# Regularity and long time behavior of one-dimensional first-order mean field games and the planning problem 

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#### Abstract

We study the regularity and long time behavior of the one-dimensional, local, first-order mean field games system and the planning problem, assuming a Hamiltonian of superlinear growth, with a nonseparated, strictly monotone dependence on the density. We improve upon the existing literature by obtaining two regularity results. The first is the existence of classical solutions without the need to assume blow-up of the cost function near small densities. The second result is the interior smoothness of weak solutions without the need to assume neither blow-up of the cost function nor that the initial density be bounded away from zero. We also characterize the long time behavior of the solutions, proving that they satisfy the turnpike property with an exponential rate of convergence, and that they converge to the solution of the infinite horizon system. Our approach relies on the elliptic structure of the system and displacement convexity estimates. In particular, we apply displacement convexity methods to obtain both global and local a priori lower bounds on the density.


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Keywords: displacement convexity; Hamilton-Jacobi equations; quasilinear elliptic equations; oblique derivative problems; Bernstein method; non-linear method of continuity.

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## 1 Introduction

The main purpose of this paper is to establish that, under very general conditions, the solutions to the onedimensional first-order mean field games system with local coupling (MFG for short) are smooth, and to fully

[^0]characterize their long time behavior. Specifically, we study the regularity of the solutions to standard MFG with a prescribed terminal condition,
\[

$$
\begin{cases}-u_{t}(x, t)+H\left(u_{x}(x, t), m(x, t)\right)=0 & (x, t) \in Q_{T}=\mathbb{T} \times(0, T)  \tag{MFG}\\ m_{t}(x, t)-\left(m(x, t) H_{p}\left(u_{x}(x, t), m(x, t)\right)\right)_{x}=0 & (x, t) \in Q_{T} \\ m(x, 0)=m_{0}(x), u(x, T)=g(m(x, T)) & x \in \mathbb{T}\end{cases}
$$
\]

as well as to the so-called planning problem with a prescribed terminal density,

$$
\begin{cases}-u_{t}(x, t)+H\left(u_{x}(x, t), m(x, t)\right)=0 & (x, t) \in Q_{T} \\ m_{t}(x, t)-\left(m(x, t) H_{p}\left(u_{x}(x, t), m(x, t)\right)\right)_{x}=0 & (x, t) \in Q_{T} \\ m(x, 0)=m_{0}(x), m(x, T)=m_{T}(x) & x \in \mathbb{T}\end{cases}
$$

(MFGP)
where $\mathbb{T}$ denotes the 1 -dimensional torus, $-H(p, m): \mathbb{R} \times(0, \infty) \rightarrow \mathbb{R}$ and $g(m):(0, \infty) \rightarrow \mathbb{R}$ are strictly increasing in $m, H$ has super-linear growth in $p$, and $m_{0}, m_{T}: \mathbb{T} \rightarrow[0,+\infty)$ are probability densities. We also show convergence of the solutions to each of these problems, as $T \rightarrow \infty$, to the solution of the infinite time horizon MFG system,

$$
\begin{cases}-v_{t}(x, t)+\lambda+H\left(v_{x}(x, t), \mu(x, t)\right)=0 & (x, t) \in \mathbb{T} \times(0, \infty)  \tag{MFGL}\\ \mu_{t}(x, t)-\left(\mu(x, t) H_{p}\left(v_{x}(x, t), \mu(x, t)\right)\right)_{x}=0 & (x, t) \in \mathbb{T} \times(0, \infty) \\ \mu(x, 0)=m_{0}(x) & x \in \mathbb{T}\end{cases}
$$

where $\lambda=-H(0,1)$.
MFG were introduced by Lasry and Lions [19, 16, and at the same time, in a particular setting, by Caines, Huang, and Malhamé [14. They are non-cooperative differential games with infinitely many players, in which the players find an optimal strategy, determined by the value function $u$, by observing the distribution $m$ of the other players.

Classical solutions to (MFG), in arbitrary dimension, were previously obtained by the second author in [21, 22], when the initial density is bounded away from 0 , and under the blow-up assumption

$$
\begin{equation*}
\lim _{m \rightarrow 0^{+}} H(p, m)=+\infty \tag{1}
\end{equation*}
$$

which, from the optimal control point of view, corresponds to placing a very strong incentive for players to occupy low-density regions and precludes the appearance of empty regions. A similar regularity result was recently obtained in [26] by A. Porretta for the case of (MFGP), when the Hamiltonian has the separated form $H(p, m) \equiv H(p)-f(m)$, and the terminal density $m_{T}$ is also bounded away from 0 .

Our first contribution is the following theorem, which shows that, in the one-dimensional problem, assumption (11) can be completely removed. We refer to Section 2 for assumptions (M), (H) (G), (E), (W) and (L) and to the notation subsection for the definition of the function spaces mentioned below.

Theorem 1.1. Let $0<\alpha<1$, and assume that (M), (H), and (E) hold. Then the following statements hold:
(i) There exists a classical solution $(u, m) \in C^{3, \alpha}\left(\overline{Q_{T}}\right) \times C^{2, \alpha}\left(\overline{Q_{T}}\right)$ to (MFGP). The function $m$ is unique, and $u$ is unique up to a constant.
(ii) There exists a unique classical solution $(u, m) \in C^{3, \alpha}\left(\overline{Q_{T}}\right) \times C^{2, \alpha}\left(\overline{Q_{T}}\right)$ to (MFG).

Our second result establishes interior smoothness of the solutions when, besides removing the assumption (11), one also weakens the lower bound assumptions for given densities $m_{0}$ and $m_{T}$, replacing the latter with the integrability conditions

$$
\begin{equation*}
\frac{1}{m_{0}^{\kappa}} \in L^{1}(\mathbb{T}), \frac{1}{m_{T}^{\kappa}} \in L^{1}(\mathbb{T}) \text { for some } \kappa>0 \tag{2}
\end{equation*}
$$

We observe that, in particular, (2) allows the initial density to vanish in a set of measure zero. In spite of this fact, our result also shows that $m$ becomes strictly positive instantly after the initial time. Moreover, in the case of (MFG), the density remains bounded below, and the solution remains smooth up to and including $t=T$. We refer to Section 6 for the definition of a weak solution.

Theorem 1.2. Let $0<\alpha<1$, and assume that (W) (H) and (E) hold. Then the following statements hold:
(i) There exists a weak solution

$$
(u, m) \in\left(\operatorname{BV}\left(Q_{T}\right) \cap L^{\infty}\left(Q_{T}\right)\right) \times\left(C\left([0, T], H^{-1}(\mathbb{T})\right) \cap L_{+}^{\infty}\left(Q_{T}\right)\right)
$$

to MFGP). Moreover, $(u, m) \in C_{\mathrm{loc}}^{3, \alpha}\left(Q_{T}\right) \times C_{\mathrm{loc}}^{2, \alpha}\left(Q_{T}\right)$ and $m>0$ in $(0, T)$. The function $m$ is unique, and $u$ is unique up to a constant.
(ii) Assume, further, that the function $H$ satisfies, for each $(p, m) \in \mathbb{R} \times(0, \infty)$,

$$
\begin{equation*}
H_{p}(p, m) p \geq 0 \tag{3}
\end{equation*}
$$

Then there exists a unique weak solution

$$
(u, m) \in\left(\operatorname{BV}\left(Q_{T}\right) \cap L^{\infty}\left(Q_{T}\right)\right) \times\left(C\left([0, T], H^{-1}(\mathbb{T})\right) \cap L_{+}^{\infty}\left(Q_{T}\right)\right)
$$

to MFG). Moreover, $(u, m) \in C_{\mathrm{loc}}^{3, \alpha}(\mathbb{T} \times(0, T]) \times C_{\mathrm{loc}}^{2, \alpha}(\mathbb{T} \times(0, T])$, and $m>0$ in $(0, T]$.
Concerning the long time behavior of (1.1), it was shown by P. Cardaliaguet and P.J. Graber in 3, Thm 5.1] that the rescaled solution $(x, s) \mapsto u(x, s T) / T$ converges, in a certain space $L^{p}(\mathbb{T} \times(\delta, 1))$, to the map $\lambda(1-s)$, while the rescaled density $(x, s) \mapsto m(x, s T)$ converges in $L^{p}(\mathbb{T} \times(0,1))$ to the invariant measure $\mu \equiv 1$. Our third result shows that, when the marginals are strictly positive, a much stronger statement holds. That is, the solutions satisfy the turnpike property with an exponential rate of convergence, and the limit as $T \rightarrow \infty$ of the pair $(u(t)-\lambda(T-t), m(t))$ can be fully characterized as the solution to (MFGL). We emphasize that this is a convergence result at the original time scale (compare with [6, Thm 2.6, Thm. 5.1] and [8, Thm 4.1, Thm. 5.3]).

Theorem 1.3. Assume that (M), (H), (G) (E), and (L), hold, and let $T>1$. Assume that $\left(u^{T}, m^{T}\right)$ is either the solution to (MFG), or the solution to (MFGP) that satisfies $\int_{\mathbb{T}} v^{T}\left(\cdot, \frac{T}{2}\right)=0$, where

$$
v^{T}(x, t):=u^{T}(x, t)-\lambda(T-t)
$$

Then the following holds:
(i) There exist constants $C, \omega>0$, independent of $T$, such that

$$
\left\|m^{T}(t)-1\right\|_{L^{\infty}(\mathbb{T})}+\left\|u_{x}^{T}(t)\right\|_{L^{\infty}(\mathbb{T})} \leq C\left(e^{-\omega t}+e^{-\omega(T-t)}\right), \quad t \in[0, T]
$$

Moreover, if ( $u^{T}, m^{T}$ ) solves (MFG), and (3) holds, we have

$$
\left\|m^{T}(t)-1\right\|_{L^{\infty}(\mathbb{T})}+\left\|u_{x}^{T}(t)\right\|_{L^{\infty}(\mathbb{T})} \leq C e^{-\omega t}, \quad t \in[0, T]
$$

(ii) There exist functions $(v, \mu)$ such that, for each $T_{0}>0$,

$$
v^{T} \rightarrow v \text { in } C^{3, \alpha}\left(\mathbb{T} \times\left[0, T_{0}\right]\right) \text { as } T \rightarrow \infty
$$

and

$$
m^{T} \rightarrow \mu \text { in } C^{2, \alpha}\left(\mathbb{T} \times\left[0, T_{0}\right]\right) \text { as } T \rightarrow \infty
$$

Moreover, one has

$$
\begin{equation*}
\lim _{t \rightarrow \infty} v(\cdot, t)=c, \quad \lim _{t \rightarrow \infty} \mu(\cdot, t)=1 \text { uniformly in } \mathbb{T} \tag{4}
\end{equation*}
$$

where

$$
c= \begin{cases}g(1) & \text { if }\left(u^{T}, m^{T}\right) \text { solves (MFG) } \\ 0 & \text { if }\left(u^{T}, m^{T}\right) \text { solves (MFGP) }\end{cases}
$$

Finally, $(v, \mu)$ is the unique classical solution to (MFGL) satisfying (4) and

$$
v \in W^{1, \infty}(\mathbb{T} \times(0, \infty)), \quad \mu^{-1} \in L^{\infty}(\mathbb{T} \times(0, \infty)), \quad \mu-1 \in L^{1}(\mathbb{T} \times(0, \infty)) \cap L^{\infty}(\mathbb{T} \times(0, \infty))
$$

In particular, since the Hamiltonian $H(p, m)$ is non-separated, our results yield well-posedness and regularity of MFG systems with congestion, such as

$$
\left\{\begin{array}{l}
-u_{t}+\frac{\left|u_{x}\right|^{2}}{2\left(m+c_{0}\right)^{\alpha}}=f(m) \text { in } Q_{T},  \tag{6}\\
m_{t}-\left(\frac{m}{\left(m+c_{0}\right)^{\alpha}} u_{x}\right)_{x}=0 \text { in } Q_{T},
\end{array}\right.
$$

where $0<\alpha<2, c_{0} \geq 0$, and $f^{\prime}>0$. Some of the key techniques used in [21, 22, 26, as well as in the present work, were developed by P.-L. Lions in his lectures at Collège de France [19], where he obtained several a priori estimates for the solutions to (MFGP), in the special case of a separated, quadratic Hamiltonian. The most important of these is the observation that the problems (MFG) and MFGP) can be transformed into a single quasilinear elliptic equation in $u$ after eliminating the variable $m$. Indeed, if one defines $H^{-1}$ by

$$
m=H^{-1}(p, H(p, m))
$$

then $m=H^{-1}\left(u_{x}, u_{t}\right)$ and the problem becomes

$$
\begin{cases}Q u:=-\operatorname{Tr}\left(A(D u) D^{2} u\right)=0 & \text { in } Q_{T}  \tag{Q}\\ N u:=B(x, t, u, D u)=0 & \text { on } \partial Q_{T}\end{cases}
$$

where $D u=\left(u_{x}, u_{t}\right)$ and, for $(x, z, p, s) \in \mathbb{T} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$
\begin{gather*}
A(p, s)=\left(H_{p}+\frac{1}{2} m H_{m p},-1\right) \otimes\left(H_{p}+\frac{1}{2} m H_{m p},-1\right)-\left(\begin{array}{cc}
\frac{1}{4} m^{2} H_{m p}^{2}+m H_{m} H_{p p} & 0 \\
0 & 0
\end{array}\right),  \tag{Q1}\\
B(x, 0, z, p, s)=-s+H\left(p, m_{0}(x)\right) \tag{B1}
\end{gather*}
$$

and

$$
B(x, T, z, p, s)= \begin{cases}s-H\left(p, g^{-1}(z)\right) & \text { in the case of (MFG) }  \tag{B2}\\ s-H\left(p, m_{T}(x)\right) & \text { in the case of (MFGP). }\end{cases}
$$

The condition for ellipticity, that is, for the matrix $A$ to be positive, is

$$
\begin{equation*}
-4 m H_{m} H_{p p}>m^{2} H_{m p}^{2} \tag{7}
\end{equation*}
$$

which is also the well-known condition for uniqueness to (MFG) that follows from the Lasry-Lions monotonicity method (see, for instance, Lions and P.E. Souganidis [20]). We remark from (77) that, in particular, the strict positivity of the density is crucial for the regularizing properties of the system. The lower bounds on $m$ obtained in Corollary 3.2 and Proposition 6.3 both heavily rely on the one-dimensionality assumption, and this is the main obstacle to generalizing our results to higher dimensions. Indeed, in dimensions $d>1$, it remains an open question whether the existence of smooth solutions to local first order MFG systems can still be established if one removes or significantly weakens (11), or if $m_{0}$ is not assumed to be bounded away from 0 . Even for $d=1$, it is still unknown whether one can allow $m_{0}$ or $m_{T}$ to vanish in a set of positive measure.

Section 2 contains all the assumptions that will be in place about the Hamiltonian $H$, as well as the initial and terminal data. In Section 3, we establish an integral displacement convexity formula (see Proposition 3.1) that will allow us to bound the density $m$ in terms of its initial and terminal values. Section 4 contains
the necessary a priori estimates that are needed to prove the existence of classical solutions. In particular, we obtain, in Section 4.1, estimates for an $\epsilon$-approximation of (MFGP) via standard MFG systems with a terminal condition of the type $u(\cdot, T)=g(\cdot, m(\cdot, T))$, which we require to prove existence for MFGP). Finally, we provide a counterexample to existence of solutions to (MFG) when the terminal cost function $g$ is also allowed to depend on the space variable (see Proposition 4.5). In Sections 56 and 7 we prove our main results, Theorems 1.1, 1.2, and 1.3, respectively.

We remark that, in the special case of a separated Hamiltonian, the displacement convexity estimates of Section 3 were first obtained by D. A. Gomes and T. Seneci in [11]. Further estimates on the density using displacement convexity were also obtained by T. Bakaryan, R. Ferreira, and Gomes in [1], and by Porretta in [26] (see also Lavenant, Santambrogio [17]). Weak solutions, as defined in Section 6, have been widely studied for both (MFG) (see [2, 3, 4, 7, 21]) and (MFGP) (see [13, 26, 23]). For classical solutions in the time-independent case we refer to Evans [9] and Gomes, Mitake [12]. Concerning the study of the long time behavior of solutions, specifically the second part of Theorem 1.3, we follow the program developed by Porretta and Cirant in [8, where a similar analysis was performed for second-order MFG systems, and, unlike the earlier work [6], does not involve the use of the master equation (see also [5, 25]).

## Notation

Let $d, k \in \mathbb{N}$. For $T>0$, we denote by $Q_{T}:=\mathbb{T} \times(0, T), \overline{Q_{T}}:=\mathbb{T} \times[0, T]$ and $\partial Q_{T}:=\mathbb{T} \times\{0, T\}$. For $\alpha \in(0,1], T>0$, and $\Omega \subset \mathbb{R}^{d}$ we denote by $C^{k+a}(\Omega)$, the standard space of $k$ times differentiable scalar functions with $\alpha$-Hölder continuous $k^{\text {th }}$ order derivatives, with the usual norm. Furthermore, we denote by $C_{l o c}^{k+\alpha}(\Omega)$ the functions $u$ that belong to $C^{k+\alpha}(K)$, for all compact sets $K \subset \Omega$. For functions $u: \mathbb{T} \times[0, T] \rightarrow \mathbb{R}$, we denote by osc $u:=\max _{(x, t) \in \mathbb{T} \times[0, T]} u(x, t)-\min _{(x, t) \in \mathbb{T} \times[0, T]} u(x, t), D u(x, t):=\left(u_{x}(x, t), u_{t}(x, t)\right)$. We denote by $H^{-1}(\mathbb{T})$ the dual space of the Sobolev space $H^{1}(\mathbb{T})$, and the space of $H^{-1}\left(\mathbb{T}^{d}\right)$-valued $\alpha$-Hölder continuous functions by $C^{0, \alpha}\left([0, T] ; H^{-1}\left(\mathbb{T}^{d}\right)\right)$. We write $C=C\left(K_{1}, K_{2}, \ldots, K_{M}\right)$ for a positive constant $C$ depending monotonically on the non-negative quantities $K_{1}, \ldots, K_{M} . \mathrm{BV}\left(Q_{T}\right)$ denotes the space of functions of bounded variation, and $L_{+}^{\infty}\left(Q_{T}\right)$ consists of the functions $m \in L^{\infty}\left(Q_{T}\right)$ such that $m \geq 0$ a.e. in $Q_{T}$.

## 2 Assumptions

In what follows, $C_{0}$ and $\gamma, \alpha$ are positive constants, with $\gamma>1$, and $0<\alpha<1$. Moreover, $\bar{C}:(0, \infty) \rightarrow(0, \infty)$ is a continuous, strictly positive function. Except when explicitly stated, assumptions (M), (H), (G), and (E) will be in place throughout the paper.
(M) (Assumptions on $m_{0}$ and $m_{T}$ for classical solutions) The given functions $m_{0}$ and $m_{T}$ satisfy

$$
\begin{equation*}
m_{0}, m_{T} \in C^{2, \alpha}(\mathbb{T}), m_{0}, m_{T}>0, \text { and } \int_{\mathbb{T}} m_{0}=\int_{\mathbb{T}} m_{T}=1 \tag{M1}
\end{equation*}
$$

(H) (Assumptions on $H$ ) The functions $H, H_{p}$, and $H_{p p}$ are in $C^{4}(\mathbb{R} \times(0, \infty))$, and $H_{m}<0$. Moreover, for $(p, m) \in \mathbb{R} \times(0, \infty)$,

$$
\begin{gather*}
\frac{1}{C_{0}}(1+|p|)^{\gamma-2} \leq H_{p p} \leq \bar{C}(m)(1+|p|)^{\gamma-2}  \tag{H1}\\
p H_{p} \geq\left(1+\frac{1}{C_{0}}\right) H-\bar{C}(m)  \tag{H2}\\
\left|H_{p p p}\right| \leq \bar{C}(m)(1+|p|)^{\gamma-3},  \tag{H3}\\
\left|H_{m}\right| \leq \bar{C}(m)(1+|p|)^{\gamma}  \tag{HM1}\\
m\left|H_{m m}\right| \leq-\bar{C}(m) H_{m}, \quad|p| \| H_{m p}\left|\leq-\bar{C}(m) H_{m}, \quad m\right| p| | H_{m m p} \mid \leq-\bar{C}(m) H_{m} \tag{HM2}
\end{gather*}
$$

$$
\begin{equation*}
\left|H_{m p p}\right| \leq \bar{C}(m)(1+|p|)^{\gamma-2} \tag{HM3}
\end{equation*}
$$

(G) (Assumptions on $g$ ) The function $g:(0, \infty) \rightarrow \mathbb{R}$ is four times continuously differentiable and satisfies, for all $m>0$,

$$
\begin{equation*}
g^{\prime}(m)>0 \tag{G1}
\end{equation*}
$$

(E) (Ellipticity of the system) The function $H$ satisfies, for $m>0$, the condition

$$
\begin{equation*}
-4 m H_{m} H_{p p} \geq\left(1+\frac{1}{C_{0}}\right) m^{2} H_{m p}^{2} \tag{E1}
\end{equation*}
$$

(W) (Assumptions on $m_{0}, m_{T}, H$, and $g$ for weak solutions) The functions $m_{0}$ and $m_{T}$ satisfy, for some $\kappa>0$,

$$
\begin{equation*}
m_{0}, m_{T} \in L^{\infty}(\mathbb{T}), m_{0}, m_{T} \geq 0, \int_{\mathbb{T}} m_{0}=\int_{\mathbb{T}} m_{T}=1, \text { and } \frac{1}{m_{0}^{\kappa}}, \frac{1}{m_{T}^{\kappa}} \in L^{1}(\mathbb{T}) \tag{MW}
\end{equation*}
$$

$H$ satisfies, for some constant $s \in(-\kappa-1, \kappa-1)$, and for $(p, m) \in \mathbb{R} \times\left(0, \frac{1}{C_{0}}\right)$,

$$
\begin{equation*}
-H_{m}(0, m) \leq C_{0} m^{s},-H_{m}(p, m) \geq \frac{1}{C_{0}} m^{s} \tag{HW}
\end{equation*}
$$

and $g$ satisfies

$$
\begin{equation*}
\lim _{m \rightarrow 0^{+}} g(m)>-\infty \tag{GW}
\end{equation*}
$$

(L) (Assumption on $H$ for the long time average) The function $H$ satisfies, for $(p, m) \in \mathbb{R} \times(0, \infty)$,

$$
\begin{equation*}
-m H_{m}(p, m) \geq \frac{1}{\bar{C}(m)} \tag{HL}
\end{equation*}
$$

## 3 Displacement convexity and estimates on the density

To obtain estimates for the density at interior times, we will prove an integral formula which, in particular, implies that the quantity

$$
\int_{\mathbb{T}} h(m(x, \cdot)) d x
$$

is a convex function in $[0, T]$ whenever $h$ is convex, provided that (7) holds.
Proposition 3.1. Let $(u, m) \in C^{2}\left(\bar{Q}_{T}\right) \times C^{1}\left(\bar{Q}_{T}\right)$ be a classical solution to

$$
\begin{cases}-u_{t}+H\left(u_{x}, m\right)=0, & \text { in } Q_{T}  \tag{8}\\ m_{t}-\left(m H_{p}\left(u_{x}, m\right)\right)_{x}=0, & \text { in } Q_{T} \\ m(\cdot, 0)=m_{0}, & \text { in } \mathbb{T},\end{cases}
$$

and let $h \in W^{2, \infty}(\mathbb{R})$. Then

$$
\begin{align*}
& \frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m(x, t)) d x=\int_{\mathbb{T}} h^{\prime \prime}(m)\left(m_{t}-m_{x}\left(H_{p}+\frac{m}{2} H_{p m}\right)\right)^{2} d x \\
&-\int_{\mathbb{T}} h^{\prime \prime}(m)\left(m_{x}\right)^{2}\left(\frac{m^{2}}{4} H_{p m}^{2}+m H_{p p} H_{m}\right) d x \tag{9}
\end{align*}
$$

Moreover, there exists $C=C\left(C_{0}\right)$ such that, if $h^{\prime \prime}>0$,

$$
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m(x, t)) d x \geq \frac{1}{C} \int_{\mathbb{T}} h^{\prime \prime}(m)\left(-m H_{m} H_{p p} m_{x}^{2}+m^{2} H_{p p}^{2} u_{x x}^{2}\right) d x
$$

Proof. Let $g: \mathbb{R} \rightarrow \mathbb{R}$, be a smooth function. Since $m$ satisfies the continuity equation, the following holds for each $t \in[0, T]$ :

$$
\begin{equation*}
\int_{\mathbb{T}}\left(m_{t}(x, t)-\left(m(x, t) H_{p}\left(u_{x}, m(x, t)\right)\right)_{x}\right)\left(\partial_{t} g(m(x, t))-\left(g(m(x, t)) H_{p}\left(u_{x}, m(x, t)\right)\right)_{x}\right) d x=0 \tag{10}
\end{equation*}
$$

Expanding equation (10), we obtain

$$
\begin{aligned}
0=\int_{\mathbb{T}}\left(m_{t}-m_{x}\left(H_{p}+\right.\right. & \left.\left.m H_{p m}\right)-m H_{p p} u_{x x}\right)\left(g^{\prime}(m) m_{t}-m_{x}\left(g^{\prime}(m) H_{p}+g(m) H_{p m}\right)-g(m) H_{p p} u_{x x}\right) d x \\
=\int_{\mathbb{T}} & g^{\prime}(m)\left(m_{t}\right)^{2}-m_{t} m_{x}\left[2 g^{\prime}(m) H_{p}+\left(g^{\prime}(m) m+g(m)\right) H_{p m}\right] \\
& +m_{x} H_{p p} u_{x x}\left[H_{p}\left(g^{\prime}(m) m+g(m)\right)+2 g(m) m H_{p m}\right] \\
& +m_{x}^{2}\left[\left(H_{p}+m H_{p m}\right)\left(g^{\prime}(m) H_{p}+g(m) H_{p m}\right)\right] \\
& -m_{t} H_{p p} u_{x x}\left[H_{p}\left(g^{\prime}(m) m+g(m)\right)+2 g(m) m H_{p m}\right] \\
& +g(m) m\left(H_{p p} u_{x x}\right)^{2} d x=A_{1}-A_{2}+A_{3}+A_{4}-A_{5}+A_{6}
\end{aligned}
$$

We split term $A_{3}$ as follows

$$
A_{3}=\int_{\mathbb{T}} m_{x} H_{p p} H_{p} u_{x x}\left(g^{\prime}(m) m+g(m)\right) d x+2 \int_{\mathbb{T}} g(m) m_{x} m H_{p m} H_{p p} u_{x x} d x=A_{3.1}+A_{3.2}
$$

From the continuity equation, we have that

$$
m H_{p p} u_{x x}=m_{t}-m_{x}\left(H_{p}+m H_{p m}\right)
$$

Hence, terms $A_{3.2}$ and $A_{6}$ can be written as

$$
\begin{gathered}
A_{3.2}=2 \int_{\mathbb{T}} m_{t} m_{x} H_{p m} g(m) d x-2 \int_{\mathbb{T}}\left(m_{x}\right)^{2} H_{p m}\left(g(m) H_{p}+m g(m) H_{p m}\right) d x=A_{3.2 .1}-A_{3.2 .2} \\
A_{6}=\int_{\mathbb{T}} \frac{g(m)}{m}\left[m_{t}-m_{x}\left(H_{p}+m H_{p m}\right)\right]^{2} d x \\
=\int_{\mathbb{T}} \frac{g(m)}{m}\left(m_{t}\right)^{2}-2 \frac{g(m)}{m} m_{t} m_{x}\left(H_{p}+m H_{p m}\right)+\frac{g(m)}{m}\left(m_{x}\right)^{2}\left(H_{p}+m H_{p m}\right)^{2} d x=A_{6.1}-A_{6.2}+A_{6.3} .
\end{gathered}
$$

From the Hamilton-Jacobi (HJ for short) equation, we have that

$$
H_{p} u_{x x}=u_{x t}-H_{m} m_{x}
$$

Therefore, $A_{3.1}$ may be written as

$$
A_{3.1}=\int_{\mathbb{T}} m_{x} H_{p p} u_{x t}\left(g^{\prime}(m) m+g(m)\right) d x-\int_{\mathbb{T}}\left(m_{x}\right)^{2} H_{p p} H_{m}\left(g^{\prime}(m) m+g(m)\right) d x=A_{3.1 .1}-A_{3.1 .2}
$$

We now begin by grouping together terms $A_{5}$, and $A_{3.1 .1}$, which yields, for $L(m)=g(m) m, L^{\prime}(m)=g(m)+$ $m g^{\prime}(m)$,

$$
\begin{aligned}
-A_{5}+A_{3.1 .1} & =\int_{\mathbb{T}} m_{x}\left(g(m)+m g^{\prime}(m)\right) H_{p p} u_{x t}-\left(g(m)+m g^{\prime}(m)\right) m_{t} H_{p p} u_{x x} d x \\
& =\int_{\mathbb{T}}-\partial_{t}(L(m))\left(H_{p}\right)_{x}+L^{\prime}(m) m_{t} H_{p m} m_{x}+(L(m))_{x} \partial_{t}\left(H_{p}\right)-L^{\prime}(m) m_{x} m_{t} H_{p m} d x \\
& =\int_{\mathbb{T}} \partial_{t}\left((L(m))_{x}\right) H_{p}+(L(m)) \partial_{t}\left(H_{p}\right) d x=\frac{d}{d t} \int_{\mathbb{T}}(L(m))_{x} H_{p} d x
\end{aligned}
$$

Next, we group together all the terms with $m_{t} m_{x}$ factor, namely $A_{2}, A_{3.2 .1}$, and $A_{6.2}$, which yields

$$
-A_{2}+A_{3.2 .1}-A_{6.2}=-\int_{\mathbb{T}} 2 m_{t} m_{x}\left(g^{\prime}(m)+\frac{g(m)}{m}\right)\left(H_{p}+\frac{m}{2} H_{p m}\right) d x .
$$

Collecting the terms involving $\left(m_{t}\right)^{2}$, namely terms $A_{1}$ and $A_{6.1}$, we obtain

$$
A_{1}+A_{6.1}=\int_{\mathbb{T}}\left(m_{t}\right)^{2}\left(g^{\prime}(m)+\frac{g(m)}{m}\right) d x
$$

Finally, we group together the terms involving $m_{x}^{2}$, namely $A_{4}, A_{3.2 .2}, A_{6.3}$, and $A_{3.1 .2}$ :

$$
\begin{aligned}
& A_{4}-A_{3.2 .2}+A_{6.3}-A_{3.1 .2}= \\
& \qquad \begin{aligned}
\int_{\mathbb{T}}\left(m_{x}\right)^{2}\left[\left(g^{\prime}(m)+\right.\right. & \left.\left.\frac{g(m)}{m}\right)\left(H_{p}+\frac{m}{2} H_{p m}\right)^{2}\right] d x \\
& -\int_{\mathbb{T}}\left(m_{x}\right)^{2}\left[\left(g^{\prime}(m)+\frac{g(m)}{m}\right)\left(\frac{m^{2}}{4} H_{p m}^{2}+m H_{p p} H_{m}\right)\right] d x .
\end{aligned}
\end{aligned}
$$

Thus, putting everything together, we obtain

$$
\begin{align*}
-\frac{d}{d t} \int_{\mathbb{T}}(L(m))_{x} H_{p} d x=\int_{\mathbb{T}}\left(g^{\prime}(m)+\frac{g(m)}{m}\right)( & \left.m_{t}-m_{x}\left(H_{p}+\frac{m}{2} H_{p m}\right)\right)^{2} d x \\
& -\int_{\mathbb{T}} m_{x}^{2}\left(g^{\prime}(m)+\frac{g(m)}{m}\right)\left(\frac{m^{2}}{4} H_{p m}^{2}+m H_{p p} H_{m}\right) d x . \tag{11}
\end{align*}
$$

Next, notice that for a smooth function $h: \mathbb{R} \rightarrow \mathbb{R}$, we have

$$
\frac{d}{d t} \int_{\mathbb{T}} h(m) d x=\int_{\mathbb{T}}(h(m))_{x} H_{p}+m h^{\prime}(m)\left(H_{p}\right)_{x} d x=\int_{\mathbb{T}}\left(h(m)-h^{\prime}(m) m\right)_{x} H_{p} d x .
$$

Thus, if we require that

$$
-L(m)=h(m)-h^{\prime}(m) m,
$$

we obtain

$$
-\frac{d}{d t} \int_{\mathbb{T}}(L(m))_{x} H_{p} d x=\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m) d x .
$$

The relation between $h, g$ is

$$
m g(m)=h^{\prime}(m) m-h(m),
$$

therefore

$$
g(m)=-\frac{h(m)}{m}+h^{\prime}(m),
$$

and, thus,

$$
g^{\prime}(m)+\frac{g(m)}{m}=-\frac{h^{\prime}(m)}{m}+\frac{h(m)}{m^{2}}+h^{\prime \prime}(m)-\frac{h(m)}{m^{2}}+\frac{h^{\prime}(m)}{m}=h^{\prime \prime}(m),
$$

from which (9) follows.
Now, setting $r=1-\frac{1}{1+C_{0}^{-1}}$, we have

$$
-\frac{m^{2}}{2} H_{p m}^{2}-m H_{m} H_{p p}=-\frac{m^{2}}{2} H_{p m}^{2}-(1-r) m H_{m} H_{p p}-r m H_{m} H_{p p},
$$

and so, applying (E), and multiplying by $h^{\prime \prime}(m) m_{x}^{2}$, (9) yields

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m(x, t)) d x \geq \int_{\mathbb{T}}-r h^{\prime \prime}(m) m H_{m} H_{p p} m_{x}^{2} \tag{12}
\end{equation*}
$$

On the other hand, we infer from (E) that

$$
\begin{align*}
&\left(m_{t}\right.\left.-m_{x}\left(H_{p}+\frac{m}{2} H_{p m}\right)\right)^{2}-m_{x}^{2}\left(\frac{m^{2}}{2} H_{p m}^{2}+m H_{m} H_{p p}\right) \\
& \geq\left(m_{t}-m_{x} H_{p}-\frac{m_{x} m}{2} H_{p m}\right)^{2}+\frac{1}{C_{0}}\left(\frac{m_{x} m}{2} H_{p m}\right)^{2}=\left(m_{t}-m_{x} H_{p}\right)^{2}-2\left(m_{t}-m_{x} H_{p}\right) \frac{m_{x} m}{2} H_{p m} \\
&+(1-r)^{-1}\left(\frac{m_{x} m}{2} H_{p m}\right)^{2}=r\left(m_{t}-m_{x} H_{p}\right)^{2}+\left((1-r)^{\frac{1}{2}}\left(m_{t}-m_{x} H_{p}\right)-(1-r)^{-\frac{1}{2}} \frac{m_{x} m}{2} H_{p m}\right)^{2} \\
& \geq r\left(m_{t}-m_{x} H_{p}\right)^{2}=r m^{2} H_{p p}^{2} u_{x x}^{2} \tag{13}
\end{align*}
$$

where the last equality follows from the equation of $m$. As before, multiplying by $h^{\prime \prime}(m)$ then yields

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m(x, t)) d x \geq \int_{\mathbb{T}} r h^{\prime \prime}(m) m^{2} H_{p p}^{2} u_{x x}^{2} \tag{14}
\end{equation*}
$$

Combining (12) and (14), we conclude that (3.1) holds.
It now follows readily that the density of the solution is bounded above and below in terms of the initial and terminal densities.

Corollary 3.2. Let $(u, m) \in C^{2}\left(\bar{Q}_{T}\right) \times C^{1}\left(\bar{Q}_{T}\right)$ be a classical solution to (MFG) or MFGP). Then, if $c_{1}:=\min \left(\min m_{0}, \min m(\cdot, T)\right), C_{1}=\max \left(\max m_{0}, \max m(\cdot, T)\right)$, we obtain that

$$
\begin{equation*}
c_{1} \leq m(x, t) \leq C_{1}, \text { for all }(x, t) \in \bar{Q}_{T} \tag{15}
\end{equation*}
$$

Proof. The proof follows directly from Proposition 3.1 above. Indeed, note that, in view of (E), for any convex function $h$, the map

$$
C(t):=\int_{\mathbb{T}} h(m(x, t)) d x
$$

is convex, and thus

$$
C(t) \leq \max (C(0), C(T)), \text { for all } t \in[0, T]
$$

Hence, setting $h_{p}(m)=m^{p}$ and letting $p \rightarrow-\infty$ yields the result for the lower bound, whereas letting $p \rightarrow+\infty$ yields the upper bound.

Remark 3.3. For dimensions $d>1$, formula (9) is no longer true. If one repeats the same argument, the issue will arise at the term $A_{6.2}$. However, in the case of a separated Hamiltonian, i.e. $H(p, m) \equiv H(p)-f(m)$, one still obtains the weaker formula

$$
\begin{align*}
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m(x, t)) d x=\int_{\mathbb{T}}\left(\left(h^{\prime \prime}(m) m^{2}\right.\right. & \left.-h^{\prime}(m) m+h(m)\right)\left(\operatorname{tr}\left(D_{p p}^{2} H D_{x x}^{2} u\right)\right)^{2} \\
& \left.+\left(h^{\prime}(m) m-h(m)\right) \operatorname{tr}\left(\left(D_{p p}^{2} H D_{x x}^{2} u\right)^{2}\right)+h^{\prime \prime}(m) m f^{\prime}(m)|D m|^{2}\right) d x \tag{16}
\end{align*}
$$

In this higher-dimensional setting, it is no longer true that the left hand side is convex whenever $h$ is convex. In particular, the statement is false for negative powers of $m$, but true for positive powers. Thus, from the proof of Corollary 3.2 we see that the upper bound on $m$ still holds (see [11]).

## 4 Estimates on the solution and the terminal density

In this section we obtain the necessary a priori $L^{\infty}$-bounds on $u, D u$, and $m(\cdot, T)$ for solutions to both (MFG) and (MFGP). Combined with the results of the previous section, this will yield global upper and lower bounds on the density. In order to treat the setting of Theorem 1.2 where the density may vanish at $\{0, T\}$, we also obtain $L^{\infty}$-bounds on $u$ that do not depend on the quantities $\left(\min m_{0}\right)^{-1},\left(\min m_{T}\right)^{-1}$.

Proposition 4.1. Let $(u, m) \in C^{2}\left(\bar{Q}_{T}\right) \times C^{1}\left(\bar{Q}_{T}\right)$ be a classical solution to (MFG), and let $c_{1}=\min m_{0}, C_{1}=$ $\max m_{0}$. Then, for each $(x, t) \in \bar{Q}_{T}$,

$$
\begin{gather*}
c_{1} \leq m(x, T) \leq C_{1} \text { for all } x \in \mathbb{T}  \tag{17}\\
H\left(0, c_{1}\right)(t-T)+g\left(c_{1}\right) \leq u(x, t) \leq H\left(0, C_{1}\right)(t-T)+g\left(C_{1}\right) \text { for all }(x, t) \in \bar{Q}_{T}, \tag{18}
\end{gather*}
$$

and

$$
\begin{equation*}
-\int_{t}^{T} H\left(0, \min _{\mathbb{T}}(m(\cdot, s)) d s+g\left(c_{1}\right) \leq u(x, t) \leq-\int_{t}^{T} H\left(0, \max _{\mathbb{T}}(m(\cdot, s)) d s+g\left(C_{1}\right)\right.\right. \tag{19}
\end{equation*}
$$

Proof. We will only show the lower bounds, since the argument for the upper bounds is completely symmetrical. We fix $\delta>0$ and let $\epsilon>0$ be such that

$$
H\left(0, c_{1}\right)-H\left(0, c_{1}-\delta\right)<-\epsilon T, \text { for all } x \in \mathbb{T}
$$

We define

$$
w^{\epsilon, \delta}(t):=H\left(0, c_{1}-\delta\right)(t-T)+\frac{\epsilon}{2}(t-T)^{2}+g\left(c_{1}-\delta\right)
$$

and note that

$$
w_{x x}=0, w_{x, t}=0, w_{t t}=\epsilon
$$

The function $v^{\epsilon, \delta}(x, t):=u(x, t)-w^{\epsilon, \delta}(t)$ has a minimum at some $\left(x_{0}, t_{0}\right) \in \bar{Q}_{T}$. If we first assume that $t_{0} \in(0, T)$, then it follows that

$$
D^{2} u-D^{2} w^{\epsilon, \delta} \geq 0
$$

which, in view of (Q), implies

$$
0=-\operatorname{Tr}\left(A D^{2} u\right) \leq-\operatorname{Tr}\left(A D^{2} w^{\epsilon, \delta}\right)=-\epsilon<0
$$

a contradiction. On the other hand, assume that $t_{0}=0$. Then,

$$
u_{t}\left(x_{0}, 0\right) \geq w_{t}^{\epsilon, \delta}\left(x_{0}, 0\right), u_{x}\left(x_{0}, 0\right)=w_{x}^{\epsilon, \delta}(0)=0
$$

and thus

$$
\begin{aligned}
& 0=-u_{t}\left(x_{0}, t_{0}\right)+H\left(0, m_{0}\left(x_{0}\right)\right) \leq-w_{t}^{\epsilon, \delta}(0)+H\left(0, m_{0}\left(x_{0}\right)\right)=-H\left(0, c_{0}-\delta\right)+H\left(0, m_{0}\left(x_{0}\right)\right)+\epsilon T \\
& \leq-H\left(0, c_{1}-\delta\right)+H\left(0, c_{1}\right)+\epsilon T<0
\end{aligned}
$$

by our choice of $\epsilon$, which is a contradiction. Hence, the minimum must be achieved at $t_{0}=T$. At that point, we have

$$
u_{t}\left(x_{0}, T\right) \leq w_{t}^{\epsilon, \delta}(T), u_{x}\left(x_{0}, T\right)=w_{x}^{\epsilon, \delta}(T)=0
$$

Consequently,

$$
\begin{gathered}
u\left(x_{0}, T\right)=g\left(H^{-1}\left(0, u_{t}\left(x_{0}, T\right)\right)\right) \geq g\left(H^{-1}\left(0, w_{t}^{\epsilon, \delta}(T)\right)\right)=g\left(H^{-1}\left(0, H\left(0, c_{1}-\delta\right)\right)\right) \\
=g\left(c_{1}-\delta\right)=w^{\epsilon, \delta}(T)
\end{gathered}
$$

We have thus shown that

$$
u(x, t) \geq w^{\epsilon, \delta}(t), \text { for all }(x, t) \in \bar{Q}_{T} .
$$

Letting $\epsilon \rightarrow 0$, and then $\delta \rightarrow 0$, yields the lower bound in (18) In particular, for $t=T$, we have

$$
g(m(x, T)) \geq g\left(c_{1}\right) \text { for all } x \text { in } \mathbb{T}
$$

which proves the lower bound in (17). Now, we define

$$
w(t)=-\int_{t}^{T} H(0, c(s)) d s+g\left(c_{1}\right)
$$

where $c(s):=\min _{\mathbb{T}}\{m(\cdot, s)\}$ is the running minimum of the density. We observe that the function $v(x, t)=$ $u(x, t)-w(t)$ satisfies $v_{t}=u_{t}-H(0, c(t)), v_{x}=u_{x}$. Thus, for any $\epsilon>0$, at any extremum point of $v-\epsilon t$, one has $v_{t}=H(0, m)-H(0, c(t))-\epsilon<0$. Letting $\epsilon \rightarrow 0$ thus implies that $v$ achieves its minimum at $t=T$. Therefore, using (17), we obtain

$$
u(x, t)-w(t) \geq \min _{\mathbb{T}} g(m(\cdot, T))-g\left(c_{1}\right) \geq 0
$$

and this is precisely the lower bound in (19).
Now, for solutions to (MFGP), we do not need to estimate the terminal density, as it is part of the given data. Concerning $u$, since the solution is only unique up to a constant, we may only bound the oscillation of $u$, and this is done in the following proposition.
Proposition 4.2. Let $(u, m) \in C^{2}\left(\bar{Q}_{T}\right) \times C^{1}\left(\bar{Q}_{T}\right)$ solve (8). There exists a constant $C>0$, with

$$
C=C\left(C_{0}, \int_{0}^{T}\left|H\left(0, \min _{\mathbb{T}} m(\cdot, s)\right)\right| d s, \bar{C}\left(\max _{\bar{Q}_{T}} m\right)\right)
$$

such that

$$
\operatorname{osc}_{\bar{Q}_{T}} u \leq C\left(\left.T+T^{-\frac{1}{\gamma-1}}+\int_{0}^{T} \right\rvert\, H\left(0, \min _{\mathbb{T}} m(\cdot, s) \mid d s\right) .\right.
$$

Proof. We define the functions $c$ and $w$, for $t \in[0, T]$, by

$$
c(t)=\min _{\mathbb{T}} m(\cdot, t), \quad w(t)=-\int_{t}^{T} H(0, c(s)) d s
$$

Arguing as in the proof of (19), we obtain

$$
\begin{equation*}
\max _{\bar{Q}_{T}}(u-w)=\max _{\mathbb{T}}\left((u(\cdot, 0)-w(0)), \min _{\bar{Q}_{T}}(u-w)=\min _{\mathbb{T}}(u(\cdot, T)-w(T)) .\right. \tag{20}
\end{equation*}
$$

Now, in view of (H1) and Proposition 4.1, $0=-u_{t}+H\left(u_{x}, m\right) \geq-u_{t}+\frac{1}{C}\left|u_{x}\right|^{\gamma}-C$. Next, we define $\gamma^{\prime}$ by $\frac{1}{\gamma}+\frac{1}{\gamma^{\prime}}=1$. By the Hopf-Lax formula, the function

$$
v(x, t)=\min _{y \in \mathbb{R}}\left(\left(\frac{C}{\gamma}\right)^{\frac{\gamma^{\prime}}{\gamma}}(T-t) \frac{|x-y| \gamma^{\gamma^{\prime}}}{\gamma^{\prime}(T-t)^{\gamma^{\prime}}}+C(T-t)+u(y, T)\right)
$$

then solves, in $\bar{Q}_{T}$,

$$
-v_{t}(x, t)+\frac{1}{C}\left|v_{x}\right|^{\gamma}-C=0, v(\cdot, T)=u(\cdot, T)
$$

and, thus, by the comparison principle,

$$
u \leq v
$$

On the other hand, up to increasing the constant $C$,

$$
v(x, 0) \leq \frac{C}{T \gamma^{\prime}-1}+C T+\min _{\mathbb{T}} u(\cdot, T)
$$

and so

$$
\max _{\mathbb{T}} u(\cdot, 0) \leq \max _{\mathbb{T}} v(\cdot, 0) \leq \frac{C}{T^{\gamma^{\prime}-1}}+C T+\min _{\mathbb{T}} u(\cdot, T)
$$

In view of (20), we obtain

$$
\operatorname{osc}_{\bar{Q}_{T}}(u-w) \leq \frac{C}{T \gamma^{\prime}-1}+C T+w(T)-w(0)
$$

and, thus,

$$
\operatorname{osc}_{\bar{Q}_{T}} u \leq \frac{C}{T^{\gamma^{\prime}-1}}+C T+2 \cdot \operatorname{osc}_{\bar{Q}_{T}} w \leq \frac{C}{T^{\gamma^{\prime}-1}}+C T+2 \int_{0}^{T}|H(0, c(s))| d s
$$

We finally obtain a priori estimates on the gradient of $u$, while simultaneously treating the case of (MFG) and (MFGP). The proof closely follows [21, Lem. 3.8] and [22, Lem 3.3], but yields a slightly stronger estimate due to the $d=1$ assumption (see (23)). For the purpose of studying the long time behavior, we will keep track of the dependence of $T$ for large values of $T$.

Proposition 4.3. Let $(u, m) \in C^{3}\left(\bar{Q}_{T}\right) \times C^{2}\left(\bar{Q}_{T}\right)$ be a classical solution to (MFG) or (MFGP). There exists a constant $C>0$, with

$$
\begin{aligned}
& C=C\left(C_{0}, T, T^{-1}, \text { osc } u, \gamma,\|m\|_{L^{\infty}\left(\bar{Q}_{T}\right)},\left\|m^{-1}\right\|_{L^{\infty}\left(\bar{Q}_{T}\right)},\right. \\
& \left.\qquad\left\|\left(m_{0}\right)_{x}\right\|_{L^{\infty}(\mathbb{T})},\left\|\left(m_{T}\right)_{x}\right\|_{L^{\infty}(\mathbb{T})},\|\bar{C}\|_{L^{\infty}[\min m, \max m]}\right)
\end{aligned}
$$

such that

$$
\|D u\|_{L^{\infty}\left(\bar{Q}_{T}\right)}^{\gamma} \leq C
$$

Proof. Since $u_{t}=H\left(u_{x}, m\right)$, and $m$ is bounded above and below, we infer from (H1) and (H2) that it is enough to show that

$$
\left\|u_{x}\right\|_{L^{\infty}\left(\bar{Q}_{T}\right)} \leq C T^{2}
$$

We let

$$
\tilde{u}=u-\min u+1-\frac{(\text { osc } u+2)}{T}(T-t)
$$

and note that the function $\tilde{u}$ has been constructed to satisfy

$$
|\tilde{u}| \leq 1+\operatorname{osc} u, \quad \tilde{u}(\cdot, 0) \leq-1, \tilde{u}(\cdot, T) \geq 1
$$

Define

$$
v(x, t)=\frac{1}{2} u_{x}^{2}+\frac{k}{2} \tilde{u}^{2},
$$

where $k=\left\|u_{x}\right\|^{\frac{3}{2}}{ }_{T}$. Let $\left(x_{0}, t_{0}\right) \in \bar{Q}_{T}$ be a point where $v$ achieves its maximum value. With no loss of generality, we may assume that $p=u_{x}\left(x_{0}, t_{0}\right)$ satisfies

$$
|p| \geq 1,|p|^{2} \geq \frac{1}{2}\left\|u_{x}\right\|^{2}
$$

We remark here that throughout the proof, the constant $C$ is subject to increase from line to line.
Case 1: $t_{0}=T$. For this case we consider the linearization of the HJ equation,

$$
T_{u} v=-v_{t}+H_{p}\left(u_{x}, m\right) v_{x}
$$

Since $v_{x}=0$ and $v_{t} \geq 0$,

$$
\begin{align*}
& 0 \geq T_{u} v=T_{u}\left(\frac{1}{2}\left|u_{x}\right|^{2}\right)+k \tilde{u}\left(-\tilde{u}_{t}+H_{p} u_{x}\right) \\
& \quad=-H_{m} u_{x} m_{x}+k \tilde{u}\left(-u_{t}+H_{p} p-C\right) \geq-H_{m} u_{x} m_{x}+k \tilde{u}\left(\frac{1}{C_{0}} H\right)-C k \tilde{u} \\
& \geq-H_{m} u_{x} m_{x}+k \tilde{u} \frac{1}{C_{0}}\left(\frac{1}{\bar{C}(m)}|p|^{\gamma}-\bar{C}(m)\right)-C|p|^{\frac{3}{2}} \geq-H_{m} u_{x} m_{x}+\frac{1}{C}|p|^{\gamma+\frac{3}{2}}-C|p|^{\frac{3}{2}} . \tag{21}
\end{align*}
$$

If $(u, m)$ solves (MFG), then

$$
-H_{m} u_{x} m_{x}=-\frac{H_{m}}{g^{\prime}}|p|^{2}>0
$$

On the other hand, if $(u, m)$ solves (MFGP), then

$$
\begin{equation*}
\left|-H_{m} u_{x} m_{x}\right| \leq C\left\|\left(m_{T}\right)_{x}\right\|_{\infty}|p|^{\gamma+1} \tag{22}
\end{equation*}
$$

In either case, (21) then implies

$$
|p| \leq C
$$

Case 2: $t_{0}=0$. Regardless of whether $(u, m)$ solves (MFG) or (MFGP), this case is dealt with in the same way as was done for $t_{0}=T$ when $(u, m)$ solved (MFGP), because, in view of HM2 we then have the bound

$$
\left|-H_{m} u_{x} m_{x}\right| \leq C\left\|\left(m_{0}\right)_{x}\right\|_{\infty}|p|^{\gamma+1}
$$

Case 3: $0<t_{0}<T$. We first observe that, since $v_{x}=0$, we have

$$
u_{x} u_{x x}=-k \tilde{u} u_{x}
$$

and, thus,

$$
\begin{equation*}
\left|u_{x x}\right| \leq C k \tag{23}
\end{equation*}
$$

We consider the linearization of (Q), namely

$$
L_{u}(w)=-\operatorname{Tr}\left(A(D u) D^{2} w\right)-D_{q} \operatorname{Tr}\left(A(D u) D^{2} u\right) \cdot D w
$$

Through direct computation, using Q1, one obtains

$$
\begin{equation*}
L_{u}\left(\frac{1}{2} u_{x}^{2}\right)=-\left|-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x x}\right|^{2}+\frac{1}{4} m^{2} H_{m p}^{2} u_{x x}^{2}-m H_{m} H_{p p} u_{x x}^{2} \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{u}\left(k \frac{1}{2} \tilde{u}^{2}\right)=-k\left|-\tilde{u}_{t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x}\right|^{2}+k \frac{1}{4} m^{2} H_{m p}^{2} u_{x}^{2}-k m H_{m} H_{p p} u_{x}^{2}+E_{1}+E_{2}+E_{3}+E_{4} \tag{25}
\end{equation*}
$$

where

$$
\begin{gathered}
E_{1}=2\left(-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x x}\right)\left(H_{p p}+\frac{1}{2} m H_{m p p}\right) k \tilde{u} u_{x}, \\
E_{2}=\left(\frac{1}{2} H_{m p} H_{m p p}+m H_{m p} H_{p p}+m H_{m} H_{p p p}\right) u_{x x} k \tilde{u} u_{x} \\
E_{3}=\left(-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x x}\right) \frac{2}{H_{m}}\left(H_{p m}+\frac{1}{2}\left(m H_{m m p}+H_{m p}\right)\right) k \tilde{u}\left(-\tilde{u}_{t}+H_{p} u_{x}\right)
\end{gathered}
$$

$E_{4}=$

$$
\frac{1}{H_{m}}\left(\frac{1}{2}\left(m H_{m p}^{2}+m^{2} H_{m p} H_{m m p}\right)+m H_{m m} H_{p p}+m H_{m} H_{m p p}+H_{m} H_{p p}\right) u_{x x} k \tilde{u}\left(-\tilde{u}_{t}+H_{p} u_{x}\right)
$$

Now we estimate each of the $E_{i}$. By Young's inequality, we obtain

$$
\left|E_{1}\right| \leq \frac{1}{4}\left|-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x x}\right|^{2}+C\left|H_{p p}+\frac{1}{2} m H_{m p p}\right|^{2} k^{2} u_{x}^{2} \tilde{u}^{2}
$$

As a result of (H1), and HM3), we thus obtain

$$
\begin{equation*}
\left|E_{1}\right| \leq \frac{1}{4}\left|-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right)\right|^{2}+C|p|^{2 \gamma+1} \tag{26}
\end{equation*}
$$

Next, to estimate $\left|E_{2}\right|$, we use (23), (H1) (H3), (HM1), (HM2), and (HM3) to obtain

$$
\begin{equation*}
\left|E_{2}\right| \leq C|p|^{2 \gamma+1} \tag{27}
\end{equation*}
$$

For $E_{3}$, we have

$$
\begin{equation*}
\left|E_{3}\right| \leq \frac{1}{4}\left|-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x x}\right|^{2}+\frac{C k^{2}}{H_{m}^{2}}\left(H_{p m}^{2}+m^{2} H_{m m p}^{2}+H_{m p}^{2}\right)\left|-\tilde{u}_{t}+H_{p} u_{x}\right|^{2} \tag{28}
\end{equation*}
$$

and therefore, in view of (H1) and (HM2), as well as the HJ equation, we obtain

$$
\begin{equation*}
\left|E_{3}\right| \leq \frac{1}{4}\left|-u_{x t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x x}\right|^{2}+C|p|^{2 \gamma+1} \tag{29}
\end{equation*}
$$

Finally, for $E_{4}$, we observe that (23), (H1), (HM2), and (HM3) yield

$$
\begin{equation*}
\left|E_{4}\right| \leq C|p|^{2 \gamma+1} \tag{30}
\end{equation*}
$$

Now, (E) implies that

$$
\begin{align*}
\mid-\tilde{u}_{t}+\left(H_{p}\right. & \left.+\frac{1}{2} m H_{m p}\right)\left.u_{x}\right|^{2}-\frac{1}{4} m^{2} H_{m p}^{2} p^{2}+m H_{m} H_{p p} p^{2} \\
\geq & \geq-\tilde{u}_{t}+\left.\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x}\right|^{2}+\frac{1}{4 C_{0}} m^{2} H_{m p}^{2} p^{2}=\left|-\tilde{u}_{t}+\left(H_{p} u_{x}+\frac{1}{2} m H_{m p}\right) p\right|^{2} \\
& +\frac{1}{C_{0}}\left(\frac{1}{2} m H_{m p} p\right)^{2} \geq \frac{1}{2}\left|-\tilde{u}_{t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x}\right|^{2}+\frac{1}{C}\left|-\tilde{u}_{t}+H_{p} u_{x}\right|^{2} \tag{31}
\end{align*}
$$

So, as a result of (25), (H1) and (H2), we get

$$
\begin{equation*}
L_{u}\left(k \frac{1}{2} \tilde{u}^{2}\right) \leq-\frac{1}{2} k\left|-\tilde{u}_{t}+\left(H_{p}+\frac{1}{2} m H_{m p}\right) u_{x}\right|^{2}-\frac{1}{C}|p|^{2 \gamma+\frac{3}{2}}+E_{1}+E_{2}+E_{3}+E_{4} \tag{32}
\end{equation*}
$$

Now, since $\left(x_{0}, t_{0}\right)$ is an interior maximum point of $v$, we have $L_{u}(v) \geq 0$. Thus, combining (26), (27), (29), (30), (24) and (32), we conclude

$$
0 \leq-\frac{1}{C}|p|^{2 \gamma+\frac{3}{2}}+C|p|^{2 \gamma+1}
$$

which implies

$$
|p| \leq C
$$

### 4.1 Estimates for MFG with $\epsilon$-penalized terminal condition

In order to obtain classical solutions to (MFGP), it will be necessary to use a natural approximation method, which was previously used in [24] to obtain weak solutions to the second-order planning problem. The solution will be obtained as the limit of solutions to standard MFG systems with a penalized terminal condition. Specifically, we will need to prove estimates for solutions $\left(u^{\epsilon}, m^{\epsilon}\right)$ to

$$
\left\{\begin{array}{l}
-u_{t}^{\epsilon}+H\left(u_{x}^{\epsilon}, m^{\epsilon}\right)=0 \text { in } Q_{T} \\
m_{t}^{\epsilon}-\left(m^{\epsilon} H_{p}\left(u_{x}^{\epsilon}, m^{\epsilon}\right)\right)_{x}=0 \text { in } Q_{T} \\
m^{\epsilon}(x, 0)=m_{0}(x), \epsilon u^{\epsilon}(x, T)=m^{\epsilon}(x, T)-m_{T}(x) \text { on } \partial Q_{T}
\end{array}\right.
$$

As long as $u^{\epsilon}$ is bounded in $L^{\infty}\left(Q_{T}\right)$, the limit is expected to solve (MFGP). This estimate is obtained in the following lemma. While treating this system, we will temporarily assume that $H(0,0)$ is finite. This assumption will be removed in the proof of Theorem 1.1.

Lemma 4.4. For $\epsilon>0$, let $\left(u^{\epsilon}, m^{\epsilon}\right) \in C^{2}\left(\overline{Q_{T}}\right) \times C^{1}\left(\overline{Q_{T}}\right)$ be a classical solution to system (MFG $)$, and set $c_{1}=\min \left\{\min _{\mathbb{T}} m_{0}, \min _{\mathbb{T}} m_{T}\right\}, C_{1}=\max \left\{\max _{\mathbb{T}} m_{0}, \max _{\mathbb{T}} m_{T}\right\}$. Assume that $H(0,0)<\infty$. Then there exists a constant $C>0$, independent of $\epsilon$, such that

$$
\begin{equation*}
\left\|u^{\epsilon}\right\|_{L^{\infty}\left(\overline{Q_{T}}\right)} \leq C \tag{33}
\end{equation*}
$$

Furthermore, for all $\epsilon<\frac{1}{C}$, we have

$$
\begin{equation*}
\frac{c_{1}}{2} \leq m^{\epsilon}(x, t) \leq 2 C_{1} \text { for all }(x, t) \in \overline{Q_{T}} \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|m^{\epsilon}(T, \cdot)-m_{T}(\cdot)\right\|_{\infty} \leq \epsilon C \tag{35}
\end{equation*}
$$

Proof. As a result of Proposition 4.2, since $H\left(0,{\underset{\bar{Q}}{T}}^{\min } m^{\epsilon}\right) \leq H(0,0)$, there exists

$$
C=C\left(C_{0}, T,|H(0,0)|,\left|H\left(0, \max _{\bar{Q}_{T}} m^{\epsilon}\right)\right|, \bar{C}\left(\max _{\bar{Q}_{T}} m^{\epsilon}\right)\right)
$$

such that

$$
\operatorname{osc}_{\bar{Q}_{T}}\left(u^{\epsilon}\right) \leq C .
$$

To make this bound on the oscillation independent of $\epsilon$, we must obtain upper bounds on the density $m^{\epsilon}$. Note that, from Corollary 3.2, it is enough to bound $m^{\epsilon}(T, \cdot)$ from above. To this end, let $M_{0}:=\max _{\mathbb{T}} m_{0}$ and, for $\delta>0$, define

$$
v^{\delta}(x, t)=u^{\epsilon}(x, t)+H\left(0, M_{0}+\delta\right)(T-t)
$$

Since $D^{2} v^{\delta}=D^{2} u^{\epsilon}$, we have that $v^{\delta}$ also solves the elliptic equation (Q) in $Q_{T}$. Therefore, the maximum of $v^{\delta}$, must occur at $t=0$ or $t=T$. If the maximum occurred at $t=0$, then at that point

$$
u_{t}^{\epsilon}-H\left(0, M_{0}\right)=v_{t}^{\delta} \leq 0, v_{x}^{\delta}=u_{x}^{\epsilon}=0
$$

and, hence,

$$
0 \geq u_{t}^{\epsilon}-H\left(0, M_{0}+\delta\right)=H\left(0, m_{0}\right)-H\left(0, M_{0}+\delta\right)
$$

which is a contradiction because $H_{m}<0$. Therefore, for every $\delta>0$, the maximum occurs at $t=T$, and, letting $\delta \rightarrow 0$, we see that the same is true for $\delta=0$. The maximum value of $v(x, t):=u^{\epsilon}(x, t)+H\left(0, M_{0}\right)(T-t)$ equals the maximum of $u^{\epsilon}(x, T)$, since $v(x, T)=u^{\epsilon}(x, T)$. Letting $x_{0} \in \mathbb{T}$ be a point at which this maximum occurs, it follows that $v_{t}\left(x_{0}, T\right) \geq 0$, and therefore

$$
H\left(0, m^{\epsilon}\left(x_{0}, T\right)\right) \geq H\left(0, M_{0}\right)
$$

which implies that

$$
m^{\epsilon}\left(x_{0}, T\right) \leq M_{0}
$$

But, since

$$
\epsilon u^{\epsilon}(x, T)=m^{\epsilon}(x, T)-m_{T}(x)
$$

we obtain, for each $x \in \mathbb{T}$,

$$
\epsilon u^{\epsilon}(x, T) \leq \epsilon u^{\epsilon}\left(x_{0}, T\right)=\left(m^{\epsilon}\left(x_{0}, T\right)-m_{T}\left(x_{0}\right)\right) \leq\left(M_{0}-m_{T}\left(x_{0}\right)\right),
$$

and, consequently,

$$
m^{\epsilon}(x, T)=\epsilon u^{\epsilon}(x, T)+m_{T}(x) \leq M_{0}+m_{T}(x)-m_{T}\left(x_{0}\right) \leq M_{0}+\operatorname{osc}_{\mathbb{T}}\left(m_{T}\right)
$$

We have thus shown that the bound on the oscillation of $u^{\epsilon}$ does not depend on $\epsilon$. Furthermore, since

$$
\epsilon u^{\epsilon}(x, T)=m^{\epsilon}(x, T)-m_{T}(x)
$$

and $m^{\epsilon}(T, \cdot), m_{T}(\cdot)$ are both probability densities, we have $\int_{\mathbb{T}} u^{\epsilon}(\cdot, T)=0$, so there must exist some $x^{\epsilon} \in \mathbb{T}$ such that

$$
u^{\epsilon}\left(x^{\epsilon}, T\right)=0
$$

This implies that, for any $(x, t) \in \bar{Q}_{T}$,

$$
-\operatorname{osc}_{\bar{Q}_{T}}\left(u^{\epsilon}\right) \leq u^{\epsilon}(x, t)-u^{\epsilon}\left(x^{\epsilon}, T\right) \leq \operatorname{osc}_{\bar{Q}_{T}}\left(u^{\epsilon}\right)
$$

which shows (33). To prove (34), we require $C$ to be large enough to satisfy $\frac{1}{C}\left\|u^{\epsilon}\right\|_{\infty}<\frac{1}{2} c_{1}$. Then for all $\epsilon<\frac{1}{C}$, we have

$$
m^{\epsilon}(x, T)=m_{T}(x)+\epsilon u^{\epsilon}(x, T) \geq m_{T}(x)-\frac{1}{2} c_{1} \geq \frac{1}{2} c_{1} .
$$

The upper bound for $m^{\epsilon}(x, T)$ is obtained similarly. We now conclude by Corollary 3.2, since the maxima and minima of $m^{\epsilon}$ both occur at $t=0, t=T$. Finally, (35) follows immediately from the terminal condition in (MFG M $_{\epsilon}$ and (33).

While the usefulness of MFG $_{\epsilon}$, will mainly be as a tool to obtain existence for (MFGP), it can also be used to provide an interesting counterexample. Indeed, one should note that (MFG ${ }_{\epsilon}$ is not itself a planning problem, but rather a special case of a standard MFG system, which would fit in the framework of (MFG) if the terminal cost function $g$ were allowed to depend on $x$. Such terminal conditions are treated in [21, 22] under the blow-up assumption (11), as well as the requirement that

$$
g(x, 0) \text { is constant, or } \lim _{m \rightarrow 0^{+}} g(x, m)=-\infty
$$

which is a slightly weaker version of (11). The following proposition illustrates the fact that, when such assumptions do not hold, the solution may fail to exist.

Proposition 4.5. Assume that $H(0,0)<\infty$, and that the condition $m_{T}>0$ in (M1) does not hold, so that $m_{T}\left(x_{0}\right)<0$ for some $x_{0} \in \mathbb{T}$. Then there exists $C>0$ such that, for all $0<\epsilon<\frac{1}{C}$, there exists no classical solution to $\mathrm{MFG}_{\epsilon}$ ).

Proof. We assume, by contradiction, that there exists a decreasing sequence $\epsilon_{n}>0$, with $\lim _{n \rightarrow \infty} \epsilon_{n}=0$, such that, for each positive integer $n$, there exists a solution $\left(u^{n}, m^{n}\right)$ to system $\left(\mathrm{MFG}_{\epsilon_{n}}\right)$. Since $H(0,0)<\infty$, the proof of Lemma 4.4 shows that, for some constant $C>0$ independent of $n \in \mathbb{N}$, we have $\left\|u^{n}\right\|_{\infty} \leq C$. However, this implies that

$$
\left\|m^{n}(T, \cdot)-m_{T}(\cdot)\right\|_{\infty} \leq C \epsilon_{n}
$$

while $m^{n}\left(x_{0}, T\right) \geq 0>m_{T}\left(x_{0}\right)$, which is a contradiction.
We finish our estimates for the $\epsilon$-penalized problem with an analogue of Proposition 4.3.
Lemma 4.6. For $\epsilon>0$, let $\left(u^{\epsilon}, m^{\epsilon}\right) \in C^{3, \alpha}\left(\overline{Q_{T}}\right) \times C^{2, \alpha}\left(\overline{Q_{T}}\right)$ be a classical solution to system MFG ${ }^{\prime}$, and assume that $H(0,0)<\infty$. Let $c_{1}$ and $C_{1}$ be as in Corollary 3.2. There exists a constant $C>0$, independent of $\epsilon$, such that, for $\epsilon<\frac{1}{C}$,

$$
\left\|D u^{\epsilon}\right\|_{\infty} \leq C
$$

Proof. We first observe that, by Corollary 3.2 and Lemma 4.4 $\left\|m^{\epsilon}\right\|_{\bar{Q}_{T}}$ and $\left\|\left(m^{\epsilon}\right)^{-1}\right\|_{\bar{Q}_{T}}$ are bounded a priori in terms of $C_{1}$ and $c_{1}^{-1}$. The proof of Proposition 4.3 may thus be repeated here, with Lemma 4.4 replacing the use of Proposition 4.2, with one exception. Namely, the term $-H_{m} u_{x}^{\epsilon} m_{x}^{\epsilon}$ in (21) should be estimated as

$$
-H_{m} u_{x}^{\epsilon} m_{x}^{\epsilon}=-\epsilon H_{m}\left(u_{x}^{\epsilon}\right)^{2}-H_{m} u^{\epsilon}\left(m_{T}\right)_{x} \geq-H_{m} u^{\epsilon}\left(m_{T}\right)_{x}
$$

which, in view of (22), yields the gradient bound in the case $t_{0}=T$. The rest of the argument follows unchanged.

## 5 Existence of classical solutions

In the previous sections, a priori $L^{\infty}$-bounds were obtained for $u, D u, m$, and $m^{-1}$. This is already sufficient to obtain classical solutions to (MFG), following the arguments of 21, 22]. The existence of solutions to (MFGP), on the other hand, is a more delicate issue, because the Neumann type boundary condition that appears in the linearization makes the latter non-invertible. Namely, the linearization of (Q) is

$$
\begin{cases}L_{u}(w)=f & \text { in } Q_{T} \\ \left(-1, H_{p}\left(u_{x}, m\right)\right) \cdot D w=g_{1}(x) & \text { at } t=0 \\ \left(1,-H_{p}\left(u_{x}, m\right)\right) \cdot D w=g_{2}(x) & \text { at } t=T\end{cases}
$$

which is an oblique boundary value problem that is only solvable for certain functions $f, g_{1}, g_{2}$ satisfying a compatibility condition that itself depends on $u$. This failure of invertibility precludes the direct use of the implicit function theorem and thus of the method of continuity, which means a different approach is needed. Indeed, we will obtain the solution as the limit as $\epsilon \rightarrow 0$ of the solution to the $\epsilon$-penalized problem (MFG ). We begin by noting, in the following lemma, that for $\epsilon$ small enough, the solutions to (MFG $\mathrm{M}_{\epsilon}$ are a priori uniformly bounded in $C^{1, \beta}\left(\bar{Q}_{T}\right)$, for some $0<\beta<1$, and that the system thus has a classical solution.

Lemma 5.1. Let $C$ be as in Lemma 4.4. For all $0<\epsilon<\frac{1}{C}$, $\mathrm{MFG}_{\epsilon}$ has a unique smooth solution $\left(u^{\epsilon}, m^{\epsilon}\right) \in C^{3, \alpha}\left(\overline{Q_{T}}\right) \times C^{2, \alpha}\left(\overline{Q_{T}}\right)$. Moreover, there exist constants $K>0,0<\beta<1$, independent of $\epsilon$, such that

$$
\begin{equation*}
\left\|u^{\epsilon}\right\|_{C^{1, \beta}} \leq K \tag{36}
\end{equation*}
$$

Proof. The a priori $C^{1}$-bounds on $u^{\epsilon}$, as well as $L^{\infty}$-bounds on $m^{\epsilon}$ and $\left(m^{\epsilon}\right)^{-1}$ (and thus on the ellipticity constants of the system), were all established in Lemmas 4.4 and 4.6. The Hölder estimate for the gradient then follows in the same way as in [21, Lem. 4.1], by directly applying the classical $C^{1, \alpha}$-estimates for quasilinear elliptic equations with oblique boundary conditions (see [18, Lem. 2.3]). Indeed, it suffices to verify that, for $(x, t, z, p, s) \in \mathbb{T} \times\{0, T\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the boundary condition

$$
B^{\epsilon}(x, 0, z, p, s)=-s+H\left(p, m_{0}(x)\right), \quad B^{\epsilon}(x, T, z, p, s)=s-H\left(p, \epsilon z+m_{T}(x)\right)
$$

is oblique. For this purpose, we let $\nu(x, t)$ denote the outward unit normal vector at $(x, t) \in \partial Q_{T}$. Then we have

$$
\begin{aligned}
& D_{(p, s)} B^{\epsilon}(x, 0, z, p, s) \cdot \nu(x, 0)=-B_{s}^{\epsilon}(x, 0, z, p, s)=1>0 \\
& D_{(p, s)} B^{\epsilon}(x, T, z, p, s) \cdot \nu(x, T)=-B_{s}^{\epsilon}(x, T, z, p, s)=1>0
\end{aligned}
$$

and thus the a priori estimate (36) follows. The proof of existence is then the same as in [21, Thm. 1.1] through the method of continuity.

We now have enough information on the $\epsilon$-penalized problem to prove our first theorem.
Proof of Theorem 1.1. We initially assume that $m_{0}, m_{T} \in C^{\infty}(\mathbb{T})$. The proof of part (ii), corresponding to (MFG), is identical to the one carried out in [21, Thm. 1.1]. We simply note that the condition $\lim _{m \rightarrow 0^{+}} H(p, m)=$ $+\infty$ in that proof was only used to guarantee the existence of a positive lower bound for the density, which in turn makes the equation (Q) uniformly elliptic. In our case, the lower bound is a consequence of Corollary 3.2 and Proposition 4.1

Now, for the case of (MFGP), we remark first that uniqueness of $u$, up to a constant, follows by the standard Lasry-Lions monotonicity method. To establish existence, we consider first the approximate system (MFG $)$, under the assumption $H(0,0)<\infty$. We assume that $\epsilon>0$ is small enough for Lemma 5.1 to guarantee the existence of solutions $\left(u^{\epsilon}, m^{\epsilon}\right)$. Letting $0<\beta<1$ be as in Lemma 5.1, we also have (36), for some constant $K>0$ independent of $\epsilon$. We infer that there exist a subsequence $\left\{u_{n}\right\}_{n} \subset\left\{u^{\epsilon}\right\}_{\epsilon}$, and $u \in C^{1, \alpha}\left(\overline{Q_{T}}\right)$, such that $u_{n} \rightarrow u$ uniformly. Furthermore, in view of Lemma 4.4, there exists $C>0$, independent of $\epsilon$, such that

$$
\frac{1}{C} \leq m^{\epsilon}(x, t) \leq C \text { for all }(x, t) \in \overline{Q_{T}}
$$

We let $(A, B)$ and $\left(A_{n}, B_{n}\right)$, be the quasilinear operators and boundary conditions corresponding, respectively, to $u$ and $u_{n}$. Then one has

$$
\begin{aligned}
\left(A_{n}, B_{n}\right) \rightarrow & (A, B) \text { locally uniformly }, \\
& D_{q} B_{n} \cdot \nu=1 .
\end{aligned}
$$

Hence, by Fiorenza's convergence theorem for elliptic equations with oblique boundary conditions (see [21, Thm. 2.5], [10, Chapter 17, Lemma 17.29]), we obtain $u_{n} \rightarrow u$ in $C^{2, \alpha}\left(\bar{Q}_{T}\right)$, and $u$ solves (Q), with the boundary condition corresponding to (MFGP). The $C^{3, \alpha}$ regularity (and, in fact, uniform convergence in $C^{3, \alpha}$ ) then follows readily from the standard Schauder estimates for linear oblique problems, as in [21, Thm. 1.1].

The last step will be to remove the assumption that $m_{0} \in C^{\infty}(\mathbb{T})$ and, for (MFGP), the assumptions that $m_{T} \in C^{\infty}(\mathbb{T})$ and $H(0,0)<\infty$. We will explain the argument for (MFGP), with the treatment of (MFG) being completely analogous. Consider, for $\delta>0$, the modified Hamiltonians $H^{\delta}(p, m):=H(p, m+\delta)$, which satisfy (H) and (E), uniformly in $\delta$, as well as $H^{\delta}(0,0)<\infty$, and a sequence of $C^{\infty}$ densities ( $m_{0}^{\delta}, m_{T}^{\delta}$ ), uniformly bounded in $C^{2, \alpha}$ and bounded away from 0 , converging uniformly to ( $m_{0}, m_{T}$ ). Let ( $u^{\delta}, m^{\delta}$ ) be the corresponding solutions to

$$
\begin{cases}-u_{t}^{\delta}+H^{\delta}\left(u_{x}^{\delta}, m^{\delta}\right)=0 & \text { in } Q_{T},  \tag{37}\\ \int_{0}^{T} \int_{\mathbb{T}} u^{\delta}=0, & \\ m_{t}^{\delta}-\left(m^{\delta} H_{p}^{\delta}\left(u_{x}^{\delta}, m^{\delta}\right)\right)_{x}=0 & \text { in } Q_{T}, \\ m^{\delta}(\cdot, 0)=m_{0}^{\delta}, m^{\delta}(\cdot, T)=m_{T}^{\delta} & \text { on } \mathbb{T} .\end{cases}
$$

Propositions 4.3 and 4.2, and Corollary 3.2, yield uniform $C^{1}$-bounds on $u^{\delta}$, and thus, as in the proof of Lemma 36], uniform $C^{1, \beta}$ bounds for some $0<\beta<1$. We may thus conclude by letting $\delta \rightarrow 0$ and applying Fiorenza's convergence result as above.

## 6 Regularity of weak solutions

We now study the existence and regularity of solutions to (MFG) and (MFGP) under the weaker assumption that, for some $\kappa>0$

$$
\int_{\mathbb{T}} \frac{1}{m_{0}^{\kappa}(x)} d x<\infty, \int_{\mathbb{T}} \frac{1}{m_{T}^{\kappa}(x)} d x<\infty
$$

We note that, in particular, the above conditions allow for the densities to vanish at a set of measure zero. This, in general, creates significant issues, because (Q) is no longer uniformly elliptic. The key estimate that will allow us to prove smoothness in this setting is an interior lower bound on the density which depends only on $t^{-1},\left\|m_{0}^{-\kappa}\right\|_{1}$ (and $(T-t)^{-1},\left\|m_{T}^{-\kappa}\right\|_{1}$, in the case of MFGP). Indeed, this yields uniform ellipticity of (Q) away from $t=0$ and $t=T$.

We begin by giving the standard definition of a weak solution (see, for instance, [3, 21, 26]).
Definition 6.1. [Definition of weak solution] A pair $(u, m) \in \operatorname{BV}\left(Q_{T}\right) \times L_{+}^{\infty}\left(Q_{T}\right)$ is called a weak solution to (MFG) (respectively (MFGP) if the following conditions hold:
(i) $u_{x} \in L^{2}\left(Q_{T}\right), u \in L^{\infty}\left(Q_{T}\right), m \in C^{0}\left([0, T] ; H^{-1}(\mathbb{T})\right)$.
(ii) $u$ satisfies the HJ inequality

$$
-u_{t}+H\left(u_{x}, m\right) \leq 0 \quad \text { in } Q_{T},
$$

in the distributional sense.
(iii) $m$ satisfies the continuity equation

$$
\begin{equation*}
m_{t}-\left(m H_{p}\left(u_{x}, m\right)\right)_{x}=0 \text { in } Q_{T}, \tag{38}
\end{equation*}
$$

in the distributional sense.
(iv) We have $m(\cdot, T) \in L^{\infty}(\mathbb{T})$. Moreover, $m(\cdot, 0)=m_{0}$ in $H^{-1}(\mathbb{T})$ and $u(\cdot, T)=g(m(\cdot, T))$ in the sense of traces (respectively, $m(\cdot, T)=m_{T}$ in $H^{-1}(\mathbb{T})$ ).
(v) The following identity holds:

$$
\iint_{Q_{T}} m(x, t)\left(H\left(u_{x}, m\right)-H_{p}\left(u_{x}, m\right) u_{x}\right) d x d t=\int_{\mathbb{T}}\left(m(x, T) u(x, T)-m_{0}(x) u(x, 0)\right) d x
$$

The following lemma will be needed to show that, for solutions to MFG), our interior regularity results may be extended up to time $t=T$.

Lemma 6.2. Let $(u, m)$ be a smooth solution to (MFG) under the assumptions of Theorem 1.1 and assume that (3) holds. Then, for every convex function $h \in C^{2}(0, \infty)$, the map

$$
t \rightarrow \int_{\mathbb{T}} h(m(x, t)) d x
$$

is decreasing. Moreover, there exists a constant $C=C\left(C_{0},\left\|\left(g^{\prime}\right)\right\|_{L^{\infty}\left(\left[\min m_{0}, \max m_{0}\right]\right)}^{-(\gamma-1)}\right)$ such that

$$
\frac{d}{d t} \int_{\mathbb{T}} h(m(x, T)) d x+\frac{1}{C} \int_{\mathbb{T}} h^{\prime \prime}(m(x, T))\left|m_{x}(x, T)\right|^{\gamma} \leq 0
$$

Proof. In view of Proposition 3.1, we have that

$$
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} h(m(x, t)) d x \geq 0
$$

and, thus, the function

$$
d(t):=\frac{d}{d t} \int_{\mathbb{T}} h(m(x, t)) d x
$$

is increasing. We then infer that the monotonicity will follow if we show that

$$
d(T) \leq 0
$$

Since $u(\cdot, T)=g(m(\cdot, T))$, and $m$ satisfies the continuity equation, we have

$$
d(T)=\int_{\mathbb{T}} h^{\prime}(m(x, T)) m_{t}(x, T) d x=\int_{\mathbb{T}} h^{\prime}(m)\left(m H_{p}\left(u_{x}, m\right)\right)_{x} d x=-\int_{\mathbb{T}} h^{\prime \prime}(m) m_{x} H_{p}\left(m_{x} g^{\prime}(m), m\right)
$$

Now, as a result of (3) and (H1),

$$
H_{p}\left(m_{x} g^{\prime}(m), m\right)\left(m_{x} g^{\prime}(m)\right) \geq \frac{1}{C}\left|m_{x} g^{\prime}(m)\right|^{\gamma}
$$

and, therefore,

$$
d(T) \leq-\frac{1}{C} \int_{\mathbb{T}} h^{\prime \prime}(m)\left|m_{x}\right|^{\gamma}
$$

We are now ready to obtain the interior lower bounds on $m$. Our method of proof relies on the displacement convexity formula (9), and uses similar techniques to [26, Prop. 5.2].

Proposition 6.3. Let $(u, m)$ be a smooth solution to (MFG) or (MFGP), under the same assumptions as in Theorem 1.1. Assume, furthermore, that (HW) holds and, in the case of (MFG), assume that (3) holds. Let

$$
\beta=\frac{2}{\kappa-s-1}
$$

and let $\delta>0$. Then, there exist a constant $C=C\left(C_{0}\left\|m_{0}^{-\kappa}\right\|_{L^{1}},\left\|m_{T}^{-\kappa}\right\|_{L^{1}}, \delta^{-1}\right)$ such that

$$
\begin{equation*}
m(x, t) \geq \frac{1}{C}\left(\frac{1}{t^{\beta+\delta}}+\frac{1}{(T-t)^{\beta+\delta}}\right)^{-1} \tag{39}
\end{equation*}
$$

Furthermore, in the case of (MFG), one has

$$
\begin{equation*}
m(x, t) \geq \frac{1}{C} t^{\beta+\delta} \tag{40}
\end{equation*}
$$

Proof. Using the displacement convexity formula (19) for $h(m)=\frac{1}{m^{\kappa}}$, we have, for each $t \in[0, T]$,

$$
\begin{equation*}
\int_{\mathbb{T}} \frac{1}{m^{\kappa}(x, t)} d x \leq \max \left(\int_{\mathbb{T}} \frac{1}{m_{0}^{\kappa}(x)} d x, \int_{\mathbb{T}} \frac{1}{m^{\kappa}(x, T)} d x\right) \tag{41}
\end{equation*}
$$

Combined with Lemma 6.2 (for the case of (MFG) where $m(\cdot, T)$ is not prescribed), this yields

$$
\begin{equation*}
\sup _{t \in[0, T]}\left\|m^{-\kappa}(t)\right\|_{1} \leq C \tag{42}
\end{equation*}
$$

Next, for any $p>1$, we define the function

$$
\phi(t):=\int_{\mathbb{T}} m^{-p \kappa}(t) d x
$$

Using Proposition 3.1 with $h(m)=m^{-p \kappa}$, as a result of (E), we obtain

$$
\begin{aligned}
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} \frac{m^{-p \kappa}(t)}{p \kappa(p \kappa+1)} d x & \geq-\frac{1}{C} \int_{\mathbb{T}} m^{-p \kappa-1} m H_{p p} H_{m}\left(m_{x}\right)^{2} d x \geq \int_{\mathbb{T}} \frac{1}{C} m^{-p \kappa-1+s}\left(m_{x}\right)^{2} d x \\
& \geq \frac{1}{C\left(\frac{-p \kappa+s+1}{2}\right)^{2}} \int_{\mathbb{T}}\left(m^{\frac{-p \kappa+s+1}{2}}\right)_{x}^{2} d x
\end{aligned}
$$

As a result, letting

$$
\begin{aligned}
C_{p} & :=\frac{C(p \kappa-s-1)^{2}}{4 p \kappa(p \kappa+1)} \\
\lambda & :=\frac{-p \kappa+s+1}{2}
\end{aligned}
$$

we have shown that

$$
\begin{equation*}
C_{p} \phi^{\prime \prime}(t) \geq \int_{\mathbb{T}}\left(m^{\lambda}\right)_{x}^{2} d x \tag{43}
\end{equation*}
$$

From (W), and the fact that $p>1$, we see that $\lambda<0$. For each $t \in[0, T]$, since $m(\cdot, t)$ is a probability measure, there exists a point $x_{0}^{t}$ such that $m\left(x_{0}^{t}, t\right)=1$. By the fundamental theorem of calculus,

$$
\begin{equation*}
\left\|m^{\lambda}(t)-1\right\|_{\infty}^{2}=\left\|m^{\lambda}(t)-m\left(x_{0}^{t}, t\right)^{\lambda}\right\|_{\infty}^{2} \leq C \int_{\mathbb{T}}\left(m^{\lambda}\right)_{x}^{2} d x \tag{44}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\left\|\frac{1}{m}\right\|_{\infty}^{2|\lambda|} \leq C\left(\int_{\mathbb{T}}\left(m^{\lambda}\right)_{x}^{2} d x+1\right) \tag{45}
\end{equation*}
$$

Now, using (42), we obtain

$$
\phi=\int_{\mathbb{T}} \frac{1}{m^{\kappa p}} \leq \int_{\mathbb{T}} \frac{1}{m^{\kappa}}\left\|\frac{1}{m}\right\|_{\infty}^{\kappa(p-1)} \leq C\left\|\frac{1}{m}\right\|_{\infty}^{\kappa(p-1)}
$$

and, consequently,

$$
\begin{equation*}
C^{-r} \phi^{r} \leq\left\|\frac{1}{m}\right\|_{\infty}^{2|\lambda|} \tag{46}
\end{equation*}
$$

where $r:=\frac{2|\lambda|}{\kappa(p-1)}$. From condition (W), we see that $r>1$. Combining (43), (45), and (46), we obtain

$$
C_{p}\left(\phi^{\prime \prime}(t)+1\right)-C^{-r} \phi(t)^{r} \geq 0
$$

that is, for some constant $C=C(p)$,

$$
\begin{equation*}
-\phi^{\prime \prime}(t)+\frac{1}{C} \phi^{r} \leq C . \tag{47}
\end{equation*}
$$

A straightforward computation then shows that the functions

$$
\begin{gathered}
\psi_{1}(t)=A_{p} t^{-p \kappa \beta}+K_{p}, \\
\psi_{2}(t)=A_{p}(T-t)^{-p \kappa \beta}+K_{p}, \\
\psi(t)=\psi_{1}(t)+\psi_{2}(t),
\end{gathered}
$$

are supersolutions of (47) for large enough $A_{p}, K_{p}$. Therefore, we have

$$
\begin{equation*}
\int_{\mathbb{T}} m^{-p \kappa}(t) \leq A_{p}\left(t^{-p \kappa \beta}+(T-t)^{-p \kappa \beta}\right)+2 K_{p} . \tag{48}
\end{equation*}
$$

Now, going back to (43) and (45), we may write

$$
\begin{equation*}
\left\|\frac{1}{m}\right\|_{\infty}^{2|\lambda|}(t) \leq C\left(\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}} m^{-p \kappa}+1\right) \tag{49}
\end{equation*}
$$

In view of (9), for $q>0$, the map

$$
\begin{equation*}
t \mapsto \int_{\mathbb{T}} m^{-q}(t) \tag{50}
\end{equation*}
$$

is convex in $[0, T]$. Thus, fixing $t_{0} \in\left(0, \frac{T}{2}\right]$, we infer that, for each $t \in\left[t_{0}, T-t_{0}\right]$,

$$
\begin{aligned}
\left(\int_{\mathbb{T}} m^{-2|\lambda| q}(t)\right)^{\frac{1}{q}} \leq \frac{2}{t_{0}} \max \left(\int_{\frac{t_{0}}{2}}^{t_{0}}\left(\int_{\mathbb{T}} m^{-2|\lambda| q}\right)^{\frac{1}{q}}, \int_{T-t_{0}}^{T-\frac{t_{0}}{2}}\left(\int_{\mathbb{T}} m^{-2|\lambda| q}\right)^{\frac{1}{q}}\right) & \\
& \leq \frac{2}{t_{0}} \int_{\frac{t_{0}}{2}}^{T-\frac{t_{0}}{2}}\left(\int_{\mathbb{T}} m^{-2|\lambda| q}\right)^{\frac{1}{q}}
\end{aligned}
$$

Letting $q \rightarrow \infty$, we obtain

$$
\begin{equation*}
\left\|m^{-1}\right\|_{L^{\infty}\left(\mathbb{T} \times\left[t_{0}, T-t_{0}\right]\right)}^{2|\lambda|} \leq \frac{2}{t_{0}} \int_{\frac{t_{0}}{2}}^{T-\frac{t_{0}}{2}}\left\|m^{-1}(t)\right\|_{\infty}^{2|\lambda|} d t . \tag{51}
\end{equation*}
$$

Now, letting $\zeta \in C^{\infty}\left(Q_{T}\right)$ be a test function, supported in $\left[\frac{t_{0}}{4}, T-\frac{t_{0}}{4}\right]$, such that $0 \leq \zeta \leq 1, \zeta \equiv 1$ in $\left[\frac{t_{0}}{2}, T-\frac{t_{0}}{2}\right]$, and $\int_{0}^{T}\left|\zeta^{\prime \prime}(t)\right| d t \leq \frac{C}{t_{0}}$, we see that (51) implies

$$
\begin{equation*}
\left\|m^{-1}\right\|_{L^{\infty}\left(\mathbb{T} \times\left[t_{0}, T-t_{0}\right]\right)}^{2| | \mid} \leq \frac{2}{t_{0}} \int_{0}^{T}\left\|m^{-1}\right\|_{\infty}^{2|\lambda|}(t) \zeta(t) d t . \tag{52}
\end{equation*}
$$

Hence, integrating by parts twice, we infer from (48) and (49) that

$$
\left\|m^{-1}\right\|_{L^{\infty}\left(\mathbb{T} \times\left[t_{0}, T-t_{0}\right]\right)}^{2|\lambda|} \leq \frac{C}{t_{0}}\left(\int_{0}^{T} \int_{\mathbb{T}}\left(m^{-p \kappa} \zeta^{\prime \prime}\right)+C T\right) \leq C\left(\frac{1}{t_{0}^{2+p \kappa \beta}}+\frac{1}{t_{0}}\right),
$$

which yields

$$
\left\|m^{-1}\right\|_{L^{\infty}\left(\mathbb{T} \times\left[t_{0}, T-t_{0}\right]\right)} \leq C\left(\frac{1}{t_{0}^{\frac{2 p k \beta}{2 \lambda \mid}}}+\frac{1}{t_{0}^{\frac{1}{2 \lambda \mid}}}\right) .
$$

Now, recalling (6), we see that

$$
\lim _{p \rightarrow \infty} \frac{1}{2|\lambda|}=0 \text { and } \lim _{p \rightarrow \infty} \frac{2+p \kappa \beta}{2|\lambda|}=\beta .
$$

Thus, we may fix $p$ chosen large enough that $\frac{2+\kappa \beta}{2|\lambda|}<\beta+\delta$, and, as a result of (6),

$$
\left\|m^{-1}\right\|_{L^{\infty}\left(\mathbb{T} \times\left[t_{0}, T-t_{0}\right]\right)} \leq C \frac{1}{t_{0}^{\beta+\delta}}
$$

This implies (39). Now, for the case of (MFG), we simply observe that, from Lemma 6.2 the map (50) is non-increasing on $[0, T]$, and, thus, (51) may be strengthened to

$$
\left\|m^{-1}\right\|_{L^{\infty}\left(\mathbb{T} \times\left[t_{0}, T\right]\right)}^{2|\lambda|} \leq \frac{2}{t_{0}} \int_{\frac{t_{0}}{2}}^{T}\left\|m^{-1}\right\|_{\infty}^{2|\lambda|}(t) d t .
$$

The following lemma is a basic computation exploiting (E1), and will be used in the proof of Theorem 1.2 to estimate the terms arising from the Lasry-Lions monotonicity method.

Lemma 6.4. There exists a constant $C=C\left(C_{0}\right)>0$ such that, given $-\infty<p_{0}<p_{1}<\infty$ and $0<m_{0}<$ $m_{1}<\infty$, we have

$$
\begin{align*}
\left(m_{1} H_{p}\left(p_{1}, m_{1}\right)-m_{0} H_{p}\left(p_{0}, m_{0}\right)\right)\left(p_{1}-p_{0}\right)-\left(H\left(p_{1}, m_{1}\right)-H\right. & \left.H\left(p_{0}, m_{0}\right)\right)\left(m_{1}-m_{0}\right) \\
& \geq \frac{m_{1}+m_{0}}{C}\left(p_{1}-p_{0}\right)^{2}+\frac{k}{C}\left(m_{1}-m_{0}\right)^{2} \tag{53}
\end{align*}
$$

where $k=\min _{\left[p_{0}, p_{1}\right] \times\left[m_{0}, m_{1}\right]}\left(-H_{m}(p, m)\right)$. Moreover, if $H$ satisfies (W), then

$$
\begin{align*}
&\left(m_{1} H_{p}\left(p_{1}, m_{1}\right)-m_{0} H_{p}\left(p_{0}, m_{0}\right)\right)\left(p_{1}-p_{0}\right)-\left(H\left(p_{1}, m_{1}\right)-H\left(p_{0}, m_{0}\right)\right)\left(m_{1}-m_{0}\right) \\
& \geq \frac{m_{1}+m_{0}}{C}\left(p_{1}-p_{0}\right)^{2}+\frac{1}{C(s+1)}\left(m_{1}^{s+1}-m_{0}^{s+1}\right)\left(m_{1}-m_{0}\right) . \tag{54}
\end{align*}
$$

Proof. Following the technique carried out in [20], for $z \in[0,1]$, we define

$$
\Delta p=p_{1}-p_{0} \Delta m=m_{1}-m_{0}, p_{z}=p_{0}+z \Delta p, m_{s}=m_{0}+z \Delta m .
$$

We then let

$$
\phi(z)=\left(m_{z} H_{p}\left(p_{z}, m_{z}\right)-m_{0} H_{p}\left(p_{0}, m_{0}\right)\right) \Delta p-\left(H\left(p_{z}, m_{z}\right)-H\left(p_{0}, m_{0}\right)\right) \Delta m,
$$

and differentiation yields

$$
\phi^{\prime}(z)=m_{z} H_{p p}(\Delta p)^{2}+m_{z} H_{m p} \Delta m \Delta p-H_{m}(\Delta m)^{2} .
$$

Now, in view of (E1), we have, for some constant $C>0$,

$$
-H_{m} \geq \frac{1}{4 H_{p p}} m_{z} H_{m p}^{2}\left(1+\frac{1}{C}\right)-\frac{1}{C} H_{m} .
$$

Therefore,

$$
\begin{align*}
\phi^{\prime}(z) \geq m_{z}\left(\frac{1}{\sqrt{1+\frac{1}{C}}} \sqrt{H_{p p}} \Delta p+\frac{\sqrt{1+\frac{1}{C}}}{2 \sqrt{H_{p p}}} H_{m p} \Delta m\right)^{2} & \\
& +m_{z} H_{p p}(\Delta p)^{2}\left(1-\frac{1}{1+\frac{1}{C}}\right)-\frac{1}{C} H_{m}(\Delta m)^{2} . \tag{55}
\end{align*}
$$

If (W) holds, then, up to increasing the constant $C>0$, as well as using (H1) and (W), we obtain

$$
\phi^{\prime}(z) \geq \frac{1}{C}\left(m_{z}(\Delta p)^{2}+m_{z}^{s}(\Delta m)^{2}\right),
$$

and integrating over $[0,1]$ then yields (54). The proof of (53) follows from (55) in the same way.

Before proving Theorem 1.2, we remind the reader that assumption (M) will not be in place, and will be instead replaced by (W).

Proof of Theorem 1.2. For $\epsilon \in(0,1)$, let $m_{0}^{\epsilon}, m_{T}^{\epsilon}$ be smooth, positive densities such that, for $\theta \in\{0, T\}$,

$$
m_{\theta}^{\epsilon} \rightarrow m_{\theta} \text { a.e. in } \mathbb{T},\left\|m_{\theta}^{\epsilon}\right\|_{\infty} \leq C \text { and }\left\|\left(m_{\theta}^{\epsilon}\right)^{-\kappa}\right\|_{1} \leq C,
$$

where $C>0$ is a constant independent of $\epsilon$. Let $\left(u^{\epsilon, 1}, m^{\epsilon, 1}\right)$ be a smooth solution to MFGP) obtained from taking $m_{0}^{\epsilon}$ and $m_{T}^{\epsilon}$, respectively, as the initial and terminal densities. Similarly, let $\left(u^{\epsilon, 2}, m^{\epsilon, 2}\right)$ be the smooth solution to (MFG) corresponding to the initial density $m_{0}^{\epsilon}$. The existence and regularity of such solutions is guaranteed by Theorem 1.1. We may further choose the $u^{\epsilon, 1}$ to be normalized so that $\int_{\mathbb{T}} u^{\epsilon, 1}(T)=0$.

As in the proof of Proposition 6.3. we obtain, for some $C>0$ independent of $\epsilon$ and for $i \in\{1,2\}$,

$$
\begin{equation*}
\left\|\left(m^{\epsilon, i}\right)^{-\kappa}\right\|_{1} \leq C \tag{56}
\end{equation*}
$$

On the other hand, Corollary 3.2 and Proposition 4.1 yield

$$
\begin{equation*}
\left\|m^{\epsilon, i}\right\|_{\infty} \leq C \tag{57}
\end{equation*}
$$

and (57), (HW) and Proposition 6.3 imply that

$$
\begin{equation*}
\int_{0}^{T} \mid H\left(0, \min _{\mathbb{T}} m^{\epsilon, i}(s) \mid d s \leq C\right. \tag{58}
\end{equation*}
$$

Thus, as a result of (GW), Proposition 4.1 and Proposition 4.2,

$$
\begin{equation*}
\left\|u^{\epsilon, i}\right\|_{\infty} \leq C . \tag{59}
\end{equation*}
$$

We will first observe that, up to a subsequence, there is convergence to a weak solution. Indeed, given $0<\epsilon, \epsilon^{\prime}<1$, applying the Lasry-Lions monotonicity method to the corresponding systems yields, for $i \in\{1,2\}$,

$$
\begin{align*}
& \int_{\mathbb{T}}\left(u^{\epsilon, i}(T)-u^{\epsilon^{\prime}, i}(T)\right)\left(m^{\epsilon, i}(T)-m^{\epsilon^{\prime}, i}(T)\right)-\int_{\mathbb{T}}\left(u^{\epsilon, i}(0)-u^{\epsilon^{\prime}, i}(0)\right)\left(m^{\epsilon, i}(0)-m^{\epsilon^{\prime}, i}(0)\right) \\
&+\iint_{Q_{T}}\left(m^{\epsilon, i} H_{p}\left(u_{x}^{\epsilon, i}, m^{\epsilon, i}\right)-m^{\epsilon^{\prime}, i} H_{p}\left(u_{x}^{\epsilon^{\prime}, i}, m^{\epsilon^{\prime}, i}\right)\right)\left(u_{x}^{\epsilon, i}-u_{x}^{\epsilon^{\prime}, i}\right) \\
& \quad-\left(H\left(u_{x}^{\epsilon, i}, m^{\epsilon, i}\right)-H\left(u_{x}^{\epsilon^{\prime}, i}, m^{\epsilon^{\prime}, i}\right)\right)\left(m^{\epsilon, i}-m^{\epsilon^{\prime}, i}\right)=0 \tag{60}
\end{align*}
$$

Lemma 6.4 therefore yields

$$
\begin{align*}
& \int_{\mathbb{T}}\left(u^{\epsilon, i}(T)-u^{\epsilon^{\prime}, i}(T)\right)\left(m^{\epsilon, i}(T)-m^{\epsilon^{\prime}, i}(T)\right)-\int_{\mathbb{T}}\left(u^{\epsilon, i}(0)-u^{\epsilon^{\prime}, i}(0)\right)\left(m^{\epsilon, i}(0)-m^{\epsilon^{\prime}, i}(0)\right) \\
&+\iint_{Q_{T}}\left(\frac{m^{\epsilon, i}+m^{\epsilon^{\prime}, i}}{C}\left(u_{x}^{\epsilon, i}-u_{x}^{\epsilon^{\prime}, i}\right)^{2}+\frac{1}{C(s+1)}\left(\left(m^{\epsilon, i}\right)^{s+1}-\left(m^{\epsilon^{\prime}, i}\right)^{s+1}\right)\left(m^{\epsilon, i}-m^{\epsilon^{\prime}, i}\right)\right) \leq 0 . \tag{61}
\end{align*}
$$

Proceeding as in [21, Thm. 1.2], it readily follows that, for $i \in\{1,2\}$, as $\epsilon \rightarrow 0,\left(u^{\epsilon, i}, m^{\epsilon, i}\right)$ converges to a weak solution $\left(u^{i}, m^{i}\right)$.

It remains to show the interior regularity. For $\delta>0$, we define

$$
I_{1, \delta}=[\delta, T-\delta], I_{2, \delta}=[\delta, T]
$$

By Proposition 6.3. there exists $C=C\left(\delta^{-1}\right)$ such that, for $t_{i} \in I_{i, \delta / 4}$,

$$
\begin{equation*}
m^{\epsilon, i}\left(\cdot, t_{i}\right) \geq \frac{1}{C} \tag{62}
\end{equation*}
$$

We must first obtain a priori gradient bounds for $u^{\epsilon, i}$ on $I_{i, \delta / 2}$. Setting

$$
\phi_{1}(t)=(t-\delta / 4)^{-2 /(\gamma-1)}+(T-\delta / 4-t)^{-2 /(\gamma-1)} \phi_{2}(t)=(t-\delta / 4)^{-2 /(\gamma-1)},
$$

we go through the steps of Proposition 4.3, replacing the function $v$ by

$$
v_{i}(x, t)=\frac{1}{2}\left(u_{x}^{\epsilon, i}\right)^{2}+\frac{1}{2}\left(\tilde{u}^{\epsilon, i}\right)^{2}-K \phi_{i}(t),
$$

where $K>0, \tilde{u}^{\epsilon, i}$ is defined as in Proposition4.3. We consider the maximum point $\left(x_{0}, t_{0}\right)$ of $v_{i}$ in $\mathbb{T} \times I_{i, \delta / 4}$. In the case of (MFGP), namely $i=1$, this maximum must be attained in the interior of $I_{i}$, since $\phi_{i}$ is unbounded near the endpoints. When $i=2$, the maximum may be attained at $t=T$, and the proof that $|p| \leq C$ in this case follows through unchanged from Case 1 of Proposition 4.3. If the maximum is achieved at an interior time, the steps of Proposition 4.3 yield that if $v_{i}\left(x_{0}, t_{0}\right)$ is large enough, then

$$
0 \leq-|p|^{2 \gamma}+|p|^{2 \gamma-2}-K\left(-\phi_{i}^{\prime \prime}+\frac{1}{C} K^{\gamma} \phi_{i}^{\gamma}-C \phi_{i}\right)
$$

Similarly to Proposition 6.3, we see that, if $K$ is chosen large enough, $\phi_{i}$ must be a supersolution to

$$
-\phi_{i}^{\prime \prime}+\frac{1}{C} K^{\gamma} \phi_{i}^{\gamma}-C \phi_{i}=0
$$

which then implies $p \leq C$, and thus $\left|u_{x}^{\epsilon, i}\right|$ is bounded on $I_{i, \delta / 2}$. In view of (62) and (57), $\left|u_{t}^{\epsilon, i}\right|=\left|H\left(u_{x}^{\epsilon, i}, m^{\epsilon, i}\right)\right|$ is also bounded on $I_{i, \delta / 2}$. That is, we have

$$
\begin{equation*}
\left\|u^{\epsilon, i}\right\|_{C^{1}\left(\mathbb{T} \times I_{i, \delta / 2}\right)} \leq C \tag{63}
\end{equation*}
$$

The interior $C^{1, \alpha}$-estimates for quasilinear elliptic equations (see [10, Chapter 13, Thm. 13.6]), followed by the interior Schauder estimates (see [15, Chapter, 2, (1.12)]) then yield, for some $C=C\left(\delta^{-1}\right)$, and for $i \in\{1,2\}$,

$$
\begin{equation*}
\left\|u^{\epsilon, i}\right\|_{C^{3+\alpha}\left(\mathbb{T} \times I_{1, \delta}\right)} \leq C \tag{64}
\end{equation*}
$$

For $i=1$, by virtue of the Arzelà-Ascoli theorem, we may finish the proof by simply letting $\epsilon \rightarrow 0$. On the other hand, for $i=2$ (that is, the case of (MFG), we require estimates up to the terminal time $T$. We first observe that (62), (57), and (64) imply that $u^{\epsilon, 2}$ solves, in $I_{2, \delta} \times \mathbb{T}$, a system of the form (MFG), where the initial density $m^{\epsilon, 2}(\cdot, \delta)$ is bounded below by a positive constant, and bounded above in $C^{2, \alpha}(\mathbb{T})$. Moreover, as in Lemma5.1. (63) implies that $u^{\epsilon, 2}$ is bounded in $C^{1, \beta}$ for some $0<\beta<1$. We may now conclude through the same convergence argument as in the proof of Theorem 1.1

Finally, by requiring some further regularity on the marginals, we establish additional Sobolev regularity for the weak solutions.

Proposition 6.5. Let $m_{0}, m_{T}$ satisfy $\left(m_{0}\right)_{x x},\left(m_{T}\right)_{x x} \in L^{1}(\mathbb{T})$. Let $(u, m)$ be a weak solution to (MFG) or (MFGP) under the assumptions of Theorem 1.2. Then, for some constant $C>0$ we have:

- In the case of (MFG),

$$
\begin{equation*}
\int_{\mathbb{T}} g^{\prime}(m(x, T))\left|m_{x}(x, T)\right|^{2}+\int_{0}^{T} \int_{\mathbb{T}} m\left(u_{x x}\right)^{2}+m^{s}\left(m_{x}\right)^{2} d x d t \leq C \tag{65}
\end{equation*}
$$

where $C=C\left(\|u\|_{\infty},\left\|\left(m_{0}\right)_{x x}\right\|_{1}, C_{0}\right)$.

- In the case of (MFGP),

$$
\begin{equation*}
\int_{0}^{T} \int_{\mathbb{T}} m\left(u_{x x}\right)^{2}+m^{s}\left(m_{x}\right)^{2} d x d t \leq C \tag{66}
\end{equation*}
$$

where $C=C\left(\|u\|_{\infty},\left\|\left(m_{0}\right)_{x x}\right\|_{1},\left\|\left(m_{T}\right)_{x x}\right\|_{1}, C_{0}\right)$.

Proof. We will show the result in the case where $(u, m)$ is smooth, since the general case follows by considering the approximations employed in the proof of Theorem [1.2. Differentiating with respect to $x$ the (MFG) or (MFGP), we obtain

$$
\left\{\begin{array}{l}
-u_{x t}+H_{p}\left(u_{x}, m\right) u_{x x}+H_{m}\left(u_{x}, m\right) m_{x}=0 \text { in } Q_{T}  \tag{67}\\
m_{x t}-\left(m_{x} H_{p}\left(u_{x}, m\right)+m H_{p p}\left(u_{x}, m\right) u_{x x}+m H_{p m}\left(u_{x}, m\right) m_{x}\right)_{x}=0 \text { in } Q_{T}
\end{array}\right.
$$

Testing against $u_{x}$ in the equation for $m_{x}$ above we obtain

$$
\begin{align*}
& \int_{\mathbb{T}} m_{x}(T) u_{x}(T) d x-\int_{\mathbb{T}} m_{x}(0) u_{x}(0) d x+\int_{0}^{T} \int_{\mathbb{T}} m_{x}\left(-u_{x t}+u_{x x} H_{p}\left(u_{x}, m\right)\right) \\
&+m u_{x x}^{2} H_{p p}\left(u_{x}, m\right)+m u_{x x} H_{p m}\left(u_{x}, m\right) m_{x} d x=0 \tag{68}
\end{align*}
$$

and, therefore,

$$
\begin{aligned}
& \int_{\mathbb{T}} m_{x}(T) u_{x}(T) d x+\int_{0}^{T} \int_{\mathbb{T}} m u_{x x}^{2} H_{p p}-H_{m}\left(m_{x}\right)^{2} d x \\
= & -\int_{\mathbb{T}} u(0)\left(m_{0}\right)_{x x} d x-\int_{0}^{T} \int_{\mathbb{T}} m u_{x x} H_{p m}\left(u_{x}, m\right) m_{x} d x .
\end{aligned}
$$

Next, we use the following bounds

$$
\left|\int_{\mathbb{T}} u(0)\left(m_{0}\right)_{x x} d x\right| \leq\|u\|_{\infty}\left\|\left(m_{0}\right)_{x x}\right\|_{1}
$$

and, for $\delta \in(0,1)$,

$$
\begin{gathered}
\left|m u_{x x} H_{p m} m_{x}\right| \leq(1-\delta) m u_{x x}^{2} H_{p} p+\frac{1}{4(1-\delta)} m\left|H_{p m}\right|^{2}\left(m_{x}\right)^{2} \\
\leq(1-\delta) m u_{x x}^{2} H_{p} p-\frac{4}{4(1-\delta)\left(1+\frac{1}{C_{0}}\right)} H_{m}\left(m_{x}\right)^{2}
\end{gathered}
$$

where in the last inequality we used (E1). Choose $\delta>0$ small enough so that

$$
\frac{1}{(1-\delta)\left(1+\frac{1}{C_{0}}\right)}<1
$$

Hence, in the case of (MFG), we have the bound

$$
\int_{\mathbb{T}} g^{\prime}(m(T))\left(m_{x}(T)\right)^{2} d x+\int_{0}^{T} \int_{\mathbb{T}} m H_{p p}\left(u_{x x}\right)^{2} d x-H_{m}\left(m_{x}\right)^{2} d x \leq C+\|u\|_{\infty}\left\|\left(m_{0}\right)_{x x}\right\|_{1}
$$

while in the case of (MFGP), we have

$$
\int_{0}^{T} \int_{\mathbb{T}} m H_{p p}\left(u_{x x}\right)^{2} d x-H_{m}\left(m_{x}\right)^{2} d x \leq C+\|u\|_{\infty}\left(\left\|\left(m_{0}\right)_{x x}\right\|_{1}+\left\|\left(m_{T}\right)_{x x}\right\|_{1}\right)
$$

## 7 Long time behavior and the infinite horizon problem

In this section, we will characterize the behavior, as $T \rightarrow \infty$, of solutions to (MFG) and (MFGP). First, we establish the turnpike property with an exponential rate of convergence. This property shows that, for large values of $T$, the players spend most of their time close to the equilibrium $m \equiv 1$.

Lemma 7.1. Let $(u, m)$ be a solution to (MFG) or (MFGP), let $T>1$, and set

$$
c_{1}=\min \left(\min m_{0}, \min m_{T}\right), C_{1}=\max \left(\max m_{0}, \max \left(m_{T}\right)\right)
$$

Then there exist constants $C, \omega>0$, with

$$
C=C\left(C_{0}, C_{1}, c_{1}^{-1},\|\bar{C}\|_{L^{\infty}\left(\left[c_{1}, C_{1}\right]\right)},\left\|\left(m_{0}\right)_{x}\right\|_{\infty},\left\|\left(m_{T}\right)_{x}\right\|_{\infty},\left\|\left(g^{\prime}\right)\right\|_{L^{\infty}\left(\left[\min m_{0}, \max m_{0}\right]\right)}^{-(\gamma-1)}\right)
$$

and

$$
\omega^{-1}=\omega^{-1}\left(C_{0}, c_{1}^{-1}, C_{1},\|\bar{C}\|_{L^{\infty}\left(\left[c_{1}, C_{1}\right]\right)}\right)
$$

such that

$$
\begin{equation*}
\|m(t)-1\|_{L^{\infty}(\mathbb{T})}+\left\|u_{x}(t)\right\|_{L^{\infty}(\mathbb{T})} \leq C\left(e^{-\omega t}+e^{-\omega(T-t)}\right), \quad t \in[0, T] \tag{69}
\end{equation*}
$$

If $(u, m)$ solves (MFG), and (3) holds, we have

$$
\begin{equation*}
\|m(t)-1\|_{L^{\infty}(\mathbb{T})}+\left\|u_{x}(t)\right\|_{L^{\infty}(\mathbb{T})} \leq C e^{-\omega t}, t \in[0, T] \tag{70}
\end{equation*}
$$

Proof. As in previous arguments, we recall that the constant $C$ may increase at each step. For each $k \in \mathbb{N}$, Proposition 3.1 yields

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}}(m-1)^{2 k} d x \geq 0 \tag{71}
\end{equation*}
$$

and, as a result of (L), (H1), and Corollary 3.2,

$$
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}}(m-1)^{2} d x \geq \frac{1}{C} \int_{\mathbb{T}}-m H_{m} m H_{p p} m_{x}^{2} d x \geq \frac{1}{C} \int_{\mathbb{T}}\left|(m-1)_{x}\right|^{2} d x
$$

Since $\int_{\mathbb{T}} m(\cdot, t) \equiv 1$, arguing in the same way as in (44), we obtain

$$
\frac{d^{2}}{d t^{2}} \int_{\mathbb{T}}(m-1)^{2} d x \geq \frac{1}{C}\|m-1\|_{\infty}^{2}
$$

Therefore, setting

$$
\phi(t):=\int_{\mathbb{T}}(m(t)-1)^{2} d x
$$

we have

$$
\begin{equation*}
-\phi^{\prime \prime}+\frac{1}{C} \phi \leq 0 \tag{72}
\end{equation*}
$$

Moreover, if $(u, m)$ solves (MFG) and (3) holds, up to increasing the value of $C$, Lemma (6.2) implies that

$$
\begin{equation*}
\phi^{\prime}(T) \leq-\frac{1}{\sqrt{C}} \phi(T) \tag{73}
\end{equation*}
$$

We now fix the choice $\omega=\frac{1}{2 \sqrt{C}}$ (the value of $C$ may still increase in subsequent steps, but the value of $\omega$ will not). The comparison principle applied to (72) then implies that, for each $t \in[0, T]$,

$$
\begin{equation*}
\phi(t) \leq \phi(0) e^{-2 \omega t}+\phi(T) e^{-2 \omega(T-t)} \leq C\left(e^{-2 \omega t}+e^{-2 \omega(T-t)}\right) \tag{74}
\end{equation*}
$$

Similarly, if $(u, m)$ solves (MFG) and (3), then (72), coupled with the Robin boundary condition (73), readily implies that

$$
\begin{equation*}
\phi(t) \leq \phi(0) e^{-2 \omega t} \leq C e^{-2 \omega t} \tag{75}
\end{equation*}
$$

By using the same convexity arguments as in (52), in view of (71), we have

$$
\begin{equation*}
\|m(t)-1\|_{\infty}^{2} \leq C \int_{t-\frac{1}{2}}^{t+\frac{1}{2}}\|m(s)-1\|_{\infty}(s)^{2} d s \leq C \int_{t-1}^{t+1} \int_{\mathbb{T}}(m-1)^{2}=C \int_{t-1}^{t+1} \phi(s) d s \tag{76}
\end{equation*}
$$

We now turn our attention to estimating $u_{x}$. Fixing $t \in[1, T-1]$, as a result of (H1), Proposition 3.1) and Corollary 3.2, we obtain, for $s \in[t-1, t+1]$,

$$
\frac{1}{C} \int_{\mathbb{T}} u_{x x}^{2}(s) \leq \frac{d^{2}}{d s^{2}} \int_{\mathbb{T}}(m(s)-1)^{2}
$$

Thus, testing against a bump function $\zeta \geq 0$, which is supported on $[t-1, t+1]$, and identically equals 1 on $\left[t-\frac{1}{2}, t+\frac{1}{2}\right]$, we get

$$
\begin{equation*}
\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \int_{\mathbb{T}} u_{x x}^{2} \leq C \int_{t-1}^{t+1} \int_{\mathbb{T}}(m-1)^{2} \zeta^{\prime \prime} \leq C \int_{t-1}^{t+1} \phi(s) d s \tag{77}
\end{equation*}
$$

Differentiating (Q) with respect to $x$, one sees that $v=u_{x}$ solves a linear elliptic equation of the form

$$
-\operatorname{Tr}\left(A(x, t) D^{2} v\right)+b(x, t) \cdot D v=0
$$

Thus, $v$ satisfies the maximum and minimum principles on compact subsets of $\bar{Q}_{T}$. Applying this observation to $\mathbb{T} \times[t-s, t+s]$, for $s \in\left(0, \frac{1}{2}\right)$, as well as the fact that, for every $t \in[0, T],\left\{x \in \mathbb{T}: u_{x}(x, t)=0\right\} \neq \emptyset$, we have

$$
\operatorname{osc}_{\mathbb{T}} v(t) \leq \operatorname{osc}_{\mathbb{T}} v(t+s)+\operatorname{osc}_{\mathbb{T}} v(t-s) \leq \int_{\mathbb{T}}\left|u_{x x}(t+s)\right|+\int_{\mathbb{T}}\left|u_{x x}(t-s)\right|
$$

Integrating in $s$ then yields

$$
\operatorname{osc}_{\mathbb{T}} u_{x}(t) \leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \int_{\mathbb{T}}\left|u_{x x}\right|
$$

and, thus, as a result of (77) and the Cauchy-Schwarz inequality,

$$
\begin{equation*}
\left\|u_{x}(t)\right\|_{\infty}^{2} \leq C \int_{t-1}^{t+1} \phi(s) d s \tag{78}
\end{equation*}
$$

Now, adding (76) and (78), followed by (74), we obtain (69) for $t \in[1, T-1]$. Similarly, when ( $u, m$ ) solves (MFG) and (3) holds, (75) yields (70) for $t \in[1, T-1]$. We observe that, for $t \in[0, T] \backslash[1, T-1]$, the bounds on $\|m(t)-1\|_{\infty}$ given by (69) and (70) hold trivially, up to increasing the value of $C$. Let us see that the same is true for the bounds on $\left\|u_{x}(t)\right\|_{\infty}$ on the interval [ 0,1$]$. Indeed, we may simply follow the proof of Proposition 4.3, applied to the MFG system on the domain $\mathbb{T} \times[0,1]$, with the only change being on Case 1 of that proof, that is, when the maximum value is attained at $t=1$. For this case, we may simply use the fact that, as a result of (69) holding for $t=1,\left|u_{x}(\cdot, 1)\right|$ is bounded. Thus, if we take $T=1$ in Proposition 4.2, this yields a bound on $\left\|u_{x}\right\|_{\mathbb{T} \times[0,1]}$ that depends only on $C_{0},\|m\|_{L^{\infty}\left(\bar{Q}_{T}\right)},\left\|m^{-1}\right\|_{L^{\infty}\left(\bar{Q}_{T}\right)},\left\|\left(m_{0}\right)_{x}\right\|_{\infty}$, and $\|\bar{C}\|_{L^{\infty}([\min m, \max m])}$. A similar argument may be followed on $\mathbb{T} \times[T-1, T]$, which completes the proof.

Having established the turnpike property, we now follow the program developed in [8 to study the long time behavior. In order to characterize the limit, as $T \rightarrow \infty$, of the functions $(u(t)-\lambda(T-t), m(t))$, we first show a uniqueness result for (MFGL).

Lemma 7.2. Assume that (L) holds. Then, up to adding a constant to $v$, there exists at most one classical solution $(v, \mu)$ to (MFGL) satisfying (5).

Proof. Assume that $\left(v^{1}, \mu^{1}\right),\left(v^{2}, \mu^{2}\right)$ are solutions to (MFGL) satisfying (5). Since $\mu^{1}-1, \mu^{2}-1 \in L^{1}(\mathbb{T} \times$ $(0, \infty))$, there exists a sequence $T_{k} \rightarrow \infty$ such that

$$
\lim _{k \rightarrow \infty} \int_{\mathbb{T}}\left(\left|\mu^{1}\left(\cdot, T_{k}\right)-1\right|+\left|\mu^{2}\left(\cdot, T_{k}\right)-1\right|\right)=0
$$

Performing the standard Lasry-Lions computation for $v^{1}, v^{2}$ on $Q_{T_{k}}$, using Lemma 6.4, and noting that

$$
\mu^{i},\left(\mu^{i}\right)^{-1}, v_{x}^{i}, \in L^{\infty}(\mathbb{T} \times(0, \infty)), \quad i \in\{1,2\}
$$

we obtain

$$
\begin{align*}
\frac{1}{C}\left(\int_{0}^{T_{k}} \int_{\mathbb{T}}\left|v_{x}^{1}-v_{x}^{2}\right|^{2}+\left|\mu^{1}-\mu^{2}\right|^{2}\right) \leq \int_{\mathbb{T}}- & \left(v^{1}\left(T_{k}\right)-v^{2}\left(T_{k}\right)\right)\left(\mu^{1}\left(T_{k}\right)-\mu^{2}\left(T_{k}\right)\right) \\
& =\int_{\mathbb{T}}-\left(v^{1}\left(T_{k}\right)-v^{2}\left(T_{k}\right)\right)\left(\left(\mu^{1}\left(T_{k}\right)-1\right)-\left(\mu^{2}\left(T_{k}\right)-1\right)\right) \tag{79}
\end{align*}
$$

Now, since $v^{1}, v^{2} \in L^{\infty}(\mathbb{T} \times(0, \infty))$, the right hand side converges to 0 as $k \rightarrow \infty$. Therefore,

$$
\int_{0}^{\infty} \int_{\mathbb{T}}\left|v_{x}^{1}-v_{x}^{2}\right|^{2}+\left|\mu^{1}-\mu^{2}\right|^{2}=0
$$

This implies that $\mu^{1}=\mu^{2}$ and $v_{x}^{1}=v_{x}^{2}$. From the HJ equations, $v_{t}^{1}=v_{t}^{2}$, which concludes the proof.
In the following lemma, we obtain uniform estimates for the solution that are independent of $T$.
Lemma 7.3. Let $\left(u^{T}, m^{T}\right)$ be a solution to MFG) or MFGP) for $T>0$, and let $\omega>0$ be the constant from Lemma 7.1. Set $v^{T}=u^{T}-\lambda(T-t)$. Then there exists a constant $C>0$, independent of $T$, such that:

- If (3) holds and $\left(u^{T}, m^{T}\right)$ solves (MFG), then

$$
\begin{equation*}
\left|v^{T}(t)-g(1)\right| \leq C e^{-\omega t} \text { for all } t \in[0, T] \tag{80}
\end{equation*}
$$

- If $\left(u^{T}, m^{T}\right)$ solves (MFGP), and

$$
\begin{equation*}
\int_{\mathbb{T}} v^{T}\left(\frac{1}{2} T\right) d x=0 \tag{81}
\end{equation*}
$$

then we have

$$
\begin{equation*}
\left\|v^{T}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq C \tag{82}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|v^{T}\right\|_{L^{\infty}\left(Q_{\frac{T}{2}}\right)} \leq C e^{-\omega t} \tag{83}
\end{equation*}
$$

Proof. First we note that in both (MFG) and (MFGP), as a result of Lemma 7.1, the function $v_{x}^{T}=u_{x}^{T}$ is bounded uniformly, independently of $T$, and, by Corollary 3.2 so are $m^{T},\left(m^{T}\right)^{-1}$. Therefore, since $H$ is smooth, and thus locally Lipschitz, we have, for some constant $C>0$ independent of $T>0$,

$$
\begin{equation*}
\left|v_{t}^{T}\right| \leq C\left(\left|v_{x}^{T}\right|+\left|m^{T}-1\right|\right) \tag{84}
\end{equation*}
$$

Assume first that $\left(u^{T}, m^{T}\right)$ solves (MFG) and (3) holds. Integrating the HJ equation in $[t, T]$ and using (84) along with (70) in Lemma 7.1 we obtain

$$
\left|v^{T}(t)-v^{T}(T)\right| \leq C \int_{t}^{T} e^{-\omega s} d s
$$

Furthermore, using the fact that

$$
v^{T}(T)=u^{T}(T)=g\left(m^{T}(T)\right)
$$

and

$$
\left|m^{T}(T)-1\right| \leq C e^{-\omega T}
$$

by increasing the constant $C$ if necessary, we obtain

$$
\left|v^{T}(t)-g(1)\right| \leq C\left(e^{-\omega T}+e^{-\omega t}\right) \leq 2 C e^{-\omega t}
$$

which proves (80). Next, we assume that $\left(u^{T}, m^{T}\right)$ solves MFGP) and (81) holds. Letting $t<\frac{T}{2}$, and integrating the HJ equation in $\left[t, \frac{T}{2}\right]$, we obtain from (84) and (69) that

$$
\begin{equation*}
\left|\int_{\mathbb{T}} v^{T}(\cdot, t)\right| \leq C \int_{t}^{\frac{T}{2}} e^{-\omega s}+e^{-\omega(T-s)} d s \leq \frac{2 C}{\omega}\left(e^{-\omega t}+e^{-\omega \frac{T}{2}}\right) \leq \frac{4 C}{\omega} e^{-\omega t} \tag{85}
\end{equation*}
$$

Similarly, for $t \geq \frac{T}{2}$ integrating the HJ equation in $\left[\frac{T}{2}, t\right]$ yields

$$
\begin{equation*}
\left|\int_{\mathbb{T}} v^{T}(\cdot, t) d x\right| \leq C \tag{86}
\end{equation*}
$$

Now, for every $t \in[0, T]$, there exists a point $x_{t} \in \mathbb{T}$ such that $v^{T}\left(x_{t}, t\right)=\int_{\mathbb{T}} v^{T}(\cdot, t)$. Therefore,

$$
\left|v^{T}(x, t)\right| \leq \operatorname{osc}_{\mathbb{T}} v^{T}(t)+\left|\int_{\mathbb{T}} v^{T}(\cdot, t)\right|
$$

As a result, in view of (69), the estimates (86) and (85) yield, respectively, (82) and (83).
We are now ready to prove our last result.
Proof of Theorem 1.3. We set

$$
v^{T}=u^{T}-\lambda(T-t),
$$

and show that $v^{T}$ is is convergent as $T \rightarrow \infty$.
In view of Lemmas 7.1 and [7.3, as well as (84), we see that $\left\|v^{T}\right\|_{W^{1, \infty}\left(Q_{T}\right)}$ and $\left\|m^{T}\right\|_{\infty}$ are bounded, independently of $T$. We may therefore apply the Arzelà-Ascoli theorem to conclude that, up to extracting a subsequence, there exist $v \in W^{1, \infty}(\mathbb{T} \times[0, \infty))$ and $\mu \in L^{\infty}(\mathbb{T} \times[0, \infty))$ such that

$$
v^{T} \rightarrow v \text { locally uniformly in } \mathbb{T} \times[0, \infty)
$$

and

$$
m^{T} \rightharpoonup \mu \text { weakly }-* \text { in } L^{\infty}(\mathbb{T} \times(0, \infty))
$$

We now fix $T_{0} \in(1, \infty)$, and assume that $T>T_{0}+1$. Then $\left(v^{T}, m^{T}\right)$ solves the system

$$
\begin{cases}-v_{t}^{T}+\lambda+H\left(v_{x}^{T}, m^{T}\right)=0 & \text { in } Q_{T_{0}}  \tag{87}\\ m_{t}^{T}-\left(m^{T} H_{p}\left(v_{x}^{T}, m^{T}\right)\right)_{x}=0 & \text { in } Q_{T_{0}} \\ m^{T}(\cdot, 0)=m_{0} & \end{cases}
$$

Moreover, as a result of the interior $C^{1, \alpha}$ estimates for quasilinear elliptic equations, and the interior Schauder estimates for linear equations, $m^{T}\left(\cdot, T_{0}\right)$ is uniformly bounded in $C^{2, \alpha+\epsilon}$, where $\epsilon>0$ is chosen such that $\alpha+\epsilon<1$. Therefore, as in the proof of Theorem 1.1, we conclude that, as $T \rightarrow \infty$,

$$
\begin{equation*}
\left(v^{T}, m^{T}\right) \rightarrow(v, \mu) \text { in } C^{3, \alpha}\left(\mathbb{T} \times\left[0, T_{0}\right]\right) \times C^{2, \alpha}\left(\mathbb{T} \times\left[0, T_{0}\right]\right) \tag{88}
\end{equation*}
$$

In particular, this implies that $(v, \mu) \in C_{\mathrm{loc}}^{3, \alpha}(\mathbb{T} \times[0, \infty)) \times C_{\mathrm{loc}}^{2, \alpha}(\mathbb{T} \times[0, \infty))$, and that $(v, \mu)$ solves (MFGL). Letting $T \rightarrow \infty$ in (69) yields

$$
\begin{equation*}
\|\mu(t)-1\|_{\infty}+\left\|v_{x}(t)\right\|_{\infty} \leq C e^{-\omega t} \tag{89}
\end{equation*}
$$

which shows that $\mu-1 \in L^{1}(\mathbb{T} \times(0, \infty))$. Moreover, since $\left\|\left(m^{T}\right)^{-1}\right\|_{\infty}$ is bounded, we conclude that (5) holds.
Now, since a subsequence was extracted, we must verify that the limit is uniquely determined. In view of Lemma 7.2 $\mu$ is uniquely determined, and $v$ is uniquely determined up to a constant. In the case of (MFG) we see from (80) that

$$
\lim _{t \rightarrow \infty}\|v(t)-g(1)\|_{\infty}=0
$$

On the other hand, in the case of (MFGP), letting $T \rightarrow \infty$ followed by $t \rightarrow \infty$ in (83), we obtain

$$
\lim _{t \rightarrow \infty}\|v(t)\|_{\infty}=0
$$

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