Fractional Fourier Transforms Meet Riesz Potentials and Image Processing

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Abstract Via chirp functions from fractional Fourier transforms, the authors introduce fractional Riesz potentials related to chirp functions, establish their relations with fractional Fourier transforms, fractional Laplace operators related to chirp functions, and fractional Riesz transforms related to chirp functions, and obtain their boundedness on rotation invariant spaces related to chirp functions. Finally, the authors give the numerical image simulation of fractional Riesz potentials related to chirp functions and their applications in image processing. The main novelty of this article is to propose a new image encryption method for the double phase coding based on the fractional Riesz potential related to chirp functions. The symbol of fractional Riesz potentials related to chirp functions essentially provides greater degrees of freedom and greatly makes the information more secure.

1 Introduction

The image processing has always been an important topic of information sciences, which plays an important role in applied sciences and hence attracts a lot of attention (see, for instance, [3, 8, 10, 11, 16, 24, 33, 38, 44, 46]).

On the other hand, it is well known that the Fourier transform is one of the most basic important tools in both pure and applied mathematics. It is also a standard and powerful tool for analyzing and processing stationary signals, but it is limited in processing and analyzing non-stationary signals. In what follows, we use $\mathscr{S}(\mathbb{R}^n)$ to denote the set of all Schwartz functions equipped with well-known topology determined by a countable family of norms, and also $\mathscr{S}'(\mathbb{R}^n)$ the set of all continous linear functionals on $\mathscr{S}(\mathbb{R}^n)$ equipped with the weak$-*$ topology.

Definition 1.1. For any $f \in \mathscr{S}(\mathbb{R}^n)$, its *Fourier transform* $\widehat{f}$ or $\mathcal{F}(f)$ is defined by setting, for any $\boldsymbol{\xi} \in \mathbb{R}^n$,

$$\widehat{f}(\boldsymbol{\xi}) := \mathcal{F}(f) := \int_{\mathbb{R}^n} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{x}$$

and the *inverse Fourier transform* $f^\vee$ of f is defined by setting, for any $\boldsymbol{x} \in \mathbb{R}^n$, $f^\vee(\boldsymbol{x}) := \widehat{f}(-\boldsymbol{x})$.

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 42B35, 42A38, 94A08.

Key words and phrases. fractional Fourier transform, fractional Riesz potential, numerical picture simulation, image encryption, image decryption.

This project is partially supported by the National Natural Science Foundation of China (Grant Nos. 12071197, 12071052 and 11971058) and the National Key Research and Development Program of China (Grant No. 2020YF A0712900).

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The fractional Fourier transform (for short, FRFT) is proposed and developed to analyze and process non-stationary signals. At present, the FRFT has been applied in many fields, such as partial differential equation (see, for instance, [22]), wavelet analysis (see, for instance, [35]), complex transmission (see, for instance, [32]), frequency filter (see, for instance, [40]), time-frequency analysis (see, for instance, [1, 14, 15, 27]), optical signal processing (see, for instance, [2, 29, 30, 31, 36]), and optical image processing (see, for instance, [7, 23]).

The FRFT originated in Wiener's work in [45]. In 1980, the FRFT was given by Namias in [28], which is mainly based on the eigenfunction expansion method. The integral expressions of the FRFT on $\mathscr{S}(\mathbb{R})$ and $L^2(\mathbb{R})$ were given by McBride-Kerr in [26] and Kerr in [21], respectively. In 2021, the behavior of the FRFT on $L^p(\mathbb{R})$ for any $p \in [1, 2)$ was established by Chen et al. in [6].

In recent years, the research of the multidimensional FRFT has attracted more and more attention. In [18], Kamalakkannan and Roopkumar gave the following definition of the multidimensional FRFT.

Definition 1.2. Let $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$. The *multidimensional fractional Fourier transform* (for short, multidimensional FRFT) $\mathcal{F}_{\boldsymbol{\alpha}}(f)$, with order $\boldsymbol{\alpha}$, of f is defined by setting, for any $\boldsymbol{u} \in \mathbb{R}^n$,

$$\mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{u}) := \int_{\mathbb{R}^n} f(\boldsymbol{x}) K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{u})\, d\boldsymbol{x},$$

where, for any $\boldsymbol{x} := (x_1, \ldots, x_n), \boldsymbol{u} := (u_1, \ldots, u_n) \in \mathbb{R}^n$,

$$K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{u}) := \prod_{k=1}^{n} K_{\alpha_k}(x_k, u_k)$$

and, for any $k \in \{1, \ldots, n\}$,

$$K_{\alpha_k}(x_k, u_k) := \begin{cases} c(\alpha_k) e^{2\pi i \{a(\alpha_k)[x_k^2 + u_k^2 - 2b(\alpha_k) x_k u_k]\}} & \text{if } \alpha_k \notin \pi\mathbb{Z}, \\ \delta(x_k - u_k) & \text{if } \alpha_k \in 2\pi\mathbb{Z}, \\ \delta(x_k + u_k) & \text{if } \alpha_k \in 2\pi\mathbb{Z} + \pi, \end{cases}$$

with $a(\alpha_k) := \frac{\cot(\alpha_k)}{2} := \frac{\cos(\alpha_k)}{2\sin(\alpha_k)}$, $b(\alpha_k) := \sec(\alpha_k) := \frac{1}{\cos(\alpha_k)}$, $c(\alpha_k) := \sqrt{1 - i\cot(\alpha_k)}$, and δ being the Dirac measure at 0.

We refer the reader to [18, 19, 20, 48, 49, 50] for some studies on the multidimensional FRFT.

Remark 1.3. Let $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \ldots, n\}$. The *chirp function* $e_{\boldsymbol{\alpha}}$ is defined by setting, for any $\boldsymbol{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$e_{\boldsymbol{\alpha}}(\boldsymbol{x}) := e^{2\pi i \sum_{k=1}^{n} a(\alpha_k) x_k^2},$$

where, for any $k \in \{1, \ldots, n\}$, $a(\alpha_k) := \frac{\cot(\alpha_k)}{2}$. The chirp function is the most common nonstationary signal in which the frequency increases (upchirp) or decreases (downchirp) with time. It is easy to show that the multidimensional FRFT $\mathcal{F}_{\boldsymbol{\alpha}}(f)$ of f can be expressed into that, for any $\boldsymbol{u} := (u_1, \ldots, u_n) \in \mathbb{R}^n$,

$$\mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{u}) = c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha}}(\boldsymbol{u}) \mathcal{F}(e_{\boldsymbol{\alpha}} f)(\widetilde{\boldsymbol{u}}), \tag{1.1}$$

where $c(\boldsymbol{\alpha}) := c(\alpha_1)\cdots c(\alpha_n)$ with $\{c(\alpha_k)\}_{k=1}^n$ the same as in Definition 1.2 and where $\widetilde{\boldsymbol{u}} := (u_1 \csc(\alpha_1), \dots, u_n \csc(\alpha_n))$ with $\csc(\alpha_k) := \frac{1}{\sin(\alpha_k)}$ for any $k \in \{1, \dots, n\}$. Using (1.1), we can easily prove that $\mathcal{F}_{\boldsymbol{\alpha}}(f)$ maps $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$. Obviously, we can rewrite that, for any $\boldsymbol{x} := (x_1, \dots, x_n), \boldsymbol{u} := (u_1, \dots, u_n) \in \mathbb{R}^n$,

$$K_{\boldsymbol{\alpha}}(\boldsymbol{x}, \boldsymbol{u}) = c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha}}(\boldsymbol{x}) e_{\boldsymbol{\alpha}}(\boldsymbol{u}) e^{-2\pi i \sum_{k=1}^n x_k u_k \csc(\alpha_k)}.$$

Thus, the multidimensional FRFT is closely related to chirp functions.

Remark 1.4. When $\boldsymbol{\alpha} = (\frac{\pi}{2} + 2k_1\pi, \dots, \frac{\pi}{2} + 2k_n\pi)$ with $\{k_j\}_{j=1}^n \subset \mathbb{Z}$, the multidimensional FRFT goes back to the classical Fourier transform.

In this article, via chirp functions from fractional Fourier transforms, the authors introduce fractional Riesz potentials related to chirp functions, establish their relations with fractional Fourier transforms, fractional Laplace operators related to chirp functions, and fractional Riesz transforms related to chirp functions, and obtain their boundedness on rotation invariant spaces related to chirp functions. Finally, the authors give the numerical image simulation of fractional Riesz potentials related to chirp functions and their applications in image processing. The main novelty of this article is to propose a new image encryption method for the double phase coding based on the fractional Riesz potential related to chirp functions. The symbol of fractional Riesz potentials related to chirp functions essentially provides greater degrees of freedom and greatly makes the information more secure.

The remainder of this article is organized as follows.

In Section 2, based on the multidimensional FRFT, we introduce the fractional Riesz potential related to chirp functions and establish the relations among the multidimensional FRFT, the fractional Riesz potential related to chirp functions, the fractional Riesz transform related to chirp functions, and the Laplace operator related to chirp functions in $\mathscr{S}'(\mathbb{R}^n)$. We show that, in the rotation invariant space, the boundedness of the fractional Riesz potential is equivalent to the boundedness of the classical Riesz potential. In addition, through the relation between the fractional Fourier transform and the Laplace operator related to chirp functions, we introduce the fractional Laplace operator related to chirp functions, which, together with the fractional Riesz potential, provides a theoretical basis of the application of image encryption.

An electronic image simulation of the fractional Riesz potential related to chirp functions in the two-dimensional case is given in Section 3.

In Section 4, we present an image encryption method of double phase coding based on fractional Riesz potentials related to chirp functions, which mainly changes the amplitude in the fractional Fourier domain. Compared with the image encryption method of double phase coding based on the FRFT, the symbol of fractional Riesz potentials related to chirp functions in the image encryption method of double phase coding based on fractional Riesz potentials related to chirp functions essentially provides more degrees of freedom and greatly improves the security of information.

A conclusion is given in Section 5.

The fractional pseudo-differential operator and the fractional generalized Sobolev space and other function spaces related to chirp functions as well as their applications will be presented in a forthcoming article.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1,2,\ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any given multi-index $\boldsymbol{\zeta} := (\zeta_1,\ldots,\zeta_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, let

$$|\boldsymbol{\zeta}| := \zeta_1 + \cdots + \zeta_n \text{ and } \partial^{\boldsymbol{\zeta}} := \left(\frac{\partial}{\partial x_1}\right)^{\zeta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\zeta_n}.$$

We use C to denote a positive constant which is independent of the main parameters involved, whose value may vary from line to line. The *symbol* $g \lesssim h$ means $g \le Ch$. For a complex number z with $\operatorname{Re} z > 0$, let the *gamma function*

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt.$$

For any $p \in (0,\infty)$, the *Lebesgue space* $L^p(\mathbb{R}^n)$ is defined to be the set of all the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p\, dx\right]^{1/p} < \infty.$$

Moreover, when we prove a theorem or the like, we always use the same symbols in the wanted proved theorem or the like.

2 Fractional Riesz potentials related to chirp functions

Recall that, for any $f \in \mathscr{S}(\mathbb{R})$, the *Hilbert transform* $H(f)$ of f, is defined by setting, for any $x \in \mathbb{R}$,

$$H(f)(x) := \frac{1}{\pi}\ \mathrm{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy,$$

which is the prototype of Calderón–Zygmund operators and plays an irreplaceable role in harmonic analysis. The Hilbert transform is also a multiplier operator, that is, for any $f \in \mathscr{S}(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\mathcal{F}(Hf)(x) = -i \operatorname{sgn}(x) \mathcal{F}(f)(x). \tag{2.1}$$

It can be seen from (2.1) that the Hilbert transform is a phase-shift converter that multiplies the positive frequency portion of the original signal by $-i$; in other words, it maintains the same amplitude and shifts the phase by $-\pi/2$, while the negative frequency portion is shifted by $\pi/2$. Note that the Riesz transform is a natural generalization of the Hilbert transform in the n-dimensional case and is also a Calderón–Zygmund operator, with properties analogous to those of the Hilbert transform on $\mathbb{R}$. For any $j \in \{1,\ldots,n\}$ and $f \in \mathscr{S}(\mathbb{R}^n)$, the *Riesz transform* $R_j(f)$ of f is defined by setting, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$R_j(f)(\boldsymbol{x}) := c_n\ \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\boldsymbol{x}-\boldsymbol{y}|^{n+1}} f(\boldsymbol{y})\, d\boldsymbol{y},$$

where $c_n := \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$. The Riesz transform is also a multiplier operator, that is, for any $j \in \{1,\ldots,n\}$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{x} := (x_1,\ldots,x_n) \in \mathbb{R}^n$,

$$\mathcal{F}(R_j f)(\boldsymbol{x}) = -\frac{i x_j}{|\boldsymbol{x}|} \mathcal{F}(f)(\boldsymbol{x});$$

therefore, the multiplier of the Riesz transform is $-ix_j/|\boldsymbol{x}|$, and hence the Riesz transform is not only a phase-shift converter, but also an amplitude attenuator.

In [47], Zayed originally introduced the fractional Hilbert transform related to chirp functions which has been widely used in signal processing (see, for instance, [40, 39, 43]). In [6], Chen et al. regarded the fractional Hilbert transform as the fractional Fourier multiplier operator. Due to the development and the wide application of the multidimensional FRFT, motivated by the relations among the Fourier transform, the Riesz transform, the multidimensional FRFT, and the fractional Hilbert transform, Fu et al. [9] introduced the following fractional Riesz transform related to chirp functions.

Definition 2.1. For any $j \in \{1, \dots, n\}$ and $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, the *jth fractional Riesz transform related to chirp functions*, $R_j^{\alpha}(f)$, of $f \in \mathscr{S}(\mathbb{R}^n)$ is defined by setting, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$R_j^{\alpha}(f)(\boldsymbol{x}) := c_n \text{ p.v. } e_{-\alpha}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{x_j - y_j}{|\boldsymbol{x} - \boldsymbol{y}|^{n+1}} f(\boldsymbol{y}) e_{\alpha}(\boldsymbol{y})\, d\boldsymbol{y},$$

where $c_n := \Gamma(\frac{n+1}{2})/\pi^{\frac{n+1}{2}}$ and e_α is the same as in Remark 1.3.

The fractional Riesz transform related to chirp functions, R_j^{α}, is also a fractional multiplier operator. That is, for any $j \in \{1, \dots, n\}$, $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{u} := (u_1, \dots, u_n) \in \mathbb{R}^n$, one has

$$\mathcal{F}_{\alpha}\left(R_j^{\alpha} f\right)(\boldsymbol{u}) = -i \frac{\widetilde{u}_j}{|\widetilde{\boldsymbol{u}}|} \mathcal{F}_{\alpha}(f)(\boldsymbol{u}), \tag{2.2}$$

where $\widetilde{\boldsymbol{u}} := (\widetilde{u}_1, \dots, \widetilde{u}_n) = (u_1 \csc(\alpha_1), \dots, u_n \csc(\alpha_n))$.

In [9], Fu et al. also gave the application of the fractional Riesz transform related to chirp functions in the edge detection. Compared with the classical Riesz transform, the fractional Riesz transform related to chirp functions can detect the information in any direction by adjusting its order.

Recall that, for any $\beta \in (0, n)$ and $f \in \mathscr{S}(\mathbb{R}^n)$, the *Riesz potential* is defined by setting, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$I_\beta(f)(\boldsymbol{x}) := \frac{1}{\gamma(\beta)} \int_{\mathbb{R}^n} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{n-\beta}}\, d\boldsymbol{y}, \tag{2.3}$$

with $\gamma(\beta) := \pi^{\frac{n}{2}} 2^{\beta} \Gamma(\frac{\beta}{2})/\Gamma(\frac{n-\beta}{2})$. The Riesz potential, also known as the fractional integral operator, is not only a crucial integral operator in Fourier analysis, but also plays a very significant role in fractional differential equations. In this article, in order to avoid the confusion with the fractional operator, we always call it the Riesz potential. It is well known that, for any $\beta \in (0, n)$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{x} \in \mathbb{R}^n$, by the property of the Fourier transform, one has

$$\mathcal{F}(I_\beta f)(\boldsymbol{x}) = (2\pi)^{-\beta} |\boldsymbol{x}|^{-\beta} \mathcal{F}(f)(\boldsymbol{x}) \tag{2.4}$$

in $\mathscr{S}'(\mathbb{R}^n)$.

Inspired by the relation between the Riesz potential and the Fourier transform, as well as the definitions of both the multidimensional FRFT and the fractional Riesz transform related to chirp functions, we introduce a new fractional Riesz potential related to chirp functions as follows.

Definition 2.2. If $\beta \in (0, n)$ and $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \ldots, n\}$, the *fractional Riesz potential related to chirp functions*, $I_\beta^{\boldsymbol{\alpha}}$, is defined by setting, for any $f \in \mathscr{S}(\mathbb{R}^n)$ and $\boldsymbol{x} \in \mathbb{R}^n$,

$$I_\beta^{\boldsymbol{\alpha}}(f)(\boldsymbol{x}) := \frac{1}{\gamma(\beta)} e_{-\boldsymbol{\alpha}}(\boldsymbol{x}) \int_{\mathbb{R}^n} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{n-\beta}} e_{\boldsymbol{\alpha}}(\boldsymbol{y})\, d\boldsymbol{y},$$

where $e_{\boldsymbol{\alpha}}$ is the same as in Remark 1.3 and $\gamma(\beta)$ is the same as in (2.3).

Remark 2.3. When $\boldsymbol{\alpha} = (\frac{\pi}{2} + k_1\pi, \ldots, \frac{\pi}{2} + k_n\pi)$ with $\{k_j\}_{j=1}^n \subset \mathbb{Z}$, the fractional Riesz potential related to chirp functions goes back to the classical Riesz potential.

The Riesz potential is closely related to the Laplace operator, the Fourier transform, and the Riesz transform. Based on this, in Subsection 2.1, we establish the relations among the fractional Riesz potential, the multidimensional FRFT, the fractional Riesz transform related to chirp functions, and the Laplace operator related to chirp functions. In Subsection 2.2, we obtain the boundedness in the rotation invariant space of the fractional Riesz potential related to chirp functions.

2.1 Relations among fractional Fourier transforms, fractional Riesz potentials related to chirp functions, fractional Laplace operators related to chirp functions, and fractional Riesz transforms related to chirp functions

First, we establish the relation between the multidimensional FRFT and the fractional Riesz potential related to chirp functions.

Definition 2.4. (see [13]) The *Fourier transform* $\widehat{u}$ of any tempered distribution u is defined by setting, for any $f \in \mathscr{S}(\mathbb{R}^n)$, $\langle \widehat{u}, f \rangle := \langle u, \widehat{f} \rangle$.

Definition 2.5. Let $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \ldots, n\}$. The *multidimensional* FRFT $\widehat{u}$ of any tempered distribution u is defined by setting, for any $f \in \mathscr{S}(\mathbb{R}^n)$, $\langle \mathcal{F}_{\boldsymbol{\alpha}} u, f \rangle := \langle u, \mathcal{F}_{\boldsymbol{\alpha}} f \rangle$.

Let $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \ldots, n\}$. Using Definition 2.4, Definition 2.5, (1.1), and properties of the Fourier transform, we can easily conclude that, for any $f \in \mathscr{S}(\mathbb{R}^n)$ and $\boldsymbol{u} \in \mathbb{R}^n$, (1.1) also holds true.

Definition 2.6. (see [13]) Let $f, g \in L^1(\mathbb{R}^n)$. The *convolution* $f * g$ is defined by setting, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$(f * g)(\boldsymbol{x}) := \int_{\mathbb{R}^n} f(\boldsymbol{y}) g(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y}.$$

Lemma 2.7. (see [13]) *For any $u \in \mathscr{S}'(\mathbb{R}^n)$ and $f \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{F}(f * u) = \widehat{f}\,\widehat{u}$.*

Lemma 2.8. (see [37]) *For any $\beta \in (0, n)$ and $\xi \in \mathbb{R}^n$, the identity*

$$\mathcal{F}\left(\frac{1}{|\boldsymbol{x}|^{n-\beta}}\right)(\boldsymbol{\xi}) = \gamma(\beta)(2\pi)^{-\beta} |\boldsymbol{\xi}|^{-\beta}$$

holds true in the distribution sense, that is, for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left(\frac{1}{|\boldsymbol{x}|^{n-\beta}}\right) \widehat{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} = \gamma(\beta) \int_{\mathbb{R}^n} (2\pi)^{-\beta} |\boldsymbol{x}|^{-\beta} \varphi(\boldsymbol{x})\, d\boldsymbol{x},$$

where $\gamma(\beta)$ is the same as in (2.3).

Theorem 2.9. *For any $\beta \in (0,n)$, $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{u} := (u_1, \dots, u_n) \in \mathbb{R}^n$,*

$$\mathcal{F}_{\boldsymbol{\alpha}} \left(I_\beta^{\boldsymbol{\alpha}} f\right)(\boldsymbol{u}) = (2\pi)^{-\beta} |\widetilde{\boldsymbol{u}}|^{-\beta} \mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{u}) \text{ in } \mathscr{S}'(\mathbb{R}^n),$$

where $\widetilde{\boldsymbol{u}} := (\widetilde{u}_1, \dots, \widetilde{u}_n) = (u_1 \csc(\alpha_1), \dots, u_n \csc(\alpha_n))$. The $(2\pi)^{-\beta}|\widetilde{\boldsymbol{u}}|^{-\beta}$ is called the symbol *of the fractional Riesz potential related to chirp functions.*

Proof. Fix an $f \in \mathscr{S}(\mathbb{R}^n)$. From (1.1) and Definition 2.2, we infer that, for any $\boldsymbol{u} \in \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{\alpha}} \left(I_\beta^{\boldsymbol{\alpha}} f\right)(\boldsymbol{u}) &= c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha}}(\boldsymbol{u}) \mathcal{F}(e_{\boldsymbol{\alpha}} I_\beta^{\boldsymbol{\alpha}} f)(\widetilde{\boldsymbol{u}}) \\
&= \frac{1}{\gamma(\beta)} c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha}}(\boldsymbol{u}) \mathcal{F}\left(e_{\boldsymbol{\alpha}} f * \left(\frac{1}{|\cdot|^{n-\beta}}\right)\right)(\widetilde{\boldsymbol{u}}).
\end{aligned}$$

By Lemma 2.7 and Lemma 2.8, we obtain, for any $\boldsymbol{u} \in \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{F}_{\boldsymbol{\alpha}} \left(I_\beta^{\boldsymbol{\alpha}} f\right)(\boldsymbol{u}) &= \frac{1}{\gamma(\beta)} c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha}}(\boldsymbol{u}) \mathcal{F}(e_{\boldsymbol{\alpha}} f)(\widetilde{\boldsymbol{u}}) \mathcal{F}\left(\frac{1}{|\cdot|^{n-\beta}}\right)(\widetilde{\boldsymbol{u}}) \\
&= \frac{1}{\gamma(\beta)} c(\boldsymbol{\alpha}) e_{\boldsymbol{\alpha}}(\boldsymbol{u}) \mathcal{F}(e_{\boldsymbol{\alpha}} f)(\widetilde{\boldsymbol{u}}) \gamma(\beta) (2\pi)^{-\beta} |\widetilde{\boldsymbol{u}}|^{-\beta} \\
&= (2\pi)^{-\beta} |\widetilde{\boldsymbol{u}}|^{-\beta} \mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{u}),
\end{aligned}$$

which completes the proof of Theorem 2.9. □

Remark 2.10. According to Theorem 2.9, we can easily observe that the fractional Riesz potential related to chirp functions has the semigroup property, that is, for any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $\beta_1, \beta_2 \in (0,n)$, and $\beta_1 + \beta_2 \in (0,n)$, $I_{\beta_1}^{\boldsymbol{\alpha}} I_{\beta_2}^{\boldsymbol{\alpha}} = I_{\beta_1+\beta_2}^{\boldsymbol{\alpha}}$.

Lemma 2.11. (FRFT inversion theorem) (see [18]) *For any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{x} \in \mathbb{R}^n$,*

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^n} \mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{u}) K_{-\boldsymbol{\alpha}}(\boldsymbol{u}, \boldsymbol{x})\, d\boldsymbol{u},$$

where $K_{-\boldsymbol{\alpha}}$ is the same as in Definition 1.2 with $\boldsymbol{\alpha}$ replace by $-\boldsymbol{\alpha}$.

From Definition 2.5 and Lemma 2.11, it follows that, for any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, and for any $f \in \mathscr{S}'(\mathbb{R}^n)$, $\mathcal{F}_{-\boldsymbol{\alpha}} \mathcal{F}_{\boldsymbol{\alpha}} f = f$. By Theorem 2.9 and Lemma 2.11, we conclude that, for any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $\beta \in (0,n)$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{x} \in \mathbb{R}^n$, the fractional Riesz potential related to chirp functions can be rewritten as

$$(I_\beta^{\boldsymbol{\alpha}} f)(\boldsymbol{x}) = \mathcal{F}_{-\boldsymbol{\alpha}} \left((2\pi)^{-\beta} |\widetilde{\boldsymbol{u}}|^{-\beta} (\mathcal{F}_{\boldsymbol{\alpha}} f)(\boldsymbol{u})\right)(\boldsymbol{x})$$

in $\mathscr{S}'(\mathbb{R}^n)$, where $\boldsymbol{u} \in \mathbb{R}^n$ and $\widetilde{\boldsymbol{u}}$ is the same as in Theorem 2.9. In what follows, let

$$m_\beta^\alpha(\boldsymbol{u}) := (2\pi)^{-\beta}|\widetilde{\boldsymbol{u}}|^{-\beta},$$

which is called the *symbol* of I_β^α.

It is easy to show that the fractional Riesz potential related to chirp functions, $I_\beta^\alpha f$, of any $f \in \mathscr{S}(\mathbb{R}^n)$ can be decomposed into the composition of the multidimensional FRFT of order $\boldsymbol{\alpha}$, the symbol $m_\beta^\alpha(\boldsymbol{u})$ of the fractional Riesz potential related to chirp functions, and the multidimensional FRFT of order $-\boldsymbol{\alpha}$, as shown in the following Fig. 1:

(i) multidimensional FRFT of order $\boldsymbol{\alpha}$, namely $g^\alpha(\boldsymbol{u}) := (\mathcal{F}_\alpha f)(\boldsymbol{u})$ for any $\boldsymbol{u} \in \mathbb{R}^n$;

(ii) multiplication by the symbol of the fractional Riesz potential related to chirp functions $m_\beta^\alpha(\boldsymbol{u})$, namely
$$h_\beta^\alpha(\boldsymbol{u}) := m_\beta^\alpha(\boldsymbol{u})g^\alpha(\boldsymbol{u}) \text{ for any } \boldsymbol{u} \in \mathbb{R}^n;$$

(iii) multidimensional FRFT of order $-\boldsymbol{\alpha}$, namely $(I_\beta^\alpha f)(\boldsymbol{x}) := (\mathcal{F}_{-\alpha}h_\beta^\alpha)(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathbb{R}^n$.

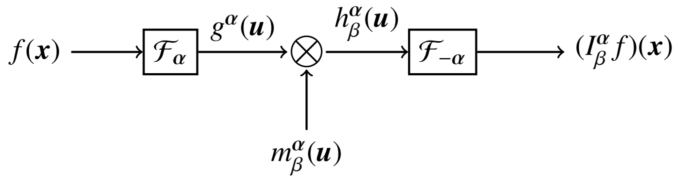


Figure 1: The decomposition of $I_\beta^\alpha(f)$

Remark 2.12. The symbol $m_\beta^\alpha(\boldsymbol{u})$ of the fractional Riesz potential related to chirp functions does not contain i, and hence there is no phase-shift effect. It can only change the amplitude, and hence the fractional Riesz potential related to chirp functions $m_\beta^\alpha(\boldsymbol{u})$ is only an amplitude modulator.

In [9], the derivative formula of the multidimensional FRFT was established as follows.

Lemma 2.13. (see [9]) (multidimensional FRFT derivative formula) *Let* $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ *with* $\alpha_k \notin \pi\mathbb{Z}$ *for any* $k \in \{1, \dots, n\}$, $f \in L^1(\mathbb{R}^n)$, *and* e_α *be the same as in Remark 1.3. If* $k \in \{1, \dots, n\}$ *and* $e_\alpha f$ *is absolutely continuous on* $\mathbb{R}^n$ *with respect to the kth variable, then, for any* $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\mathcal{F}_\alpha\left(e_{-\alpha}(\boldsymbol{y})\frac{\partial[e_\alpha(\boldsymbol{y})f(\boldsymbol{y})]}{\partial y_k}\right)(\boldsymbol{x}) = 2\pi i x_k \csc(\alpha_k)\mathcal{F}_\alpha(f)(\boldsymbol{x}).$$

Then the relations among the fractional Riesz potential related to chirp functions, the multidimensional FRFT, and the fractional Riesz transform related to chirp functions can be established as follows.

Theorem 2.14. *For any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, and for $f \in \mathscr{S}(\mathbb{R}^n)$, one has*

$$f = I_1^{\alpha}\left(\sum_{j=1}^n R_j^{\alpha}\left(e_{-\alpha}\partial_j(e_{\alpha} f)\right)\right) = \sum_{j=1}^n I_1^{\alpha}\left(R_j^{\alpha}\left(e_{-\alpha}\partial_j(e_{\alpha} f)\right)\right)$$

in $\mathscr{S}'(\mathbb{R}^n)$, where e_{α} is the same as in Remark 1.3, R_j^{α} for any $j \in \{1, \dots, n\}$ is the jth fractional Riesz transform, and I_1^{α} is the fractional Riesz potential I_{β}^{α} with $\beta = 1$.

Proof. From Theorem 2.9, (2.2), and Lemma 2.13, we deduce that, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}
&\mathcal{F}_{\alpha}\left(I_1^{\alpha}\left(\sum_{j=1}^n R_j^{\alpha}\left(e_{-\alpha}\partial_j(e_{\alpha} f)\right)\right)\right)(\boldsymbol{x}) \\
&\quad = (2\pi)^{-1}|\widetilde{\boldsymbol{x}}|^{-1}\, \mathcal{F}_{\alpha}\left(\sum_{j=1}^n R_j^{\alpha}\left(e_{-\alpha}\partial_j(e_{\alpha} f)\right)\right)(\boldsymbol{x}) \\
&\quad = (2\pi)^{-1}|\widetilde{\boldsymbol{x}}|^{-1} \sum_{j=1}^n -i\frac{\widetilde{x}_j}{|\widetilde{\boldsymbol{x}}|}\mathcal{F}_{\alpha}\left(e_{-\alpha}\partial_j(e_{\alpha} f)\right)(\boldsymbol{x}) \\
&\quad = (2\pi)^{-1}|\widetilde{\boldsymbol{x}}|^{-1} \sum_{j=1}^n -i\frac{\widetilde{x}_j}{|\widetilde{\boldsymbol{x}}|}2\pi i\widetilde{x}_j\mathcal{F}_{\alpha}(f)(\boldsymbol{x}) = \mathcal{F}_{\alpha}(f)(\boldsymbol{x}).
\end{aligned}$$

Then, by taking the multidimensional FRFT of order $-\boldsymbol{\alpha}$ and using Lemma 2.11, we obtain the desired identity, which completes the proof of Theorem 2.14. $\square$

Remark 2.15. When $\boldsymbol{\alpha} = (\frac{\pi}{2} + k_1\pi, \dots, \frac{\pi}{2} + k_n\pi)$ with $\{k_j\}_{j=1}^n \subset \mathbb{Z}$, in this case, Theorem 2.9 goes back to (2.4) of [25, pp. 124] and Theorem 2.14 goes back to (3.0.3) of [25, pp. 124] .

From Lemma 2.13, we can easily deduce the following theorem; we omit the details.

Theorem 2.16. *Let $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $\boldsymbol{\varsigma} \in \mathbb{Z}_+^n$, and e_{α} be the chirp function. For any $f \in \mathscr{S}(\mathbb{R}^n)$ and $\boldsymbol{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$, one has*

$$\mathcal{F}_{\alpha}\left(e_{-\alpha}(\boldsymbol{y})\partial^{\boldsymbol{\varsigma}}[e_{\alpha}(\boldsymbol{y})f(\boldsymbol{y})]\right)(\boldsymbol{x}) = (2\pi i\widetilde{\boldsymbol{x}})^{\boldsymbol{\varsigma}}\mathcal{F}_{\alpha}(f)(\boldsymbol{x}),$$

where $\widetilde{\boldsymbol{x}} := (\widetilde{x}_1, \dots, \widetilde{x}_n) := (x_1 \csc(\alpha_1), \dots, x_n \csc(\alpha_n))$.

From Lemma 2.13, it follows that, for any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $f \in \mathscr{S}(\mathbb{R}^n)$, and $\boldsymbol{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\mathcal{F}_{\alpha}\left(-e_{-\alpha}\Delta[e_{\alpha} f]\right)(\boldsymbol{x}) = 4\pi^2|\widetilde{\boldsymbol{x}}|^2\mathcal{F}_{\alpha}(f)(\boldsymbol{x}), \tag{2.5}$$

where $\widetilde{\boldsymbol{x}} := (\widetilde{x}_1, \dots, \widetilde{x}_n) := (x_1 \csc(\alpha_1), \dots, x_n \csc(\alpha_n))$, e_{α} is the same as in Remark 1.3, and Δ is the *Laplace operator* $\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

Motivated by (2.5), we replace the exponent 2 in (2.5) by a positive real number to introduce fractional Laplace operators related to chirp functions.

Definition 2.17. Suppose $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $e_{\boldsymbol{\alpha}}$ is the same as in Remark 1.3, $z \in \mathbb{R}$, and $z \in (0, \infty)$. The *fractional Laplace operator related to chirp functions*, Δ_z, is defined by setting, for any $f \in \mathscr{S}(\mathbb{R}^n)$ and $\boldsymbol{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$\mathcal{F}_{\boldsymbol{\alpha}}\left(\left[-e_{-\boldsymbol{\alpha}}(\boldsymbol{y})\Delta_z[e_{\boldsymbol{\alpha}}(\boldsymbol{y})f(\boldsymbol{y})]\right]^{\frac{z}{2}}\right)(\boldsymbol{x}) := (2\pi)^z|\widetilde{\boldsymbol{x}}|^z\mathcal{F}_{\boldsymbol{\alpha}}(f)(\boldsymbol{x}),$$

where $\widetilde{\boldsymbol{x}} := (\widetilde{x}_1, \dots, \widetilde{x}_n) = (x_1 \csc(\alpha_1), \dots, x_n \csc(\alpha_n))$. The $(2\pi)^z|\widetilde{\boldsymbol{x}}|^z$ is called the *symbol* of the fractional Laplace operators related to chirp functions, Δ_z.

Obviously, the fractional Laplace operator related to chirp functions, Δ_z, can be viewed as the inverse operator of the fractional Riesz potential related to chirp functions and plays a key role in the decryption process of the image encryption in Section 4.

2.2 Boundedness of fractional Riesz potentials related to chirp functions

In this subsection, we establish the boundedness of the fractional Riesz potential related to chirp functions on the rotation invariant space related to chirp functions, which is equivalent to the boundedness of the classical Riesz potential on the same rotation invariant space related to chirp functions.

Definition 2.18. Let $(X, \|\cdot\|_X)$ be a Banach space. Then X is called the *rotation invariant space related to chirp functions* if, for any $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, and for any $f \in X$,

$$\|e_{\boldsymbol{\alpha}} f\|_X = \|f\|_X,$$

where $e_{\boldsymbol{\alpha}}$ is the same as in Remark 1.3.

Remark 2.19. The well-konwn rotation invariant space includes the Lebesgue space, the Morrey space, and the Herz space.

Theorem 2.20. *If X, Y are two rotation invariant spaces related to chirp functions, $\beta \in (0, n)$, and $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, then I_β is bounded from X to Y if and only if $I_\beta^{\boldsymbol{\alpha}}$ is bounded from X to Y.*

Proof. Let $\|I_\beta\|_{X\to Y} < \infty$. By Definition 2.2 and Definition 2.18, we conclude that, for any $f \in X$,

$$\|I_\beta^{\boldsymbol{\alpha}}(f)\|_Y = \|I_\beta(e_{\boldsymbol{\alpha}} f)\|_Y \lesssim \|e_{\boldsymbol{\alpha}} f\|_X = \|f\|_X.$$

Conversely, if $\|I_\beta^{\boldsymbol{\alpha}}\|_{X\to Y} < \infty$, then we obtain, for any $f \in X$,

$$\|I_\beta(f)\|_Y = \|e_{-\boldsymbol{\alpha}} I_\beta(f)\|_Y = \|I_\beta^{\boldsymbol{\alpha}}(e_{-\boldsymbol{\alpha}} f)\|_Y \lesssim \|e_{-\boldsymbol{\alpha}} f\|_X = \|f\|_X.$$

This finishes the proof of Theorem 2.20. □

From Theorem 2.20, we can easily deduce the following corollary; we omit the details.

Corollary 2.21. (Hardy–Littlewood–Sobolev theorem) *Let $\beta \in (0, n)$, $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_k \notin \pi\mathbb{Z}$ for any $k \in \{1, \dots, n\}$, $p \in [1, \frac{n}{\beta})$, and $\frac{1}{q} := \frac{1}{p} - \frac{n}{\beta}$.*

(i) *If* $p \in (1, \frac{n}{\beta})$, *then there exists a positive constant* C *such that, for any* $f \in L^p(\mathbb{R}^n)$,

$$\|I_\beta^\alpha f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)};$$

(ii) *There exists a positive constant* C *such that, for* $f \in L^1(\mathbb{R}^n)$ *and any* $\lambda \in (0, \infty)$,

$$|\{x \in \mathbb{R}^n : |I_\beta^\alpha f(\boldsymbol{x})| > \lambda\}| \leq C \left[\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}\right]^{\frac{n}{n-\beta}}.$$

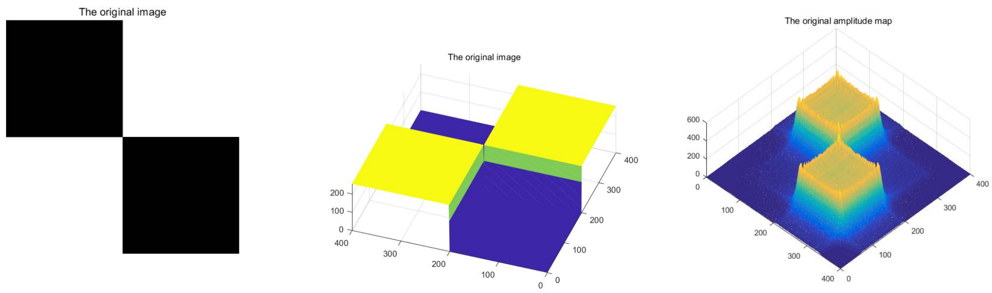
3 Simulation of fractional Riesz potentials related to chirp functions

In this section, we apply the fractional Riesz potential related to chirp functions to an image with the help of the FRFT discrete algorithm (see, for instance, [4, 5, 41]).

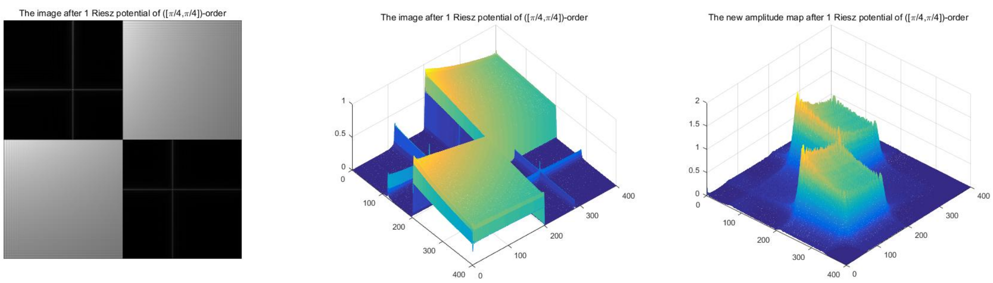
In the following Fig. 2, we give the numerical simulation of $I_\beta^{(\pi/4,\pi/4)}$. In the continuous case, the following Fig. 2(a) can be regarded as the function

$$f(x_1, x_2) := \begin{cases} 0, & \forall\,(x_1, x_2) \in [0, 200]^2 \cup [200, 400]^2, \\ 255, & \text{otherwise} \end{cases}$$

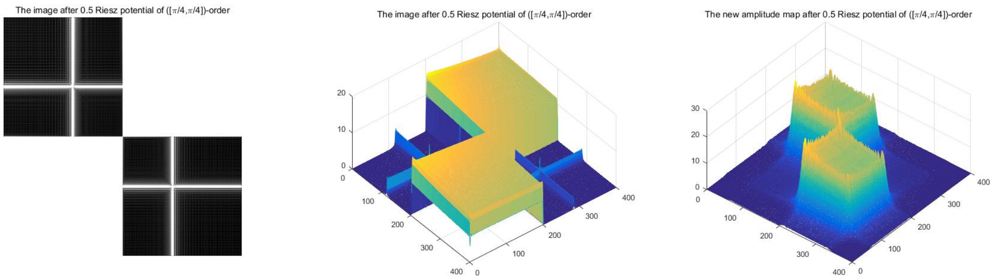
on $\mathbb{R}^2$.



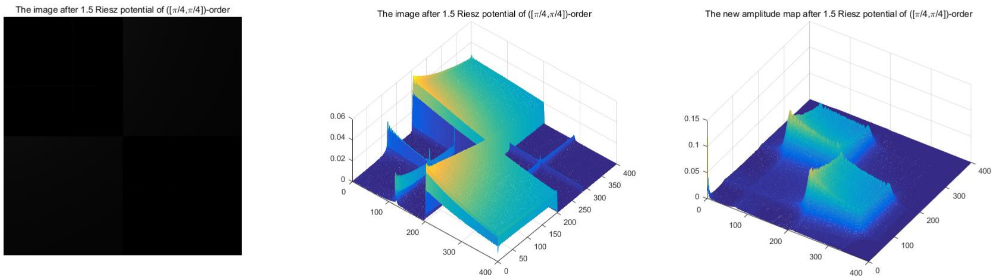
(a) (b) (c)



(d) (e) (f)



(g) (h) (i)

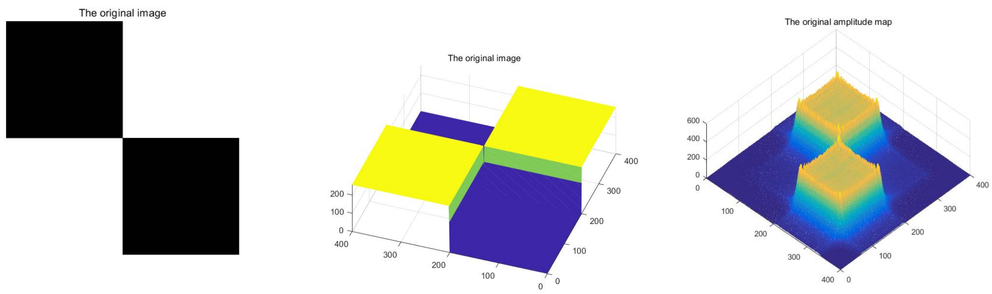


(j) (k) (l)

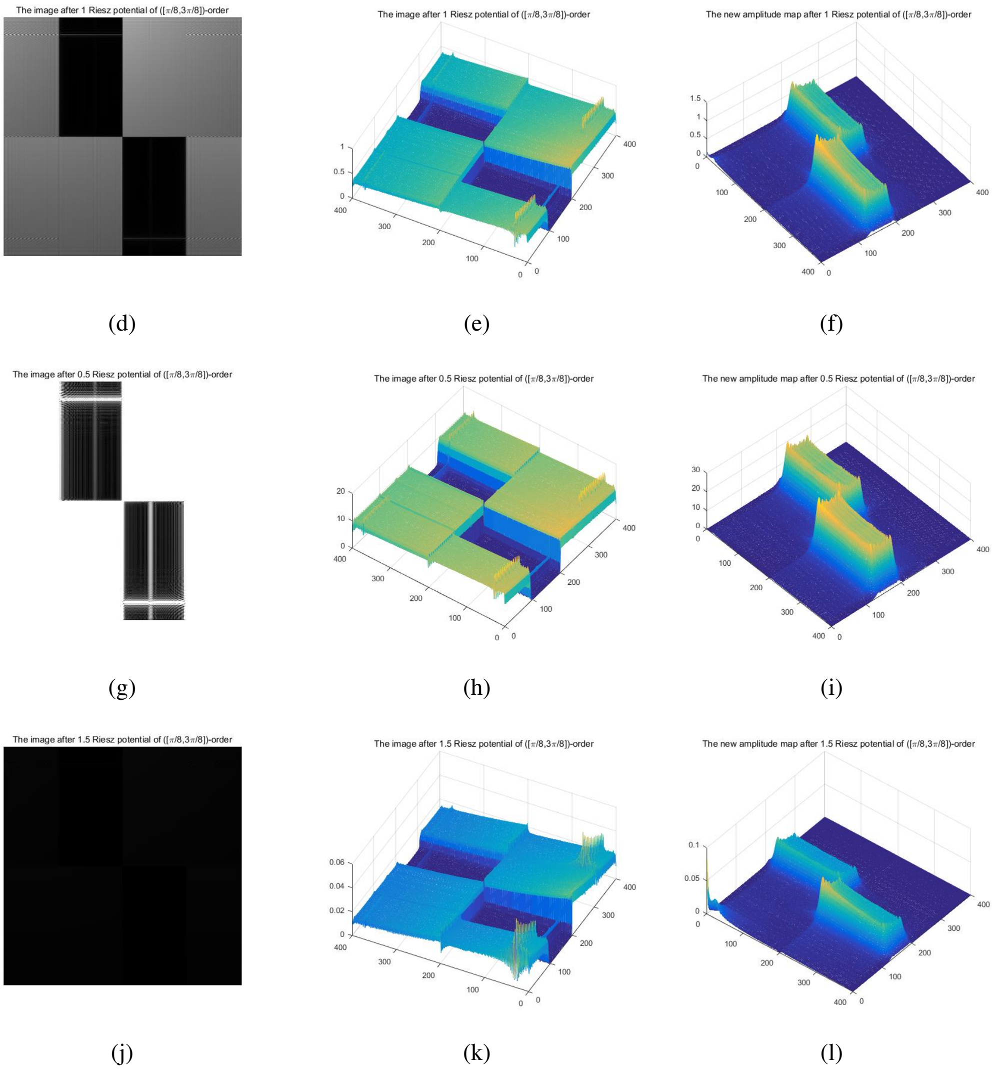
Figure 2: Numerical simulation of $I_\beta^{(\pi/4,\pi/4)}$

As is shown in the above Fig. 2, (a) is the original 2-dimensional grayscale image with 400 pixels $\times$ 400 pixels; (d), (g), and (k) are the 2-dimensional grayscale images, respectively, after $I_1^{(\pi/4,\pi/4)}f$, $I_{0.5}^{(\pi/4,\pi/4)}f$, and $I_{1.5}^{(\pi/4,\pi/4)}f$. Graphs (b), (e), (h), and (k) of Fig. 2 are the 3-dimensional color graphs of f, $I_1^{(\pi/4,\pi/4)}f$, $I_{0.5}^{(\pi/4,\pi/4)}f$, and $I_{1.5}^{(\pi/4,\pi/4)}f$, respectively. Graphs (c), (f), (i), and (l) of Fig. 2 are the amplitude images in the fractional Fourier domain of order $\alpha = (\pi/4, \pi/4)$ of f, $I_1^{(\pi/4,\pi/4)}f$, $I_{0.5}^{(\pi/4,\pi/4)}f$, and $I_{1.5}^{(\pi/4,\pi/4)}f$, respectively.

In the following Fig. 3, we give the numerical simulation of $I_\beta^{(\pi/8,3\pi/8)}$.



(a) (b) (c)



(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

Figure 3: Numerical simulation of $I_\beta^{(\pi/8,3\pi/8)}$

As is shown in the above Fig. 3, (a) is the original 2-dimensional grayscale image with 400 pixels $\times$ 400 pixels; (d), (g), and (k) are the 2-dimensional grayscale images, respectively, after $I_1^{(\pi/8,3\pi/8)}f$, $I_{0.5}^{(\pi/8,3\pi/8)}f$, and $I_{1.5}^{(\pi/8,3\pi/8)}f$. Graphs (b), (e), (h), and (k) of Fig. 3 are the 3-dimensional color graphs of f, $I_1^{(\pi/8,3\pi/8)}f$, $I_{0.5}^{(\pi/8,3\pi/8)}f$, and $I_{1.5}^{(\pi/8,3\pi/8)}f$. Graphs (c), (f), (i), and (l) of Fig. 3 are the amplitude images in the fractional Fourier domain of order $\alpha = (\pi/8, 3\pi/8)$ of f, $I_1^{(\pi/8,3\pi/8)}f$, $I_{0.5}^{(\pi/8,3\pi/8)}f$, and $I_{1.5}^{(\pi/8,3\pi/8)}f$, respectively.

Comparing Fig. 2(i) with Fig. 3(i), Fig. 2(f) with Fig. 3(f), and Fig. 2(l) with Fig. 3(l), we find that, when α changes and β remains unchanged, the image amplitude changes dramatically. Comparing (f), (i), and (l) of both Fig. 2 and Fig. 3, we conclude that, when β changes and α remains unchanged, the picture amplitude also changes dramatically. Graphs (c), (f), (i), and (l) of both Fig. 2 and Fig. 3 indicate that the symbol of the fractional Riesz potential can correspondingly dramatically change the amplitude in the fractional Fourier domain by adjusting parameters α and β. To sum up, Fig. 2 and Fig. 3 show that the fractional Riesz potential is an amplitude modulator. This is quite different from the fractional multiplier of the fractional Riesz transform, which is not only a phase-shift converter but also an amplitude attenuator.

In conclusion, we known that the fractional Hilbert transform related to chirp functions has a phase-shift effect in the fractional Fourier domain. However, the fractional Riesz transform related to chirp functions not only has a phase-shift in the fractional Fourier domain, but also can attenuate amplitude. Moreover, the fractional Riesz potential related to chirp functions can change the amplitude in the fractional Fourier domain. Since these three transforms behave quite different due to their different multipliers or symbols, we predict that these three transforms will play quite different important roles in signal processing.

4 Image encryption with double phase coding based on fractional Riesz potentials related to chirp functions

With the development of broadband network and multimedia technology, the acquisition, transmission, and processing of image data spread to all corners of the digital era. Security issues are also becoming increasingly serious. Many image datum need to be transmitted and stored confidentially, such as photographs taken by satellites, architectural drawings from financial institutions, and, in the telemedicine system, patient records and medical images.

In [12, 42], Goudail et al. and Unnikrishnan et al. proposed a double phase coding image encryption method based on the FRFT. Compared with the double phase coding image encryption method based on the Fourier transform in [17, 34], in addition to the phase mask, the improved double phase coding encryption key increases the order of the FRFT twice, and hence expands the key space. When the order is unknown, it will not be normally decrypted. Now, we propose a new image encryption method based on the fractional Riesz potential related to chirp functions with double phase coding. That is, we change the amplitude of the FRFT domain through the symbol of the fractional Riesz potential related to chirp functions, whose symbol, together with the order of the FRFT, provides greater degrees of freedom, expands the key space, and improves the security of the protected information. The following Fig. 4 exactly explains this encryption processing.

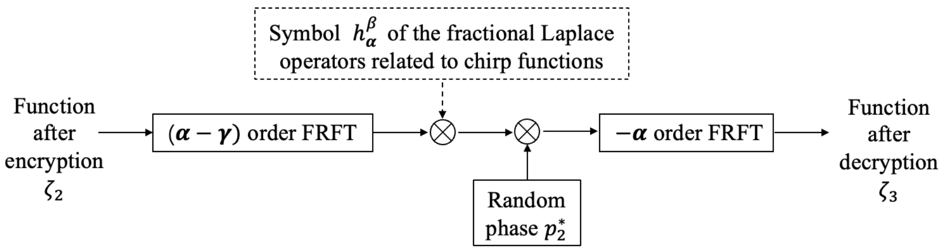


Figure 4: Encryption processing with double phase coding based on I_{β}^{α}

As is shown in Fig. 4, the input function $\zeta_1(x)$ represents the image to be encrypted after the normalization and the pixel value range is $[0, 1]$. The function after encryption $\zeta_2(\boldsymbol{x})$ represents the encryption image. The random phases p_1 and p_2 are given, respectively, by $p_1(\boldsymbol{x}) := e^{2\pi i n_1(\boldsymbol{x})}$ and $p_2(\boldsymbol{x}) := e^{2\pi i n_2(\boldsymbol{x})}$ for any $\boldsymbol{x} \in \mathbb{R}^2$, which are called the random phase masks, where $n_1(\boldsymbol{x})$ and $n_2(\boldsymbol{x})$ are two statistically independent white sequences uniformly distributed on $[0, 1]$. If the symbol of the fractional Riesz potential related to chirp functions is not added in the above encryption processing, this image encryption processing goes back to the image encryption with double phase coding based on FRFT.

Conversely, we next give the following decryption processing with double phase coding based on the fractional Riesz potential related to chirp functions.

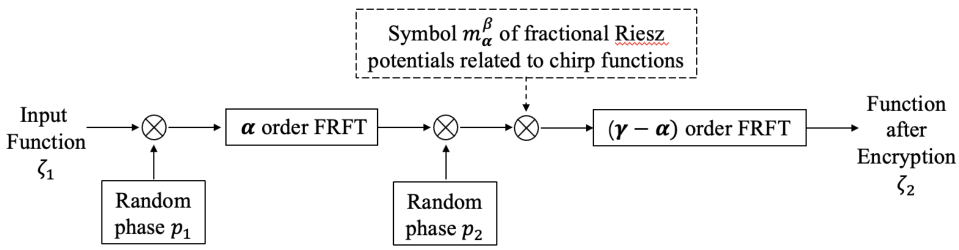
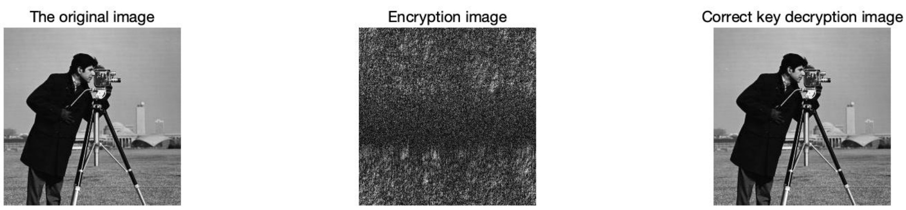


Figure 5: Decryption processing with double phase coding based on I_{β}^{α}

As is shown in Fig. 5, the decryption processing is also the inverse of the encryption processing. From Section 2 we deduce that the symbol of the fractional Laplace operator related to chirp functions multiplied by the symbol of the fractional Riesz potential related to chirp functions is 1, that is, the fractional Laplace operator related to chirp functions can be regarded as the inverse operation of the fractional Riesz potential related to chirp functions. The random phase p_2^* is defined by setting $p_2^*(\boldsymbol{x}) := e^{-2\pi i n_2(\boldsymbol{x})}$ for any $\boldsymbol{x} \in \mathbb{R}^2$, which is the conjugate of the random phase mask p_2 in Fig. 4, and the function after decryption $\zeta_3(\boldsymbol{x})$ represents the image after decryption.

Now, we study the simulation of both the image encryption and the image decryption. The following Fig. 6 illustrates the digital simulation of both the encryption and the decryption using classical pictures.



(a) (b) (c)

(d) (e) (f)

Figure 6: Simulation of image encryption and image decryption

As is shown in Fig. 6, (a) is the test image with 256 pixels × 256 pixels of image encryption; (b) is the image after encryption; (c) is the image after decryption with correct key; (d), (e), and (f) are the images after decryption with wrong key, respectively. The simulation passwords, parameter, and images of both the image encryption and the image decryption are presented in the following Table 1.

Password	**Parameters:** $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, β, $\boldsymbol{\gamma} = (\theta_1, \theta_2)$	**Image**
correct password	$\boldsymbol{\alpha} = (\frac{7\pi}{8}, \frac{5\pi}{8})$, $\beta = 0.75$, $\boldsymbol{\gamma} = (\frac{\pi}{4}, \frac{3\pi}{8})$	(c) in Fig. 6
error password	$\boldsymbol{\alpha} = (\frac{7\pi}{8}, \frac{5\pi}{8})$, $\beta = 0.75$, $\boldsymbol{\gamma} = (\frac{(1+0.05)\pi}{4}, \frac{3\pi}{8})$	(d) in Fig. 6
error password	$\boldsymbol{\alpha} = (\frac{(7+0.1)\pi}{8}, \frac{5\pi}{8})$, $\beta = 0.75$, $\boldsymbol{\gamma} = (\frac{\pi}{4}, \frac{3\pi}{8})$	(e) in Fig. 6
error password	$\boldsymbol{\alpha} = (\frac{7\pi}{8}, \frac{5\pi}{8})$, $\beta = 0.85$, $\boldsymbol{\gamma} = (\frac{(1+0.05)\pi}{4}, \frac{3\pi}{8})$	(f) in Fig. 6

Table 1: Decryption images corresponding to different keys

As is shown in Fig. 6 and Table 1, even if one knows that the double phase coding image encryption method based on the fractional Riesz potential related to chirp functions is used, even if one changes only one parameter of the correct keys in (c) and (d) in Fig. 6, one cannot obtain the encrypted original image, let alone we have five parameters. In other words, our parameters and symbols provide more degrees of freedom and make the information more secure.

The following Fig. 7 presents the mean square error (for short, MSE) curves of the decrypted image and the original image when there are different deviations of the keys for both the double phase coding image encryption based on the FRFT and the double phase coding image encryption based on the fractional Riesz potential related to chirp functions, respectively.

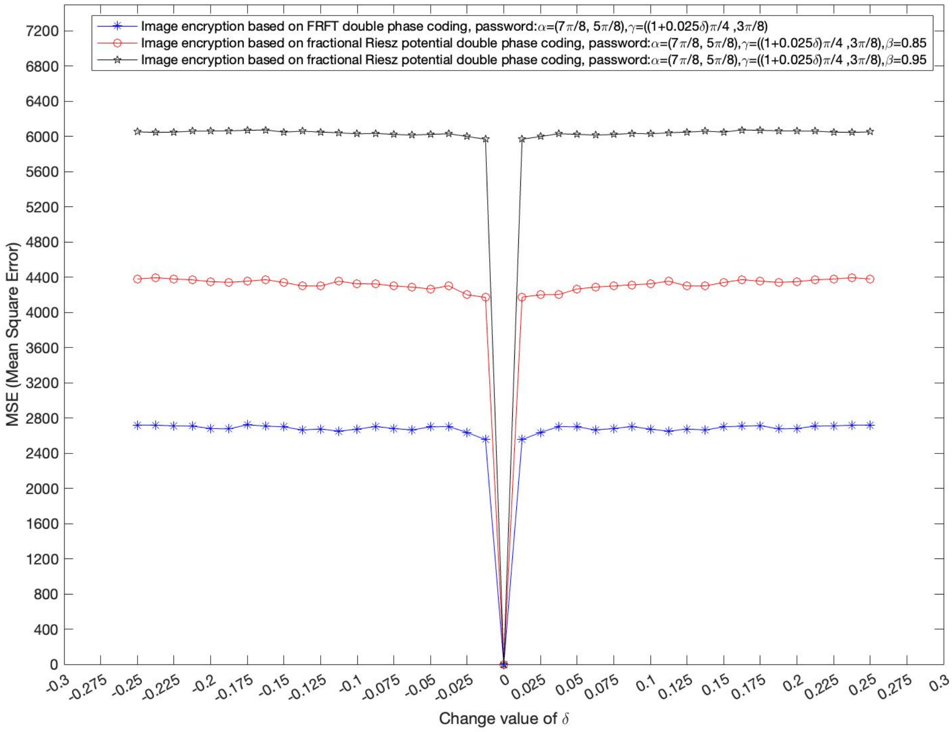


Figure 7: When the key is wrong, the MSE curve of the double phase coded image decryption based on the FRFT and based on I^{α}_{β} compared to the original image, respectively.

The correct passwords for the image encryption with double phase coding based on the FRFT are $\alpha = (7\pi/8, 5\pi/8)$ and $\gamma = (\pi/4, 3\pi/8)$; the correct passwords for the image encryption with double phase coding based on the fractional Riesz potentials related to chirp functions are $\alpha = (7\pi/8, 5\pi/8)$, $\gamma = (\pi/4, 3\pi/8)$, and $\beta = 0.75$. From Fig. 7, we infer that the parameter β of the double phase coded image encryption based on the fractional Riesz transform related to chirp functions greatly improves the security compared with the double phase coded image encryption based on the FRFT, and we also deduce that the key is much more secure when blindly decrypting.

To sum up, the above discussions reveal that the fractional Riesz potential related to chirp functions can be applied to image encryption, and the encryption effect is powerful.

5 Conclusions

In this article, we introduce fractional Riesz potentials related to chirp functions, establish their relations with the FRFT, the fractional Laplace operator related to chirp functions, and the fractional Riesz transform related to chirp functions. We apply the fractional Riesz potential related to chirp functions to the image encryption. Our experiments show that the symbol of fractional Riesz potential related to chirp functions essentially expands the key space and greatly improves the security of images.

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