# SHARP NON-UNIQUENESS OF SOLUTIONS TO STOCHASTIC NAVIER-STOKES EQUATIONS 

WEIQUAN CHEN, ZHAO DONG, AND XIANGCHAN ZHU


#### Abstract

In this paper we establish a sharp non-uniqueness result for stochastic $d$-dimensional $(d \geq 2)$ incompressible Navier-Stokes equations. First, for every divergence free initial condition in $L^{2}$ we show existence of infinite many global in time probabilistically strong and analytically weak solutions in the class $L^{\alpha}\left(\Omega, L_{t}^{p} L^{\infty}\right)$ for any $1 \leq p<2, \alpha \geq 1$. Second, we prove the above result is sharp in the sense that pathwise uniqueness holds in the class of $L_{t}^{p} L^{q}$ for some $p \in[2, \infty], q \in(2, \infty]$ such that $\frac{2}{p}+\frac{d}{q} \leq 1$, which is a stochastic version of Ladyzhenskaya-Prodi-Serrin criteria. Moreover, for stochastic $d$-dimensional incompressible Euler equation, existence of infinitely many global in time probabilistically strong and analytically weak solutions is obtained. Compared to the stopping time argument used in [HZZ19, HZZ21a], we developed a new stochastic version of the convex integration. More precisely, we introduce expectation during convex integration scheme and construct directly solutions on the whole time interval $[0, \infty)$.


## 1. Introduction

The Navier-Stokes/Euler equations are fundamental models in fluid dynamics. Existence of global strong solutions to the three dimensional incompressible Navier-Stokes system is one of the Millennium Prize Problems. An intimately related question is that of uniqueness of solutions, which has been studied a lot in the literature. In 2D case, existence and uniqueness of solution is well-known. In higher dimensions, since existence of weak solutions is known [Ler34, Hop51], there are a number of literature on the uniqueness of weak solutions. For brevity, we summarize the classical Ladyzhenskaya-Prodi-Serrin uniqueness criteria as follows:

Theorem 1.1 ([FJR72, Kat84, FLRT00, LM01, CL22]). Let $d \geq 2$ and $u$ be a weak solution to the incompressible Navier-Stokes equations such that u belongs to

$$
X_{T}^{p, q}:=\left\{\begin{array}{cc}
L^{p}\left(0, T ; L^{q}\right) & 1 \leq p<\infty  \tag{1.1}\\
C\left([0, T] ; L^{q}\right) & p=\infty
\end{array}\right.
$$

for some $2 \leq p \leq \infty$ and $d \leq q \leq \infty$ such that $2 / p+d / q \leq 1$. Then $u$ is unique in this class of weak solutions and is Leray-Hopf in the sense that $u \in C_{w}\left([0, T] ; L^{2}\right) \cap L^{2}\left(0, T ; H^{1}\right)$ and satisfies energy inequality.

In the literature the space $X^{p, q}$ is called sub-critical when $2 / p+d / q<1$, critical when $2 / p+d / q=1$, and super-critical when $2 / p+d / q>1$. It is natural to ask what would happen in the supercritical regime

[^0]$2 / p+d / q>1$. i.e. whether uniqueness holds in this case. Recently Cheskidov and Luo [CL21,CL22] proved the solutions are not unique in $L^{p}\left(0, T ; L^{\infty}\right), 1 \leq p<2, d \geq 2$ nor in $C\left([0, T], L^{p}\right), p<2, d=2$ by using the method of convex integration. Convex integration was introduced into fluid dynamics by De Lellis and Székelyhidi Jr. [DS09, DS10, DS13, DS14]. This method has already led to a number of groundbreaking results: Isett [Ise18] proved Onsager's conjecture, see also [BDSV19]. Non-uniqueness of weak solutions to the incompressible Navier-Stokes equations was obtained by Buckmaster and Vicol [BV19], see also Buckmaster, Colombo and Vicol [BCV18]. Burczak, Modena and Székelyhidi Jr. [BMS21] then obtained illposedness for power-law flows and also, in particular, non-uniqueness of weak solutions to the Navier-Stokes equations for every given divergence free initial condition in $L^{2}$. We refer to the reviews [BV19, BV21] for more details and references. We also mention by a different method, a first non-uniqueness result for Leray solutions was obtained in [ABC21] for the Navier-Stokes system with a special force.

In view of these negative developments, a suitable stochastic perturbation may provide a regularizing effect on problems. In the deterministic case, a selection of solutions depending continuously on the initial condition has not been obtained. However, the probabilistic counterpart, i.e. the Feller property and even the strong Feller property which corresponds to a smoothing with respect to the initial condition, ware established by Da Prato and Debussche [DD03] and by Flandoli and Romito [FR08]. A transport noise and linear multiplicative noise prevent blow up of strong solutions have been obtained by Flandoli, Gubinelli and Priola [FGP10] and Flandoli and Luo [FL21] and Glatt-Holtz and Vicol [GHV14] and Röckner, Zhu and Zhu [RZZ14]. One would naturally ask whether the non-uniqueness result still holds in the stochastic case. Recently in [HZZ19, HZZ21a, HZZ21b] non-uniqueness in law and even non-uniqueness of Markov family for the stochastic Navier-Stokes/Euler equations have been established by a stochastic counterpart of the convex integration method. One may further ask whether the noise makes the critical regularity of uniqueness different, i.e. whether non-uniqueness still holds in the supercritical regime.

In this paper we prove that sharp non-uniqueness holds for the stochastic $d$-dimensional $(d \geq 2)$ NavierStokes system driven by an additive noise. In [HZZ19, HZZ21a, HZZ21b] the stopping time is introduced to control the noise uniformly in $\omega$ and it was removed by a suitable extension of solutions. However, such extensions require the solution at stopping time belongs to $L^{2}$-space which is not applicable in our case as the solution is only in $L^{2}$ space for a.e. $t$. Instead we introduce expectation during convex integration scheme which can be viewed as a new stochastic version of the convex integration. Since the nonlinear term is quadratic, we have to estimate higher moments at step $q$ than the moment bound we required at step $q+1$. Then it seems we have to bound all the finite moment at each step which may blow up during iterations. The key point that this method works is that the higher moments at step $q$ only depends on the parameters up the step $q$ and we could choose the parameters at the step $q+1$ to guarantee smallness. Moreover, to construct the solutions directly on the whole time interval $[0, \infty)$, we introduced the norm of the following form:

$$
\sup _{s \geq 0}\left(\mathbb{E}\|u\|_{L^{p}\left([s, s+1] ; L^{\infty}\right)}^{\alpha}\right)^{1 / \alpha}
$$

with $p, \alpha \geq 1$. This requires the stochastic part also has finite norm of this form. To this end we introduced a damping term in the linear equation and subtract the extra term in the nonlinear equation (see (1.4) and (1.5) below for more details.)
1.1. Main Results. In this paper we are concerned with stochastic Navier-Stokes equations on $\mathbb{T}^{d}, d \geq 2$ driven by an additive stochastic noise. The equations govern the time evolution of the fluid velocity $u$ and read as

$$
\left\{\begin{array}{l}
\mathrm{d} u(t)=(-\operatorname{div}(u(t) \otimes u(t))+\Delta u(t)-\nabla p(t)) \mathrm{d} t+\mathrm{dW}(t)  \tag{1.2}\\
\operatorname{div} u(t)=0 \\
u(0)=u_{0}
\end{array}\right.
$$

Here $\mathrm{W}=\{\mathrm{W}(t) ; 0 \leq t<\infty\}$ is a $G G^{*} \mathrm{~W}$ iener process on a given probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $G$ is a Hilbert-Schmidt operator from $L^{2}$ to $L^{2}$. Let $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ denote the normal filtration generated by W, that is, the canonical right-continuous filtraton augmented by all the $\mathbb{P}$-negligible events.

Compared to the deterministic case, the stochastic equations possess additional structural features. First of all, we distinguish between probabilistically strong and probabilistically weak (also called martingale) solutions. Probabilistically strong solutions are constructed on a given probability space and are adapted with respect to the given noise. Probabilistically weak solutions do not have this property: they are typically obtained by the method of compactness where the noise as well as the probability space becomes part of the construction. For the Navier-Stokes equations only probabilistically weak solutions are obtained by a compactness argument. In fact it is necessary to take expectation to control the noise and obtain uniform energy estimates, which then leads to probabilistically weak solutions. Due to the lack of uniqueness in higher dimensions we cannot apply Yamada-Watanabe's theorem to obtain probabilistically strong solutions. Moreover, if we analyze the equation $\omega$-wise, then the converging subsequence from compactness argument may depend on $\omega$ which destroys adaptedness. Consequently, it has been a long standing open problem to construct probabilistically strong solutions to the Navier-Stokes system (1.2) in higher dimensions, see page 84 in [Fla08]. In [HZZ21a], Hofmanová, Zhu and the third named author solved this problem and proved existence of global-in-time non-unique probabilistically strong and analytically weak solutions for every given divergence free initial condition in $L^{2}$ in 3D case by using a stochastic convex integration method. Our main result also extends this result to higher dimensional case.

Now we recall the definition of probabisitcally strong and analytically weak solution to the system (1.2).
Definition 1.2. $A W^{s, p}$-valued $(s \in \mathbb{R}, 1 \leq p \leq \infty)$ continuous $\mathscr{F}_{t}$-adapted proccess $u=\{u(t) ; t \in[0, \infty)\}$ is said to be a global in time probabilistically strong and analytically weak solution to system (1.2) with initial data $u_{0} \in L_{\sigma}^{2}$, if $\mathbb{P}-$ a.s. :
(i) div $u \equiv 0$ in the sense of distribution;
(ii) $u \in L_{l o c}^{2}\left([0, \infty) ; L^{2}\right)$;
(iii) for any test function $\varphi \in C_{\sigma}^{\infty}\left(\mathbb{T}^{d}\right)$, we have for each $t \in[0, \infty)$ that

$$
\begin{equation*}
\langle u(t), \varphi\rangle=\left\langle u_{0}, \varphi\right\rangle+\int_{0}^{t}\langle-\operatorname{div}(u(s) \otimes u(s))+\Delta u(s), \varphi\rangle \mathrm{d} s+\langle\mathrm{W}(t), \varphi\rangle . \tag{1.3}
\end{equation*}
$$

To state our main result, we first decompose $u=v+z$ with $z$ solving the linear stochastic equation:

$$
\left\{\begin{array}{l}
\mathrm{d} z(t)=(\Delta-\mathrm{I}) z(t) \mathrm{d} t+\nabla P_{1}+\mathrm{dW}(t) \quad t \in[0, \infty)  \tag{1.4}\\
\operatorname{div} z(t)=0 \quad t \in[0, \infty) \\
z(0)=u_{0}
\end{array}\right.
$$

and $v$ solving the non-linear equations:

$$
\left\{\begin{array}{l}
\partial_{t} v=-\operatorname{div}[(v(t)+z(t)) \otimes(v(t)+z(t))]+\Delta v(t)+z(t)-\nabla p(t) \quad t \in[0, \infty)  \tag{1.5}\\
\operatorname{div} v(t)=0 \quad t \in[0, \infty) \\
v(0)=0
\end{array}\right.
$$

By adding a new damping term, we could obtain a uniform in time bound (see Theorem 2.1.) The following is our main result and it is proved in Section 2.

Theorem 1.3. For any $\varepsilon>0$, any $1 \leq \alpha, r<\infty, 1 \leq p<2$, any $u_{0} \in L_{\sigma}^{2}$ and any smooth vector field $w \in \mathcal{C}_{0, \sigma}^{1}$, there exists a probabilistically strong and analytically weak global solution $u$ to (1.2) with initial data $u_{0}$, such that

$$
\begin{equation*}
u-z \in \bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right) \bigcap \bar{L}^{\alpha}\left(\Omega ; Z^{p, r}\right) \bigcap \mathbf{E}_{p} \tag{1.6}
\end{equation*}
$$

and $u$ is close to $w+z$ in the following sense:

$$
\begin{equation*}
\|u-(w+z)\|_{\bar{L}^{\alpha}\left(\Omega ; Z^{p, r}\right)}+\|u-(w+z)\|_{\mathbf{E}_{p}}<\varepsilon \tag{1.7}
\end{equation*}
$$

In particular, there exists infinitely many different probabilistically strong and analytically weak global solutions to (1.2). For the definition of spaces we refer to (1.14), (1.16), (1.17) and (1.18).
Remark 1.4. It is well-known that the martingale solutions constructed by Galerkin approximation satisfy the energy inequality:

$$
\begin{aligned}
{\left[\mathbb{E}\left(\|u(t)\|_{L^{2}}^{2}\right)\right]^{1 / 2} } & \leq\left(\left\|u_{0}\right\|_{L^{2}}^{2}+T \cdot \operatorname{Tr}\left[G^{*} G\right]\right)^{1 / 2} \\
& \leq\left\|u_{0}\right\|_{L^{2}}+T^{1 / 2} \operatorname{Tr}\left[G^{*} G\right]^{1 / 2}, \forall t \in[0, T]
\end{aligned}
$$

However, for $d \geq 3$, due to lack of uniqueness, such solutions are not probabilistically strong solutions. By choosing different special $w$ and using the $\mathbf{E}_{p}$-closeness in (1.7), we can obtain infinitely many different solutions that break the energy inequality on $[0, T]$, i.e.

$$
\begin{equation*}
\left\{\int_{0}^{T}\left[\mathbb{E}\left(\|u(t)\|_{L^{2}}^{2}\right)\right]^{p / 2} d t\right\}^{1 / p}>T^{1 / p}\left(\left\|u_{0}\right\|_{L^{2}}+T^{1 / 2} \operatorname{Tr}\left[G^{*} G\right]^{1 / 2}\right) \tag{1.8}
\end{equation*}
$$

The above result implies there exists infinitely many solutions such that $u-z \in L_{s}^{p} L^{\infty}$. If $z$ also have such regularity we could obtain infinitely many solutions $u \in L_{s}^{p} L^{\infty}$. The result then reads as follows and its proof is given in Sestion 2.
Corollary 1.5. Let $1 \leq \alpha<\infty, 1 \leq p<2$. Assume that $\operatorname{Tr}\left[G^{*}(I-\Delta)^{\lambda} G\right]<\infty$ and $u_{0} \in L^{p_{1}}, p_{1} \geq 2$ with

$$
\begin{equation*}
\lambda>\frac{d}{2}-1 \text { and } p_{1}>\frac{p d}{2} \tag{1.9}
\end{equation*}
$$

then there exist infinitely many different probabilistically strong and analytically weak global solutions to (1.2) in $\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$ with the same initial data $u_{0}$.

The non-uniqueness of regularity $L_{s}^{p} L^{\infty}, 1 \leq p<2$ is sharp in the sense that the solution is unique in the space $L^{2}\left([0, T] ; L^{\infty}\right)$. This result is a stochastic extension of Theorem 1.1 and its proof is given in Section 5.

Theorem 1.6. (Pathwise Uniqueness for the Stochastic N-S System) Let $d \geq 2$, and $0<T<\infty$ be arbitrarily fixed. There exists at most one solution $u$ to (1.2) satisfying

$$
\begin{equation*}
u \in X_{T}^{p, q}, \mathbb{P}-\text { a.s. } \tag{1.10}
\end{equation*}
$$

for some $p \in[2, \infty]$ and $q \in(2, \infty]$ such that

$$
\begin{equation*}
\frac{2}{p}+\frac{d}{q} \leq 1 \tag{1.11}
\end{equation*}
$$

Moreover, the solution $u$ is Leray-Hopf and $u \in L^{2}\left(\Omega ; C_{[0, T]} L^{2}\right) \bigcap L^{2}\left(\Omega ; L_{[0, T]}^{2} H^{1}\right)$ and satisfies the energy inequality:

$$
\begin{equation*}
\frac{1}{2} \mathbb{E}\|u(t)\|_{L^{2}}^{2}+\int_{0}^{t} \mathbb{E}\|\nabla u(s)\|_{L^{2}}^{2} d s \leq \frac{1}{2} \mathbb{E}\left\|u_{0}\right\|_{L^{2}}^{2}+\frac{t}{2} \operatorname{Tr}\left[G^{*} G\right], t \in[0, T] \tag{1.12}
\end{equation*}
$$

1.2. Application to the stochastic Euler equations. Theorem 1.3 also holds for the following stochastic Euler equation:

$$
\left\{\begin{array}{l}
\mathrm{d} u(t)=(-\operatorname{div}(u(t) \otimes u(t))-\nabla p(t)) \mathrm{d} t+\mathrm{dW}(t)  \tag{1.13}\\
\operatorname{div} u(t)=0 \\
u(0)=u_{0}
\end{array}\right.
$$

Theorem 1.7. Under the same assumption as Theorem 1.3, there exists infinitely many different probabilistically strong and analytically weak solutions to (1.13) with the same given $L^{2}$-initial data.

In the two-dimensional case stochastic Euler equations have been studied in [Bes99, BF99, BFM16, BP01, GHV14]. The three-dimensional case has been treated in [CFH19, GHV14, Kim01, MV00, HZZ21b]. In particular, Glatt-Holtz and Vicol [GHV14] obtained local well-posedness of strong solutions to stochastic Euler equations in two and three dimensions, global well-posedness in two dimensions for additive and linear multiplicative noise. Hofmanová, Zhu and the third named author [HZZ21b] established existence and non-uniqueness of global dissipative martingale solutions for additive noise. We emphasize that before our work there's no result for the existence of probabilistically strong solutions to the stochastic Euler equations in higher dimensions. Our result is the first one on this point.

Moreover, by the interpolation $L_{s}^{3 / 2-} C^{1 / 3} \supset L_{s}^{p} L^{\infty} \cap L_{s}^{1} C^{1-}$ for some $1 \leq p<2$, we obtain as a byproduct the existence of infinitely many non-conserving solutions in $L^{\alpha}\left(\Omega ; L_{l o c}^{3 / 2-}\left([0, \infty) ; C^{1 / 3}\right)\right) \cap$ $L^{\alpha}\left(\Omega ; L_{l o c}^{1}\left([0, \infty) ; C^{1-}\right)\right)$ to the stochastic Euler system (1.13). This gives the first stochastic version of the Onsagar's conjecture in negative direction with an exact " $1 / 3$-Hölder regularity" in spacial variables.
Definition 1.8. A probabilistically strong and analytically weak solution (in the sense of Definition 1.2 without the Laplacian term) $u$ to (1.13) is said to be conserving, if $\mathbb{P}$ - a.s.,

$$
\|u(t)\|_{L^{2}}=\left\|u_{0}\right\|_{L^{2}}+2 \int_{0}^{t}\langle u(t), \mathrm{d} W(t)\rangle_{L^{2}}+t \cdot \operatorname{Tr}\left[G^{*} G\right] \quad \text { for all } t \in[0, \infty)
$$

Otherwise, we say that the solution $u$ is non-conserving.
Theorem 1.9. Let $d \geq 2,1 \leq \alpha<\infty, \varepsilon>0$ and $u_{0} \in W^{1, \infty}$ be arbitrarily given. And let $G$ satisfies $\operatorname{Tr}\left[G^{*}(I-\Delta)^{\frac{d}{2}+1} G\right]<\infty$. Then there exist infinitely many different non-conserving probabilistically strong and analytically weak solutions $L^{\alpha}\left(\Omega ; L_{\text {loc }}^{3 / 2-\varepsilon}\left([0, \infty) ; C^{1 / 3}\right)\right) \cap L^{\alpha}\left(\Omega ; L_{l o c}^{1}\left([0, \infty) ; C^{1-\varepsilon}\right)\right)$ to the stochastic Euler system (1.13), with initial data $u_{0}$.
1.3. Notations. Throughout the paper, we employ the notation $a \lesssim b$ if there exists a constant $c>0$ such that $a \leq c b . \mathbb{N}_{0}=\mathbb{N} \cup\{0\}$. Let $\mathcal{S}^{\prime}$ be the space of distributions on $\mathbb{T}^{d}$. For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{s, p}=\left\{f \in \mathcal{S}^{\prime}:\|f\|_{W^{s, p}}:=\left\|(I-\Delta)^{s / 2} f\right\|_{L^{p}}<\infty\right\}$.

Given a Banach space $\left(Y,\|\cdot\|_{Y}\right)$ and $I \subset \mathbb{R}$ we write $L^{p}(I ; Y)$ for the space of $L^{p}$-integrable functions from $I$ to $\mathbb{R}$, equipped with the usual $L^{p}$-norm. We also use $L_{I}^{p} Y$ to denote the space of functions $f$ from $[0, \infty)$ to $Y$ satisfying $\left.f\right|_{I} \in L^{p}(I, Y)$. We also use $L_{l o c}^{p}([0, \infty) ; Y)$ to denote the space of functions $f$ from $[0, \infty)$ to $Y$ satisfying $f \in L_{[0, T]}^{p} Y$ for all $T>0$. We also write $C(I ; Y)$ for the space of continuous
functions from $I$ to $Y$ equipped with the supremum norm in a bounded subset. $C_{I} Y$ is similar as $L_{I}^{p} Y$ with $\left.f\right|_{I} \in C(I, Y)$. We also use $L^{\alpha}(\Omega, Y)$ to denote the space of functions on $\Omega$ with finite $\alpha$ moment, equipped with the usual $L^{\alpha}$-norm. Whenever $I=[s, s+1]$, we simply write $L_{s}^{p} Y:=L_{[s, s+1]}^{p} Y$ and $C_{s} Y:=C_{[s, s+1]} Y$.

For smooth tensor fields, we use the following notations:

$$
C_{c, \sigma}^{k}:=C_{c, \sigma}^{k}\left([0, \infty) \times \mathbb{T}^{d}\right), C_{\sigma}^{k}:=C_{\sigma}^{k}\left([0, \infty) \times \mathbb{T}^{d}\right), C_{t, x}^{k}:=C^{k}\left([0, \infty) \times \mathbb{T}^{d}\right)
$$

where $k \in \mathbb{N}_{0} \cup\{\infty\}$ and the indices " $c$ " and " $\sigma$ " mean "compact support in time" and "divergence-free", respectively. And for $C^{N}$-norms and semi-norms of function $f \in C_{t, x}^{N}$, we write

$$
\begin{aligned}
{[f(t)]_{m} } & :=\sum_{|\alpha|=m, \alpha \in \mathbb{N}_{0}^{3}} \sup _{x \in \mathbb{T}^{d}}\left|\partial_{x}^{\alpha} f(t, x)\right|, t \in \mathbb{R} ; \\
{[f]_{m,[a, b]} } & :=\sum_{|\alpha|+k=m, \alpha \in \mathbb{N}_{0}^{3}, k \in \mathbb{N}_{0}} \sup _{(t, x) \in[a, b] \times \mathbb{T}^{d}}\left|\partial_{t}^{k} \partial_{x}^{\alpha} f(t, x)\right|,-\infty<a<b<\infty ;
\end{aligned}
$$

for $0 \leq m \leq N$, and

$$
\begin{aligned}
\|f(t)\|_{N} & :=\sum_{m=0}^{N}[f(t)]_{m}, t \in \mathbb{R} \\
\|f\|_{N,[a, b]} & :=\sum_{m=0}^{N}[f]_{m,[a, b]},-\infty<a<b<\infty
\end{aligned}
$$

For simplicity, we write $[f]_{m, s}:=[f]_{m,[s, s+1]}$ and $\|f\|_{N, s}:=\|f\|_{N,[s, s+1]}$ for $s \geq 0$.
We define the spaces for $1 \leq p<2$ and $\alpha, r \geq 1$ :

$$
\begin{gather*}
\mathcal{C}_{0, \sigma}^{1}:=\left\{u \in C_{\sigma}^{1} \mid\|u\|_{C_{\sigma}^{1}\left([0, \infty) \times \mathbb{T}^{d}\right)}<\infty, u(0)=0\right\}  \tag{1.14}\\
Z_{s}^{p, r}:=C_{s} W^{-1,1} \bigcap L_{s}^{p} L^{\infty} \bigcap L_{s}^{1} W^{1, r} \\
\text { with } \quad\|u\|_{p, r ; s}:=\|u\|_{C_{s} W^{-1,1}}+\|u\|_{L_{s}^{p} L^{\infty}}+\|u\|_{L_{s}^{1} W^{1, r}}, s \geq 0  \tag{1.15}\\
\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right):=\left\{u: \Omega \times[0, \infty) \longrightarrow W^{-1,1} \bigcap L^{\infty} \bigcap W^{1, r} \mid\|u\|_{\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right)}:=\sup _{s \geq 0}\left(\mathbb{E}\|u\|_{p, r ; s}^{\alpha}\right)^{1 / \alpha}<\infty\right\} . \tag{1.16}
\end{gather*}
$$

For stochastic processes $u: \Omega \times[0, \infty) \longrightarrow Y$, we introduce the following notations for $\alpha, p \geq 1$ :

$$
\begin{align*}
\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} Y\right) & :=\left\{u: \Omega \times[0, \infty) \longrightarrow Y \mid\|u\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} Y\right)}:=\sup _{s \geq 0}\left(\mathbb{E}\left\|_{u}\right\|_{L_{s}^{p} Y}^{\alpha}\right)^{1 / \alpha}<\infty\right\}  \tag{1.17}\\
\bar{L}^{\alpha}\left(\Omega, C_{s} Y\right) & :=\left\{u: \Omega \times[0, \infty) \longrightarrow Y \mid\|u\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} Y\right)}:=\sup _{s \geq 0}\left(\mathbb{E}\|u\|_{C_{s} Y}^{\alpha}\right)^{1 / \alpha}<\infty\right\} \\
\mathbf{E}_{p} & :=\left\{u: \Omega \times[0, \infty) \longrightarrow Y \mid\|u\|_{\mathbf{E}_{p}}:=\sup _{s \geq 0}\|u\|_{L_{s}^{p} L^{2}\left(\Omega ; L^{\infty}\right)}<\infty\right\} \tag{1.18}
\end{align*}
$$

It's easy to see that $\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} Y\right) \subset L^{\alpha}\left(\Omega, L_{l o c}^{p}([0, \infty) ; Y)\right)$ and $\bar{L}^{\alpha}\left(\Omega, C_{s} Y\right) \subset L^{\alpha}(\Omega, C([0, \infty) ; Y))$.
1.4. Organization of the paper. In Section 2, we give the proof of Theorem 1.3 and Corollary 1.5 assuming the main iteration Proposition 2.2. The proof of Proposition 2.2 is given in Section 3 and 4: We construct the velocity perturbation and the new Reynold Stress in Section 3. Estimate of the perturbation and Reynold stress error is presented in Section 4. Section 5 is devoted to the proof of Theorem 1.6. In Appendix we collect several auxiliary results.

## 2. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3 and Corollary 1.5. More precisely, by means of the convex integration method we construct infinitely many global-in-time probablistically strong solutions to the Navier-Stokes system (1.2). Different from the previous work using convex integration in stochastic setting by introducing suitable stopping times, we take expectation in the convex integration iterative estimates and construct directly solutions on the whole time interval $[0, \infty)$. The key point is that the $m$-th moment of approximate velocity and the error at step $q$ only depends on $m$ and the parameters up to the step $q$ of the iteration. Then we could choose the parameters at the level $q+1$ to guarantee smallness of the velocity perturbations and the error at step $q+1$.

As the first step, we recall the following regularity result for the linear equation. The linear system (1.4) is known to be well-posed and the solution $z$ can be represented as follows:

$$
\begin{equation*}
z(t)=e^{t(\Delta-\mathrm{I})} u_{0}+W_{c o n}(t), \quad t \in[0, \infty) \tag{2.1}
\end{equation*}
$$

where $W_{\text {con }}(t):=\mathbf{P} \int_{0}^{t} e^{(t-s)(\Delta-\mathrm{I})} \mathrm{dW}(s)$ with the Helmholtz projection $\mathbf{P}$. The following result is well known and can be obtained by using the method in [ Da 04 ].
Theorem 2.1. Let $W_{\text {con }}(t):=\mathbf{P} \int_{0}^{t} e^{(t-s)(\nu \Delta-\mathrm{I})} \mathrm{dW}(s)$ with $\nu=0$, 1 . For $0<\delta<1 / 2$ and $1 \leq m<\infty$, there exists finite constant $C_{m, \delta}>0$ such that

$$
\begin{equation*}
\sup _{s \geq 0} \mathbb{E}\left(\left\|W_{\text {con }}\right\|_{C_{[s, s+1]}^{1 / 2-\delta} L^{2}}^{m}\right) \leq C_{m, \delta} \tag{2.2}
\end{equation*}
$$

Let us now explain how the convex integration is set up. More precisely, we intend to develop an iteration procedure leading to the proof of Theorem 1.3. The iteration is indexed by a parameter $q \in \mathbb{N}_{0}$. At each step $q$, a pair $\left(v^{(q)}, \stackrel{\circ}{R}^{(q)}\right)$ is constructed solving the following system

$$
\left\{\begin{array}{l}
\partial_{t} v^{(q)}+\operatorname{div}\left[\left(v^{(q)}+z\right) \otimes\left(v^{(q)}+z\right)\right]=-\nabla p^{(q)}+\Delta v^{(q)}+z+\operatorname{div} \stackrel{\circ}{R}^{(q)}  \tag{2.3}\\
\operatorname{div} v^{(q)} \equiv 0 \\
v^{(q)}(0)=0
\end{array}\right.
$$

Here the trace-free $d \times d$ matrix $\stackrel{\circ}{R}^{(q)}(t, x)$ is the so-called Reynold stress term.
The iteration starts at

$$
\begin{equation*}
v^{(0)}=w, \stackrel{\circ}{R}^{(0)}=\mathcal{R}\left(\partial_{t} w-\Delta w-z\right)+(w+z) \dot{\otimes}(w+z) \tag{2.4}
\end{equation*}
$$

where $w \in \mathcal{C}_{0, \sigma}^{1}$ is the pre-given vector field as in Proposition 1.3 and $\mathcal{R}$ denotes the reverse-divergence operator which we recall in Appendix B for convenience. Note that by definition, $v^{(0)}(0)=w(0, \cdot)=0$. By (2.2), we have $\mathbb{P}-a . s . \stackrel{\circ}{R}^{(0)} \in C_{s} L^{1}$ for all $s \geq 0$ and

$$
\begin{aligned}
\|\overbrace{}^{(0)}\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} & \lesssim \sup _{s \geq 0}\left\|\partial_{t} w\right\|_{L_{s}^{1} L^{1}}+\sup _{s \geq 0}\|w\|_{L_{s}^{1} H^{1}}+\|z\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\sup _{s \geq 0}\|w\|_{L_{s}^{2} L^{2}}^{2}+\|z\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}^{2} \\
& \lesssim 1+\sup _{s \geq 0}\|w\|_{1, s}^{2}+\|z\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}^{2}<\infty
\end{aligned}
$$

The main ingredient in the proof of Theorem 1.3 is the following iteration.
Proposition 2.2. (Main Iteration) Let $1 \leq \alpha, r<\infty, 1 \leq p<2, u_{0} \in L_{\sigma}^{2}$. Let also $\delta>0$ be arbitrarily given. If $\left(v^{(q)}, \stackrel{\circ}{R}^{(q)}\right) \in C_{\sigma}^{\infty} \times C_{[0, \infty)} L^{1}$ is an $\left(\mathcal{F}_{t}\right)_{t \geq 0^{-}}$-adapted solution to (2.3), then there exists an $\left(\mathcal{F}_{t}\right)_{t \geq 0^{-}}$ adapted process $\left(v^{(q+1)}, \stackrel{\circ}{R}^{(q+1)}\right) \in C_{\sigma}^{\infty} \times C_{[0, \infty)} L^{1}$ which solves $(2.3)$ and satisfies

$$
\begin{equation*}
\left\|\dot{R}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2} \leq \delta ; \tag{2.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|v^{(q+1)}-v^{(q)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \leq C\left(\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-q}\right)+\delta \tag{2.6}
\end{equation*}
$$

with some constant $C>0$ independent of $q$, and

$$
\begin{equation*}
\left\|v^{(q+1)}-v^{(q)}\right\|_{\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right)}+\left\|v^{(q+1)}-v^{(q)}\right\|_{\mathbf{E}_{p}} \leq \delta \tag{2.7}
\end{equation*}
$$

The proof of this result is presented in Section 3 and 4 below. Now we have all in hand to complete the proof of Theorem 1.3 and Corollary 1.5.

Proof of Theorem 1.3. Let $1 \leq \alpha, r<\infty, 1 \leq p<2$, the initial data $u_{0} \in L_{\sigma}^{2}$, the smooth vector field $w \in \mathcal{C}_{0, \sigma}^{1}$ and $\varepsilon>0$ be arbitrarily given as stated in Theorem 1.3. We start the iteration from (2.4). We repeatedly apply Proposition 2.2 with

$$
\begin{equation*}
\delta_{q+1}:=\min \left\{\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}, \frac{\varepsilon}{2^{q+1}}\right\}, \quad q \in \mathbb{N}_{0} \tag{2.8}
\end{equation*}
$$

and obtain $\left(\mathcal{F}_{t}\right)_{t \geq 0 \text {-adapted process }}\left(v^{(q)}, \stackrel{\circ}{R^{(q)}}\right) \in C_{\sigma}^{\infty} \times C_{[0, \infty)} L^{1}$ such that for all $q \geq 1$ that

$$
\begin{equation*}
\left\|\check{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \leq \delta_{q}^{2} \tag{2.9}
\end{equation*}
$$

and for $\varpi_{q}=v^{(q)}-v^{(q-1)}$

$$
\begin{equation*}
\left\|\varpi_{q}\right\|_{\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right)}+\left\|\varpi_{q}\right\|_{\mathbf{E}_{p}} \leq \delta_{q} \tag{2.10}
\end{equation*}
$$

and

$$
\left\|\varpi_{q}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \lesssim\left\|\stackrel{\circ}{R}^{(q-1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-(q-1)}
$$

Here the implicit constant is deterministic and independent of $q$ and $\varepsilon$. Moreover, by (2.9) for all $q \geq 2$

$$
\begin{equation*}
\left\|\varpi_{q}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \lesssim \delta_{q-1}+2^{-(q-1)} . \tag{2.11}
\end{equation*}
$$

Hence, $\stackrel{\circ}{R}(q) \rightarrow 0$ and there exists some $v \in \bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right) \bigcap \bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right) \bigcap \mathbf{E}_{p}$ such that

$$
v^{(q)} \longrightarrow v \text { in } \bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right) \bigcap \bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right) \bigcap \mathbf{E}_{p}
$$

Taking the limit as $q \rightarrow \infty$, one sees that $v$ satisfies the system (1.5). Thus the process $u:=v+z$ is a solution to (1.2) with initial data $u_{0}$ in the sense of Definition 1.2.

Finally, by (2.8) and (2.10), we have the closeness in $L^{\alpha}\left(\Omega ; Z^{p, r}\right)$ and $\mathbf{E}_{p}$-norms :

$$
\begin{aligned}
\|u-(w+z)\|_{\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right)}+\|u-(w+z)\|_{\mathbf{E}_{p}} & =\|v-w\|_{\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right)}+\|v-w\|_{\mathbf{E}_{p}} \\
& \leq \sum_{q=1}^{\infty}\left(\left\|\varpi_{q}\right\|_{\bar{L}^{\alpha}\left(\Omega, Z^{p, r}\right)}+\left\|\varpi_{q}\right\|_{\mathbf{E}_{p}}\right)
\end{aligned}
$$

$$
\leq \sum_{q=1}^{\infty} \frac{\varepsilon}{2^{q}}<\varepsilon
$$

Thus the proof is complete.

Proof of Corollary 1.5. First we choose a sequence $\left\{w_{n} ; n \in \mathbb{N}\right\} \subset \mathcal{C}_{0, \sigma}^{\infty}$ such that

$$
\begin{equation*}
\sup _{s \geq 0}\left\|w_{i}-w_{j}\right\|_{L_{s}^{p} L^{\infty}} \geq 1, \text { for each pair } i \neq j \tag{2.12}
\end{equation*}
$$

Applying Theorem 1.3 to $u_{0}, \varepsilon=1 / 3$ and each $w_{n}$ respectively then yields a sequence $\left\{u_{n}\right\}$ of solutions to (1.2) such that $u_{n}-z \in \bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right) \bigcap \bar{L}^{\alpha}\left(\Omega ; Z^{p, r}\right) \bigcap \mathbf{E}_{p}$ and

$$
\begin{equation*}
\left\|u_{n}-\left(w_{n}+z\right)\right\|_{\bar{L}^{\alpha}\left(\Omega ; Z^{p, r}\right)}+\left\|u_{n}-\left(w_{n}+z\right)\right\|_{\mathbf{E}_{p}}<1 / 3 \tag{2.13}
\end{equation*}
$$

Then any two of the solutions are different:

$$
\begin{align*}
& \left\|u_{i}-u_{j}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)} \\
& \geq \sup _{s \geq 0}\left\|w_{i}-w_{j}\right\|_{L_{s}^{p} L^{\infty}}-\left\|u_{i}-\left(w_{i}+z\right)\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)}-\left\|u_{j}-\left(w_{j}+z\right)\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)} \\
& \geq 1-1 / 3-1 / 3=1 / 3, \forall i \neq j \tag{2.14}
\end{align*}
$$

Now we show that $u_{n} \in \bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$. It suffices to prove $z \in \bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$. For this, we split $z$ into two parts by (2.1). For the first part, by standard estimates of heat kernel, we have for $2 \leq p_{1}<\infty$ that

$$
\left\|e^{t(\Delta-I)} u_{0}\right\|_{L^{\infty}} \lesssim e^{-\frac{t}{2}} t^{-\frac{d}{2 p_{1}}}\left\|u_{0}\right\|_{L^{p_{1}}}, \quad t \in(0, \infty)
$$

Hence, for $e^{t(\Delta-I)} u_{0} \in \bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$, we only need $1-\frac{p d}{2 p_{1}}>0$, i.e. $p_{1}>\frac{p d}{2}$. For the stochastic convolution part, note that $\operatorname{Tr}\left[G^{*}(I-\Delta)^{\lambda} G\right]<\infty$ implies $W_{\text {con }} \in \bar{L}^{\alpha}\left(\Omega, C_{s} L^{\infty}\right) \subset \bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$ for all $1 \leq \alpha<\infty$.

## 3. Construction of the Iteration

This section is devoted to the construction of $v^{(q+1)}$ and $\stackrel{\circ}{R}^{(q+1)}$. To this end, we employ a two-step approach. In the first step we do mollification to avoid a loss of derivative during convex integration scheme. In the second step we proceeds along the lines of [CL22, Section 4] and do space-time convex integration. To keep the initial condition for the mollification equation we introduce a time cutoff function for the perturbation which leads to an extra error (see [BMS21, HZZ21a]).

### 3.1. Mollification.

We intend to replace $\left(v_{q}, \stackrel{\circ}{R}^{(q)}\right)$ by a mollified field $\left(v_{\ell}^{(q)}, \stackrel{\circ}{R}_{\ell}^{(q)}\right)$. To this end, let $\phi \in C_{c}^{\infty}\left(\mathbb{R}^{d} ; \mathbb{R}_{+}\right)$with supp $\phi \subset B_{1}(0)$, and $\varphi \in C_{c}^{\infty}\left(\mathbb{R} ; \mathbb{R}_{+}\right)$with $\operatorname{supp} \varphi \subset[0,1]$. And we define the mollifiers as follows:

$$
\begin{aligned}
\phi_{\ell_{q+1}} & :=\ell_{q+1}^{-d} \phi\left(\cdot / \ell_{q+1}\right) \\
\varphi_{\ell_{q+1}} & :=\ell_{q+1}^{-1} \varphi\left(\cdot / \ell_{q+1}\right)
\end{aligned}
$$

The one sided mollifier here is used in order to preserve adaptedness. Here, $\ell_{q+1} \in(0,1)$ is a small parameter and will be set later. If there's no confusion, we will simply write $\ell$ for $\ell_{q+1}$. Now we extend $v^{(q)}, z, p_{q}$ and $\stackrel{\circ}{R}^{(q)}$ to $t<0$ by taking them equal to the value at $t=0$. Then $v^{(q)}, z, p_{q}$ and $\stackrel{\circ}{R}^{(q)}$ also satisfies equation for $t<0$ as $\partial_{t} v^{(q)}=0$ from construction.

We define a mollification of $v^{(q)}, \stackrel{\circ}{R}^{(q)}, z$ in space and time by convolution as follows:

$$
\begin{aligned}
& z_{\ell}:=\left(z *_{x} \phi_{\ell}\right) *_{t} \varphi_{\ell} \\
& v_{\ell}^{(q)}:=\left(v^{(q)} *_{x} \phi_{\ell}\right) *_{t} \varphi_{\ell} \\
& \stackrel{\circ}{R}_{\ell}^{(q)}:=\left(\stackrel{\circ}{R}^{(q)} *_{x} \phi_{\ell}\right) *_{t} \varphi_{\ell}
\end{aligned}
$$

Then a slight calculation shows that $\left(v_{\ell}^{(q)}, \stackrel{\circ}{R}_{\ell}^{(q)}\right)$ satisfies on $[0, \infty) \times \mathbb{T}^{d}$

$$
\left\{\begin{array}{l}
\partial_{t} v_{\ell}^{(q)}+\operatorname{div}\left[\left(v_{\ell}^{(q)}+z_{\ell}\right) \otimes\left(v_{\ell}^{(q)}+z_{\ell}\right)\right]=-\nabla \hat{p}_{\ell}^{(q)}+\nu \Delta v_{\ell}^{(q)}+z_{\ell}+\operatorname{div}\left(\stackrel{\circ}{R}_{\ell}^{(q)}+\stackrel{\circ}{R}_{c o m}^{\ell}\right)  \tag{3.1}\\
\operatorname{div} v_{\ell}^{(q)} \equiv 0 \\
v_{\ell}^{(q)}(0)=0
\end{array}\right.
$$

with $\stackrel{\circ}{R}_{\text {com }}^{\ell}$ and $\hat{p}_{\ell}^{(q)}$ given by

$$
\begin{align*}
\stackrel{\circ}{R}_{c o m}^{\ell}:= & \left(v_{\ell}^{(q)}+z_{\ell}\right) \dot{\otimes}\left(v_{\ell}^{(q)}+z_{\ell}\right)-\left[\left(\left(v^{(q)}+z\right) \dot{\otimes}\left(v^{(q)}+z\right)\right) *_{x} \phi_{\ell}\right] *_{t} \varphi_{\ell}  \tag{3.2}\\
& \hat{p}_{\ell}^{(q)}:=p_{\ell}^{(q)}-\frac{1}{d}\left|v_{\ell}^{(q)}+z_{\ell}\right|^{2}+\frac{1}{d}\left(\left|v^{(q)}+z\right|^{2} *_{x} \phi_{\ell}\right) *_{t} \varphi_{\ell}
\end{align*}
$$

It is easy to see that $z_{\ell}$ is $\left(\mathcal{F}_{t}\right)_{t \geq 0}$-adapted and so are $v_{\ell}^{(q)}$ and $\stackrel{\circ}{R}_{\ell}^{(q)}$.

### 3.2. Construction of the Main Perturbation $\omega_{q+1}$ •

Let us now proceed with the construction of the perturbation $\omega_{q+1}$ which then defines the next iteration by $v^{(q+1)}:=v_{\ell}^{(q)}+\omega_{q+1}$. To this end we employ the stationary Mikado flows introduced in [DS17] and presented in [CL22], which we recall in Appendix A. In particular, the building blocks $\mathbb{W}_{k}^{(q+1)}(k \in \Lambda)$ is given by (A.3) with $\mu=\mu_{q+1}$, the spatial concentration parameter whose value will be given in Section 4.1. Here, $\Lambda \subset \mathbb{Z}^{d}$ introduced in Lemma A. 1 is a finite set. Now we introduce a spacial oscillation parameter $\sigma_{q+1} \in \mathbb{N}$ and $\mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} x\right)$ is $\left(\sigma_{q+1}^{-1} \mathbb{T}\right)^{d}$-periodic.

Following [CL22], we then use a temporal smooth function to oscillate the building blocks intermittently in time. We choose a function $g \in C_{c}^{\infty}((0,1))$ with $\|g\|_{L^{2}}=1$, and define

$$
\begin{equation*}
g_{\kappa}(t):=\kappa_{q+1}^{1 / 2} \cdot g\left(\kappa_{q+1} t\right), t \in[0,1] . \tag{3.3}
\end{equation*}
$$

Here, $\kappa_{q+1}>0$ is a large constant and will be specified later. Next, we extend the function $g_{\kappa}$ periodically onto $[0, \infty)$ and still denote it by $g_{\kappa}$. One can show that for all $1 \leq m \leq \infty$ and all $s \in[0, \infty)$,

$$
\begin{align*}
& \left\|g_{\kappa}\right\|_{L^{m}[s, s+1]} \lesssim \kappa_{q+1}^{1 / 2-1 / m}  \tag{3.4}\\
& \left\|g_{\kappa}\right\|_{L^{2}[s, s+1]}=1 \tag{3.5}
\end{align*}
$$

We also introduce a temporal oscillation parameter $\varsigma_{q+1} \in \mathbb{N}$ so that the rescaled function $g_{\kappa}\left(\varsigma_{q+1} \cdot\right)$ is $\varsigma_{q+1}^{-1}$-periodic.

The parameters $\mu_{q+1}, \sigma_{q+1}, \kappa_{q+1}$ and $\varsigma_{q+1}$ are assumed to be sufficiently large for the moment and will be set in Section 4.1.

As the next step, we shall define certain amplitude functions used in the definition of the perturbation $\omega_{q+1}$. To this end, let $\chi_{q+1} \in C^{\infty}\left(\mathbb{R}^{d \times d} ; \mathbb{R}_{+}\right)$be such that

$$
\begin{align*}
& \text { (i) } \chi_{q+1}(R) \text { is monotonically increasing with respect to }|R| \\
& \text { (ii) } \chi_{q+1}(R)=\left\{\begin{array}{c}
4^{-(q+1)}, 0 \leq|R| \leq 4^{-(q+1)} ; \\
|R|, 2 \leq|R|<\infty ;
\end{array}\right.  \tag{3.6}\\
& \text { (iii) }|R| / 2 \leq \chi_{q+1}(R) \leq 2|R|, \text { when } 4^{-(q+1)}<|R|<2
\end{align*}
$$

It is easy to see that the function $\chi_{q+1}$ has bounded partial derivatives of all orders. Then we define $\varrho_{q+1}$ by

$$
\begin{equation*}
\varrho_{q+1}:=4 \chi_{q+1}\left(\stackrel{\circ}{R}_{\ell}^{(q)}\right) . \tag{3.7}
\end{equation*}
$$

It follows that $2\left|\stackrel{\circ}{R}_{\ell}^{(q)}\right| \leq \varrho_{q+1}$ and hence Id $-\stackrel{\circ}{R}_{\ell}^{(\mathrm{q})} / \varrho_{\mathrm{q}+1} \in \mathrm{~B}_{1 / 2}$ (Id), which by Lemma A. 1 and (A.2), (A.3) implies that

$$
\begin{align*}
\varrho_{q+1} \mathrm{Id}-\stackrel{\mathrm{R}}{\ell}_{(\mathrm{q})} & =\sum_{k \in \Lambda} \varrho_{q+1} \Gamma_{k}^{2}\left(\operatorname{Id}-\frac{\stackrel{\circ}{\mathrm{R}}_{\ell}^{(\mathrm{q})}}{\varrho_{\mathrm{q}+1}}\right) \mathbf{e}_{k} \otimes \mathbf{e}_{k} \\
& =\sum_{k \in \Lambda} \varrho_{q+1} \Gamma_{k}^{2}\left(\operatorname{Id}-\frac{\stackrel{\circ}{\mathrm{R}}_{\ell}^{(\mathrm{q})}}{\varrho_{\mathrm{q}+1}}\right) \int_{\mathbb{T}^{d}} \mathbb{W}_{k}^{(q+1)}(x) \otimes \mathbb{W}_{k}^{(q+1)}(x) \mathrm{d} x \\
& =\sum_{k \in \Lambda} \varrho_{q+1} \Gamma_{k}^{2}\left(\operatorname{Id}-\frac{\stackrel{\circ}{\mathrm{R}}_{\ell}^{(\mathrm{q})}}{\varrho_{\mathrm{q}+1}}\right) \int_{\mathbb{T}^{d}} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} x\right) \otimes \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} x\right) \mathrm{d} x \tag{3.8}
\end{align*}
$$

Now we define the amplitude function

$$
\begin{equation*}
a_{k}^{(q+1)}(t, x):=g_{\kappa}\left(\varsigma_{q+1} t\right) \varrho_{q+1}^{1 / 2}(t, x) \Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{\circ}{\mathrm{R}}_{\ell}^{(\mathrm{q})}}{\varrho_{\mathrm{q}+1}}\right), k \in \Lambda . \tag{3.9}
\end{equation*}
$$

Note that $a_{k}^{(q+1)}$ is $\left(\mathcal{F}_{t}\right)_{t \geq 0}$-adapted. With these preparation in hand, we define the principal perturbation as

$$
\begin{equation*}
\omega_{q+1}^{(p)}(t, x):=\sum_{k \in \Lambda} a_{k}^{(q+1)}(t, x) \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} x\right) \tag{3.10}
\end{equation*}
$$

We also define the incompressibility corrector

$$
\begin{equation*}
\omega_{q+1}^{(c)}(t, x):=\sigma_{q+1}^{-1} \sum_{k \in \Lambda} \nabla a_{k}^{(q+1)}(t, x): \mathbb{V}_{k}^{(q+1)}\left(\sigma_{q+1} x\right) \tag{3.11}
\end{equation*}
$$

where $\mathbb{V}_{k}^{(q+1)}$ is given in (A.4) with $\mu=\mu_{q+1}$. By (A.6) and a direct computation we deduce that

$$
\begin{equation*}
\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}=\sigma_{q+1}^{-1} \operatorname{div} \sum_{k \in \Lambda} a_{k}^{(q+1)}(t, x) \mathbb{V}_{k}^{(q+1)}\left(\sigma_{q+1} x\right) \tag{3.12}
\end{equation*}
$$

and hence $\operatorname{div}\left(\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}\right)=0$ since $a_{k}^{(q+1)}(t, x) \mathbb{V}_{k}^{(q+1)}$ is skew-symmetric.
Next we introduce the temporal corrector. To this end, we define the function

$$
\begin{equation*}
h_{\kappa}(t):=\int_{0}^{t}\left(g_{\kappa}(s)^{2}-1\right) d s, t \in \mathbb{R} \tag{3.13}
\end{equation*}
$$

Then by properties of the function $g_{\kappa}$ one can easily see that $h_{\kappa}$ is periodic with period 1 and smooth on $[0, \infty)$ and obeys the bound

$$
\begin{equation*}
\left|h_{\kappa}(t)\right| \leq 1, t \in[0, \infty) \tag{3.14}
\end{equation*}
$$

Then the divergence-free temporal corrector is defined by

$$
\begin{equation*}
\omega_{q+1}^{(t)}(t, x):=\varsigma_{q+1}^{-1} \cdot h_{\kappa}\left(\varsigma_{q+1} t\right) \mathbf{P} \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}(t, x) \tag{3.15}
\end{equation*}
$$

Here, $\mathbf{P}=\operatorname{Id}-\nabla \Delta^{-1}$ div is the Helmholtz projection.
To keep the initial condition, we introduce a smooth cut-off $\Theta_{q+1} \in C^{\infty}([0, \infty) ;[0,1])$ such that

$$
\Theta_{q+1}(t)=\left\{\begin{array}{ll}
0 & \text { when } t \leq \ell_{q+1}^{1 / 2} / 2,  \tag{3.16}\\
1 & \text { when } t \geq \ell_{q+1}^{1 / 2},
\end{array} \quad \text { and } \quad\left\|\Theta_{\mathrm{q}+1}^{(\mathrm{n})}\right\|_{0} \lesssim \ell_{\mathrm{q}+1}^{-\mathrm{n} / 2}\right.
$$

And we define

$$
\widetilde{\omega}_{q+1}^{(p)}:=\Theta_{q+1} \omega_{q+1}^{(p)}, \quad \widetilde{\omega}_{q+1}^{(c)}:=\Theta_{q+1} \omega_{q+1}^{(c)}, \quad \widetilde{\omega}_{q+1}^{(t)}:=\Theta_{q+1}^{2} \omega_{q+1}^{(t)}
$$

Finally, the total perturbation $\omega_{q+1}$ is defined by

$$
\omega_{q+1}=\widetilde{\omega}_{q+1}^{(p)}+\widetilde{\omega}_{q+1}^{(c)}+\widetilde{\omega}_{q+1}^{(t)}
$$

which is mean zero, divergence free and $\left(\mathcal{F}_{t}\right)_{t \geq 0}$-adapted. The new velocity $v^{(q+1)}$ is defined as

$$
\begin{equation*}
v^{(q+1)}:=v_{\ell}^{(q)}+\omega_{q+1}=v^{(q)}+\varpi_{q+1} \tag{3.17}
\end{equation*}
$$

with $\varpi_{q+1}:=v^{(q+1)}-v^{(q)}=\left(v_{\ell}^{(q)}-v^{(q)}\right)+\omega_{q+1}$.

### 3.3. The new Reynold Stress $\stackrel{\circ}{R}^{(q+1)}$.

In this subsection we give the new Reynold Stress $\stackrel{\circ}{R}^{(q+1)}$. First, according to (A.5), (3.8) and (3.10), it follows that

$$
\begin{align*}
\operatorname{div} & \left(\widetilde{\omega}_{q+1}^{(p)} \otimes \widetilde{\omega}_{q+1}^{(p)}\right)+\operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)} \\
= & \Theta_{q+1}^{2} \cdot\left[\operatorname{div}\left(\omega_{q+1}^{(p)} \otimes \omega_{q+1}^{(p)}\right)+\operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}\right]+\left(1-\Theta_{q+1}^{2}\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)} \\
= & \Theta_{q+1}^{2} \cdot\left[\operatorname{div} \sum_{k \in \Lambda}\left(a_{k}^{(q+1)}\right)^{2} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right) \stackrel{\circ}{\otimes} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)+\operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}\right] \\
& +\left(1-\Theta_{q+1}^{2}\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}+\operatorname{div} \stackrel{\circ}{R}_{f a r}^{(q+1)}+\nabla\left(\frac{1}{d} \Theta_{q+1}^{2}\left|\omega_{q+1}^{(p)}\right|^{2}\right) \\
= & \Theta_{q+1}^{2} \operatorname{div} \sum_{k \in \Lambda}\left(a_{k}^{(q+1)}\right)^{2}\left(\mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right){\left.\stackrel{\otimes}{\otimes} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)-\int_{\mathbb{T}^{d}} \mathbb{W}_{k}^{(q+1)}(x) \otimes \mathbb{W}_{k}^{(q+1)}(x) \mathrm{d} x\right)} \quad+\Theta_{q+1}^{2} \cdot\left(1-g_{\kappa}^{2}\left(\varsigma_{q+1} t\right)\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}+\left(1-\Theta_{q+1}^{2}\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}+\operatorname{div} \stackrel{\circ}{R}_{f a r}^{(q+1)}\right. \\
& +\nabla\left[\Theta_{q+1}^{2}\left(\frac{1}{d}\left|\omega_{q+1}^{(p)}\right|^{2}+g_{\kappa}\left(\varsigma_{q+1} t\right)^{2} \cdot \varrho_{q+1}\right)\right] \\
= & \nabla p_{q+1}^{(p)}+\Theta_{q+1}^{2} \cdot\left(1-g_{\kappa}^{2}\left(\varsigma_{q+1} t\right)\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}+\left(1-\Theta_{q+1}^{2}\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}+\operatorname{div}\left(\stackrel{\circ}{R}_{f a r}^{(q+1)}+\stackrel{\circ}{R}_{o s c, x}^{(q+1)}\right)
\end{align*}
$$

where

$$
p_{q+1}^{(p)}:=\Theta_{q+1}^{2}\left[g_{\kappa}\left(\varsigma_{q+1} t\right)^{2} \varrho_{q+1}+\frac{1}{d}\left|\omega_{q+1}^{(p)}\right|^{2}-\frac{1}{d} \sum_{k \in \Lambda}\left|a_{k}^{(q+1)}\right|^{2} \int_{\mathbb{T}^{d}}\left|\mathbb{W}_{k}^{(q+1)}\right|^{2} d x\right]
$$

$$
\begin{align*}
& \stackrel{\circ}{R}_{f a r}^{(q+1)}:=\Theta_{q+1}^{2} \sum_{k \neq k^{\prime}} a_{k}^{(q+1)} a_{k^{\prime}}^{(q+1)} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right){\stackrel{\circ}{\otimes} \mathbb{W}_{k^{\prime}}^{(q+1)}\left(\sigma_{q+1} \cdot\right)}^{\stackrel{\circ}{R}_{o s c, x}^{(q+1)}}:=\Theta_{q+1}^{2} \sum_{k \in \Lambda} \mathcal{B}\left(\nabla\left(a_{k}^{(q+1)}\right)^{2}, \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right){\left.\stackrel{\circ}{\otimes} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)-\int_{\mathbb{T}^{d}} \mathbb{W}_{k}^{(q+1)}(x) \stackrel{\circ}{\otimes} \mathbb{W}_{k}^{(q+1)}(x) \mathrm{d} x\right)}^{\text {(3.19) }} .\right. \tag{3.19}
\end{align*}
$$

Here $\mathcal{B}$ denotes the bilinear anti-divergence operator defined in Appendix B.
Moreover, using (3.18), (3.13), (3.15), one can see that

$$
\begin{align*}
& \partial_{t} \widetilde{\omega}_{q+1}^{(t)}+\operatorname{div}\left(\widetilde{\omega}_{q+1}^{(p)} \otimes \widetilde{\omega}_{q+1}^{(p)}\right)+\operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)} \\
& =\nabla p_{q+1}^{(p, t)}+\operatorname{div}\left(\stackrel{\circ}{R}_{f a r}^{(q+1)}+\stackrel{\circ}{R}_{o s c, x}^{(q+1)}+\stackrel{\circ}{R}_{o s c, t}^{(q+1)}\right)+\left(1-\Theta_{q+1}^{2}\right) \operatorname{div} \stackrel{\circ}{R}_{\ell}^{(q)}+2 \Theta_{q+1} \Theta_{q+1}^{\prime} \omega_{q+1}^{(t)}, \tag{3.21}
\end{align*}
$$

with

$$
\begin{align*}
p_{q+1}^{(p, t)} & :=p_{q+1}^{(p)}-\varsigma_{q+1}^{-1} \Delta^{-1} \operatorname{divdiv} \partial_{t}\left(\Theta_{q+1}^{2} h_{\kappa}\left(\varsigma_{q+1} t\right) \stackrel{\circ}{R}_{\ell}^{(q)}\right), \\
\stackrel{\circ}{R}_{o s c, t}^{(q+1)} & :=\varsigma_{q+1}^{-1} \cdot \Theta_{q+1}^{2} h_{\kappa}\left(\varsigma_{q+1} t\right) \partial_{t} \stackrel{\circ}{R}_{\ell}^{(q)} \tag{3.22}
\end{align*}
$$

Then by (3.1) and (3.21), we have that $v^{(q+1)}$ solves the random Reynold system

$$
\left\{\begin{array}{l}
\partial_{t} v^{(q+1)}+\operatorname{div}\left[\left(v^{(q+1)}+z\right) \otimes\left(v^{(q+1)}+z\right)\right]=-\nabla p_{q+1}+\nu \Delta v^{(q+1)}+z+\operatorname{div} \stackrel{\circ}{R}^{(q+1)} \\
\operatorname{div} v^{(q+1)} \equiv 0 \\
v^{(q+1)}(0)=0
\end{array}\right.
$$

on $[0, \infty) \times \mathbb{T}^{d}$, with some gradient pressure $p_{q+1}$ and the new Reynold stress

$$
\begin{equation*}
\stackrel{\circ}{R}^{(q+1)}:=\stackrel{\circ}{R}_{c o m}^{\ell}+\stackrel{\circ}{R}_{c o m}^{(q+1)}+\stackrel{\circ}{R}_{\text {far }}^{(q+1)}+\stackrel{\circ}{R}_{o s c, x}^{(q+1)}+\stackrel{\circ}{R}_{o s c, t}^{(q+1)}+\stackrel{\circ}{R}_{l i n}^{(q+1)}+\stackrel{\circ}{R}_{c o r}^{(q+1)}+\stackrel{\circ}{R}_{c u t}^{(q+1)} \tag{3.23}
\end{equation*}
$$

given by (3.2), (3.19), (3.20), (3.22), and
$\stackrel{\circ}{R}_{\text {lin }}^{(q+1)}:=\mathcal{R}\left[\Theta_{q+1} \partial_{t}\left(\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}\right)-\nu \Delta \omega_{q+1}+\left(z_{\ell}-z\right)\right]+\left(v_{\ell}^{(q)}+z_{\ell}\right) \dot{\otimes} \omega_{q+1}+\omega_{q+1} \stackrel{\circ}{\otimes}\left(v_{\ell}^{(q)}+z_{\ell}\right)$,
$\stackrel{\circ}{R}_{c o r}^{(q+1)}:=\left(\widetilde{\omega}_{q+1}^{(c)}+\widetilde{\omega}_{q+1}^{(t)}\right) \stackrel{\circ}{\otimes} \omega_{q+1}+\widetilde{\omega}_{q+1}^{(p)} \stackrel{\circ}{\otimes}\left(\widetilde{\omega}_{q+1}^{(c)}+\widetilde{\omega}_{q+1}^{(t)}\right)$,
$\dot{R}_{c o m}^{(q+1)}:=\left(v^{(q+1)}+z_{\ell}\right) \dot{\otimes}\left(z-z_{\ell}\right)+\left(z-z_{\ell}\right) \dot{\otimes}\left(v^{(q+1)}+z_{\ell}\right)+\left(z-z_{\ell}\right) \dot{\otimes}\left(z-z_{\ell}\right)$,
$\stackrel{\circ}{R}_{\text {cut }}^{(q+1)}:=\left(1-\Theta_{q+1}^{2}\right) \stackrel{\circ}{R}_{\ell}^{(q)}+\Theta_{q+1}^{\prime} \mathcal{R}\left(\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}+2 \Theta_{q+1} \omega_{q+1}^{(t)}\right)$.
Here $\mathcal{R}$ is the anti-divergence we recall in Appendix B.

## 4. Proof of Proposition 2.2

This section is devoted to the proof for the Main Iteration Proposition 2.2. In the following $\alpha, r, p$ are fixed and given in the statement of Proposition 2.2. First of all, we start the proof by fixing the parameters in Section 4.1. In Section 4.2 we prove two propositions. One gives the finite moment bound for the approximate velocity and the error at step $q$ is independent of parameters at step $q+1$, which is the key point for the whole proof. The other shows the mollification convergence which is required for various subsequent
estimates. Section 4.3 is the main part of the proof and contains inductive estimates of approximate velocity and error.

### 4.1. Choice of Parameters.

In the sequel the mollification, concentration and oscillation parameters : $\ell_{q+1}, \mu_{q+1}, \sigma_{q+1}, \kappa_{q+1}$ and $\varsigma_{q+1}$ have to be carefully chosen in order to respect all the conditions appearing in the estimates below. To this end, we first choose an universal and sufficiently small constant $0<\vartheta<\frac{1}{2 d+9}$ such that

$$
(d+3) \vartheta \leq \min \left\{2\left(\frac{1}{p}-\frac{1}{2}\right), \frac{d-1}{4 r}\right\}
$$

which leads to

$$
\left\{\begin{array}{l}
\frac{1}{2 \vartheta} \geq(4 d+7) \vartheta+\frac{d-1}{2}  \tag{4.1}\\
\left(\frac{1}{p}-\frac{1}{2}\right)(1+d-(5 d+17) \vartheta) \geq \frac{d+3}{2} \vartheta \\
\left(\frac{1}{p}-\frac{1}{2}\right) \frac{1}{\vartheta} \geq \frac{d-1}{2} \\
\frac{d-1}{r} \geq(4 d+12) \vartheta
\end{array}\right.
$$

Now we choose the parameters using $\vartheta$ as follows:

$$
\begin{align*}
& \text { Mollification }: \ell_{q+1}=\lambda_{q+1}^{-\vartheta} \\
& \text { Spacial Concentration : } \mu_{q+1}=\lambda_{q+1} \\
& \text { Spacial Oscillation : } \sigma_{q+1}=\left\lceil\lambda_{q+1}^{\frac{1}{2 \vartheta}}\right\rceil  \tag{4.2}\\
& \text { Temporal Concentration : } \kappa_{q+1}=\lambda_{q+1}^{1+d-(5 d+17) \vartheta+\frac{1}{\vartheta}} ; \\
& \text { Temporal Oscillation : } \varsigma_{q+1}=\left\lceil\lambda_{q+1}^{(d+6) \vartheta}\right\rceil
\end{align*}
$$

where " $\lceil x\rceil$ " means the smallest integer larger than $x \in \mathbb{R}$. As one will see in the final subsection, the final control of $\varpi_{q+1}$ and $\stackrel{\circ}{R}^{(q+1)}$ is small by choosing $\lambda_{q+1}$ large enough.

### 4.2. Preparations.

In this subsection we first show that the moment estimates of $v^{(q)}$ and $\AA^{(q)}$ is finite and independent of $\lambda_{q+1}$, i.e. parameters at level $q+1$. Since the equation has quadratic nonlinearity, the estimates of moments at step $q+1$ contain higher moments of step $q$. By using the following proposition the higher moments only depends on parameters up to step $q$ and we could choose parameters at step $q+1$ to guarantee smallness in the proof.
Proposition 4.1. (Finiteness of All Moments) For each $q \in \mathbb{N}_{0}$ and any $0<m<\infty$ and $N \in \mathbb{N}_{0}$, there exists a finite constant $C_{m, N, \lambda_{1}, \ldots, \lambda_{q}}>0$ independent of $\lambda_{q+1}$ such that

$$
\begin{equation*}
\sup _{s \geq 0} \mathbb{E}\left\|v^{(q)}\right\|_{N, s}^{m}+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{m} \leq C_{m, N, \lambda_{1}, \ldots, \lambda_{q}} \tag{4.3}
\end{equation*}
$$

To prove Proposition 4.1, we need the following $C^{N}$-estimates for the coefficients $a_{k}^{(q+1)}(k \in \Lambda)$, which will be used also in section 4.3.

Lemma 4.2. For each $m, N \in \mathbb{N}_{0}$ and $k \in \Lambda$, there exists a sequence $\left\{C_{N}\right\}$ of deterministic constants such that for all $s \geq 0$ and $t \in[s, s+1]$ we have

$$
\begin{equation*}
\left\|\partial_{t}^{m} a_{k}^{(q+1)}(t)\right\|_{N} \lesssim 4^{(m+N) q} \sum_{j=0}^{m} \varsigma_{q+1}^{j}\left|g_{\kappa}^{(j)}\left(\varsigma_{q+1} t\right)\right| \ell_{q+1}^{-(N+m-j)-\left(N+m+\frac{1}{2}\right)(d+1)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+m+3 / 2}\right) \tag{4.4}
\end{equation*}
$$

Here the implicit constant is deterministic and independent of $\lambda_{q+1}$.

Proof of Lemma 4.2. By the Sobolev embedding $W^{d+1,1} \hookrightarrow C$ and mollification estimates, we obtain

$$
\begin{equation*}
\left\|\stackrel{\circ}{R}_{\ell}^{(q)}\right\|_{N, s} \lesssim \ell^{-(d+1)-N}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}} \tag{4.5}
\end{equation*}
$$

By Leibniz rule, we get

$$
\left[\partial_{t}^{m} a_{k}^{(q+1)}(t)\right]_{N} \lesssim \sum_{j=0}^{m} \varsigma_{q+1}^{j}\left|g_{\kappa}^{(j)}\left(\varsigma_{q+1} t\right)\right| \cdot\left[\varrho_{q+1}^{1 / 2} \cdot \Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right)\right]_{N+m-j, s}
$$

Then it suffices to show for all $N \in \mathbb{N}_{0}$ that

$$
\begin{equation*}
\left[\varrho_{q+1}^{1 / 2} \cdot \Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right)\right]_{N, s} \lesssim 4^{N q} \cdot \ell_{q+1}^{-N-\left(N+\frac{1}{2}\right)(d+1)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+3 / 2}\right) \tag{4.6}
\end{equation*}
$$

Using Leibniz rule again, we obtain

$$
\left[\varrho_{q+1}^{1 / 2} \cdot \Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right)\right]_{N, s} \lesssim \sum_{j=0}^{N}\left[\begin{array}{c}
\varrho_{q+1}^{1 / 2}
\end{array}\right]_{j, s}\left[\Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right)\right]_{N-j, s}
$$

In the following we prove for all $N \in \mathbb{N}_{0}$ that

$$
\left[\begin{array}{c}
\left.\varrho_{q+1}^{1 / 2}\right]_{N, s}  \tag{4.7}\\
\lesssim \\
\ell_{q+1}^{-\frac{d+1}{2}}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{1 / 2}\right) \quad N=0 \\
4^{N q} \cdot \ell_{q+1}^{-N-N(d+1)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N}\right) \quad N \geq 1
\end{array}\right.
$$

and

$$
\begin{equation*}
\left[\Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right)\right]_{N, s} \lesssim 4^{N q} \cdot \ell_{q+1}^{-N-(d+1) N}\left(1+\left\|\check{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+1}\right) \tag{4.8}
\end{equation*}
$$

which implies (4.6) and the final result. In the following, we write $[\cdot]_{N} "$ instead of " $[\cdot]_{N, s}$ " if there's no confusion.

First, note that the case $N=0$ of (4.7) and (4.8) is immediate by (3.7) and the definition of $\chi_{q+1}$ and $\Gamma_{k}$. So we only show the case $N \geq 1$. This is achieved by using the $C^{N}$-estimates (C.1) and (C.2) for compositions given in [BLIS15] which we recall in Appendix C.

By (C.1), (3.7) and (4.5), we have

$$
\left[\varrho_{q+1}\right]_{N} \lesssim\left[\chi_{q+1}\right]_{1}\left[\stackrel{\circ}{R}_{\ell}^{(q)}\right]_{N}+\left\|\mathrm{D} \chi_{q+1}\right\|_{N-1}\left\|\stackrel{\circ}{R}_{\ell}^{(q)}\right\|_{0}^{N-1}\left[\stackrel{\circ}{R}_{\ell}^{(q)}\right]_{N}
$$

$$
\begin{equation*}
\lesssim \ell_{q+1}^{-N-(d+1) N}\left(1+\left\|\check{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N}\right), N \geq 1 \tag{4.9}
\end{equation*}
$$

Then, we apply (C.2) to the function $\Psi_{1}(y)=y^{1 / 2}\left(y \in\left(4^{-q} / 2,+\infty\right)\right)$ and by (4.9) we obtain that

$$
\begin{aligned}
{\left[\varrho_{q+1}^{1 / 2}\right]_{N} } & \lesssim\left[\Psi_{1}\right]_{1}\left[\varrho_{q+1}\right]_{N}+\left\|\mathrm{D} \Psi_{1}\right\|_{N-1}\left[\varrho_{q+1}\right]_{1}^{N} \\
& \lesssim\left(2^{q}\left\|\mathrm{D} \chi_{q+1}\right\|_{N-1}+4^{(N-1 / 2) q}\left\|\mathrm{D} \chi_{q+1}\right\|_{0}^{N}\right) \cdot \ell_{q+1}^{-N-(d+1) N}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N}\right) \\
& \lesssim 4^{N q} \ell_{q+1}^{-N-(d+1) N}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N}\right), N \geq 1
\end{aligned}
$$

which implies (4.7). Similarly, we apply (C.2) to the function $\Psi_{2}(y)=y^{-1}\left(y \in\left(4^{-q} / 2,+\infty\right)\right)$ and use (4.9) to deduce that

$$
\begin{align*}
{\left[\varrho_{q+1}^{-1}\right]_{N} } & \lesssim\left[\Psi_{2}\right]_{1}\left[\varrho_{q+1}\right]_{N}+\left\|\mathrm{D} \Psi_{2}\right\|_{N-1}\left[\varrho_{q+1}\right]_{1}^{N} \\
& \lesssim 4^{(N+1) q} \ell_{q+1}^{-N-(d+1) N}\left(1+\left\|\AA^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N}\right), N \geq 1 \tag{4.10}
\end{align*}
$$

Now we proceed with a bound for $\Gamma_{k}\left(\operatorname{Id}-\frac{\stackrel{R}{R}^{(q)}}{\varrho_{q+1}}\right)$. By (C.1), we have to estimate the following:

$$
\left[\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right]_{N}+\left\|\frac{\stackrel{\circ}{\ell}_{\ell}^{(q)}}{\varrho_{q+1}}\right\|_{0}^{N-1}\left[\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right]_{N} \lesssim\left[\frac{\stackrel{\circ}{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right]_{N}
$$

Then by (4.5), (4.10) and (3.7), we have

$$
\begin{aligned}
{\left[\frac{\check{R}_{\ell}^{(q)}}{\varrho_{q+1}}\right]_{N} \lesssim } & \lesssim \sum_{m=1}^{N}\left[\varrho_{q+1}^{-1}\right]_{m}\left[\stackrel{\circ}{R}_{\ell}^{(q)}\right]_{N-m}+\left[\varrho_{q+1}^{-1}\right]_{0}\left[\stackrel{\circ}{R}_{\ell}^{(q)}\right]_{N} \\
\lesssim & \sum_{m=1}^{N}\left[4^{(m+1) q} \cdot \ell_{q+1}^{-N-N(d+1)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{m+1}\right)\right. \\
& \left.\quad+4^{(m+1) q} \cdot \ell_{q+1}^{-N-N(d+1)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}\right)\right] \\
\lesssim & 4^{(N+1) q} \cdot \ell_{q+1}^{-N-N(d+1)}\left(1+\left\|\stackrel{R}{R}_{\ell}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+1}\right), N \geq 1 .
\end{aligned}
$$

Thus we obtain (4.8) and the proof is complete.

Prooffor Proposition 4.1. By the construction we see that $v^{(q+1)}$ and $\stackrel{\circ}{R}^{(q+1)}$ only depends on $\lambda_{1}, \ldots, \lambda_{q+1}$. Hence, we only need to show the right hand side of (4.3) is finite.

We prove the estimate by induction. For $q=0$, (4.3) follows directly from (2.4) and (2.2). Now assume (4.3) holds for some $q \in \mathbb{N}_{0}$ by induction. We are then prove (4.3) for $q+1$. By (4.4) we have

$$
\begin{equation*}
\left\|a_{k}^{(q+1)}\right\|_{N, s} \lesssim 1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+3 / 2}, \forall s \geq 0 ; N \in \mathbb{N}_{0}, k \in \Lambda \tag{4.11}
\end{equation*}
$$

Here, the implicit finite constant is deterministic and only depends on $N, q$ and $\lambda_{q+1}$.
We start with the estimate of perturbations.

## - Estimates of perturbations :

By (3.10), (3.11), (A.8), (A.9) and (4.11), we have for all $s \geq 0$ that

$$
\begin{align*}
\left\|\omega_{q+1}^{(p)}\right\|_{N, s} & \lesssim \sum_{k \in \Lambda}\left\|a_{k}^{(q+1)}\right\|_{N, s}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{W^{N, \infty}} \\
& \lesssim 1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+3 / 2} ;  \tag{4.12}\\
\left\|\omega_{q+1}^{(c)}\right\|_{N, s} & \lesssim \sum_{k \in \Lambda}\left\|\nabla a_{k}^{(q+1)}\right\|_{N, s}\left\|\nabla_{k}^{(q+1)}\right\|_{W^{N, \infty}} \\
& \lesssim 1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+5 / 2} \tag{4.13}
\end{align*}
$$

and by (3.15), mollification estimate and Sobolev embedding, we have for all $s \geq 0$ that

$$
\begin{equation*}
\left\|\omega_{q+1}^{(t)}\right\|_{N, s} \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}} \tag{4.14}
\end{equation*}
$$

For $\tilde{\omega}_{q+1}^{(p)}, \tilde{\omega}_{q+1}^{(c)}, \tilde{\omega}_{q+1}^{(t)}$ we see that the $n$-th derivative of $\Theta_{q+1}$ behaves like $\ell_{q+1}^{-n / 2}$ does not pose any problems as the $C^{N}$ norm of $\omega_{q+1}^{(p)}, \omega_{q+1}^{(c)}, \omega_{q+1}^{(t)}$ has more powers of $\ell_{q+1}^{-1}$. Taking $m$ th-moment and then supremum for $s \in[0, \infty)$ in the above estimates, we have for $m \in \mathbb{N}$ that

$$
\begin{align*}
\sup _{s \geq 0} \mathbb{E}\left\|v^{(q+1)}\right\|_{N, s}^{m} & \lesssim \sup _{s \geq 0} \mathbb{E}\left\|v_{\ell_{q+1}}^{(q)}\right\|_{N, s}^{m}+\sup _{s \geq 0} \mathbb{E}\left\|\omega_{q+1}\right\|_{N, s}^{m} \\
& \lesssim \sup _{s \geq 0} \mathbb{E}\left\|v^{(q)}\right\|_{N, s}^{m}+1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{\left(N+\frac{5}{2}\right) m}<\infty . \tag{4.15}
\end{align*}
$$

## - Estimates of Reynold stress errors :

By (3.2) and mollification estimate, we have for all $s \geq 0$ that

$$
\left\|\stackrel{\circ}{R}_{c o m}^{\ell_{q+1}}\right\|_{C_{s} L^{1}} \lesssim\left\|v^{(q)}\right\|_{0,[s-1, s+1]}^{2}+\|z\|_{C_{[s-1, s+1]} L^{2}}^{2}
$$

Moreover, by (3.19), (3.20), (3.16), (B.3), (A.8) and (4.11), we have for all $s \geq 0$ that

$$
\begin{aligned}
\left\|\AA_{f a r}^{(q+1)}\right\|_{C_{s} L^{1}} & \lesssim\left\|\Theta_{q+1}\right\|_{0}^{2} \sum_{k \neq k^{\prime}}\left\|a_{k}^{(q+1)}\right\|_{0, s}\left\|a_{k^{\prime}}^{(q+1)}\right\|_{0, s}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{L^{2}}\left\|\mathbb{W}_{k^{\prime}}^{(q+1)}\right\|_{L^{2}} \\
& \lesssim 1+\left\|\dot{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{3}, \\
\left\|\dot{R}_{o s c, s}^{(q+1)}\right\|_{C_{s} L^{1}} & \lesssim\left\|\Theta_{q+1}\right\|_{0}^{2} \sum_{k \in \Lambda}\left\|\nabla\left|a_{k}^{(q+1)}\right|^{2}\right\|_{C_{s} C^{1}}\left\|\mathbb{W}_{k}^{(q+1)} \dot{\otimes} \mathbb{W}_{k}^{(q+1)}\right\|_{L^{1}} \\
& \lesssim \sum_{k \in \Lambda}\left\|a_{k}^{(q+1)}\right\|_{1, s}\left\|a_{k}^{(q+1)}\right\|_{2, s}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{L^{2}}^{2} \\
& \lesssim 1+\left\|\dot{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{6} .
\end{aligned}
$$

By (3.22), (3.16) and mollification estimate, we have for all $s \geq 0$ that

$$
\left\|\stackrel{\circ}{R}_{o s c, t}^{(q+1)}\right\|_{C_{s} L^{1}} \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}
$$

and by (3.24), (3.16), $L^{2}$-boundedness of the operators $\mathcal{R}$ and $\mathcal{R}$ div, (3.25), (4.12), (4.13) and (4.14), we have for all $s \in[0, \infty)$ that

$$
\begin{aligned}
\left\|\stackrel{R}{R}_{l i n}^{(q+1)}\right\|_{C_{s} L^{1}} & \lesssim\left\|\Theta_{q+1}\right\|_{0}\left\|\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}\right\|_{1, s}+\left\|\omega_{q+1}\right\|_{1, s}+\|z\|_{C_{[s-1, s+1]} L^{2}} \\
& +\left\|\omega_{q+1}\right\|_{C_{s} L^{2}}\left(\left\|v^{(q)}\right\|_{C_{[s-1, s+1]} L^{2}}+\|z\|_{C_{[s-1, s+1]} L^{2}}\right) \\
& \lesssim 1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5}+\left\|v^{(q)}\right\|_{0,[s-1, s+1]}^{2}+\|z\|_{C_{[s-1, s+1]} L^{2}}^{2} \\
\left\|\stackrel{\circ}{R}_{c o r}^{(q+1)}\right\|_{C_{s} L^{1}} & \lesssim\left\|\Theta_{q+1}\right\|_{0}^{2}\left\|\omega_{q+1}^{(p)}\right\|_{0, s}^{2}+\left\|\Theta_{q+1}\right\|_{0}^{2}\left\|\omega_{q+1}^{(c)}\right\|_{0, s}^{2}+\left\|\Theta_{q+1}\right\|_{0}^{4}\left\|\omega_{q+1}^{(t)}\right\|_{0, s}^{2} \\
& \lesssim 1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5}
\end{aligned}
$$

Finally by (3.26), and mollification estimate, we have for all $s \geq 0$ that

$$
\begin{aligned}
\left\|\stackrel{R}{R}_{\text {com }}^{(q+1)}\right\|_{C_{s} L^{1}} & \lesssim\left\|v^{(q+1)}\right\|_{0, s}^{2}+\|z\|_{C_{[s-1, s+1]} L^{2}}^{2} \\
& \lesssim 1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5}+\left\|v^{(q)}\right\|_{0, s}^{2}+\|z\|_{C_{[s-1, s+1]}^{2} L^{2}}^{2}
\end{aligned}
$$

and by 3.27), (3.16), standard mollification estimate, $L^{1}$-boundedness of the operator $\mathcal{R}$, (4.12), (4.13) and (4.14), we have for all $s \in[0, \infty)$ that

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{\text {cut }}^{(q+1)}\right\|_{C_{s} L^{1}} & \lesssim\left\|\stackrel{\circ}{R}_{\ell}^{(q)}\right\|_{C_{s} L^{1}}+\left\|\omega_{q+1}^{(p)}\right\|_{0, s}+\left\|\omega_{q+1}^{(c)}\right\|_{0, s}+\left\|\omega_{q+1}^{(t)}\right\|_{0, s} \\
& \lesssim 1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5 / 2}
\end{aligned}
$$

Taking $m$ th-moment and then supremum for $s \geq 0$ in the above estimates, and together by (4.15), we have for $m \in \mathbb{N}$ that

$$
\begin{equation*}
\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q+1)}\right\|_{C_{s} L^{1}}^{m} \lesssim 1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{6 m}+\sup _{s \geq 0} \mathbb{E}\left\|v^{(q)}\right\|_{0, s}^{2 m}+\sup _{s \geq 0} \mathbb{E}\|z\|_{C_{s} L^{2}}^{2 m}<\infty \tag{4.16}
\end{equation*}
$$

This completes the proof.

As the next step, we show the following mollification convergence which are used in the estimate of approximate velocity and the Reynold error.

Proposition 4.3. Assume $u_{0} \in L_{\sigma}^{2}$ and $\operatorname{Tr}\left(G^{*} G\right)<\infty$. Then, for each $q \in \mathbb{N}_{0}$,

$$
\begin{gather*}
\left\|z-z_{\ell}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \longrightarrow 0, \text { as } \ell \rightarrow 0  \tag{4.17}\\
\left\|v^{(q)}-v_{\ell}^{(q)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\left\|v^{(q)}-v_{\ell}^{(q)}\right\|_{L^{\alpha}\left(\Omega, \mathbb{Z}^{p, r}\right)}+\left\|v^{(q)}-v_{\ell}^{(q)}\right\|_{\mathbf{E}_{p}} \lesssim C_{q} \ell  \tag{4.18}\\
\left\|v^{(q)} \stackrel{\circ}{\otimes} z-\left(v^{(q)} \dot{\otimes} z\right)_{\ell}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \longrightarrow 0, \text { as } \ell \rightarrow 0 \tag{4.19}
\end{gather*}
$$

Here $C_{q}$ is a constant depending only on $\lambda_{1}, \ldots \lambda_{q}$ and $\left(v^{(q)} \dot{\otimes} z\right)_{\ell}:=\left(v^{(q)} \dot{\otimes} z\right) *_{x} \phi_{\ell} *_{t} \varphi_{\ell}$.

To prove Proposition 4.3, we give a lemma. To this end, we introduce the following notations for stochastic vector fields $u \in \bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)$ and tensor fields $R \in \bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)$ :

$$
\begin{aligned}
& \mathbb{T}_{\ell}\left(u ; s_{0}\right):=\sup _{s \geq s_{0}} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}^{2}}\|u(t)-u(t-\tau)\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t \\
& \mathbb{S}_{\ell}\left(u ; s_{0}\right):=\sup _{s \geq s_{0}} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}^{d}}\left\|u(t)-\tau_{-y} u(t)\right\|_{L^{2}}^{2} \phi_{\ell}(y) \mathrm{d} y \mathrm{~d} t \\
& \mathbb{T}_{\ell}^{\prime}\left(R ; s_{0}\right):=\sup _{s \geq s_{0}} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}^{2}}\|R(t)-R(t-\tau)\|_{L^{1}} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t \\
& \mathbb{S}_{\ell}^{\prime}\left(R ; s_{0}\right):=\sup _{s \geq s_{0}} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}^{d}}\left\|R(t)-\tau_{-y} R(t)\right\|_{L^{1}} \phi_{\ell}(y) \mathrm{d} y \mathrm{~d} t
\end{aligned}
$$

where $s_{0} \in \mathbb{R}, \tau_{y} u:=u(\cdot+y)$. For simplicity we write $\mathbb{T}_{\eta}(u):=\mathbb{T}_{\eta}(u ; 0)$.
Lemma 4.4. We have the following estimates:

$$
\begin{align*}
& \sup _{s \geq 0} \mathbb{E} \int_{s}^{s+1}\left\|u(t)-u_{\ell}(t)\right\|_{L^{2}}^{2} \mathrm{~d} t \lesssim \mathbb{T}_{\ell}(u)+\mathbb{S}_{\ell}(u ;-1),  \tag{4.20}\\
& \sup _{s \geq 0} \mathbb{E} \int_{s}^{s+1}\left\|R(t)-R_{\ell}(t)\right\|_{L^{1}} \mathrm{~d} t \leq \mathbb{T}_{\ell}^{\prime}(R)+\mathbb{S}_{\ell}^{\prime}(R ;-1), \tag{4.21}
\end{align*}
$$

In particular, we have

$$
\begin{align*}
& \left\|v^{(q)} \dot{\otimes} z-\left(v^{(q)} \dot{\otimes} z\right)_{\ell}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}  \tag{4.22}\\
& \lesssim\left(\left\|v^{(q)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\|z\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\left\|u_{0}\right\|_{L^{2}}\right)\left(\mathbb{T}_{\ell}\left(v^{(q)}\right)+\mathbb{S}_{\ell}\left(v^{(q)}\right)+\mathbb{T}_{\ell}(z)+\mathbb{S}_{\ell}(z ;-1)\right)^{1 / 2}
\end{align*}
$$

Proof of Lemma 4.4. A straightforward application of Hölder's inequality and Fubbini’s Theorem gives

$$
\begin{aligned}
& \int_{s}^{s+1}\left\|u(t)-u_{\ell}(t)\right\|_{L^{2}}^{2} \mathrm{~d} t \\
& \lesssim \int_{s}^{s+1}\left\|u(t)-\left(\varphi_{\ell} *_{t} u\right)(t)\right\|_{L^{2}}^{2} \mathrm{~d} t+\int_{s-1}^{s+1}\left\|u(t)-\left(\phi_{\ell} *_{x} u\right)(t)\right\|_{L^{2}}^{2} \mathrm{~d} t \\
& \lesssim \int_{s}^{s+1} \int_{\mathbb{R}}\|u(t)-u(t-\tau)\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t+\int_{s-1}^{s+1} \int_{\mathbb{R}^{d}}\left\|u(t)-\tau_{-y} u(t)\right\|_{L^{2}}^{2} \phi_{\ell}(y) \mathrm{d} y \mathrm{~d} t
\end{aligned}
$$

which implies (4.20). Similarly (4.21) holds.
Now we apply (4.21) to $v^{(q)} \dot{\otimes} z$. Note that for $u, v \in L_{s}^{2} L^{2}$

$$
\begin{gathered}
\|(u \dot{\otimes} v)(t)-(u \dot{\otimes} v)(t-\tau)\|_{L^{1}} \lesssim\|u(t)\|_{L^{2}}\|v(t)-v(t-\tau)\|_{L^{2}}+\|v(t-\tau)\|_{L^{2}}\|u(t)-u(t-\tau)\|_{L^{2}} \\
\left\|(u \dot{\otimes} v)(t)-\tau_{-y}(u \dot{\otimes} v)(t)\right\|_{L^{1}} \lesssim\|u(t)\|_{L^{2}}\left\|v(t)-\tau_{-y} v(t)\right\|_{L^{2}}+\left\|\tau_{-y} v(t)\right\|_{L^{2}}\left\|u(t)-\tau_{-y} u(t)\right\|_{L^{2}}
\end{gathered}
$$

Applying Hölder's inequality we obtain
$\mathbb{T}_{\ell}^{\prime}\left(v^{(q)} \dot{\otimes} z\right) \lesssim\left\|v^{(q)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \cdot \mathbb{T}_{\ell}(z)^{1 / 2}+\left(\sup _{s \geq 0} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}}\|z(t-\tau)\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{d} t\right)^{1 / 2} \mathbb{T}_{\ell}\left(v^{(q)}\right)^{1 / 2}$,
$\mathbb{S}_{\ell}^{\prime}\left(v^{(q)} \dot{\otimes} z\right) \lesssim\left\|v^{(q)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \cdot \mathbb{S}_{\ell}(z ;-1)^{1 / 2}+\left(\sup _{s \geq-1} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}^{d}}\left\|\tau_{-y} z(t)\right\|_{L^{2}}^{2} \phi_{\ell}(y) \mathrm{d} y \mathrm{~d} t\right)^{1 / 2} \mathbb{S}_{\ell}\left(v^{(q)}\right)^{1 / 2}$.
Moreover, we have

$$
\begin{aligned}
\sup _{s \geq 0} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}}\|z(t-\tau)\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t & =\sup _{s \geq 0} \int_{\mathbb{R}}\left(\mathbb{E} \int_{s-\tau}^{s-\tau+1}\|z(t)\|_{L^{2}}^{2} \mathrm{~d} t\right) \varphi_{\ell}(\tau) \mathrm{d} \tau \\
& \lesssim\|z\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}^{2}+\left\|u_{0}\right\|_{L^{2}}^{2} \\
\sup _{s \geq-1} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}^{d}}\left\|\tau_{-y} z(t)\right\|_{L^{2}}^{2} \phi_{\ell}(y) \mathrm{d} y \mathrm{~d} t & \leq\|z\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}^{2}+\left\|u_{0}\right\|_{L^{2}}^{2}
\end{aligned}
$$

which implies the final result.

With Lemma 4.4 in hand, we are ready to prove Proposition 4.3.
Proof of Proposition 4.3. We first note that

$$
\mathbb{T}_{\ell}\left(v^{(q)}\right)+\mathbb{S}_{\ell}\left(v^{(q)} ;-1\right) \lesssim \ell_{q+1}^{2} \cdot \sup _{s \in \mathbb{R}} \mathbb{E}\left\|v^{(q)}\right\|_{1, s}^{2}
$$

which implies (4.18) holds. Then by lemma 4.4 it suffices to prove

$$
\begin{equation*}
\mathbb{T}_{\ell}(z)+\mathbb{S}_{\ell}(z ;-1) \longrightarrow 0 \text { as } \ell \rightarrow 0 \tag{4.23}
\end{equation*}
$$

For the first term it is easy to see that

$$
\mathbb{T}_{\ell}(z) \lesssim \mathbb{T}_{\ell}\left(W_{\text {con }}\right)+\mathbb{T}_{\ell}\left(z^{i n}\right)
$$

and

$$
\mathbb{T}_{\ell}\left(W_{c o n}\right) \lesssim \ell_{q+1}^{1-2 \delta} \sup _{s \geq 0} \mathbb{E}\left\|W_{c o n}\right\|_{C_{[s-1, s+1]}^{1 / 2-\delta} L^{2}}^{2}
$$

Here $z^{i n}=e^{t(\Delta-I)} u_{0}$.
In the following we first consider $\mathbb{T}_{\ell}\left(z^{i n}\right)$. Using the dominated convergence theorem and the fact that $z^{i n}$ is $\mathbb{P}$ - a.s. uniformly continuous in $L^{2}$ on $[-2,2]$ we obtain

$$
\begin{aligned}
& \sup _{-1 \leq s \leq 1} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}}\left\|z^{i n}(t)-z^{i n}(t-\tau)\right\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t \\
& \lesssim \mathbb{E} \int_{-1}^{2} \int_{\mathbb{R}}\left\|z^{i n}(t)-z^{i n}(t-\tau)\right\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t \longrightarrow 0 \text { as } \ell \rightarrow 0
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
& \sup _{s \geq 1} \mathbb{E} \int_{s}^{s+1} \int_{\mathbb{R}}\left\|z^{i n}(t)-z^{i n}(t-\tau)\right\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t \\
& =\sup _{s \geq 1} \int_{s}^{s+1} \int_{\mathbb{R}}\left\|e^{(t-\tau)(\Delta-I)}\left(e^{\tau(\Delta-I)} u_{0}-u_{0}\right)\right\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \mathrm{~d} t \\
& \leq \int_{\mathbb{R}}\left\|e^{\tau(\Delta-I)} u_{0}-u_{0}\right\|_{L^{2}}^{2} \varphi_{\ell}(\tau) \mathrm{d} \tau \longrightarrow 0 \text { as } \ell \rightarrow 0
\end{aligned}
$$

Thus $\mathbb{T}_{\ell}\left(z^{i n}\right) \longrightarrow 0$.

Now we consider $\mathbb{S}_{\ell}(z ;-1)$ and we also seperate into $z^{i n}$ and $W_{c o n}$ part. First we have for all $t>0$ and $y \in \mathbb{R}^{d}$ that

$$
\begin{aligned}
\mathbb{E}\left\|W_{c o n}(t)-\tau_{-y} W_{c o n}(t)\right\|_{L^{2}}^{2} & =\mathbb{E}\left\|\left(I-\tau_{-y}\right) W_{c o n}(t)\right\|_{L^{2}}^{2} \\
& =\int_{0}^{t}\left\|\left(I-\tau_{-y}\right) \circ \mathbf{P} e^{(t-r)(\Delta-I)} G\right\|_{L_{2}}^{2} \mathrm{~d} r \\
& \leq\left\|\left(I-\tau_{-y}\right) \circ G\right\|_{L_{2}}^{2} \cdot \int_{0}^{t} e^{-2(t-r)} \mathrm{d} r \\
& \leq \frac{1}{2}\left\|\left(I-\tau_{-y}\right) \circ G\right\|_{L_{2}}^{2}
\end{aligned}
$$

Moreover, we have

$$
\left\|\varphi_{\ell} * z^{i n}-z^{i n}\right\|_{L^{2}} \lesssim\left\|\varphi_{\ell} * u_{0}-u_{0}\right\|_{L^{2}}
$$

Then, by Fubini's Theorem and the dominated convergence theorem, we have

$$
\begin{align*}
\mathbb{S}_{\ell}(z ;-1) & \lesssim \int_{\mathbb{R}^{d}}\left\|\left(I-\tau_{-y}\right) \circ G\right\|_{L_{2}}^{2} \phi_{\ell}(y) \mathrm{d} y+\left\|\varphi_{\ell} * u_{0}-u_{0}\right\|_{L^{2}}  \tag{4.24}\\
& =\int_{\mathbb{R}^{d}}\left\|\left(I-\tau_{-\ell y}\right) \circ G\right\|_{L_{2}}^{2} \phi(y) \mathrm{d} y+\left\|\varphi_{\ell} * u_{0}-u_{0}\right\|_{L^{2}} \longrightarrow 0 \text { as } \ell \rightarrow 0
\end{align*}
$$

since for each $y \in \mathbb{R}^{d}$,

$$
\begin{aligned}
& \left\|\left(I-\tau_{-\ell y}\right) \circ G\right\|_{L_{2}}^{2}=\sum_{n \in \mathbb{N}}\left\|G e_{n}-\tau_{-\ell y} G e_{n}\right\|_{L^{2}}^{2} \longrightarrow 0 \text { as } \ell \rightarrow 0 \\
& \left\|\left(I-\tau_{-\ell y}\right) \circ G\right\|_{L_{2}}^{2} \leq 4\|G\|_{L_{2}}^{2} \forall \ell>0
\end{aligned}
$$

This completes the proof.

### 4.3. Proof of Proposition 2.2.

To conclude the proof of Proposition 2.2 we shall verify (2.5)-(2.7). In the following we use $C_{q}$ to denote deterministic constant that may depend on $\lambda_{1}, \cdots, \lambda_{q}, \varphi, \phi, \chi_{1}, \cdots, \chi_{q+1}$ and $\Gamma_{k}$. Note that $C_{q}$ is independent of $\lambda_{q+1}$ and the end point $s \geq 0$ of time intervals $\left[s, s+1\right.$ ]. In the following estimates, $C_{q}$ may change from line to line.

First, we recall the following result proved in [MS18, Lemma 2.1] (see also [CL22, Lemma B.1]).
Lemma 4.5 (Improved Hölder's Inequality on $\left.\mathbb{T}^{d}\right)$. Let $1 \leq p \leq \infty$ and $a, f \in C^{\infty}\left(\mathbb{T}^{d}\right)$. Then for any $\sigma \in \mathbb{N}$,

$$
\begin{equation*}
\left|\|a f(\sigma \cdot)\|_{L^{p}}-\|a\|_{L^{p}}\|f\|_{L^{p}}\right| \lesssim \sigma^{-1 / p}\|a\|_{1}\|f\|_{L^{p}} \tag{4.25}
\end{equation*}
$$

This result is applied to bound $\omega_{q+1}^{(p)}$ in $L^{2}$. By taking $m=0$ and $m=1$ respectively in Lemma 4.2, we have for all $s \geq 0$ and $t \in[s, s+1]$ that

$$
\begin{equation*}
\left\|a_{k}^{(q+1)}(t)\right\|_{N} \leq C_{N, q} \cdot\left|g_{\kappa}\left(\varsigma_{q+1} t\right)\right| \cdot \ell_{q+1}^{-N-(N+1 / 2)(d+1)}\left(1+\left\|\circ^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{N+3 / 2}\right) \tag{4.26}
\end{equation*}
$$

$$
\begin{equation*}
\left\|\partial_{t} a_{k}^{(q+1)}(t)\right\|_{N} \leq C_{N, q}\left(\varsigma_{q+1}\left|g_{\kappa}^{\prime}\left(\varsigma_{q+1} t\right)\right|+\left|g_{\kappa}\left(\varsigma_{q+1} t\right)\right| \ell_{q+1}^{-1}\right) \ell_{q+1}^{-N-(N+3 / 2)(d+1)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\substack{[s-1, s+1] \\(1)]}}^{N+5 / 2}\right) \tag{4.27}
\end{equation*}
$$

Here the deterministic constant $C_{N, q}>0$ is independent of the choice of $\lambda_{q+1}$ and of $s$.
4.3.1. Inductive Estimate of $v^{(q+1)}$.

- $\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)$-Estimate :

By (3.10), (A.8), (4.26) and (4.25), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{align*}
\left\|\omega_{q+1}^{(p)}(t)\right\|_{L^{2}} & \lesssim \sum_{k \in \Lambda}\left(\left\|a_{k}^{(q+1)}(t)\right\|_{L^{2}}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{L^{2}}+\sigma_{q+1}^{-1 / 2}\left\|a_{k}^{(q+1)}(t)\right\|_{1}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{L^{2}}\right) \\
& \lesssim\left|g_{\kappa}\left(\varsigma_{q+1} t\right)\right|[\left\|\varrho_{q+1}(t)\right\|_{L^{1}}^{1 / 2}+C_{q} \cdot \sigma_{q+1}^{-1 / 2} \ell_{q+1}^{-\frac{3 d+5}{2}}(1+\|\overbrace{}^{(q)}\|_{C_{[s-1, s+1]} L^{1}}^{5 / 2})] . \tag{4.28}
\end{align*}
$$

Notice that

$$
\begin{aligned}
\left\|\varrho_{q+1}(t)\right\|_{L^{1}}^{1 / 2} & =2\left(\int_{\mathbb{T}^{d}} \chi_{q+1}\left(\stackrel{\circ}{R}_{\ell}^{(q)}(t, x)\right) \mathrm{d} x\right)^{1 / 2} \\
& \leq C_{q}\left(1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{1 / 2}\right), \forall t \in[s, s+1]
\end{aligned}
$$

and

$$
\begin{equation*}
\left\|\left\|\varrho_{q+1}\right\|_{L_{x}^{1}}^{1 / 2}\right\|_{C^{1}[s, s+1]} \leq C_{q} \cdot \ell_{q+1}^{-1}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}\right) \tag{4.29}
\end{equation*}
$$

Taking $L^{2}[s, s+1]$-norm in (4.28), and using (4.25) again, and by (3.5), (4.29) and mollification estimate, we have

$$
\begin{equation*}
\left\|\omega_{q+1}^{(p)}\right\|_{L_{s}^{2} L^{2}} \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{L_{[s-\ell, s+1]}^{1} L^{1}}^{1 / 2}+2^{-q}+C_{q}\left(\varsigma_{q+1}^{-\frac{1}{2}} \ell_{q+1}^{-1}+\sigma_{q+1}^{\left.-\frac{1}{2} \ell_{q+1}^{-\frac{3 d+5}{2}}\right)\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5 / 2}\right) . . . ~ . ~ . ~}\right. \tag{4.30}
\end{equation*}
$$

Moreover, we obtain

$$
\begin{align*}
\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{L_{[s-\ell, s+1]}^{1} L^{1}} & \leq \sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{L_{[s-\ell, s]}^{1} L^{1}}+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \\
& \leq \ell_{q+1} \cdot \sup _{s \geq 0} \mathbb{E}\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{s} L^{1}}+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \tag{4.31}
\end{align*}
$$

Taking the second moment and then supremum for $s \geq 0$ in (4.30), by (4.31), (4.3) and (4.2), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(p)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} & \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-q}+C_{q}\left(\varsigma_{q+1}^{-\frac{1}{2}} \ell_{q+1}^{-1}+\sigma_{q+1}^{-\frac{1}{2}} \ell_{q+1}^{-\frac{3 d+5}{2}}+\ell_{q+1}^{\frac{1}{2}}\right)\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{5}\right)^{\frac{1}{2}} \\
& \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-q}+\mathfrak{C}_{q}\left(\lambda_{q+1}^{-\left(\frac{d}{2}+2\right) \vartheta}+\lambda_{q+1}^{-\left(\frac{1}{4 \vartheta}-\frac{3 d+5}{2} \vartheta\right)}+\lambda_{q+1}^{-\vartheta / 2}\right) \\
& \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-q}+\mathfrak{C}_{q} \cdot \lambda_{q+1}^{-\vartheta / 2} \tag{4.32}
\end{align*}
$$

Here we use the first constraint of (4.1) to deduce that

$$
\frac{1}{4 \vartheta}-\frac{3 d+5}{2} \vartheta>\frac{4 d+7}{2} \vartheta-\frac{3 d+5}{2} \vartheta>\vartheta
$$

Then by (3.11), (A.9) and (4.26), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{align*}
\left\|\omega_{q+1}^{(c)}(t)\right\|_{L^{\infty}} & \lesssim \sigma_{q+1}^{-1} \cdot \sum_{k \in \Lambda}\left\|\nabla a_{k}^{(q+1)}(t)\right\|_{0}\left\|\mathbb{V}_{k}^{(q+1)}\right\|_{L^{\infty}} \\
& \leq C_{q} \cdot\left|g_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right| \cdot \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}} \ell_{q+1}^{-\frac{3 d+5}{2}}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5 / 2}\right) \tag{4.33}
\end{align*}
$$

Taking $L^{2}[s, s+1]$-norm in time, then the second moment and finally supremum for $s \geq 0$, by (3.5), (4.3) and (4.2), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(c)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} & \lesssim\left\|\omega_{q+1}^{(c)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{\infty}\right)} \\
& \leq C_{q} \cdot \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}} \ell_{q+1}^{-\frac{3 d+5}{2}}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{3}\right)^{1 / 2} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\left(\frac{1}{2 \vartheta}-\frac{3 d+5}{2} \vartheta-\frac{d-3}{2}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} . \tag{4.34}
\end{align*}
$$

Here we use the first constraint of (4.1) to deduce that

$$
\frac{1}{2 \vartheta}-\frac{3 d+5}{2} \vartheta-\frac{d-3}{2}>\frac{5 d+9}{2} \vartheta+1>\vartheta
$$

For $\omega_{q+1}^{(t)}$ we estimate $W^{1, a}$ bound with $a>1$ for later use. By (3.15), (3.14) and using $L^{a}$-boundedness of the Helmholtz projection, mollification estimates and the Sobolev embedding $W^{2+\left(1-\frac{1}{a}\right) d, 1} \hookrightarrow W^{2, a}$, we obtain for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{align*}
\left\|\omega_{q+1}^{(t)}(t)\right\|_{W^{1, a}} & \lesssim\left|h_{\kappa}\left(\varsigma_{q+1} t\right)\right| \cdot \varsigma_{q+1}^{-1}\left\|\stackrel{\circ}{R}_{\ell}^{(q)}(t)\right\|_{W^{2, a}} \\
& \lesssim \varsigma_{q+1}^{-1}\left\|\stackrel{\circ}{R}_{\ell}^{(q)}(t)\right\|_{W^{2+\left(1-\frac{1}{a}\right) d, 1}} \\
& \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-2-\left(1-\frac{1}{a}\right) d}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}} \\
& \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-(d+2)}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}} \tag{4.35}
\end{align*}
$$

Then taking $L^{2}$-norm in time and in probability, and finally supremum for $s \geq 0$, by (3.5), (4.3) and (4.2), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} & \lesssim\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} W^{1,2}\right)} \\
& \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-(d+2)} \cdot\left(\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{2}\right)^{1 / 2} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.36}
\end{align*}
$$

Hence, by (3.17), (3.16), (4.32), (4.34), (4.36) and Proposition 4.3, we have

$$
\begin{align*}
\left\|\varpi_{q+1}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} & \leq\left\|v_{\ell_{q+1}^{(q)}}^{(q)} v^{(q)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\left\|\omega_{q+1}^{(p)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\left\|\omega_{q+1}^{(c)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \\
& \lesssim\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-q}+C_{q} \cdot \lambda_{q+1}^{-\vartheta / 2} \tag{4.37}
\end{align*}
$$

Choose $\lambda_{q+1}$ sufficiently large and we get (2.6).

As the next step, we shall verify (2.7). To this end, we estimate each norms in the definition of $Z_{s}^{p, r}$ and $\mathbf{E}_{p}$.

- $\bar{L}^{\alpha}\left(\Omega, C_{s} W^{-1,1}\right)$-Estimate :

By (3.12), (A.9), (4.26) and (3.3), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\omega_{q+1}^{(p)}(t)+\omega_{q+1}^{(c)}(t)\right\|_{W^{-1,1}} & \leq \sigma_{q+1}^{-1} \cdot\left\|\sum_{k \in \Lambda} a_{k}^{(q+1)}(t) \mathbb{V}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)\right\|_{L^{m}} \\
& \leq \sigma_{q+1}^{-1} \cdot \sum_{k \in \Lambda}\left\|a_{k}^{(q+1)}(t)\right\|_{0}\left\|\mathbb{V}_{k}^{(q+1)}\right\|_{L^{m}} \\
& \leq C_{q} \cdot \kappa_{q+1}^{1 / 2} \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}-\frac{d-1}{m}} \ell_{q+1}^{-\frac{d+1}{2}}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{3 / 2}\right)
\end{aligned}
$$

Here we take $m>1$ and close to 1 . Taking supremum norm on $[s, s+1]$, then $\alpha$-th moment, and finally supremum for $s \geq 0$, by (4.3) and (4.2), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}\right\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} W^{-1,1}\right)} & \leq C_{q} \cdot \kappa_{q+1}^{1 / 2} \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}-\frac{d-1}{m}} \ell_{q+1}^{-\frac{d+1}{2}}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{\alpha / 2}\right)^{1 / \alpha} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-(3 d+8) \vartheta} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.38}
\end{align*}
$$

For $\omega_{q+1}^{(t)}$ by (3.15), (3.14) and the Sobolev embedding $W^{d / 2,1} \hookrightarrow L^{2}$, we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\omega_{q+1}^{(t)}(t)\right\|_{H^{-1}} & \leq \varsigma_{q+1}^{-1}\left|h_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right|\left\|\stackrel{\circ}{R}_{\ell}^{(q)}(t)\right\|_{L^{2}} \\
& \lesssim \varsigma_{q+1}^{-1}\left\|\stackrel{R}{R}_{\ell}^{(q)}(t)\right\|_{W^{d / 2,1}} \\
& \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-d / 2}\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}
\end{aligned}
$$

Taking supremum norm on $[s, s+1]$ and then $\alpha$-th moment, and finally supremum for $s \geq 0$, by (4.3) and (4.2), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} H^{-1}\right)} & \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-d / 2}\left(\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{\alpha}\right)^{1 / \alpha} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} . \tag{4.39}
\end{align*}
$$

Hence by (3.16), (4.38), (4.39) and Proposition 4.3, we have

$$
\begin{align*}
\left\|\varpi_{q+1}\right\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} W^{-1,1}\right)} & \lesssim\left\|v_{\ell_{q+1}}^{(q)}-v^{(q)}\right\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} L^{\infty}\right)}+\left\|\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}\right\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} W^{-1,1}\right)}+\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{\alpha}\left(\Omega, C_{s} H^{-1}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.40}
\end{align*}
$$

- $\mathbf{E}_{p}$ - and $\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$ - Estimate :

The estimates of these two norms are similar and we only give the $\mathbf{E}_{p}$-estimate here.
By (3.10), (A.8) and (4.26), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\left\|\omega_{q+1}^{(p)}(t)\right\|_{L^{\infty}} \leq \sum_{k \in \Lambda}\left\|a_{k}^{(q+1)}(t)\right\|_{0}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{L^{\infty}}
$$

$$
\begin{equation*}
\leq C_{q} \cdot\left|g_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right| \cdot \mu_{q+1}^{\frac{d-1}{2}} \ell_{q+1}^{-\frac{d+1}{2}}\left(1+\left\|\check{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{3 / 2}\right) \tag{4.41}
\end{equation*}
$$

Taking second moment, then $L^{p}[s, s+1]$-norm in time and finally supremum for $s \geq 0$, by (3.4), (4.3), (4.2), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(p)}\right\|_{\mathbf{E}_{p}} & \leq C_{q} \cdot \kappa_{q+1}^{\frac{1}{2}-\frac{1}{p}} \mu_{q+1}^{\frac{d-1}{2}} \ell_{q+1}^{-\frac{d+1}{2}}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{3}\right)^{1 / 2} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\left[\left(\frac{1}{p}-\frac{1}{2}\right)\left(1+d-(5 d+17) \vartheta+\frac{1}{\vartheta}\right)-\frac{d+1}{2} \vartheta-\frac{d-1}{2}\right]} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.42}
\end{align*}
$$

Here, we use the fact that $1 \leq p<2$, and the second and the third constraint of (4.1) to deduce that

$$
\left(\frac{1}{p}-\frac{1}{2}\right)\left(1+d-(5 d+17) \vartheta+\frac{1}{\vartheta}\right)-\frac{d+1}{2} \vartheta-\frac{d-1}{2} \geq \vartheta
$$

For $\omega_{q+1}^{(c)}$ and $\omega_{q+1}^{(t)}$, we use (4.33) and (4.35) with $a=d+1$ and obtain

$$
\begin{align*}
\left\|\omega_{q+1}^{(c)}\right\|_{\mathbf{E}_{p}} & \leq C_{q} \cdot \kappa_{q+1}^{\frac{1}{2}-\frac{1}{p}} \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}} \ell_{q+1}^{-\frac{3 d+5}{2}}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{5}\right)^{1 / 2} \\
& \leq C_{q} \cdot \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}} \ell_{q+1}^{-\frac{3 d+5}{2}} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta}  \tag{4.43}\\
\left\|\omega_{q+1}^{(t)}\right\|_{\mathbf{E}_{p}} & \lesssim \sup _{s \geq 0}\left\|\omega_{q+1}^{(t)}\right\|_{L_{s}^{p} L^{2}\left(\Omega ; W^{1, d+1}\right)} \\
& \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-(d+2)}\left(\sup _{s \geq 0} \mathbb{E}\left\|\dot{R}^{(q)}\right\|_{C_{s} L^{1}}^{2}\right)^{1 / 2} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.44}
\end{align*}
$$

Hence, by (3.16), (4.42), (4.43), (4.44) and Proposition 4.3, we have

$$
\begin{align*}
\left\|\varpi_{q+1}\right\|_{\mathbf{E}_{p}} & \leq\left\|v_{\ell_{q+1}}^{(q)}-v^{(q)}\right\|_{\mathbf{E}_{p}}+\left\|\omega_{q+1}^{(p)}\right\|_{\mathbf{E}_{p}}+\left\|\omega_{q+1}^{(c)}\right\|_{\mathbf{E}_{p}}+\left\|\omega_{q+1}^{(t)}\right\|_{\mathbf{E}_{p}} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.45}
\end{align*}
$$

- $\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)$-Estimate :

By (3.10), (A.8) and (4.26), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\omega_{q+1}^{(p)}(t)\right\|_{W^{1, r}} & \lesssim \sum_{k \in \Lambda}\left\|a_{k}^{(q+1)}(t)\right\|_{1}\left\|\mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)\right\|_{W^{1, r}} \\
& \leq C_{q} \cdot\left|g_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right| \cdot \sigma_{q+1} \mu_{q+1}^{1+\frac{d-1}{2}-\frac{d-1}{r}} \ell_{q+1}^{-\frac{3 d+5}{2}}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5 / 2}\right)
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time, then $\alpha$-th moment, and finally supremum for $s \geq 0$, by (3.4), (4.2) and (4.3), we have

$$
\left\|\omega_{q+1}^{(p)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)} \leq C_{q} \cdot \kappa_{q+1}^{-\frac{1}{2}} \sigma_{q+1} \mu_{q+1}^{1+\frac{d-1}{2}-\frac{d-1}{r}} \ell_{q+1}^{-\frac{3 d+5}{2}}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{5 \alpha / 2}\right)^{1 / \alpha}
$$

$$
\begin{align*}
& \leq C_{q} \cdot \lambda_{q+1}^{-\left(-(4 d+11) \vartheta+\frac{d-1}{r}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.46}
\end{align*}
$$

Here we use the last constraint of (4.1) to deduce that

$$
-(4 d+11) \vartheta+\frac{d-1}{r} \geq \vartheta
$$

Then by (3.11), (A.9) and (4.26), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\omega_{q+1}^{(c)}(t)\right\|_{W^{1, r}} & \lesssim \sigma_{q+1}^{-1} \sum_{k \in \Lambda}\left\|\nabla a_{k}^{(q+1)}(t)\right\|_{1}\left\|\mathbb{V}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)\right\|_{W^{1, r}} \\
& \leq C_{q} \cdot\left|g_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right| \cdot \mu_{q+1}^{\frac{d-1}{2}-\frac{d-1}{r}} \ell_{q+1}^{-\frac{5 d+9}{2}}\left(1+\left\|\stackrel{\circ}{R^{(q)}}\right\|_{C_{[s-1, s+1]} L^{1}}^{7 / 2}\right)
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time, then $\alpha$-th moment, and finally supremum for $s \geq 0$, by (3.4), (4.1) and (4.3), we have

$$
\begin{align*}
\left\|\omega_{q+1}^{(c)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)} & \leq C_{q} \cdot \kappa_{q+1}^{-\frac{1}{2}} \mu_{q+1}^{\frac{d-1}{2}-\frac{d-1}{r}} \ell_{q+1}^{-\frac{5 d+9}{2}}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{7 \alpha / 2}\right)^{1 / \alpha} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\left(1+\frac{1}{2 \vartheta}-(5 d+13) \vartheta+\frac{d-1}{r}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.47}
\end{align*}
$$

Here we use the last and the first constraint of (4.1) to deduce

$$
1+\frac{1}{2 \vartheta}-(5 d+13) \vartheta+\frac{d-1}{r} \geq 1+\frac{1}{2 \vartheta}-(d+1) \vartheta>\vartheta
$$

For $\omega_{q+1}^{(t)}$, we simply use (4.35) to have

$$
\begin{align*}
\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)} & \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-(d+2)}\left(\sup _{s \geq 0} \mathbb{E}\left\|\AA^{(q)}\right\|_{C_{s} L^{1}}^{\alpha}\right)^{1 / \alpha} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} . \tag{4.48}
\end{align*}
$$

Hence, by (3.16), (4.46), (4.47), (4.48) and Proposition 4.3, we have

$$
\begin{align*}
& \left\|\varpi_{q+1}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)} \\
& \leq\left\|v_{\ell_{q+1}}^{(q)}-v^{(q)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)}+\left\|\omega_{q+1}^{(p)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)}+\left\|\omega_{q+1}^{(c)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)}+\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.49}
\end{align*}
$$

Combining (4.37), (4.40), (4.45) and (4.49), and choosing $\lambda_{q+1}$ sufficiently large, we get (2.7).
4.3.2. Inductive Estimate of $\stackrel{\circ}{R}^{(q+1)}$.

To conclude the proof of Proposition, we shall verify (2.5). To this end, we estimate each term in (3.23) separately.
(1) $\stackrel{\circ}{R}_{\text {com }}^{\ell}$ and $\stackrel{\circ}{R}_{\text {com }}^{(q+1)}$

- Estimate of $\stackrel{\circ}{R}_{\text {com }}^{\ell}$ :

By (3.2) and mollification estimate, we have

$$
\left\|\stackrel{\circ}{R}_{c o m}^{\ell}\right\|_{L_{s}^{1} L^{1}} \lesssim\left\|v^{(q)}+z\right\|_{L_{[s-1, s+1]}^{2} L^{2}}\left(\left\|v_{\ell}^{(q)}-v^{(q)}\right\|_{L_{s}^{2} L^{2}}+\left\|z_{\ell}-z\right\|_{L_{s}^{2} L^{2}}\right)
$$

$$
+\left\|\left(\left(v^{(q)}+z\right) \dot{\otimes}\left(v^{(q)}+z\right)\right)_{\ell}-\left(v^{(q)}+z\right) \dot{\otimes}\left(v^{(q)}+z\right)\right\|_{L_{s}^{1} L^{1}}, s \geq 0
$$

Here, $\left(\left(v^{(q)}+z\right) \dot{\otimes}\left(v^{(q)}+z\right)\right)_{\ell}:=\left[\left(\left(v^{(q)}+z\right) \dot{\otimes}\left(v^{(q)}+z\right)\right) *_{x} \phi_{\ell}\right] *_{t} \varphi_{\ell}$. Taking expectation, then by Hölder's inequality, (4.3) and Proposition 4.3, we have

$$
\left\|\stackrel{\circ}{R}_{c o m}^{\ell}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \longrightarrow 0
$$

as $\ell=\ell_{q+1} \rightarrow 0$. Hence, choose $\lambda_{q+1}$ sufficiently large so that $\ell_{q+1}=\lambda_{q+1}^{-\vartheta}$ sufficiently small and we obtain

$$
\left\|\stackrel{\circ}{R}_{c o m}^{\ell}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \leq \frac{\delta^{2}}{4}
$$

- Estimate of $\stackrel{\circ}{R}_{\text {com }}^{(q+1)}$ :

By (3.26) and mollification estimate, we have for all $s \geq 0$ that

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{\text {com }}^{(q+1)}\right\|_{L_{s}^{1} L^{1}} & \lesssim\left\|v^{(q+1)}+z_{\ell}\right\|_{L_{s}^{2} L^{2}}\left\|z-z_{\ell}\right\|_{L_{s}^{2} L^{2}}+\left\|z-z_{\ell}\right\|_{L_{s}^{2} L^{2}}^{2} \\
& \lesssim\left(\left\|v^{(q)}\right\|_{0,[s-1, s+1]}+\left\|\omega_{q+1}\right\|_{L_{s}^{2} L^{2}}+\|z\|_{L_{[s-1, s+1]}^{2} L^{2}}\right)\left\|z-z_{\ell}\right\|_{L_{s}^{2} L^{2}} .
\end{aligned}
$$

Taking expectation and supremum for $s \geq 0$, then by Hölder's inequality, (2.2), (4.3) and the previous $\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)$-estimates for perturbations, we have

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{c o m}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} & \lesssim\left(\sup _{s \geq 0} \mathbb{E}\left\|v^{(q)}\right\|_{0, s}^{2}+C_{q}+\sup _{s \geq 0} \mathbb{E}\|z\|_{L_{s}^{2} L^{2}}^{2}\right)^{1 / 2}\left\|z-z_{\ell}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \\
& \leq C_{q} \cdot\left\|z-z_{\ell}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \leq \frac{\delta^{2}}{4} .
\end{aligned}
$$

Here, in the last inequality, by (4.17), we choose $\lambda_{q+1}$ large enough and obtain

$$
C_{q} \cdot\left\|z-z_{\ell}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \leq \frac{\delta^{2}}{4}
$$

(2) Oscillation Errors $\stackrel{\circ}{R}_{f a r}^{(q+1)}, \stackrel{\circ}{R}(q+1)$ asc,x a $\stackrel{\circ}{R}_{o s c, t}^{(q+1)}$

- Estimate of $\stackrel{\circ}{R}_{f a r}^{(q+1)}$ :

By (3.19), (3.16), (A.7) and (4.26), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{f a r}^{(q+1)}(t)\right\|_{L^{1}} & \lesssim \sum_{k \neq k^{\prime}}\left\|a_{k}^{(q+1)}(t)\right\|_{0}\left\|a_{k^{\prime}}^{(q+1)}(t)\right\|_{0}\left\|\mathbb{W}_{k}^{(q+1)} \dot{\otimes} \mathbb{W}_{k^{\prime}}^{(q+1)}\right\|_{L^{1}} \\
& \leq C_{q} \cdot\left|g_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right|^{2} \cdot \mu_{q+1}^{-1} \ell_{q+1}^{-(d+1)}\left(1+\left\|\stackrel{R}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{3}\right) .
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time, then expectation and finally supremum for $s \geq 0$, by (3.5), (4.3) and (4.2), we have

$$
\begin{aligned}
\left\|\stackrel{R}{R}_{f a r}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} & \leq C_{q} \cdot \mu_{q+1}^{-1} \ell_{q+1}^{-(d+1)}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{3}\right) \\
& \leq C_{q} \cdot \lambda_{q+1}^{-(1-(d+1) \vartheta)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta}
\end{aligned}
$$

Here we have used the fact that $0<\vartheta<\frac{1}{d+2}$.

- Estimate of $\stackrel{\circ}{R}_{\text {osc }, x}^{(q+1)}$ :

By (3.20), (3.16), (B.3), $L^{1}$-boundedness of the operator $\mathcal{R}$, (A.8), (B.2) and (4.26), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{o s c, x}^{(q+1)}(t)\right\|_{L^{1}} & \lesssim \sum_{k \in \Lambda}\left\|\nabla\left|a_{k}^{(q+1)}(t)\right|^{2}\right\|_{1}\left\|\mathcal{R}\left[\mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right) \dot{\otimes} \mathbb{W}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)\right]\right\|_{L^{1}} \\
& \lesssim \sigma_{q+1}^{-1} \sum_{k \in \Lambda}\left\|\nabla a_{k}^{(q+1)}(t)\right\|_{1}\left\|a_{k}^{(q+1)}(t)\right\|_{1}\left\|\mathbb{W}_{k}^{(q+1)}\right\|_{L^{2}}^{2} \\
& \leq C_{q} \cdot\left|g_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right|^{2} \cdot \sigma_{q+1}^{-1} \ell_{q+1}^{-(4 d+7)}\left(1+\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{6}\right) .
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time, and expectation and finally supremum for $s \geq 0$, by the normalized $L^{2}$-norm of $g_{\kappa}$, (4.3) and (4.2), we have

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{o s c, x}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} & \leq C_{q} \cdot \sigma_{q+1}^{-1} \ell_{q+1}^{-(4 d+7)}\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}^{6}\right) \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\left(\frac{1}{2 \vartheta}-(4 d+7) \vartheta\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} .
\end{aligned}
$$

Here we use the first constraint of (4.1) to deduce that

$$
\frac{1}{2 \vartheta}-(4 d+7) \vartheta \geq \frac{d-1}{2}>\vartheta
$$

- Estimate of $\stackrel{\circ}{R}(q+1)$ :

By (3.22), (3.16), (3.14) and mollification estimate, we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
\left\|\stackrel{\circ}{R}_{o s c, t}^{(q+1)}(t)\right\|_{L^{1}} & \leq \varsigma_{q+1}^{-1}\left|h_{\kappa_{q+1}}\left(\varsigma_{q+1} t\right)\right|\left\|\partial_{t} \stackrel{\circ}{R}_{\ell_{q+1}}^{(q)}(t)\right\|_{L^{1}} \\
& \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-1}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time, then expectation and finally supremum for $s \geq 0$, by (4.3) and (4.2), we have

$$
\begin{aligned}
\left\|\check{R}_{o s c, t}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} & \lesssim \varsigma_{q+1}^{-1} \ell_{q+1}^{-1} \cdot \sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta}
\end{aligned}
$$

(3) Linear Error $\stackrel{\circ}{R}_{\text {lin }}^{(q+1)}$ :

In the following we choose $\gamma:=\frac{2(d-1)}{2(d-1)-\vartheta}>1$, i.e. $(d-1)-\frac{d-1}{\gamma}=\frac{\vartheta}{2}$. We estimate each term in (3.24) separately. First, using (3.12), (3.16), and $L^{\gamma}$-boundedness of the operator $\mathcal{R} \operatorname{div}$, (A.9), (4.27), we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\left\|\mathcal{R}\left[\Theta_{q+1}(t) \partial_{t}\left(\omega_{q+1}^{(p)}(t)+\omega_{q+1}^{(c)}(t)\right)\right]\right\|_{L^{\gamma}}
$$

$$
\begin{aligned}
& \lesssim \sigma_{q+1}^{-1} \sum_{k \in \Lambda}\left\|\partial_{t} a_{k}^{(q+1)}(t) \mathbb{V}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)\right\|_{L^{\gamma}} \\
& \lesssim \sigma_{q+1}^{-1} \sum_{k \in \Lambda}\left\|\partial_{t} a_{k}^{(q+1)}(t)\right\|_{0}\left\|\mathbb{V}_{k}^{(q+1)}\left(\sigma_{q+1} \cdot\right)\right\|_{L^{\gamma}} \\
& \leq C_{q} \cdot \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}-\frac{d-1}{\gamma}}\left(\varsigma_{q+1}\left|g_{\kappa}^{\prime}\left(\varsigma_{q+1} t\right)\right|+\left|g_{\kappa}\left(\varsigma_{q+1} t\right)\right| \ell_{q+1}^{-1}\right) \ell_{q+1}^{-\frac{3 d+3}{2}}\left(1+\left\|\dot{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}^{5 / 2}\right)
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time, then expectation and finally supremum for $s \geq 0$, by (3.4), (4.3) and (4.2), we have

$$
\begin{align*}
& \left\|\mathcal{R}\left[\Theta_{q+1} \partial_{t}\left(\omega_{q+1}^{(p)}+\omega_{q+1}^{(c)}\right)\right]\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \\
& \leq C_{q} \cdot \sigma_{q+1}^{-1} \mu_{q+1}^{-1+\frac{d-1}{2}-\frac{d-1}{\gamma}} \ell_{q+1}^{-\frac{3 d+3}{2}}\left(\kappa_{q+1}^{\frac{1}{2}} \varsigma_{q+1}+\kappa_{q+1}^{-\frac{1}{2}} \ell_{q+1}^{-1}\right)\left(1+\sup _{s \geq 0} \mathbb{E}\left\|\dot{R}^{(q)}\right\|_{C_{s} L^{1}}^{5 / 2}\right) \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\left(\vartheta-(d-1)+\frac{d-1}{\gamma}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta / 2} . \tag{4.50}
\end{align*}
$$

Here, we use the definition of $\gamma$ and to deduce that

$$
\vartheta-(d-1)+\frac{d-1}{\gamma}=\vartheta-\frac{\vartheta}{2}=\frac{\vartheta}{2} .
$$

For the second part, we use (B.1) and the previous $\bar{L}^{\alpha}\left(\Omega, L_{s}^{1} W^{1, r}\right)$-estimates for perturbations to obtain

$$
\begin{align*}
\left\|\mathcal{R}\left(\nu \Delta \omega_{q+1}+\left(z_{\ell}-z\right)\right)\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} & \lesssim\left\|\omega_{q+1}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} H^{1}\right)}+\left\|z_{\ell}-z\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta}+\frac{\delta^{2}}{4} \tag{4.51}
\end{align*}
$$

Here we used similar argument as above and choose $\lambda_{q+1}$ sufficiently large so that

$$
\left\|z_{\ell}-z\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)} \leq \frac{\delta^{2}}{4}
$$

For the last part, by using (2.2), (4.3) and the previous $\bar{L}^{\alpha}\left(\Omega, L_{s}^{p} L^{\infty}\right)$-estimates for perturbations we obtain

$$
\begin{align*}
& \left\|\left(v_{\ell}^{(q)}+z_{\ell}\right) \dot{\otimes} \omega_{q+1}+\omega_{q+1} \dot{\otimes}\left(v_{\ell}^{(q)}+z_{\ell}\right)\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \\
& \lesssim\left(\sup _{s \geq 0} \mathbb{E}\left\|v^{(q)}\right\|_{0,[s, s+2]}^{2}+\sup _{s \geq 0} \mathbb{E}\|z\|_{C_{[s, s+2]} L^{2}}^{2}\right)^{1 / 2}\left\|\omega_{q+1}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{p} L^{\infty}\right)} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta} \tag{4.52}
\end{align*}
$$

Hence, by (4.50), (4.51) and (4.52), we obtain

$$
\left\|\stackrel{\circ}{R}_{l i n}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta / 2}+\frac{\delta^{2}}{4}
$$

(4) Correction Error $\stackrel{\circ}{R}_{\text {cor }}{ }^{(q+1)}$ :

By (3.25), (3.16), (4.32), (4.34), (4.36) and (4.3), we have

$$
\left\|\stackrel{\circ}{R}_{\text {cor }}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \lesssim\left(\left\|\omega_{q+1}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}+\left\|\omega_{q+1}^{(p)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}\right)\left\|\omega_{q+1}^{(c)}+\omega_{q+1}^{(t)}\right\|_{\bar{L}^{2}\left(\Omega, L_{s}^{2} L^{2}\right)}
$$

$$
\begin{aligned}
& \lesssim\left(\left\|\stackrel{\circ}{R}^{(q)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)}^{1 / 2}+2^{-q}+C_{q} \cdot \lambda_{q+1}^{-\vartheta}\right) \cdot C_{q} \cdot \lambda_{q+1}^{-\vartheta} \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta}
\end{aligned}
$$

(5) Cutoff Error $\stackrel{\circ}{R}_{\text {cut }}^{(q+1)}$ :

By (3.27), (3.16), standard mollification estimate and $L^{1}$-boundedness of the operator $\mathcal{R}$, we have for all $t \in[s, s+1](s \geq 0)$ that

$$
\begin{aligned}
& \left\|\stackrel{\circ}{R}_{c u t}^{(q+1)}(t)\right\|_{L^{1}} \\
& \lesssim\left(1-\Theta_{q+1}^{2}(t)\right)\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{[s-1, s+1]} L^{1}}+\left\|\Theta_{q+1}^{\prime}\right\|_{0}\left(\left\|\omega_{q+1}^{(p)}(t)\right\|_{L^{1}}+\left\|\omega_{q+1}^{(c)}(t)\right\|_{L^{1}}+\left\|\omega_{q+1}^{(t)}(t)\right\|_{L^{1}}\right)
\end{aligned}
$$

Taking $L^{1}[s, s+1]$-norm in time and expectation and supremum for $s \geq 0$, by (4.3), Hölder's inequality, the previous $\bar{L}^{1}\left(\Omega, L_{s}^{1} W^{1, r}\right)$-estimates of perturbations and (3.16), we have

$$
\begin{aligned}
& \left\|\stackrel{\circ}{R}_{c u t}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \\
& \lesssim \ell_{q+1}^{1 / 2} \cdot \sup _{s \geq 0} \mathbb{E}\left\|\stackrel{\circ}{R}^{(q)}\right\|_{C_{s} L^{1}}+\left\|\Theta_{q+1}^{\prime}\right\|_{0}\left(\left\|\omega_{q+1}^{(p)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} W^{1, r}\right)}+\left\|\omega_{q+1}^{(c)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} W^{1, r}\right)}+\left\|\omega_{q+1}^{(t)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} W^{1, r}\right)}\right) \\
& \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta / 2}+\ell_{q+1}^{-1 / 2} \cdot C_{q} \cdot \lambda_{q+1}^{-\vartheta} \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta / 2}
\end{aligned}
$$

Hence, we finally have

$$
\left\|\stackrel{\circ}{R}^{(q+1)}\right\|_{\bar{L}^{1}\left(\Omega, L_{s}^{1} L^{1}\right)} \leq C_{q} \cdot \lambda_{q+1}^{-\vartheta / 2}+\frac{3 \delta^{2}}{4} .
$$

Choosing $\lambda_{q+1}$ sufficiently large, we get (2.5). The proof of Proposition 2.2 is complete.

## 5. PRoof of Theorem 1.6

In this section we give the proof of Theorem 1.6 by extending the classical result [FJR72, Kat84, FLRT00, LM01, CL22] in PDE case to the stochastic case. For simplicity we consider (1.2) on $[0, T]$ for any $T>0$. In the following we first prove that a $X_{T}^{p, q}$-valued solution is a Leray-Hopf solution. Then we prove pathwise uniqueness. To this end, we introduced the following stochastic linearized N-S system:

$$
\left\{\begin{array}{l}
\mathrm{d} \chi(t)=(-\operatorname{div}(u(t) \otimes \chi(t))+\Delta \chi(t)-\nabla p(t)) \mathrm{d} t+\mathrm{dW}(t)  \tag{5.1}\\
\operatorname{div} \chi=0
\end{array}\right.
$$

Theorem 5.1. Let $u \in X_{T}^{p, q} P$-a.s. with $p, q$ satisfying (1.11) be a solution to system (1.2). Then for any given $\chi_{0} \in L_{\sigma}^{2}$, there exists a probabilistically strong and analytically weak solution $\chi \in L^{2}\left(\Omega ; C_{[0, T]} L^{2}\right)$ $\bigcap L^{2}\left(\Omega ; L_{[0, T]}^{2} H^{1}\right)$ to system (5.1) with initial data $\chi_{0}$ such that $\mathbb{P}-$ a.s.,

$$
\begin{equation*}
\frac{1}{2} \mathbb{E}\|\chi(t)\|_{L^{2}}^{2}+\nu \int_{0}^{t} \mathbb{E}\|\nabla \chi(s)\|_{L^{2}}^{2} d s \leq \frac{1}{2} \mathbb{E}\left\|\chi_{0}\right\|_{L^{2}}^{2}+\frac{t}{2} \operatorname{Tr}\left[G^{*} G\right], t \in[0, T] \tag{5.2}
\end{equation*}
$$

Existence follows by a standard Galerkin method and we omit the details. The uniqueness of probabilistically strong solutions can be shown by a similar but simpler argument as in the following pathwise uniqueness proof. Now, due to the regularity of $u$ and $\chi$, we have $\mathbb{P}-a . s$. that $u \otimes \chi \in L_{[0, T]}^{2} L^{2}$ and hence $\operatorname{div}(u \otimes \chi), \partial_{t} \chi \in L_{[0, T]}^{2} H^{-1}$. Thus we obtain $\chi \in C_{[0, T]} L^{2}$ by [LR15, Theorem 4.2.5].

We also introduce the following backward system on $[0, T] \times \mathbb{T}^{d}$ for the pair $(\Phi, \gamma)$ :

$$
\left\{\begin{array}{l}
-\partial_{t} \Phi-u \cdot \nabla \Phi-\nu \Delta \Phi+\nabla \gamma=F  \tag{5.3}\\
\operatorname{div} \Phi=0 \\
\Phi(T)=0
\end{array}\right.
$$

Theorem 5.2 ([CL22, Theorem A.3]). Let $d \geq 2, \nu>0$ and $0<T<\infty$ be arbitrarily fixed. Let $u \in X_{T}^{p, q}$ with some $p \in[2, \infty]$ and $q \in(2, \infty]$ satisfying (1.11). Then for any given $F \in C_{c}^{\infty}\left([0, T] \times \mathbb{T}^{d}\right)$, the system (5.3) has a weak solution $\Phi \in L_{[0, T]}^{\infty} H^{1} \bigcap L_{[0, T]}^{2} H^{2}$ with some $\nabla \gamma \in L_{[0, T]}^{2} L^{2}$, such that it can be used as a test function in the weak formulation

$$
\int_{0}^{T} \int_{\mathbb{T}^{d}} \eta \cdot\left(\partial_{t} \Phi+u \cdot \nabla \Phi+\nu \Delta \Phi\right) d x d t=0
$$

for $\eta \in L_{[0, T]}^{\infty} L^{2} \bigcap L_{[0, T]}^{2} H^{1}$.

Now, with Theorem 5.1 and 5.2 in hand, we are ready to prove Theorem 1.6.
Prooffor Theorem 1.6. Let $u$ be a solution to (1.2) $)_{\nu}$ as stated in Theorem 1.6, with initial data $u_{0}$. By Theorem 5.1, there exists a probabilistically strong and analytically weak solution $\chi \in L^{2}\left(\Omega ; C_{[0, T]} L^{2}\right)$ $\bigcap L^{2}\left(\Omega ; L_{[0, T]}^{2} H^{1}\right)$ to (5.1) with initial data $\chi_{0}=u_{0}$ and satisfying (5.2). We first show that $\mathbb{P}-$ a.s. , $\chi=u$. Let $\eta:=u-\chi$ and we can write the equations for $\eta$ :

$$
\partial_{t} \eta+u \cdot \nabla \eta-\nu \Delta \eta+\nabla q=0
$$

This equation is understood in the sense of analytic weak formulation as follows:

$$
\begin{equation*}
\int_{0}^{T} \int_{\mathbb{T}^{d}} \eta \cdot\left(\partial_{t} \phi+u \cdot \nabla \phi+\Delta \phi\right) d x d t=0 \tag{5.4}
\end{equation*}
$$

for all divergence-free test function $\phi \in C^{\infty}\left([0, T] \times \mathbb{T}^{d}\right)$ such that $\phi(T)=0$. Now for $u(\omega) \in X_{T}^{p, q}$ and arbitrarily given $F \in C_{c}^{\infty}\left([0, T] \times \mathbb{T}^{d}\right)$ applying Theorem 5.2, we obtain a solution $\Phi(\omega)$ to (5.3) with regularity stated in Theorem 5.2. Let $\phi=\Phi(\omega)$ in (5.4) we obtain

$$
\begin{aligned}
0 & =-\int_{0}^{T} \int_{\mathbb{T}^{d}} \eta \cdot\left(\partial_{t} \Phi+u \cdot \nabla \Phi+\Delta \Phi\right) d x d t \\
& =\int_{0}^{T} \int_{\mathbb{T}^{d}} \eta(t, x) \cdot F(t, x) d x d t
\end{aligned}
$$

which implies $\chi=u$ by continuity of $u$ and $\chi$. (1.12) follows directly from (5.2).
Next, we show pathwise uniqueness. Let $u_{1}, u_{2}$ be two solutions to (1.2) with the same initial data $u_{0}$, as stated in Theorem 1.6. Then by the first part of the proof, they are Leray-Hopf and belong to
$C_{[0, T]} L^{2} \bigcap L_{[0, T]}^{2} H^{1} \bigcap X_{T}^{p, q}, \mathbb{P}-$ a.s.. Now, let $\Upsilon:=u_{1}-u_{2}$. Fix $\mathbb{P}-$ a.s. $\omega$, we have that $\Upsilon(\omega) \in$ $C_{[0, T]} L^{2} \bigcap L_{[0, T]}^{2} H^{1} \bigcap X_{T}^{p, q}$ is a weak solution (in the sense of space-time distribution) to the Stokes system

$$
\left\{\begin{array}{l}
\partial_{t} \Upsilon=\Delta \Upsilon-\nabla p+\operatorname{div} F \\
\operatorname{div} \Upsilon \equiv 0 \\
\Upsilon(0)=0
\end{array}\right.
$$

with $F:=-\left(\Upsilon \otimes u_{1}+u_{2} \otimes \Upsilon\right) \in L_{[0, T]}^{2} L^{2}$. Taking $L^{2}$-inner product on the both side of equation, we obtain

$$
\begin{align*}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\|\Upsilon(t)\|_{L^{2}}^{2}+\nu\|\nabla \Upsilon(t)\|_{L^{2}}^{2} & =\langle\operatorname{div} F(t), \Upsilon(t)\rangle \\
& =\left\langle\Upsilon(t) \otimes u_{1}(t), \nabla \Upsilon(t)\right\rangle_{L^{2}}, t \in[0, T] \tag{5.5}
\end{align*}
$$

- Case1 : $d<q \leq \infty$.

By Hölder's inequality with $\frac{1}{r}+\frac{1}{q}+\frac{1}{2}=1$, the Sobolev embedding $H^{d / q} \hookrightarrow L^{r}$, the interpolation and Young's inequality, we have for all $t \in[0, T]$ that

$$
\begin{aligned}
\left|\left\langle\Upsilon(t) \otimes u_{1}(t), \nabla \Upsilon(t)\right\rangle_{L^{2}}\right| & \leq\left\|u_{1}(t)\right\|_{L^{q}}\|\Upsilon(t)\|_{L^{r}}\|\nabla \Upsilon(t)\|_{L^{2}} \\
& \leq C\left\|u_{1}(t)\right\|_{L^{q}}\|\Upsilon(t)\|_{H^{d / q}}\|\nabla \Upsilon(t)\|_{L^{2}} \\
& \leq C\left\|u_{1}(t)\right\|_{L^{q}}\|\Upsilon(t)\|_{L^{2}}^{1-d / q}\|\nabla \Upsilon(t)\|_{L^{2}}^{1+d / q} \\
& \leq C\left\|u_{1}(t)\right\|_{L^{q}}^{\frac{2}{1-d / p}}\|\Upsilon(t)\|_{L^{2}}^{2}+\frac{1}{2}\|\nabla \Upsilon(t)\|_{L^{2}}^{2} .
\end{aligned}
$$

Using (1.11), we have for all $t \in[0, T]$ that

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\|\Upsilon(t)\|_{L^{2}}^{2}+\|\nabla \Upsilon(t)\|_{L^{2}}^{2} \leq C\left(\left\|u_{1}(t)\right\|_{L^{q}}^{p}+1\right)\|\Upsilon(t)\|_{L^{2}}^{2}
$$

Here, $C>0$ is a deterministic and finite constant. Applying Gronwall's inequality, then by the facts that $\left\|u_{1}\right\|_{L_{[0, T]}^{p} L^{q}}<\infty$ and $\Upsilon(0)=0$ for $\mathbb{P}-a . s . \omega$, we obtain the desired uniqueness.

- Case2 : $d=q<\infty$.

Since $u_{i} \in C_{[0, T]} L^{d}$, we have for any $\varepsilon>0$ there exist $\bar{u}_{1}, u_{\varepsilon}$ such that $u_{1}=\bar{u}_{1}+u_{\varepsilon}$ with $\left\|\bar{u}_{1}\right\|_{C_{[0, T]} L^{d}}<$ $\varepsilon$ and $\left\|u_{\varepsilon}\right\|_{L^{\infty}\left([0, T] \times \mathbb{T}^{d}\right)}<\infty$.

Then for the $\bar{u}_{1}$ part, by Hölder's inequality with $\frac{1}{r}+\frac{1}{d}+\frac{1}{2}=1$, the Sobolev imbedding $H^{1} \hookrightarrow L^{r}$, we have for all $t \in[0, T]$ that

$$
\begin{aligned}
\left|\left\langle\Upsilon(t) \otimes \bar{u}_{1}(t), \nabla \Upsilon(t)\right\rangle_{L^{2}}\right| & \leq\left\|\bar{u}_{1}(t)\right\|_{L^{d}}\|\Upsilon(t)\|_{L^{r}}\|\nabla \Upsilon(t)\|_{L^{2}} \\
& \leq C \varepsilon\|\nabla \Upsilon(t)\|_{L^{2}}^{2},
\end{aligned}
$$

and for the $u_{\varepsilon}$ part, by Hölder's inequality and Young's inequality, we have for all $t \in[0, T]$ that

$$
\begin{aligned}
\left|\left\langle\Upsilon(t) \otimes u_{\varepsilon}(t), \nabla \Upsilon(t)\right\rangle_{L^{2}}\right| & \leq\left\|u_{\varepsilon}\right\|_{L^{\infty}}\|\Upsilon(t)\|_{L^{2}}\|\nabla \Upsilon(t)\|_{L^{2}} \\
& \leq 4\left\|u_{\varepsilon}\right\|_{L^{\infty}}^{2}\|\Upsilon(t)\|_{L^{2}}^{2}+\frac{1}{4}\|\nabla \Upsilon(t)\|_{L^{2}}^{2}
\end{aligned}
$$

Here, $C>0$ is a deterministic and finite constant that only depends on $d$. Then by choosing $\varepsilon=\frac{1}{4 C}$, and together with (5.5), we have for all $t \in[0, T]$ that

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\|\Upsilon(t)\|_{L^{2}}^{2}+\|\nabla \Upsilon(t)\|_{L^{2}}^{2} \leq 8\left\|u_{\varepsilon}\right\|_{L^{\infty}}^{2}\|\Upsilon(t)\|_{L^{2}}^{2}
$$

Then using Gronwall's inequality together with the fact that $\Upsilon(0)=0$ implies the desired uniqueness.

## Appendix A. Stationary Mikado Flows and its div-Potential

In this part we recall the construction of stationary Mikado flows from [CL22]. We point out that the construction is entirely deterministic. Let us begin with the following geometric lemma (cf. [Nas54, Lemma 1] [DS17, Lemma 2.4]). Recall that $S_{+}^{d \times d}$ is the set of all positive definite and symmetric $d \times d$ matrices.

Lemma A. 1 (Geometric Lemma). For any compact subset $\mathcal{K} \subset S_{+}^{d \times d}$, there exist a finite set $\Lambda \subset \mathbb{Z}^{d}$ and smooth functions $\Gamma_{k} \in C^{\infty}(\mathcal{K} ; \mathbb{R})$ such that for all $R \in \mathcal{K}$,

$$
\begin{equation*}
R=\sum_{k \in \Lambda} \Gamma_{k}^{2}(R) \mathbf{e}_{k} \otimes \mathbf{e}_{k} \tag{A.1}
\end{equation*}
$$

where $\mathbf{e}_{k}:=k /|k|$ for each $k \in \Lambda$.
In the following we always choose $\mathcal{K}=B_{1 / 2}$ (Id).
The construction of stationary Mikado flows is as follows. We first choose a point $p_{k} \in(0,1)^{d}$ for each $k \in \Lambda$ such that $p_{k} \neq p_{-k}$ if both $k,-k \in \Lambda$. For each $k \in \Lambda$ we denote a periodic line $l_{k}:=\left\{s k+p_{k} \in\right.$ $\left.\mathbb{T}^{d}: s \geq 0\right\}$ that passes through $p_{k}$ in direction $\mathbf{e}_{k}$.

Now let $\Psi, \Phi \in C_{c}^{\infty}((1 / 2,1) ; \mathbb{R})$ and $c_{k}>0$ be set such that the functions

$$
\begin{aligned}
& \Psi_{k}(x):=\mu^{\frac{d-1}{2}} c_{k} \Psi\left(\mu \operatorname{dist}\left(l_{k}, x\right)\right), x \in \mathbb{T}^{d} \\
& \Phi_{k}(x):=\mu^{\frac{d-1}{2}-2} c_{k} \Phi\left(\mu \operatorname{dist}\left(l_{k}, x\right)\right), x \in \mathbb{T}^{d}
\end{aligned}
$$

satisfying

$$
\begin{align*}
& \left\|\Psi_{k}\right\|_{L^{2}}=1  \tag{A.2}\\
& \Delta \Phi_{k}=\Psi_{k} \text { on } \mathbb{T}^{d}
\end{align*}
$$

Here, the constant $\mu>0$ is the so-called concentration parameter and always set to be sufficiently large for special use. Then the stationary Mikado flows and their div-potential are defined by

$$
\begin{align*}
\mathbb{W}_{k} & :=\Psi_{k} \mathbf{e}_{k}  \tag{A.3}\\
\mathbb{V}_{k} & :=\mathbf{e}_{k} \otimes \nabla \Phi_{k}-\nabla \Phi_{k} \otimes \mathbf{e}_{k} \tag{A.4}
\end{align*}
$$

For convenience, we recall the following equality and bounds from [CL22, Theorem 4.3].

$$
\begin{align*}
& \operatorname{div} \mathbb{W}_{k}=0 \\
& \operatorname{div}\left(\mathbb{W}_{k} \otimes \mathbb{W}_{k}\right)=0  \tag{A.5}\\
& \operatorname{div} \mathbb{V}_{k}=\mathbb{W}_{k} \tag{A.6}
\end{align*}
$$

and

$$
\begin{equation*}
\left\|\mathbb{W}_{k} \otimes \mathbb{W}_{k^{\prime}}\right\|_{L^{\alpha}} \lesssim \mu^{(d-1)-\frac{d}{\alpha}}, \text { for all } k \neq k^{\prime} \tag{A.7}
\end{equation*}
$$

$$
\begin{align*}
& \mu^{-m}\left\|\nabla^{m} \mathbb{W}_{k}\right\|_{L^{\alpha}} \lesssim m \mu^{\frac{d-1}{2}-\frac{d-1}{\alpha}}, \text { for all } k  \tag{A.8}\\
& \mu^{-m}\left\|\nabla^{m} \mathbb{V}_{k}\right\|_{L^{\alpha}} \lesssim m \mu^{-1+\frac{d-1}{2}-\frac{d-1}{\alpha}}, \text { for all } k \tag{A.9}
\end{align*}
$$

for all $m \in \mathbb{N}_{0}$ and all $1 \leq \alpha \leq \infty$.

## Appendix B. The Operators $\mathcal{R}$ and $\mathcal{B}$

In this section we recall anti-divergence operator $\mathcal{R}$ and $\mathcal{B}$ from [CL22, Appendix B].

## - Anti-divergence $\mathcal{R}$

The operator $\mathcal{R}: C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \longrightarrow C^{\infty}\left(\mathbb{T}^{d} ; \mathcal{S}_{0}^{d \times d}\right)$

$$
(\mathcal{R} v)^{i j}:=\mathcal{R}_{k}^{i j} v^{k}:=\frac{2-d}{d-1} \Delta^{-2} \partial_{i} \partial_{j} \partial_{k} v^{k}-\frac{\delta_{i j}}{d-1} \Delta^{-1} \partial_{k} v^{k}+\Delta^{-1} \partial_{i} \delta_{j k} v^{k}+\Delta^{-1} \partial_{j} \delta_{i k} v^{k}
$$

Here, $\mathcal{S}_{0}^{d \times d}$ denotes the set of $d \times d$ trace-free symmetric matrices. A direct computation (see [CL22, Appendix B.2]) gives that

$$
\begin{align*}
\operatorname{tr}(\mathcal{R} v) & =0 \\
\operatorname{div}(\mathcal{R} v) & =v-f_{\mathbb{T}^{d}} v \\
\mathcal{R} \Delta v & =\nabla v+\nabla^{\mathrm{T}} v \tag{B.1}
\end{align*}
$$

It can be shown that $\mathcal{R}$ is $L^{p}$-bounded for $1 \leq p \leq \infty$ ( see [CL22, Theorem B.3]) and

$$
\begin{equation*}
\|\mathcal{R} f(\sigma \cdot)\|_{L^{p}} \lesssim \sigma^{-1}\|f\|_{L^{p}} \tag{B.2}
\end{equation*}
$$

for $f$ with mean zero.

## - Bilinear anti-divergence $\mathcal{B}$

The operator $\mathcal{B}: C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right) \times C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}^{d \times d}\right) \longrightarrow C^{\infty}\left(\mathbb{T}^{d} ; \mathcal{S}_{0}^{d \times d}\right)$ is defined by

$$
(\mathcal{B}(v, M))_{i j}:=v^{l} \mathcal{R}_{k}^{i j} M_{l}^{k}+\mathcal{R}\left(\partial_{i} v^{l} \mathcal{R}_{k}^{i j} M_{l}^{k}\right)
$$

Let $C_{0}^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}^{d \times d}\right)$ be the set of periodic smooth matrix-valued functions with zero mean. Then for $v \in$ $C^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}^{d}\right)$ and $M \in C_{0}^{\infty}\left(\mathbb{T}^{d} ; \mathbb{R}^{d \times d}\right)$, we have (see [CL22, Theorem B.4])

$$
\begin{align*}
\operatorname{div}(\mathcal{B}(v, M)) & =v M-f_{\mathbb{T}^{d}} v M \\
\|(\mathcal{B}(v, M))\|_{L^{p}} & \lesssim\|v\|_{1}\|M\|_{L^{p}}, \forall 1 \leq p \leq \infty \tag{B.3}
\end{align*}
$$

## Appendix C. $C^{N}$-Estimates for Compositions

We recall the following lemma from [BLIS15, Proposition C.1].
Lemma C.1. Let $\Psi: \Omega \longrightarrow \mathbb{R}$ and $f: \mathbb{R}^{n} \longrightarrow \Omega$ be two smooth functions, with $\Omega \subset \mathbb{R}^{m}$. Then, for every $N \in \mathbb{N}$, there is a constant $C=C(n, m, N)>0$ such that

$$
\begin{equation*}
[\Psi \circ f]_{N} \leq C\left([\Psi]_{1}[f]_{N}+\|\mathrm{D} \Psi\|_{N-1}\|f\|_{0}^{N-1}[f]_{N}\right) \tag{C.1}
\end{equation*}
$$

$$
\begin{equation*}
[\Psi \circ f]_{N} \leq C\left([\Psi]_{1}[f]_{N}+\|\mathrm{D} \Psi\|_{N-1}[f]_{1}^{N}\right) \tag{C.2}
\end{equation*}
$$

## References

[ABC21] Dallas Albritton, Elia Brué, and Maria Colombo. Non-uniqueness of Leray solutions of the forced Navier-Stokes equations. arXiv:2112.03116v1, 122021.
[BCV18] Tristan Buckmaster, Maria Colombo, and Vlad Vicol. Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1. arXiv:1809.00600, 2018.
[BDSV19] Tristan Buckmaster, Camillo De Lellis, László Székelyhidi, Jr., and Vlad Vicol. Onsager’s conjecture for admissible weak solutions. Comm. Pure Appl. Math., 72(2):229-274, 2019.
[Bes99] H. Bessaih. Martingale solutions for stochastic Euler equations. Stochastic Anal. Appl., 17(5):713-725, 1999.
[BF99] H. Bessaih and F. Flandoli. 2-D Euler equation perturbed by noise. NoDEA Nonlinear Differential Equations Appl., 6(1):3554, 1999.
[BFM16] Z. Brzézniak, F. Flandoli, and M. Maurelli. Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity. Arch. Rational Mech. Anal., 221(1):107-142, 2016.
[BLIS15] Tristan Buckmaster, Camillo De Lellis, Philip Isett, and László Székelyhidi, Jr. Anomalous dissipation for 1/5-Hölder Euler flows. Ann. of Math., 182(1):127-172, 2015.
[BMS21] Jan Burczak, Stefano Modena, and László Székelyhidi, Jr. Non uniqueness of power-law flows. Communications in Mathematical Physics, 388:199-243, 2021.
[BP01] Z. Brzézniak and S. Peszat. Stochastic two dimensional Euler equations. Ann. Probab., 29(4):1796-1832, 2001.
[BV19] Tristan Buckmaster and Vlad Vicol. Convex integration and phenomenologies in turbulence. arXiv:1901.09023 (math), 01 2019.
[BV21] Tristan Buckmaster and Vlad Vicol. Convex integration constructions in hydrodynamics. Bull. Amer. Math. Soc. (N.S.), 58(1):1-44, 2021.
[CFH19] D. Crisan, F. Flandoli, and D. D. Holm. Solution properties of a 3D stochastic Euler fluid equation. J. Nonlinear Sci., 29(3):813-870, 2019.
[CL21] Alexey Cheskidov and Xiaoyutao Luo. $L^{2}$-critical nonuniqueness for the 2D Navier-Stokes equations. arXiv:2105.12117v1, 052021.
[CL22] Alexey Cheskidov and Xiaoyutao Luo. Sharp nonuniqueness for the Navier-Stokes equations. Invent. Math., 229:987-1054, 052022.
[Da 04] Giuseppe Da Prato. Kolmogorov Equations for Stochastic PDEs. Springer Basel AG, 2004.
[DD03] Giuseppe Da Prato and A. Debussche. Ergodicity for 3D stochastic Navier-Stokes equations. J. Math. Pures Appl., 82:877947, 2003.
[DS09] Camillo De Lellis and László Székelyhidi, Jr. The Euler equations as a differential inclusion. Ann. of Math. (2), 170(3):14171436, 2009.
[DS10] Camillo De Lellis and László Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. Arch. Rational Mech. Anal., 195(1):225-260, 2010.
[DS13] Camillo De Lellis and László Székelyhidi, Jr. Dissipative continuous Euler flows. Invent. Math., 193(2):377-407, 2013.
[DS14] Camillo De Lellis and László Székelyhidi, Jr. Dissipative Euler flows and Onsager's conjecture. J. Eur. Math. Soc. (JEMS), 16(7):1467-1505, 2014.
[DS17] Sara Daneri and László Székelyhidi, Jr. Non-uniqueness and h-principle for hölder-continuous weak solutions of euler equations. Arch. Rational Mech. Anal., 224(2):471-514, 2017.
[FGP10] Franco Flandoli, M. Gubinelli, and E. Priola. Well posedness of the transport equation by stochastic perturbation. Invent. Math., 180:1-53, 2010.
[FJR72] E. B. Fabes, B. F. Jones, and N. M. Rivière. The initial value problem for the Navier-Stokes equations with data in $L^{p}$. Arch. Rational Mech. Anal., 45:222-240, 1972.
[FL21] Franco Flandoli and Dejun Luo. High mode transport noise improves vorticity blow-up control in 3D Navier-Stokes equations. Probability Theory and Related Fields, 180(1-2):309-363, 032021.
[Fla08] Franco Flandoli. An introduction to 3D stochastic fluid dynamics. In SPDE in hydrodynamic: recent progress and prospects, volume 1942 of Lecture Notes in Math., pages 51-150. Springer, Berlin, 2008.
[FLRT00] Giulia Furioli, Pierre G. Lemarié-Rieusset, and Elide Terraneo. Unicité dans $L^{3}\left(\mathbb{R}^{3}\right)$ et d'autres espaces fonctionnels limites pour Navier-Stokes. Rev. Mat. Iberoamericana, 16(3):605-667, 2000.
[FR08] Franco Flandoli and Marco Romito. Markov selections for the 3D stochastic Navier-Stokes equations. Probability Theory and Related Fields, 140(3-4):407-458, 2008.
[GHV14] Nathan E. Glatt-Holtz and Vlad Vicol. Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. The Annals of Probability, 42(1):80-145, 012014.
[Hop51] E. Hopf. Über die anfangswertaufgabe für die hydrodynamischen grundgleichungen. erhard schmidt zu seinem 75. geburtstag gewidmet. Math. Nachr., 4(1-6):213-231, 1951.
[HZZ19] Martina Hofmanová, Rongchan Zhu, and Xiangchan Zhu. Non-uniqueness in law of stochastic 3D Navier-Stokes equations. arXiv:1912.11841v2, 122019.
[HZZ21a] Martina Hofmanová, Rongchan Zhu, and Xiangchan Zhu. Global-in-time probabilistically strong and Markov solutions to stochastic 3D Navier-Stokes equations: existence and non-uniqueness. arXiv:2104.09889v1, 042021.
[HZZ21b] Martina Hofmanová, Rongchan Zhu, and Xiangchan Zhu. On ill- and well-posedness of dissipative martingale solutions to stochastic 3d euler equations. Communications on Pure and Applied Mathematics, page To appear, 92021.
[Ise18] Philip Isett. A proof of Onsager's conjecture. Ann. of Math. (2), 188(3):871-963, 2018.
[Kat84] Tosio Kato. Strong $L^{p}$-solutions of the Navier-Stokes equation in $\mathbf{R}^{\mathbf{m}}$, with applications to weak solutions. Math. Z., 187(4):471-480, 1984.
[Kim01] J. U. Kim. Existence of a local smooth solution in probability to the stochastic Euler equations in $\mathbb{R}^{3}$. J. Funct. Anal., 256(11):3660-3687, 2001.
[Ler34] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math, 63(1):193-248, 1934.
[LM01] P.-L. Lions and N. Masmoudi. Uniqueness of mild solutions of the Navier-Stokes system in $L^{N}$. Comm. Partial Differential Equations, 26(11-12):2211-2226, 2001.
[LR15] Wei Liu and Michael Röckner. Stochastic Partial Differential Equations: An Introduction. Universtext. Springer Science and Business Media, 2015.
[MS18] Stefano Modena and László Székelyhidi, Jr. Non-uniqueness for the transport equation with Sobolev vector fields. Ann. PDE, 4(2):Art.18, 38, 2018.
[MV00] R. Mikulevicius and G. Valiukevicius. On stochastic Euler equation in $\mathbb{R}^{d}$. Electron. J. Probab., 2(6):1-20, 2000.
[Nas54] John Nash. $C^{1}$ isometric imbeddings. Ann. of Math. (2), 60:383-396, 1954.
[RZZ14] Michael Röckner, Rongchan Zhu, and Xiangchan Zhu. Local existence and non-explosion of solutions for stochastic frational partial differential equations driven by multiplicative noise. Stochastic Process. Appl., 124(5):1974-2002, 2014.
(W. Chen) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email address: chenweiquan@amss.ac.cn
(Z. Dong) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email address: dzhao@amt.ac.cn
(X. Zhu) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Email address: zhuxiangchan@126.com


[^0]:    2010 Mathematics Subject Classification. 60H15; 35R60; 35Q30.
    Key words and phrases. stochastic Navier-Stokes equations, stochastic Euler equations, probabilistically strong solutions, sharp non-uniqueness, convex integration.

    Research supported by National Key R\&D Program of China (No. 2020YFA0712700) and the NSFC (No. 11931004, 12090014, 12288201) and the support by key Lab of Random Complex Structures and Data Science, Youth Innovation Promotion Association (2020003), Chinese Academy of Science. The financial support by the DFG through the CRC 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is greatly acknowledged.

