# GRÖBNER BASES OF RADICAL LI-LI TYPE IDEALS ASSOCIATED WITH PARTITIONS 

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#### Abstract

For a partition $\lambda$ of $n$, the Specht ideal $I_{\lambda} \subset K\left[x_{1}, \ldots, x_{n}\right]$ is the ideal generated by all Specht polynomials of shape $\lambda$. In their unpublished manuscript, Haiman and Woo showed that $I_{\lambda}$ is a radical ideal, and gave its universal Gröbner basis (Murai et al. published a quick proof of this result). On the other hand, an old paper of Li and Li studied analogous ideals, while their ideals are not always radical. The present paper introduces a class of ideals generalizing both Specht ideals and radical Li-Li ideals, and studies their radicalness and Gröbner bases.


## 1. Introduction

Let $S=K\left[x_{1}, \ldots, x_{n}\right]$ be a polynomial ring over an infinite field $K$. For a subset $A=\left\{a_{1}, a_{2}, \ldots, a_{m}\right\}$ of $[n]:=\{1,2, \ldots, n\}$, let

$$
\Delta(A):=\prod_{1 \leq i<j \leq m}\left(x_{a_{i}}-x_{a_{j}}\right) \in S
$$

be the difference product. For a sequence of subsets $\mathcal{Y}=\left(Y_{1}, Y_{2}, \ldots, Y_{k-1}\right)$ with $[n] \supset Y_{1} \supset Y_{2} \supset \cdots \supset Y_{k-1}$, Li and Li [8] studied the ideal

$$
\begin{equation*}
I_{\mathcal{Y}}:=\left(\prod_{i=1}^{k-1} \Delta\left(X_{i}\right) \mid X_{i} \supset Y_{i} \text { for all } i, \bigcup_{i=1}^{k-1} X_{i}=[n]\right) \tag{1.1}
\end{equation*}
$$

of $S$ (more precisely, the polynomial ring in [8] is $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$ ). Among other things, they showed the following.

Theorem 1.1 (c.f. Li-Li [8, Theorem 2]). With the above notation, $I_{\mathcal{Y}}$ is a radical ideal if and only if $\# Y_{2} \leq 1$.

A partition of a positive integer $n$ is a non-increasing sequence of positive integers $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right)$ with $\lambda_{1}+\cdots+\lambda_{p}=n$. Let $P_{n}$ be the set of all partitions of $n$. A partition $\lambda$ is frequently represented by its Young diagram. For example, $(4,2,1)$ is represented as $\boxminus \square$. A (Young) tableau of shape $\lambda \in P_{n}$ is a bijective filling of the

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squares of the Young diagram of $\lambda$ by the integers in $[n]$. For example,

is a tableau of shape $(4,2,1)$. Let $\operatorname{Tab}(\lambda)$ be the set of all tableaux of shape $\lambda$. Recall that the Specht polynomial $f_{T}$ of $T \in \operatorname{Tab}(\lambda)$ is $\prod_{j=1}^{\lambda_{1}} \Delta(T(j))$, where $T(j)$ is the set of the entries of the $j$-th column of $T$ (here the entry in the $i$-th row is the $i$-th element of $T(j))$. For example, if $T$ is the above tableau, then $f_{T}=$ $\left(x_{4}-x_{5}\right)\left(x_{4}-x_{6}\right)\left(x_{5}-x_{6}\right)\left(x_{3}-x_{2}\right)$.

We call the ideal

$$
I_{\lambda}:=\left(f_{T} \mid T \in \operatorname{Tab}(\lambda)\right) \subset S
$$

the Specht ideal of $\lambda$. These ideals have been studied from several points of view (and under several names and characterizations), see for example, [1, 9, 10, 13]. The following is an unpublished result of Haiman and Woo (6]), to which Murai, Ohsugi and the second author ([11]) published a quick proof.

Theorem 1.2 (Haiman-Woo [6], see also [11]). If $\mathcal{F} \subset P_{n}$ is a lower filter with respect to the dominance order $\unlhd$, then $I_{\mathcal{F}}:=\sum_{\lambda \in \mathcal{F}} I_{\lambda}$ is a radical ideal, for which $\left\{f_{T} \mid T \in \operatorname{Tab}(\mu), \mu \in \mathcal{F}\right\}$ forms a universal Gröbner basis (i.e., a Gröbner basis with respect to all monomial orders). In particular, $I_{\lambda}$ is a radical ideal, for which $\left\{f_{T} \mid T \in \operatorname{Tab}(\mu), \mu \unlhd \lambda\right\}$ forms a universal Gröbner basis.

Let us explain why the second assertion follows form the first. Since $\lambda \unrhd \mu$ for $\lambda, \mu \in P_{n}$ implies $I_{\lambda} \supset I_{\mu}$ (c.f. Lemma [2.5), we have $I_{\lambda}=I_{\mathcal{F}}$ for the lower filter $\mathcal{F}:=\left\{\mu \in P_{n} \mid \mu \unlhd \lambda\right\}$.

The Li-Li ideals $I_{\mathcal{Y}}$ and the Specht ideals $I_{\lambda}$ share common examples. In fact, for $\mathcal{Y}=\left(Y_{1}, Y_{2}, \ldots, Y_{k-1}\right)$ with $\# Y_{1} \leq 1, Y_{2}=\cdots=Y_{k-1}=\emptyset$ and $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right) \in P_{n}$ with $\lambda_{1}=\cdots=\lambda_{p-1}=k-1$, we have $I_{\mathcal{Y}}=I_{\lambda}$ by [8, Corollary 3.2].

In this paper, we study a common generalization of the radical $\mathrm{Li}-\mathrm{Li}$ ideals and the Specht ideals, for which almost direct analogs of Theorem 1.2 hold. For example, in Sections 2 and 3, we take a positive integer $l$, and a partition $\lambda \in P_{n+l-1}$ with $\lambda_{1} \geq l$, and consider tableaux like

| 1 | 1 | 1 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 5 | 8 |  |  |  |
| 6 | 7 |  |  |  |  |

( $l=4$ in this case). Using these tableau, we define the ideal $I_{l, \lambda}$.
The symmetric group $\mathcal{S}_{n-1}$ of the set $\{2, \ldots, n\}$ still acts on $I_{l, \lambda}$, so our ideals have representation theoretic interest. The following are other motivations of the present paper.
(1) Recently, the defining ideals of subspace arrangements have been intensely studied (c.f. [2, 4, [14]). Our $I_{l, \lambda}$ and its generalization $\sqrt{I_{l, m, \lambda}}$ introduced in Section 4 give new classes of these ideals. Note that $I_{l, m, \lambda}$ is not a radical ideal in general, while Corollary 4.5 gives the generators of its radical explicitly.
(2) A universal Gröbner basis is very important, since it is closely related to the Gröbner fan. While we can use a computer for explicit examples, it is extremely difficult to construct universal Gröbner bases for some (infinite) family of ideals. Theorem 1.2 gives universal Gröbner bases of Specht ideals $I_{\lambda}$. However, since $I_{\lambda}$ are symmetric, this case is exceptional. So it must be very interesting, if the Gröbner bases of non-symmetric ideals $I_{l, \lambda}$ given in Theorem 2.6 are universal. Corollary 3.11 is an affirmative evidence.
(3) One of the motivations of the paper [8] of Li and Li is an application to graph theory (see [5] for further connection to Gröbner bases theory). We expect that the present paper gives a new inslight to this direction.
In the present paper, for the convention and notation of the Gröbner bases theory, we basically follow [7, Chapter 1].

## 2. A generalization of the case $\# Y_{1}=\cdots=\# Y_{l}=1$

We keep the same notation as Introduction, and fix a positive integer $l$. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l}:=\left\{\lambda \in P_{n+l-1} \mid \lambda_{1} \geq l\right\}$, we consider a bijective filling of the squares of the Young diagram of $\lambda$ by the multiset $\{\overbrace{1, \ldots, 1}^{l \text { copies }}, 2, \ldots, n\}$ such that no two copies of 1 are contained in the same column. Let $\operatorname{Tab}(l, \lambda)$ be the set of such tableaux. For example, the tableau (1.2) above is an element of $\operatorname{Tab}(4, \lambda)$ for $\lambda=(6,3,2)$ (moreover, this is a standard tableau defined below). The Specht polynomial $f_{T}$ of $T \in \operatorname{Tab}(l, \lambda)$ is defined by the same way as in the classical case. For example, if $T$ is the one in (1.2), then $f_{T}=\left(x_{1}-x_{4}\right)\left(x_{1}-x_{6}\right)\left(x_{4}-x_{6}\right)\left(x_{1}-x_{5}\right)\left(x_{1}-x_{7}\right)\left(x_{5}-\right.$ $\left.x_{7}\right)\left(x_{1}-x_{8}\right)$. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$, consider the ideal

$$
I_{l, \lambda}:=\left(f_{T} \mid T \in \operatorname{Tab}(l, \lambda)\right)
$$

of $S$. Clearly, $\operatorname{Tab}(1, \lambda)=\operatorname{Tab}(\lambda)$ and $I_{1, \lambda}=I_{\lambda}$.
For $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right), \mu=\left(\mu_{1}, \ldots, \mu_{q}\right) \in P_{m}$, we write $\lambda \unrhd \mu$ if $\lambda$ is equal to or larger than $\mu$ with respect to the dominance order, that is,

$$
\lambda_{1}+\cdots+\lambda_{i} \geq \mu_{1}+\cdots+\mu_{i} \quad \text { for } i=1,2, \ldots, \min \{p, q\} .
$$

In what follows, we regard $\left[P_{n+l-1}\right]_{\geq l}$ as a subposet of $P_{n+l-1}$.
For $\lambda \in P_{m}$ and $j$ with $1 \leq j \leq \lambda_{1}$, let $\lambda_{j}^{\perp}$ be the length of the $j$-th column of the Young diagram of $\lambda$. Then $\lambda^{\perp}=\left(\lambda_{1}^{\perp}, \lambda_{2}^{\perp}, \ldots\right)$ is a partition of $m$ again. It is a classical result that $\lambda \unrhd \mu$ if and only if $\lambda^{\perp} \unlhd \mu^{\perp}$.

Remark 2.1. By [3, Proposition 2.3], if $\lambda$ covers $\mu$ (i.e., $\lambda \triangleright \mu$, and there is no other partition between them), then there are two integers $i, i^{\prime}$ with $i<i^{\prime}$ such that $\mu_{i}=\lambda_{i}-1, \mu_{i^{\prime}}=\lambda_{i^{\prime}}+1$, and $\mu_{k}=\lambda_{k}$ for all $k \neq i, i^{\prime}$, equivalently, there are two integers $j, j^{\prime}$ with $j<j^{\prime}$ such that $\mu_{j}^{\perp}=\lambda_{j}^{\perp}+1, \mu_{j^{\prime}}^{\perp}=\lambda_{j^{\prime}}^{\perp}-1$, and $\mu_{k}^{\perp}=\lambda_{k}^{\perp}$ for all $k \neq j, j^{\prime}$. Clearly, $\mu_{j}^{\perp} \geq \mu_{j^{\prime}}^{\perp}+2$ in this case. Here, we allow the case $i^{\prime}$ is larger than the length $p$ of $\lambda$, where we set $\lambda_{i^{\prime}}=0$. Similarly, the case $\mu_{j^{\prime}}^{\perp}=0$ might occur.
Remark 2.2. By a similar argument to the proof of Lemma 2.3, to generate $I_{l, \lambda}$, it suffices to use $T \in \operatorname{Tab}(l, \lambda)$ such that the left most $l$ squares in the first row are
filled by 1. So, in manner of (1.1), the ideal $I_{l, \lambda}$ can be represented as follows.

$$
I_{l, \lambda}=\left(\prod_{i=1}^{\lambda_{1}} \Delta\left(X_{i}\right) \mid 1 \in X_{i} \text { for } 1 \leq i \leq l, \# X_{i}=\lambda_{i}^{\perp} \text { for all } i, \bigcup_{i=1}^{\lambda_{1}} X_{i}=[n]\right)
$$

Convention. Throughout this paper, when we consider the Gröbner bases, we use the lexicographic order with $x_{1}<\cdots<x_{n}$ unless otherwise specified (see Lemma 3.10 below, which states that only the order among the variables $x_{1}, \ldots, x_{n}$ matters for our Gröbner bases), and the initial monomial in $_{<}(f)$ of $0 \neq f \in S$ will be simply denoted by in $(f)$.

For $T \in \operatorname{Tab}(l, \lambda)$, recall that $T(j)$ is the set of the entries of the $j$-th column of $T$. If $\sigma$ is a permutation on $T(j)$, we have $f_{\sigma T}=\operatorname{sgn}(\sigma) f_{T}$ for each $j$. In this sense, to consider $f_{T}$, we may assume that $T$ is column standard, that is, all columns are increasing from top to bottom (in particular, all 1's appear in the 1st row).

If $T$ is column standard and the number $i$ is in the $d_{i}$-th row of $T$, we have

$$
\begin{equation*}
\operatorname{in}\left(f_{T}\right)=\prod_{i=1}^{n} x_{i}^{d_{i}-1} \tag{2.1}
\end{equation*}
$$

(recall our convention on the monomial order).
If a column standard tableau $T \in \operatorname{Tab}(l, \lambda)$ is also row semi-standard (i.e., all rows are non-decreasing from left to right), we say $T$ is standard. Let $\operatorname{STab}(l, \lambda)$ be the set of standard tableaux in $\operatorname{Tab}(l, \lambda)$. We simply denote $\operatorname{STab}(1, \lambda)$ by $\operatorname{STab}(\lambda)$. The next result is very classical when $l=1$.
Lemma 2.3. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l},\left\{f_{T} \mid T \in \operatorname{STab}(l, \lambda)\right\}$ forms a basis of the vector space $V$ spanned by $\left\{f_{T} \mid T \in \operatorname{Tab}(l, \lambda)\right\}$. Hence $\left\{f_{T} \mid T \in \operatorname{STab}(l, \lambda)\right\}$ is a minimal system of generators of $I_{l, \lambda}$.

Proof. In the classical case (i.e., when $l=1$ ), we can rewrite $f_{T}$ for $T \in \operatorname{Tab}(\lambda)$ as a linear combination of $f_{T_{i}}$ 's for $T_{i} \in \operatorname{STab}(\lambda)$ repeatedly using the relations given by Garnir elements (see [12, §2.6]). Such a relation concerns the $j$-th and the ( $j+1$ )-st columns of $T$. The classical argument directly works in our case unless both of these columns contain 1. So we assume that both columns have 1. Since $f_{T}=\prod_{j=1}^{\lambda_{1}} \Delta(T(j))$, we can concentrate on the $j$-th and $(j+1)$-st columns of $T$, and may assume that $T$ consists of two columns (i.e., $\lambda$ is of the form $\left(2, \lambda_{2}, \ldots, \lambda_{p}\right) \in$ $\left.P_{n+2-1}=P_{n+1}\right)$ and $l=2$. Set $\widetilde{\lambda}:=\left(\lambda_{2}, \ldots, \lambda_{p}\right) \in P_{n-1}$. Removing the first row from $T \in \operatorname{Tab}(2, \lambda)$, we have $\widetilde{T} \in \operatorname{Tab}(\widetilde{\lambda})$ (the set of the entries of $\widetilde{T}$ is $\{2, \ldots, n\}$ ). The converse operation $\operatorname{Tab}(\widetilde{\lambda}) \ni \widetilde{T} \longmapsto T \in \operatorname{Tab}(2, \lambda)$ also makes sense. Clearly, $f_{T}=\left(\prod_{i=2}^{n}\left(x_{1}-x_{i}\right)\right) \cdot f_{\widetilde{T}}$. Multiplying $\prod_{i=2}^{n}\left(x_{1}-x_{i}\right)$ to both sides of a Garnir relation $f_{\widetilde{T}}=\sum_{i=1}^{k} \pm f_{\widetilde{T}_{i}}\left(\widetilde{T}, \widetilde{T}_{i} \in \operatorname{Tab}(\widetilde{\lambda})\right)$, we have the relation $f_{T}=\sum_{i=1}^{k} \pm f_{T_{i}}$ $\left(T, T_{i} \in \operatorname{Tab}(2, \lambda)\right)$. As in the classical case, $T_{i}$ need not to be standard, but is closer to standard than $T$. Using these relations, the argument in [12, §2.6] is applicable to our case, and we can show that $\left\{f_{T} \mid T \in \operatorname{STab}(l, \lambda)\right\}$ spans $V$.

As we have seen in (2.1), $\operatorname{in}\left(f_{T}\right) \neq \operatorname{in}\left(f_{T^{\prime}}\right)$ holds for distinct $T, T^{\prime} \in \operatorname{STab}(l, \lambda)$. So $\left\{f_{T} \mid T \in \operatorname{STab}(l, \lambda)\right\}$ is linearly independent.

For $\boldsymbol{a}=\left(a_{1}, \ldots, a_{n}\right) \in K^{n}$, there are distinct $\alpha_{1}, \ldots, \alpha_{p} \in K$ with $\left\{\alpha_{1}, \ldots, \alpha_{p}\right\}=$ $\left\{a_{1}, \ldots, a_{n}\right\}$ as sets. Now we can define the partition $\mu=\left(\mu_{1}, \ldots, \mu_{p}\right) \in P_{n}$ such that $\alpha_{i}$ appears $\mu_{i}$ times in $\left(a_{1}, \ldots, a_{n}\right)$ for each $i$. This partition $\mu$ will be denoted by $\Lambda(\boldsymbol{a})$. For example, $\Lambda((1,0,2,1,2,2))=(3,2,1)$.

For $\boldsymbol{a} \in K^{n}$, set $\boldsymbol{a}_{(l)}:=(\overbrace{a_{1}, \ldots, a_{1}}^{l \text { copies }}, a_{2}, \ldots, a_{n}) \in K^{n+l-1}$ and $\Lambda_{l}(\boldsymbol{a}):=\Lambda\left(\boldsymbol{a}_{(l)}\right) \in$ $\left[P_{n+l-1}\right]_{\geq l}$. For example, if $\boldsymbol{a}=(1,0,2,1,2,2)$, then $\boldsymbol{a}_{(3)}=(1,1,1,0,2,1,2,2)$ and $\Lambda_{3}(\boldsymbol{a})=(4,3,1)$. When $l=1$, the following result is classical.

Lemma 2.4 (c.f. [11, Lemma 2.1.]). Let $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$ and $T \in \operatorname{Tab}(l, \lambda)$. For $\boldsymbol{a} \in K^{n}$ with $\Lambda_{l}(\boldsymbol{a}) \nexists \lambda$, we have $f_{T}(\boldsymbol{a})=0$.

Proof. For $\boldsymbol{a}=\left(a_{1}, \ldots, a_{n}\right) \in K^{n}$, replacing $i$ with $a_{i}$ for each $i$ in $T$, we have a tableau $T(\boldsymbol{a})$, whose entries are elements in $K$. It is easy to see that $f(\boldsymbol{a}) \neq 0$ if and only if the entries in each column of $T(\boldsymbol{a})$ are all distinct. So the assertion follows from the same argument as [11, Lemma 2.1].

Lemma 2.5 (c.f. [10, Theorem 1.1]). For $\lambda, \mu \in\left[P_{n+l-1}\right]_{\geq l}$ with $\lambda \unrhd \mu$, we have $I_{l, \lambda} \supset I_{l, \mu}$.

Proof. The proof is essentially same as the classical case, while we have to care about one point. First, we will recall a basic property of difference products. For subsets $A=\left\{a_{1}, a_{2}, \ldots, a_{k}\right\}$ and $B=\left\{b_{1}, b_{2}, \ldots, b_{k^{\prime}}\right\}$ of $[n]$ with $k \geq k^{\prime}+2$, we have
$\Delta(A) \cdot \Delta(B)=\sum_{k-k^{\prime} \leq i \leq k}(-1)^{i-k+k^{\prime}}\left[\Delta\left(A \backslash\left\{a_{i}\right\}\right) \cdot \Delta\left(B \cup\left\{a_{i}\right\}\right) \cdot \prod_{1 \leq i^{\prime}<k-k^{\prime}}\left(x_{a_{i^{\prime}}}-x_{a_{i}}\right)\right]$
by [8, Proposition 3.1], where we regard $a_{i}$ as the last element of $B \cup\left\{a_{i}\right\}$.
Let us start with the main body of the proof. To prove the assertion, we may assume that $\lambda$ covers $\mu$. By Remark [2.1, there are $j, j^{\prime}$ with $j<j^{\prime}$ such that $\mu_{j}^{\perp}=\lambda_{j}^{\perp}+1, \mu_{j^{\prime}}^{\perp}=\lambda_{j^{\prime}}^{\perp}-1$, and $\mu_{i}^{\perp}=\lambda_{i}^{\perp}$ for all $i \neq j, j^{\prime}$. Take $T \in \operatorname{Tab}(l, \mu)$, and let $A=\left\{a_{1}, \ldots, a_{k}\right\}$ (resp. $B=\left\{b_{1}, \ldots, b_{k^{\prime}}\right\}$ ) be the set of the contents of the $j$-th (resp. $j^{\prime}$-th) column of $T$. For $i$ with $k-k^{\prime} \leq i \leq k$, consider the tableau $T_{i}$ whose $j$-th (resp. $j^{\prime}$-th) column consists of the elements of $A \backslash\left\{a_{i}\right\}$ (resp. $B \cup\left\{a_{i}\right\}$ ) and the other columns are same as those of $T$. Since $a_{i} \geq 2$ for $i \geq 2$, we have $T_{i} \in \operatorname{Tab}(l, \lambda)$. By (2.2), we have

$$
\begin{equation*}
f_{T}=\sum_{k-k^{\prime} \leq i \leq k}(-1)^{i-k+k^{\prime}}\left[f_{T_{i}} \cdot \prod_{1 \leq i^{\prime}<k^{\prime}-k}\left(x_{a_{i^{\prime}}}-x_{a_{i}}\right)\right] \in I_{l, \lambda}, \tag{2.3}
\end{equation*}
$$

and it means that $I_{l, \lambda} \supset I_{l, \mu}$.
We say that $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$ is a lower (resp. upper) filter if $\lambda \in \mathcal{F}, \mu \in\left[P_{n+l-1}\right]_{\geq l}$ and $\mu \unlhd \lambda$ (resp. $\mu \unrhd \lambda$ ) imply $\mu \in \mathcal{F}$. For a lower filter $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$, set

$$
G_{l, \mathcal{F}}:=\left\{f_{T} \mid T \in \operatorname{STab}(l, \lambda) \text { for } \lambda \in \mathcal{F}\right\}
$$

and let $I_{l, \mathcal{F}} \subset S$ be the ideal generated by $G_{l, \mathcal{F}}$, equivalently,

$$
I_{l, \mathcal{F}}:=\sum_{\lambda \in \mathcal{F}} I_{l, \lambda} .
$$

In particular, for $\lambda \in\left[P_{n+l-1}\right]_{\geq l}, \mathcal{F}_{\lambda}:=\left\{\mu \in\left[P_{n+l-1}\right]_{\geq l} \mid \mu \unlhd \lambda\right\}$ is a lower filter, and we have $I_{l, \lambda}=I_{l, \mathcal{F}_{\lambda}}$ by Lemma 2.5, For convenience, set $G_{l, \emptyset}=\emptyset$ and $I_{l, \emptyset}=(0)$.

For an upper filter $\emptyset \neq \mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$, we consider the ideal

$$
J_{l, \mathcal{F}}:=\left(f \in S \mid f(\boldsymbol{a})=0 \text { for } \forall \boldsymbol{a} \in K^{n} \text { with } \Lambda_{l}(\boldsymbol{a}) \in \mathcal{F}\right) .
$$

Clearly, $J_{l, \mathcal{F}}$ is a radical ideal.
Theorem 2.6. Let $\mathcal{F} \subsetneq\left[P_{n+l-1}\right]_{\geq l}$ be a lower filter, and $\mathcal{F}^{c}:=\left[P_{n+l-1}\right]_{\geq l} \backslash \mathcal{F}$ its complement (note that $\mathcal{F}^{\text {c }}$ is an upper filter). Then $G_{l, \mathcal{F}}$ is a Gröbner basis of $J_{l, \mathcal{F c}}$.

The following corollary is immediate from the theorem.
Corollary 2.7. With the above situation, we have $I_{l, \mathcal{F}}=J_{l, \mathcal{F c}}$, and $I_{l, \mathcal{F}}$ is a radical ideal. In particular, $I_{l, \lambda}$ is a radical ideal with

$$
I_{l, \lambda}=\left(f \in S \mid f(\boldsymbol{a})=0 \text { for } \forall \boldsymbol{a} \in K^{n} \text { with } \Lambda_{l}(\boldsymbol{a}) \nexists \lambda\right),
$$

for which $\left\{f_{T} \mid T \in \operatorname{STab}(l, \mu), \mu \unlhd \lambda\right\}$ forms a Gröbner basis.
The strategy of the proof of Theorem [2.6 is essentially same as that of [11, Theorem 1.1], but we repeat it here for the reader's convenience. For a partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right) \in P_{m}$ and a positive integer $i$, we write $\lambda+\langle i\rangle$ for the partition of $m+1$ obtained by rearranging the sequence $\left(\lambda_{1}, \ldots, \lambda_{i}+1, \ldots, \lambda_{p}\right)$, where we set $\lambda+\langle i\rangle=\left(\lambda_{1}, \ldots, \lambda_{p}, 1\right)$ when $i>p$. For example $(4,2,2,1)+\langle 2\rangle=(4,2,2,1)+\langle 3\rangle=$ $(4,3,2,1)$, and $(4,2,2,1)+\langle i\rangle=(4,2,2,1,1)$ for all $i \geq 5$. Since $\lambda \unlhd \mu$ implies $\lambda+\langle i\rangle \unlhd \mu+\langle i\rangle$ for all $i$, if $\mathcal{F} \subset P_{m}$ is an upper (resp. lower) filter, then so is

$$
\mathcal{F}_{i}:=\left\{\mu \in P_{m-1} \mid \mu+\langle i\rangle \in \mathcal{F}\right\} .
$$

Example 2.8. Even if a lower filter $\mathcal{F}$ has a unique maximal element, $\mathcal{F}_{i}$ does not in general. For example, if $\mathcal{F}:=\left\{\lambda \in\left[P_{7}\right]_{\geq l} \mid \lambda \unlhd(3,2,2)\right\}$ for $l=1,2$, then $\mathcal{F}_{2}$ has two maximal elements $(3,1,1,1)$ and $(2,2,2)$.

Lemma 2.9 (c.f. [11, Lemma 3.3]). Let $\emptyset \neq \mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$ be an upper filter, and let $f$ be a polynomial in $J_{l, \mathcal{F}}$ of the form

$$
f=g_{d} x_{n}^{d}+\cdots+g_{1} x_{n}+g_{0}
$$

where $g_{0}, \ldots, g_{d} \in K\left[x_{1}, \ldots, x_{n-1}\right]$ and $g_{d} \neq 0$. Then $g_{0}, \ldots, g_{d}$ belong to $J_{l, \mathcal{F}_{d+1}}$.
Proof. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right) \in \mathcal{F}_{d+1}$, and take $\boldsymbol{a}=\left(a_{1}, \ldots, a_{n-1}\right) \in K^{n-1}$ with $\Lambda_{l}(\boldsymbol{a})=\lambda$. Then there are distinct elements $\alpha_{1}, \ldots, \alpha_{p} \in K$ such that $\alpha_{i}$ appears $\lambda_{i}$ times in $\boldsymbol{a}_{(l)}$ for $i=1, \ldots, p$. Since $\mathcal{F}$ is an upper filter, we have $\lambda+\langle i\rangle \in \mathcal{F}$ for $i=1,2, \ldots, d+1$. We will consider two cases as follows (in the sequel, for $\alpha \in K$, $(\boldsymbol{a}, \alpha)$ means the point in $K^{n}$ whose coordinate is $\left(a_{1}, \ldots, a_{n-1}, \alpha\right)$ ): (i) If $p<d+1$, then $\lambda+\langle d+1\rangle=\left(\lambda_{1}, \ldots, \lambda_{p}, 1\right)$. Thus, for any $\alpha \in K \backslash\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{p}\right\}$, we have $\Lambda_{l}(\boldsymbol{a}, \alpha)=\lambda+\langle d+1\rangle \in \mathcal{F}$, and hence $f(\boldsymbol{a}, \alpha)=0$. (ii) If $p \geq d+1$, then we have
$\Lambda_{l}\left(\boldsymbol{a}, \alpha_{i}\right)=\lambda+\langle i\rangle \in \mathcal{F}$ for any $i=1, \ldots, d+1$ (note that $\lambda+\langle i\rangle \unrhd \lambda+\langle d+1\rangle \in \mathcal{F}$ for these $i$ ), and hence $f\left(\boldsymbol{a}, \alpha_{i}\right)=0$.

In both cases, it follows that the polynomial $f\left(\boldsymbol{a}, x_{n}\right)=\sum_{i=0}^{d} g_{i}(\boldsymbol{a}) x_{n}^{i} \in K\left[x_{n}\right]$ has at least $d+1$ zeros. Since the degree of $f\left(\boldsymbol{a}, x_{n}\right)$ is $d, f\left(\boldsymbol{a}, x_{n}\right)$ is the zero polynomial in $K\left[x_{n}\right]$. Thus, $g_{i}(\boldsymbol{a})=0$ for $i=0,1, \ldots, d$. Hence, $g_{0}, \ldots, g_{d} \in J_{l, \mathcal{F}_{d+1}}$.
The proof of Theorem 2.6. First, we show that $G_{l, \mathcal{F}} \subset J_{l, \mathcal{F c}}$. Take $T \in \operatorname{STab}(l, \lambda)$ for $\lambda \in \mathcal{F}$, and $\boldsymbol{a} \in K^{n}$ with $\Lambda_{l}(\boldsymbol{a}) \in \mathcal{F}^{c}$ (i.e., $\left.\Lambda_{l}(\boldsymbol{a}) \notin \mathcal{F}\right)$. Since $\mathcal{F}$ is a lower filter, we have $\Lambda_{l}(\boldsymbol{a}) \nexists \lambda$, and hence $f_{T}(\boldsymbol{a})=0$ by Lemma [2.4. So $f_{T} \in J_{l, \mathcal{F} \mathrm{c}}$.

For $\mu \in\left[P_{n+l-2}\right]_{\geq l}$, it is easy to see that

$$
\mu \notin\left(\mathcal{F}^{c}\right)_{i} \Longleftrightarrow \mu+\langle i\rangle \notin \mathcal{F}^{c} \Longleftrightarrow \mu+\langle i\rangle \in \mathcal{F} \Longleftrightarrow \mu \in \mathcal{F}_{i},
$$

so we have $\left[P_{n+l-2}\right]_{\geq l} \backslash\left(\mathcal{F}^{\mathrm{c}}\right)_{i}=\mathcal{F}_{i}$.
To prove the theorem, it suffices to show that the initial monomial in $(f)$ for all $0 \neq f \in J_{l, \mathcal{F c}}$ can be divided by $\operatorname{in}\left(f_{T}\right)$ for some $f_{T} \in G_{l, \mathcal{F}}$. We will prove this by induction on $n$. The case $n=1$ is trivial. For $n \geq 2$, let $f=g_{d} x_{n}^{d}+\cdots+g_{1} x_{n}+g_{0} \in$ $J_{l, \mathcal{F c}}$, where $g_{i} \in K\left[x_{1}, \ldots, x_{n-1}\right]$ and $g_{d} \neq 0$. By Lemma 2.9, one has $g_{d} \in J_{l,(\mathcal{F c})_{d+1}}$. By the induction hypothesis, we have $G_{l, \mathcal{F}_{d+1}}\left(=G_{l,\left[P_{n+l-2}\right]>l} \backslash\left(\mathcal{F}^{c}\right)_{d+1}\right)$ is a Gröbner basis of $J_{l,(\mathcal{F c})_{d+1}}$. Then there is $T \in \operatorname{STab}(l, \mu)$ for $\mu \in \mathcal{F}_{d+1}$ such that $\operatorname{in}\left(f_{T}\right)$ divides $\operatorname{in}\left(g_{d}\right)$. Set $\lambda:=\mu+\langle d+1\rangle \in \mathcal{F}$. Let us consider the tableau $T^{\prime} \in \operatorname{STab}(l, \lambda)$ such that the image of each $i=1,2, \ldots, n-1$ is same for $T$ and $T^{\prime}$. So $n$ is in the newly added square. Since $\lambda=\mu+\langle d+1\rangle, n$ is in the $(q+1)$-st row of $T^{\prime}$ for some $q \leq d$. Since we have in $(f)=\operatorname{in}\left(g_{d} x_{n}^{d}\right)=x_{n}^{d} \cdot \operatorname{in}\left(g_{d}\right)$ and $\operatorname{in}\left(f_{T^{\prime}}\right)=x_{n}^{q} \cdot \operatorname{in}\left(f_{T}\right)$ by (2.1), in $\left(f_{T^{\prime}}\right)$ divides $\operatorname{in}(f)$. Hence, the proof is completed.

For $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right) \in\left[P_{n+l-1}\right]_{\geq l}$, set $H_{l, \lambda}:=\left\{\boldsymbol{a} \in K^{n} \mid \Lambda_{l}(\boldsymbol{a})=\lambda\right\}$. Then we have the decomposition $K^{n}=\bigsqcup_{\lambda \in\left[P_{n+l-1] \geq l}\right.} H_{l, \lambda}$, and the dimension of $H_{l, \lambda}$ equals the length $p$ of $\lambda$. For an upper filter $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}, S / J_{l, \mathcal{F}}\left(=S / I_{l, \mathcal{F c}}\right)$ is the coordinate ring of $\bigsqcup_{\lambda \in \mathcal{F}} H_{l, \lambda}$.
Proposition 2.10. The codimension of the ideal $I_{l, \lambda}$ is $\lambda_{1}-l+1$.
Proof. By the above remark, the algebraic set defined by $I_{l, \lambda}$ is the union of $H_{l, \mu}$ for all $\mu \in\left[P_{n+l-1}\right]_{\geq l}$ with $\mu \nexists \lambda$. Among these partitions, $\mu^{\prime}=\left(\lambda_{1}+1,1,1, \ldots\right)$ has the largest length $n+l-1-\lambda_{1}$, and hence $\operatorname{codim} I_{l, \lambda}=n-\operatorname{dim} S / I_{l, \lambda}=$ $n-\left(n+l-1-\lambda_{1}\right)=\lambda_{1}-l+1$.
Example 2.11. For $\lambda=(3,3,1), \operatorname{STab}(2, \lambda)$ consists of the following 11 elements

| 1 | 1 | 2 |  | 1 | 1 | 2 |  | 1 | 1 | 2 |  | 1 | 1 |  |  | 1 | 1 |  |  | 1 | 1 | 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 4 | 5 |  | 3 | 4 | 6 |  | 3 | 5 | 6 |  | 2 |  |  |  | 2 | 4 |  |  | 2 | 5 | 6 |  |
| 6 |  |  |  | 5 |  |  |  | 4 |  |  |  | 6 |  |  |  | 5 |  |  |  | 4 |  |  |  |
|  |  | 1 | 1 | 4 |  | 1 | 1 | 4 |  | 1 | 1 |  |  | 1 |  | 5 |  | 1 | 1 | 5 |  |  |  |
|  |  | 2 | 3 | 5 |  | 2 | 3 | 6 |  | 2 | 5 |  |  | 2 |  |  |  | 2 | 4 |  |  |  |  |
|  |  | 6 |  |  |  | 5 |  |  |  | 3 |  |  |  | 4 |  |  |  | 3 |  |  |  |  |  |

so $I_{2, \lambda}$ is minimally generated by 11 elements. For a non-empty subset $F \subset[n]$, consider the ideal $P_{F}=\left(x_{i}-x_{j} \mid i, j \in F\right)$. Clearly, $P_{F}$ is a prime ideal of
codimension $\# F-1$. By Corollary 2.7, $I_{3, \lambda}$ is a radical ideal whose minimal primes are $P_{F}$ for $F \subset[n]$ either (i) $1 \in F$ and $\# F=3$, or (ii) $1 \notin F$ and $\# F=4$.

## 3. Under the opposite monomial order

Philosophically, we next treat the Gröbner basis of $I_{l, \lambda}$ with respect to the lexicographic order with $x_{1}>x_{2}>\cdots>x_{n}$, which is opposite to the one used in the previous section. However, for notational simplicity, we keep using the lexicographic order with $x_{1}<\cdots<x_{n}$, but we consider tableaux whose squares are bijectively $l$ copies
filled by the multiset $\{1, \ldots, n-1, \overbrace{n, \ldots, n}\}$. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$, let $\operatorname{Tab}(\lambda, l)$ be the set of such tableaux of shape $\lambda$. As in the previous section, we can define the standard-ness of $T \in \operatorname{Tab}(\lambda, l)$. For example, the tableau $T$ in (3.2) below is standard. Let $\operatorname{STab}(\lambda, l)$ be the subset of $\operatorname{Tab}(\lambda, l)$ consisting of standard tableaux. By the same argument as Lemma 2.3, we have the following.

Lemma 3.1. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l},\left\{f_{T} \mid T \in \operatorname{STab}(\lambda, l)\right\}$ forms a basis of the vector space spanned by $\left\{f_{T} \mid T \in \operatorname{Tab}(\lambda, l)\right\}$.

For $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$, consider the ideal

$$
I_{\lambda, l}:=\left(f_{T} \mid T \in \operatorname{Tab}(\lambda, l)\right)=\left(f_{T} \mid T \in \operatorname{STab}(\lambda, l)\right)
$$

For $\boldsymbol{a} \in K^{n}$, set $\boldsymbol{a}^{(l)}:=(a_{1}, \ldots, a_{n-1}, \overbrace{a_{n}, \ldots, a_{n}}^{l \text { copies }}) \in K^{n+l-1}$ and $\Lambda^{l}(\boldsymbol{a}):=\Lambda\left(\boldsymbol{a}^{(l)}\right) \in$ $\left[P_{n+l-1}\right]_{\geq l}$. Up to the automorphism of $S$ exchanging $x_{1}$ and $x_{n}$, the ideal $I_{\lambda, l}$ coincides with $I_{l, \lambda}$ treated in the previous section. Hence we have $I_{\mu, l} \subset I_{\lambda, l}$ for $\mu, \lambda \in\left[P_{n+l-1}\right]_{\geq l}$ with $\mu \unlhd \lambda$, and

$$
\begin{equation*}
I_{\lambda, l}=\left(f \in S \mid f(\boldsymbol{a})=0 \text { for } \forall \boldsymbol{a} \in K^{n} \text { with } \Lambda^{l}(\boldsymbol{a}) \nexists \lambda\right) . \tag{3.1}
\end{equation*}
$$

In $T \in \operatorname{STab}(\lambda, l)$, all $n$ 's are in the bottom of their columns. Let $w(T)$ denote the number of squares which locate above some $n$. For example, if

$$
T=\begin{array}{|l|l|l|l|l|l|}
\hline 1 & 2 & 3 & 5 & 8 & 8  \tag{3.2}\\
\hline 4 & 6 & 8 & & & \\
\cline { 1 - 4 } & 8 & & & \\
\hline
\end{array}
$$

( $n=8$ and $l=4$ in this case), then $w(T)=3$. In fact, the squares filled by 2,6 , and 3 are counted. For $T \in \operatorname{Tab}(\lambda, l)$, the degree of $\operatorname{in}\left(f_{T}\right)$ with respect to $x_{n}$ is $w(T)$. Let $\operatorname{sh}_{<n}(T) \in P_{n-1}$ denote the shape of $T^{\prime}$, where $T^{\prime}$ is the tableau obtained by removing all squares filled by $n$ from $T$. For example, if $T$ is the above one, we have $\operatorname{sh}_{<8}(T)=(4,2,1)$.

For $\mu \in P_{n-1}$, set

$$
\langle\mu\rangle^{l}:=\left\{\lambda \in\left[P_{n+l-1}\right]_{\geq l} \mid \exists T \in \operatorname{STab}(\lambda, l) \text { with } \operatorname{sh}_{<n}(T)=\mu\right\} .
$$

For $\lambda \in\langle\mu\rangle^{l}$ and $T \in \operatorname{STab}(\lambda, l)$ with $\operatorname{sh}_{<n}(T)=\mu$, the positions of the squares filled by $n$ only depend on $\lambda$ and $\mu$. We call these squares $n$-squares of $\lambda$. Similarly, $w(T)$ does not depend on a particular choice of $T$, and we denote this value by $w_{\mu}(\lambda)$.

Lemma 3.2. Let $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$, and $\mathcal{F}:=\left\{\rho \in\left[P_{n+l-1}\right]_{\geq l} \mid \rho \unlhd \lambda\right\}$ the lower filter of $\left[P_{n+l-1}\right]_{\geq l}$. For $\mu \in P_{n-1}$, if $X:=\langle\mu\rangle^{l} \cap \mathcal{F}$ is non-empty, then there is the element $\widetilde{\lambda} \in X$ satisfying $w_{\mu}(\rho)>w_{\mu}(\widetilde{\lambda})$ for all $\rho \in X \backslash\{\widetilde{\lambda}\}$.
Proof. We determine $\widetilde{\lambda}=\left(\widetilde{\lambda}_{1}, \widetilde{\lambda}_{2}, \ldots, \widetilde{\lambda}_{p}\right)$ inductively from $\widetilde{\lambda}_{1}$. First, we set

$$
\widetilde{\lambda}_{1}:=\max \left\{\rho_{1} \mid \rho=\left(\rho_{1}, \ldots, \rho_{q}\right) \in X\right\} \quad \text { and } \quad X_{1}:=\left\{\rho \in X \mid \rho_{1}=\widetilde{\lambda}_{1}\right\}
$$

and next $\widetilde{\lambda}_{2}:=\max \left\{\rho_{2} \mid \rho \in X_{1}\right\}$ and $X_{2}:=\left\{\rho \in X_{1} \mid \rho_{2}=\widetilde{\lambda}_{2}\right\}$. We repeat this procedure until the sum $\widetilde{\lambda}_{1}+\widetilde{\lambda}_{2}+\cdots$ reaches $n+l-1$.

We will show that $\widetilde{\lambda}$ has the expected property. For $\rho \in X \backslash\{\widetilde{\lambda}\}$, set $i_{0}:=\min \{i \mid$ $\left.\rho_{i} \neq \lambda_{i}\right\}$. Then we have $\rho_{i_{0}}<\widetilde{\lambda}_{i_{0}}$, and $\rho$ has an $n$-square in the $i$-th row for some $i>i_{0}$. Let $i_{1}$ be the smallest $i$ with this property. Raising up the right most square in the $i_{1}$-th row to the right end of $i_{0}$-th row, we get $\rho^{\prime} \in X$ (here we use the present form of $\mathcal{F}$ ). It is clear that $w_{\mu}(\rho)>w_{\mu}\left(\rho^{\prime}\right)$ and $\rho \triangleleft \rho^{\prime}$. Repeating this argument until our partition reaches $\widetilde{\lambda}$, we get the expected inequality.
Example 3.3. In the above lemma, the case $\lambda \neq \widetilde{\lambda}$ might happen. For example, if $\lambda=(4,2,1)$ and $\mu=(3,3)$ (so $l=1$ now), we have $\widetilde{\lambda}=(3,3,1)$.

For a lower filter $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$ and a non-negative integer $k$, set

$$
\mathcal{F}^{k}:=\left\{\mu \in P_{n-1} \mid \exists \lambda \in\langle\mu\rangle^{l} \cap \mathcal{F} \text { with } w_{\mu}(\lambda) \leq k\right\} .
$$

Example 3.4. Consider the case $n=6, l=2, \lambda=(3,3,1)$ and $\mathcal{F}:=\left\{\rho \in\left[P_{7}\right]_{\geq 2} \mid\right.$ $\rho \unlhd \lambda\}$. Then $\mathcal{F}^{3}, \mathcal{F}^{2}, \mathcal{F}^{1}, \mathcal{F}^{0}$ are the lower filters of $P_{5}$ whose unique maximal elements are $(3,2),(3,1,1),(2,1,1,1)$ and $(1,1,1,1,1)$, respectively. In the following diagrams, $\star$ 's represent the positions of $n$-squares of the corresponding partitions of $n+l-1(=7)$. It is also easy to see that $\mathcal{F}^{k}=\mathcal{F}^{3}$ for all $k \geq 3$.


Lemma 3.5. If $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$ is a lower filter, then $\mathcal{F}^{k}$ is a lower filter of $P_{n-1}$.
Proof. It suffices to show that if $\mu \in \mathcal{F}^{k}$ covers $\nu \in P_{n-1}$ then $\nu \in \mathcal{F}^{k}$. In this situation, there are two integers $j, j^{\prime}$ with $j<j^{\prime}$ such that $\nu_{j}^{\perp}=\mu_{j}^{\perp}+1, \nu_{j^{\prime}}^{\perp}=\mu_{j^{\prime}}^{\perp}-1$, and $\nu_{i}^{\perp}=\mu_{i}^{\perp}$ for all $i \neq j, j^{\prime}$. In other words, moving a square in the $j^{\prime}$-th column of $\mu$ to the $j$-th column, we get $\nu$. Anyway, we can take $\lambda \in \mathcal{F} \cap\langle\mu\rangle^{l}$ with $w_{\mu}(\lambda) \leq k$, and we want to construct $\rho \in \mathcal{F} \cap\langle\nu\rangle^{l}$ with $w_{\nu}(\rho) \leq k$.

For each $i$, we have $\mu_{i}^{\perp} \leq \lambda_{i}^{\perp} \leq \mu_{i}^{\perp}+1$, and there is an $n$-square in the $i$-th column of $\lambda$ if and only if $\lambda_{i}^{\perp}=\mu_{i}^{\perp}+1$. We have the following four cases.
(1) $\lambda_{j}^{\perp}=\mu_{j}^{\perp}$ and $\lambda_{j^{\prime}}^{\perp}=\mu_{j^{\prime}}^{\perp}$.
(2) $\lambda_{j}^{\perp}=\mu_{j}^{\perp}+1$ and $\lambda_{j^{\prime}}^{\perp}=\mu_{j^{\prime}}^{\perp}+1$.
(3) $\lambda_{j}^{\perp}=\mu_{j}^{\perp}$ and $\lambda_{j^{\prime}}^{\perp}=\mu_{j^{\prime}}^{\perp}+1$.
(4) $\lambda_{j}^{\perp}=\mu_{j}^{\perp}+1$ and $\lambda_{j^{\prime}}^{\perp}=\mu_{j^{\prime}}^{\perp}$.

In the case (4), we set $\rho=\lambda$, that is, exchanging the $n$-square in the $j$-th column and the bottom square of $j^{\prime}$-th column, we get $\rho$ and $\nu$ from $\lambda$ and $\mu$. In the other cases, we first move the $n$-squares in the $j$-th and $j^{\prime}$-th columns of $\lambda$ (their existence depends on the cases (1)-(3)) vertically along the change from $\mu$ to $\nu$. For example, in the case (2), the above operation is


Furthermore, if necessary, we apply a suitable column permutation as the following figure (in this situation, since $\mu_{j-1}^{\perp}=\nu_{j-1}^{\perp} \geq \nu_{j}^{\perp}=\mu_{j}^{\perp}+1$, we have $\mu_{j-1}^{\perp}>\mu_{j}^{\perp}$ and there is no $n$-square in the $(j-1)$-st column of the left and the middle diagrams). In any cases, we have $\rho \unlhd \lambda \in \mathcal{F}$ and $w_{\nu}(\rho) \leq w_{\mu}(\lambda) \leq k$, that is, $\rho$ satisfies the expected property.


Proposition 3.6. Let $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$, and $\mathcal{F}:=\left\{\rho \in\left[P_{n+l-1}\right]_{\geq l} \mid \rho \unlhd \lambda\right\}$ the lower filter of $\left[P_{n+l-1}\right]_{\geq l}$. If $f \in I_{\lambda, l}$ is of the form $f=g_{d} x_{n}^{d}+\cdots+g_{1} x_{n}+g_{0}$ with $g_{0}, \ldots, g_{d} \in K\left[x_{1}, \ldots, x_{n-1}\right]$ and $g_{d} \neq 0$, then $g_{0}, \ldots, g_{d}$ belong to $I_{\mathcal{F}^{d}}$.
Proof. Assume that $g_{m} \notin I_{\mathcal{F}^{d}}$ for some $m$. By the classical case (i.e., when $l=1$ ) of Corollary 2.7, there are some $\boldsymbol{a} \in K^{n-1}$ such that $\mu:=\Lambda(\boldsymbol{a}) \notin \mathcal{F}^{d}$ and $g_{m}(\boldsymbol{a}) \neq 0$. If $\mu=\left(\mu_{1}, \ldots, \mu_{p}\right)$, there are distinct elements $\alpha_{1}, \ldots, \alpha_{p} \in K$ such that $\alpha_{i}$ appears $\mu_{i}$ times in $\boldsymbol{a}$ for $i=1, \ldots, p$.

We have

$$
\begin{equation*}
f=\sum_{\substack{T \in \operatorname{STab}\left(\lambda^{\prime}, l\right) \\ \lambda^{\prime} \in \mathcal{F}}} h_{T} \cdot f_{T} \tag{3.4}
\end{equation*}
$$

for some $h_{T} \in S$. For $T \in \operatorname{STab}\left(\lambda^{\prime}, l\right)$, replacing $i$ with $a_{i}$ in $T$ for all $1 \leq i \leq n-1$, and $n$ with $x_{n}$, we get the tableau $T(\boldsymbol{a})$ whose entries are elements of $K \cup\left\{x_{n}\right\}$.

Take $\rho \in\langle\nu\rangle^{l}$ for some $\nu \in P_{n-1}$. We call a bijective filling $\mathcal{T}$ of the squares of the Young diagram of $\rho$ by the multiset

$$
\{\overbrace{\alpha_{1}, \ldots, \alpha_{1}}^{\mu_{1} \text { copies }}, \overbrace{\alpha_{2}, \ldots, \alpha_{2}}^{\mu_{2} \text { copies }}, \cdots, \overbrace{\alpha_{p}, \ldots, \alpha_{p}}^{\mu_{p} \text { copies }}, \overbrace{x_{n}, \ldots, x_{n}}^{l \text { copies }}\}
$$

such that all $n$-squares are filled by $x_{n}$ is called an $\boldsymbol{a}$-tableau. We call $\rho$ the shape of $\mathcal{T}$, and denote it by $\operatorname{sh}(\mathcal{T})$. We also denote $\nu$ by $\operatorname{sh}_{<n}(\mathcal{T})$. A typical example of an $\boldsymbol{a}$-tableau is $T(\boldsymbol{a})$ given above. We say an $\boldsymbol{a}$-tableau $\mathcal{T}$ is regular, if the entries
in the $j$-th column of $\mathcal{T}$ are all distinct for each $j$. Note that $T(\boldsymbol{a})$ is regular if and only if $f_{T}\left(\boldsymbol{a}, x_{n}\right) \neq 0$.

For all $\alpha \in K$, we can show that $\bar{\mu}:=\Lambda^{l}(\boldsymbol{a}, \alpha)$ belongs to $\langle\mu\rangle^{l}$ (recall that $\mu=$ $\Lambda(\boldsymbol{a}))$. For example, if $\alpha \notin\left\{\alpha_{1}, \ldots, \alpha_{p}\right\}$, we have $\bar{\mu}=\left(\mu_{1}, \ldots, \mu_{j}, l, \mu_{j+1}, \ldots, \mu_{p}\right)$, where $j:=\max \left\{i \mid \mu_{i} \geq l\right\}$, and hence $\bar{\mu}_{i} \geq \mu_{i}$ and $\mu_{i}^{\perp} \leq \bar{\mu}_{i}^{\perp} \leq \mu_{i}^{\perp}+1$ for all $i$. The case $\alpha \in\left\{\alpha_{1}, \ldots, \alpha_{p}\right\}$ can be shown by a similar argument. Anyway, if $\langle\mu\rangle^{l} \cap \mathcal{F}=\emptyset$, then $\Lambda^{l}(\boldsymbol{a}, \alpha) \notin \mathcal{F}$, and hence $f(\boldsymbol{a}, \alpha)=0$ by (3.1). So it implies that $f\left(\boldsymbol{a}, x_{n}\right)=0$ and $g_{m}(\boldsymbol{a})=0$. This is a contradiction. So $\langle\mu\rangle^{l} \cap \mathcal{F}$ is non-empty, and it has the element $\tilde{\lambda}$ with the minimum $w_{\mu}(-)$ by Lemma3.2, Let $\widetilde{\mathcal{T}}$ be an $\boldsymbol{a}$-tableau of shape $\widetilde{\lambda}$ with $\operatorname{sh}_{<n}(\widetilde{\mathcal{T}})=\mu$ such that all squares in the $i$-th row of $\mu$ are filled by $\alpha_{i}$. Assume that, for each $i$ with $1 \leq i \leq p, \alpha_{i}$ appears $d_{i}$ times in squares above some $n$-squares in $\tilde{\mathcal{T}}$. See Example 3.7 below. We have $\sum_{i=1}^{p} d_{i}=w_{\mu}(\lambda)>d$, where the inequality follows from that $\mu \notin \mathcal{F}^{d}$.

Claim. Let $\mathcal{T}$ be a regular $\boldsymbol{a}$-tableau with $\operatorname{sh}(\mathcal{T}) \in \mathcal{F}$. For all $i$ with $1 \leq i \leq p, \alpha_{i}$ appears at least $d_{i}$ times in squares above some $n$-squares in $\mathcal{T}$.

Proof of Claim. Set $\nu:=\operatorname{sh}_{<n}(\mathcal{T}) \in P_{n-1}$. We will prove the assertion by induction on $\nu$ with respect to the dominance order. Since $\mathcal{T}$ is regular, it is easy to see that $\mu \unlhd \nu$ by the classical case (i.e., when $l=1$ ) of Corollary 2.7. If $\mu=\nu$, applying suitable actions of column stabilizers (i.e., permutations of entries in the same column), we may assume that each square in the $i$-th row of $\mathcal{T}$ is filled by $\alpha_{i}$ or $x_{n}$. So the assertion can be shown by an argument similar to the proof of Lemma 3.2. Next consider the case $\mu \triangleleft \nu$. As the induction hypothesis, we assume that the assertion holds for $\mathcal{T}^{\prime}$ with $\mu \unlhd \operatorname{sh}_{<n}\left(\mathcal{T}^{\prime}\right) \triangleleft \nu$.

To proceed with proof by contradiction, assume that $\mathcal{T}$ does not satisfy the expected condition, that is, there is some $s$ such that $\alpha_{s}$ appears less than $d_{s}$ times in squares above some $n$-squares in $\mathcal{T}$. Since $\mu \triangleleft \nu$ now, there are some $t$ and $j, j^{\prime}$ with $j<j^{\prime}$ such that $\alpha_{t}$ appears in the $j^{\prime}$-th column of $\mathcal{T}$, but does not appear in the $j$-th column. If $\alpha_{s}$ has this property, we take $s$ as $t$. We move the square in the $j^{\prime}$-th column filled by $\alpha_{t}$ to the $j$-th column, and get the partition $\nu^{\prime} \in P_{n-1}$ (a suitable column permutation might be required). The following condition is crucial.
(*) $s=t$ holds, and the bottom of the $j$-th column of $\mathcal{T}$ is an $n$-square, and that of the $j^{\prime}$-th column is not.
In the case $(*)$ is not satisfied, we move the $n$-squares in these columns (if they exist) vertically like (3.3), then apply a suitable column permutation if necessary (sometimes, we have to move the $j$-th column to left and/or the $j^{\prime}$-th column to right). Finally, we get an $\boldsymbol{a}$-tableau $\mathcal{T}^{\prime}$ with $\operatorname{sh}_{<n}\left(\mathcal{T}^{\prime}\right)=\nu^{\prime}$. On the other hand, if $(*)$ holds, we move the $n$-square in the $j$-th column to below the bottom of the $j^{\prime}$-th column. Of course, do not forget to move the $\alpha_{s}$-square in the $j^{\prime}$-th column to the $j$-th column. Applying a suitable column permutation if necessary, we get $\mathcal{T}^{\prime}$ with $\operatorname{sh}_{<n}\left(\mathcal{T}^{\prime}\right)=\nu^{\prime}$. In this case, we have $\operatorname{sh}(\mathcal{T})=\operatorname{sh}\left(\mathcal{T}^{\prime}\right)$.

In both cases, $\mathcal{T}^{\prime}$ is regular, and $\alpha_{s}$ appears less than $d_{s}$ times in squares above some $n$-squares in $\mathcal{T}^{\prime}$. Moreover, we have $\operatorname{sh}\left(\mathcal{T}^{\prime}\right) \unlhd \operatorname{sh}(\mathcal{T})$, and hence $\operatorname{sh}\left(\mathcal{T}^{\prime}\right) \in \mathcal{F}$. Since $(\mu \unlhd) \nu^{\prime} \triangleleft \nu$, it contradicts the induction hypothesis.

We back to the proof of the proposition itself. In (3.4), we have

$$
f\left(\boldsymbol{a}, x_{n}\right)=\sum h_{T}\left(\boldsymbol{a}, x_{n}\right) \cdot f_{T}\left(\boldsymbol{a}, x_{n}\right)
$$

If $f_{T}\left(\boldsymbol{a}, x_{n}\right) \neq 0$, then $T(\boldsymbol{a})$ is regular. So, by Claim, $f_{T}\left(\boldsymbol{a}, x_{n}\right)$ can be divided by

$$
R\left(x_{n}\right)=\prod_{1 \leq i \leq p}\left(x_{n}-\alpha_{i}\right)^{d_{i}}
$$

and $f\left(\boldsymbol{a}, x_{n}\right)$ itself can be divided by $R\left(x_{n}\right)$. While the degree of $f\left(\boldsymbol{a}, x_{n}\right)$ is at most $d$, we have $\operatorname{deg} f\left(\boldsymbol{a}, x_{n}\right) \geq \operatorname{deg} R\left(x_{n}\right)=\sum_{i=1}^{p} d_{i}>d$. This is a contradiction.

Example 3.7. Consider the case $n=8, l=3$, and take the lower filter given by $\mathcal{F}=\left\{\lambda \in\left[P_{10}\right]_{\geq 3} \mid \lambda \unlhd(4,4,2)\right\}$. If $\boldsymbol{a}=\left(\alpha_{1}, \alpha_{1}, \alpha_{1}, \alpha_{2}, \alpha_{2}, \alpha_{3}, \alpha_{3}\right)$ (hence $\mu=(3,2,2)$ ), the $\boldsymbol{a}$-tableau $\overline{\mathcal{T}}$ given in the proof of Proposition 3.6 is as follows

\[

\]

Above three $n(=8)$-boxes, there are two copies of $\alpha_{1}$, so we have $d_{1}=2$. Similarly, since there is one $\alpha_{2}$ (resp. $\alpha_{3}$ ) above three $n$-boxes, we have $d_{2}=1$ (resp. $d_{3}=1$ ).

The following are examples of regular $\boldsymbol{a}$-tableaux whose shape belong to $\mathcal{F}$. In each case, there are at least 2 (resp. 1) $\alpha_{1}$ (resp. $\alpha_{2}$ and $\alpha_{3}$ ) above $n$-squares.


Theorem 3.8. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$, set $\mathcal{F}:=\left\{\rho \in\left[P_{n+l-1}\right]_{\geq l} \mid \rho \unlhd \lambda\right\}$ be the lower filter. Then $\left\{f_{T} \mid T \in \operatorname{STab}(\rho, l), \rho \in \mathcal{F}\right\}$ is a Gröbner basis of $I_{\lambda, l}$.
Proof. It suffices to show that the initial monomial in $(f)$ for all $0 \neq f \in I_{\lambda, l}$ can be divided by in $\left(f_{T}\right)$ for some $T \in \operatorname{STab}(\rho, l)$ with $\rho \in \mathcal{F}$. Let $f=g_{d} x_{n}^{d}+\cdots+g_{1} x_{n}+g_{0}$, where $g_{i} \in K\left[x_{1}, \ldots, x_{n-1}\right]$ and $g_{d} \neq 0$. By Proposition [3.6, one has $g_{d} \in I_{\mathcal{F}^{d}}$. By Theorem 1.2, $\left\{f_{T} \mid T \in \operatorname{STab}(\mu), \mu \in \mathcal{F}^{d}\right\}$ is a Gröbner basis of $I_{\mathcal{F}^{d}}$ (since we fix the monomial order, it is enough to consider standard tableaux, see [11, Remark 3.5]), and there is a tableau $T \in \operatorname{STab}(\mu)$ for some $\mu \in \mathcal{F}^{d}$ such that in $\left(f_{T}\right)$ divides $\operatorname{in}\left(g_{d}\right)$. So we can take $\rho \in\langle\mu\rangle^{l} \cap \mathcal{F}$ with $e:=w_{\mu}(\rho) \leq d$. Let us consider the tableau $T^{\prime} \in \operatorname{STab}(\rho, l)$ such that the image of each $i=1, \ldots, n-1$ is same for $T$ and $T^{\prime}$. Since we have $\operatorname{in}(f)=x_{n}^{d} \cdot \operatorname{in}\left(g_{d}\right)$ and $\operatorname{in}\left(f_{T^{\prime}}\right)=x_{n}^{e} \cdot \operatorname{in}\left(f_{T}\right), \operatorname{in}\left(f_{T^{\prime}}\right)$ divides $\operatorname{in}(f)$.

Example 3.9. Contrary to Theorem 2.6, Theorem 3.8 cannot be generalized to the ideal $I_{\mathcal{F}, l}:=\left(f_{T} \mid T \in \operatorname{Tab}(\lambda, l), \lambda \in \mathcal{F}\right)$ for a lower filter $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$. For example, if $\mathcal{F} \subset\left[P_{8}\right]_{\geq 2}$ is the lower filter whose maximal elements are $(4,2,1,1)$ and $(3,3,2)$, then $x_{4}^{2} x_{5}^{3} x_{6} x_{7}^{2}$ is a minimal generator of $\operatorname{in}\left(I_{\mathcal{F}, 2}\right)$, but this cannot be represented in the form of $\operatorname{in}\left(f_{T}\right)$ for $T \in \operatorname{STab}(\lambda, 2)$.

The following fact might be well-known to the specialist, and is stated in [11] without proof. This time, we give a proof for the reader's convenience.

Lemma 3.10. Let $I \subset S=K\left[x_{1}, \ldots, x_{n}\right]$ be a graded ideal, and $G \subset I$ a Gröbner basis of $I$ with respect to the lexicographic order $<$ with $x_{1}<x_{2}<\cdots<x_{n}$. If all elements of $G$ are products of linear forms, then $G$ is a Gröbner basis of I with respect to any monomial order $\prec$ with $x_{1} \prec x_{2} \prec \cdots \prec x_{n}$.

The assumption that $I$ is graded is unnecessary, but we add it here for the simplicity.
Proof. Since $g \in G$ is a product of linear forms, we have $\mathrm{in}_{\prec}(g)=\mathrm{in}_{<}(g)$, and hence

$$
\operatorname{in}_{<}(I)=\left(\operatorname{in}_{<}(g) \mid g \in G\right)=\left(\operatorname{in}_{\prec}(g) \mid g \in G\right) \subset \operatorname{in}_{\prec}(I)
$$

Since $\mathrm{in}_{\prec}(I)$ and $\mathrm{in}_{<}(I)$ have the same Hilbert function (in fact, they have the same Hilbert function as $I$ itself), we have $\mathrm{in}_{\prec}(I)=\mathrm{in}_{<}(I)$. It implies that $G$ is a Gröbner basis of $I$ with respect to $\prec$.
Corollary 3.11. Let $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$. With respect to a monomial order in which $x_{1}$ is either the smallest or the largest among the variables $x_{1}, \ldots, x_{n},\left\{f_{T} \mid T \in\right.$ $\operatorname{Tab}(l, \rho), \rho \unlhd \lambda\}$ is a Gröbner basis of $I_{l, \lambda}$.

Since we consider several monomial orders, we have to treat $\operatorname{Tab}(l, \lambda)$, not $\operatorname{STab}(l, \lambda)$.
Proof. First, we consider the case $x_{1}$ is the smallest among $x_{1}, \ldots, x_{n}$. Since the ideal $I_{l, \lambda}$ is symmetric for variables $x_{2}, \ldots, x_{n}$, and Specht polynomials are products of linear forms, we may assume that our monomial order is the lexicographic order with $x_{1}<\cdots<x_{n}$ by Lemma 3.10, and the assertion follows from Corollary [2.7. Similarly, if $x_{1}$ is the largest, we may assume that our monomial order is the lexicographic order with $x_{1}>\cdots>x_{n}$, and the assertion follows from Theorem 3.8.

Example 3.12. For $\lambda=(3,3), I_{2, \lambda}$ is generated by 3 elements of degree 3. With respect to a monomial order in which $x_{1}$ is the smallest, $\operatorname{in}\left(I_{2, \lambda}\right)$ is minimally generated by 3 elements of degree 3 and 2 elements of degree 4 . On the other hand, with respect to an order in which $x_{1}$ is the largest, $\operatorname{in}\left(I_{2, \lambda}\right)$ is minimally generated by 3 elements of degree 3,3 elements of degree 4 , and an element of degree 6 . Computer experiment suggests that $\operatorname{in}\left(I_{l, \lambda}\right)$ with respect to an order in which $x_{1}$ is the smallest requires fewer generators.

Problem 3.13. With the notation of Corollary 3.11, is $\left\{f_{T} \mid T \in \operatorname{Tab}(l, \rho), \rho \unlhd \lambda\right\}$ a universal Gröbner basis of $I_{l, \lambda}$ ?

We have computed several partitions $\lambda$ up to $n=8$ using SageMath and Macaulay2, and we have not found a counter example yet.

## 4. A generalization of the case $\# Y_{1} \geq 2$ and $\# Y_{2}=\cdots=\# Y_{l}=1$

In this section, we fix a positive integer $m$ with $1 \leq m \leq n$, and set

$$
\Delta_{m}:=\Delta(\{1, \ldots, m\})=\prod_{1 \leq i<j \leq m}\left(x_{i}-x_{j}\right) .
$$

For $T \in \operatorname{Tab}(l, \lambda)$ with $\left[P_{n+l-1}\right]_{\geq l}$, set

$$
f_{m, T}:=\operatorname{lcm}\left\{f_{T}, \Delta_{m}\right\} \in S \quad \text { and } \quad I_{l, m, \lambda}:=\left(f_{m, T} \mid T \in \operatorname{Tab}(l, \lambda)\right) \subset S
$$

Note that $I_{l, 1, \lambda}=I_{l, \lambda}$ and $I_{l, n, \lambda}=\left(\Delta_{n}\right)$.
Example 4.1. Even if $l=1, I_{l, m, \lambda}$ is not a radical ideal in general, while their generators are squarefree products of linear forms $\left(x_{i}-x_{j}\right)$. For example, if $\lambda=$ $(2,2)$, we have

$$
I_{1,3, \lambda}=\left(\Delta_{3} \cdot\left(x_{1}-x_{4}\right), \Delta_{3} \cdot\left(x_{2}-x_{4}\right), \Delta_{3} \cdot\left(x_{3}-x_{4}\right)\right)
$$

where $\Delta_{3}=\left(x_{1}-x_{2}\right)\left(x_{1}-x_{3}\right)\left(x_{2}-x_{3}\right)$. (Note that an analog of Lemma 2.3 does not hold here. So we have to consider a non-standard tableau also to generate $I_{l, m, \lambda}$.) Clearly, $\Delta_{3} \notin I_{1,3, \lambda}$, but we can show that $\Delta_{3} \in \sqrt{I_{1,3, \lambda}}$ by Lemma 4.2 below. Moreover, the statement corresponding to Lemma 2.5 does not hold for $I_{l, m, \lambda}$. In fact, if $\lambda=(2,2)$ and $\mu=(2,1,1)$, then $\mu \triangleleft \lambda$, but $I_{1,3, \mu}=\left(\Delta_{3}\right) \not \subset I_{1,3, \lambda}$.

However, we have the following.
Lemma 4.2. For $\lambda, \mu \in\left[P_{n+l-1}\right]_{\geq l}$ with $\lambda \unrhd \mu$, we have $\sqrt{I_{l, m, \lambda}} \supset I_{l, m, \mu}$.
Proof. It suffices to show that $f_{m, T} \in \sqrt{I_{l, m, \lambda}}$ for all $T \in \operatorname{Tab}(l, \mu)$. By Lemma 2.5, there are some $k \in \mathbb{N}, T_{1}, \ldots, T_{k} \in \operatorname{Tab}(l, \lambda)$ and $g_{1}, \ldots, g_{k} \in S$ such that $f_{T}=$ $\sum g_{i} f_{T_{i}}$. Multiplying $\Delta_{m}$ to both sides, we have

$$
\Delta_{m} \cdot f_{T}=\sum g_{i} \cdot\left(\Delta_{m} \cdot f_{T_{i}}\right)
$$

Since $f_{m, T_{i}}$ divides $\Delta_{m} \cdot f_{T_{i}}$, we have $\Delta_{m} \cdot f_{T} \in I_{l, m, \lambda}$. However, since $\Delta_{m} \cdot f_{T}$ divides $\left(f_{m, T}\right)^{2}$, we have $\left(f_{m, T}\right)^{2} \in I_{l, m, \lambda}$.

For a lower filter $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$, set

$$
G_{l, m, \mathcal{F}}:=\left\{f_{m, T} \mid T \in \operatorname{Tab}(l, \lambda), \lambda \in \mathcal{F}\right\} \quad \text { and } \quad I_{l, m, \mathcal{F}}:=\left(G_{l, m, \mathcal{F}}\right)=\sum_{\lambda \in \mathcal{F}} I_{l, m, \lambda}
$$

For an upper filter $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$ with the lower filter $\mathcal{F}^{c}:=\left[P_{n+l-1}\right]_{\geq l} \backslash \mathcal{F}$, we consider the ideal

$$
J_{l, m, \mathcal{F}}:=\left(\Delta_{m}\right) \cap J_{l, \mathcal{F}}\left(=\left(\Delta_{m}\right) \cap I_{l, \mathcal{F c}}\right) .
$$

Since both $\left(\Delta_{m}\right)$ and $J_{l, \mathcal{F}}$ are radical ideals, so is $J_{l, m, \mathcal{F}}$. Since $J_{l, m, \mathcal{F}} \subset\left(\Delta_{m}\right)$, the codimension of $J_{l, m, \mathcal{F}}$ for $m \geq 2$ is 1 (unless $\mathcal{F}=\left[P_{n+l-1}\right]_{\geq l}$, equivalently, $J_{l, m, \mathcal{F}}=0$ ).
Theorem 4.3. Let $\mathcal{F} \subsetneq\left[P_{n+l-1}\right]_{\geq l}$ be a lower filter, and $\mathcal{F}^{c}:=\left[P_{n+l-1}\right]_{\geq l} \backslash \mathcal{F}$ its complement. Then $G_{l, m, \mathcal{F}}$ is a Gröbner basis of $J_{l, m, \mathcal{F}^{c}}$. Hence $J_{l, m, \mathcal{F}^{c}}=I_{l, m, \mathcal{F}}$, and $I_{l, m, \mathcal{F}}$ is a radical ideal.

Let us prepare the proof of Theorem 4.3.
Lemma 4.4. Let $\mathcal{F} \subset\left[P_{n+l-1}\right]_{\geq l}$ be an upper filter, and let $f$ be a polynomial in $J_{l, m, \mathcal{F}}$ of the form

$$
f=g_{d} x_{n}^{d}+\cdots+g_{1} x_{n}+g_{0}
$$

where $g_{0}, \ldots, g_{d} \in K\left[x_{1}, \ldots, x_{n-1}\right]$ and $g_{d} \neq 0$. If $m<n$, then $g_{0}, \ldots, g_{d}$ belong to $J_{l, m, \mathcal{F}_{d+1}}$.

Proof. Here we use the same notation as in the proof of Lemma 2.9, Take $\boldsymbol{a}=$ $\left(a_{1}, \ldots, a_{n-1}\right) \in K^{n-1}$. Since $f \in\left(\Delta_{m}\right)$, if $a_{i}=a_{j}$ for some $1 \leq i<j \leq m$, then $f(\boldsymbol{a}, \alpha)=0$ for all $\alpha \in K$, and hence $g_{i}(\boldsymbol{a})=0$ for all $i$. It means that each $g_{i}$ can be divided by $\Delta_{m}$ in $K\left[x_{1}, \ldots, x_{n-1}\right]$. So it remains to show that $g_{i} \in J_{l, \mathcal{F}_{d+1}}$, but it follows from Lemma [2.9, since $f \in J_{l, \mathcal{F}}$.
The proof of Theorem 4.3. First, we show that $G_{l, m, \mathcal{F}} \subset J_{l, m, \mathcal{F} c}$. For any $f_{m, T} \in$ $G_{l, m, \mathcal{F}}$, it is clear that $f_{m, T} \in\left(\Delta_{m}\right)$, and we have $f_{m, T} \in\left(f_{T}\right) \subset J_{l, \mathcal{F c}}$ by Theorem 2.6. Hence $f_{m, T} \in J_{l, m, \mathcal{F}^{c}}$.

So it remains to show that, for any $0 \neq f \in J_{l, m, \mathcal{F c}}$, there is some $f_{m, T} \in G_{l, m, \mathcal{F}}$ such that $\operatorname{in}\left(f_{m, T}\right)$ divides $\operatorname{in}(f)$, but it can be done by induction on $n-m$ (we fix $m$ ) in the same way as in the proof of Theorem [2.6, while we use Lemma 4.4 instead of Lemma 2.9 .

The following corollary immediately follows from Theorem 4.3,
Corollary 4.5. For $\lambda \in\left[P_{n+l-1}\right]_{\geq l}$,

$$
\bigcup_{\substack{\mu \in\left[P_{n+l-1]} \\ \mu \unlhd \lambda\right.}} G_{l, m, \mu}
$$

is a Gröbner basis of $\sqrt{I_{l, m, \lambda}}=J_{l, m, \mathcal{F}}$, where $\mathcal{F}$ is the upper filter $\left\{\nu \in\left[P_{n+l-1}\right]_{\geq l} \mid\right.$ $\nu \not \Perp \lambda\}$. In particular,

$$
\sqrt{I_{l, m, \lambda}}=\sum_{\substack{\mu \in\left[P_{n+l-1}\right]_{l} \\ \mu \unlhd \lambda}} I_{l, m, \mu} .
$$

Remark 4.6. If $\lambda=\left(\lambda_{1}, \ldots, \lambda_{p}\right) \in P_{n+l-1}$ is of the form $\lambda_{1}=\cdots=\lambda_{p-1}=k-1$ for some $k>l$, then our $\sqrt{I_{l, m, \lambda}}\left(=\sum_{\mu \unlhd \lambda} I_{l, m, \mu}\right)$ coincides with the Li-Li ideal $I_{\mathcal{Y}}$ for $\mathcal{Y}=\left(Y_{1}, Y_{2}, \ldots, Y_{k-1}\right)$ with $Y_{1}=\{1,2, \ldots, m\}, Y_{2}=\cdots=Y_{l}=\{1\}$ and $Y_{l+1}=\cdots=Y_{k-1}=\emptyset$ in the notation of the Introduction.
Proposition 4.7. $I_{l, m, \lambda}$ is a radical ideal for $m \leq 2$.
Proof. The case $m=1$ follows from Theorem 2.6. So we treat the case $m=2$. By Theorem 4.3, it suffices to show that $f_{2, T} \in I_{l, 2, \lambda}$ for all $T \in \operatorname{Tab}(l, \mu)$ with $\mu \unlhd \lambda$. If the letters 2 and (some copy of) 1 are not in the same column of $T$, then we have $f_{2, T}=\left(x_{1}-x_{2}\right) f_{T}$, and if they are in the same column, then we have $f_{2, T}=f_{T}$. We first treat the former case. Since $I_{l, \mu} \subset I_{l, \lambda}$ by Lemma 2.5, there are $g_{1}, \ldots, g_{k} \in S$ and $T_{1}, \ldots, T_{k} \in \operatorname{Tab}(l, \lambda)$ such that $f_{T}=\sum_{i=1}^{k} g_{i} f_{T_{i}}$. Multiplying $\left(x_{1}-x_{2}\right)$ to the both sides, we have

$$
f_{2, T}=\left(x_{1}-x_{2}\right) f_{T}=\sum_{i=1}^{k} g_{i} \cdot\left(x_{1}-x_{2}\right) f_{T_{i}} .
$$

Since $f_{2, T_{i}}$ divides $\left(x_{1}-x_{2}\right) f_{T_{i}}$, we have $f_{2, T} \in I_{l, 2, \lambda}$. So the case when 1 and 2 are in the same column (equivalently, $f_{2, T}=f_{T}$ ) remains. We may assume that $\lambda$ covers $\mu$, and we want to modify the argument of the proof of Lemma 2.5, which shows that $I_{l, \mu} \subset I_{l, \lambda}$. In the sequel, we use the same notation as there.

The crucial case is that $1,2 \in A$ (we may assume that $a_{1}=1, a_{2}=2$ ) and $1 \notin B$. In fact, in other cases, it is easy to see that $f_{2, T_{i}}=f_{T_{i}}$ for all $i$. By (2.3), we have

$$
f_{T}=\sum_{k-k^{\prime} \leq i \leq k}(-1)^{i-k+k^{\prime}}\left(x_{1}-x_{a_{i}}\right) f_{T_{i}}
$$

and $T_{i} \in \operatorname{Tab}(l, \lambda)$ for all $i$. For $i \geq 3$, the letters 1 and 2 stay in the same column of $T_{i}$, and we have $f_{2, T_{i}}=f_{T_{i}}$. So the case $k-k^{\prime} \geq 3$ is easy, and we may assume that $k-k^{\prime}=2$. Then, among $T_{2}, \ldots, T_{k}$, only $T_{2}$ does not have 1 and 2 in the same column. Hence

$$
\begin{aligned}
f_{2, T}=f_{T} & =\left(x_{1}-x_{2}\right) f_{T_{2}}+\sum_{3 \leq i \leq k}(-1)^{i}\left(x_{1}-x_{a_{i}}\right) f_{T_{i}} \\
& =f_{2, T_{2}}+\sum_{3 \leq i \leq k}(-1)^{i}\left(x_{1}-x_{a_{i}}\right) f_{2, T_{i}} \in I_{l, 2, \lambda} .
\end{aligned}
$$

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