standard. Let $\operatorname{STab}(\lambda, l)$ be the subset of $\operatorname{Tab}(\lambda, l)$ consisting of standard tableaux. By the same argument as Lemma 2.3, we have the following.

Lemma 3.1. For $\lambda \in [P_{n+l-1}]_{\geq l}$, $\{f_T \mid T \in \operatorname{STab}(\lambda, l)\}$ forms a basis of the vector space spanned by $\{f_T \mid T \in \operatorname{Tab}(\lambda, l)\}$.

For $\lambda \in [P_{n+l-1}]_{\geq l}$, consider the ideal

$$I_{\lambda,l} := (f_T \mid T \in \operatorname{Tab}(\lambda, l)) = (f_T \mid T \in \operatorname{STab}(\lambda, l)).$$

For $\boldsymbol{a} \in K^n$, set $\boldsymbol{a}^{(l)} := (a_1, \ldots, a_{n-1}, \overbrace{a_n, \ldots, a_n}^{l \text{ copies}}) \in K^{n+l-1}$ and $\Lambda^l(\boldsymbol{a}) := \Lambda(\boldsymbol{a}^{(l)}) \in [P_{n+l-1}]_{\geq l}$. Up to the automorphism of S exchanging x_1 and x_n , the ideal $I_{\lambda,l}$ coincides with $I_{l,\lambda}$ treated in the previous section. Hence we have $I_{\mu,l} \subset I_{\lambda,l}$ for $\mu, \lambda \in [P_{n+l-1}]_{\geq l}$ with $\mu \leq \lambda$, and

(3.1)
$$I_{\lambda,l} = (f \in S \mid f(\boldsymbol{a}) = 0 \text{ for } \forall \boldsymbol{a} \in K^n \text{ with } \Lambda^l(\boldsymbol{a}) \not\leq \lambda).$$

In $T \in \operatorname{STab}(\lambda, l)$, all n's are in the bottom of their columns. Let w(T) denote the number of squares which locate above some n. For example, if

$$(3.2) T = \begin{bmatrix} 1 & 2 & 3 & 5 & 8 & 8 \\ 4 & 6 & 8 & \\ \hline 7 & 8 & \\ \end{bmatrix}$$

(n = 8 and l = 4 in this case), then w(T) = 3. In fact, the squares filled by 2, 6, and 3 are counted. For $T \in \text{Tab}(\lambda, l)$, the degree of $\text{in}(f_T)$ with respect to x_n is w(T). Let $\text{sh}_{\leq n}(T) \in P_{n-1}$ denote the shape of T', where T' is the tableau obtained by removing all squares filled by n from T. For example, if T is the above one, we have $\text{sh}_{\leq 8}(T) = (4, 2, 1)$.

For $\mu \in P_{n-1}$, set

$$\langle \mu \rangle^l := \{ \lambda \in [P_{n+l-1}]_{\geq l} \mid \exists T \in \operatorname{STab}(\lambda, l) \text{ with } \operatorname{sh}_{< n}(T) = \mu \}.$$

For $\lambda \in \langle \mu \rangle^l$ and $T \in \operatorname{STab}(\lambda, l)$ with $\operatorname{sh}_{\langle n}(T) = \mu$, the positions of the squares filled by *n* only depend on λ and μ . We call these squares *n*-squares of λ . Similarly, w(T)does not depend on a particular choice of *T*, and we denote this value by $w_{\mu}(\lambda)$. **Lemma 3.2.** Let $\lambda \in [P_{n+l-1}]_{\geq l}$, and $\mathcal{F} := \{\rho \in [P_{n+l-1}]_{\geq l} \mid \rho \leq \lambda\}$ the lower filter of $[P_{n+l-1}]_{\geq l}$. For $\mu \in P_{n-1}$, if $X := \langle \mu \rangle^l \cap \mathcal{F}$ is non-empty, then there is the element $\widetilde{\lambda} \in X$ satisfying $w_{\mu}(\rho) > w_{\mu}(\widetilde{\lambda})$ for all $\rho \in X \setminus \{\widetilde{\lambda}\}$.

Proof. We determine $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_p)$ inductively from $\widetilde{\lambda}_1$. First, we set

 $\widetilde{\lambda}_1 := \max\{ \rho_1 \mid \rho = (\rho_1, \dots, \rho_q) \in X \} \text{ and } X_1 := \{ \rho \in X \mid \rho_1 = \widetilde{\lambda}_1 \},\$

and next $\widetilde{\lambda}_2 := \max\{\rho_2 \mid \rho \in X_1\}$ and $X_2 := \{\rho \in X_1 \mid \rho_2 = \widetilde{\lambda}_2\}$. We repeat this procedure until the sum $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 + \cdots$ reaches n + l - 1.

We will show that λ has the expected property. For $\rho \in X \setminus {\{\lambda\}}$, set $i_0 := \min\{i \mid \rho_i \neq \lambda_i\}$. Then we have $\rho_{i_0} < \tilde{\lambda}_{i_0}$, and ρ has an *n*-square in the *i*-th row for some $i > i_0$. Let i_1 be the smallest *i* with this property. Raising up the right most square in the i_1 -th row to the right end of i_0 -th row, we get $\rho' \in X$ (here we use the present form of \mathcal{F}). It is clear that $w_{\mu}(\rho) > w_{\mu}(\rho')$ and $\rho \triangleleft \rho'$. Repeating this argument until our partition reaches $\tilde{\lambda}$, we get the expected inequality.

Example 3.3. In the above lemma, the case $\lambda \neq \tilde{\lambda}$ might happen. For example, if $\lambda = (4, 2, 1)$ and $\mu = (3, 3)$ (so l = 1 now), we have $\tilde{\lambda} = (3, 3, 1)$.

For a lower filter $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$ and a non-negative integer k, set

$$\mathcal{F}^k := \{ \mu \in P_{n-1} \mid \exists \lambda \in \langle \mu \rangle^l \cap \mathcal{F} \text{ with } w_\mu(\lambda) \le k \}.$$

Example 3.4. Consider the case $n = 6, l = 2, \lambda = (3, 3, 1)$ and $\mathcal{F} := \{ \rho \in [P_7]_{\geq 2} \mid \rho \leq \lambda \}$. Then $\mathcal{F}^3, \mathcal{F}^2, \mathcal{F}^1, \mathcal{F}^0$ are the lower filters of P_5 whose unique maximal elements are (3, 2), (3, 1, 1), (2, 1, 1, 1) and (1, 1, 1, 1, 1), respectively. In the following diagrams, \star 's represent the positions of *n*-squares of the corresponding partitions of n + l - 1 (= 7). It is also easy to see that $\mathcal{F}^k = \mathcal{F}^3$ for all $k \geq 3$.



Lemma 3.5. If $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$ is a lower filter, then \mathcal{F}^k is a lower filter of P_{n-1} .

Proof. It suffices to show that if $\mu \in \mathcal{F}^k$ covers $\nu \in P_{n-1}$ then $\nu \in \mathcal{F}^k$. In this situation, there are two integers j, j' with j < j' such that $\nu_j^{\perp} = \mu_j^{\perp} + 1, \nu_{j'}^{\perp} = \mu_{j'}^{\perp} - 1$, and $\nu_i^{\perp} = \mu_i^{\perp}$ for all $i \neq j, j'$. In other words, moving a square in the j'-th column of μ to the j-th column, we get ν . Anyway, we can take $\lambda \in \mathcal{F} \cap \langle \mu \rangle^l$ with $w_{\mu}(\lambda) \leq k$, and we want to construct $\rho \in \mathcal{F} \cap \langle \nu \rangle^l$ with $w_{\nu}(\rho) \leq k$.

For each *i*, we have $\mu_i^{\perp} \leq \lambda_i^{\perp} \leq \mu_i^{\perp} + 1$, and there is an *n*-square in the *i*-th column of λ if and only if $\lambda_i^{\perp} = \mu_i^{\perp} + 1$. We have the following four cases.

(1) $\lambda_{j}^{\perp} = \mu_{j}^{\perp}$ and $\lambda_{j'}^{\perp} = \mu_{j'}^{\perp}$. (2) $\lambda_{j}^{\perp} = \mu_{j}^{\perp} + 1$ and $\lambda_{j'}^{\perp} = \mu_{j'}^{\perp} + 1$. (3) $\lambda_{j}^{\perp} = \mu_{j}^{\perp}$ and $\lambda_{j'}^{\perp} = \mu_{j'}^{\perp} + 1$. (4) $\lambda_{j}^{\perp} = \mu_{j}^{\perp} + 1$ and $\lambda_{j'}^{\perp} = \mu_{j'}^{\perp}$. In the case (4), we set $\rho = \lambda$, that is, exchanging the *n*-square in the *j*-th column and the bottom square of *j'*-th column, we get ρ and ν from λ and μ . In the other cases, we first move the *n*-squares in the *j*-th and *j'*-th columns of λ (their existence depends on the cases (1)-(3)) vertically along the change from μ to ν . For example, in the case (2), the above operation is



Furthermore, if necessary, we apply a suitable column permutation as the following figure (in this situation, since $\mu_{j-1}^{\perp} = \nu_{j-1}^{\perp} \ge \nu_j^{\perp} = \mu_j^{\perp} + 1$, we have $\mu_{j-1}^{\perp} > \mu_j^{\perp}$ and there is no *n*-square in the (j-1)-st column of the left and the middle diagrams). In any cases, we have $\rho \trianglelefteq \lambda \in \mathcal{F}$ and $w_{\nu}(\rho) \le w_{\mu}(\lambda) \le k$, that is, ρ satisfies the expected property.



Proposition 3.6. Let $\lambda \in [P_{n+l-1}]_{\geq l}$, and $\mathcal{F} := \{\rho \in [P_{n+l-1}]_{\geq l} \mid \rho \leq \lambda\}$ the lower filter of $[P_{n+l-1}]_{\geq l}$. If $f \in I_{\lambda,l}$ is of the form $f = g_d x_n^d + \cdots + g_1 x_n + g_0$ with $g_0, \ldots, g_d \in K[x_1, \ldots, x_{n-1}]$ and $g_d \neq 0$, then g_0, \ldots, g_d belong to $I_{\mathcal{F}^d}$.

Proof. Assume that $g_m \notin I_{\mathcal{F}^d}$ for some m. By the classical case (i.e., when l = 1) of Corollary 2.7, there are some $\mathbf{a} \in K^{n-1}$ such that $\mu := \Lambda(\mathbf{a}) \notin \mathcal{F}^d$ and $g_m(\mathbf{a}) \neq 0$. If $\mu = (\mu_1, \ldots, \mu_p)$, there are distinct elements $\alpha_1, \ldots, \alpha_p \in K$ such that α_i appears μ_i times in \mathbf{a} for $i = 1, \ldots, p$.

We have

(3.4)
$$f = \sum_{\substack{T \in \operatorname{STab}(\lambda',l)\\\lambda' \in \mathcal{F}}} h_T \cdot f_T$$

for some $h_T \in S$. For $T \in \operatorname{STab}(\lambda', l)$, replacing *i* with a_i in *T* for all $1 \leq i \leq n-1$, and *n* with x_n , we get the tableau $T(\boldsymbol{a})$ whose entries are elements of $K \cup \{x_n\}$.

Take $\rho \in \langle \nu \rangle^l$ for some $\nu \in P_{n-1}$. We call a bijective filling \mathcal{T} of the squares of the Young diagram of ρ by the multiset

$$\{\overbrace{\alpha_1,\ldots,\alpha_1}^{\mu_1 \text{ copies}},\overbrace{\alpha_2,\ldots,\alpha_2}^{\mu_2 \text{ copies}},\ldots,\overbrace{\alpha_p,\ldots,\alpha_p}^{\mu_p \text{ copies}},\overbrace{x_n,\ldots,x_n}^{l \text{ copies}}\}$$

such that all *n*-squares are filled by x_n is called an **a**-tableau. We call ρ the shape of \mathcal{T} , and denote it by $\operatorname{sh}(\mathcal{T})$. We also denote ν by $\operatorname{sh}_{< n}(\mathcal{T})$. A typical example of an **a**-tableau is $T(\mathbf{a})$ given above. We say an **a**-tableau \mathcal{T} is regular, if the entries

(3.3)

in the *j*-th column of \mathcal{T} are all distinct for each *j*. Note that $T(\boldsymbol{a})$ is regular if and only if $f_T(\boldsymbol{a}, x_n) \neq 0$.

For all $\alpha \in K$, we can show that $\bar{\mu} := \Lambda^l(\boldsymbol{a}, \alpha)$ belongs to $\langle \mu \rangle^l$ (recall that $\mu = \Lambda(\boldsymbol{a})$). For example, if $\alpha \notin \{\alpha_1, \ldots, \alpha_p\}$, we have $\bar{\mu} = (\mu_1, \ldots, \mu_j, l, \mu_{j+1}, \ldots, \mu_p)$, where $j := \max\{i \mid \mu_i \geq l\}$, and hence $\bar{\mu}_i \geq \mu_i$ and $\mu_i^{\perp} \leq \bar{\mu}_i^{\perp} \leq \mu_i^{\perp} + 1$ for all i. The case $\alpha \in \{\alpha_1, \ldots, \alpha_p\}$ can be shown by a similar argument. Anyway, if $\langle \mu \rangle^l \cap \mathcal{F} = \emptyset$, then $\Lambda^l(\boldsymbol{a}, \alpha) \notin \mathcal{F}$, and hence $f(\boldsymbol{a}, \alpha) = 0$ by (3.1). So it implies that $f(\boldsymbol{a}, x_n) = 0$ and $g_m(\boldsymbol{a}) = 0$. This is a contradiction. So $\langle \mu \rangle^l \cap \mathcal{F}$ is non-empty, and it has the element λ with the minimum $w_{\mu}(-)$ by Lemma 3.2. Let $\widetilde{\mathcal{T}}$ be an \boldsymbol{a} -tableau of shape λ with $\operatorname{sh}_{\leq n}(\widetilde{\mathcal{T}}) = \mu$ such that all squares in the *i*-th row of μ are filled by α_i . Assume that, for each *i* with $1 \leq i \leq p$, α_i appears d_i times in squares above some *n*-squares in $\widetilde{\mathcal{T}}$. See Example 3.7 below. We have $\sum_{i=1}^p d_i = w_{\mu}(\lambda) > d$, where the inequality follows from that $\mu \notin \mathcal{F}^d$.

Claim. Let \mathcal{T} be a regular \boldsymbol{a} -tableau with $\operatorname{sh}(\mathcal{T}) \in \mathcal{F}$. For all i with $1 \leq i \leq p$, α_i appears at least d_i times in squares above some n-squares in \mathcal{T} .

Proof of Claim. Set $\nu := \operatorname{sh}_{\langle n}(\mathcal{T}) \in P_{n-1}$. We will prove the assertion by induction on ν with respect to the dominance order. Since \mathcal{T} is regular, it is easy to see that $\mu \leq \nu$ by the classical case (i.e., when l = 1) of Corollary 2.7. If $\mu = \nu$, applying suitable actions of column stabilizers (i.e., permutations of entries in the same column), we may assume that each square in the *i*-th row of \mathcal{T} is filled by α_i or x_n . So the assertion can be shown by an argument similar to the proof of Lemma 3.2. Next consider the case $\mu \triangleleft \nu$. As the induction hypothesis, we assume that the assertion holds for \mathcal{T}' with $\mu \leq \operatorname{sh}_{\langle n}(\mathcal{T}') \triangleleft \nu$.

To proceed with proof by contradiction, assume that \mathcal{T} does not satisfy the expected condition, that is, there is some s such that α_s appears less than d_s times in squares above some n-squares in \mathcal{T} . Since $\mu \triangleleft \nu$ now, there are some t and j, j' with j < j' such that α_t appears in the j'-th column of \mathcal{T} , but does not appear in the j-th column. If α_s has this property, we take s as t. We move the square in the j'-th column filled by α_t to the j-th column, and get the partition $\nu' \in P_{n-1}$ (a suitable column permutation might be required). The following condition is crucial.

(*) s = t holds, and the bottom of the *j*-th column of \mathcal{T} is an *n*-square, and that of the *j'*-th column is not.

In the case (*) is not satisfied, we move the *n*-squares in these columns (if they exist) vertically like (3.3), then apply a suitable column permutation if necessary (sometimes, we have to move the *j*-th column to left and/or the *j'*-th column to right). Finally, we get an *a*-tableau \mathcal{T}' with $\operatorname{sh}_{< n}(\mathcal{T}') = \nu'$. On the other hand, if (*) holds, we move the *n*-square in the *j*-th column to below the bottom of the *j'*-th column. Of course, do not forget to move the α_s -square in the *j'*-th column to the *j*-th column. Applying a suitable column permutation if necessary, we get \mathcal{T}' with $\operatorname{sh}_{< n}(\mathcal{T}') = \nu'$. In this case, we have $\operatorname{sh}(\mathcal{T}) = \operatorname{sh}(\mathcal{T}')$.

In both cases, \mathcal{T}' is regular, and α_s appears less than d_s times in squares above some *n*-squares in \mathcal{T}' . Moreover, we have $\operatorname{sh}(\mathcal{T}') \leq \operatorname{sh}(\mathcal{T})$, and hence $\operatorname{sh}(\mathcal{T}') \in \mathcal{F}$. Since $(\mu \leq) \nu' < \nu$, it contradicts the induction hypothesis. We back to the proof of the proposition itself. In (3.4), we have

$$f(\boldsymbol{a}, x_n) = \sum h_T(\boldsymbol{a}, x_n) \cdot f_T(\boldsymbol{a}, x_n)$$

If $f_T(\boldsymbol{a}, x_n) \neq 0$, then $T(\boldsymbol{a})$ is regular. So, by Claim, $f_T(\boldsymbol{a}, x_n)$ can be divided by

$$R(x_n) = \prod_{1 \le i \le p} (x_n - \alpha_i)^{d_i},$$

and $f(\boldsymbol{a}, x_n)$ itself can be divided by $R(x_n)$. While the degree of $f(\boldsymbol{a}, x_n)$ is at most d, we have deg $f(\boldsymbol{a}, x_n) \ge \deg R(x_n) = \sum_{i=1}^p d_i > d$. This is a contradiction. \Box

Example 3.7. Consider the case n = 8, l = 3, and take the lower filter given by $\mathcal{F} = \{\lambda \in [P_{10}]_{\geq 3} \mid \lambda \leq (4, 4, 2)\}$. If $\boldsymbol{a} = (\alpha_1, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3)$ (hence $\mu = (3, 2, 2)$), the \boldsymbol{a} -tableau $\widetilde{\mathcal{T}}$ given in the proof of Proposition 3.6 is as follows

α_1	α_1	α_1	x_8
α_2	α_2	x_8	
α_3	α_3		
x_8		_	

Above three n (= 8)-boxes, there are two copies of α_1 , so we have $d_1 = 2$. Similarly, since there is one α_2 (resp. α_3) above three *n*-boxes, we have $d_2 = 1$ (resp. $d_3 = 1$).

The following are examples of regular *a*-tableaux whose shape belong to \mathcal{F} . In each case, there are at least 2 (resp. 1) α_1 (resp. α_2 and α_3) above *n*-squares.

$\alpha_1 \alpha_1 \alpha_1 \alpha_1 x_8$	$\alpha_1 \alpha_1 \alpha_1 \alpha_2$	$\alpha_1 \alpha_1 \alpha_1 \alpha_2$
$\alpha_2 \alpha_2$	$\alpha_2 \alpha_3 x_8 x_8$	$\alpha_2 \alpha_3 \alpha_3 \alpha_3 x_8$
$\alpha_3 \alpha_3$	$\alpha_3 x_8$	$x_8 x_8$
$x_8 x_8$		

Theorem 3.8. For $\lambda \in [P_{n+l-1}]_{\geq l}$, set $\mathcal{F} := \{\rho \in [P_{n+l-1}]_{\geq l} \mid \rho \leq \lambda\}$ be the lower filter. Then $\{f_T \mid T \in \operatorname{STab}(\rho, l), \rho \in \mathcal{F}\}$ is a Gröbner basis of $I_{\lambda,l}$.

Proof. It suffices to show that the initial monomial $\operatorname{in}(f)$ for all $0 \neq f \in I_{\lambda,l}$ can be divided by $\operatorname{in}(f_T)$ for some $T \in \operatorname{STab}(\rho, l)$ with $\rho \in \mathcal{F}$. Let $f = g_d x_n^d + \cdots + g_1 x_n + g_0$, where $g_i \in K[x_1, \ldots, x_{n-1}]$ and $g_d \neq 0$. By Proposition 3.6, one has $g_d \in I_{\mathcal{F}^d}$. By Theorem 1.2, $\{f_T \mid T \in \operatorname{STab}(\mu), \mu \in \mathcal{F}^d\}$ is a Gröbner basis of $I_{\mathcal{F}^d}$ (since we fix the monomial order, it is enough to consider standard tableaux, see [11, Remark 3.5]), and there is a tableau $T \in \operatorname{STab}(\mu)$ for some $\mu \in \mathcal{F}^d$ such that $\operatorname{in}(f_T)$ divides $\operatorname{in}(g_d)$. So we can take $\rho \in \langle \mu \rangle^l \cap \mathcal{F}$ with $e := w_\mu(\rho) \leq d$. Let us consider the tableau $T' \in \operatorname{STab}(\rho, l)$ such that the image of each $i = 1, \ldots, n-1$ is same for T and T'. Since we have $\operatorname{in}(f) = x_n^d \cdot \operatorname{in}(g_d)$ and $\operatorname{in}(f_{T'}) = x_n^e \cdot \operatorname{in}(f_T)$, $\operatorname{in}(f_{T'})$ divides $\operatorname{in}(f)$. \Box

Example 3.9. Contrary to Theorem 2.6, Theorem 3.8 cannot be generalized to the ideal $I_{\mathcal{F},l} := (f_T \mid T \in \text{Tab}(\lambda, l), \lambda \in \mathcal{F})$ for a lower filter $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$. For example, if $\mathcal{F} \subset [P_8]_{\geq 2}$ is the lower filter whose maximal elements are (4, 2, 1, 1) and (3, 3, 2), then $x_4^2 x_5^3 x_6 x_7^2$ is a minimal generator of $\text{in}(I_{\mathcal{F},2})$, but this cannot be represented in the form of $\text{in}(f_T)$ for $T \in \text{STab}(\lambda, 2)$.

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The following fact might be well-known to the specialist, and is stated in [11] without proof. This time, we give a proof for the reader's convenience.

Lemma 3.10. Let $I \subset S = K[x_1, \ldots, x_n]$ be a graded ideal, and $G \subset I$ a Gröbner basis of I with respect to the lexicographic order < with $x_1 < x_2 < \cdots < x_n$. If all elements of G are products of linear forms, then G is a Gröbner basis of I with respect to any monomial order \prec with $x_1 \prec x_2 \prec \cdots \prec x_n$.

The assumption that I is graded is unnecessary, but we add it here for the simplicity.

Proof. Since $g \in G$ is a product of linear forms, we have $\operatorname{in}_{\prec}(g) = \operatorname{in}_{\prec}(g)$, and hence $\operatorname{in}_{\prec}(I) = (\operatorname{in}_{\prec}(g) \mid g \in G) = (\operatorname{in}_{\prec}(g) \mid g \in G) \subset \operatorname{in}_{\prec}(I)$.

Since $\operatorname{in}_{\prec}(I)$ and $\operatorname{in}_{\prec}(I)$ have the same Hilbert function (in fact, they have the same Hilbert function as I itself), we have $\operatorname{in}_{\prec}(I) = \operatorname{in}_{\prec}(I)$. It implies that G is a Gröbner basis of I with respect to \prec .

Corollary 3.11. Let $\lambda \in [P_{n+l-1}]_{\geq l}$. With respect to a monomial order in which x_1 is either the smallest or the largest among the variables x_1, \ldots, x_n , $\{f_T \mid T \in \text{Tab}(l, \rho), \rho \leq \lambda\}$ is a Gröbner basis of $I_{l,\lambda}$.

Since we consider several monomial orders, we have to treat $\text{Tab}(l, \lambda)$, not $\text{STab}(l, \lambda)$.

Proof. First, we consider the case x_1 is the smallest among x_1, \ldots, x_n . Since the ideal $I_{l,\lambda}$ is symmetric for variables x_2, \ldots, x_n , and Specht polynomials are products of linear forms, we may assume that our monomial order is the lexicographic order with $x_1 < \cdots < x_n$ by Lemma 3.10, and the assertion follows from Corollary 2.7. Similarly, if x_1 is the largest, we may assume that our monomial order is the lexicographic order graphic order with $x_1 > \cdots > x_n$, and the assertion follows from Theorem 3.8. \Box

Example 3.12. For $\lambda = (3,3)$, $I_{2,\lambda}$ is generated by 3 elements of degree 3. With respect to a monomial order in which x_1 is the smallest, $in(I_{2,\lambda})$ is minimally generated by 3 elements of degree 3 and 2 elements of degree 4. On the other hand, with respect to an order in which x_1 is the largest, $in(I_{2,\lambda})$ is minimally generated by 3 elements of degree 3, 3 elements of degree 4, and an element of degree 6. Computer experiment suggests that $in(I_{l,\lambda})$ with respect to an order in which x_1 is the smallest requires fewer generators.

Problem 3.13. With the notation of Corollary 3.11, is $\{f_T \mid T \in \text{Tab}(l, \rho), \rho \leq \lambda\}$ a universal Gröbner basis of $I_{l,\lambda}$?

We have computed several partitions λ up to n = 8 using SageMath and Macaulay2, and we have not found a counter example yet.

4. A GENERALIZATION OF THE CASE $\#Y_1 \ge 2$ and $\#Y_2 = \cdots = \#Y_l = 1$

In this section, we fix a positive integer m with $1 \le m \le n$, and set

$$\Delta_m := \Delta(\{1, \dots, m\}) = \prod_{1 \le i < j \le m} (x_i - x_j).$$

For $T \in \operatorname{Tab}(l, \lambda)$ with $[P_{n+l-1}]_{>l}$, set

 $f_{m,T} := \operatorname{lcm} \{ f_T, \Delta_m \} \in S \text{ and } I_{l,m,\lambda} := (f_{m,T} \mid T \in \operatorname{Tab}(l,\lambda)) \subset S.$

Note that $I_{l,1,\lambda} = I_{l,\lambda}$ and $I_{l,n,\lambda} = (\Delta_n)$.

Example 4.1. Even if l = 1, $I_{l,m,\lambda}$ is not a radical ideal in general, while their generators are squarefree products of linear forms $(x_i - x_j)$. For example, if $\lambda = (2, 2)$, we have

$$I_{1,3,\lambda} = (\Delta_3 \cdot (x_1 - x_4), \Delta_3 \cdot (x_2 - x_4), \Delta_3 \cdot (x_3 - x_4)),$$

where $\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. (Note that an analog of Lemma 2.3 does not hold here. So we have to consider a non-standard tableau also to generate $I_{l,m,\lambda}$.) Clearly, $\Delta_3 \notin I_{1,3,\lambda}$, but we can show that $\Delta_3 \in \sqrt{I_{1,3,\lambda}}$ by Lemma 4.2 below. Moreover, the statement corresponding to Lemma 2.5 does not hold for $I_{l,m,\lambda}$. In fact, if $\lambda = (2, 2)$ and $\mu = (2, 1, 1)$, then $\mu \triangleleft \lambda$, but $I_{1,3,\mu} = (\Delta_3) \notin I_{1,3,\lambda}$.

However, we have the following.

Lemma 4.2. For $\lambda, \mu \in [P_{n+l-1}]_{\geq l}$ with $\lambda \supseteq \mu$, we have $\sqrt{I_{l,m,\lambda}} \supset I_{l,m,\mu}$.

Proof. It suffices to show that $f_{m,T} \in \sqrt{I_{l,m,\lambda}}$ for all $T \in \text{Tab}(l,\mu)$. By Lemma 2.5, there are some $k \in \mathbb{N}, T_1, \ldots, T_k \in \text{Tab}(l,\lambda)$ and $g_1, \ldots, g_k \in S$ such that $f_T = \sum g_i f_{T_i}$. Multiplying Δ_m to both sides, we have

$$\Delta_m \cdot f_T = \sum g_i \cdot (\Delta_m \cdot f_{T_i}).$$

Since f_{m,T_i} divides $\Delta_m \cdot f_{T_i}$, we have $\Delta_m \cdot f_T \in I_{l,m,\lambda}$. However, since $\Delta_m \cdot f_T$ divides $(f_{m,T})^2$, we have $(f_{m,T})^2 \in I_{l,m,\lambda}$.

For a *lower* filter $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$, set

$$G_{l,m,\mathcal{F}} := \{ f_{m,T} \mid T \in \operatorname{Tab}(l,\lambda), \lambda \in \mathcal{F} \} \text{ and } I_{l,m,\mathcal{F}} := (G_{l,m,\mathcal{F}}) = \sum_{\lambda \in \mathcal{F}} I_{l,m,\lambda}.$$

For an *upper* filter $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$ with the lower filter $\mathcal{F}^{\mathsf{c}} := [P_{n+l-1}]_{\geq l} \setminus \mathcal{F}$, we consider the ideal

$$J_{l,m,\mathcal{F}} := (\Delta_m) \cap J_{l,\mathcal{F}} \ (= (\Delta_m) \cap I_{l,\mathcal{F}^c}).$$

Since both (Δ_m) and $J_{l,\mathcal{F}}$ are radical ideals, so is $J_{l,m,\mathcal{F}}$. Since $J_{l,m,\mathcal{F}} \subset (\Delta_m)$, the codimension of $J_{l,m,\mathcal{F}}$ for $m \geq 2$ is 1 (unless $\mathcal{F} = [P_{n+l-1}]_{\geq l}$, equivalently, $J_{l,m,\mathcal{F}} = 0$).

Theorem 4.3. Let $\mathcal{F} \subseteq [P_{n+l-1}]_{\geq l}$ be a lower filter, and $\mathcal{F}^{\mathsf{c}} := [P_{n+l-1}]_{\geq l} \setminus \mathcal{F}$ its complement. Then $G_{l,m,\mathcal{F}}$ is a Gröbner basis of $J_{l,m,\mathcal{F}^{\mathsf{c}}}$. Hence $J_{l,m,\mathcal{F}^{\mathsf{c}}} = I_{l,m,\mathcal{F}}$, and $I_{l,m,\mathcal{F}}$ is a radical ideal.

Let us prepare the proof of Theorem 4.3.

Lemma 4.4. Let $\mathcal{F} \subset [P_{n+l-1}]_{\geq l}$ be an upper filter, and let f be a polynomial in $J_{l,m,\mathcal{F}}$ of the form

$$f = g_d x_n^d + \dots + g_1 x_n + g_0,$$

where $g_0, \ldots, g_d \in K[x_1, \ldots, x_{n-1}]$ and $g_d \neq 0$. If m < n, then g_0, \ldots, g_d belong to $J_{l,m,\mathcal{F}_{d+1}}$.

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Proof. Here we use the same notation as in the proof of Lemma 2.9. Take $\boldsymbol{a} = (a_1, \ldots, a_{n-1}) \in K^{n-1}$. Since $f \in (\Delta_m)$, if $a_i = a_j$ for some $1 \leq i < j \leq m$, then $f(\boldsymbol{a}, \alpha) = 0$ for all $\alpha \in K$, and hence $g_i(\boldsymbol{a}) = 0$ for all i. It means that each g_i can be divided by Δ_m in $K[x_1, \ldots, x_{n-1}]$. So it remains to show that $g_i \in J_{l,\mathcal{F}_{d+1}}$, but it follows from Lemma 2.9, since $f \in J_{l,\mathcal{F}}$.

The proof of Theorem 4.3. First, we show that $G_{l,m,\mathcal{F}} \subset J_{l,m,\mathcal{F}^c}$. For any $f_{m,T} \in G_{l,m,\mathcal{F}}$, it is clear that $f_{m,T} \in (\Delta_m)$, and we have $f_{m,T} \in (f_T) \subset J_{l,\mathcal{F}^c}$ by Theorem 2.6. Hence $f_{m,T} \in J_{l,m,\mathcal{F}^c}$.

So it remains to show that, for any $0 \neq f \in J_{l,m,\mathcal{F}^c}$, there is some $f_{m,T} \in G_{l,m,\mathcal{F}}$ such that $in(f_{m,T})$ divides in(f), but it can be done by induction on n-m (we fix m) in the same way as in the proof of Theorem 2.6, while we use Lemma 4.4 instead of Lemma 2.9.

The following corollary immediately follows from Theorem 4.3.

Corollary 4.5. For $\lambda \in [P_{n+l-1}]_{\geq l}$,

$$\bigcup_{\substack{\mu \in [P_{n+l-1}] \ge l \\ \mu \le \lambda}} G_{l,m,\mu}$$

is a Gröbner basis of $\sqrt{I_{l,m,\lambda}} = J_{l,m,\mathcal{F}}$, where \mathcal{F} is the upper filter $\{\nu \in [P_{n+l-1}]_{\geq l} \mid \nu \not \leq \lambda\}$. In particular,

$$\sqrt{I_{l,m,\lambda}} = \sum_{\substack{\mu \in [P_{n+l-1}] \ge l \\ \mu \le \lambda}} I_{l,m,\mu}.$$

Remark 4.6. If $\lambda = (\lambda_1, \ldots, \lambda_p) \in P_{n+l-1}$ is of the form $\lambda_1 = \cdots = \lambda_{p-1} = k-1$ for some k > l, then our $\sqrt{I_{l,m,\lambda}}$ (= $\sum_{\mu \leq \lambda} I_{l,m,\mu}$) coincides with the Li-Li ideal $I_{\mathcal{Y}}$ for $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_{k-1})$ with $Y_1 = \{1, 2, \ldots, m\}, Y_2 = \cdots = Y_l = \{1\}$ and $Y_{l+1} = \cdots = Y_{k-1} = \emptyset$ in the notation of the Introduction.

Proposition 4.7. $I_{l,m,\lambda}$ is a radical ideal for $m \leq 2$.

Proof. The case m = 1 follows from Theorem 2.6. So we treat the case m = 2. By Theorem 4.3, it suffices to show that $f_{2,T} \in I_{l,2,\lambda}$ for all $T \in \text{Tab}(l,\mu)$ with $\mu \leq \lambda$. If the letters 2 and (some copy of) 1 are not in the same column of T, then we have $f_{2,T} = (x_1 - x_2)f_T$, and if they are in the same column, then we have $f_{2,T} = f_T$. We first treat the former case. Since $I_{l,\mu} \subset I_{l,\lambda}$ by Lemma 2.5, there are $g_1, \ldots, g_k \in S$ and $T_1, \ldots, T_k \in \text{Tab}(l,\lambda)$ such that $f_T = \sum_{i=1}^k g_i f_{T_i}$. Multiplying $(x_1 - x_2)$ to the both sides, we have

$$f_{2,T} = (x_1 - x_2)f_T = \sum_{i=1}^k g_i \cdot (x_1 - x_2)f_{T_i}.$$

Since f_{2,T_i} divides $(x_1 - x_2)f_{T_i}$, we have $f_{2,T} \in I_{l,2,\lambda}$. So the case when 1 and 2 are in the same column (equivalently, $f_{2,T} = f_T$) remains. We may assume that λ covers μ , and we want to modify the argument of the proof of Lemma 2.5, which shows that $I_{l,\mu} \subset I_{l,\lambda}$. In the sequel, we use the same notation as there.

The crucial case is that $1, 2 \in A$ (we may assume that $a_1 = 1, a_2 = 2$) and $1 \notin B$. In fact, in other cases, it is easy to see that $f_{2,T_i} = f_{T_i}$ for all *i*. By (2.3), we have

$$f_T = \sum_{k-k' \le i \le k} (-1)^{i-k+k'} (x_1 - x_{a_i}) f_{T_i}$$

and $T_i \in \text{Tab}(l, \lambda)$ for all *i*. For $i \geq 3$, the letters 1 and 2 stay in the same column of T_i , and we have $f_{2,T_i} = f_{T_i}$. So the case $k - k' \geq 3$ is easy, and we may assume that k - k' = 2. Then, among T_2, \ldots, T_k , only T_2 does *not* have 1 and 2 in the same column. Hence

$$f_{2,T} = f_T = (x_1 - x_2) f_{T_2} + \sum_{3 \le i \le k} (-1)^i (x_1 - x_{a_i}) f_{T_i}$$

= $f_{2,T_2} + \sum_{3 \le i \le k} (-1)^i (x_1 - x_{a_i}) f_{2,T_i} \in I_{l,2,\lambda}.$

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