SPATIAL RESTRICTED N+1-BODY PROBLEM WITH REPULSIVE MANEV POTENTIAL

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Abstract. We study the spatial Maxwell restricted $(N+1)-$body problem, which consists of the motion of an infinitesimal particle attracted by the gravitational field of N bodies. These bodies are arranged in a planar ring configuration. This configuration consists of $N-1$ primaries of equal masses m located at the vertices of a regular polygon that is rotating on its own plane about its center of mass with a constant angular velocity ω. Another primary of mass $m_0 = \beta m$ $(\beta > 0)$ is placed at the center of the ring. Moreover, we assume that the central body may be an ellipsoid, or radiation source, which is modeled through the Manev potential $(-1/r + e/r^2)$, $e \neq 0$, where e is a parameter related to the obliticity or radiation source (according to the sign of the parameter e) of the central mass. We study the dynamics of a infinitesimal mass under the gravitational attraction of the primaries plus the influence of the central mass through the Manev potential. Specifically, we investigate the relative equilibria of the infinitesimal mass and their linear stability as functions of the mass parameter β, the ratio of mass of the central body to the mass of one of $N-1$ remaining bodies, and e parameter. We also prove the nonexistence of binary collisions between the central body and the infinitesimal mass.

1. Introduction and statement of the problem

Quasi–homogeneous potential of the form $-(a/r - e/r^2)$, where r is the distance between particles, and a, e are real constants, was considered by Newton in his work Philosophiae Naturalis Principia Mathematica (Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2). One of the reasons to add the term e/r^2 to the gravitational attraction $(-a/r)$ was the impossibility to explain the Moons apsidal motion within the framework of the inverse-square force law. Nevertheless, the model was abandoned in favor of the classical potential. Many years later Manev (1924) [8] proposed a similar corrective term in order to maintain dynamical astronomy within the framework of classical mechanics and offering at the same time equally good justifications of the observed phenomena as in the relativity theory. For instance, when a is positive and e is negative, the corrective term provides a justification of the perihelion advance of Mercury.

In this work we consider the motion in a three-dimensional space of an infinitesimal mass P under the gravitational attraction of $N = n+1$ point masses, P_0, P_i, $i = 1, \ldots, n$ called *primaries*. Assume that the potential generated by the primary P_0 is a Manev potential $(-1/r + e/r^2)$, with parameter e, and that the gravitational attraction due to P_i, $i = 1, \ldots, n$ is Newtonian $-1/r$.

We emphasize that the parameter $e \in \mathbb{R}$ models several problems, for example, when the central body of the ring is no longer spherical, but an ellipsoid of revolution (spheroid). According to [5], [6] the parameter e is associated with flattening, in natural bodies like planets, the spheroid is flattened $e < 0$, but also we can think of artificial bodies and assume they are prolates, in that case $e > 0$. In general, this fact is seen more used in potentials of the Schwarzschild type ($A/r - e/r^3$, introduced in 1998 by Mioc and Savinski in [9]). We consider that the central body is a source of radiation, repulsive if $e > 0$ and attractive if $e < 0$, and then the effect of radiation can be modeled in a similar way to the flattened ellipsoid (see, for example, [7]).

We also shall assume that the n-primaries P_i ($i = 1, \dots, n$) are in a n-gon configuration, that is, the bodies P_i, $i = 1, \dots, n$ have the same mass $m_i = m$, for all $i = 1, \dots, n$, and are located symmetrically with respect to the central body P_0, of mass $m_0 = \beta m$, which is at the center of mass of the system. P_0 will also be called the central body, and P_i, $i = 1, \dots, n$ the peripherals, as in the Maxwell ring model. In an inertial reference system the peripheral bodies move in a circular orbit around P_0 with angular velocity ω. This problem will be called Maxwell's ring restricted $(N+1)$-body problem with Manev potential or shortly, Manev R$(N+1)$BP, note that $N = n+1$.

Dynamical important aspects of the case $n = 2$ (the dynamics of the Spatial Restricted Four Body Problem with repulsive Manev potential from an analytical point of view) were given in [4]. For the planar case, a particular numerical study ($n = 7$) is made on the number of equilibria and the bifurcations that depend on the Manev parameter in [2]. Other studies about the existence of equilibria and permitted region of motion can be found in [5] and [6]. We found that in [6] it was studied the existence of some symmetric periodic solutions in the planar case using numerical methods.

The main purpose is to study important aspects of the dynamics of the spatial restricted $(N+1)$-body problem with repulsive or attractive Manev potential from an analytical, mainly for any quantity of peripherals n.

In Section 2 we pose the problem and study the main features in particular, we deduce the Manev R$(N+1)$BP model and characterize the symmetries of the Hamiltonian system. For the repulsive case, that is, $e > 0$ we prove that, due to the repulsive force emanating from the central body, it is not possible to have a binary collision between the infinitesimal mass and the central body in the Manev R$(N+1)$BP.

In Section 3 we study the number and location of equilibria. In particular we observe that any equilibrium must lie on the lines of symmetries of the regular polygon that the peripheral bodies form. Using this information we are able to determine the type of equilibrium points and the number of them as functions of the parameters β and e. Bifurcation parameters are characterized.

In Section 4 several general results concerning the stability are proved analytically.

2. Statement of the problem and preliminary results

Although the equations of motion of the Manev R$(N+1)$BP are already known, and derived by [5]. Here we present a different approach. Once we have the model, we present some basic features. Specially, the model depend on a parametric e, for the repulsive case, that is, $e > 0$ will be prove that it is not possible to have a binary collision between the infinitesimal mass and the central body in the Manev

R$(N+1)$BP.

We consider one $(N{+}1)$-body problem in inertial frame, where $q_i = (q_i^{(1)}, q_i^{(2)}, q_i^{(3)})$, with $i = 0, \dots, N$ are the position of $(N{+}1)$ bodies with positive masses $m_0, m_1, \dots m_N$, respectively. If one of the bodies, in particular the one at position q_0 has a Manev effect, the potential $U = U(q_0, \dots, q_N)$ is given by

$$U = \sum_{0\le i<j\le N} \frac{\mathcal{G} m_i m_j}{||q_i - q_j||} - \sum_{j=1}^{N} \frac{\mathcal{G} m_0 m_j B}{||q_0 - q_j||^2}, \tag{1}$$

where $\mathcal{G}$ is the Gaussian constant of gravitation and $U = U(q_0, \dots, q_N)$ is the potential and B is the corrective coefficient term corresponding to the Manev potential. The equations of motion will be given by

$$\begin{aligned} m_0 \ddot{q}_0 &= \sum_{j=1}^{N} \left(\frac{\mathcal{G} m_0 m_j (q_j - q_0)}{||q_0 - q_j||^3} - \frac{2\mathcal{G} m_0 m_j B (q_j - q_0)}{||q_0 - q_j||^4} \right), \\ m_i \ddot{q}_i &= \sum_{j=0,\ j\neq i}^{N} \frac{\mathcal{G} m_i m_j (q_j - q_i)}{||q_i - q_j||^3} - \frac{2\mathcal{G} m_0 m_i B (q_0 - q_i)}{||q_0 - q_i||^4},\ i = 1, \dots, N, \end{aligned} \tag{2}$$

If we consider that the particle at position q_N is small, that is, with mass $m_N \approx 0$, its influence on the other bodies can be neglected. We obtain a restricted problem of $N+1$-bodies as follows

$$\begin{aligned} m_0 \ddot{q}_0 &= \sum_{j=1}^{N-1} \left(\frac{\mathcal{G} m_0 m_j (q_j - q_0)}{||q_0 - q_j||^3} - \frac{2\mathcal{G} m_0 m_j B (q_j - q_0)}{||q_0 - q_j||^4} \right), \\ m_i \ddot{q}_i &= \sum_{j=0,\ j\neq i}^{N-1} \frac{\mathcal{G} m_i m_j (q_j - q_i)}{||q_i - q_j||^3} - \frac{2\mathcal{G} m_0 m_i B (q_0 - q_i)}{||q_0 - q_i||^4},\ i = 1, \dots, N-1, \\ \ddot{q}_N &= \sum_{j=0}^{N-1} \frac{\mathcal{G} m_j (q_j - q_N)}{||q_N - q_j||^3} - \frac{2\mathcal{G} m_0 B (q_0 - q_N)}{||q_N - q_0||^4}. \end{aligned} \tag{3}$$

The system of equations (3) is uncoupled, in the sense that the first N equations can be solved independently of the last one. In order to solve the last equation, a solution of the first N equations is needed.

The primary P_0 has mass m_0 and is located at the origin, and the primaries P_i, $i = 1, \dots, N-1 = n$ (also known as peripherals) have all equal masses m and are located at the vertices of a regular polygon with center at P_0 and rotate around it with angular velocity ω.

From the first $n+1$ equation of (3), the equations of motion of the primaries are given by

$$\begin{aligned} 0 &= \sum_{j=1}^{n} \frac{\mathcal{G} m q_j}{||q_j||^3} - \frac{2\mathcal{G} m B q_j}{||q_j||^4}, \\ \ddot{q}_i &= \sum_{j=1,\ j\neq i}^{n} \frac{\mathcal{G} m (q_j - q_i)}{||q_i - q_j||^3} - \frac{\mathcal{G} m_0 q_i}{||q_i||^3} + \frac{2\mathcal{G} m_0 B q_i}{||q_i||^4},\ i = 1, \dots, n. \end{aligned} \tag{4}$$

While that from the last equation of (3), the equation of motion of the small particle is given by

$$\ddot{q}_{n+1} = \sum_{j=1}^{n} \frac{\mathcal{G}m(q_j - q_{n+1})}{||q_{n+1} - q_j||^3} - \frac{\mathcal{G}m_0 q_{n+1}}{||q_{n+1}||^3} + \frac{2\mathcal{G}m_0 B q_{n+1}}{||q_{n+1}||^4}. \tag{5}$$

As the location of the peripherals is on a plane, we will consider that $q_i^{(3)} = 0$, for all $i = 1, \ldots, n$. Thus, we will forget about the third component and we can use complex coordinates in (4), so that

$$q_j(t) = de^{i\omega t} e^{i\frac{2\pi(j-1)}{n}}, \ j = 0, \ldots, n,$$

where d is the radius of the polygon. In particular $\ddot{q}_1(t) = -\omega^2 d e^{i\omega t}$. Thus, the first two equations are as follows:

$$0 = de^{i\omega t}\mathcal{G}m \left(\sum_{j=1}^{n} \frac{e^{i\frac{2\pi(j-1)}{n}}}{d^3} - \frac{2Be^{i\frac{2\pi(j-1)}{n}}}{d^4} \right) \tag{6}$$

$$-\omega^2 de^{i\omega t} = \mathcal{G}mde^{i\omega t} \left(\sum_{j=2}^{n} \frac{e^{i\frac{2\pi(j-1)}{n}} - 1}{d_j^3} - \frac{\beta}{d^3} + \frac{2B\beta}{d^4} \right), \tag{7}$$

where $\beta = \frac{m_0}{m}$ denotes the mass parameter and d_j is the distance P_1P_j, with $j = 2, \ldots, n$.

Equation (6) is satisfied trivially. Using the geometry of our model (see Figure 1) we have the following general relation for the angles of the formed regular polygon as a function of n.

$$\varphi = \frac{(n-2)\pi}{n}, \ \ \psi = \frac{2\pi}{n} \text{ and } \theta = \frac{\psi}{2} = \frac{\pi}{n} = \frac{\varphi}{n-2}.$$

Also, from the law of sinus, the distance between P_0 and P_1 (clearly is the same distance between P_0 and P_i, $i = 1, \ldots, n$) is

$$d = a\frac{\sin(\varphi/2)}{\sin(\psi)} = \frac{a}{2\sin(\frac{\pi}{n})},$$

where a is the side of the regular n-gnon and the distance between P_1 and P_i, $i = 1, \ldots, n$ is $d_i = a\frac{\sin((n-i+1)\theta)}{\sin\theta}$. We define the corrective coefficient term as $B := ea$, where e is denoted as the Manev parameter. Thus, the equation (7) is satisfied when

$$\frac{\mathcal{G}m}{a^3\omega^2} = \frac{1}{\Delta}, \tag{8}$$

where $\Delta = \rho(\Lambda + \beta\rho^2 - 2\beta e\rho^3)$, with

$$\Lambda = \sum_{i=2}^{n} \frac{\sin^2(\pi/n)}{\sin[(i-1)(\pi/n)]} \text{ and } \rho = 2\sin(\pi/n).$$

From identity (8), Δ must always be positive, i.e., $e < \frac{\Lambda + \beta\rho^2}{2\beta\rho^3}$.

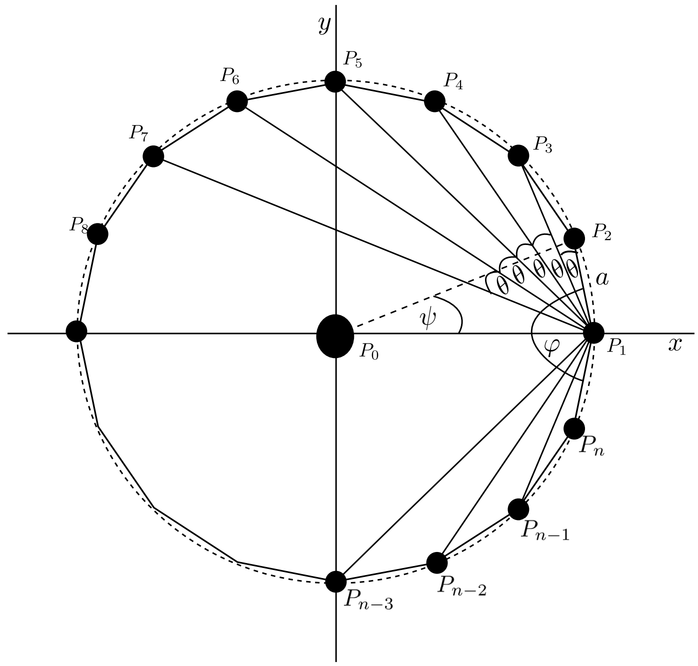


FIGURE 1. The configuration of the primaries in the "ring" problem of $n+1$ bodies.

Definition 2.1. *We call the admissible value of e, when e satisfies*

$$e < e_0 := \frac{\Lambda + \beta\rho^2}{2\beta\rho^3}. \tag{9}$$

Using the identity (8) in (5), the motion of the small particle P in the inertial coordinate system is given by

$$\ddot{q}_{n+1} = \mathcal{G}m\left(-\frac{\beta q_{n+1}}{r_0^3} + \frac{2e\beta q_{n+1}}{r_0^4} + \sum_{i=1}^{n}\frac{q_i - q_{n+1}}{r_i^3}\right), \tag{10}$$

where r_i, $i = 0, \ldots, n$ denote the distance between P and primaries.

By scaling the physical variables of (10) using the transformations $q^*_{n+1} = \frac{q_{n+1}}{a}$, $q^*_i = \frac{q_i}{a}$, $i = 0, \ldots, n$ and $t^* = \omega t$ (for simplicity, we will use the same notation as in (10) so we will drop *) and using the identity (8) is obtained

$$\ddot{q}_{n+1} = \frac{1}{\Delta}\left(-\frac{\beta q_{n+1}}{r_0^3} + \frac{2e\beta a q_{n+1}}{r_0^4} + \sum_{i=1}^{n}\frac{q_i - q_{n+1}}{r_i^3}\right), \tag{11}$$

Without loss of generality we assume that $\omega^2 = \Delta$. The motion of the small particle P in a rotating coordinate system $Oxyz$ (see Figure 2) is then described by the following system of second-order differential equations,

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} + 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \tag{12}$$

where the function Ω is defined by

$$\Omega = \Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0} - \frac{e}{r_0^2}\right) + \sum_{i=1}^{n} \frac{1}{r_i}\right], \tag{13}$$

with

$$r_0 = (x^2 + y^2 + z^2)^{1/2} \text{ and } r_i = [(x - x_i)^2 + (y - y_i)^2 + z^2]^{1/2}. \tag{14}$$

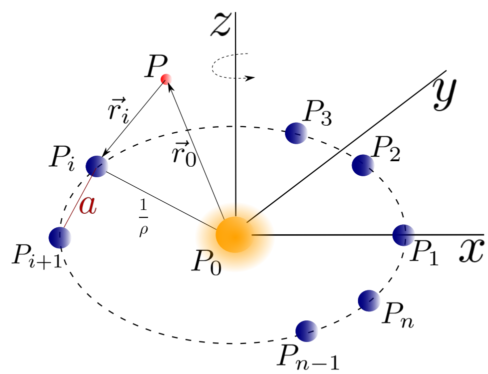


FIGURE 2. The configuration of the problem. P is the small body and P_i, $i = 0, 1, 2, \ldots, n$ are the primaries.

The phase space associated to system (12) (as a first order differential system) is given by

$$\mathcal{M} \;=\; \Big\{(x, y, z, \dot{x}, \dot{y}, \dot{z}) \;\in\; \big(\mathbb{R}^3 \setminus \{(0,0,0), (x_i, y_i, 0) : i = 1, \ldots, n)\}\big) \;\times\; \mathbb{R}^3\Big\}.$$

where $x_i = \frac{1}{\rho}\cos\left(\frac{2\pi(i-1)}{n}\right)$ and $y_i = \frac{1}{\rho}\sin\left(\frac{2\pi(i-1)}{n}\right)$, with $i = 1, \ldots, n$.

Finally making the change of variables $(x, y, z, \dot{x}, \dot{y}, \dot{z}) \to (x, y, z, X+y, Y-x, Z)$, we obtains that system (12) can be written as a Hamiltonian system of first-order differential equations, where the associated Hamiltonian function is given by

$$H = \frac{1}{2}(X^2 + Y^2 + Z^2) + yX - xY - V, \tag{15}$$

where the potential V is

$$V = \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0} - \frac{e}{r_0^2}\right) + \sum_{i=1}^{n} \frac{1}{r_i}\right], \tag{16}$$

The problem has two invariant subspaces, the plane $z = \dot{z} = 0$, named Planar Manev R(N+1)BP and the z-axis, named Rectilinear Manev R(N+1)BP. This two subproblems can be studied separately.

2.1. **Symmetries.** The system (12) admits the following time reversal symmetries:

$$\begin{aligned} S_1: &\quad (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, -z, -\dot{x}, \dot{y}, \dot{z}, -t), \\ S_2: &\quad (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t), \end{aligned} \tag{17}$$

for all n, and

$$\begin{aligned} S_3: &\quad (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (-x, y, -z, \dot{x}, -\dot{y}, \dot{z}, -t), \\ S_4: &\quad (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (-x, y, z, \dot{x}, -\dot{y}, -\dot{z}, -t), \end{aligned} \tag{18}$$

for n even.

In particular, if $\gamma(t) \in \mathcal{M}$ is a solution, then $\tilde{\gamma}(t) = S_j(\gamma(-t))$, for $j = 1, 2, 3, 4$ is also a solution. Of course, the composition of theses symmetries give us new symmetries.

The fixed sets of these four symmetries are the subspaces

$$L_1 = \{y = z = \dot{x} = 0\}, \quad L_2 = \{y = \dot{x} = \dot{z} = 0\},$$
$$L_3 = \{x = z = \dot{y} = 0\}, \quad L_4 = \{x = \dot{y} = \dot{z} = 0\}.$$

If a solution starts in one Lagrangian subplane L_i at time $t = 0$ and returns at a later time $T/2$, then the solution is T-periodic and the orbit is S_i-symmetric. If a solution starts in one Lagrangian subplane L_i at time $t = 0$ and hits another L_j $(i \neq j)$ at a later time $t = T/4$, then the solution is T-periodic and the orbit of this solution is carried into itself by both symmetries. We call such a periodic solution doubly symmetric.

In addition, the following rotation, composed with the previous symmetries, verifies that the symmetry axes that the regular polygon of the configuration possesses determine reversibility symmetries.

$$R : (x, y, z, \dot{x}, \dot{y}, \dot{z}) \to \begin{array}{l} (\cos(\alpha)x - \sin(\alpha)y, \sin(\alpha)x - \cos(\alpha)y, z, \\ \cos(\alpha)\dot{x} - \sin(\alpha)\dot{y}, \sin(\alpha)\dot{x} + \cos(\alpha)\dot{y}, \dot{z}), \end{array} \tag{19}$$

with $\alpha = 2\pi/n$.

2.2. **Jacobi constant.** Similarly to the classical circular R$(N+1)$BP, the system (12) possesses the first integral, known as Jacobi constant, given by

$$C = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{20}$$

In the repulsive case, it is not possible to have a binary collision between the infinitesimal mass and the central body in the Manev R$(N+1)$BP. This is consequence of the following result.

Theorem 2.1. *For any $\beta > 0$ and an admissible $e > 0$, a solution of the Manev R(N+1)BP (12) must satisfy*

$$\liminf_{t \to \pm\infty} r_0(t) > 0,$$

where r_0 is given in (14).

Proof. Consider $\gamma(t)$ a solution of the Manev R(N+1)BP. Then by (20), there exists a constant $C \in \mathbb{R}$ such that $C(\gamma(t)) = C$, $\forall\, t$. Suppose that $\liminf_{t\to\infty} r_0(t) = 0$ (analogously when $t \to -\infty$). Then, there exists a sequence $t_n \underset{n\nearrow\infty}{\longrightarrow} \infty$ such that

$$\lim_{n\to\infty} C(\gamma(t_n)) = -\infty,$$

the above happens because the term $\frac{1}{r_0(t_n)} - \frac{e}{r_0(t_n)^2}$ of the effective potential Ω tends to $-\infty$ when $e > 0$, which is a contradiction. $\square$

3. Equilibrium points

The equilibrium points of (12) correspond to the points $(x, y, z, 0, 0, 0) \in \mathcal{M}$ such that

$$
\begin{aligned}
x - \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right)x + \sum_{i=1}^{n}\frac{x - x_i}{r_i^3}\right] &= 0,\\
y - \frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right)y + \sum_{i=1}^{n}\frac{y - y_i}{r_i^3}\right] &= 0,\\
z\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right) + \sum_{i=1}^{n}\frac{1}{r_i^3}\right] &= 0.
\end{aligned}
\tag{21}
$$

Since any equilibrium point is determined by the position (x, y, z) of the infinitesimal mass, from now we represent the equilibrium points of (12) only by the position vector.

In the following result we are going to characterize the location of the equilibrium points.

Theorem 3.1. *For any fixed value of n and for any $\beta > 0$ and an admissible e, the equilibrium points of the Manev R(N+1)BP (12) in the $z = 0$ plane must lie on the symmetry axes of the configuration, i.e., are located in the lines $y = \tan(\frac{i\pi}{n})x$, $i = 1, \dots, n$. In the case $z \neq 0$ (spatial case) for positive admissible value e the equilibrium points are on the z-axis, while for $e \leq 0$ there are no equilibrium points.*

Proof. Recall that, in the planar case $z = 0$, the equations of the equilibrium points given in (21) are reduced to

$$
\begin{aligned}
x + V_x &= 0\\
y + V_y &= 0.
\end{aligned}
\tag{22}
$$

The system (22) implies the following equation

$$
yV_x - xV_y = 0, \tag{23}
$$

where V is given by (16), $V_x = \frac{\partial V}{\partial x}$ and $V_y = \frac{\partial V}{\partial y}$. From equation (23) it is obtained that

$$
\sum_{i=1}^{n}\frac{xy_i - yx_i}{r_i^3} = 0. \tag{24}
$$

The position of the peripherals is given by (x_i, y_i) where $x_i = \frac{1}{\rho}\cos\varphi_i$ and $y_i = \frac{1}{\rho}\sin\varphi_i$, with $\varphi_i = \frac{2\pi(i-1)}{n}$, $i = 1, \dots, n$. Let us introduce polar coordinates to the position of the infinitesimal particle, $x = -r\cos\theta$, $y = r\sin\theta$, so for fixed $r > 0$ the equation (24) can be rewrite as

$$
F(\theta) := \sum_{i=1}^{n}\frac{\sin(\theta + \varphi_i)}{r_i^3} = 0,
$$

where $r_i^2 = r^2 + 2r\cos(\theta+\varphi_i) + 1$. Now,

$$\begin{aligned} F(\theta) &= \frac{\sin(\theta)}{r_1^3} + \sum_{i=2}^{n} \frac{\sin(\theta+\varphi_i)}{r_i^3} \\ &= \frac{\sin(\theta)}{r_1^3} + \sum_{i=1}^{n-1} \frac{\sin(\theta+\varphi_{i+1})}{r_{i+1}^3}. \end{aligned}$$

If we consider r_i as function of θ, that is, $r_i = r_i(\theta)$, we notice that $r_{i+1}^2(\theta) = r_1^2(\theta+\varphi_{i+1})$. Acording to Lemma 7.1 (see Appendix 7)

$$F(\theta) = f(\theta) + \sum_{i=1}^{n-1} f(\theta + iT),$$

with $f(\theta) = \frac{\sin(\theta)}{r_1^3(\theta)}$ and $T = \frac{2\pi}{n}$. It is clear that $F(\theta)$ satisfies the hypothesis of Lemma 7.1, and therefore, $F(\theta) = 0$, if and only if, $\theta = \frac{k\pi}{n}$, $k \in \mathbb{Z}$, which completes the proof in the case $z = 0$.

Next we consider $z \neq 0$, the system (21) can be rewritten as

$$\begin{aligned} \Delta x\, Q &= -\sum_{i=1}^{n} \frac{x_i}{r_i^3}, \\ \Delta y\, Q &= -\sum_{i=1}^{n} \frac{y_i}{r_i^3}, \\ z(1-Q) &= 0, \end{aligned} \tag{25}$$

with $Q = 1 - \dfrac{1}{\Delta}\left(\beta\left(\dfrac{1}{r_0^3} - \dfrac{2e}{r_0^4}\right) + \displaystyle\sum_{i=1}^{n}\dfrac{1}{r_i^3}\right)$. Since $z \neq 0$, then $Q = 1$, so the system (25) is reduced to the system

$$\begin{aligned} \Delta x &= -\sum_{i=1}^{n} \frac{x_i}{r_i^3}, \\ \Delta y &= -\sum_{i=1}^{n} \frac{y_i}{r_i^3}. \end{aligned} \tag{26}$$

The second equation of the system (26) can be rewritten as

$$\begin{aligned} \Delta y &= -\frac{1}{\rho}\sum_{i=1}^{n} \frac{\sin(\varphi_i)}{r_i^3} \\ &= \frac{1}{\rho}\sum_{i=2}^{\lfloor \frac{n+1}{2} \rfloor} \sin(\varphi_i)\left(\frac{1}{r_{n+2-i}^3} - \frac{1}{r_i^3}\right). \end{aligned} \tag{27}$$

Suppose that $y \geq 0$, then $r_i \leq r_{n+2-i}$, for all $i = 2, \ldots, \lfloor \frac{n+1}{2} \rfloor$, now

$$\frac{1}{r_{n+2-i}^3} - \frac{1}{r_i^3} \leq 0.$$

Therefore, the right member of (27) is not positive, so $y \leq 0$, for the hypothesis ($y \geq 0$) we have $y = 0$. Now we can assume $y = 0$ and suppose that $x > 0$ in the

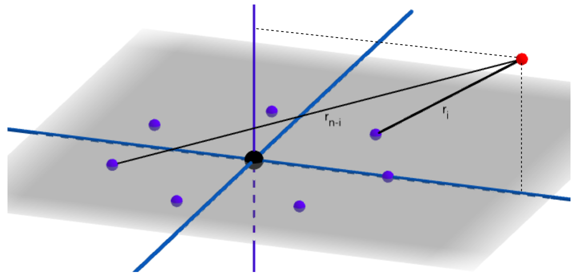


FIGURE 3. Distance between the small particle in position $(x, 0, z)$ and the peripheries P_i and P_{n-i}, respectively.

first equation of the system (26). The equation can be rewritten as

$$\Delta x = -\frac{1}{\rho}\sum_{i=1}^{n}\frac{\cos(\varphi_i)}{r_i^3}. \tag{28}$$

Using trigonometric identities we have $\cos(\varphi_i) = \dfrac{\sin(\varphi_i+\theta)}{\sin\theta} - \cot\theta\sin(\varphi_i)$, with $\theta = \frac{2\pi}{n}$. Now the equation (28) is given by

$$\Delta x = -\frac{1}{\rho}\left(\frac{1}{\sin(\theta)}\sum_{i=1}^{n}\frac{\sin(\varphi_{i+1})}{r_i^3} - \cot(\theta)\sum_{i=1}^{n}\frac{\sin(\varphi_i)}{r_i^3}\right), \tag{29}$$

but as $y = 0$, then $\displaystyle\sum_{i=1}^{n}\frac{\sin(\varphi_i)}{r_i^3} = 0$, so the equation (29) is transformed into

$$\begin{aligned}\Delta x &= -\frac{1}{\rho\sin(\theta)}\sum_{i=1}^{n}\frac{\sin(\varphi_{i+1})}{r_i^3}\\ &= \frac{1}{\rho}\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\sin(\varphi_{i+1})\left(\frac{1}{r_{n-i}^3}-\frac{1}{r_i^3}\right).\end{aligned} \tag{30}$$

The same is true if we start considering $y \leq 0$.

For $y = 0$ and $x > 0$, $r_i \leq r_{n-i}$ (see Figure 3), for all $i = 1, \ldots, \lfloor\frac{n-1}{2}\rfloor$ and $n \geq 3$ (in the case $n = 2$ is evident). In effect,

$$\begin{aligned} r_i \leq r_{n-i} \Leftrightarrow r_i^2 &< r_{n-i}^2,\ \ \forall i = 1,\ldots,\lfloor\tfrac{n-1}{2}\rfloor\\ \Leftrightarrow x^2+z^2+\tfrac{1}{\rho}-\tfrac{2x}{\rho}\cos(\varphi_i) &< x^2+z^2+\tfrac{1}{\rho}-\tfrac{2x}{\rho}\cos(\varphi_{n-i}),\ \ \forall i = 1,\ldots,\lfloor\tfrac{n-1}{2}\rfloor\\ \Leftrightarrow \cos(\varphi_{n-i}) &< \cos(\varphi_i),\ \ \forall i = 1,\ldots,\lfloor\tfrac{n-1}{2}\rfloor. \end{aligned}$$

Note that $\cos(\varphi_{n-i}) = \cos(\varphi_i + \frac{4\pi}{n})$, also $\cos(\varphi_i)$ is a decreasing function in $[0,\pi]$ for all $i = 1,\ldots,\lfloor\frac{n-1}{2}\rfloor$, therefore $\cos(\varphi_{n-i}) < \cos(\varphi_i)$, $\forall i = 1,\ldots,\lfloor\frac{n-1}{2}\rfloor$ is true. So, $r_i \leq r_{n-i}$, for all $i = 1,\ldots,\lfloor\frac{n-1}{2}\rfloor$. Now $\frac{1}{r_{n-i}^3} - \frac{1}{r_i^3} < 0$, the right member of the equation (30) is negative, therefore $x < 0$, which contradicts the assumption. On the contrary, if we assume $x < 0$, $r_i \geq r_{n-i}$, for all $i = 1,\ldots,\lfloor\frac{n-1}{2}\rfloor$ it is easy to check with the same above argument, then $\frac{1}{r_{n-i}^3} - \frac{1}{r_i^3} > 0$, the right member of the equation (30) is positive. Therefore, $x > 0$ which contradicts the assumption again, so $x = 0$. Thus, the equilibrium points with $z \neq 0$ must be on the z-axis. Moreover, from (25) there are not equilibrium points on the z-axis when $e < 0$. This completes the proof.

□

Next our purpose is to characterize the localization and number of equilibrium points for a fixed value of $\beta > 0$ and admissible e. As we saw in Theorem 3.1, the points of planar equilibria exist only on the axes of symmetry. In addition, given the symmetries (17) and (18) and the rotational symmetry (19), we can note that it is enough to study the equilibria that lie on the positive x-axis, those that lie on the line $y = \tan(\pi/n)x$, with $x > 0$ and those that are found on the positive z-axis. We will denote the equilibrium points as:

Definition 3.1. *We will denote the equilibrium points that lie on the x-axis, with $x > 0$ by L_x^+, the equilibria that lie on the line $y = \tan(\pi/n)x$ with $x > 0$ by L_m^+ (m is for mediatrix) and those who are on the z-axis, with $z > 0$ by L_z^+.*

The sets where we will study the location of the equilibrium points on the xy-plane will be defined below

Definition 3.2. *The set $\mathcal{R} = \{y =, x > 0\} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1 = \{y = 0, x > 1/\rho\}$ and $\mathcal{R}_2 = \{y = 0, 0 < x < 1/\rho\}$. And the set $\mathcal{L} = \{y = \tan(\pi/n)x, x > 0\}$.*

3.1. **Equilibrium points on the z-axis with $e > 0$.** From (21), an equilibrium point on the positive z-axis, L_z^+, is a solution $z > 0$ of

$$\beta\left(\frac{1}{z^3} - \frac{2e}{z^4}\right) + \frac{n}{(1/\rho^2 + z^2)^{3/2}} = 0. \tag{31}$$

The following result shows the existence of the only equilibrium point on the z-axis with $z > 0$ and a bound on its location.

Theorem 3.2. *For any fixed value of n, for any $\beta > 0$ and a positive admissible e, there exists a unique equilibrium point on the positive z-axis, $L_z^+ = (0, 0, \bar{z})$. Furthermore, $0 < \bar{z} < 2e$.*

Proof. Consider the auxiliary functions

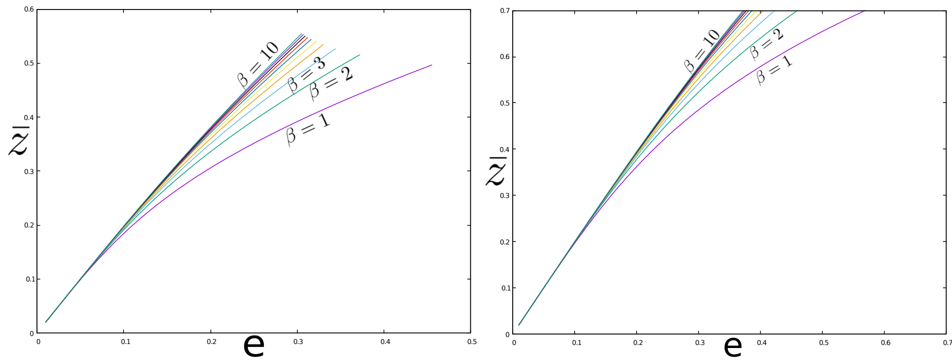
$$h_1(z) = \beta\left(\frac{1}{z^3} - \frac{2e}{z^4}\right) \quad \text{and} \quad h_2(z) = -\frac{n}{(1/\rho^2 + z^2)^{3/2}}.$$

Then, an equilibrium point on the positive z axis is a solution of the equation $h_1(z) = h_2(z)$. On one hand, we have that $\lim_{z\to 0^+} h_1(z) = -\infty$, $h_1(z) < 0$ and $h_1'(z) > 0$ for $0 < z < 2e$, and $h_1(z) > 0$ for $z > 2e$. On the other hand, $h_2(z) < 0$ and $h_2'(z) > 0$ for $z > 0$. Then, it is straightforward that there exists a unique positive solution of $h_1(z) = h_2(z)$.

□

Remark 3.1. *For any $\beta > 0$ and admissible e, let L_z^+ be the equilibrium point given in Theorem 3.2. Then,*

$$\lim_{e\to 0} \bar{z} = 0, \quad \lim_{\beta\to 0} \bar{z} = 0, \quad \textit{and} \quad \lim_{\beta\to\infty} \bar{z} = 2e.$$



(a) $n = 3$ (b) $n = 10$

FIGURE 4. Graph of the equilibrium point $\overline{z} = \overline{z}(e, \beta)$ fixed β, for $\beta = 1, 2, \ldots, 10$.

The first limit is obtained from the upper and lower bounds of $\bar{z}$. To obtain the second limit, notice that using (31), we can written (for any fixed value of e) β as a function of $\bar{z}$ as

$$\beta = \frac{n\bar{z}^4}{(2e - \bar{z})(1/\rho^2 + \bar{z}^2)^{3/2}}.$$

Using Taylor expansion we get $\beta = \frac{\rho^3 n}{2e}\bar{z}^4 + O(\bar{z}^5)$. Finally the third limit is obtained directly dividing equation (31) by β.

Remark 3.2. *If $n = 2$, $n = 3$ or $n = 4$, then $\min\{e, \beta e\} < \bar{z} < 2e$. In effect, to obtain the upper and lower bounds of the solution, notice that $h_2(0) = -n\rho^3$ and*

$$\begin{aligned} h_1(e) &= \frac{-\beta}{e^3} < -n\rho^3 \quad \Leftrightarrow \quad e < \frac{\sqrt[3]{\beta}}{\rho\sqrt[3]{n}}, \\ h_1(\beta e) &= \frac{\beta - 2}{\beta^3 e^3} < -n\rho^3 \quad \Leftrightarrow \quad e < \frac{\sqrt[3]{2-\beta}}{\beta\rho\sqrt[3]{n}}. \end{aligned}$$

For $\beta \geq 1$, we have that $e_0 < \dfrac{\sqrt[3]{\beta}}{\rho\sqrt[3]{n}}$, whereas for $\beta < 1$, we have that $e_0 < \dfrac{\sqrt[3]{2-\beta}\rho}{\beta}$. Using the fact that $e < e_0$ (recall (9)), the claim is proved.

3.2. **Equilibrium points on the positive x-axis.** From (21), an equilibrium point on the positive x-axis, L_x^+, is a solution of

$$\Delta x^3 + \frac{2\beta e}{x} - \beta = x^2 \sum_{i=1}^{n} \frac{x - x_i}{((x - x_i)^2 + y_i^2)^{3/2}}. \tag{32}$$

In order to calculate the equilibrium points we use the auxiliary functions

$$f_1(x) = \Delta x^3 + \frac{2\beta e}{x} - \beta, \tag{33}$$

and

$$f_2(x) = x^2 \sum_{i=1}^{n} \frac{x - x_i}{((x - x_i)^2 + y_i^2)^{3/2}}, \tag{34}$$

defined for $x > 0$. It is clear that solving equation (32) is equivalent to solving the equation given by $f_1(x) = f_2(x)$ for $x > 0$.

Definition 3.3. *Let $x^* = x^*(e) = \left(\frac{2\beta e}{3\Delta}\right)^{1/4}$ be the minimum point from the function f_1, when $0 < e < e_0$.*

Now we see how many equilibria exist on the x-axis, with $x > 1/\rho$, that is, to the right of the first peripheral.

Theorem 3.3. *For any fixed value of n, for any $\beta > 0$, consider $\mathcal{R}_1$ the positive x-axis with $x > 1/\rho$.*

(1) *If $0 < e < e_0$, there exists at least one equilibrium point on $\mathcal{R}_1$, and only one if $n \leq 472$, denoted by $L^+_{x_1} = (\bar{x}_1, 0, 0)$. Moreover, $\bar{x}_1 \geq \max\{1/\rho, x^*\}$, where x^* is given in Definition 3.3.*
(2) *If $e \leq 0$, there is exactly one equilibrium point on $\mathcal{R}_1$, $L^+_{x_1}$.*

Proof. An equilibrium point on $\mathcal{R}_1$ satisfies the equation $f_1(x) = f_2(x)$. First, for the case $0 < e < e_0$, with the description of Lemma 7.3 for $x > 1/\rho$ and with the first three items (2.i), 2.ii) and 2.iii)) of Lemma 7.2, we can guarantee the existence of an equilibrium point L_{x_1}. Also, using item 2.iv) from Lemma 7.2, where $h(n) := f_1(1/\rho)$, and the fact that $h(472) \approx 471.957$ and $h(473) \approx 473.116$, we have that L_{x_1} is unique for $n \leq 472$. Using that $f_1(x^*) < f_1(1/\rho) < f_2(x)$ for any $x > 1/\rho$, we obtain the lower bound. Second, for the case $e \leq 0$ using point 1 from Lemma 7.2 and what has already been used from Lemma 7.3, it is obtained that there is a unique equilibrium in this case. □

Remark 3.3. *For any $\beta > 0$ and admissible e, let $L^+_{x_1}$ be the equilibrium point given in Theorem 3.3. Then*

$$\lim_{e\to 0} \bar{x}_1 = \bar{x}_1(\beta) \text{ exists, and } \lim_{\beta\to 0} \bar{x}_1 = \bar{x}_{1_0},$$

where $\bar{x}_{1_0}$ does not depend on e and coincides with the x coordinate of the equilibrium of the Maxwell's Ring R(N+1)BP with equal masses. When $e \to 0$, we can write the equation $f_1(x) = f_2(x)$ as

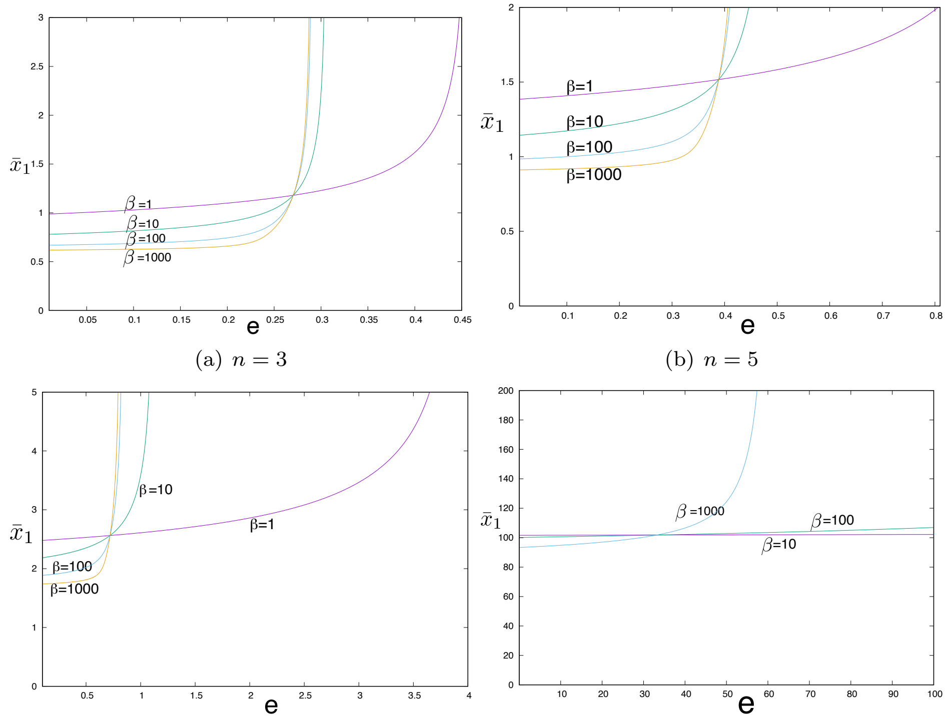
$$\frac{\rho(\Lambda + \beta\rho^2)}{\beta} x^3 = 1 + \frac{1}{\beta} x^2 \sum_{i=1}^{n} \frac{(x - x_i)}{((x - x_i)^2 + y_i^2)^{3/2}},$$

which clearly has one solution for $x > 1/\rho$ (remember, by Lemma 7.3, $f_2(x)$ is positive, when $x > 1/\rho$).

When $\beta \to 0$ the equation $f_1(x) = f_2(x)$ transforms into

$$\rho\Lambda x^3 = x^2 \sum_{i=1}^{n} \frac{(x - x_i)}{((x - x_i)^2 + y_i^2)^{3/2}}.$$

Thus, the equilibrium point $(\bar{x}_{1_0}, 0, 0)$ coincides with the equilibrium of the restricted $(N+1)$-body problem (see [3]*, case $m_0 = 0$, the authors called him R_1). In the next result, we can see that it exists an admissible value of e such that for all values of β, the equilibrium point $L^+_{x_1}$ coincide with $(\bar{x}_{1_0}, 0, 0)$. We can see this point as the intersection of curves in Figure 5, we have, for example, the cases $n = 3$, $n = 5$, $n = 10$ and $n = 500$.*



(a) $n = 3$ (b) $n = 5$

(c) $n = 10$ (d) $n = 500$

Figure 5. Evolution of the coordinates $\bar{x}_1$ as function of e.

Proposition 3.1. *There exists an admissible value of e, such that the equilibrium point $L^+_{x_1} = (\bar{x}_{1_0}, 0, 0)$ for all $\beta > 0$, where $\bar{x}_{1_0}$ is given in Remark 3.3.*

Proof. Recall that $\bar{x}_1$ is the only positive solution of the equation (32). This equation can be written as

$$x^2 \left(\rho \Lambda x - \sum_{i=1}^{n} \frac{(x - x_i)}{((x - x_i)^2 + y_i^2)^{3/2}} \right) + \frac{\beta}{x} \left(\rho^4 \left(\frac{1}{\rho} - 2e \right) x^4 - x + 2e \right) = 0.$$

Substituting $x = \bar{x}_{1_0}$ in the above equation, the first term vanishes and we get that

$$\rho^4 \left(\frac{1}{\rho} - 2e \right) \bar{x}_{1_0}^4 - \bar{x}_{1_0} + 2e = 0.$$

Solving for e,

$$e = \frac{\bar{x}_{1_0}(\rho^2 \bar{x}_{1_0}^2 + \rho \bar{x}_{1_0} + 1)}{2(1 + \rho \bar{x}_{1_0})(1 + \rho^2 \bar{x}_{1_0}^2)} < \frac{1}{2\rho},$$

which is an admissible value. $\square$

The approximate value of e for which $L^+_{x_1} = (\bar{x}_{1_0}, 0, 0)$ for some n values are given in Table 1.

Theorem 3.4. *For any $\beta > 0$, there exists a value $e^* = e^*(\beta) > 0$ such that the number of equilibrium points along the $\mathcal{R}_2$ (x-axis for $0 < x < 1/\rho$) is*

(1) 0 *if* $e \in (e^*, e_0)$,

n	$\bar{x}_{1_0}$	e
3	1.1799984049	0.27099478169
5	1.4548950111	0.36616775409
10	2.5629997052	0.50888405339
500	101.8255392116	0.59920105662

TABLE 1. The approximate value of e for which $L^+_{\bar{x}_1} = (\bar{x}_{1_0}, 0, 0)$.

(2) 1 *if* $e = e^*$,
(3) 2 *if* $0 < e < e^*$,
(4) 1 *if* $e \leq 0$.

Furthermore, $e^* < 3e_0/4$, *where* e_0 *is given in (9).*

Proof. From Equation (32), the number of equilibrium points with $0 < x < 1/\rho$ is equivalent to the number of solutions of $f_1(x) = f_2(x)$ for $0 < x < 1/\rho$. First, we consider $0 < e < e_0$. Fix a value of $\beta > 0$. Recall that f_1 has a unique minimum at $x^* = x^*(e)$ (see Lemma 7.2). Notice also that f_2 does not depend on e, for Lemma 7.3 that is a decreasing function, $f_2(0) = 0$ and $\lim_{x\to 1/\rho^-} f_2(x) = -\infty$. Then, on one hand if $e > 3e_0/4$, $x^*(e) > 1/\rho$, $f_1(1/\rho) = \frac{1}{4}\sum_{i=2}^{n} 1/\sin(\pi(i-1)/n)$ and the two functions do not intersect. On the other hand, $\lim_{e\to 0} f_1(x^*(e)) = -\beta$, so that for small values of e, $f_1(x^*) < f_2(x^*)$ for $0 < x < 1/\rho$ and the two functions intersect twice.

Finally, by continuity, there exists a value of e such that f_1 and f_2 coincide tangentially only once for $0 < x < 1/\rho$.

The case $e \leq 0$, is simpler, $f_1(x)$ is an increasing function, f_1 tends to $-\infty$ when $e < 0$ and f_1 tends to $-\beta$ when $e = 0$. Clearly, in both cases f_1 and f_2 intersect at only one point. For the values of $e \in (-\infty, e^*]$ we denote the equilibrium points $L^+_{\bar{x}_i} = (\bar{x}_i, 0, 0)$, $i = 2, 3$, where $0 < \bar{x}_3 \leq \bar{x}_2 < 1/\rho$, and the equality holds when $e = e^*$ or $e \leq 0$.

□

From the properties shown in Theorem 3.4 the following result can be proved.

Proposition 3.2. *For any fixed value of* n, *for any* β, *let* e^* *and* x^* *be as in Theorem 3.4 and Definition 3.3, respectively. Then, for any* $e < e^*$, *the equilibrium point* $L^+_{\bar{x}_3}$ *satisfies that* $0 < \bar{x}_3 < x^*$.

In the Figure 6 we can see the evolution of e^* for different values of n and in the Figure 7 we can see the regions where there are 0 and 2 equilibria, where the curve $e = e^*$, as we know, represents the bifurcation that occurs, that is, when there is only one equilibrium.

3.3. **Equilibrium points on the line** $y = \tan\left(\frac{\pi}{n}\right)$**x, with** $x > 0$**.** The equilibrium point on the straight line $y = \tan(\pi/n)x$ is of the form $L^+_m = re^{i\pi/n}$, in complex coordinates, with $r = \sqrt{x^2 + y^2}$. The first two equations of (21) can be represented by the equation

$$h(r) = \Delta r^3 - \beta + \frac{2e\beta}{r} - \sum_{j=1}^{n} \frac{1 - \frac{1}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{(1 + \frac{1}{(\rho r)^2} - \frac{2}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}} = 0. \tag{35}$$

Definition 3.4. *Let* $\mathcal{C}$ *be the circumference containing the peripherals.*

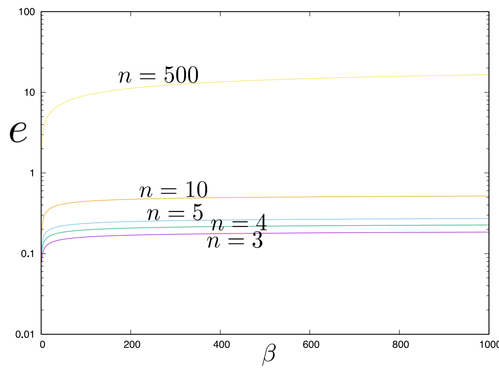
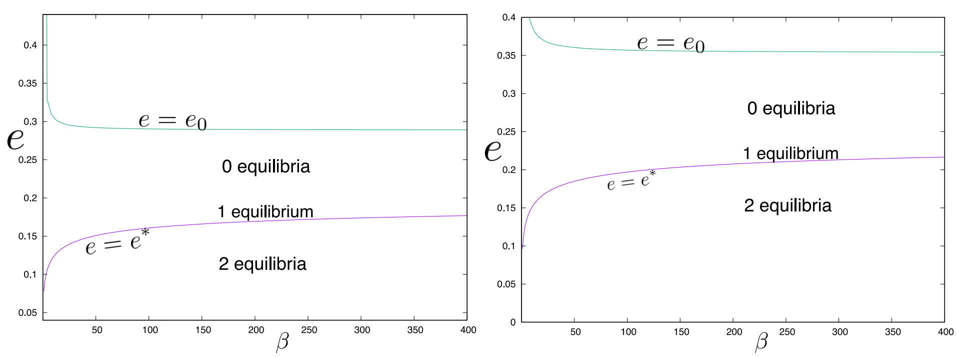


FIGURE 6. Evolution of the e^* in function of β, for different values of n.



(a) $n = 3$ (b) $n = 4$

FIGURE 7. Evolution of the e^* as a function of β, for the admissible regions with their respective amount of equilibria, for $x \in (0, 1/\rho)$.

The following result shows the existence of equilibrium points on the line $y = \tan(\pi/n)x$, $x > 0$.

Theorem 3.5. *For any fixed value of n, for any $\beta > 0$ and admissible e, consider $\mathcal{L}$ the line $y = \tan(\pi/n)x$, with $x > 0$.*

(1) *If $0 < e < e_0$, there exist at least two equilibrium points on $\mathcal{L}$. One of them is inside the circumference $\mathcal{C}$ and the other is outside of $\mathcal{C}$.*
(2) *If $e \leq 0$, there exists at least one equilibrium point on $\mathcal{L}$. It is outside the circumference $\mathcal{C}$.*

The proof is directly from Lemma 7.4. Note that for the case $0 < e < e_0$, the equilibrium points found are, one outside of the circumference $\mathcal{C}$ and the other inside of $\mathcal{C}$, let's call them L_{m_1} and L_{m_2}, respectively.

4. Linear Stability of the equilibrium solutions

The linearization of the the Hamiltonian system (12) is given by the matrix

$$A = A(x,y,z) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ V_{xx} & V_{xy} & V_{xz} & 0 & 1 & 0 \\ V_{xy} & V_{yy} & V_{yz} & -1 & 0 & 0 \\ V_{xz} & V_{yz} & V_{zz} & 0 & 0 & 0 \end{pmatrix},$$

Next, we study the eigenvalues and eigenvectors of the matrix A evaluated on each equilibrium point. Due to the symmetries, we will only study the stability of the equilibrium points L_z^+ and L_ξ, $\xi \in \{x_i, m_j\}$, $i = 1,2,3$, $j = 1,2$.

4.1. **Stability of the equilibrium points on the z-axis.** Consider the equilibrium point $L_z^+ = (0,0,\bar{z})$ $(n \geq 3)$ (see Theorem 3.2). Using the fact that $\bar{z}$ must satisfy the relation (31), it is not difficult to see that

$$\begin{aligned} & V_{xy}(L_z^+) = V_{xz}(L_z^+) = V_{yz}(L_z^+) = 0, \\ & V_{xx}(L_z^+) = V_{yy}(L_z^+) = \frac{3\beta(2e - \bar{z})}{2\rho^2 \Delta \left(\bar{z}^2 + \frac{1}{\rho^2}\right) \bar{z}^4}, \\ & V_{zz}(L_z^+) = \tfrac{\beta}{\Delta(\rho^2+\bar{z}^2)z^4} \left(3\bar{z} - 8e - 2e\rho^2\bar{z}^2\right). \end{aligned} \tag{36}$$

Proposition 4.1. *For any $\beta > 0$ and an admissible $e > 0$. Then, the eigenvalues associated to the the equilibrium point L_z^+ are $\pm\lambda_3 = \pm w\mathrm{i}$, $w > 0$, and*

$$\pm\lambda_{1,2} = \pm a \pm b\mathrm{i}, \;\; a > 0, \;\; b > 0.$$

Proof. Using (36), the eigenvalues of the matrix $A(L_z^+)$ are $\pm\lambda_3 = \pm\sqrt{V_{zz}(L_z^+)}$ and the solutions $\pm\lambda_{1,2}$ of

$$p(\lambda) = \lambda^4 - (2\gamma - 2)\lambda^2 + (1+\gamma)^2,$$

where $\gamma = V_{xx}(L_z^+) > 0$.

On one hand, using the fact that $\bar{z} < 2e$ (see Theorem 3.2), we have that

$$3\bar{z} - 2e\rho^2\bar{z}^2 - 8e < 3\bar{z} - 8e < 6e - 8e = -2e < 0.$$

Therefore, $V_{zz}(L_z^+) < 0$ and two of the eigenvalues are pure imaginary.

On the other hand, the solutions of $p(\lambda) = 0$ are

$$\lambda_\pm^2 = \gamma - 1 \pm 2\mathrm{i}\sqrt{\gamma}.$$

Therefore, the equilibrium point L_z^+ is of type *center* $\times$ *saddle* $\times$ *saddle*, that is, unstable. This completes the proof. □

4.2. **Stability of planar equilibrium points.** Consider the equilibrium points $L_{x_i}^+ = (\bar{x}_i, 0, 0)$, $i = 1,2,3$ and $L_{m_j} = (\bar{x}_j, \tan\frac{\pi}{n}\bar{x}_j, 0)$, $j = 1,2$ (see Theorems 3.3, 3.4 and 3.5 respectively). It is verified that the second partial derivatives of the potential satisfies

$$V_{xz}(L_\xi^+) = V_{yz}(L_\xi^+) = 0,$$

and

$$V_{zz}(L_\xi^+) = -\frac{1}{\Delta}\left[\beta\left(\frac{1}{r_0^3} - \frac{2e}{r_0^4}\right) + \sum_{j=1}^{n} \frac{1}{r_j^3}\right],$$

where $\xi \in \{x_i, m_j\}$, $i = 1,2,3$, $j = 1,2$. The characteristic polynomial can be written as

$$\lambda^4 + (2 - V_{xx} - V_{yy})\lambda^2 + V_{xx}V_{yy} + V_{xx} - V_{xy}^2 + V_{yy} + 1.$$

Then, the eigenvalues of the matrix $A(L_\xi^+)$ are

$$\begin{aligned} \pm\lambda_1 &= \pm\tfrac{1}{\sqrt{2}}\left(\Gamma + \sqrt{\Lambda}\right)^{1/2}, \\ \pm\lambda_2 &= \pm\tfrac{1}{\sqrt{2}}\left(\Gamma - \sqrt{\Lambda}\right)^{1/2}, \\ \pm\lambda_3 &= \pm\sqrt{V_{zz}}, \end{aligned} \tag{37}$$

where

$$\Lambda = (V_{xx} - V_{yy})^2 - 8(V_{xx} + V_{yy}) - 4V_{xy}^2, \qquad \Gamma = V_{xx} + V_{yy} - 2, \tag{38}$$

and the derivatives are evaluated at the corresponding equilibrium point. Notice that

$$\Lambda < \Gamma^2 \Leftrightarrow (1 + V_{xx})(1 + V_{yy}) - 4V_{xy}^2 > 0. \tag{39}$$

Now we will study the stability of the equilibrium points, separately, the spatial stability (in the direction of z) and the planar stability. For the first part it is only enough to know the sign of $V_{zz}(L_\xi^+)$. When $e \le 0$ it is clear that $V_{zz}(L_\xi^+) < 0$. If $0 < e < e_0$, we will use the equations of the equilibrium points (32) for $L_{x_i}^+$ and (47) for $L_{m_j}^+$.

$$V_{zz}(L_{x_i}^+) = -\frac{1}{\Delta}\left(\Delta + \frac{1}{\rho x}\sum_{j=1}^{n} \frac{\cos(\frac{2\pi j}{n})}{(x^2 + \frac{1}{\rho^2} - \frac{2x}{\rho} + \cos(\frac{2\pi j}{n}))^{3/2}}\right). \tag{40}$$

So that we know the sign $V_{zz}(L_{m_j}^+)$, $L_{m_j}^+$ we will write it in polar coordinates, that is, $L_{m_j} = re^{i\frac{\pi}{n}}$, with $r = \sqrt{x^2 + y^2}$. Then,

$$\begin{aligned} V_{zz}(L_{m_j}^+) &= -\frac{1}{\Delta}\left[\left(\frac{1}{r^3} - \frac{2e}{r^4}\right)\beta + \sum_{j=1}^{n} \frac{1}{(r^2 + \frac{1}{\rho^2} - \frac{2r}{\rho} + \cos(\frac{2\pi j}{n} + \frac{\pi}{n})^{3/2}}\right] \\ &= -\frac{1}{\Delta}\left(\Delta + \frac{1}{\rho r}\sum_{j=1}^{n} \frac{\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{(r^2 + \frac{1}{\rho^2} - \frac{2r}{\rho} + \cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}}\right). \end{aligned} \tag{41}$$

Clearly, $V_{zz}(L_{x_i}^+) < 0$ and $V_{zz}(L_{m_j}^+) < 0$. Therefore, the eigenvalues associated to the equilibrium points L_{x_i} and L_{m_j}, with $i = 1,2,3$ and $j = 1,2$ are pure imaginary. The planar stability will be studied below.

4.3. **Planar stability of the equilibrium points on the x-axis, with $x > 0$.** We consider the points L_{x_i}, $i = 1,2,3$. Using (21), we have that

$$\bar{x}_i - \frac{1}{\Delta}\left(\beta\left(\frac{1}{\bar{x}_i^2} - \frac{2e}{\bar{x}_i^3}\right) + \sum_{j=1}^{n} \frac{\bar{x}_i - x_j}{((\bar{x}_i - x_j)^2 + y_j^2)^{3/2}}\right) = 0,$$

The equation of motion (restricted to planar problem, that is, only the first two equations seen in (12)) in complex variables, given by

$$\ddot{W} + 2i\dot{W} = \Omega_W, \tag{42}$$

where $W = x + iy$ and

$$\Omega_W = \Omega_x + i\Omega_y = W - \frac{1}{\Delta}\sum_{j=1}^{n}\frac{W-\omega_j}{|W-\omega_j|^3} - \frac{\beta}{\Delta}\frac{W}{|W|^3} + \frac{2e\beta}{\Delta}\frac{W}{|W|^4},$$

with $\omega_j = \frac{1}{\rho}e^{i\varphi_j}$ and $\varphi_j = 2\pi j/n$, $j = 1, \ldots, n$. In the neighborhood of an equilibrium w_0 (w_0 being the affix of one of the points L_{x_1} , L_{x_2} or L_{x_3}), the following expansions hold $W = w_0 + \epsilon$.

So we get the linear equations in ϵ and δ, with $\delta = \overline{\epsilon}$

$$\begin{aligned} \ddot{\epsilon} + 2i\dot{\epsilon} &= \epsilon + A\epsilon + B\delta \\ \ddot{\delta} - 2i\dot{\delta} &= \delta + A\delta + \overline{B}\epsilon, \end{aligned} \tag{43}$$

where

$$\begin{aligned} A &= \frac{1}{2\Delta}\sum_{j=1}^{n}\frac{1}{|w_0-\omega_j|^3} + \frac{\beta}{2\Delta}\frac{1}{|w_0|^3} - \frac{2e\beta}{\Delta}\frac{1}{|w_0|^4} \\ B &= \frac{3}{2\Delta}\sum_{j=1}^{n}\frac{1}{|w_0-\omega_j|^3}\frac{w_0-\omega_j}{\overline{w_0-\omega_j}} + \frac{3\beta}{2\Delta}\frac{1}{|w_0|^3}\frac{w_0}{\overline{w_0}} - \frac{4e\beta}{\Delta}\frac{1}{|w_0|^4}\frac{w_0}{\overline{w_0}}. \end{aligned} \tag{44}$$

In a matricial form

$$\begin{pmatrix} \dot{\epsilon} \\ \dot{\delta} \\ \ddot{\epsilon} \\ \ddot{\delta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A+1 & B & -2i & 0 \\ \overline{B} & A+1 & 0 & 2i \end{pmatrix}$$

with the characteristic polynomial for the coefficient matrix

$$\chi(\lambda) = (1+\lambda^2)^2 + 2A(1-\lambda^2) + A^2 - |B|^2,$$

denoting by $\nu = \lambda^2$, the characteristic polynomial assumes the form

$$\hat{\chi} = \nu^2 + 2(1-A)\nu + (1+A)^2 - |B|^2.$$

Note that the eigenvalues of the linearized system will be pure imaginary (i.e., the equilibrium point of the planar system is linearly stable), if and only if, the roots of the previous polynomial are non-positive. This condition is equivalent to

$$\begin{aligned} l_1 &= |B|^2 - 4A > 0, \\ l_2 &= 1 - A > 0, \\ l_3 &= 1 + A - |B| > 0. \end{aligned} \tag{45}$$

Remark 4.1. *The conditions for l_1, l_2 and l_3 are equivalent to $\Lambda > 0$, $\Gamma < 0$ and $\Lambda < \Gamma^2$, respectively (see (38)).*

Lemma 4.1. *For each $\beta > 0$,*

1. *If $e < e_0$ and $x = w_0 \in (\frac{1}{\rho}, +\infty)$ (equilibrium solution), $B(x = \omega_0) = |B(x = \omega_0)|$.*
2. *If $e \leq 0$ or $e \to 0^+$ and $x = w_0 \in (0, \frac{1}{\rho})$, $B(x = \omega_0) = |B(x = \omega_0)|$.*

Proof. $B(x)$ admits a symmetry when changing $x \to 1/x$, so we can assume $x = \frac{1}{\rho s}$. Thus, $B(x) = \dfrac{3}{2\Delta}\displaystyle\sum_{j=1}^{n} \dfrac{1}{|x-\omega_j|^3}\dfrac{x-\omega_j}{\overline{x-\omega_j}} + \dfrac{3\beta}{2\Delta}\dfrac{1}{|x|^3} - \dfrac{4e\beta}{\Delta}\dfrac{1}{|x|^4}\dfrac{x}{\overline{x}}$ which is equivalent to

$$B(x) = \frac{3\rho^3 s^3}{2\Delta}\sum_{j=1}^{n} \frac{1-s\bar{\omega}_j}{1-s\bar{\omega}_{-j}}\left(1+s^2-2s\cos\left(\frac{2\pi j}{n}\right)\right)^{-3/2} + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4,$$

with $\bar{\omega}_j = e^{\frac{i\,2\pi j}{n}}$. We introduce the notation

$$\{f(u)\}_n = \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{2\pi j}{n}\right)$$

$$B(x) = \frac{3n\rho^3 s^3}{2\Delta}\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4$$

Note that the equation (32) (using $s = 1/(\rho x)$) is equivalent to

$$s^3 = 2e\rho s^4 + \frac{\Delta}{\beta\rho^3} - \frac{n}{\beta}s^3 h_n(s), \tag{46}$$

then

$$\begin{aligned} B &= \frac{3n\rho^3 s^3}{2\Delta}\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n + \tfrac{3}{2} - \tfrac{4e\beta\rho^4}{\Delta}s^4 - \tfrac{3n}{2\Delta}s^3 h_n(s) \\ &= \frac{3n\rho^3 s^3}{2\Delta}\left[\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n - h_n(s)\right] + \tfrac{3}{2} - \tfrac{4e\beta\rho^4}{\Delta}s^4. \end{aligned}$$

Let $B_1 = \left\{\dfrac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n - h_n(s)$,

$$\begin{aligned} B_1 &= \left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}} - \frac{1-se^{iu}}{(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n \\ &= s\left\{\frac{e^{-iu}(1-se^{iu})}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n \\ &= s\left\{e^{-iu}\frac{1}{(1-se^{-iu})^2(1-se^{-iu})^{1/2}(1-se^{iu})^{1/2}}\right\}_n \end{aligned}$$

Using the expansion $\frac{1}{(1-z)^{1/2}} = \sum_{k=1}^{\infty} a_k z^k$, with $a_k > 0$, we obtain then

$$B_1 = s\left\{e^{-iu}\sum_{k=0}^{\infty}(k+1)s^k e^{-iku}\sum_{k=0}^{\infty}a_k s^k e^{-iku}\sum_{k=0}^{\infty}a_k s^k e^{iku}\right\}_n = \left\{\sum_{p=0}^{\infty}\{P(e^{iu},e^{-iu})s^p\}_n\right\},$$

where $P(e^{iu},e^{-iu})$ is polynomial with positive coefficients. Then $B_1 > 0$, thus $B > 0$, when $0 < s < 1$, for all $0 < e < e_0$. In the case when $e < 0$, we consider, again

$$B(x) = \frac{3n\rho^3 s^3}{2\Delta}\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n + \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4,$$

where the term

$$\left\{\frac{1-se^{iu}}{(1-se^{-iu})(1-se^{iu})^{3/2}(1-se^{-iu})^{3/2}}\right\}_n$$

is positive, the proof is similar to the proof $B_1 > 0$. Then $B > 0$, when $s \in (0, +\infty) - \{1\}$ and $e < 0$. Note that if $e \to 0^+$, $B > 0$, thus, for continuity on e, we have that for values of e close to 0, $B > 0$, when $s > 1$. □

The following technical lemma will be used later, the proof can be seen in [3].

Lemma 4.2. *For every $s \in (0, +\infty) - \{1\}$,*

$$\left\{3\frac{1+s^2e^{2iu}-2se^{iu}}{(1+s^2-2s\cos u)^{5/2}} - \frac{1}{(1+s^2-2s\cos u)^{3/2}} - 2\frac{1-se^{iu}}{(1+s^2-2s\cos u)^{3/2}}\right\}_n > 0$$

Now we see what is the stability characteristic of the equilibrium point L_{x_1}.

Proposition 4.2. *For each β and e admissible, L_{x_1} is unstable.*

Proof. Using Lemma 4.1, that is, $|B| = B$ and the equation (46), that is,

$$1 = \frac{\beta}{\Delta}s^3 + \frac{ns^3}{\Delta}h_n(s) - \frac{2\beta e}{\Delta}s^4,$$

then

$$\begin{aligned} |B| - A - 1 &= B - A - 1 \\ &= \frac{n\rho^3 s^3}{2\Delta}\left\{3\frac{1+s^2e^{2iu}-2se^{iu}}{(1+s^2-2s\cos u)^{5/2}} - \frac{1}{(1+s^2-2s\cos u)^{3/2}} - 2\frac{1-se^{iu}}{(1+s^2-2s\cos u)^{3/2}}\right\}_n \\ &+ \frac{3\beta\rho^3}{2\Delta}s^3 - \frac{4e\beta\rho^4}{\Delta}s^4 - \frac{\beta\rho^3}{2\Delta}s^3 + \frac{2e\beta\rho^4}{\Delta}s^4 - \frac{\beta}{\Delta\rho^3}s^3 + \frac{2\beta e\rho^4}{\Delta}s^4 \\ &= \frac{n\rho^3 s^3}{2\Delta}\left\{3\frac{1+s^2e^{2iu}-2se^{iu}}{(1+s^2-2s\cos u)^{5/2}} - \frac{1}{(1+s^2-2s\cos u)^{3/2}} - 2\frac{1-se^{iu}}{(1+s^2-2s\cos u)^{3/2}}\right\}_n \end{aligned}$$

Now, using Lemma 4.2 is obtained that $|B| - A - 1 > 0$, thus $l_3 < 0$. Therefore there must be a root of the characteristic polynomial $\chi(\lambda)$ with a non-zero real part. Thus, L_{x_1} is unstable. □

Proposition 4.3. *For each β and $e \le 0$, L_{x_2} is unstable.*

Proof. Using Lemma 4.1 and Lemma 4.2 the result is obtained in a similar form as in Proposition 4.2. □

Proposition 4.4. *For each $\beta > 0$ and $0 < e < e^* < \frac{3e_0}{4}$, with e^* bifurcation parameter (as in Theorem 3.4), $L^+_{x_2}$ ($x_2 \in (x^*, 1/\rho)$, with x^* as in Lemma 7.2) is unstable.*

Proof. Remember, from Lemma 4.1

$$B = \frac{3n\rho^3 s^3}{2\Delta}s\left\{e^{-iu}\frac{1}{(1-se^{-iu})^2(1-se^{-iu})^{1/2}(1-se^{iu})^{1/2}}\right\}_n + \frac{3}{2} - \frac{4e\beta\rho^4}{\Delta}s^4.$$

Notice that $x^* < x < 1/\rho$ is equivalent to $1 < s < s^*$, with $s^* = 1/(\rho x^*) = (3\Delta/(2\beta e))^{1/4}/\rho$, then $\frac{3}{2} - \frac{4e\beta\rho^4}{\Delta}s^4 > 0$. Thus, $B > 0$. To prove that $l_3 < 0$, proceed in a similar way to the proof of Proposition 4.2. □

4.4. **Planar stability of the equilibrium points on the straight line $y = \tan(\pi/n)x$, with $x > 0$.** The equilibrium points on the straight line $y = \tan(\pi/n)x$ in the complex variable are of the form $L_{m_j} = re^{i\pi/n}$, with $j = 1, 2$. Recall that these equilibrium points satisfies the equation

$$\Delta r^3 - \beta + \frac{2e\beta}{r} - \sum_{j=1}^{n}\frac{1 - \frac{1}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n})}{(1 + \frac{1}{(\rho r)^2} - \frac{2}{\rho r}\cos(\frac{2\pi j}{n} + \frac{\pi}{n}))^{3/2}} = 0. \tag{47}$$

Note that equation (47) (using $s=\frac{1}{\rho r}$) is equivalent to

$$\frac{\Lambda}{\rho^2}+\beta-2\beta e\rho-\beta s^3+2e\beta\rho s^4-s^3h_n(s,\pi/n)=0, \tag{48}$$

with $h_n(s,\pi/n)=\sum_{j=1}^n\dfrac{1-s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)}{(1+s^2-2s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right))^{3/2}}$. If we divide the equation (48) by β and make β tend to infinity, it is clear that for large β, s tends to 1 or s tend to $\bar{s}$, where $\bar{s}$ satisfies the equation $2\rho e-1+(2\rho e-1)s+(2\rho e-1)s^2+2\rho e s^3=0$, the second case happens only if $e>0$. From the equation (48) is obtained

$$\beta=\frac{s^3h_n(\pi/n,s)-\frac{\Lambda}{\rho^2}}{1-s^3-2e\rho(1-s^4)}, \tag{49}$$

and

$$1=\frac{\rho^3}{\Delta}(\beta s^3+s^3h_n(s,\pi/n)-2\beta e\rho s^4). \tag{50}$$

Equations (49) and (50) we will use later.

To study planar linear stability, we can use what was seen in the previous section, that is, we can analyse the values l_1, l_2 l_3 over the equilibria L_{m_i}. For this, we must calculate A and B defined in the previous section.

$$\begin{aligned}
A &= \frac{1}{2\Delta}\sum_{j=1}^n\frac{1}{|w_0-\omega_j|^3}+\frac{\beta}{2\Delta}\frac{1}{|w_0|^3}-\frac{2e\beta}{\Delta}\frac{1}{|w_0|^4}\\
&=\frac{\rho^3s^3}{2\Delta}\sum_{j=1}^n\frac{1}{\left(1+s^2-2s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)\right)^{3/2}}+\frac{\beta\rho^3}{2\Delta}s^3-\frac{2e\beta\rho^4}{\Delta}s^4\\
&=\frac{\rho^3s^3}{2\Delta(1-s^3-2e\rho(1-s^4))}\\
&\times\left(\sum_{j=1}^n\frac{1-s^4\cos(\frac{2\pi j}{n}+\frac{\pi}{n})-2e\rho(1-s^4)-4e\rho s^4(1-s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))}{(1+s^2-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{3/2}}-(1-4\rho e s)\frac{\Lambda}{\rho^2}\right)\\
B &= \frac{3}{2\Delta}\sum_{j=1}^n\frac{1}{|w_0-\omega_j|^3}\frac{w_0-\omega_j}{\overline{w_0-\omega_j}}+\frac{3\beta}{2\Delta}\frac{1}{|w_0|^3}\frac{3}{2\Delta}\sum_{j=1}^n\frac{1}{|w_0-\omega_j|^3}\frac{w_0-\omega_j}{\overline{w_0-\omega_j}}\\
&+\frac{3\beta}{2\Delta}\frac{1}{|w_0|^3}\frac{w_0}{\overline{w_0}}-\frac{4e\beta}{\Delta}\frac{1}{|w_0|^4}\frac{w_0}{\overline{w_0}}-\frac{4e\beta}{\Delta}\frac{1}{|w_0|^4}\frac{w_0}{\overline{w_0}}\\
&=\frac{3\rho^3s^3}{2\Delta}e^{\frac{i2\pi}{n}}\sum_{j=1}^n\frac{1-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n})+s^2\cos(\frac{4\pi j}{n}+\frac{2\pi}{n})}{(1+s^2-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{5/2}}+\frac{3\beta\rho^3}{2\Delta}e^{\frac{i2\pi}{n}}s^3-\frac{4e\beta\rho^4}{\Delta}e^{\frac{i2\pi}{n}}s^4
\end{aligned}$$

Proposition 4.5. *If β is sufficiently large and $\frac{1}{8\rho}<e<\frac{3}{8\rho}$, then L_{m_1} is linearly stable.*

Proof. Note that if, s tend to 1^-, then by (49) β is sufficiently large, and so e_0 tends to $e_0=\frac{1}{2\rho}$. On the other hand,

$$|B|=\left|\frac{3\rho^3s^3}{2\Delta}\sum_{j=1}^n\frac{1-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n})+s^2\cos(\frac{4\pi j}{n}+\frac{\pi}{n})}{(1+s^2-2s\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{5/2}}+\frac{3\beta\rho^3}{2\Delta}s^3-\frac{4e\beta\rho^4}{\Delta}s^4\right|.$$

If s tends to 1^- and $e < \frac{3}{8\rho}$, then the argument inside $||$ is positive. Substituting relation (49) in l_2 and l_3, we obtain

$$\begin{aligned} l_2 &= 1 - A \\ &= \tfrac{\beta\rho^3 s^3}{2\Delta} + \tfrac{\rho^3 s^3}{2\Delta} \textstyle\sum_{j=1}^n \frac{1-2s\cos\left(\frac{2\pi j+\pi}{n}\right)}{2-2\cos\left(\frac{2\pi j+\pi}{n}\right)^(3/2)}, \\ l_3 &= A + 1 - |B| \\ &= \frac{\rho^3 s^3}{2\Delta}\sum_{j=1}^n \left(\frac{3-2s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)}{(1+s^2-2s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right))^{3/2}} - \frac{3-6s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right)-3s^2\cos\left(\frac{4\pi j}{n}+\frac{2\pi}{n}\right)}{(1+s^2-2s\cos\left(\frac{2\pi j}{n}+\frac{\pi}{n}\right))^{5/2}}\right), \end{aligned}$$

Thus, when s tends to 1^-

$$l_1 = \frac{3\rho^3}{2\Delta}\sum_{j=1}^n \frac{1-2\cos(\frac{2\pi j}{n}+\frac{\pi}{n})+\cos(\frac{4\pi j}{n}+\frac{2\pi}{n})}{(2-2\cos(\frac{2\pi j}{n}+\frac{\pi}{n}))^{5/2}} + \frac{\beta\rho^3}{\Delta}\left(-\frac{1}{2}+4e\rho\right) > 0,$$

$$l_2 = \frac{\rho^3}{2\Delta}\left(\beta + \sum_{j=1}^n \frac{1-2s\cos\left(\frac{2\pi j+\pi}{n}\right)}{2-2\cos\left(\frac{2\pi j+\pi}{n}\right)^(3/2)}\right) > 0,$$

$$l_3 \cdot \frac{2\Delta}{\rho^3} = \sum_{j=1}^n \frac{\left(\cos\left(\frac{2\pi j+\pi}{n}\right)+3\right)\csc^2\left(\frac{2\pi j+\pi}{2n}\right)}{4\sqrt{2-2\cos\left(\frac{2\pi j+\pi}{n}\right)}} > 0,$$

since β is large enough. Finally, l_1, l_2 and l_3 are all positive numbers when $s \to 1^-$ (equivalently $r \to 1/\rho^+$) or when β sufficiently large. With these conditions L_{m_1} is linearly stable.

□

5. Global dynamics of the rectilinear Manev R($N+1$)BP

As we said in Section 2, the $(z, \dot{z})$ plane is an invariant plane of the Manev R$(N+1)$BP. The equations (12) reduces to the first order system

$$\begin{aligned} \dot{z} &= Z, \\ \dot{Z} &= -\frac{z}{\Delta}\left[\beta\left(\frac{1}{|z|^3}-\frac{2e}{z^4}\right)+\frac{2}{(1/4+z^2)^{3/2}}\right]. \end{aligned} \tag{51}$$

Furthermore, due to the symmetry $(z, Z) \to (-z, -Z)$, it is enough to study the problem for $z > 0$. We consider in this section the rectilinear Manev R$(N+1)$BP, that is, the subproblem given by (51) for $z > 0$.

The problem can be written in Hamiltonian form with Hamiltonian function associated

$$H = H(z, Z) = \frac{1}{2}Z^2 - V(z), \tag{52}$$

where the potential $V(z)$, can be written for $z > 0$ as

$$V(z) = \frac{1}{\Delta}\left[\beta\left(\frac{1}{z}-\frac{e}{z^2}\right)+\frac{2}{(1/4+z^2)^{1/2}}\right].$$

$V(z)$ possesses a unique critical point, which is a local maximum, that coincides with the equilibrium point $\bar{z}$ (see equation (31)). See the graph of V in Figure 8.

The constant value of $H = h$ it is also called *energy* and is related to the Jacobi constant by $H = -C/2$. Clearly, the energy of the equilibrium point L_z^+ is negative.

Analogously to the 2-body problem, we will say that a solution of (51), $(z(t), Z(t))$, is hyperbolic if it comes from (and arrive at) infinity with positive velocity, and it is parabolic if it comes from (and arrive at) infinity with zero velocity. Next result

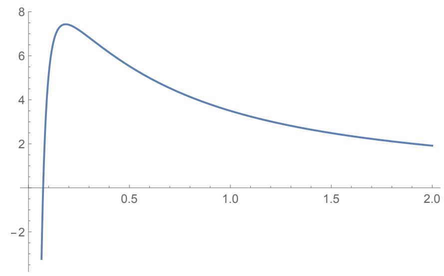


FIGURE 8. Graph of the potential $V(z)$ for $n = 3$, $\beta = 1$ and $e = 0.3$.

states that the solutions of the rectilinear Manev R$(N + 1)$BP are similar to the solutions of the one dimensional two body problem: periodic (bounded), parabolic and hyperbolic orbits.

Theorem 5.1. *For any $\beta > 0$ and a positive admissible e, let $L_z^+ = (0, 0, \bar{z})$ be the equilibrium point of the Manev R($N + 1$)BP given in Theorem 3.2. Let H be the Hamiltonian (52) and $\bar{h} = H(\bar{z}, 0)$. If $(z(t), Z(t))$ is a solution of the rectilinear Manev R($N + 1$)BP problem with $h = H(z(t), Z(t))$, then*

(i) *it is periodic, if and only if, $\bar{h} < h < 0$,*
(ii) *it is parabolic, if and only if, $h = 0$,*
(iii) *it is hyperbolic, if and only if, $h > 0$.*

Proof. Since this restricted problem is given by an autonomous Hamiltonian with one-degree of freedom, any solution of system of (51) lies on a level curve of $H = h$. Clearly, $H \geq -V(\bar{z}) = \bar{h}$.

For $h \in (\bar{h}, 0)$, the level curve $H = h$ in the (z, Z) plane cuts the positive z-axis in two points. Using the symmetry $(z, Z) \to (z, -Z)$, the crossings are perpendicular, so the level curve is symmetric with respect the z axis, and the solution is periodic. If $h = 0$, the corresponding level curve is $Z^2 = 2V(z)$, which tends to zero as $z \nearrow \infty$. That correspond to the parabolic solution. Finally, if $h > 0$, the level curve is $2h + 2V(z) = Z^2 \longrightarrow 2h$ when $z \nearrow \infty$. That correspond to the hyperbolic orbits. □

The phase portrait of the rectilinear Manev R4BP is shown (for a specific values of β and e) in Figure 9. Notice that, as stated in Theorem 2.1, solutions cannot accumulate at $z = 0$.

Recall that we have seen that the equilibrium point L_z^+ of the Manev R(N + 1)BP has a center in the z direction (see Section 4.1). The Lyapunov's Center Theorem ensures that if the ratios between the associated eigenvalues of L_z^+ are not integers, there exists a family of periodic orbits emanating from the equilibrium point. Theorem 5.1 shows that this family of periodic orbits (p.o.) exists for all values of β and admissible e (vertical p.o. from now on).

6. Concluding remarks

7. Appendix

The following technical lemma characterizes the roots of a particular type of function (see its proof in [1]).

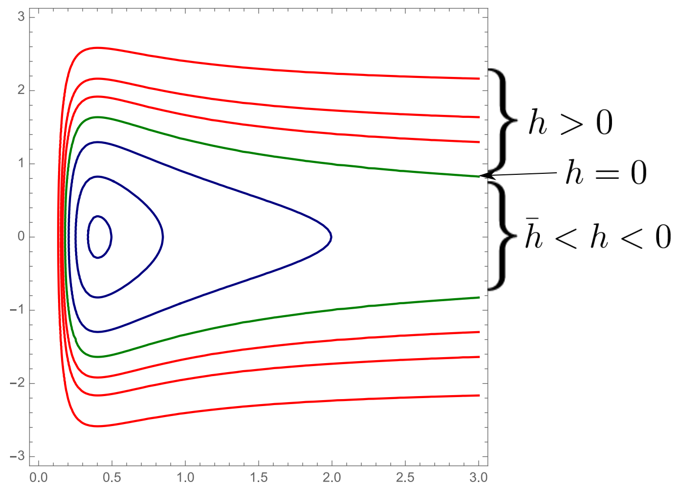


FIGURE 9. Phase portrait of the Hamiltonian system (51) for $n = 3$, $\beta = 1$ and $e = 0.3$.

Lemma 7.1. *Let T be a positive constant, n a natural number, and*

$$F(p) = f(p) + \sum_{j=1}^{n-1} f(p + jT) \tag{53}$$

where f is a function such that

i) $f(p + nT) = f(p)$,
ii) $f(p) = 0$, *if and only if*, $p = \frac{knT}{2}$, *for all* $k \in \mathbb{Z}$,
iii) $f(-p) = -f(p)$.

Then $F(p) = 0$, if and only if, $p = \frac{kT}{2}$, $k \in \mathbb{Z}$.

Some properties of the function f_1, defined in (33) we will resume them in the next Lemma.

Lemma 7.2. *For any fixed value of $\beta > 0$ and e admissible, the function $f_1(x)$, $x > 0$, defined in (33) has the following properties:*

(1) *Case $e \leq 0$.*
  (i) *$f_1(x)$ is an increasing function.*
  (ii) $\lim_{x \to +\infty} f_1(x) = +\infty$ *and* $\lim_{x \to 0^+} f_1(x) = -\infty$.
  (iv) *$f_1(0) = -\beta$, when $e = 0$.*
(2) *Case $0 < e < e_0$.*
  (i) *It has only one critical point, which is a minimum, at*

$$x^* = x^*(e) = \left(\frac{2\beta e}{3\Delta}\right)^{1/4}, \tag{54}$$

  where e_0 is given in (9).
  (ii) *$x^*(e)$ is an increasing function of e and $x^*(3e_0/4) = 1/\rho$.*
  (iii) *$f_1(x^*(e)) = 4\Delta^{1/4}\left(\frac{2\beta e}{3}\right)^{3/4} - \beta$ as a function of e has only one critical point, which is a maximum, at $3e_0/4$.*

(iv) $f_1(1/\rho)=\dfrac{1}{4}\displaystyle\sum_{i=2}^{n}\frac{1}{\sin\left(\frac{\pi(i-1)}{n}\right)}=\frac{\Lambda}{\rho^2}$, *is an increasing function in n.*

Proof. The proof of complete part 1 and the proof of part 2- i), 2- ii) and 2-ii) are straightforward calculations. For part iv), we will just prove that $h(n)=f_1(1/\rho)$ is an increasing function.

Consider the case where n is even; the odd case is similar. Then

$$\begin{aligned} h(n) &= \frac{1}{4}\sum_{j=1}^{n-1}\frac{1}{\sin\left(\frac{\pi\, j}{n}\right)} \\ &= \frac{1}{4}\left(2\sum_{j=1}^{\frac{n}{2}-1}\frac{1}{\sin\left(\frac{\pi j}{n}\right)}+1\right) \end{aligned}$$

and

$$\begin{aligned} h(n+1) &= \frac{1}{4}\sum_{j=1}^{n}\frac{1}{\sin\left(\frac{\pi\, j}{n+1}\right)} \\ &= \frac{1}{4}\left(2\sum_{j=1}^{\frac{n}{2}}\frac{1}{\sin\left(\frac{\pi j}{n+1}\right)}\right). \end{aligned}$$

Now,

$$h(n+1)-h(n)=\frac{1}{4}\left(2\sum_{j=1}^{\frac{n}{2}-1}\left(\frac{1}{\sin\left(\frac{\pi j}{n+1}\right)}-\frac{1}{\sin\left(\frac{\pi j}{n}\right)}\right)+\frac{2}{\sin\left(\frac{n\pi}{2(n+1)}\right)}-1\right).$$

Since $\frac{1}{\sin\left(\frac{\pi j}{n+1}\right)}-\frac{1}{\sin\left(\frac{\pi j}{n}\right)}>0$, for all $j=1,\dots,\frac{n}{2}-1$ and $\frac{2}{\sin\left(\frac{n\pi}{2(n+1)}\right)}-1>0$, then $h(n+1)-h(n)>0$. Therefore, $h(n)$ is an increasing function in n. □

Now, some properties of the function f_2, defined in (34), we summarise them in the next Lemma.

Lemma 7.3. *The function $f_2(x)$, $x>0$, defined in (34) has the following properties:*

(1) *$f_2(x)>0$, when $x>1/\rho$ and $f_2(x)<0$, when $x\in(0,\frac{1}{\rho})$, and in both cases f_2 is decreasing.*
(2) $\lim_{x\to\infty} f_2(x)=n$, $\lim_{x\to\frac{1}{\rho}^+} f_2(x)=+\infty$, $\lim_{x\to\frac{1}{\rho}^-} f_2(x)=-\infty$ *and* $\lim_{x\to 0^+} f_2(x)=0$.

Proof. The proof of the item 2 and the first two properties of item 1 are easy to verify, using straightforward calculations. For the last two properties we will use the results of Bang and Elmabsout (2004) in [3] and Moeckel and Simo (1995) in [10]. Now, using that $x_i=\frac{1}{\rho}\cos(\varphi_i)$ and $y_i=\frac{1}{\rho}\sin(\varphi_i)$, with $\varphi_i=\frac{2\pi(i-1)}{n}$, for $i=1,\dots,n$, now

$$\begin{aligned} f_2(x) &= x^2\sum_{i=1}^{n}\frac{x-\frac{1}{\rho}\cos(\varphi_i)}{((x^2-\frac{2x}{\rho}\cos(\varphi_i)+\frac{1}{\rho^2})^{3/2}} \\ &= (\rho x)^2\sum_{i=1}^{n}\frac{\rho x-\cos(\varphi)}{((\rho x)^2-2x\rho\cos(\varphi_i)+1)^{3/2}}. \end{aligned} \tag{55}$$

Making the change of variable $t = \dfrac{1}{\rho x}$, we obtain

$$f_2(t) = \sum_{i=1}^{n} \frac{1 - t\cos(\varphi_i)}{(t^2 - 2t\cos(\varphi_i) + 1)^{3/2}}. \tag{56}$$

We need to see that $f_2(t)$ is increasing for $t \in (0,1)$ and $f_2(t)$ is increasing for $t > 1$. In order to [3], we notice that $f(t) = (tV(t))'$, where

$$V(t) = \sum_{i=1}^{n} \frac{1}{(t^2 - 2t\cos(\varphi_i) + 1)^{1/2}}. \tag{57}$$

For $0 < t < 1$, $V(t)$ is a series in t with all its Taylor coefficients are positive (see [10]). So V and all its derivatives are positive. So, $f_2'(t) = V(t) + tV'(t) > 0$. Then $f_2(t)$ is an increasing function for $0 < t < 1$. Using that $f_2(t) = -\frac{1}{t^2}V'(1/t)$, then $f_2'(t) = \frac{2}{t^3}V'(\frac{1}{t}) + \frac{1}{t^4}V''(t)$, it follows that f_2 is increasing for $t > 1$. □

Some properties of the function h defined in (35) are listed in the next lemma.

Lemma 7.4. *For any fixed value of $\beta > 0$ and e admissible, the function $h(r)$, defined in (35) has the following properties.*

(1) *Case $e \le 0$.*
  (i) $\lim_{r\to+\infty} h(r) = +\infty$
  (ii) $\lim_{r\to 0^+} h(r) = -\infty$, *when $e < 0$.*
  (iii) $\lim_{r\to 0^+} h(r) = -\beta$, *when $e = 0$.*
  (iv) $h\left(\frac{1}{\rho}\right) < 0$.

(2) *Case $0 < e < e_0$.*
  (i) $\lim_{r\to+\infty} h(r) = +\infty$.
  (ii) $\lim_{r\to 0^+} h(r) = +\infty$.
  (iii) $h\left(\frac{1}{\rho}\right) < 0$.

Proof. The proof of the case $e \le 0$ part 1-i), 1-ii) and 1-ii) and the case $e < 0$ and part 2-i), 2-ii) are straightforward calculations. On the other hand,

$$\begin{aligned} h\left(\tfrac{1}{\rho}\right) &= \frac{\Lambda}{\rho^2} - \sum_{j=1}^{n} \frac{1 - \cos\left(\frac{\pi}{n} + \frac{2\pi j}{n}\right)}{\left(2 - 2\cos\left(\frac{\pi}{n} + \frac{2\pi j}{n}\right)\right)} \\ &= \frac{1}{4}\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)} - \sum_{j=1}^{n} \frac{1 - \cos\left(\frac{\pi}{n} + \frac{2\pi j}{n}\right)}{\left(2 - 2\cos\left(\frac{\pi}{n} + \frac{2\pi j}{n}\right)\right)}. \end{aligned} \tag{58}$$

Let's define $h_1(n) = \dfrac{1}{4}\displaystyle\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)}$ and $h_2(n) = \displaystyle\sum_{j=1}^{n} \frac{1 - \cos\left(\frac{\pi}{n} + \frac{2\pi j}{n}\right)}{\left(2 - 2\cos\left(\frac{\pi}{n} + \frac{2\pi j}{n}\right)\right)}$. Then,

$$\begin{aligned} h\left(\tfrac{1}{\rho}\right) &= h_1(n) - h_2(n) = 2h_1(n) - (h_2(n) + h_1(n)) \\ &= 2(h_1(n) - h_1(2n)) < 0, \end{aligned}$$

because $h_1(n)$ is increasing function with respect to n according to Lemma 7.2. □

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